ASSIGNMENT 1 — DYNAMICAL SYSTEMS AND ODEs

For all questions below, show your work (e.g., use an equation editor), as well as providing all programming code and plots in the report.

1. $dy/dt = y - t^2 + 1$; Initial conditions (IC): $y_0 = 0.5, t_0 = 0$; $h = 0.5$; (6 marks)

   a. Solve this equation for 3 time steps using Euler integration by hand and show with a table (e.g., $n|t_n|y_n|f_n|h \cdot f_n|y_{n+1}$). (3 marks)

   b. Solve this equation for 3 time steps using RK4 integration by hand and show with a table (e.g., $n|t_n|y_n|k1t|k1y|...|k4y|y_{n+1}$). (3 marks)

2. The following set of coupled differential equation are known as the Lotka-Volterra equations, which can be used to model predator-prey relationships in nature:

$$dx/dt = \alpha x - \beta xy$$

$$dy/dt = \delta xy - \gamma y$$

x and y represent the population of the prey and predator, respectively. Use $\alpha = 1.0$, $\beta = 0.5$, $\delta = 0.5$, and $\gamma = 1.0$, length of time (60 days), step size (0.001 days), and initial conditions for your prey ($x_0 = 20$) and predator ($y_0 = 10$). (13 marks | 18 marks)

   a. Find the solution to this set of equations by programming both Euler integration and Runge-Kutta integration schemes (**i.e., do not use the built in integrator**). Remember to include the code for both Euler and RK4 in your report. (10 marks)

   b. Compare the Euler and Runge-Kutta algorithms when plotting the states over time. Include a title, x-label, y-label, and legend in your plot. (2 marks)

   c. Describe what you observe in terms of the predator-prey relationship over time. (1 mark)

   d. Describe what each term in the equation represents. **(Graduates only)** (1 mark)

   e. What are potential limitations of the model? **(Graduates only)** (1 mark)

   f. Increase the value of α. What happens and why? **(Graduates only)** (1 mark)

   g. Set ($\alpha, \beta, \delta, \gamma$) = (0.2, 0.2, 0.2, 0.2). What happens and why? **(Graduates only)** (1 mark)

   h. Set ($\alpha, \beta, \delta, \gamma$) = (0.2, 0.2, 0.02, 0.0). What happens and why? **(Graduates only)** (1 mark)

3. For the following 2nd order differential equation: $3\ddot{x} - 4\dot{x} + x = 0$ (3 marks)

   a. Convert into a system of 1st order, ODEs. (2 marks)

   b. Express in matrix form. (1 mark)

4. For the following 4rth order differential equation: $2x'''' - 4x'' - cos(t)x' + 9x = t^2$
Graduates only (2 marks)

   a. Convert into a system of 1st order, ODEs. (1 mark)

   b. Express in matrix form. (1 mark).

5. For the following mass-damper-spring differential equation: $m\ddot{x} = -b\dot{x} - kx + mg$. Start off by using $k = 1.5$, $b = 0.5$, and $m = 1.5$, length of time of 50 seconds, step size of 0.01 seconds, and initial conditions for position and velocity ($x = 0.0, \dot{x} = 0.0$). (9 marks | 12 marks)

   a. Convert into a system of 1st order, ODEs. (2 mark)

   b. Solve using a built-in numerical integrator (e.g., odeint in Python). (5 marks)

   c. Plot the states over time. Include a title, x-label, y-label, and legend in your plot. (1 mark)

   d. Plot the state-space plot and describe what you observe. Include a title, x-label, y-label, and legend in your plot. (1 mark)

   e. What is the undamped angular frequency, $w_0 = \sqrt{\frac{k}{m}}$, of your system? (**Graduates only**). (1 mark)

   f. Calculate, $\zeta = \frac{b}{2\sqrt{mk}}$, to find out whether your system is overdamped ($\zeta > 1$), critically damped ($\zeta = 1$), or underdamped ($\zeta < 1$). **(Graduates only)** (1 mark)

   g. Change the b in your system such that it becomes critically damped and replot your states over time and state-space plots. What do you notice? (**Graduates only**). (1 mark)