ASSIGNMENT 1 — DYNAMICAL SYSTEMS AND ODEs

For all questions below, show your work (e.g., use an equation editor), as well as providing all programming code and plots in the report.

1. $dy/dt = y - t^2 + 1$; Initial conditions (IC): $y_0 = 0.5, t_0 = 0$; $h = 0.5$; (6 marks)

   a. Solve this equation for 3 time steps using Euler integration by hand and show with a table (e.g., $n|t_n|y_n|f_n|h \cdot f_n|y_{n+1}$). (3 marks)

   b. Solve this equation for 3 time steps using RK4 integration by hand and show with a table (e.g., $n|t_n|y_n|k1t|k1y|...|k4y|y_{n+1}$). (3 marks)

2. The following set of coupled differential equation are known as the Lotka-Volterra equations, which can be used to model predator-prey relationships in nature:

$$dx/dt = \alpha x - \beta xy$$

$$dy/dt = \delta xy - \gamma y$$

x and y represent the population of the prey and predator, respectively. Decide on values for α, β, δ, and γ, length of time, step size, initial conditions, and what animal your prey and predator are. (13 marks | 18 marks)

   a. Find the solution to this set of equations by programming both Euler integration and Runge-Kutta integration schemes (**i.e., do not use the built in integrator**). Remember to include the code for both Euler and RK4 in your report. (10 marks)

   b. Compare the Euler and Runge-Kutta algorithms when plotting the states over time. Include a title, x-label, y-label, and legend in your plot. (2 marks)

   c. Describe what you observe in terms of the predator-prey relationship over time. (1 mark)

   d. Describe what each term in the equation represents. **(Graduates only)** (1 mark)

   e. What are potential limitations of the model? **(Graduates only)** (1 mark)

   f. Increase the value of α. What happens and why? **(Graduates only)** (1 mark)

   g. Set ($\alpha, \beta, \delta, \gamma$) = (0.2, 0.2, 0.2, 0.2). What happens and why? **(Graduates only)** (1 mark)

   h. Set ($\alpha, \beta, \delta, \gamma$) = (0.2, 0.2, 0.02, 0.0). What happens and why? **(Graduates only)** (1 mark)

3. For the following 2nd order differential equation: $3\ddot{x} - 4\dot{x} + x = 0$ (3 marks)

   a. Convert into a system of 1st order, ODEs. (2 marks)

   b. Express in matrix form. (1 mark)

4. For the following 4rth order differential equation: $2x'''' - 4x'' - cos(t)x' + 9x = t^2$ **Graduates only** (2 marks)

   a. Convert into a system of 1st order, ODEs. (1 mark)

   b. Express in matrix form. (1 mark).

5. For the following mass-damper-spring differential equation: $m\ddot{x} = -b\dot{x} - kx + mg$ (9 marks | 12 marks)

   a. Convert into a system of 1st order, ODEs. (1 mark)

   b. Decide on values for k, b, and m (other than zero), length of time, step size, initial conditions. g = 9.81. (1 mark)

   c. Solve using a built-in numerical integrator (e.g., odeint in Python). (5 marks)

   d. Plot the states over time. Include a title, x-label, y-label, and legend in your plot. (1 mark)

   e. Plot the state-space plot and describe what you observe. Include a title, x-label, y-label, and legend in your plot. (1 mark)

   f. What is the undamped angular frequency, $w_0 = \sqrt{\frac{k}{m}}$, of your system? **(Graduates only)**. (1 mark)

   g. Calculate, $\zeta = \frac{b}{2\sqrt{mk}}$, to find out whether your system is overdamped ($\zeta > 1$), critically damped ($\zeta = 1$), or underdamped ($\zeta < 1$). **(Graduates only)** (1 mark)

   h. Change the b in your system such that it becomes critically damped and replot your states over time and state-space plots. What do you notice? **(Graduates only)**. (1 mark)