GEF4510 PROBLEM SET 10

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Storm Surge

a) We have the follwoing non-linear shallow water equations supplied by the exercise:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla_H \cdot \left(\frac{\mathbf{U}\mathbf{U}}{h} \right) + f\mathbf{k} \times \mathbf{U} = -gh\nabla_H \zeta + \frac{\boldsymbol{\tau}_s - \boldsymbol{\tau}_b}{\rho_0} \tag{1}$$

$$\frac{\partial h}{\partial t} + \nabla_H \cdot \mathbf{U} = 0 \tag{2}$$

Firstly we note that (2) already is and when we execute the dot product and utilize the fact the we are looking at a straight coast that we treat as a 1D (in space) problem ($\partial/\partial y = 0$) we get

$$\frac{\partial h}{\partial t} + \nabla_H \cdot \mathbf{U} = \frac{\partial h}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \frac{\partial h}{\partial t} + \frac{\partial U}{\partial x} = 0$$

Then we look at (1). As linear a part of the linearization and since $|\mathbf{U^2}| \ll |\mathbf{U}|$, we assume that the second term on the LHS of (1) is neglectable. For the first term on the RHS we have that $\zeta = h - H$ and if we use the insignificance of variations in the equuillibrium depth H we can write

$$gh\nabla_H \zeta = gh\nabla_H(h - H) = gh(\nabla_H h - \nabla_H H) = gh\nabla_H h \approx gH\nabla_H h = gH\left(\frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j} \right) = gH\frac{\partial h}{\partial x}\mathbf{i}$$

where in the last step we have used no variations in y and the assumption that the surface level deviation ζ is much less than the equillibrium depth H, i.e. $\zeta \ll H$, which implies that $h = \zeta + H \approx H$ is a good approximation for this factor (we can't do this with the terms involving changes in h as h very much does change with time and space while H does not (at least not significantly)). Then we write out the components of the various vectors in (1) to obtain

$$\frac{\partial U}{\partial t}\mathbf{i} + \frac{\partial V}{\partial t}\mathbf{j} + fU\mathbf{j} - fV\mathbf{i} = -gH\frac{\partial h}{\partial x}\mathbf{i} + \frac{\tau_s^x - \tau_b^x}{\rho_0}\mathbf{i} + \frac{\tau_s^y - \tau_b^y}{\rho_0}\mathbf{j}$$

Here we have a vector equation which must be satsified for both the components/directions seperately so this brings us to the linearized versions

$$\frac{\partial U}{\partial t} - fV = -gH\frac{\partial h}{\partial x} + \frac{\tau_s^x - \tau_b^x}{\rho_0} \tag{3}$$

$$\frac{\partial V}{\partial t} + fU = \frac{\tau_s^y - \tau_b^y}{\rho_0} \tag{4}$$

$$\frac{\partial h}{\partial t} + \frac{\partial U}{\partial x} = 0 \tag{5}$$

where (3) and (4) are the linearized versions of (1) and (5) comes from (2).

b) Going forward we now use $H = H_0 =$ const. in our expressions. The Ekman solution is the natural open boundary condition because as $x \to \infty$ we don't expect the effects due ot the boundary (coast) at $x = 0$ to have any effect on the solution. Thus we don't expect any changes with x in our dependent variables and we have a situation (at $x \to \infty$) where we really only experience the surface wind stress generating steady flow and therefore and unchanging water surface at a constant equillibrium height. To this end the Ekman solution provides a good boundary condition as $x \to \infty$. Initially we have specified that $h = H_0, \ \forall x$ (ocean at rest and in equillibrium height) and since the Ekman solution is steady ($\partial/\partial t = 0$) we see that (with the Ekman BC) at $x \to \infty$ we don't allow any changes with time of the surface level. Meaning that the initial condition will prevail for all t and thus imply the BC

$$h = H_0 \quad \Rightarrow \quad \frac{\partial h}{\partial x} = 0 \ , \qquad x \to \infty, \ \forall t$$

When it comes to the no flow condition at the coast, this can be formulated as $u = 0, \ x = 0, \ \forall t$ (the actual velocity in x-direction) Using the definition of $\mathbf{U}$ we can write

$$\mathbf{U} = \int_{-H}^{\zeta} \mathbf{u}\, dz \qquad \Rightarrow \qquad U\mathbf{i} + V\mathbf{j} = \mathbf{i}\int_{-H}^{\zeta} u\, dz + \mathbf{j}\int_{-H}^{\zeta} v\, dz$$

and again we have a vector equation where each component equation must be satisfied. This means if we have $u = 0, \ x = 0, \ \forall t$, the integrand will be zero and thus $U = 0, \ x = 0, \ \forall t$. As for the initial condition we are specified with a state at rest meaning $\mathbf{u} = \mathbf{0}, \ t = 0, \ \forall x$. If we again use the definition of $\mathbf{U}$, for $t = 0, \ \forall x$, we see that

$$\mathbf{U} = \int_{-H}^{\zeta} \mathbf{u}\, dz = \mathbf{0} \qquad \Rightarrow \qquad U = V = 0 \ , \quad t = 0, \ \forall x$$

The initial condition for h is $h = H_0, \ t = 0, \ \forall x$ and is such, as mentioned above, due to the specified initial state at rest where the surface is in equillibrium height. Finally, in the interest of completeness, we can write down the Ekman solutions using the necessary assumptions ($\partial/\partial t = 0, \ \partial/\partial x = 0, \ \tau_b^x = \tau_b^y = 0$):

$$U = -\frac{\tau_s^x}{\rho_0 f} \ , \qquad V = \frac{\tau_s^y}{\rho_0 f}$$

where the above mention assumptions have been inserted into (3)-(5).

c) If we neglect $\partial U/\partial t$ and solve (3) with respect to V (see very similar calculations in exercise **d)**) we get

$$V = \frac{gH_0}{f}\frac{\partial h}{\partial x} - \frac{\tau_s^x - \tau_b^x}{\rho_0 f}$$

and then we insert this into (4) using (5) for $\partial h/\partial t$ to have

$$-\frac{gH_0}{f}\frac{d^2U}{dx^2} + fU = \frac{\tau_s^y - \tau_b^y}{\rho_0} \qquad \Rightarrow \qquad \frac{d^2U}{dx^2} - \frac{1}{L_R^2}U = -\frac{f(\tau_s^y - \tau_b^y)}{\rho_0 g H_0}$$

Here we see that we have an ordinary differential equation for U (U is only a function of x since the time-rate-of-change was neglected) and since inertial oscillations are a wave-like feature, we can't have this with the time-indepentent nature of U. So we have avoided the inertial oscillations, but having taken away the time varying part of U.

d) The assumption that $\partial U/\partial t$ is small compared to the coriolis acceleration means it's justifiable to neglect the former term when comparing it to fV in (3). Furthermore using the assumptions (in the exercise text) about the stress terms, we can formulate the new set of equations:

$$fV - gH_0\frac{\partial h}{\partial x} = 0 \tag{6}$$

$$\frac{\partial V}{\partial t} + fU = \frac{\tau_s^y}{\rho_0} \tag{7}$$

$$\frac{\partial h}{\partial t} + \frac{\partial U}{\partial x} = 0 \tag{8}$$

Here we see that (6) immediately gives geostrophic balance for the flow V along the coast. The flow in the y-direction is balanced by the pressure gradient force (represented here by the sea surface height). The pressure term contains the factor H_0 because the left side doesn't contain the actual velocity v, but the depth-integrated-velocity V. Then we move on to find the analytic solutions of (6)-(8). We begin solving (6) with respect to V and inserting this into (7) to get

$$V = \frac{gH_0}{f}\frac{\partial h}{\partial x} \qquad \Rightarrow \qquad \frac{gH_0}{f}\frac{\partial^2 h}{\partial x \partial t} + fU = \frac{\tau_s^y}{\rho_0}$$

Next we substitute $\partial h/\partial t$ from (8) into the above obtained expression whilst remembering the assumption that $\partial U/\partial t$ was small comapred to the coriolis acceleration, meaning $U \approx U(x)$ with no time-dependency further giving ordinary derivatives for U in what follows:

$$\frac{\partial h}{\partial t} = -\frac{\partial U}{\partial x} \qquad \Rightarrow \qquad -\frac{gH_0}{f}\frac{d^2 U}{dx^2} + fU = \frac{\tau_s^y}{\rho_0} \qquad \Rightarrow \qquad \frac{d^2 U}{dx^2} - \frac{1}{L_R^2}U + \frac{U_E}{L_R^2} = 0$$

where the last implication is done by rearrangement and usage of the definitions $L_R = \sqrt{gH_0}/f$ and $U_E = \tau_s^y/\rho_0 f$. So now we have a second order linear ordinary differential equation for U. This can be solved by finding the solution to the homogenous version of the equation in addition to finding a particular solution to the full equation. The homogenous is a typical second order equation with exponential solutions (can be solved step-by-step using the method of characteristic equation to find the factor in the exponent (in this case $1/L_R$)) and we get

$$\frac{d^2 U_h}{dx^2} - \frac{1}{L_R^2}U_h = 0 \qquad \Rightarrow \qquad U_h = C_1 e^{\frac{x}{L_R}} + C_2 e^{-\frac{x}{L_R}} \ , \quad C_1, C_2 \in \mathbb{R}$$

For our case we have domain in which x gets increasingly negative when we move away from the coast. This means we must require $C_2 = 0$ in order to have a finite flow (otherwise we have an unphysical solution). When hunting for a particular solution U_p it is a good idea to try a solution that has the same form as the inhomogenous part of the equation, so in this case a constant $U_p = A$. Inserting this into the full ODE gives

$$-\frac{A}{L_R^2} + \frac{U_E}{L_R^2} = 0 \qquad \Rightarrow \qquad A = U_E$$

This in turn gives the total general solution

$$U(x) = U_h(x) + U_p(x) = C_1 e^{\frac{x}{L_R}} + U_E$$

To determine C_1 we apply the coast no flow boundary condition. This gives

$$U(0) = 0 = C_1 + U_E \qquad \Rightarrow \qquad C_1 = -U_E$$

which in turn produces the following analytical result for the flow in the x-direction:

$$U(x) = -U_E e^{\frac{x}{L_R}} + U_E = U_E \left(1 - e^{\frac{x}{L_R}}\right) \tag{9}$$

Next we move on to finding the flow in the y-direction. To accomplish this we insert our newly found (9) into (7). We get

$$\frac{\partial V}{\partial t} = \frac{\tau_s^y}{\rho_0} - fU = \frac{\tau_s^y}{\rho_0} - fU_E \left(1 - e^{\frac{x}{L_R}}\right) = fU_E e^{\frac{x}{L_R}} \qquad \Rightarrow \qquad V = ftU_E e^{\frac{x}{L_R}} + C_3$$

where $C_3 \in \mathbb{R}$ is the integration constant. Then we have the initial condition (state at rest) which gives us the complete expression for V:

$$V(x,0) = 0 = C_3 \qquad \Rightarrow \qquad V(x,t) = ftU_E e^{\frac{x}{L_R}} \tag{10}$$

Finally we find h by integrating (6) using our result in (10):

$$\frac{\partial h}{\partial x} = \frac{f^2}{gH_0} tU_E e^{\frac{x}{L_R}} = \frac{tU_E}{L_R^2} e^{\frac{x}{L_R}} \qquad \Rightarrow \qquad h = \frac{tU_E}{L_R} e^{\frac{x}{L_R}} + C_4$$

Here both the initial condition and the Ekman boundary condition for h gives us the same answer for C_4 so lets use the latter:

$$\lim_{x \to -\infty} h(x,t) = H_0 = C_4 \qquad \Rightarrow \qquad h(x,t) = H_0 \left(1 + \frac{tU_E}{L_R H_0} e^{\frac{x}{L_R}}\right) \tag{11}$$

So we have now reached the analytic solutions (9), (10) and (11) to the set (6)-(8) of equations. We note that due to the negelcting of the $\partial U/\partial t$ we see that U is a function of x only whilst the two tothers V, h are both functions of time and space. The two latter both increase in time and increase as we approach the coastline. However U decrease towards the coast in line with the boundary condition here.

e) Based on the previous assumptions here we list the stress terms: $\tau_s^x = 0$, $\tau_s^y =$ const., $\tau_b^x = 0$, $\tau_b^y = \rho_0 RV/H_0$. If we diffenerntiate (7) (but now also including the additional term of the bottom stress) and solve for dU/dx we get

$$\frac{\partial^2 V}{\partial x \partial t} + f\frac{dU}{dx} = -\frac{R}{H_0}\frac{\partial V}{\partial x} \qquad \Rightarrow \qquad \frac{dU}{dx} = -\frac{1}{f}\frac{\partial^2 V}{\partial x \partial t} - \frac{R}{fH_0}\frac{\partial V}{\partial x}$$

and further insertion of this result into (8) yields

$$\frac{\partial h}{\partial t} - \frac{1}{f}\frac{\partial^2 V}{\partial x \partial t} - \frac{R}{fH_0}\frac{\partial V}{\partial x} = 0$$

and then inserting the expression for V from (6) into the above equation we end up we the following equation

$$\frac{\partial^3 h}{\partial x^2 \partial t} + \frac{R}{H_0}\frac{\partial^2 h}{\partial x^2} - \frac{\partial h}{\partial t} = 0$$

namely a third order PDE for h. Unfortunately I was unable to find the solution to this equation.

f) See attached `visualize_results.py` for calculations of the analytical solution. Below in Figure 1 is plots of all three dependent variables U, V and h in a x-t diagram. More specific; the contours of the variables are plotted. As reflected by (9), and now this plot, we see that U is time-independent and only changes with x We find the largest values away a long away from the coast and then the velocity decreases more rapidly as we get closer to the coast. This is in large part due to the boundary condition of no flow through the coast.

As for V we see both the time- and space-dependency in the plot. We note increasing flow parallel to the coast as we move both closer to the coast and forward in time. This is consistent with what we would expect from the analytic expression in (10), i.e. sort of a linear increase with time and exponential increase with x.

For the water surface level h we se a similar structure as to V, which again is expected based on the solution in (11). We also note that the water surface is more or less flat a long way from the coast (satisfying the Ekman solution as $x \to \infty$) and the gradient gets steeper (contours closer together) as we get closer to the coast (and forward in time).

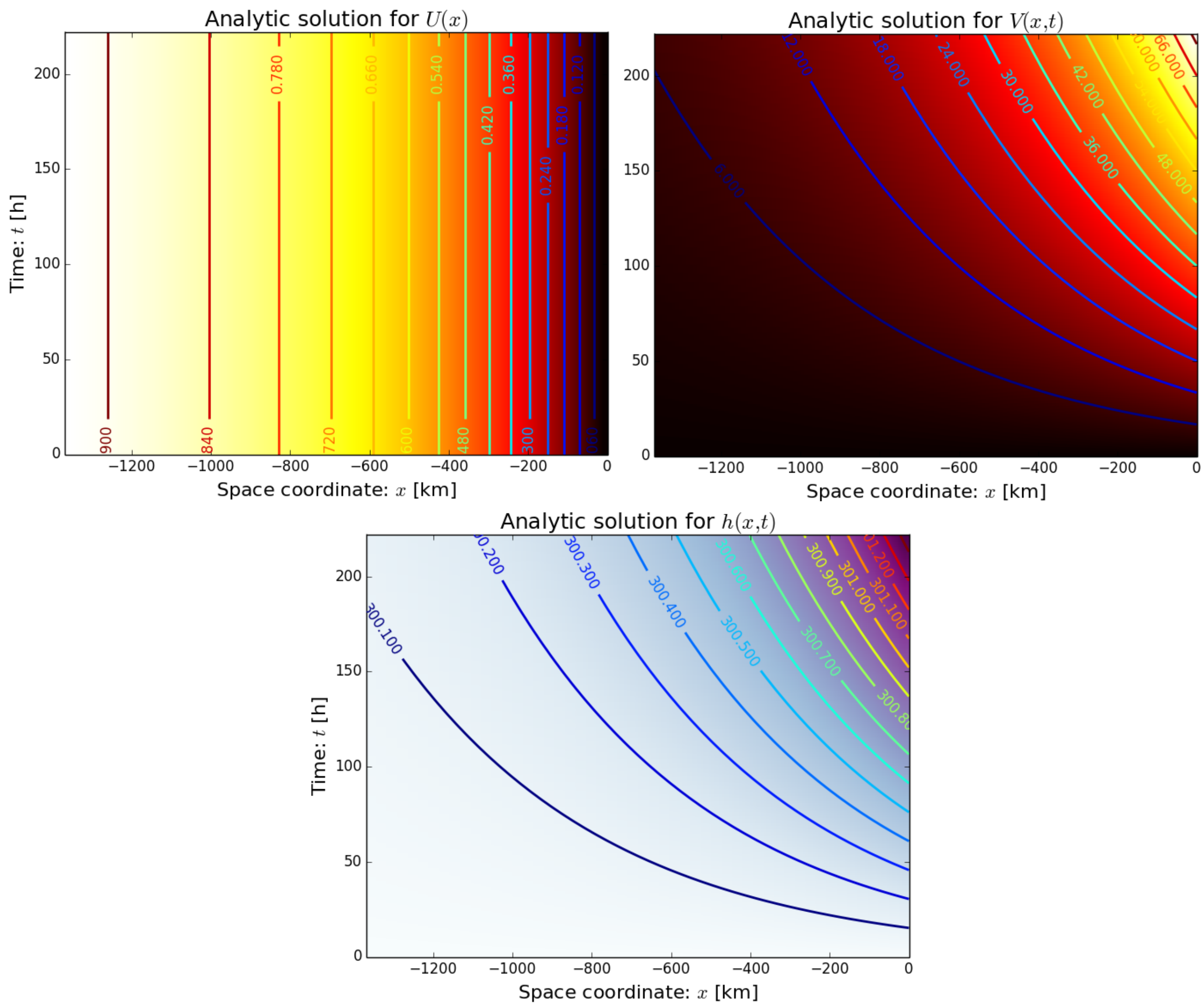

Figure 1: Plot of the analytic solutions found in **d)** using the values from **Part 2** of the exercise.

g) According to the calculations done in **a)** er would end up with and additional term in (3) that comes from the changes in H. We neglected this term in **a)**. Doing similar analysis as in **a)** for the first term on the RHS of (1) we can write

$$gh\nabla_H\zeta = gh\nabla_H(h - H) = gh(\nabla_H h - \nabla_H H) \approx gH\left(\frac{\partial h}{\partial x} - \frac{\partial H}{\partial x}\right)\mathbf{i}$$

where we again used the approximation that $h \approx H$ in the non-channging terms. Again the variations in y (for all variables) are dropped as brefore. So we get an additional term in the x-component of the momentum momentum equation giving

$$\frac{\partial U}{\partial t} - fV = -gH\left(\frac{\partial h}{\partial x} - \frac{\partial H}{\partial x}\right) + \frac{\tau_s^x - \tau_b^x}{\rho_0} \tag{12}$$

$$\frac{\partial V}{\partial t} + fU = \frac{\tau_s^y - \tau_b^y}{\rho_0} \tag{13}$$

$$\frac{\partial h}{\partial t} + \frac{\partial U}{\partial x} = 0 \tag{14}$$

as the set of shallow water equations instead of (3)-(5). Possible additional changes although baked in and not mentioned explicitly in (12)-(14)) is the bottom friction. One might need to make adjustments to the expressions for the bottom friction since the bottom is no longer flat and one would might need to consider the component of the flow parallel to the non-flat bottom instead of just the flow itself.

h) When constructing the FBTCS we use recall the figure supplied by the exercise illustrating the staggering of the grid. Here we see that when centering around a U, V-point (in space) we use h_{j+1} and h_j and when centering around an h-point we use U_j and U_{j-1}. In both these cases the distance is Δx insetad of the normal $2\Delta x$ we are used to in centered schemes. This leads to the following FDA's for the terms in (3)-(5) (listed on lines corresponding to where we aim to use them in correspondence with (3)-(5)):

$$\left[\frac{\partial U}{\partial t}\right]_j^n = \frac{U_j^{n+1} - U_j^n}{\Delta t}, \qquad \left[\frac{\partial h}{\partial x}\right]_j^n = \frac{h_{j+1}^n - h_j^n}{\Delta x}$$

$$\left[\frac{\partial V}{\partial t}\right]_j^n = \frac{V_j^{n+1} - V_j^n}{\Delta t}$$

$$\left[\frac{\partial h}{\partial t}\right]_j^n = \frac{h_j^{n+1} - h_j^n}{\Delta t}, \qquad \left[\frac{\partial U}{\partial x}\right]_j^{n+1} = \frac{U_j^{n+1} - U_{j-1}^{n+1}}{\Delta x}$$

Then we insert these FDA's into (3)-(5) and keep in mind the spirit of the forward-backward in time element of the scheme, i.e. as soon as we compute updated values of a dependent variable we use this when computing updated values for the other dependent variables. Insertion and some rearrangment gives the follwoing scheme

$$U_j^{n+1} = U_j^n + \Delta t \left[fV_j^n - \frac{gH_0}{\Delta x}(h_{j+1}^n - h_j^n) + \frac{\tau_s^x - \tau_b^x}{\rho_0} \right] \tag{15}$$

$$V_j^{n+1} = V_j^n - \Delta t \left[fU_j^{n+1} - \frac{\tau_s^y - \tau_b^y}{\rho_0} \right] \tag{16}$$

$$h_j^{n+1} = h_j^n - \frac{\Delta t}{\Delta x} \left[U_j^{n+1} - U_{j-1}^{n+1} \right] \tag{17}$$

where (15) is valid for $j = 1(1)j_{max} - 1$ since we specify the boundary condition (no flow through the coast) at $j = j_{max}$. Furthermore (16) is valid for $j = 1(1)j_{max}$, i.e. all j's since there our no BC for V. Finally (17) is valid for $j = 2(1)j_{max}$ since we specify the Ekman BC at $j = 1$ ($x = -L$). All three (15)-(17) are valid for $n = 1(1)n_{max}$ in time; we specify the initial rest condition on all three variables at $n = 0$.

i) As suggested in Von Neumann's method we atrt by defining discrete Fourier components for our dependent variables. We define these as

$$U_j^n = U_n e^{i\alpha(j+\frac{1}{2})\Delta x} \ , \qquad V_j^n = V_n e^{i\alpha(j+\frac{1}{2})\Delta x} \ , \qquad h_j^n = h_n e^{i\alpha j \Delta x}$$

where we have implemented the staggering of the grid, i.e. U, V-points are staggered $\Delta x/2$ with respect to h-points. Next we insert these components into (15)-(17), divide through by common factors and do some rearrangement to get

$$U_{n+1} - U_n = f\Delta t V_n - 2i\gamma g H_0 h_n \tag{18}$$

$$V_{n+1} - V_n = -f\Delta t U_{n+1} \tag{19}$$

$$h_{n+1} - h_n = -2i\gamma U_{n+1} \tag{20}$$

where we have defined

$$\gamma = \frac{\Delta t}{\Delta x} \sin\left(\frac{\alpha \Delta x}{2}\right)$$

If we rewrite (18) where we increase the time-counter n by one we have

$$U_{n+2} - U_{n+1} = f\Delta t V_{n+1} - 2i\gamma g H_0 h_{n+1}$$

and then we subtract this result from (18) to see

$$U_{n+2} - 2U_{n+1} + U_n = f\Delta t(V_{n+1} - V_n) - 2i\gamma g H_0 (h_{n+1} - h_n)$$

and here we have an equation in which we can insert both (19) and (20). Doing this gives

$$U_{n+2} - 2U_{n+1} + U_n = -f^2\Delta t^2 U_{n+1} + 4gH_0\gamma^2 U_{n+1}$$

$$U_{n+2} - 2\lambda U_{n+1} + U_n = 0 \tag{21}$$

where the final step is accomplished thorugh factoring out U_{n+1} and defining

$$\lambda = 1 - 2gH_0\gamma^2 - \frac{f^2\Delta t^2}{2}$$

Then if we define the growth factor as $G = U_{n+1}/n$ and insert this into (21) we get the follwing quadratic equation (with the accompanying solution) for G

$$G^2 - 2\lambda G + 1 = 0 \qquad \Rightarrow \qquad G = \lambda \pm \sqrt{\lambda^2 - 1}$$

and we see that if the root is real, i.e. $\lambda^2 > 1$, then $|G^+| > 1$ and we end up with and unstable scheme. However if we assume $\lambda^2 \leq 1$ we get a imaginary root which leads to the following result:

$$G = \lambda \pm i\sqrt{1 - \lambda^2} \qquad \Rightarrow \qquad |G| = \sqrt{\lambda^2 + 1 - \lambda^2} = 1$$

So given the condtition $\lambda^2 \leq 1$ we get a neutrally stable scheme. Then we can investigate what limitations this puts on the time step Δt. Writing out the inequality after taking the square root gives

$$-1 \leq 1 - 2gH_0\gamma^2 - \frac{f^2\Delta t^2}{2} \leq 1 \qquad \Rightarrow \qquad 2gH_0\frac{\Delta t^2}{\Delta x^2} - \frac{f^2\Delta t^2}{2} \leq 2$$

where we have used the fact the the right inequality in the first expression is automatically satisfied since we subtract some positiv evalues from 1. Also to arrive at the implied expression we have written out γ and substituted in the maximum value for the sin-factor (this maximum value is 1 and can safely be done as this just makes Δt smaller and therefopre doesn't violate the inequality). Solving for the time step gives

$$\Delta t^2 \leq \frac{2}{\frac{2gH_0}{\Delta x^2} + \frac{f^2}{2}} = \frac{\Delta x^2/gH_0}{1 + \frac{f^2\Delta x^2}{4gH_0}} \qquad \Rightarrow \qquad \Delta t \leq \frac{\Delta x}{c_0\sqrt{1 + \left(\frac{\Delta x}{2L_R}\right)^2}}$$

where we have applied the definitions of $L_R = c_0/f$ and $c_0 = \sqrt{gH_0}$. So we must satisfy the above condition for Δt in order for our scheme to be stable, and if we do so, we hav a neutrally stable scheme where the growth factor is exactly equal to 1 in absolute value.

A simpler condition arises if we were to use a Δx which is much smaller than twice the Rossby's deformation radius, i.e. $\Delta x \ll 2L_R$ or even just $\Delta x \ll L_R$. Then the second term in the square root is approximately zero and we may write

$$\Delta t \leq \frac{\Delta x}{c_0} \qquad \Rightarrow \qquad c_0\frac{\Delta t}{\Delta x} = C \leq 1$$

So for most applications, which in general tries to resolve motion on the L_R-scale (it's even the dominant length scale for shallow water problems), it's safe to use this simpler condition. In fact for the program witten in exercise **k)**, this condition is used.

Finally we can note that since the growth factor $G = U_{n+1}/U_n$ is equal to 1 in absolute value, we see by $U_{n+1} = GU_n$ that we have now loss of amplitude goning from one time level to the next, and hence our scheme is not dissipative, this is a result wich applies to all neutrally stable schemes.

j) I order to show that the scheme (15)-(17) is consistent we make use of Taylor series. To this end we therefore need the Taylor series of all terms referring to the previous/next time- or space-point. This can be found using the standard formula for Taylor series for all the terms except U_{j-1}^{n+1} located in (17). Here we have an increment in both space and time. Thus we need to make use of the two-dimensional Taylor generalization. Below we list all the required Taylor series (explicitly to first order)

$$U_j^{n+1} = U_j^n + \frac{\partial U}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta t^2) \tag{22}$$

$$V_j^{n+1} = V_j^n + \frac{\partial V}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta t^2) \tag{23}$$

$$h_j^{n+1} = h_j^n + \frac{\partial h}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta t^2) \tag{24}$$

$$h_{j+1}^{n} = h_j^n + \frac{\partial h}{\partial x}\bigg|_j^n \Delta x + \mathcal{O}(\Delta x^2) \tag{25}$$

$$U_{j-1}^{n+1} = U_j^n - \frac{\partial U}{\partial x}\bigg|_j^n \Delta x + \frac{\partial U}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta x^2, \Delta t^2) \tag{26}$$

The next step is to insert these into our scheme (18)-(20). Using (22) and (25) in (15), (23) and (22) in (16) and (24), (22) and (26) in (17) gives

$$U_j^n + \frac{\partial U}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta t^2) = U_j^n + \Delta t \left\{ fV_j^n - \frac{gH_0}{\Delta x}\left[h_j^n + \frac{\partial h}{\partial x}\bigg|_j^n \Delta x + \mathcal{O}(\Delta x^2) - h_j^n \right] + \frac{\tau_s^x - \tau_b^x}{\rho_0} \right\}$$

$$V_j^n + \frac{\partial V}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta t^2) = V_j^n - \Delta t \left\{ f\left[U_j^n + \frac{\partial U}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta t^2) \right] - \frac{\tau_s^y - \tau_b^y}{\rho_0} \right\}$$

$$h_j^n + \frac{\partial h}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta t^2) = h_j^n - \frac{\Delta t}{\Delta x}\left[U_j^n + \frac{\partial U}{\partial t}\bigg|_j^n \Delta t - U_j^n + \frac{\partial U}{\partial x}\bigg|_j^n \Delta x - \frac{\partial U}{\partial t}\bigg|_j^n \Delta t + \mathcal{O}(\Delta x^2, \Delta t^2) \right]$$

Then some terms eliminate each other and we can do some further rearrangment to get

$$\frac{\partial U}{\partial t}\bigg|_j^n + \mathcal{O}(\Delta t) - fV_j^n = -gH_0 \frac{\partial h}{\partial x}\bigg|_j^n + \mathcal{O}(\Delta x) + \frac{\tau_s^x - \tau_b^x}{\rho_0} \tag{27}$$

$$\frac{\partial V}{\partial t}\bigg|_j^n + \mathcal{O}(\Delta t) + f\left[U_j^n + \frac{\partial U}{\partial t}\bigg|_j^n \Delta t \right] = \frac{\tau_s^y - \tau_b^y}{\rho_0} \tag{28}$$

$$\frac{\partial h}{\partial t}\bigg|_j^n + \frac{\partial U}{\partial x}\bigg|_j^n + \mathcal{O}(\Delta x, \Delta t^2) = 0 \tag{29}$$

and if we now let Δx and Δt go towards zero independently we see that we arrive back at our original set of equations (3)-(5). We can then conclude that our scheme (15)-(17) is consistent. Since we now know that the scheme is stable (from **i)** and consistent, we further know, through the Lax equivalence theorem, that the scheme is convergent, i.e. it will converge towards the actual solution of the equations.

k) The numerical solution together with the scheme (15)-(17) is implemented in the attached program `storm_surge.f90`. The choice of $\Delta x \ll L_R$ is important in part because it is convenient to use the simplified stability condition $C \leq 1$, but mainly because Rossby's deformations radius L_R is the dominant length scale for shallow water problems, meaning the main part of the dynamics take place on this scale, and we wish to resolve these dynamics in order to see what's going on. If we were to choose a Δx close to L_R or even bigger, we would not be able to resolve these dynamics and a huge point of even doing the simulation in the first place, will be lost.

l) In Figure 2 there is plots of the numerical solutions U, V and h in a x-t diagram. A value of $C = 0.8$ was used when generating these solutions. Also the maximum distance away from the coast was set to $x = -L = -10L_R$ and worth noting is that for these numerical solutions, the plots extend to $-L/2$ on the x-axis whereas for the analytical plots, the x-axis extended to $L/4$. The choice of plotting a region closer to the coast than the entire domain, is that this is where the interesting stuff happens. Far out the the sea surface is nearly flat (Ekman BC) and not much to talk about. The choice of having slightly different regions plottet in the analytical solution was due to diffuculties in getting the contours look nice for the analytical solutions, but the important thing is to look at the values on the contour lines and to what x, t-point they correspond to.

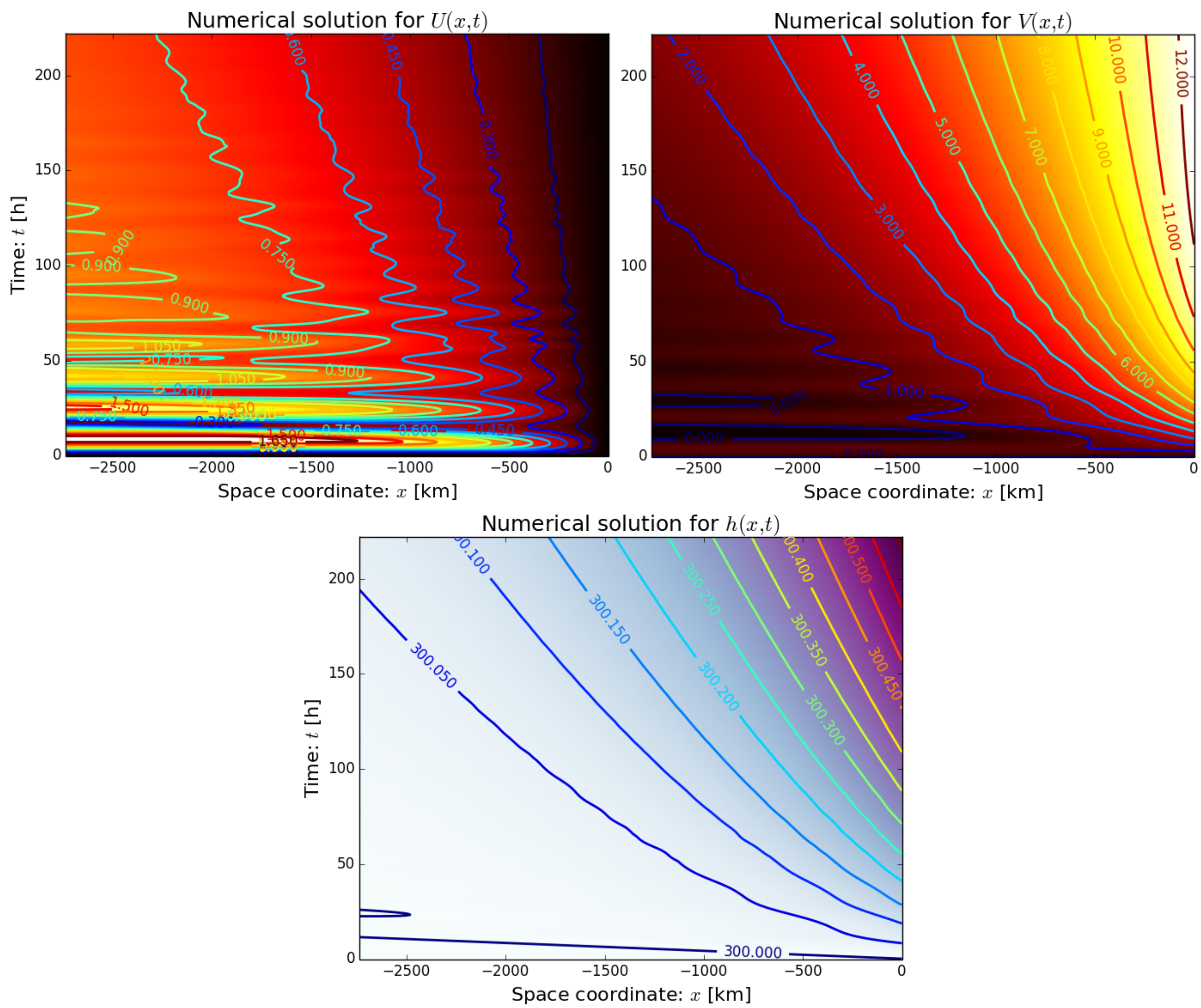

Figure 2: Plot of the numerical solutions found in **k)** using $C = 0.8$

Regarding the plots above, almost all of the comments made for the analytical plots in **f)** also applies here to the numerical plots, and `visualize_results.py` once again operates as the plotting script. That being said, there are definetly some differences. throughout all three variables we see a smoother and less fluctuating trend for the analytical solutions. This is in large part due to the fact that while solving the analytical case we had to make even further simplifications and thus leaving out some features that we do see in the numerical solutions. A main factor here is the $\partial U/\partial t$-term. In the analytical solutions we see straight vertical lines meaning no change with time, or in other words, time-independence (we also see this in the actual analytical expression for U in (9)). But in the numerical solutions we clearly see a time depency for U and a plot that looks much more interesting. We have fluctuating wave-like fetaures emerging. The difference in U is maybe the most striking difference between the analytical and numerical approach. There are differences also in V, but here the analytical solutions is much more similar than for U. For h the solutions are very similar (although take into account the different values on the x-axis) and the analytical one seems to have done a good job here. For all three variables, as with the analyitcal case, we see that for times further into the future, the solution becomes less time-dependent and we might see a trend towards a steady state eventually. As a final note we can see that in general, for all three variables, that the values in the analytical solution, is somewhat larger than for the numerical solution, especially for V, but also slightly for the other two. This may come from neglecting the effect of bottom friciton in **d)** and therefore loosing the effect of the sea-bottom slowing the flow down. But in the numerical simulation, this effect is present in our scheme.