Inventing Sequence Models as Vectorized Signal Processors

Julius Smith
West Coast Machine Learning

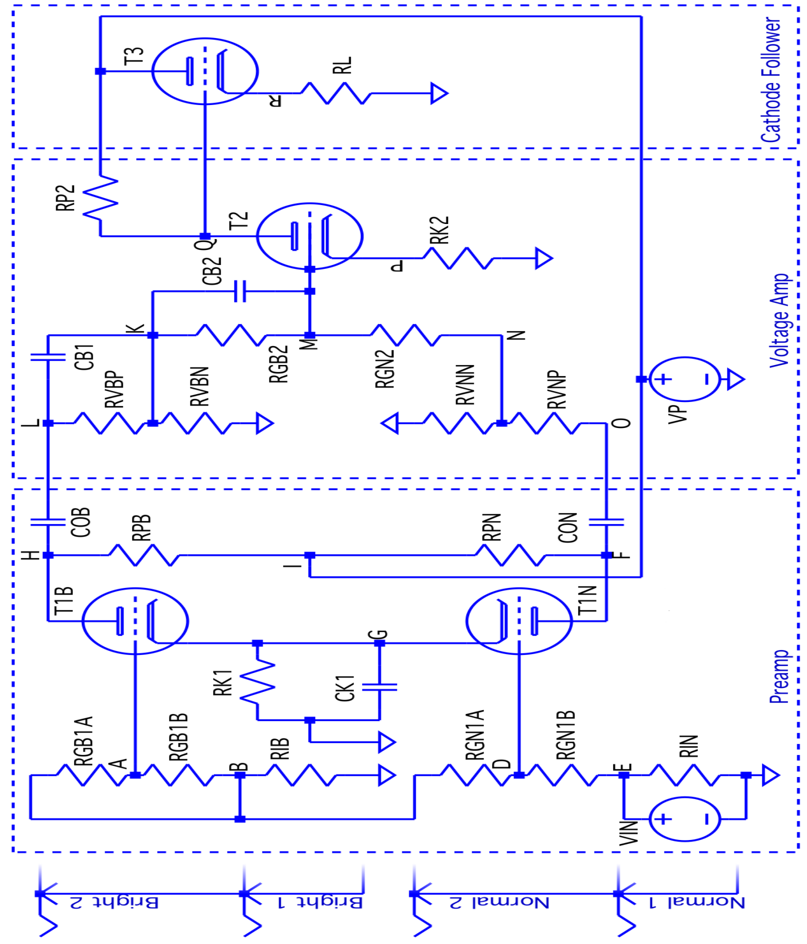
June 20, 2024

Background

My Path to CCRMA (Center for Computer Research in Music and Acoustics)

Musician : Math : Physics : EE : Control : DSP : System ID : SAIL/CCRMA

(a) Some Gig



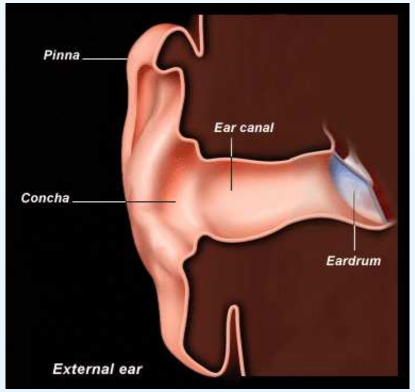
(b) Tube Amp

Courses Developed for CCRMA

- **Music 320A**: AUDIO SPECTRUM ANALYSIS
- **Music 320B**: AUDIO FILTER ANALYSIS AND STRUCTURES
- **Music 420A**: **PHYSICAL** AUDIO SIGNAL PROCESSING
- **Music 421A**: **TIME-FREQUENCY** AUDIO SIGNAL PROCESSING

420A

$\longrightarrow$

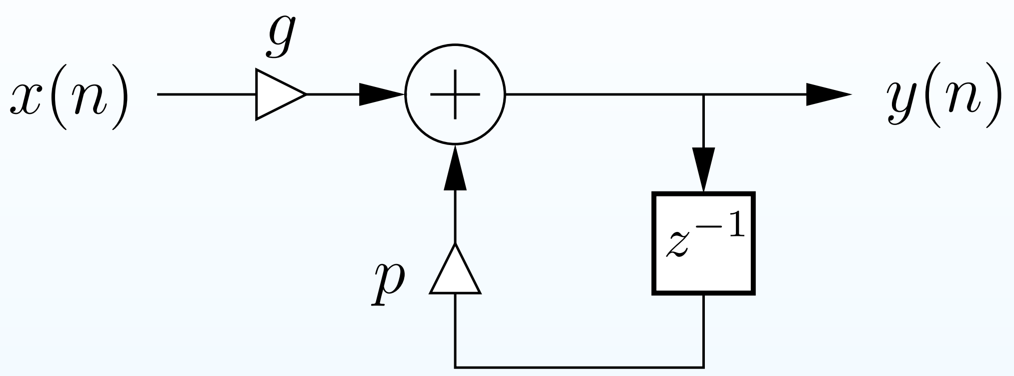


421A

All four textbooks **free online**

Music 320 Project Idea

Assignment: Do Something Cool with a *One-Pole Recursive Digital Filter*



One Pole at $z = p$

$$y(n) = g\,x(n) + p\,y(n-1),\; n = 0, 1, 2, \ldots$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{g}{1 - p\,z^{-1}}$$

Idea: Let's Make an Associative Memory!

- $x(n)$ can be a *long vector* $\underline{x}(n) \in \mathbb{R}^N$ representing *anything we want* — any "label"
- Set $\underline{g} = \underline{1}$ and $\underline{p} = \underline{1}$ to make $\underline{y}(n)$ a *sum of all input vectors* ("integrator")
- Choose the dimension N so large that *vectors in the sum are mostly orthogonal*
- Retrieve similar vectors using a *matched inner product* $\underline{w}^T\underline{x} > b$,
  for some suitable threshold b (Hey! That's a simulated neuron! ("Perceptron"))

Vector Retrieval by Inner Product

Given the sum of vectors

$$\underline{y}(n) = \sum_{m=0}^{n} \underline{x}(m)$$

and a “query vector” $\underline{w} = \underline{x}(k)$,
find the query in the sum using an *inner product:*

$$\underline{w}^T \underline{y}(n) \;=\; \sum_{m=0}^{n} \underline{w}^T \underline{x}(m) \;\approx\; \underline{x}^T(k)\,\underline{x}(k) \;=\; \|\underline{x}(k)\|^2 \;>\; b(k)$$

where $b(k)$ is the *detection threshold* for $\underline{x}(k)$

- This works because the spatial dimension is so large that $\underline{x}^T(j)\,\underline{x}(k) \approx \epsilon$ for $j \neq k$
- Retrieval threshold $b(k)$ depends on $\|\underline{x}(k)\|^2$
  $\Rightarrow$ **suggestion:** *reserve the radial dimension for similarity scoring*
- *I.e., only populate the* ***hypersphere*** in $\mathbb{R}^N$: $\|\,\underline{x}(k)\,\| = 1,\ \forall k$
- We just invented `RMSNorm`, used extensively in neural networks (not `LayerNorm`)

Orthogonality in High Dimensions

Let $\mathbf{a} \in \mathbb{R}^N$ and $\mathbf{b} \in \mathbb{R}^N$ be two normally random, real, unit-norm vectors in N dimensions. $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$.
The dot-product of $\mathbf{a}^T = [a_1, a_2, \ldots, a_N]$ and $\mathbf{b}^T = [b_1, b_2, \ldots, b_N]$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \sum_{i=1}^{N} a_i b_i.$$

The squared dot product is

$$(\mathbf{a} \cdot \mathbf{b})^2 = \left(\sum_{i=1}^{N} a_i b_i\right)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j b_i b_j.$$

Expected value (average):

$$E\left[(\mathbf{a} \cdot \mathbf{b})^2\right] = \sum_{i=1}^{N} \sum_{j=1}^{N} E[a_i a_j]\, E[b_i b_j] = \sum_{i=1}^{N} \frac{1}{N} \frac{1}{N} = \boxed{\frac{1}{N}}$$

Orthogonality in High Dimensions, Continued

We just showed the *expected squared dot product of two normally random unit vectors in* $\mathbb{R}^N$ *is* $1/N$, *i.e.*,

$$E\left[(\mathbf{a}\cdot\mathbf{b})^2\right] = \boxed{\frac{1}{N}}$$

since $E[a_i b_j] = 0$ for $i \neq j$, $E[a_i^2] = E[b_i]^2 = 1/N$, and $\mathbf{a}$ and $\mathbf{b}$ are independent.

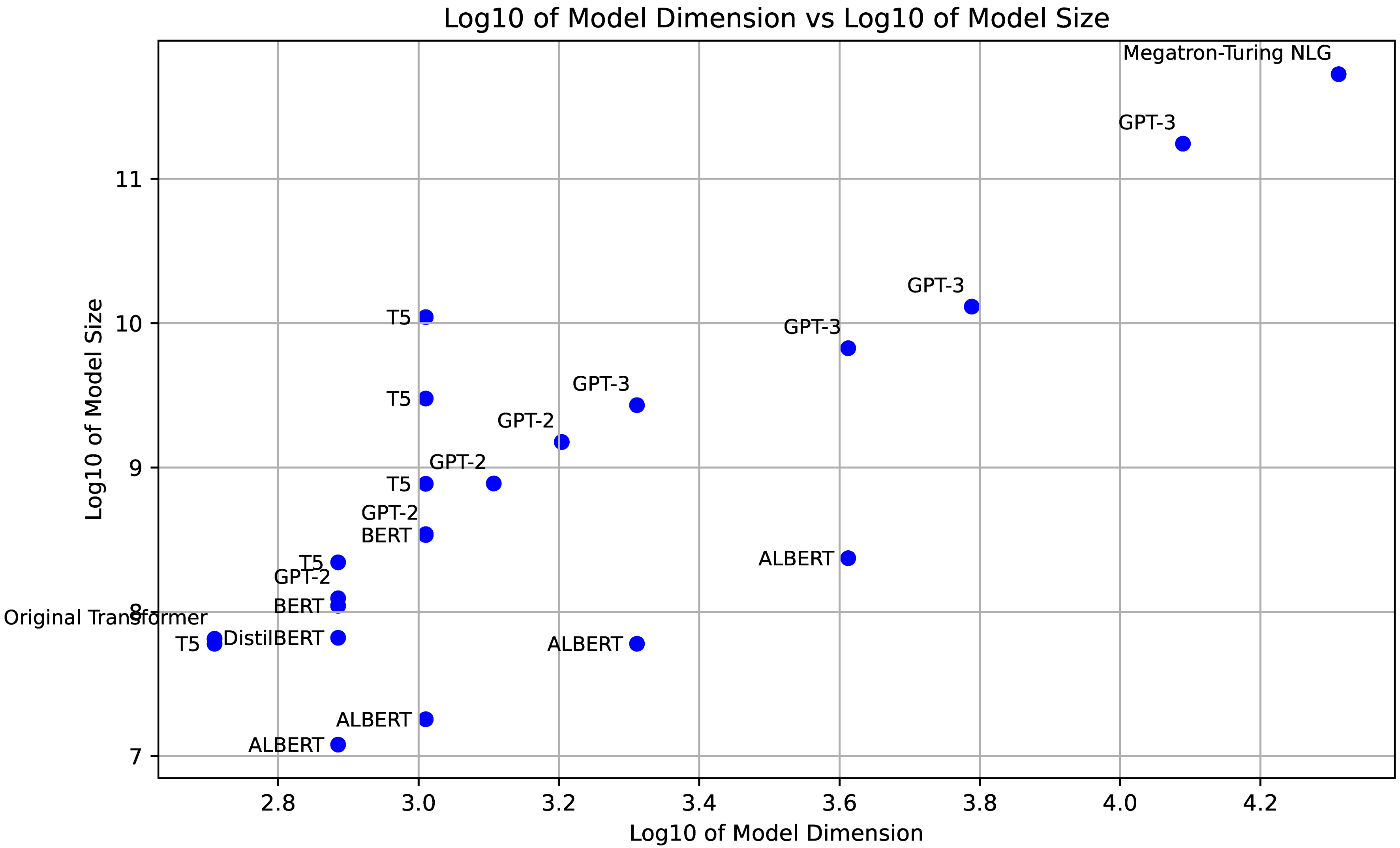
Suggestions:

- *Initialize biases larger than* $1/N$
- *Divide the sum of* M *vectors by* $\sqrt{M}$*:*
  - "power normalization"
  - "`RMSNorm`-preserving"
  - Done in Hawk & Griffin, *e.g.*
  - "Keep vector sums near the unit sphere"
- Apply `RMSNorm` when *training* the initial *vocabulary embedding* ("`word2sphere`")
- Set the *model dimension* just sufficient for the *layer width* at each level

Model Dimension vs. Model & Vocab Size (ChatGPT-4o, not checked)

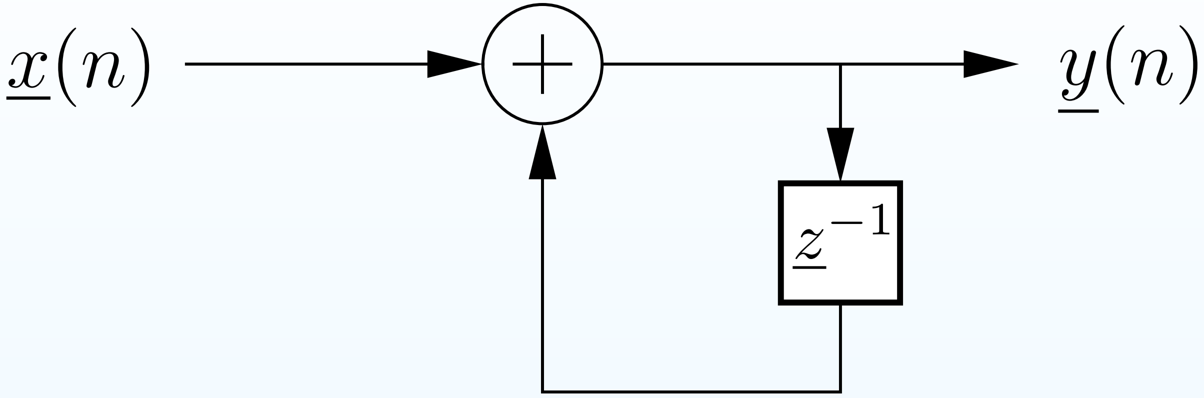
- **Original Transformer**
  - $d = 512$ (65 M)
- **BERT**
  - $d = 768$ (110 M, 30 K)
  - $d = 1024$ (340 M, 30 K)
- **GPT-2**
  - $d = 768$ (124 M, 50 K)
  - $d = 1024$ (345 M, 50 K)
  - $d = 1280$ (774 M, 50 K)
  - $d = 1600$ (1.5 B, 50 K)
- **GPT-3**
  - $d = 2048$ (2.7 B, 50 K)
  - $d = 4096$ (6.7 B, 50 K)
  - $d = 6144$ (13 B, 50 K)
  - $d = 12288$ (175 B, 50 K)
- **T5**
  - $d = 512$ (60 M, 32 K)
  - $d = 768$ (220 M, 32 K)
  - $d = 1024$ (770 M, 32 K)
  - $d = 1024$ (3 B, 32 K)
  - $d = 1024$ (11 B, 32 K)
- **ALBERT**
  - $d = 768$ (12 M, 30 K)
  - $d = 1024$ (18 M, 30 K)
  - $d = 2048$ (60 M, 30 K)
  - $d = 4096$ (235 M, 30 K)
- **DistilBERT**
  - $d = 768$ (66 M, 30 K)
- **Megatron-Turing NLG**
  - $d = 20480$ (530 B, 50 K)

Log10 Model Dimension versus Log10 Model Size

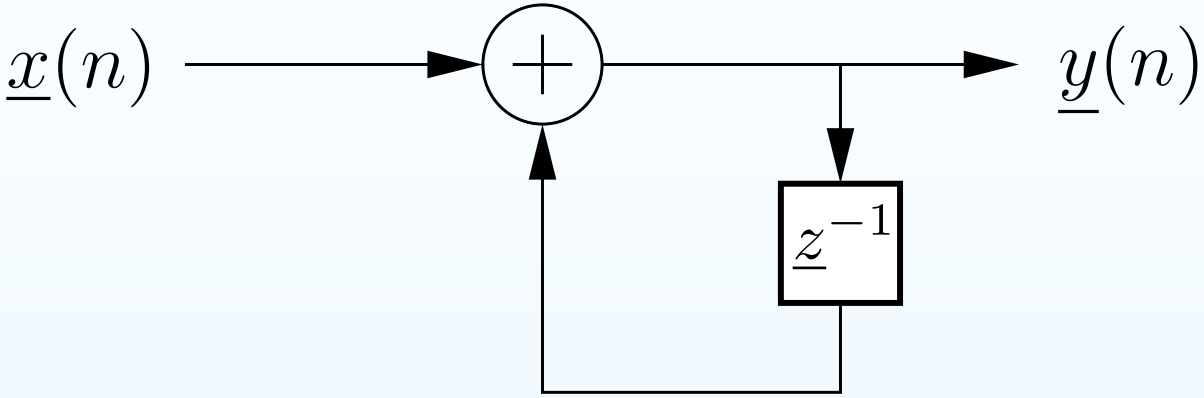


Architectures

Cumulative Vector Memory



Gated Vector Memory



Input Vector Summer

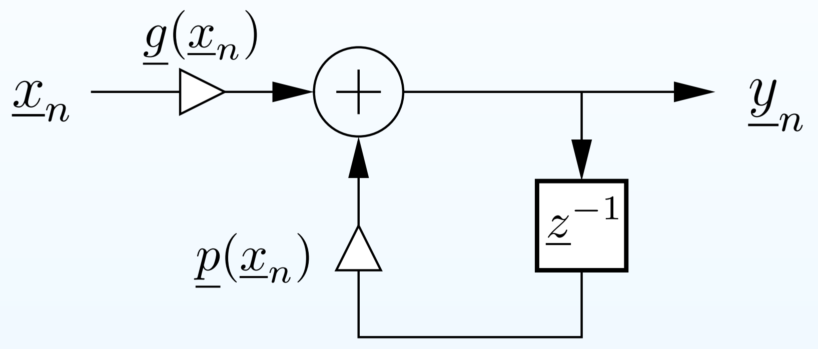
- **Problem:** Need a *memory reset*
- **Solution:** Set *feedback gain to zero* for one step to clear the memory

- **Problem:** Need an *input gate* to suppress unimportant inputs
- **Solution:** Set *input gain to zero* for unimportant inputs

- We just invented **gating**, used extensively in neural sequence models

Gated Recurrent Network

Idea: *Learn* the input and feedback gates as functions of input $\underline{x}_n$ based on many input-output examples $(\underline{x}_n, \underline{y}_n)$ ("training data"):



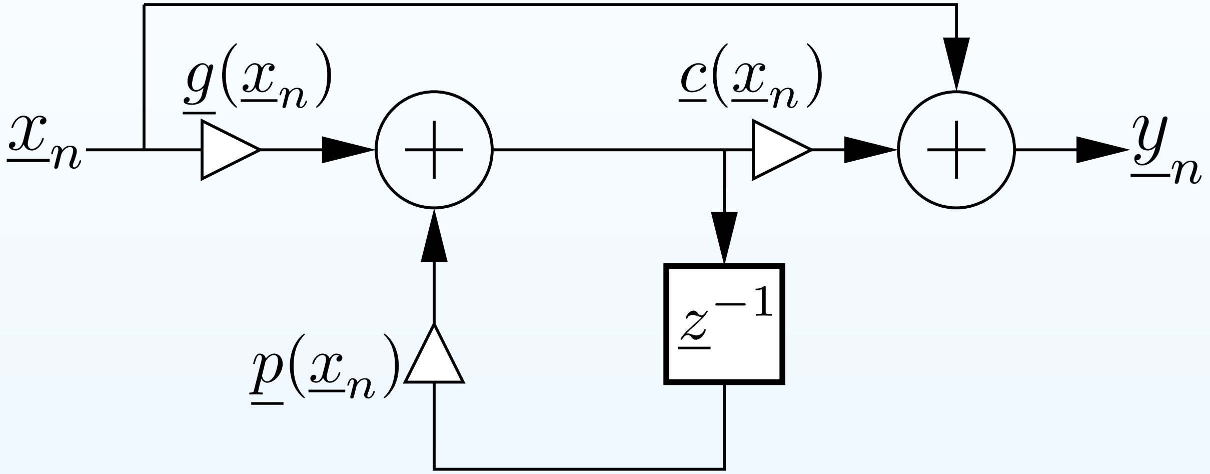
Vector Memory with Learned Input and Feedback Gates

Suggestions:

- Use learned, input-based, *activations* for gating (LSTM, GRU, Mamba smoothed input)
- While activated, *optionally* set *memory duration* via $\underline{p}$ magnitude (SSMs, Mamba)
  - *Initialize* $\underline{p}$ for desired initial memory duration (exponential fade time)
  - Learn $\underline{p}(\underline{x}_n)$ as $\mathbf{I} \cdot e^{-\Delta} \approx \mathbf{I} - \mathbf{I}\Delta$, where $\Delta = \text{softPlus}(\text{parameter}(\underline{x}_n, \underline{y}_n))$ (guaranteed stable — no "exploding gradients") [Also multiply $\underline{g}(\underline{x}_n)$ by Δ]
  - Consider *separate meaning-driven activation* multiplying feedback: $\sigma(\mathbf{L}\underline{x})\underline{p}(\underline{x})$

Output Gating

Idea: Since we have input and feedback gates, why not an ***output gate*** **and bypass?**



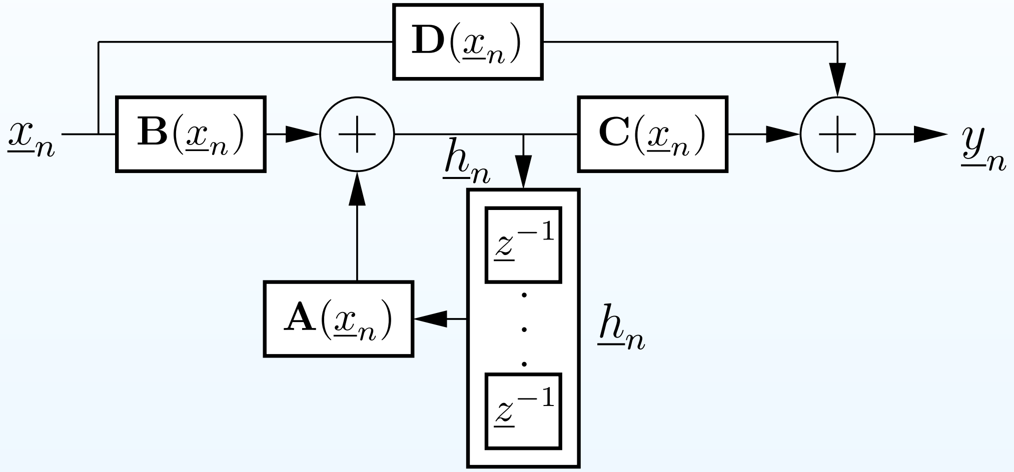
Gated RNN with **Skip Connection**

Output gating allows network to be "bypassed" when not helpful.

- **"Obvious" Suggestion:** The bypass path should be scaled for *power normalization*
- **Better yet:** Don't scale the bypass and use `RMSNorm` at the input of the next layer (prevents a "bad layer" from isolating deeper layers from the input with garbage)

State Expansion

Idea: *Expand* vector-memory dimension to an integer multiple of the model dimension:



"Structured State-Space Models" (SSM) look like this (*e.g.*, Mamba)

- Increased storage capacity
- Feedback matrix $\mathbf{A}$ typically *diagonal* since 2022 (see "S4D")
  $\Rightarrow$ Parallel bank of vector one-poles (*"linearly" gated, state-expanded RNNs*)
- In Mamba-2, $\mathbf{A} = p\,\mathbf{I}$, *i.e.*, *shared memory duration* across expanded state
- Gating matrices in Mamba[-2] are simple linear input projections:
  $[\mathbf{B}(\underline{x}_n), \mathbf{C}(\underline{x}_n)] = \mathbf{L}\,\underline{x}_n$

Memory Access

Detecting Multiple Vectors in Parallel

Idea: Detect multiple memory vectors in *parallel* using an array of *"Perceptrons"*

- Each Perceptron detects one or more memory vectors similar to its weight vectors

$$y_i(n) = \underline{w}_i^T \underline{h}(n) > b_i$$

  where $y_i(n)$ denotes the ith output at time n, i=0,. . . ,M-1,
  and $\underline{h}(n)$ denotes the ("hidden" [expanded-] state) vector memory
- Note that the $\mathbf{C}$ matrix can provide these weights:

$$\underline{y}(n) = \mathbf{C}\,\underline{h}(n) > \underline{b}$$

- The Perceptrons indicate *which weight-vectors* $\underline{w}_i$ *are present* in the vector memory $\underline{h}$
- **Idea:** (Backpropagation—1980s?): To facilitate *learning* $\underline{w}_i$ via gradient descent, replace ">" by something smoother, such as $1 + \tanh[\mathbf{C}\,\underline{h}(n) - \underline{b}]$

Sequence Modeling

- If each vector represents a *word,* a vector sum is simply a *bag of words*
- To model a *sequence* of words, we have various *sequence-position-encoding* options:

1. *Amplitude Decay* - Multiply the sum by a *forgetting factor* each sequence step (RNNs) - *poor choice* (conflates with angular distance on the hypersphere)
2. *Sinusoidal Amplitude Modulation* - Add a sinusoid with *increasing frequency* to each vector summing into the history
(used in the original Transformer)
3. *Phase Shift* - Multiply by the sum by $e^{j\Delta}$ each sample
("RoPE") - *apparently most used today*

RoPE and WRoPE

- Rotational Positional Encoding (RoPE) owns *one arc direction* along the hypersphere
- We can thus rotate our vector memory $\underline{h}(n)$ by Δ radians each time step to “age” it:

$$\underline{h}_a(n) = e^{j\Delta}\underline{h}(n), \quad \text{with } \Delta = \frac{2\pi}{L}$$

when our maximum sequence length (before reset) is L

- **Idea:** “Warped RoPE” (WRoPE) for *arbitrarily long sequences*:

$$\Delta_n = \frac{2\pi n}{n+L}, \quad n = 0, 1, 2, \ldots$$

(inspired by the *bilinear transform* used in digital filter design)

- A *blend* of *uniform* and *warped* rotations can be used:

$$\Delta_n = \begin{cases} \frac{\pi n}{L}, & n = 0, 1, 2, \ldots, L-1 \\ \pi + \frac{\pi n}{n+1}, & n = L, L+1, L+2, \ldots \end{cases}$$

where L is now the *typical* sequence length (giving it more “space” in recall)

Reserving Angular Dimensions

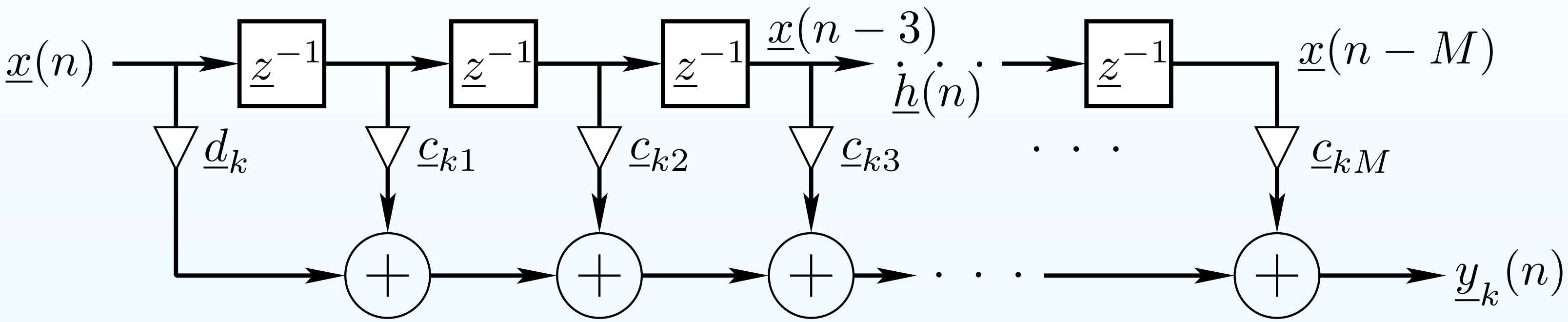
Reserving the radial dimension looks easy (*e.g.*, `RMSNorm` every k gradient-descent steps)
How to reserve an *arc dimension* for [W]RoPE and perhaps other purposes?

- RoPE: Effectively reserves *complex angle:* $\underline{h}_a(n) = e^{j\Delta}\underline{h}(n)$ (data size doubled: $\underline{x} \to (\underline{x}, \underline{0})$)
- Use *quaternions* for two more angles $\Rightarrow$ data size quadrupled: $\underline{x} \to (\underline{x}, \underline{0}, \underline{0}, \underline{0})$ $\underline{h}_a(n) = \mathbf{q}\underline{h}(n)$, where, from Wikipedia:
  $$\mathbf{q} = e^{\frac{\Delta}{2}(\underline{a}_x\mathbf{i}+\underline{a}_y\mathbf{j}+\underline{a}_z\mathbf{k})} = \cos\frac{\Delta}{2} + (\underline{a}_x\mathbf{i} + \underline{a}_y\mathbf{j} + \underline{a}_z\mathbf{k})\sin\frac{\Delta}{2} = \cos\frac{\Delta}{2} + \underline{a}\sin\frac{\Delta}{2}$$
- See the Cayley-Dickson Construction for $2^k - 1$ angles, $k = 1, 2, 3, \ldots$
- Trivial in *polar coordinates* ("write-protect" reserved dimensions during gradient step), but then back-propagation must be rewritten for polar coordinates
- Find a suitable method of *constrained gradient descent*
- Train (or calculate) a *pointwise MLP* that "de-rotates" $\underline{x} + \mu\underline{\nabla}$ given also $\underline{x}$
- Use *reserved translational dimension(s)* instead ("TraPE") (omitted from `RMSNorm`)

Attention

Attention Layer

Idea: Also use *FIR Filtering* (SSM State Expansion Factor $N \geq M$, $\mathbf{A}$ subdiagonal):



Separately learnable FIR coefficient matrices $\underline{d}_k[\underline{x}(n)], \underline{c}_j[\underline{x}(n-j), k]$, depending on:

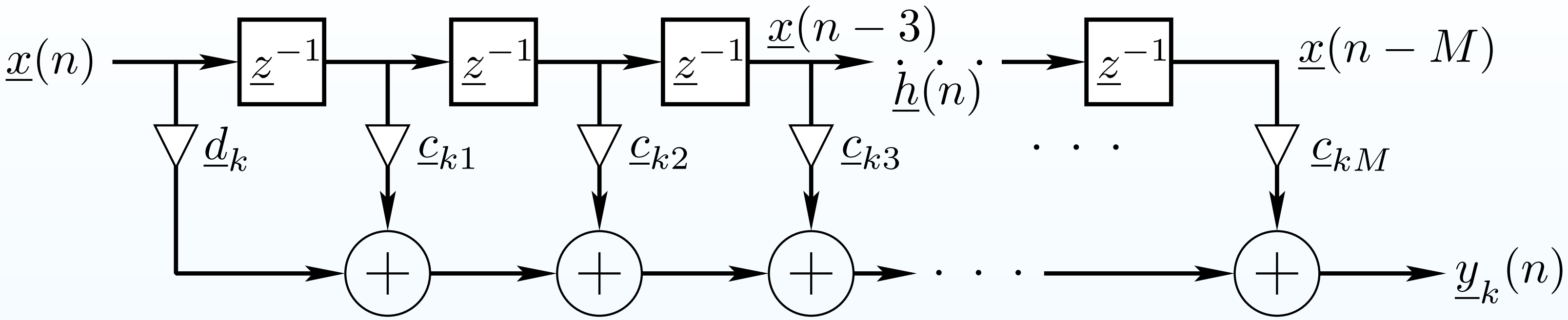
1. input *position* j in the input sequence ("context buffer" or "expanded state" + [W]RoPE)
2. input *vector* $\underline{x}(n-j),\ j = 0, 1, 2, \ldots, M$
3. *output-position* k being computed, $k = 0, 1, 2, \ldots, M$ ($M + 1$ outputs)

Idea: Add *relevance gating* suppressing unimportant inputs to each output ("attention")

Idea: Create *new embedding vectors* as *sums* of relevant input vectors ("attention")

Idea: Measure relevance using an *inner product* between the output and input positions ("dot-product attention")

Dot-Product Attention



Relevance Gating

Let $\underline{x}_k$ denote $\underline{x}(n-k)$

The contribution from input $\underline{x}_j$ to the nonlinear FIR sum for output $\underline{y}_k$ can be calculated as

$$\underline{c}_{kj}\underline{x}_j = \left[\left(\sum_{m\in\mathcal{R}(\underline{x}_k)}\underline{x}_m\right)^T \underline{x}_j\right]\underline{x}_j$$

or more generally $\boxed{\underline{c}_{kj} = Q_k^T\underline{x}_j}$, where

$Q_k(\underline{x}_k, k)$ is called the *query* vector for position k in the input sequence

The query Q_k can be a sum of *all vectors supported in the attention sum*:

$Q_k = \underline{x}_k + \underline{x}_{m_1} + \cdots + \underline{x}_{m_k}$

$\Rightarrow (Q_k^T\underline{x}_j)\underline{x}_j \approx \underline{x}_j$, if $\underline{x}_j$ is similar to *any vector* in the query sum.

Multi-Head Attention

Idea: To support multiple meaning possibilities, *partition the model space* into parallel independent *attention calculations* (“multi-head attention”)

- Each *attention head* can form an independent input interpretation
- Useful for *ambiguous* sequences, especially in the lower layers
- Also introduced in the Transformer paper (2017)

Now we need *down-projections* of the relevance-calculation components
$\Rightarrow$ relevance of input j to output k in attention-head l becomes proportional to

$$\underline{c}_{kj}\underline{x}_j = (Q_k^T\underline{x}_j)\underline{x}_j \;\longrightarrow\; \underline{c}_{lkj}\underline{x}_{lj} = \left[Q_{lk}^T(\underline{x}_k)K_{lj}(\underline{x}_j)\right] V_{lj}(\underline{x}_j)$$

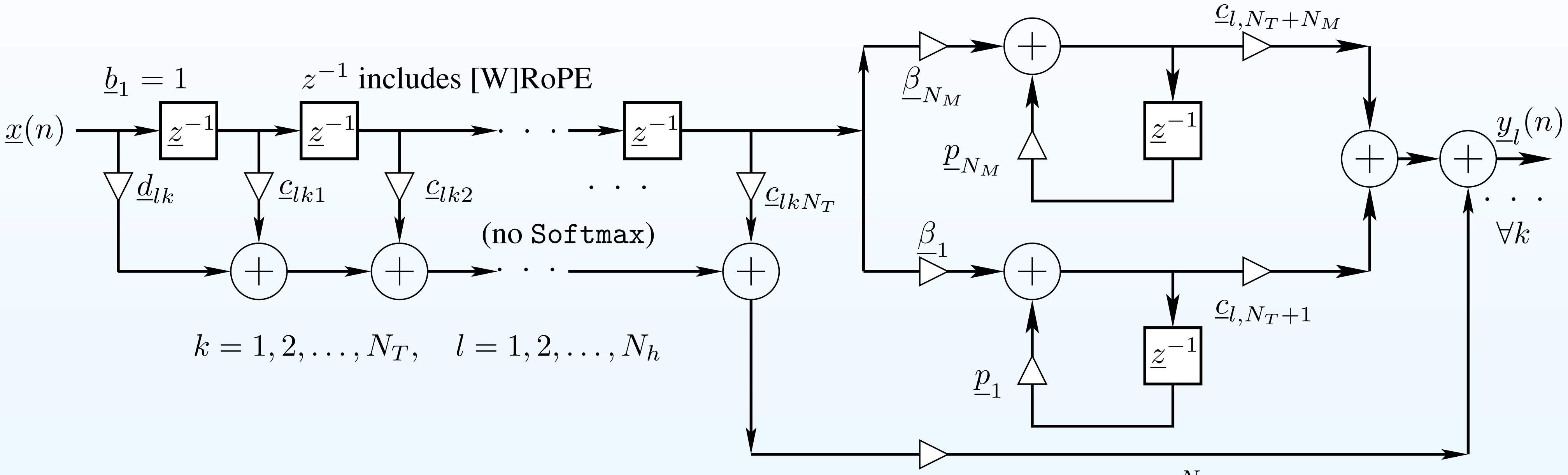
where Q_{lk} (“query”), K_{lj} (“key”), and V_{lj} (“value”) vectors are learned *down-projections* of the input $\underline{x}_j$ for each attention-head l and for all sequence indices j and k in the context buffer (“Transformer”)
Other useful generalizations can be imagined for these learned (Q,K,V) vectors, such as grouping grammatical functions, creating new model-space regions, etc.

State Space Unification of Transformers and GRNNs

$$\mathbf{A_T} = \underbrace{\underline{e}^{j\Delta_n}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \underline{1} & 0 & \cdots & 0 & 0 \\ 0 & \underline{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \underline{1} & 0 \end{pmatrix}}_{\text{Transformer}} \qquad \mathbf{A_M} = \underbrace{\underline{a}_n\underline{e}^{j\Delta_n}\begin{pmatrix} \underline{1} & 0 & \cdots & 0 \\ 0 & \underline{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{1} \end{pmatrix}}_{\text{Mamba-2 style RNN + [W]RoPE}}$$

$$\mathbf{A_{TM}} = \underline{e}^{j\Delta_n}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \underline{1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \underline{1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \underline{\beta}_1(n) & \underline{a}_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \underline{\beta}_{N_M}(n) & 0 & \cdots & \underline{a}_n \end{pmatrix} \qquad \mathbf{B_{TM}} = \begin{pmatrix} \underline{b}_1(n) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Transformer followed by GRNN with 2x State Expansion (like Mamba)



Consider $\dfrac{1}{\sqrt{\sum_{j=1}^{N_T}(Q_k^T K_j)^2}} \Rightarrow \sum_{j=1}^{N_T} \underline{c}_{lkj}^2 = 1$

TransMamba

Next Steps

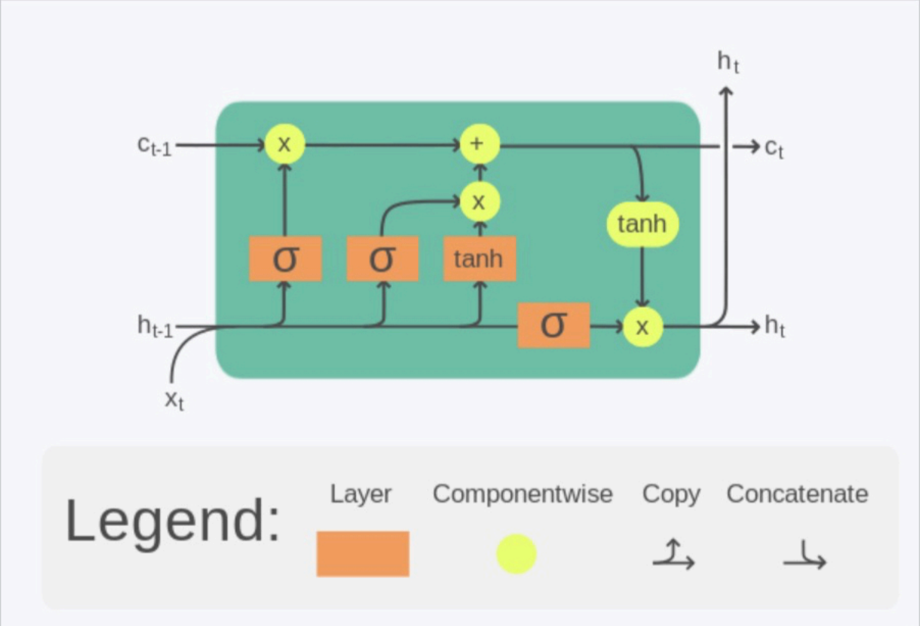
- Try to improve [Trans]**Mamba**[-2] on small synthetic datasets testing *memory*
  - Vocabulary embeddings trained to the unit hypersphere (*e.g.*, `word2sphere`)
  - Memory *duration* and *reset* functions separately trained and implemented
  - Initial *biases* at $\underline{0}$ versus $1/N$ or larger
  - Do *power normalization* in place of `RMSNorm` where possible (efficiency)
  - Try *power normalized attention* in place of $1/\sqrt{d_h}$ and `Softmax` (efficiency)
  - Adapt *model dimension* to *layer width* at each level (efficiency)
  - Warped Rotational Positional Encoding (WRoPE)
  - Translational Positional Encoding (TraPE) in its own head (no `RMSNorm`)
  - Explore other “Control Heads” that flow along purely for “conditioning” like TraPE
- Progress to date:
  - New synthetic benchmarks analogous to “needle in a haystack”
  - Adapted Andrej Karpathy’s `makemore` code, adding Mamba and new benchmarks
  - Four papers started, aiming for Arxiv, GitHub, “AI social media,” blog
- Feel free to take over any of these! (and LMK so I can do something else)

Thanks for your Attention!

Sequence Modeling Snapshots

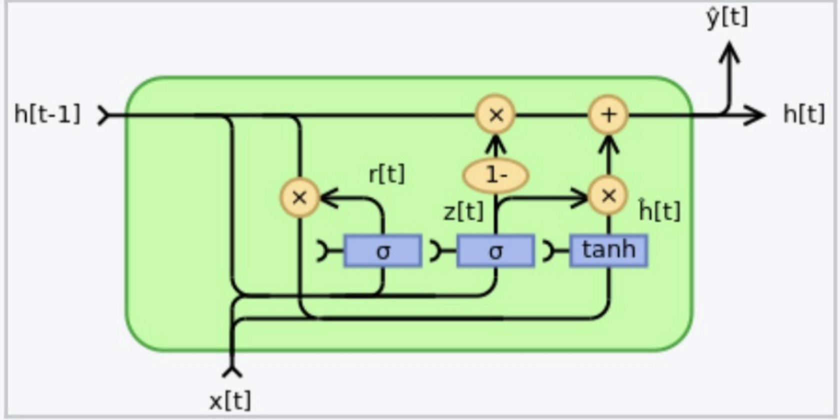
LSTM and GRU

1997: LSTM



The Long Short-Term Memory (LSTM) cell can process data sequentially and keep its hidden state through time.

2014: GRU



Gated Recurrent Unit, fully gated version

Structured State Space and Mamba

2023: Mamba (S6)

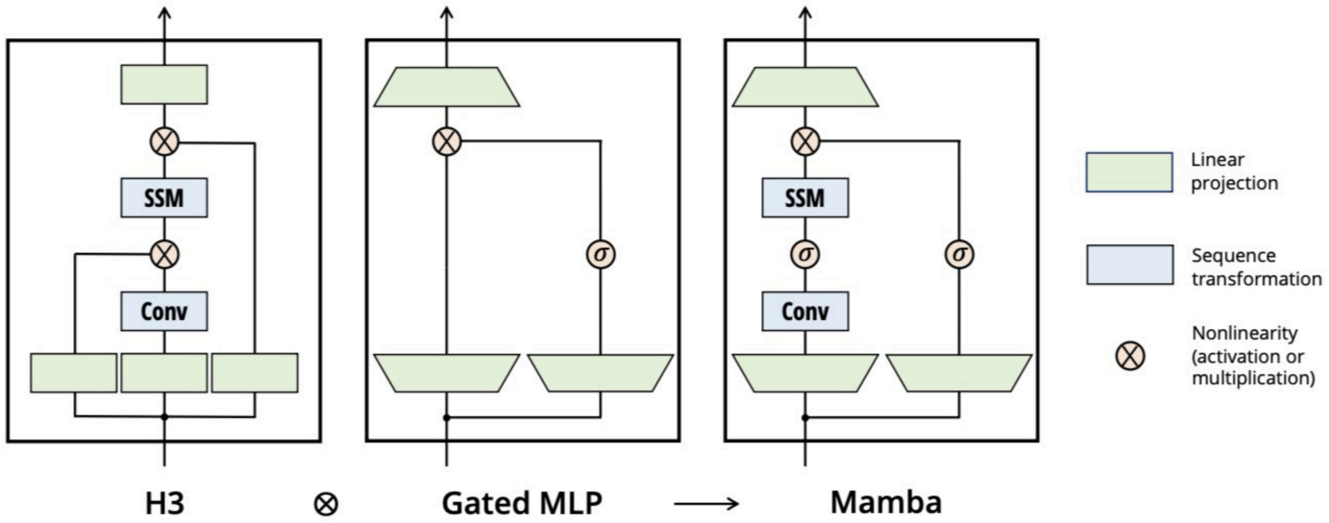


Figure 2: (**Architecture.**) Our simplified block design combines the H3 block, which is the basis of most SSM architectures, with the ubiquitous MLP block of modern neural networks. Instead of interleaving these two blocks, we simply repeat the Mamba block homogenously. Compared to the H3 block, Mamba replaces the first multiplicative gate with an activation function. Compared to the MLP block, Mamba adds an SSM to the main branch. For σ we use the SiLU / Swish activation (Hendrycks & Gimpel, 2016; Ramachandran et al., 2017).

Hawk and Griffin

Griffin: Mixing Gated Linear Recurrences with Local Attention for Efficient Language Models

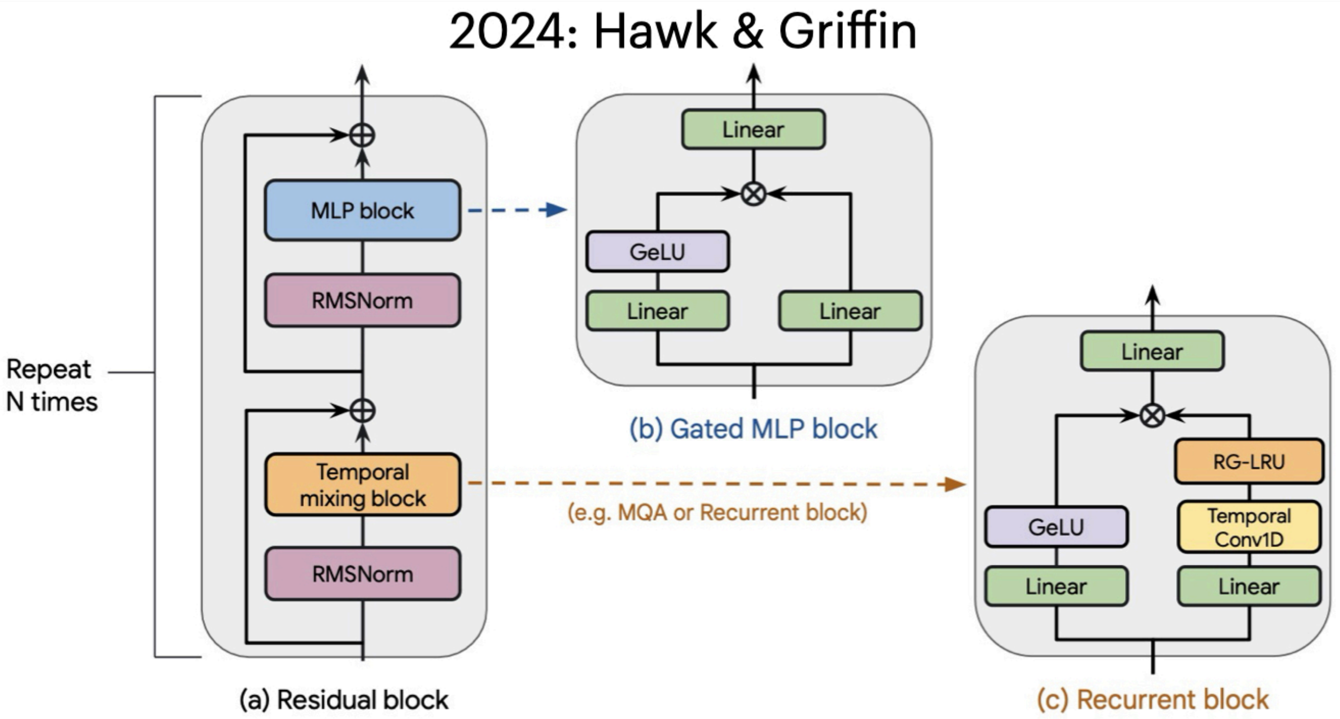


Figure 2 | a) The main backbone of our mode architecture is the residual block, which is stacked N times. b) The gated MLP block that we use. c) The recurrent block that we propose as an alternative to Multi Query Attention (MQA). It uses our proposed RG-LRU layer, defined in Section 2.4.

Gated "Linear" RNNs with State Expansion

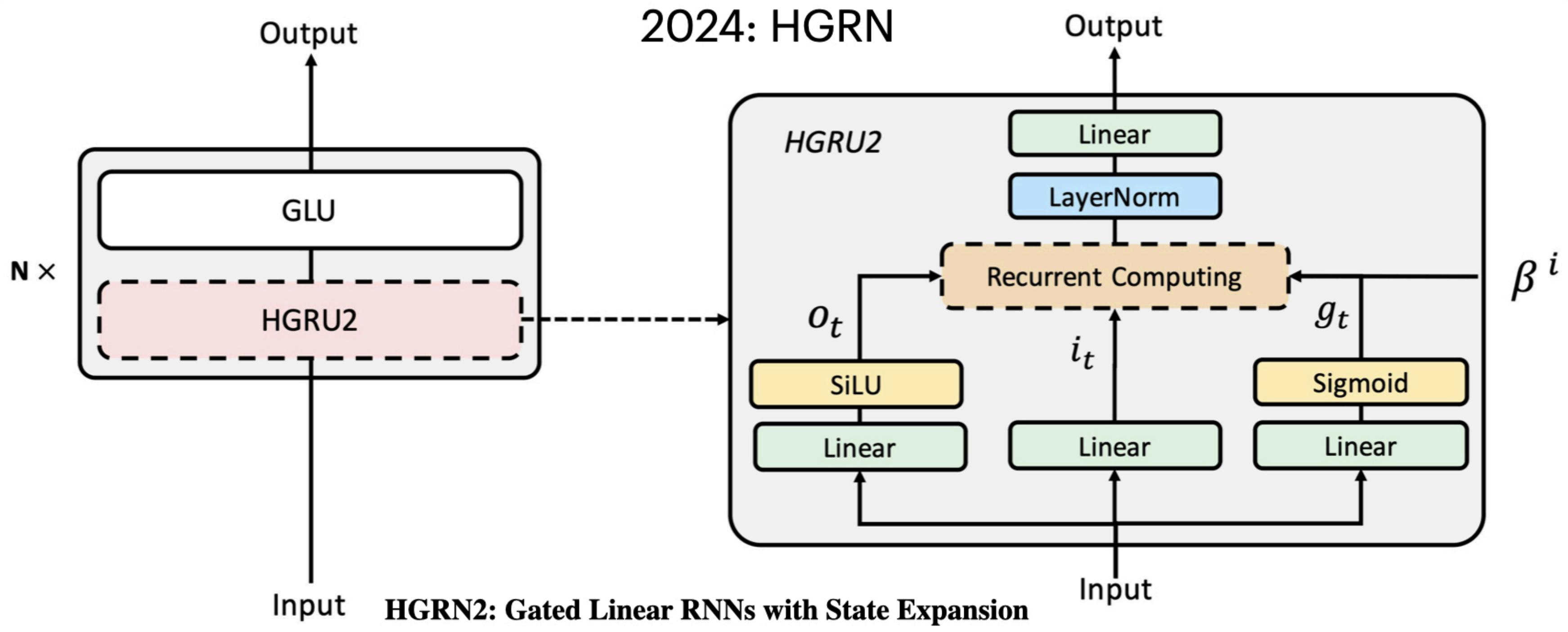


Figure 1: **The neural architecture of HGRN2.** Each HGRN2 layer includes a token mixer layer HGRU2 and a channel mixerlayer GLU. HGRU2 employs recurrent computation through Eq. 3, where $\mathbf{i}_t$ is the input vector, $\mathbf{g}_t$ is the forget gate (not lower bounded), β^i is the lower bound of the forget gate value, $\mathbf{o}_t$ is the output gate for layer i.

RWKV, Eagle, Finch

2024: Eagle-Finch RWKV

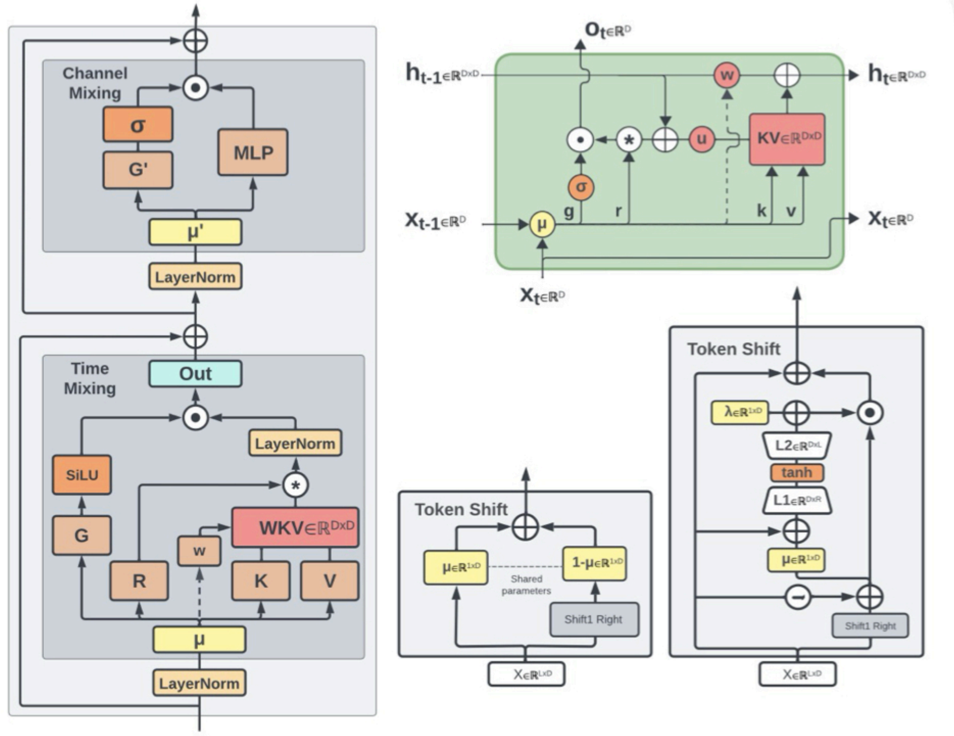


Figure 1: RWKV architecture overview. **Left:** time-mixing and channel-mixing blocks; **top-right:** RWKV time-mixing block as RNN cell; **center-bottom:** token-shift module in FeedForward module and Eagle time-mixing; **bottom-right:** token-shift module in Finch time-mixing. All shape annotations assume a single head for simplicity. Dashed arrows (left, top-right) indicate a connection in Finch, but not in Eagle.