Monsters in the hollow: Counting Naiki braid patterns using de Bruijn's Monster Theorem

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ABSTRACT
The Japanese braids known as Naiki, which are distinguished by their hollow interior, have a simple structure shared by many other fiber arts and crafts. The way in which this structure forms a cylindrical braid imposes a particular set of symmetries on the final product. This paper uses enumerative combinatorics, including de Bruijn's Monster Theorem, to count the number of two-color Naiki braids under equivalence by this natural set of symmetries.

KEYWORDS
braiding, Japanese crafts, fiber arts, weaving, counting, patterns, enumerative combinatorics, generating functions, Monster Theorem, symmetry

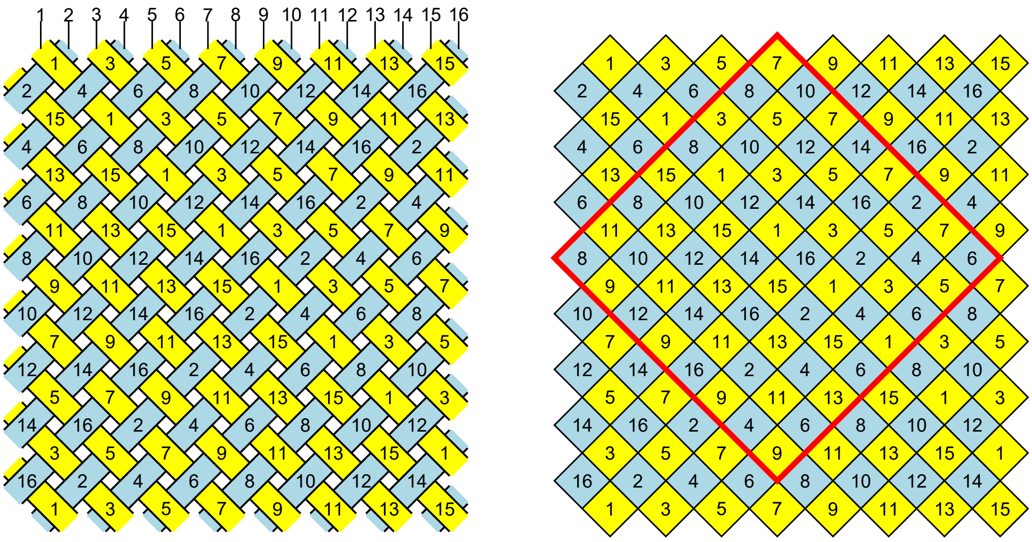
1. Introduction

Kumihimo, the Japanese word for braid, is used in Western countries to describe a collection of styles of Japanese braiding, usually using relatively simple looms to braid between 4 and 32 strands or bunches of fiber. This paper focuses on Naiki, a traditional braid usually made with 16 strands which has a hollow interior which can either be filled with a core or squeezed flat. Some examples are shown in Figure 1. The meaning of Naiki is not clear; according to Rebecca Combs (Combs, 2016, p. 73) it is probably named after the Naikidai, a braiding machine from the Edo period (1603–1868) in Japan. It is not known how the Naikidai got its name. The same braid with 8 strands is known as Edo Yatsu; Edo from the former name for Tokyo and Yatsu meaning eight (Combs, 2016, p. 15). Instructions for the braid may be found in Combs (2016); we will be focusing on the final result.

The structure of Naiki is a simple over-and-under interlacement, as shown in Figure 2a. It is the same structure as that of plain weave, with the bias oriented along the axis of the braid. It is also the same as the structure produced by the maypole dance known as Grand Chain (Tian, 2019). The numbering in the figure corresponds to the numbering of the strands as they are placed around the disk, as shown in the row of numbers above the braid. Odd-numbered threads become oriented in the S (lower right to upper left, like a "backslash") direction within the braid, while even-numbered threads become oriented in the Z (lower left to upper right, like a "forward slash") direction. In Figure 2b, we abstract the structure into a grid. Each square of the grid in the figure corresponds to a crossing of an odd-numbered thread with an

Figure 1. Examples of Naiki braids. Braiding and photography by Rosalie Neilson.

even-numbered thread, with the odd and even threads visible in alternate squares. We will refer to these as odd squares and even squares, for short.



(a) (b)

Figure 2. (a) The structure of Naiki. Strands going off one side are assumed to wrap around to the other. (b) An abstracted version of the structure, with a fundamental region marked.

This paper will focus on the situation with 16 strands, each having one of two colors. We will refer to the "spots" of the pattern as being the grid squares showing the color used in fewer strands, when that makes sense. The number of spots will be counted in the 8-by-8 fundamental region shown in Figure 2b, and will be four times the corresponding number of strands. (If there are equal numbers of strands of each color we will refer to 32 spots without identifying which color they correspond to.)

In 2011, Rosalie Neilson published *The Twenty-Four Interlacements of Edo Yatsu Gumi* (Neilson, 2011). This book gave a complete inventory of the Edo Yatsu braids with 8 strands, each of one of two colors, up to equivalence of color pattern under rotations and translations. That work was done by exhaustive search; the goal of this paper is to do a similar inventory of 16-strand Naiki patterns, but guided by theorems in enumerative combinatorics. Similar work was done by the author for the Kongō Gumi kumihimo technique

2. The Group of Permutations

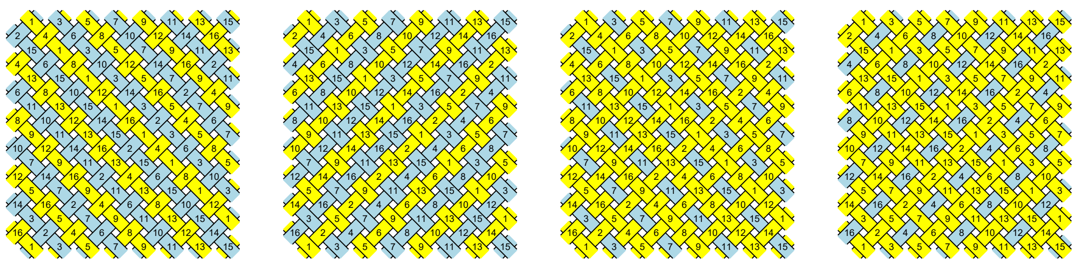
In the author's previous work on Kongō Gumi kumihimo (Holden, 2022a), the group of permutations of threads which left the pattern invariant was just the dihedral group. In the case of Naiki braids, the group is more complicated due to the independence of the two sets of threads and the lack of inherent chirality in the braid.

The group of symmetries of a Naiki braid, like that of the Kongō Gumi braid, is a three-dimensional line group, which can be thought of as a wallpaper group wrapped around a cylinder. We will consider the braid to be divided into grid squares as above. Recall that the grid squares alternate between threads with an S-orientation and threads with a Z-orientation in a checkerboard pattern. *A priori*, we need to decide whether or not we will consider two braids to be equivalent if their color patterns are equivalent, even if the thread orientations are not equivalent under the same symmetry. However, we will show that with a single exception, this distinction does not occur.

Suppose a symmetry fixes the color pattern of a braid but not the thread orientation. A symmetry must either fix the parity of each grid square, and thus fix the thread orientations, or else interchange the even squares with the odd ones. Without loss of generality, suppose thread 1 has color A and crosses over threads 2, 6, 10, 14, as in Figure 2a. If another pattern has the same colors but opposite thread orientation, then threads 2, 6, 10, 14 must have color A. Similarly, thread 3 crosses over threads 4, 8, 12, 16 in the first pattern, so those threads must have the same color in the second pattern, and likewise with thread 2 crossing over threads 3, 7, 11, 15 and thread 4 crossing over threads 1, 5, 9, 13.

Now there are four possibilities. If all four groups of threads are the same color, then every symmetry fixes the pattern however you look at it. Likewise, if the even threads are one color and the odd threads are another color, you get a checked pattern (as shown in Figure 2a) which every symmetry fixes up to a color reversal, regardless of how you consider the thread orientations. If the even threads and the odd threads both alternate colors, then you get stripes which every symmetry fixes up to a color reversal and/or glide plane reflection, as in Figures 3a and 3b.

Finally, there is the interesting case, where three of the four groups of threads are the same color and the fourth is different. In this case, you get a regularly spaced grid of spots, all of which have the same thread orientation. This thread orientation could be in either direction, as shown in Figures 3c and 3d, so this is the only case where two patterns are equivalent under color symmetry but not thread orientation symmetry. It will prove to be convenient to consider only symmetries which preserve thread orientation, and therefore to count these two patterns as distinct but equivalent under a glide plane reflection. (Note that the same proof shows that aside from the first two cases just mentioned, the set of symmetries that reverse the chirality of a



(a) (b) (c) (d)

Figure 3. (a) A stripe pattern in the S direction. (b) A stripe pattern in the Z direction. (c) Regularly spaced spots in the S direction. (d) Regularly spaced spots in the Z direction.

pattern is exactly the set of symmetries that swap the even and odd sets of threads.)

If every strand is the same color and n is divisible by 4, the symmetries which preserve thread orientation are now generated by the $n/2$ rotations around the axis, the $n/2$ distinct translations along the axis, a 180° rotation around a line perpendicular to the axis, and a glide plane reflection, parallel to the axis. (P$\overline{(n/2)}$2c in Hermann-Mauguin crystallographic notation (Radaelli, 2011, Secs. 8.1 and 10.2).)

Let's consider what each of these symmetries does to the set of threads for $n = 16$. The rotations rotate the braid by a multiple of two threads around its axis, so that even-numbered threads stay even-numbered and odd-numbered threads stay odd-numbered, as shown in Figure 4a. These rotations are generated by the permutation $(1, 3, 5, 7, 9, 11, 13, 15)(2, 4, 6, 8, 10, 12, 14, 16)$. A minimal translation followed by a rotation by 2 threads, as shown in Figure 4b, gives a helical transformation which has the effect of leaving the odd threads unchanged and permuting the even threads $(2, 6, 10, 14)(4, 8, 12, 16)$. The rotation perpendicular to the axis, shown in Figure 5a, reverses the order of both sets of threads, thus giving the permutation $(1, 15)(3, 13)(5, 11)(7, 9)(2, 16)(4, 14)(6, 12)(8, 10)$. And the glide plane reflection swaps the even and odd threads and also translates, as shown in Figure 5b, with permutation $(1, 14, 3, 12, 5, 10, 7, 8, 9, 6, 11, 4, 13, 2, 15, 16)$. These permutations generate a permutation group of degree 16 with 128 elements.

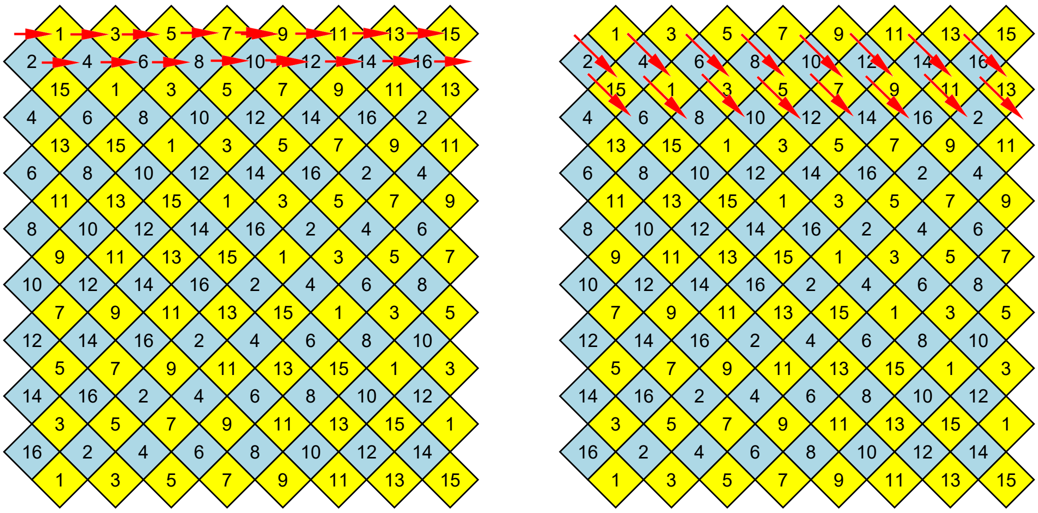
3. A Version of de Bruijn's Theorem

We can use the version of de Bruijn's Theorem from (Holden, 2022a) to count the number of Naiki patterns with a given number of threads of each color, up to the equivalences above and also up to changing the colors:

Theorem 3.1 (de Bruijn (de Bruijn, 1959, Special case of Thm. 1))**.** *If an object has m locations which can each be colored with one of q different colors, and a group G of symmetries with cycle index $H_m(x_1, \ldots, x_m)$, then the number of non-equivalent ways to color the object with the first color in m_1 locations, the second color in m_2 locations, and so on, is the coefficient of $w_1^{m_1} w_2^{m_2} \ldots$ in*

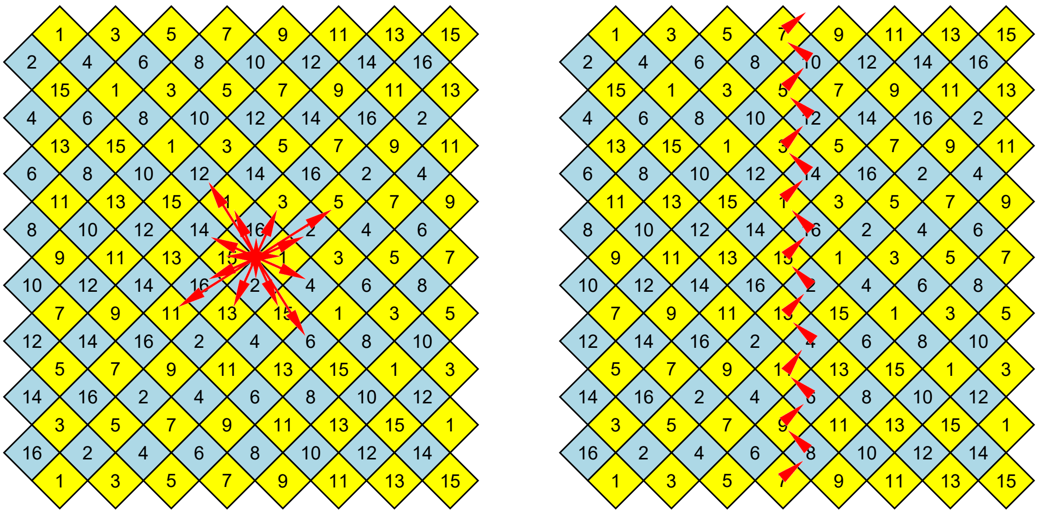
$$H_m\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}\right) S_q\left(\eta_1, \ldots \eta_q\right)$$

where $S_q(x_1, \cdots, x_q)$ is the cycle index of all ways to permute q colors, and η_k is the



(a) (b)

Figure 4. (a) A rotation of the braid around its axis. (b) A helical transformation of the braid fixing the even-numbered threads.



(a) (b)

Figure 5. (a) A rotation of the braid around a line perpendicular to its axis. (b) A glide plane reflection of the braid.

power series expansion in x of

$$e^{k(z_k x + z_{2k} x^2 + \dots)}$$

with x^j replaced by w_j^t for each j, evaluated at $z_1 = z_2 = \cdots = z_m = 0$.

If G is the group of symmetries described above, then the cycle index of G can be

Table 1. Inventory of patterns given by de Bruijn's Theorem.

Spots:	0	4	8	12	16	20	24	28	32	Total
Patterns:	1	1	5	11	30	52	95	120	91	406

calculated to be

$$H_{16}(x_1,\ldots,x_{16}) = \frac{1}{128}x_1^{16}+\frac{1}{64}x_1^8x_2^4+\frac{1}{32}x_1^8x_4^2+\frac{1}{8}x_1^4x_2^6+\frac{17}{128}x_2^8+\frac{1}{32}x_2^4x_4^2+\frac{1}{32}x_4^4+\frac{1}{8}x_8^2+\frac{1}{2}x_{16}.$$

With $q = 2$ colors, the cycle index S_q is (Gilbert and Riordan, 1961)

$$S_2(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2\right).$$

Then

$$H_{16}\left(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_{16}}\right) S_2(\eta_1,\eta_2)$$

$$= \frac{1}{128}\frac{\partial^{16}S_2}{\partial {z_1}^{16}} + \frac{1}{64}\frac{\partial^{12}S_2}{\partial {z_2}^4\partial {z_1}^8} + \frac{1}{32}\frac{\partial^{10}S_2}{\partial {z_4}^2\partial {z_1}^8} + \frac{1}{8}\frac{\partial^{10}S_2}{\partial {z_2}^6\partial {z_1}^4} + \frac{17}{128}\frac{\partial^8 S_2}{\partial {z_2}^8}$$

$$+ \frac{1}{32}\frac{\partial^6 S_2}{\partial {z_4}^2\partial {z_2}^4} + \frac{1}{32}\frac{\partial^4 S_2}{\partial {z_4}^4} + \frac{1}{8}\frac{\partial^2 S_2}{\partial {z_8}^2} + \frac{1}{2}\frac{\partial S_2}{\partial z_{16}}$$

which evaluated at $z_1 = z_2 = \cdots = z_{16} = 0$ gives

$$w_0w_{16}+w_1w_{15}+5w_2w_{14}+11w_3w_{13}+30w_4w_{12}+52w_5w_{11}+95w_6w_{10}+120w_7w_9+91w_8^2$$

Thus there are a total of 406 patterns using one or two colors, with colors distributed as in Table 1. Of these, 404 have mirror images which are not otherwise equivalent.

4. The Monster Theorem

At this point it seemed logical to further break down the classification according to how many even-numbered threads were of each color and how many odd-numbered threads had each color. Looking for further generalizations of de Bruijn's Theorem, I found:

Theorem 4.1 (de Bruijn's Monster Theorem (de Bruijn, 1971, Thm. 8.2, somewhat simplified))**.** *Suppose an object has m locations partitioned into k sets $D_1,\ldots,D_k$, and each location can each be colored with one of q different colors, with two colorings being considered equivalent if there are related by a permutation h in a group H of degree q, and the locations have a group G of symmetries such that g takes D_i to itself for all i. Let $D_{i,g}^{(j)}$, $j = 1,\ldots,J$, be the orbits of the action of g restricted to D_i, and let $R_h^{(\ell)}$, $\ell = 1,\ldots,L$, be the orbits of the action of h on the colors. Then the number of* *non-equivalent ways to color the object with the first color in $m_{1,i}$ locations of D_i, the second color in $m_{2,i}$ locations of D_i, and so on, is the coefficient of*

$$w_{1,r_1,m_{1,1}} \cdots w_{k,r_1,m_{1,k}} w_{1,r_2,m_{2,1}} \cdots w_{k,r_2,m_{2,k}} \cdots w_{1,r_q,m_{q,1}} \cdots w_{k,r_q,m_{q,k}},$$

where the r_ℓ are chosen subject to the constraint that $r_\ell = r_{\ell'}$ if there is an h in H such that h takes ℓ to ℓ', in

$$|G|^{-1}\,|H|^{-1} \sum_{g\in G} \sum_{h\in H} \prod_{i=1}^{k} \left(\prod_{j=1}^{J} \frac{\partial}{\partial z_{\left|D_{i,g}^{(j)}\right|}} \right) \left(\prod_{\ell=1}^{L} \eta_{\left|R_h^{(\ell)}\right|,i,j} \right),$$

where $\eta_{s,i,j,\ell}$ is the power series expansion in x of

$$e^{s(z_s x + z_{2s} x^2 + \dots)}$$

with x^t replaced by $w_{i,r_\ell,t}^s$ for each t, evaluated at $z_1 = z_2 = \cdots = z_m = 0$.

(In an abuse of notation, we are using ℓ in the theorem to represent both a color and an orbit of colors. Since two colors in the same orbit must have the same r_ℓ, this should be harmless.)

This is obviously a very powerful theorem. It does have a few limitations for our purposes, however. First, note that we would like D_1 to be the odd threads and D_2 to be the even threads, but the theorem does not cover the permutations which swap these sets. Since we have determined exactly how those symmetries behave, this is a surmountable problem.

The other limitation comes from the fact that colors in the same orbit of H must have the same variables in the generating function. Therefore, it will not be possible to separate the set of patterns with a odd threads of color A, b odd threads of color B, c even threads of color A, and d odd threads of color B, from the set with a odd threads of color A, b odd threads of color B, d even threads of color A, and c odd threads of color B, as both will correspond to the monomial $w_{1,1,a}w_{1,1,b}w_{2,1,a}w_{2,1,b}$. (Indeed, it is difficult to see how any generating function could make this distinction while allowing colors to be exchanged.)

Again, a close look at the particular situation will rescue us. If $a = b = c = d$, then the two sets above coincide. If not, it is not possible for a member of one set to be equivalent to a member of the other, and there is an equivalence-respecting bijection between the sets induced by exchanging the colors of either the even or odd threads, but not both. (Since exchanging colors on both parities gives a pattern equivalent to the original, it does not matter which side we exchange.) Therefore, we will merely divide the coefficient by 2 when a, b, c, d, are not all equal.

Applying the theorem with $m = 16$, $k = 2$, $q = 2$, G, D_1, D_2 as described above, and H being all permutations of 2 colors (and suppressing the values of r_ℓ which are all equal to 1), we get the polynomial

$$2w_{1,0}w_{1,8}w_{2,0}w_{2,8} + 2w_{1,0}w_{1,8}w_{2,1}w_{2,7} + 8w_{1,0}w_{1,8}w_{2,2}w_{2,6} + 10w_{1,0}w_{1,8}w_{2,3}w_{2,5}$$

$$+ 8w_{1,0}w_{1,8}w_{2,4}^2 + 2w_{1,1}w_{1,7}w_{2,0}w_{2,8} + 4w_{1,1}w_{1,7}w_{2,1}w_{2,7} + 12w_{1,1}w_{1,7}w_{2,2}w_{2,6}$$

Table 2. Inventory of patterns given by the Monster Theorem.

		even thread spots				
		0	4	8	12	16
	0	1				
	4	1	1			
	8	4	6	12		
odd	12	5	10	34	29	
thread	16	8	13	52	79	48
spots	20	5	10	34	29	
	24	4	6	12		
	28	1	1			
	32	1				

$$+ 20w_{1,1}w_{1,7}w_{2,3}w_{2,5} + 13w_{1,1}w_{1,7}w_{2,4}^2 + 8w_{1,2}w_{1,6}w_{2,0}w_{2,8} + 12w_{1,2}w_{1,6}w_{2,1}w_{2,7}$$

$$+ 48w_{1,2}w_{1,6}w_{2,2}w_{2,6} + 68w_{1,2}w_{1,6}w_{2,3}w_{2,5} + 52w_{1,2}w_{1,6}w_{2,4}^2 + 10w_{1,3}w_{1,5}w_{2,0}w_{2,8}$$

$$+ 20w_{1,3}w_{1,5}w_{2,1}w_{2,7} + 68w_{1,3}w_{1,5}w_{2,2}w_{2,6} + 116w_{1,3}w_{1,5}w_{2,3}w_{2,5} + 79w_{1,3}w_{1,5}w_{2,4}^2$$

$$+ 8w_{1,4}^2w_{2,0}w_{2,8} + 13w_{1,4}^2w_{2,1}w_{2,7} + 52w_{1,4}^2w_{2,2}w_{2,6} + 79w_{1,4}^2w_{2,3}w_{2,5} + 96w_{1,4}^2w_{2,4}^2.$$

As noted, we divide all of the coefficients aside from the last one by 2 to account for the color changes. We divide by 2 again in the cases where swapping even with odd threads gives the same or opposite color distributions, namely $w_{1,a}w_{1,8-a}w_{2,a}w_{2,8-a}$ for $a = 1, 2, 3, 4$. Arbitrarily choosing representatives such that there are at least as many spots from the odd threads as from the even, we arrive at Table 2.

With the help of a computer, we can generate diagrams for the complete set of patterns corresponding to each element of the table. These diagrams are available on GitHub (Holden, 2022b), along with some notes on how they were systematically generated.

5. Future Work

Since the structure of Naiki is the same as that of plain weave, it seems reasonable to ask if we can classify plain weave patterns using the same techniques. The symmetries that produce equivalent patterns are not the same, however, since the fabric is not usually oriented on the bias and there may be no distinguished axis. In particular, it is not clear how to deal with the symmetry that rotates the fabric 90°. Like the glide plane reflection, this rotation swaps the odd and even threads. However, there are many more possible patterns fixed by the rotation that would need to be dealt with as special cases. Possibly there is a further extension of the Monster Theorem that can deal with the situation where elements of G are allowed to permute the D_i as well as preserving them.

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Disclosure Statement

The author reports that there are no competing interests to declare.

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