Inventing Sequence Models as Vectorized Signal Processors

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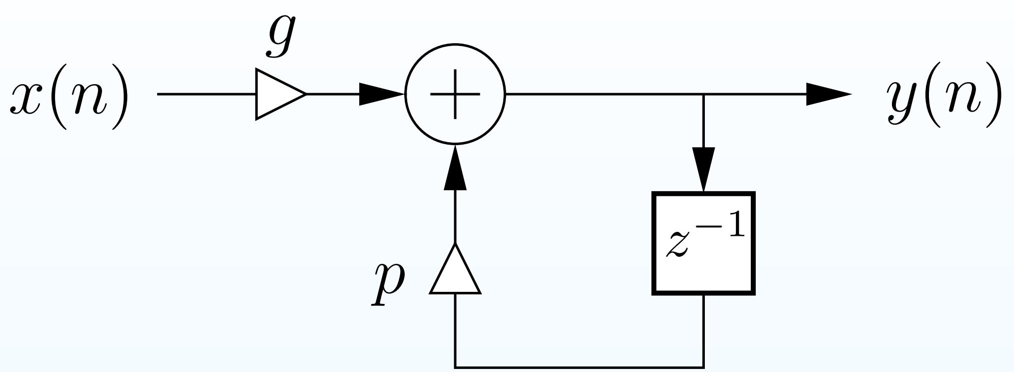
Abstract

Today's sequence models (such as large language models) in machine learning (AI) arose from a blend of principle-based design and empirical discovery, spanning several fields. This talk describes how the ideas could have emerged from an elementary signal-processing approach. This viewpoint offers some features:

1. Signal processing folks can quickly learn what is happening in a motivated way
2. Machine-learning experts might benefit from signal-processing insights
3. Obvious suggestions for things to try next naturally arise

Signal-Processing Class-Project Idea

Assignment: Do Something Cool with a *One-Pole Recursive Digital Filter*



One Pole at $z = p$

$$y(n) = g\,x(n) + p\,y(n-1),\, n = 0, 1, 2, \ldots \quad \text{[difference equation]}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{g}{1 - p\,z^{-1}} \quad \text{[transfer function]}$$

Idea: Let's Make an Associative Memory!

- $x(n)$ can be a *long vector* $\underline{x}(n) \in \mathbb{R}^N$ representing *anything we want* — any "label"
- Set $\underline{g} = \underline{1}$ and $\underline{p} = \underline{1}$ to make $\underline{y}(n)$ a *sum of all input vectors* ("integrator")
- Choose the dimension N so large that *vectors in the sum are mostly orthogonal*
- Let $\underline{x}(n)$ be **embedded vectors** (*e.g.*, `word2vec`) so that *closeness = similarity*
- Retrieve similar vectors using a *matched inner product* $\underline{w}^T \underline{x} > b$, for some suitable threshold b (Hey! That's a simulated neuron! ("Perceptron"))

Vector Retrieval by Inner Product

Given the sum of vectors

$$\underline{y}(n) = \sum_{m=0}^{n} \underline{x}(m)$$

and a "query vector" $\underline{w} = \underline{x}(k)$,
find the query in the sum using an *inner product:*

$$\underline{w}^T \underline{y}(n) \;=\; \sum_{m=0}^{n} \underline{w}^T \underline{x}(m) \;\approx\; \underline{x}^T(k)\,\underline{x}(k) \;=\; \|\underline{x}(k)\|^2 \;>\; b(k)$$

where $b(k)$ is the *detection threshold* for $\underline{x}(k)$

- This works because the spatial dimension is so large that $\underline{x}^T(j)\,\underline{x}(k) \approx \epsilon$ for $j \neq k$
- Retrieval threshold $b(k)$ depends on $\|\underline{x}(k)\|^2$

  $\Rightarrow$ **suggestion:** *reserve the radial dimension for similarity scoring*
- *I.e., only populate the* ***hypersphere*** in $\mathbb{R}^N$: $\|\,\underline{x}(k)\,\|^2 = 1$ (or N), $\forall k$
- We just invented `RMSNorm`, used extensively in neural networks (not `LayerNorm`)

Decaying Vector Retrieval by Inner Product

RNNs typically have a *forgetting factor* $p < 1$.
Given the sum of vectors

$$\underline{y}(n) = \sum_{m=0}^{n} p^{n-m} \underline{x}(m)$$

and a "query vector" $\underline{w} = \underline{x}(k)$,
the inner product now gives

$$\underline{w}^T \underline{y}(n) \;=\; \sum_{m=0}^{n} \underline{w}^T p^{n-m} \underline{x}(m) \;\approx\; p^{n-k} \underline{x}^T(k)\, \underline{x}(k) \;=\; p^{n-k} \;>\; b$$

where b is the detection threshold for $\underline{x}(k)$, independent of k if $\|\underline{x}(k)\| = 1$

- *Cannot retrieve* when $p^{n-k} < b$, setting an upper limit on n
- We need $p > b^{1/n}$ or $n_{\text{max}} \leq \log(b)/\log(p)$
- Lower $b \Rightarrow$ more memory, but also more false detections from "interference" ("other vectors in the sum")

Orthogonality in High Dimensions

Let $\mathbf{a} \in \mathbb{R}^N$ and $\mathbf{b} \in \mathbb{R}^N$ be two normally random, real, unit-norm vectors in N dimensions, with $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$
The dot-product (inner product) of $\mathbf{a}^T = [a_1, a_2, \ldots, a_N]$ and $\mathbf{b}^T = [b_1, b_2, \ldots, b_N]$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \sum_{i=1}^{N} a_i b_i.$$

The squared dot product is

$$(\mathbf{a} \cdot \mathbf{b})^2 = \left(\sum_{i=1}^{N} a_i b_i \right)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j b_i b_j.$$

Expected value (average):

$$E\left[(\mathbf{a} \cdot \mathbf{b})^2 \right] = \sum_{i=1}^{N} \sum_{j=1}^{N} E[a_i a_j]\, E[b_i b_j] = \sum_{i=1}^{N} \frac{1}{N} \frac{1}{N} = \boxed{\frac{1}{N}}$$

Orthogonality in High Dimensions, Continued

We just showed the *expected squared dot product of two normally random unit vectors in* $\mathbb{R}^N$ *is* $1/N$, *i.e.*,

$$E\left[(\mathbf{a}\cdot\mathbf{b})^2\right] = \boxed{\frac{1}{N}}$$

since $E[a_i b_j] = 0$ for $i \neq j$, $E[a_i^2] = E[b_i]^2 = 1/N$, and $\mathbf{a}$ and $\mathbf{b}$ are independent.

Suggestions:

- *Initialize biases (detection thresholds) larger than* $1/N$
- *Divide the sum of* M *vectors by* $\sqrt{M}$*:*
  - “power normalization”
  - “`RMSNorm`-preserving”
  - Done in Hawk & Griffin, *e.g.*
  - “Keep vector sums near the unit sphere”
- Apply `RMSNorm` when *training* the initial *vocabulary embedding* (“`word2sphere`”)
- Set the *model dimension* just sufficient for the *layer width* at each level
- **Caveat:** We are only considering *one mechanism* here — *others are definitely going on*

Orthogonality of Random Sums

Similarly,

$$
\begin{aligned}
E\left[\left(\underline{w}^T\underline{y}_n\right)^2\right] &= E\left[\left(\sum_{m=0}^{n}\underline{w}^T\underline{x}_m\right)^2\right] = \sum_{l=0}^{n}\sum_{m=0}^{n}E\left[\underline{w}^T\underline{x}_l\underline{x}_m^T\underline{w}\right] \\
&= \sum_{m=0}^{n}E\left[\underline{w}^T\underline{x}_m\underline{x}_m^T\underline{w}\right] = \sum_{m=0}^{n}E\left[\left(\underline{w}^T\underline{x}_m\right)^2\right] = \boxed{\frac{n}{N}}
\end{aligned}
$$

assuming $\underline{w} \notin \underline{y}$ and $\|\underline{w}\| = \|\underline{x}_m\| = 1$ for all m. Thus, *retrieval becomes unreliable when the number of summed vectors n nears the model dimension N.*

- N is of course the number of exactly orthogonal vectors possible in N dimensions
- If L vectors are typically in the sum, our Perceptron “bias” (detection threshold) should be higher than L/N
- **Suggestion:** *Keep the number of active vectors in memory well below N*

How Many Summed Vectors Needed for Language Parsing? (BERT style)

It is well known that *phone numbers* were limited to *7 digits* due to our *limited short-term memory* for *unrelated* objects. Can *language* be parsed using 7 vectors or less at each level? [Original Transformer paper had 8 attention heads and 6 layers (like neocortex)]

Layers (*e.g.*):

1. Base vocabulary = characters (26 for English)
2. Syllable in 7 chars or less (44 syllables in English; 107 in Int'l Phonetic Alphabet)
3. Word in 7 syllables or less
4. Noun + 6 or less modifying adjectives
5. Verb + up to 6 adverbs
6. Noun phrase
7. Direct or indirect object
8. Prepositional phrase
9. Subject, verb, [indirect object], object
10. Sentence
11. Paragraph
12. Section
13. Chapter
14. Book
15. Subject Area Hierarchy . . .

Different cortical areas (6 layers each) needed for this many levels.

Complex Sentence Diagram Examples:

https://www.quora.com/In-regards-to-diagramming-sentences-which-one-is-the-most-difficult-youve-ever-come-across

Pictorial Examples

- 6 dot-patterns on a die
- 7 LED segments for numbers
- 4 or fewer strokes to draw a letter

Hierarchy used to keep the number of components per concept *small*

Suggestions:

- *Ready signal* when symbol is recognized (whole letter, word, phrase, etc.)
- *Reset memory* after symbol recognition
- Memory can be *small!*

Maybe these signals are happening in the layers already?

Orthogonality of Exponentially Decaying Random Sums

RNNs typically have a *forgetting factor* $p < 1$, in which case we have, defining $\mu = n - m$ and $\lambda = n - l$:

$$
\begin{aligned}
E\left[\left(\underline{w}^T \underline{y}_n\right)^2\right] &= E\left[\left(\sum_{m=0}^{n} \underline{w}^T p^\mu \underline{x}_m\right)^2\right] = \sum_{l=0}^{n}\sum_{m=0}^{n} E\left[\underline{w}^T p^\lambda \underline{x}_l p^\mu \underline{x}_m^T \underline{w}\right] \\
&= \sum_{m=0}^{n} p^{2\mu} E\left[\underline{w}^T \underline{x}_m \underline{x}_m^T \underline{w}\right] = \sum_{m=0}^{n} p^{2\mu} E\left[\left(\underline{w}^T \underline{x}_m\right)^2\right] \\
&= \boxed{\frac{1}{N}\frac{1-p^{2(n+1)}}{1-p^2}} \to \frac{1}{N}\frac{1}{1-p^2} \quad \text{(as } n \to \infty\text{)}
\end{aligned}
$$

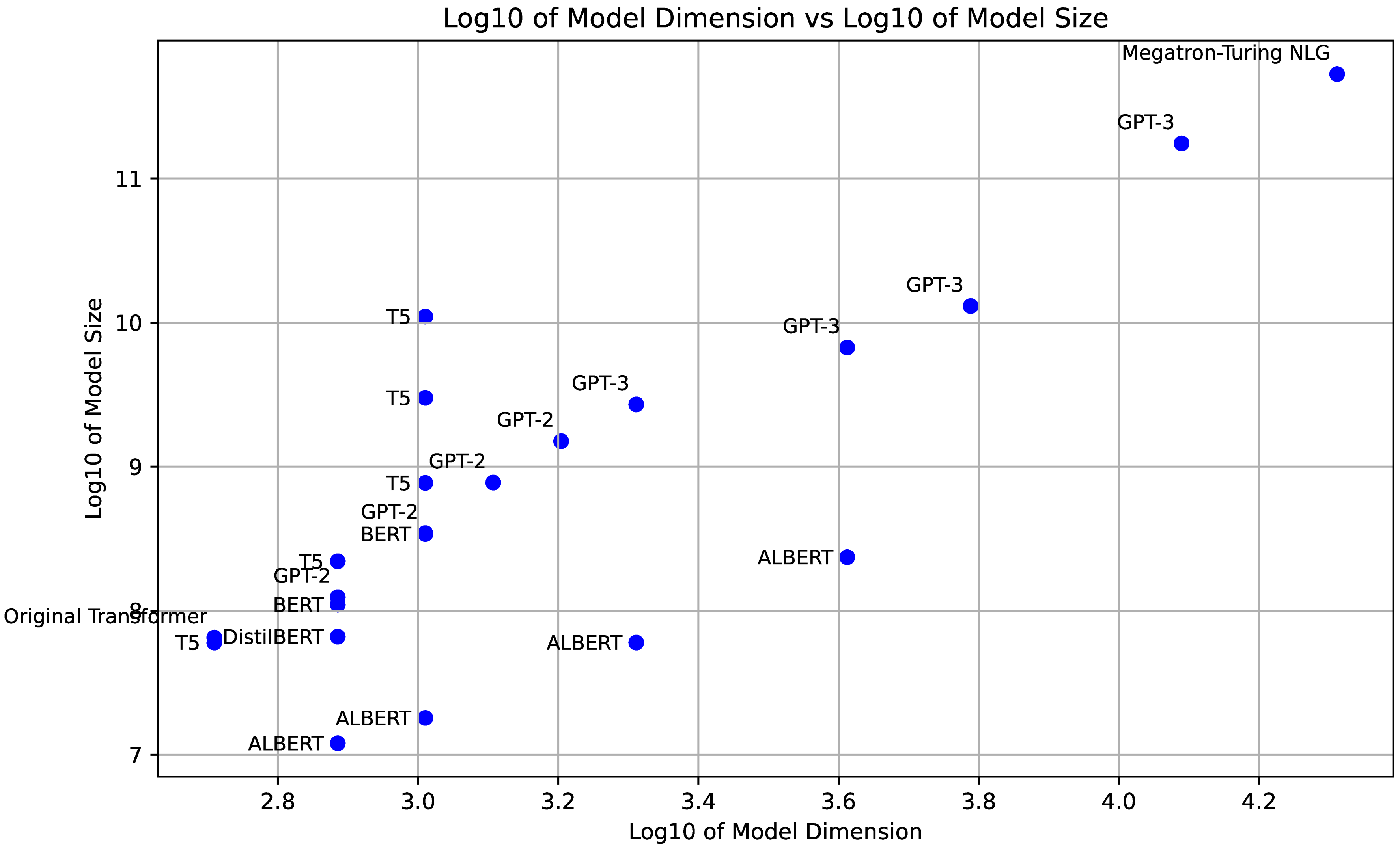
- For $1/(1-p^2) < N$, keep $p < \sqrt{(N-1)/N}$
- For $1/(1-p^2) < N/2$, keep $p < \sqrt{(N-2)/N}$

and so on. *This gives us one way to calculate a maximum feedback coefficient p in RNNs*

Model Dimension vs. Model & Vocab Size (ChatGPT-4o, not checked)

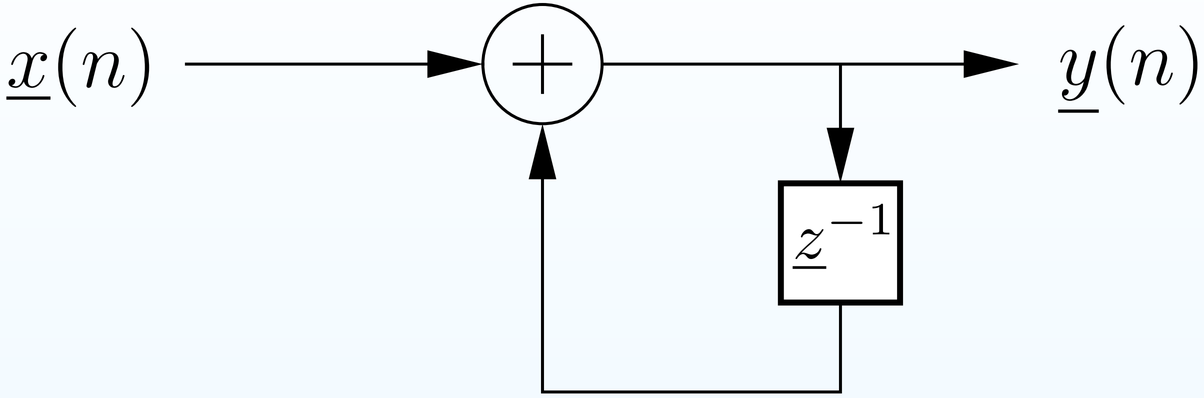
- **Original Transformer**
  - $N = 512$ (Model Size 65 M)
- **BERT**
  - $N = 768$ (110 M, 30 K)
  - $N = 1024$ (340 M, 30 K)
- **GPT-2**
  - $N = 768$ (124 M, 50 K)
  - $N = 1024$ (345 M, 50 K)
  - $N = 1280$ (774 M, 50 K)
  - $N = 1600$ (1.5 B, 50 K)
- **GPT-3**
  - $N = 2048$ (2.7 B, 50 K)
  - $N = 4096$ (6.7 B, 50 K)
  - $N = 6144$ (13 B, 50 K)
  - $N = 12288$ (175 B, 50 K)
- **T5**
  - $N = 512$ (60 M, 32 K)
  - $N = 768$ (220 M, 32 K)
  - $N = 1024$ (770 M, 32 K)
  - $N = 1024$ (3 B, 32 K)
  - $N = 1024$ (11 B, 32 K)
- **ALBERT**
  - $N = 768$ (12 M, 30 K)
  - $N = 1024$ (18 M, 30 K)
  - $N = 2048$ (60 M, 30 K)
  - $N = 4096$ (235 M, 30 K)
- **DistilBERT**
  - $N = 768$ (66 M, 30 K)
- **Megatron-Turing NLG**
  - $N = 20480$ (530 B, 50 K)

Log10 Model Dimension versus Log10 Model Size



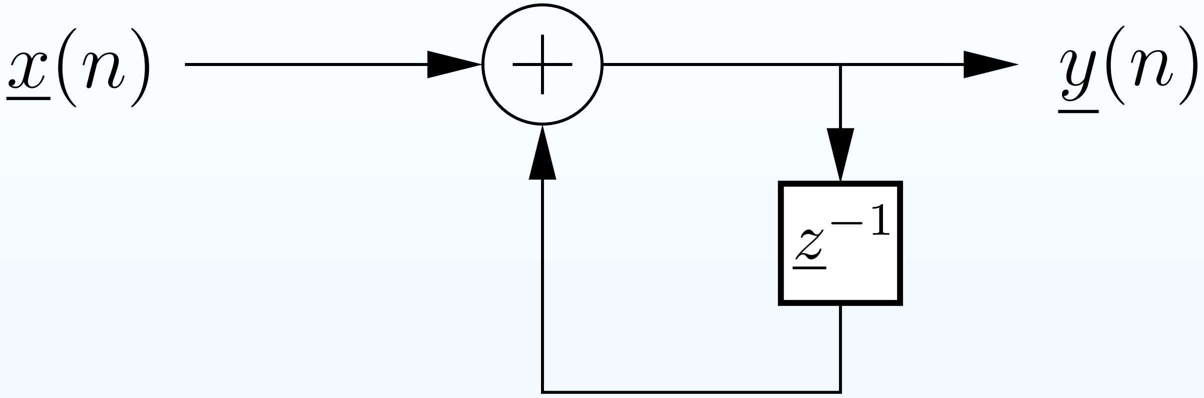
Architectures

Cumulative Vector Memory



MidJourney

Gated Vector Memory



Input Vector Summer

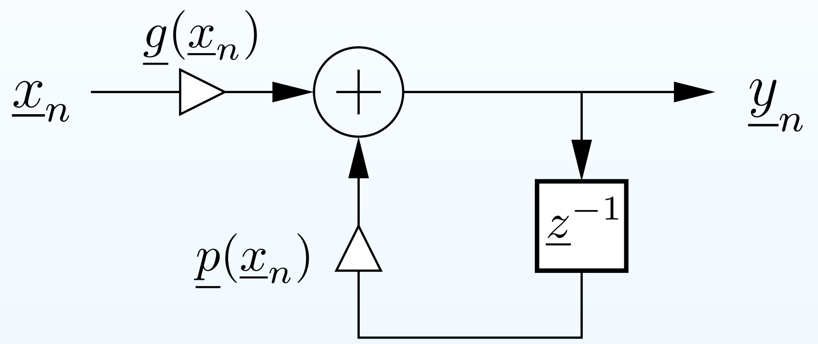
- **Problem:** Need a *memory reset*
- **Solution:** Set *feedback gain to zero* for one step to clear the memory

- **Problem:** Need an *input gate* to suppress unimportant inputs
- **Solution:** Set *input gain to zero* for unimportant inputs

- We just invented **gating**, used extensively in neural sequence models

Gated Recurrent Network

Idea: *Learn* the input and feedback gates as functions of input $\underline{x}_n$ based on many input-output examples $(\underline{x}_n, \underline{y}_n)$ ("training data"):



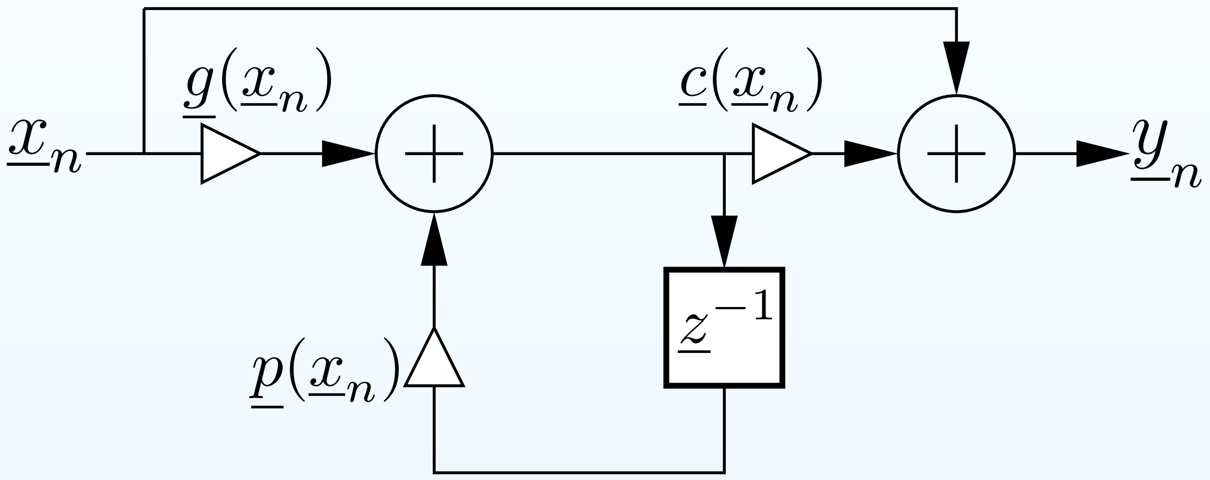
Vector Memory with Learned Input and Feedback Gates

Suggestions:

- Use learned, input-based, *activations* for gating (LSTM, GRU, Mamba smoothed input)
- While activated, *optionally* set *memory duration* via $\underline{p}$ magnitude (SSMs, Mamba)
  - *Initialize* $\underline{p}$ for desired initial memory duration (exponential fade time)
  - Learn $\underline{p}(\underline{x}_n)$ as $\mathbf{I} \cdot e^{-\Delta} \approx \mathbf{I} - \mathbf{I}\Delta$, where $\Delta = \text{softPlus}(\text{parameter}(\underline{x}_n, \underline{y}_n))$ (guaranteed stable — no "exploding gradients") [Also multiply $\underline{g}(\underline{x}_n)$ by Δ]
  - Consider *separate meaning-driven activation* multiplying feedback: $\sigma(\mathbf{L}\underline{x})\underline{p}(\underline{x})$

Output Gating

Idea: Since we have input and feedback gates, why not an ***output gate*** **and bypass?**



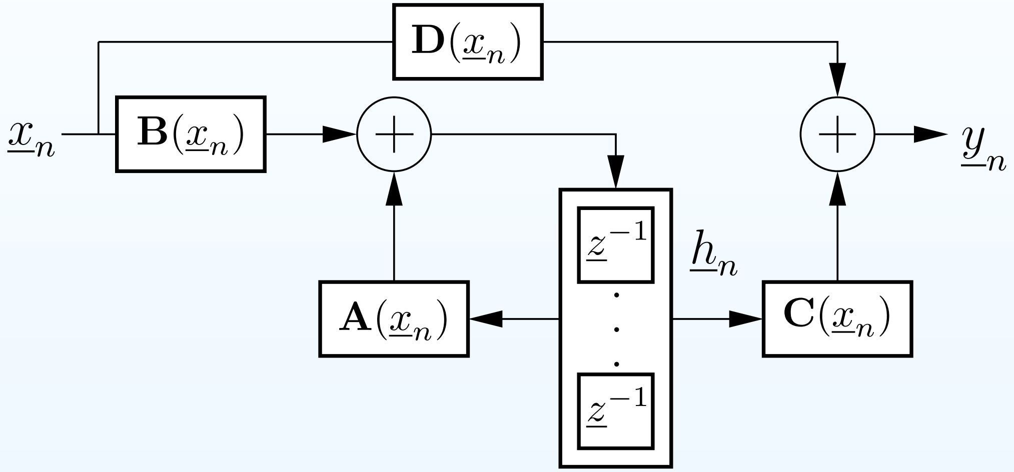
Gated RNN with **Skip Connection**

Output gating allows network to be “bypassed” when not helpful.

- **“Obvious” Suggestion:** The bypass path should be scaled for *power normalization*
- **Better yet:** Don’t scale the bypass and use `RMSNorm` at the input of the next layer (prevents a “bad layer” from isolating deeper layers from the input with garbage, and equalizes gradient backpropagation to all layers)

State Expansion

Idea: *Expand* vector-memory dimension to an integer multiple of the model dimension:



"Structured State-Space Models" (SSM) look like this (*e.g.*, Mamba)

- Increased storage capacity (more vectors can be summed and later retrieved)
- Feedback matrix $\mathbf{A}$ typically *diagonal* since 2022 (see "S4D")
  $\Rightarrow$ Parallel bank of vector one-poles (*"linearly" gated, state-expanded RNNs*)
- In Mamba-2, $\mathbf{A} = p\,\mathbf{I}$, *i.e.*, *shared memory duration* across expanded state
- Gating matrices in Mamba[-2] are simple linear input projections:
  $[\mathbf{B}(\underline{x}_n), \mathbf{C}(\underline{x}_n)] = \mathbf{L}\,\underline{x}_n$

Memory Access

Detecting Multiple Vectors in Parallel

Idea: Detect multiple memory vectors in *parallel* using an array of *"Perceptrons"*

- Each Perceptron detects one or more memory vectors similar to its weight vectors

$$y_i(n) = \underline{w}_i^T \underline{h}(n) > b_i$$

  where $y_i(n)$ denotes the ith output at time n, i=0,. . . ,M-1,
  and $\underline{h}(n)$ denotes the ("hidden" [expanded-] state) vector memory
- Note that the $\mathbf{C}$ matrix can provide these weights:

$$\underline{y}(n) = \mathbf{C}\,\underline{h}(n) > \underline{b}$$

- The Perceptrons indicate *which weight-vectors* $\underline{w}_i$ *are present* in the vector memory $\underline{h}$
- **Idea:** (Backpropagation—1980s?): To facilitate *learning* $\underline{w}_i$ via gradient descent, replace "$>$" by something smoother, such as $1 + \tanh[\mathbf{C}\,\underline{h}(n) - \underline{b}]$

Sequence Modeling

- If each vector represents a *word,* a vector sum is simply a *bag of words*
- To model a *sequence* of words, we have various *sequence-position-encoding* options:

1. *Amplitude Decay* - Multiply the sum by a *forgetting factor* each sequence step (RNNs) - *poor choice* (conflates with angular distance on the hypersphere)
2. *Sinusoidal Amplitude Modulation* - Add a sinusoid with *increasing frequency* to each vector summing into the history (used in the original Transformer)
3. *Phase Shift* - Multiply by the sum by $e^{j\Delta}$ each sample ("RoPE") - *apparently most used today*

RoPE and WRoPE

- Rotational Positional Encoding (RoPE) owns *one arc direction* along the hypersphere
- We can thus rotate our vector memory $\underline{h}(n)$ by Δ radians each time step to “age” it:

$$\underline{h}_a(n) = e^{j\Delta}\underline{h}(n), \quad \text{with } \Delta = \frac{2\pi}{L}$$

  when our maximum sequence length (before reset) is L
- **Idea:** “Warped RoPE” (WRoPE) for *arbitrarily long sequences* (processed in reverse):

$$\Delta_n = \frac{2\pi n}{n+L}, \quad n = 0, 1, 2, \dots$$

  (inspired by the *bilinear transform* used in digital filter design)
- A *blend* of *uniform* and *warped* rotations can be used:

$$\Delta_n = \begin{cases} \frac{\pi n}{L}, & n = 0, 1, 2, \dots, L-1 \\ \pi + \frac{\pi n}{n+1}, & n = L, L+1, L+2, \dots \end{cases}$$

  where L is now the *typical* sequence length (giving it more “space” in recall)

WRoPE Memory

- WRoPE sequences are naturally reversed because we can only change all stored angles by the same delta:

$$\underline{h}_a(n) = e^{j\Delta_n}\underline{h}(n), \quad n = 0, 1, 2, \ldots$$

- This makes inference non-autoregressive (more expensive)
- One improvement is to *store* past hidden states so that positional encodings can be updated arbitrarily when accessed:

$$\underline{h}_a(n, m) = e^{j\Delta_{n-m}}\underline{h}(m), \quad m = n - L, \ldots, n - 1, n$$

  (mth hidden state vector needed for inference at time n)
- This is the same amount of storage needed for the Truncated Infinite Impulse Response (TIIR) technique which provides a recursively computed sliding-window of memory
- In the TIIR case (fixed length L), might as well use normal RoPE
- WRoPE maybe competitive for encoding “journalistic style” into a vector

Truncated Infinite Impulse Response (TIIR) RNNs

A *sliding rectangular window* can be obtained as an integrator minus a *delayed* integrator:

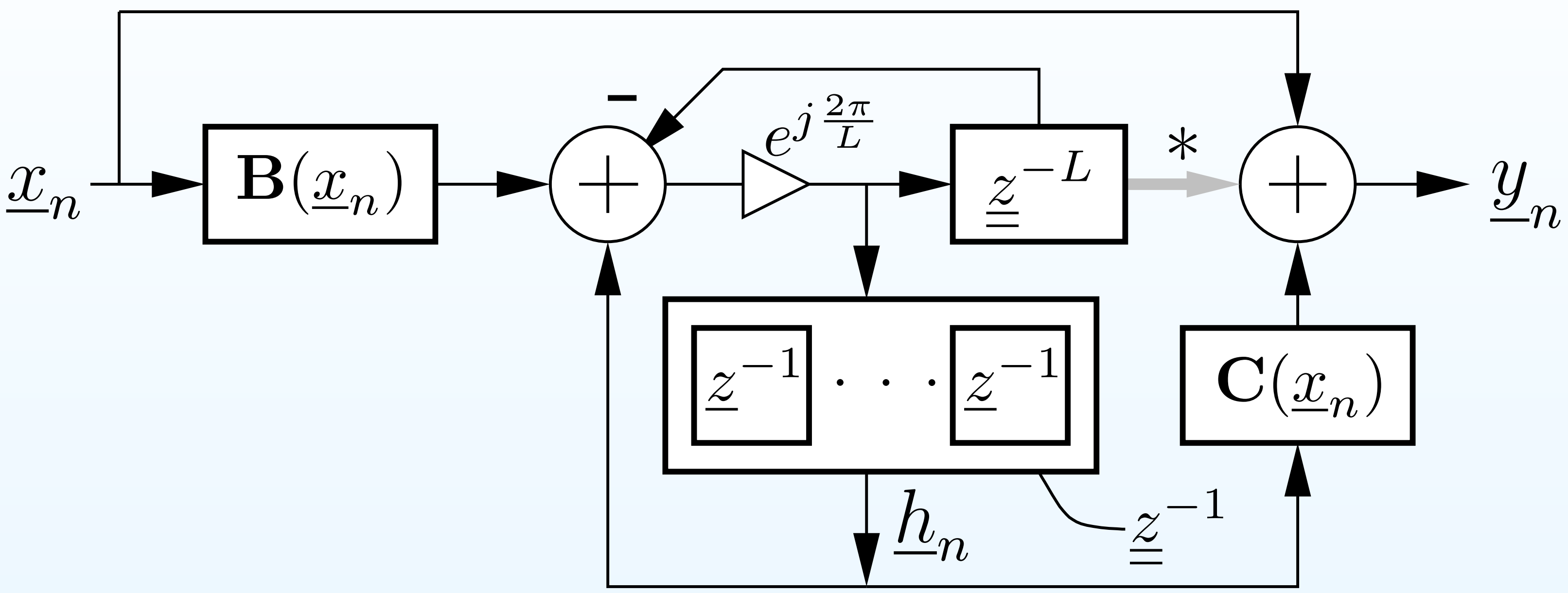
$$[1, 1, \ldots, 1] \quad \longleftrightarrow \quad \sum_{n=0}^{N-1} z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}} = \boxed{\frac{1}{1 - z^{-1}} - z^{-N} \frac{1}{1 - z^{-1}}}$$

- Thus, two identical RNNs can be *differenced* to provide a non-fading, linearly RoPEd memory of any length L
- A *real* memory of length L is needed for the *hidden state update:* $\underline{dh}(n) = \underline{h}(n+1) - \underline{h}(n) = \mathbf{B}_n \underline{x}(n)$
- Hidden state update becomes

$$\begin{aligned} \underline{h}(n+1) &= \underline{h}(n) + \underline{dh}_n \\ &= \underline{h}(n) + \mathbf{B}_n \underline{x}(n) - \mathbf{B}_{n-L}\, \underline{x}(n-L) \end{aligned}$$

- **Problem:** Accumulating floating-point round-off error (variance increases linearly)

TIIR RNN with Sliding-Window Memory and Linear RoPE



* Optional Attention Sum

Basic Idea

Architectures

Processing

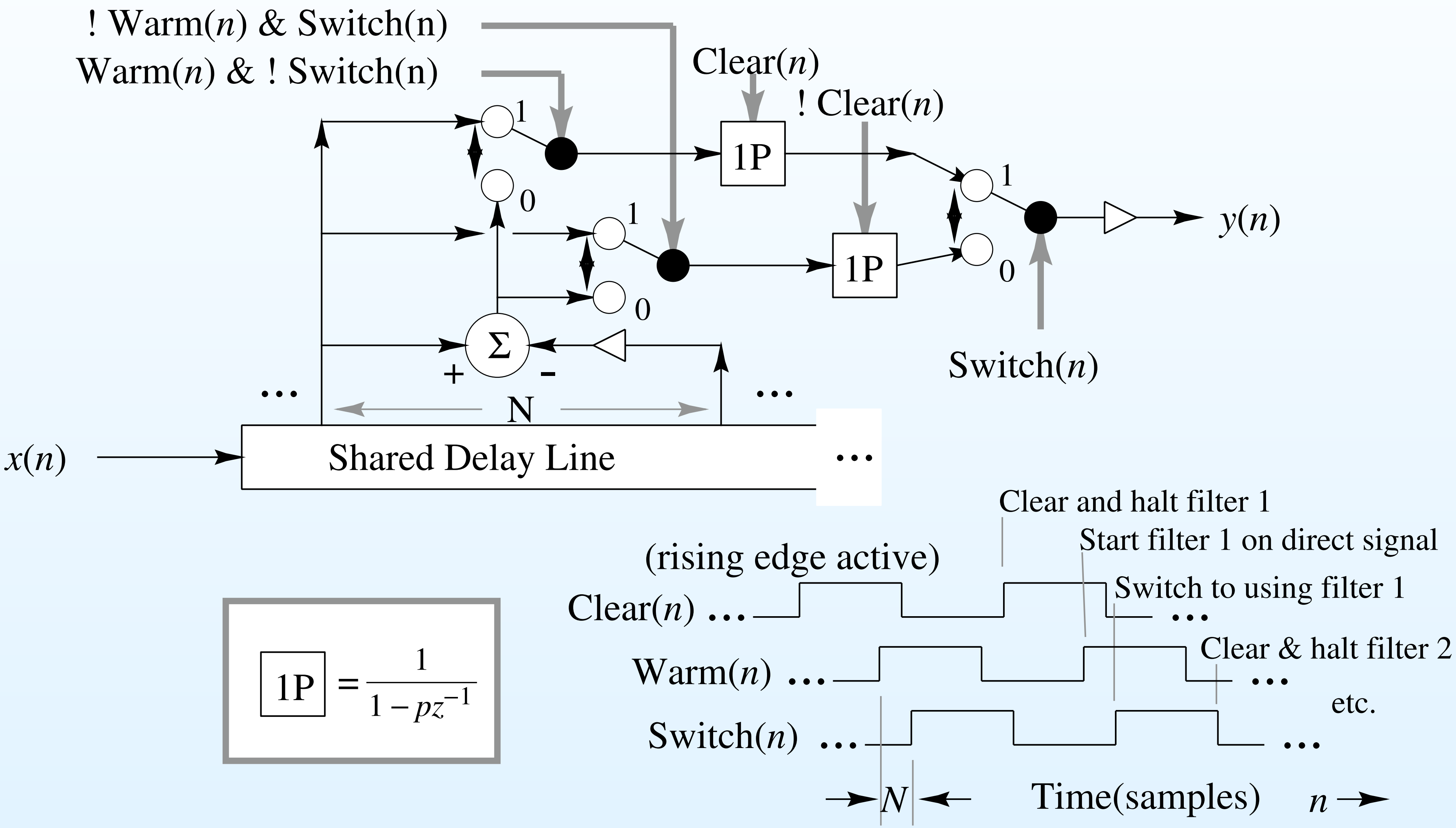
- Perceptrons
- Sequences
- WRoPE
- WRoPE Memory
- TIIR RNNs
- TIIR Sliding Window
- TIIR Resets
- Compressed Time
- Reservations

Attention

History Samples

Ping-Ponging Periodically Resetting RNNs (Classic TIIR Trick)

Needed when there is a *forgetting factor* p in the hidden-state feedback:



$$\boxed{1P} = \frac{1}{1 - pz^{-1}}$$

Searching a Time Range

- Reserving complex angle for positional encoding $\Leftrightarrow$ *real* vocabulary embedding vectors
- Downstream weights and biases must be *complex*
- Complex angle discarded by absolute value when no longer needed
- Exact orthogonality is obtained for each of the L RoPE time-steps when there is no amplitude decay as in normal RNNs, *i.e.*, $e^{jn\Delta}\underline{x}(n) \perp e^{jm\Delta}\underline{x}(n)\,\forall n \neq m$
- RoPE query vector for a vocabulary vector (“token”) $\underline{w}$ any time between n_1 to n_2 is

$$\underline{q} \;=\; \sum_{n=n_1}^{n_2} e^{jn\Delta}\underline{w} \;=\; \underline{w}\,\frac{e^{j(n_2+1)\Delta}-e^{jn_1\Delta}}{e^{j\Delta}-1} \;=\; \underline{w}\,e^{j\theta_{12}}\,\frac{\sin\left(\frac{n_2-n_1+1}{2}\Delta\right)}{\sin(\Delta/2)}$$

Compressed Time Retrieval

- Recall that a vector $\underline{w}$ is detected in the vector memory $\underline{h}$ by the inner product $\underline{w}^T\underline{x}$ exceeding some detection threshold b, *i.e.*, $\underline{w}^T\underline{x} > b$
- The threshold b can be lowered to capture a wider range of similar vectors
- The captured vectors lie in a cone at angle θ centered on the vector $\underline{w}$, and $\cos(\theta) = b$
- In WRoPE, the angular rotation decreases as $1/n$ or the like
- ...

This causes inner products to increase as vectors get squeezed together along the complex-angle arc.

Reserving Angular Dimensions

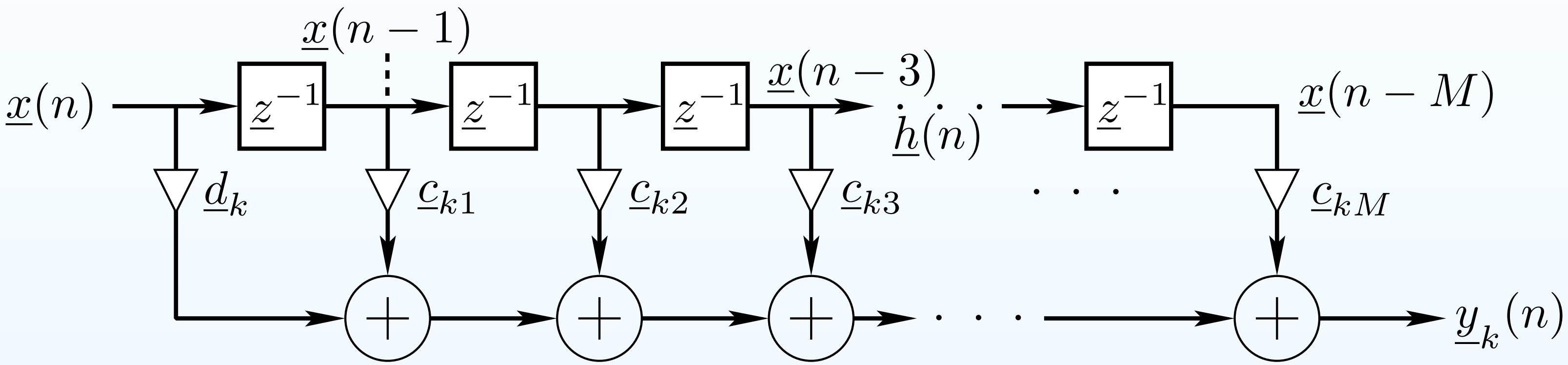
Reserving the radial dimension looks easy (*e.g.*, `RMSNorm` every k gradient-descent steps)
How to reserve an *arc dimension* for [W]RoPE and perhaps other purposes?

- RoPE: Effectively reserves *complex angle:* $\underline{h}_a(n) = e^{j\Delta}\underline{h}(n)$
(data size doubled: $\underline{x} \to (\underline{x}, \underline{0})$)
- A zero imaginary part in the vocabulary embedding reserves complex angle
- *Quaternions* give two more angles ⇔ embedding size quadrupled: $\underline{x} \to (\underline{x}, \underline{0}, \underline{0}, \underline{0})$
  - $\underline{h}_a(n) = \mathbf{q}\,\underline{h}(n)$, where, from Wikipedia:
  $\mathbf{q} = e^{\frac{\Delta}{2}(\underline{a}_x\mathbf{i}+\underline{a}_y\mathbf{j}+\underline{a}_z\mathbf{k})} = \cos\frac{\Delta}{2} + (\underline{a}_x\mathbf{i} + \underline{a}_y\mathbf{j} + \underline{a}_z\mathbf{k})\sin\frac{\Delta}{2} = \cos\frac{\Delta}{2} + \underline{a}\sin\frac{\Delta}{2}$
- See the Cayley-Dickson Construction for $2^k - 1$ angles, $k = 1, 2, 3, \ldots$
- Trivial in *polar coordinates* (“write-protect” reserved dimensions during gradient step), but then back-propagation must be rewritten for polar coordinates
- Find a suitable method of *constrained gradient descent*
- Train (or calculate) a *pointwise MLP* that “de-rotates” $\underline{x} + \mu\underline{\nabla}$ given also $\underline{x}$
- Use *reserved translational dimension(s)* instead (“TraPE”) (omitted from `RMSNorm`)

Attention

Attention Layer

Idea: Also use *FIR Filtering* (SSM State Expansion Factor M, $\mathbf{A}$ subdiagonal):



Separately learnable FIR coefficient matrices $\underline{d}_k[\underline{x}(n)], \underline{c}_j[\underline{x}(n-j), k]$, depending on:

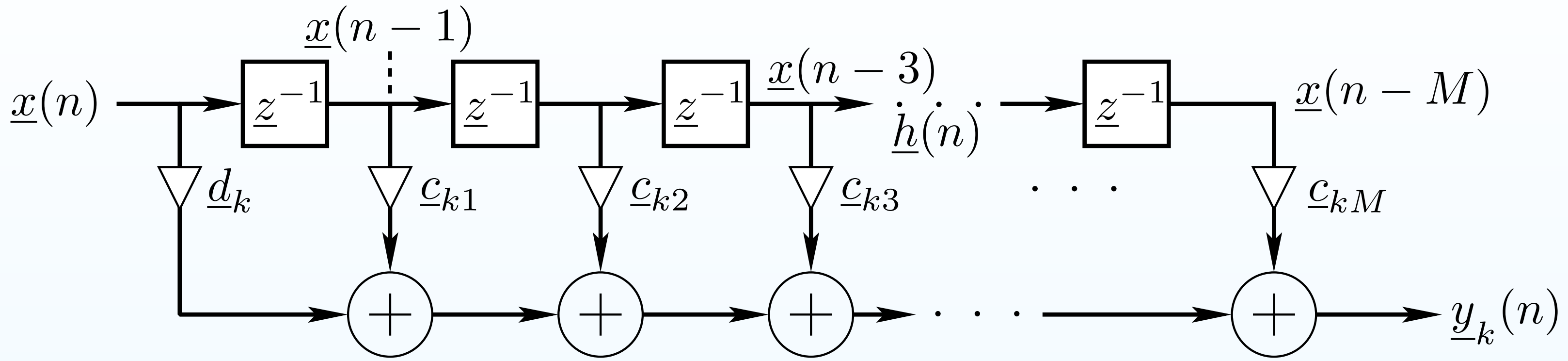
1. input *position* j in the input sequence (“context buffer” or “expanded state” + [W]RoPE)
2. input *vector* $\underline{x}(n-j),\ j = 0, 1, 2, \dots, M$
3. *output-position* k being computed, $k = 0, 1, 2, \dots, M$ ($M + 1$ outputs)

Idea: Add *relevance gating* suppressing unimportant inputs to each output (“attention”)

Idea: Create *new embedding vectors* as *sums* of relevant input vectors (“attention”)

Idea: Measure relevance using an *inner product* between the output and input positions (“dot-product attention”)

Dot-Product Attention



Relevance Gating

Let $\underline{x}_k$ denote $\underline{x}(n-k)$

The contribution from input $\underline{x}_j$ to the nonlinear FIR sum for output $\underline{y}_k$ can be calculated as

$$\underline{c}_{kj}\underline{x}_j = \left[\left(\sum_{m\in\mathcal{R}(\underline{x}_k)}\underline{x}_m\right)^T\underline{x}_j\right]\underline{x}_j$$

or more generally $\boxed{\underline{c}_{kj} = Q_k^T\underline{x}_j}$, where

$Q_k(\underline{x}_k, k)$ is called the *query* vector for position k in the input sequence

The query Q_k can be a sum of *all vectors supported in the attention sum*:

$Q_k = \underline{x}_k + \underline{x}_{m_1} + \cdots + \underline{x}_{m_k}$

$\Rightarrow (Q_k^T\underline{x}_j)\underline{x}_j \approx \underline{x}_j$, if $\underline{x}_j$ is similar to *any vector* in the query sum.

Multi-Head Attention

Idea: To support multiple meaning possibilities, *partition the model space* into parallel independent *attention calculations* (“multi-head attention”)

- Each *attention head* can form an independent input interpretation
- Useful for *ambiguous* sequences, especially in the lower layers
- Also introduced in the Transformer paper (2017)

Now we need *down-projections* of the relevance-calculation components
$\Rightarrow$ relevance of input j to output k in attention-head l becomes proportional to

$$\underline{c}_{kj}\underline{x}_j = (Q_k^T\underline{x}_j)\underline{x}_j \;\longrightarrow\; \underline{c}_{lkj}\underline{x}_{lj} = \left[Q_{lk}^T(\underline{x}_k)K_{lj}(\underline{x}_j)\right]V_{lj}(\underline{x}_j)$$

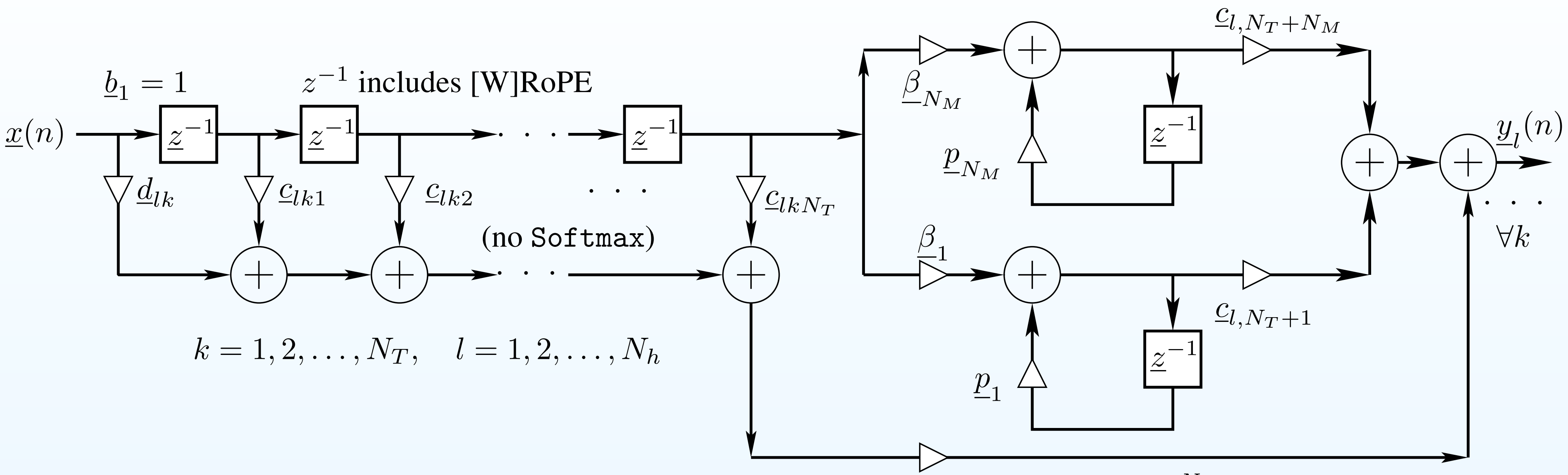
where Q_{lk} (“query”), K_{lj} (“key”), and V_{lj} (“value”) vectors are learned *down-projections* of the input $\underline{x}_j$ for each attention-head l and for all sequence indices j and k in the context buffer (“Transformer”)
Other useful generalizations can be imagined for these learned (Q,K,V) vectors, such as grouping grammatical functions, creating new model-space regions, etc.

State Space Unification of Transformers and GRNNs

$$\underbrace{\mathbf{A}_{\mathbf{T}} = \underline{e}^{j\Delta_n}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \underline{1} & 0 & \cdots & 0 & 0 \\ 0 & \underline{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \underline{1} & 0 \end{pmatrix}}_{\text{Transformer}} \qquad \underbrace{\mathbf{A}_{\mathbf{M}} = \underline{a}_n\underline{e}^{j\Delta_n}\begin{pmatrix} \underline{1} & 0 & \cdots & 0 \\ 0 & \underline{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{1} \end{pmatrix}}_{\text{Mamba-2 style RNN + [W]RoPE}}$$

$$\mathbf{A}_{\mathbf{TM}} = \underline{e}^{j\Delta_n}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \underline{1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \underline{1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \underline{\beta}_1(n) & \underline{a}_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \underline{\beta}_{N_M}(n) & 0 & \cdots & \underline{a}_n \end{pmatrix} \qquad \mathbf{B}_{\mathbf{TM}} = \begin{pmatrix} \underline{b}_1(n) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

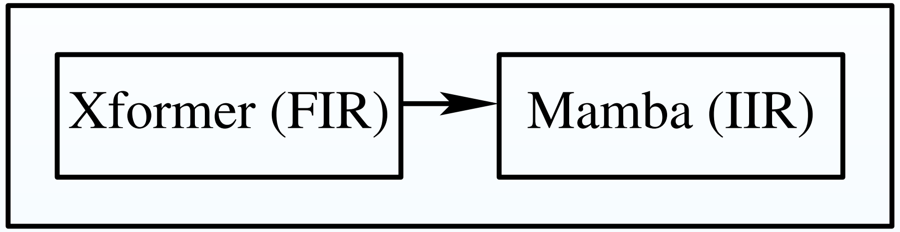
Transformer followed by GRNN with 2x State Expansion (like Mamba)

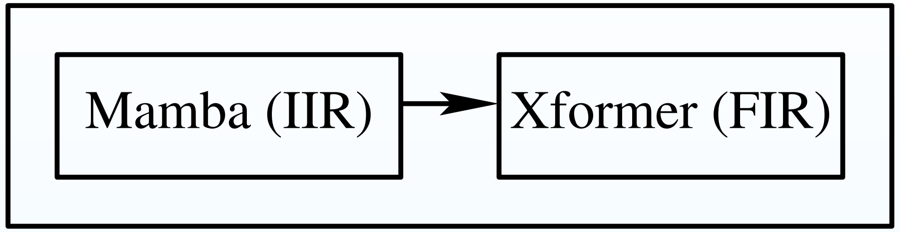


Consider $\dfrac{1}{\sqrt{\sum_{j=1}^{N_T}(Q_k^T K_j)^2}} \Rightarrow \sum_{j=1}^{N_T} \underline{c}_{lkj}^2 = 1$

TransMamba

"Direct Form I" or "Direct Form II"?





XMamba ("DF-I") versus MambaX ("DF-II")

- Perfect short-term memory
- Fuzzy, fading, long-term memory
- Like Infini-Attention

- Fading short-term memory
- Limited long-term memory
- Also like the brain

Maybe use both?

Next Steps

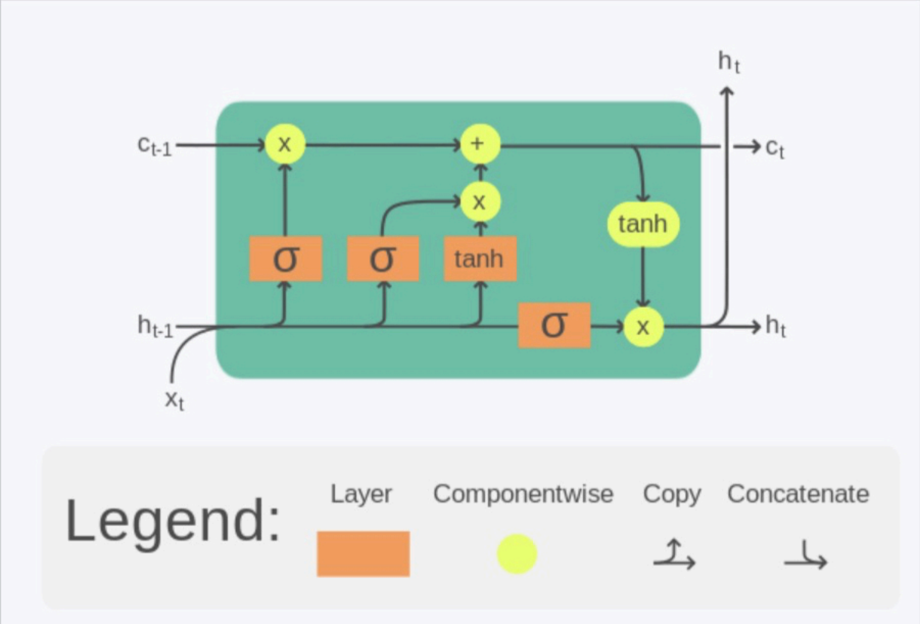
- Try to improve [Trans]**Mamba**[-2] on small synthetic datasets testing *memory*
  - Vocabulary embeddings trained to the unit hypersphere (*e.g.*, `word2sphere`)
  - Memory *duration* and *reset* functions separately trained and implemented
  - Initial *biases* at $\underline{0}$ versus L/N etc.
  - Do *power normalization* in place of `RMSNorm` where possible (efficiency)
  - Try *power normalized attention* in place of $1/\sqrt{d_h}$ and `Softmax` (efficiency)
  - Adapt *model dimension* to *layer width* at each level (efficiency)
  - Truncated Infinite Impulse Response (TIIR) sliding-window memory + linear RoPE
  - Translational Positional Encoding (TraPE) in its own head (no `RMSNorm`)
  - Explore other “Control Heads” that flow along purely for “conditioning” like TraPE
- Progress to date:
  - New synthetic benchmarks analogous to “needle in a haystack”
  - Adapted Andrej Karpathy’s `makemore` code, adding Mamba and new benchmarks
  - Four papers started, aiming for Arxiv, GitHub, “AI social media,” blog
- Feel free to take over any of these! (and LMK so I can do something else)

Thanks for your Attention!

Sequence Modeling Snapshots

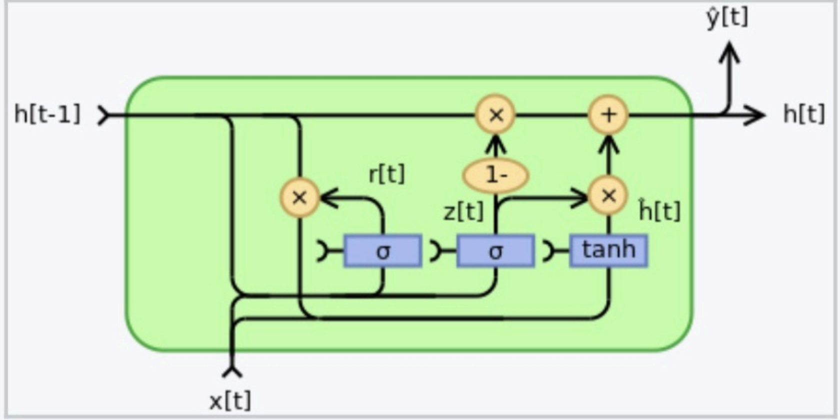
LSTM and GRU

1997: LSTM



The Long Short-Term Memory (LSTM) cell can process data sequentially and keep its hidden state through time.

2014: GRU



Gated Recurrent Unit, fully gated version

Structured State Space and Mamba

2023: Mamba (S6)

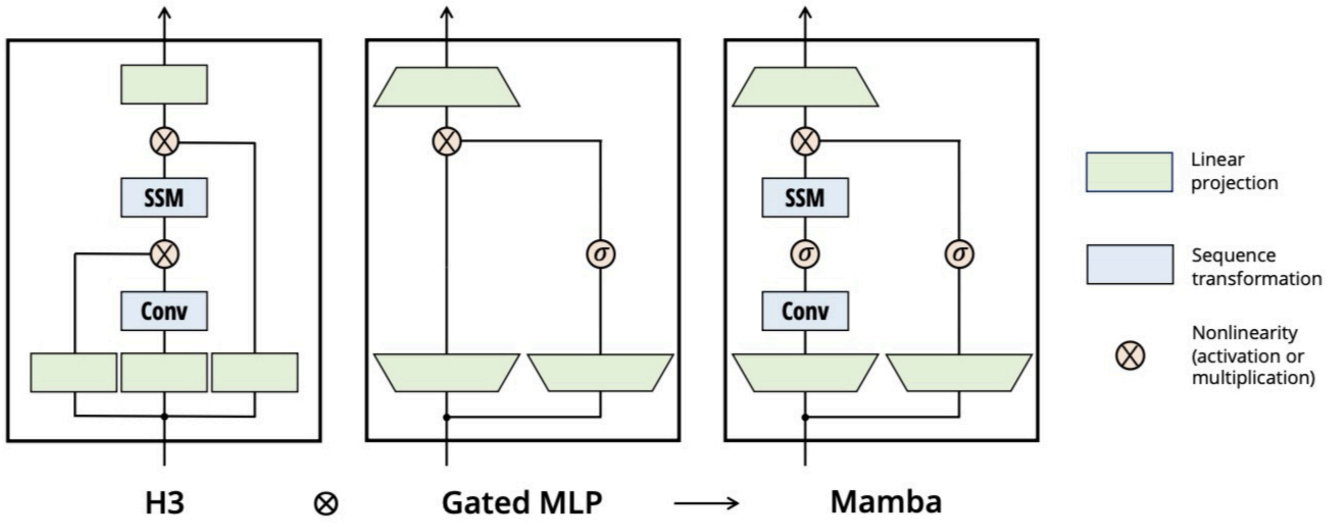


Figure 2: (**Architecture**.) Our simplified block design combines the H3 block, which is the basis of most SSM architectures, with the ubiquitous MLP block of modern neural networks. Instead of interleaving these two blocks, we simply repeat the Mamba block homogenously. Compared to the H3 block, Mamba replaces the first multiplicative gate with an activation function. Compared to the MLP block, Mamba adds an SSM to the main branch. For σ we use the SiLU / Swish activation (Hendrycks & Gimpel, 2016; Ramachandran et al., 2017).

Hawk and Griffin

Griffin: Mixing Gated Linear Recurrences with Local Attention for Efficient Language Models

2024: Hawk & Griffin

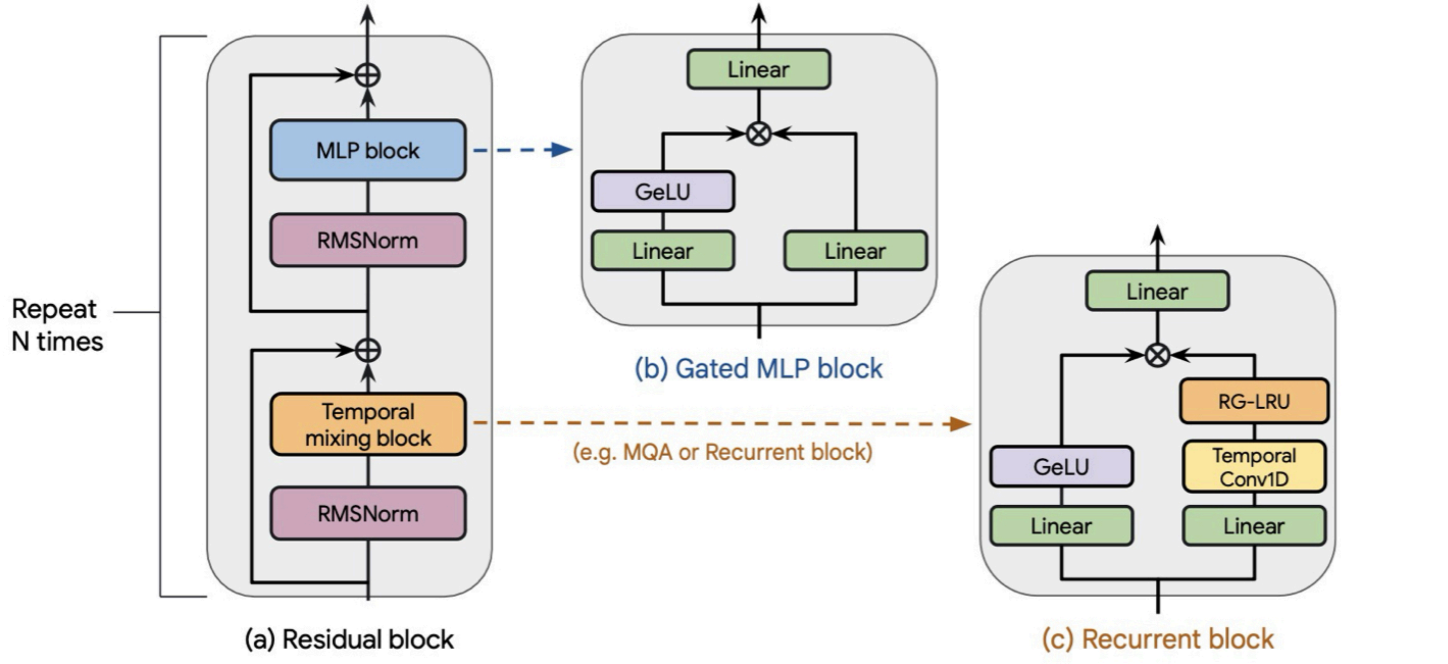
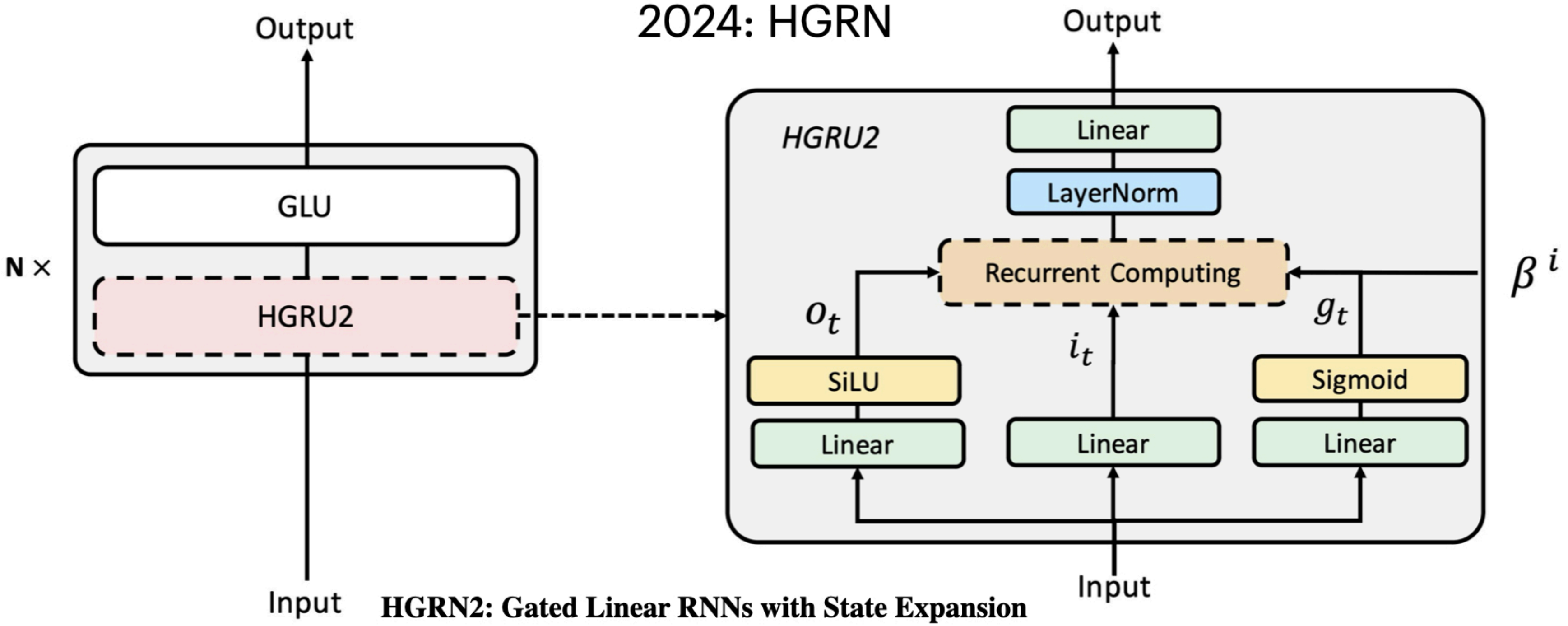


Figure 2 | a) The main backbone of our mode architecture is the residual block, which is stacked *N* times. b) The gated MLP block that we use. c) The recurrent block that we propose as an alternative to Multi Query Attention (MQA). It uses our proposed RG-LRU layer, defined in Section 2.4.

Gated "Linear" RNNs with State Expansion

2024: HGRN



HGRN2: Gated Linear RNNs with State Expansion

Figure 1: **The neural architecture of HGRN2.** Each HGRN2 layer includes a token mixer layer HGRU2 and a channel mixerlayer GLU. HGRU2 employs recurrent computation through Eq. 3, where $\mathbf{i}_t$ is the input vector, $\mathbf{g}_t$ is the forget gate (not lower bounded), β^i is the lower bound of the forget gate value, $\mathbf{o}_t$ is the output gate for layer i.

RWKV, Eagle, Finch

2024: Eagle-Finch RWKV

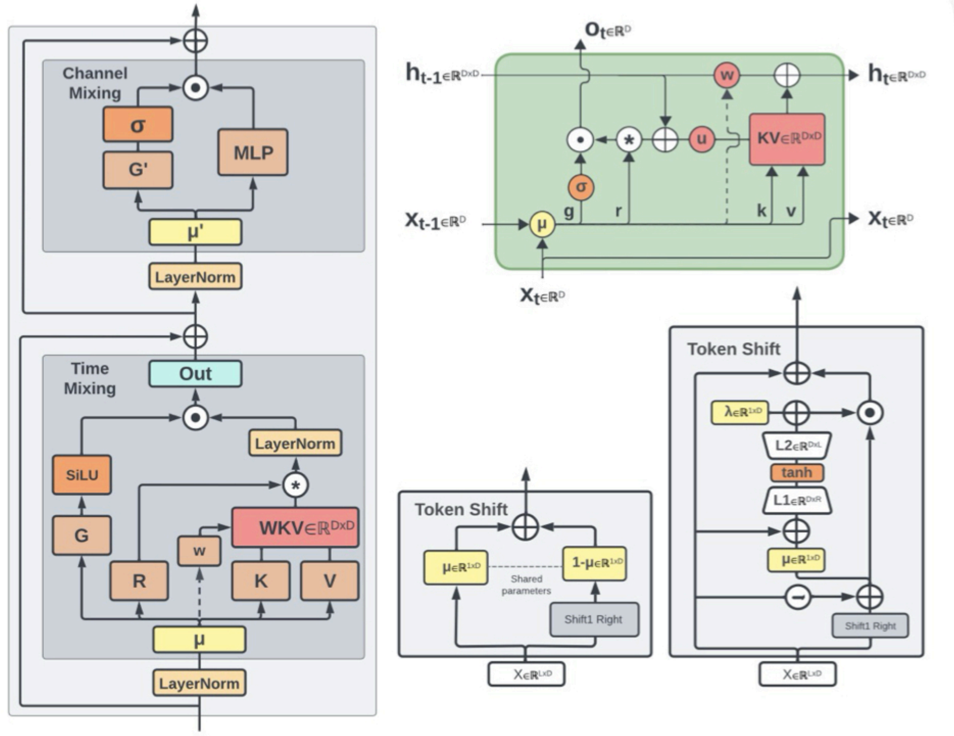


Figure 1: RWKV architecture overview. **Left:** time-mixing and channel-mixing blocks; **top-right:** RWKV time-mixing block as RNN cell; **center-bottom:** token-shift module in FeedForward module and Eagle time-mixing; **bottom-right:** token-shift module in Finch time-mixing. All shape annotations assume a single head for simplicity. Dashed arrows (left, top-right) indicate a connection in Finch, but not in Eagle.

Jamba, Zamba, & Samba Hybrid Architectures (Mamba then Attention)