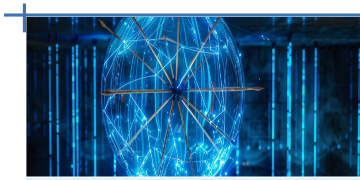


RoPE and WRoPE

- Rotational Positional Encoding (RoPE) owns *one arc direction* along the hypersphere
- We can thus rotate our vector memory $\underline{h}(n)$ by Δ radians each time step to "age" it:

$$\underline{h}_a(n) = e^{j\Delta}\underline{h}(n), \quad \text{with } \Delta = \frac{2\pi}{L}$$

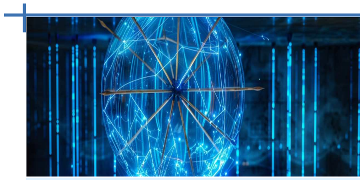
  when our maximum sequence length (before reset) is L
- **Idea:** "Warped RoPE" (WRoPE) for *arbitrarily long sequences* (processed in reverse):

$$\Delta_n = \frac{2\pi n}{n+L}, \quad n = 0,1,2,\ldots$$

  (inspired by the *bilinear transform* used in digital filter design)
- A *blend* of *uniform* and *warped* rotations can be used:

$$\Delta_n = \begin{cases} \frac{\pi n}{L}, & n = 0,1,2,\ldots,L-1 \\ \pi + \frac{\pi n}{n+1}, & n = L, L+1, L+2, \ldots \end{cases}$$

  where L is now the *typical* sequence length (giving it more "space" in recall)



WRoPE Memory

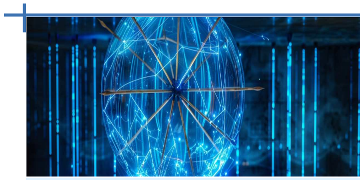
- WRoPE sequences are naturally reversed because we can only change all stored angles by the same delta:

$$\underline{h}_a(n) = e^{j\Delta_n}\underline{h}(n), \quad n = 0, 1, 2, \ldots$$

- This makes inference non-autoregressive (more expensive)
- One improvement is to *store* past hidden states so that positional encodings can be updated arbitrarily when accessed:

$$\underline{h}_a(n,m) = e^{j\Delta_{n-m}}\underline{h}(m), \quad m = n-L, \ldots, n-1, n$$

(mth hidden state vector needed for inference at time n)
- This is the same amount of storage needed for the Truncated Infinite Impulse Response (TIIR) technique which provides a recursively computed sliding-window of memory
- In the TIIR case (fixed length L), might as well use normal RoPE
- WRoPE maybe competitive for encoding "journalistic style" into a vector



Truncated Infinite Impulse Response (TIIR) RNNs

A *sliding rectangular window* can be obtained as an integrator minus a *delayed* integrator:

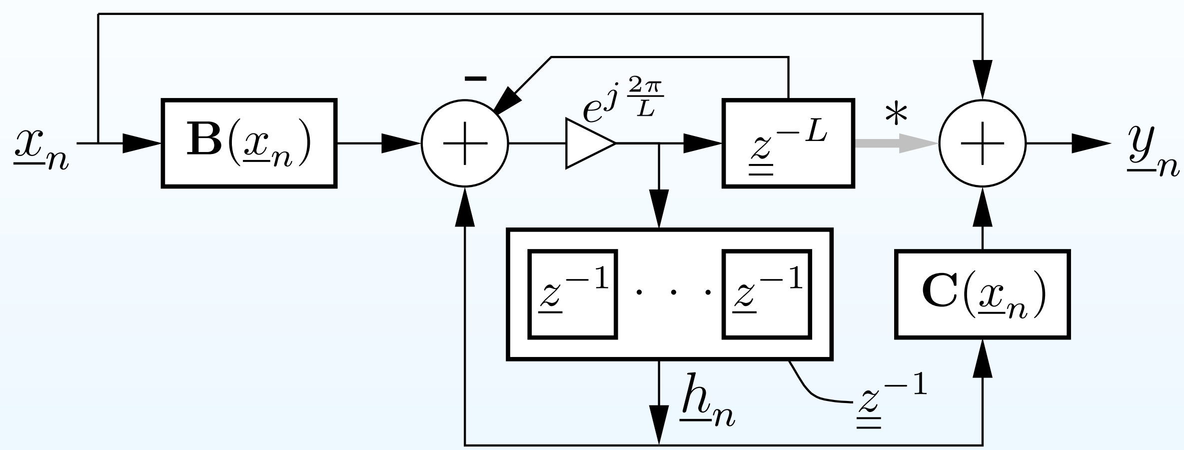
$$[1, 1, \ldots, 1] \quad \longleftrightarrow \quad \sum_{n=0}^{N-1} z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}} = \boxed{\frac{1}{1 - z^{-1}} - z^{-N} \frac{1}{1 - z^{-1}}}$$

- Thus, two identical RNNs can be *differenced* to provide a non-fading, linearly RoPEd memory of any length L
- A *real* memory of length L is needed for the *hidden state update:* $\underline{dh}(n) = \underline{h}(n+1) - \underline{h}(n) = \mathbf{B}_n \underline{x}(n)$
- Hidden state update becomes

$$\begin{aligned} \underline{h}(n+1) &= \underline{h}(n) + \underline{dh}_n \\ &= \underline{h}(n) + \mathbf{B}_n \underline{x}(n) - \mathbf{B}_{n-L}\, \underline{x}(n-L) \end{aligned}$$

- **Problem:** Accumulating floating-point round-off error (variance increases linearly)

TIIR RNN with Sliding-Window Memory and Linear RoPE



* Optional Attention Sum