GEF4510 PROBLEM SET 11

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Method of characteristics applied to a non-linear system

a) Using the specified pressure $gh\nabla_H h$ term (where ultimately we neglect the varations in y, but keep the del operator for now), we have the equations specified by the exericse:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla_H \cdot \left(\frac{\mathbf{UU}}{h}\right) + f\mathbf{k} \times \mathbf{U} = -gh\nabla_H h + \frac{\boldsymbol{\tau}_s - \boldsymbol{\tau}_b}{\rho_0} \tag{1}$$

$$\frac{\partial h}{\partial t} + \nabla_H \cdot \mathbf{U} = 0 \tag{2}$$

Then we wish to rewrite these shallow water equations in terms of the actual velocity $\mathbf{u}$ using the definition $\mathbf{U} = h\mathbf{u}$. In the following derivation we keep in mind that we are neglecting variations in y, i.e. $\partial/\partial y = 0$. For readability we write out each individual term in (1) and (2) by itself. Starting with the first term in (1) we have

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial}{\partial t}(h\mathbf{u}) = h\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial h}{\partial t} = \left(h\frac{\partial u}{\partial t} + u\frac{\partial h}{\partial t}\right)\mathbf{i} + \left(h\frac{\partial v}{\partial t} + v\frac{\partial h}{\partial t}\right)\mathbf{j} \tag{3}$$

Then we move on t the non-linear term on the LHS in (1). Here we employ the product rule of differentation to rewrite the term as it stands in the equation:

$$\begin{aligned}
\nabla_H \cdot \left(\frac{\mathbf{UU}}{h}\right) &= \nabla_H \cdot (h\mathbf{uu}) = \mathbf{u}\nabla_H \cdot (h\mathbf{u}) + (h\mathbf{u} \cdot \nabla_H)\mathbf{u} \\
&= (u\mathbf{i} + v\mathbf{j})\left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (hu\mathbf{i} + hv\mathbf{j})\right] \\
&= (u\mathbf{i} + v\mathbf{j})\frac{\partial}{\partial x} + hu\frac{\partial}{\partial x}(u\mathbf{i} + v\mathbf{j}) \\
&= \left(2hu\frac{\partial u}{\partial x} + u^2\frac{\partial h}{\partial x}\right)\mathbf{i} + \left(hv\frac{\partial u}{\partial x} + hu\frac{\partial v}{\partial x} + uv\frac{\partial h}{\partial x}\right)\mathbf{j}
\end{aligned} \tag{4}$$

Next the coriolis expands into the following terms

$$f\mathbf{k} \times \mathbf{U} = hf\mathbf{k} \times \mathbf{u} = hf\mathbf{k} \times (u\mathbf{i} + u\mathbf{j}) = hfu\mathbf{j} - hfv\mathbf{i} \tag{5}$$

furthermore the pressure term on the RHS of (1) results in

$$gh\nabla_H h = gh\frac{\partial h}{\partial x}\mathbf{i} \tag{6}$$

and finally, if we assume stress terms is made up of components where $\boldsymbol{\tau}_s = \tau_s^x\mathbf{i} + \tau_s^y\mathbf{j}$ and $\boldsymbol{\tau}_b = \tau_b^x\mathbf{i} + \tau_b^y\mathbf{j}$, we have

$$\frac{\boldsymbol{\tau}_s - \boldsymbol{\tau}_b}{\rho_0} = \frac{\tau_s^x - \tau_b^x}{\rho_0}\mathbf{i} + \frac{\tau_s^y - \tau_b^y}{\rho_0}\mathbf{j} \tag{7}$$

Then we can write out the two resulting equations for the two directions x and y using the results from (3)-(7) to get

$$h\frac{\partial u}{\partial t} + u\frac{\partial h}{\partial t} + 2hu\frac{\partial u}{\partial x} + u^2\frac{\partial h}{\partial x} - hfv = -gh\frac{\partial h}{\partial x} + \frac{\tau_s^x - \tau_b^x}{\rho_0} \tag{8}$$

$$h\frac{\partial v}{\partial t} + v\frac{\partial h}{\partial t} + hv\frac{\partial u}{\partial x} + hu\frac{\partial v}{\partial x} + uv\frac{\partial h}{\partial x} + hfu = \frac{\tau_s^y - \tau_b^y}{\rho_0} \tag{9}$$

So (8) and (9) is the written out form of (1), but this can be simplified further. To see this we initally write out (2) as well to get

$$\frac{\partial h}{\partial t} + \nabla_H \cdot \mathbf{U} = \frac{\partial h}{\partial t} + \nabla_H \cdot (h\mathbf{u}) = \frac{\partial h}{\partial t} + h\nabla_H \cdot \mathbf{u} + \mathbf{u} \cdot \nabla_H h = \frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x} = 0 \tag{10}$$

and if we know take a closer look at some of the terms in (8) and (9), we see that from the second, third and fourth term on the LHS of (8), we get the simplification

$$u\frac{\partial h}{\partial t} + 2hu\frac{\partial u}{\partial x} + u^2\frac{\partial h}{\partial x} = u\left(\frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x}\right) = hu\frac{\partial h}{\partial x}$$

using continuity equation (10) to remove the first three terms in the parenthesis. Similarly for the second, third and fifth term on the LHS in (9) we have

$$v\frac{\partial h}{\partial t} + hv\frac{\partial u}{\partial x} + uv\frac{\partial h}{\partial x} = v\left(\frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x}\right) = 0$$

where we again use (10) to get rid of all these terms. Then, when using these two simplifications as well as diving through by h in (8) and (9), the resulting versions, together with (10), become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g\frac{\partial h}{\partial x} + \frac{\tau_s^x - \tau_b^x}{\rho_0 h} \tag{11}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = \frac{\tau_s^y - \tau_b^y}{\rho_0 h} \tag{12}$$

$$\frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x} = 0 \tag{13}$$

Now we are in a position to find the compatibility equations. We multiply (13) with a provisionally unknown function λ and add the result to (11) which gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x}\right) + g\frac{\partial h}{\partial x} = fv + \frac{\tau_s^x - \tau_b^x}{\rho_0 h}$$

$$\Rightarrow \qquad \left[\frac{\partial}{\partial t} + (u + \lambda h)\frac{\partial}{\partial x}\right]u + \lambda\left[\frac{\partial}{\partial t} + \left(u + \frac{g}{\lambda}\right)\frac{\partial}{\partial x}\right]h = fv + \frac{\tau_s^x - \tau_b^x}{\rho_0 h} \tag{14}$$

Then we require the two operators in (14) (in front of u and h) to be the same, i.e.

$$\left[\frac{\partial}{\partial t} + (u + \lambda h)\frac{\partial}{\partial x}\right] = \left[\frac{\partial}{\partial t} + \left(u + \frac{g}{\lambda}\right)\frac{\partial}{\partial x}\right] = \left[\frac{\partial}{\partial t} + \frac{D_{1,2}^* x}{dt}\frac{\partial}{\partial x}\right] = \frac{D_{1,2}^*}{dt}$$

where we, in the first equality, we renamed the operator (where 1 refers to the positive solution of λ and 2 to the negative), We must then have the following result for λ

$$\frac{D^* x}{dt} = u + \lambda h = u + \frac{g}{\lambda} \qquad \Rightarrow \qquad \lambda = \pm\sqrt{\frac{g}{h}}$$

Then we make use of the chain rule of differentation to rewrite the second term in (14) as

$$\lambda\left[\frac{\partial}{\partial t} + \left(u + \frac{g}{\lambda}\right)\frac{\partial}{\partial x}\right]h = \lambda\frac{D_{1,2}^*}{dt}\left(\frac{c^2}{g}\right) = \frac{2c\lambda}{g}\frac{D_{1,2}^* c}{dt} = \pm 2\frac{D_{1,2}^* c}{dt}$$

Furthermore we insert this result and the result for λ into (14) to obtain

$$\left[\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right](u \pm 2c) = \frac{D_{1,2}^*}{dt}(u \pm 2c) = fv + \frac{\tau_s^x - \tau_b^x}{\rho_0 h} \qquad \text{along} \qquad \frac{D_{1,2}^* x}{dt} = u \pm c \tag{15}$$

So we have here found two compatibility equations together with their charcateristics, but since the set (11)-(13) is three we need one more compatibility equaion. We get this directly from (12). We define an operator

$$\frac{D_3^*}{dt} \equiv \frac{\partial}{\partial t} + \frac{D_3^* x}{dt}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

and then observe that (12) now becomes

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)v = \frac{D_3^* v}{dt} = -fu + \frac{\tau_s^y - \tau_b^y}{\rho_0 h} \qquad \text{along} \qquad \frac{D_3^* x}{dt} = u \tag{16}$$

And we have found a third compatibility equation together with its characteristic. So the set (15) (which consists of two equations) and (16) now constitute our set of equations we later are to solve numerically.

b) All code that implements the numerical method discussed below can be found in the attached file `nonlinear_swe.f90`. In order to solve (15) and (16) numerically using the method of characteristics, we need to arrive at a scheme which we can implement. From the equations we have relationships which tells us the values of the Riemann invariants $u \pm 2c$, v along their respective characteristics, i.e. the RHS of these three equations are the values of the Riemann invariants along the characteristics. Using this, we can formulate the following finite difference approximations (epmloying an ordinary forward difference) to of the three equations:

$$\frac{u_j^{n+1} + 2c_j^{n+1} - (u_P^n + 2c_P^n)}{\Delta t} = \frac{f}{2}(v_j^{n+1} + v_P^n) \tag{17}$$

$$\frac{u_j^{n+1} - 2c_j^{n+1} - (u_Q^n - 2c_Q^n)}{\Delta t} = \frac{f}{2}(v_j^{n+1} + v_Q^n) \tag{18}$$

$$\frac{v_j^{n+1} - v_R^n}{\Delta t} = -\frac{f}{2}(u_j^{n+1} + u_R^n) \tag{19}$$

where u_P, v_P, c_P and all other "large-letter-subscripted" symbols are the dependent variables values, at the crossing point point between the characteristics going through point x_j, t^{n+1} and the previous time level n. The discretization mean used on the RHS's of (17)-(19) are done along the characteristics. Point P is in reference to the "positive" $u + c$ characteristic in (15), Q to the "negative" $u - c$ and R to the characteristic u in (16). From the theory of the method of characteristics we know that x_P should lie between x_j and x_{j-1} if $(u_j^n + c_j^n)\Delta t \leq \Delta x$. Similarly Q should lie between x_j and x_{j+1} provided that $|(u_j^n - c_j^n)|\Delta t \leq \Delta x$ and lastly x_R lies between x_j and x_{j-1} or x_{j-1} (depending on if $u_j^n \geq 0$ or not) if $|u_j^n|\Delta t \leq \Delta x$. These conditions are checked in the program so that we are sure we can use these adjacent points when interpolating at P, Q, R:

$$u_P = (1 - C_{Pj}^n)u_j^n + C_{Pj}^n u_{j-1}^n\ , \qquad C_{Pj}^n = \frac{\Delta t}{\Delta x}(u_j^n + c_j^n) \tag{20}$$

$$u_Q = (1 - C_{Qj}^n)u_j^n + C_{Qj}^n u_{j+1}^n\ , \qquad C_{Qj}^n = \frac{\Delta t}{\Delta x}|u_j^n - c_j^n| \tag{21}$$

$$u_R = (1 - C_{Rj}^n)u_j^n + C_{Rj}^n f_j^n\ , \quad f_j^n = \begin{cases} u_{j-1}, & \text{if } u_j^n \geq 0 \\ u_{j+1}, & \text{if } u_j^n < 0 \end{cases} \ , \qquad C_{Rj}^n = \frac{\Delta t}{\Delta x}|u_j^n| \tag{22}$$

These are three of the eight needed. Similar interpolation is done for v_P, v_Q, v_R, c_P and c_Q. The set (17)-(19) is a system of three equations with three unknowns $u_j^{n+1}, v_j^{n+1}, c_j^{n+1}$. Solving the system explicitly for u_j^{n+1} is all we need as we on the computer can utilise this value in (18) and (19) to find the latter two. Solving for u_j^{n+1} yields the following rewrite of the system:

$$u_j^{n+1} = \frac{u_P^n + 2c_P^n + u_Q^n - 2c_Q^n + \frac{f\Delta t}{2}(2v_R^n - f\Delta t u_R^n + v_P^n + v_Q^n)}{2 + f^2\Delta t^2/2} \tag{23}$$

$$v_j^{n+1} = v_R^n - \frac{f\Delta t}{2}(u_j^{n+1} + u_R^n) \tag{24}$$

$$c_j^{n+1} = \frac{1}{2}\left[u_j^{n+1} - u_Q^n + 2c_Q^n - \frac{f\Delta t}{2}(v_j^{n+1} + v_Q^n)\right] \tag{25}$$

Using the scheme (23)-(25) together with (20)-(22) (and the 5 non-written interpolations) we have our method for solving (15)-(16), or equivalently (11)-(13), in the interior domain. A quick note about the absolute value use in (21) and (22); since the C's are used to computed a weighted mean it is supposed to act as a fraction between 0 and 1 and thus we require a positive value (the value C_{Pj}^n in (20) is basically always positive so no absoilute value needed). In addition to solving in the interior (and at the open boundaries) we also employ the Flux Relaxation Scheme for the open boundary condition, meaning we extend our computational domain outside each of the two open boaundaries where we relax our computed solution towards an exterior solution. A somewhat resonable assumption for the exterior solution could for example be $u = v = 0$ and $h = H$, i.e. outside our domain we have a situation at rest (no flow) and the fluid free surface is constant and equal to H (same H as defined by the exercise). In the two FRS-zones there is used a hyperbolic tangent to relax the solution. In particular

$$\alpha_j = 1 - \tanh\left(\frac{j-1}{2}\right), \qquad j = 1(1)F_J \tag{26}$$

$$\alpha_j = 1 - \tanh\left(\frac{J-j}{2}\right), \qquad j = J - F_J + 1(1)J \tag{27}$$

where F_J is the number of points in the FRS-sone and J is the total number of points in the computational domain. As a final remark there is worth noting that the if-statement

```
if (C_Pjn > 1 .or. C_Qjn > 1 .or. C_Rjn > 1) then
    print*, "x_P, x_Q or x_R outside! Exiting..."
    call exit(1)
end if
```

found in the code are used to check whether x_P, x_Q and x_R are between the earlier discussed points. If this is not the case, the program terminates. A trial and error approach can then be used to choose appropriate parameters ($\Delta x, \Delta t$ etc.) in order to avoid falling outside the regions discussed. So if the program does not terminate in this if-statement, then all is good on this front and the interpolation is done as illustrated above.

c) Below in figure 1 there is a Hovmöller diagram (or contour plots) for the three dependent variables u, v, h as computed in the numerical solution. We observe a solution for which, in all three variables, the system starts out very active, but later on settels a bit down an we experience less rapid changes. This could perhaps be caused by the systems initial respons to the initial condition. Evening out the fluid surface and thus initiating motion. We see this in particular in the u plot. In very early times we see the greatest (in magnitude) velocities aorund $x = 0$ which makes intuitively sense considering the shape of the initial condition. In ascossiation with the motion we see oscillations going forward in time, especially for v and h, but also in terms of the shading in the u plot. Looking closer at the period of these oscillations, i.e. counting the amount of time between two troughs for example, we recognice these as inertial oscillations with period $T = 2\pi/f \approx 17\text{h}$ (or frequency $f = 10^{-4}\text{s}$). In addition we also see oscillating values in the x direction indicating we have wave motion in the fluid, which may be expected given that waves are inherent in the shallow water equations.

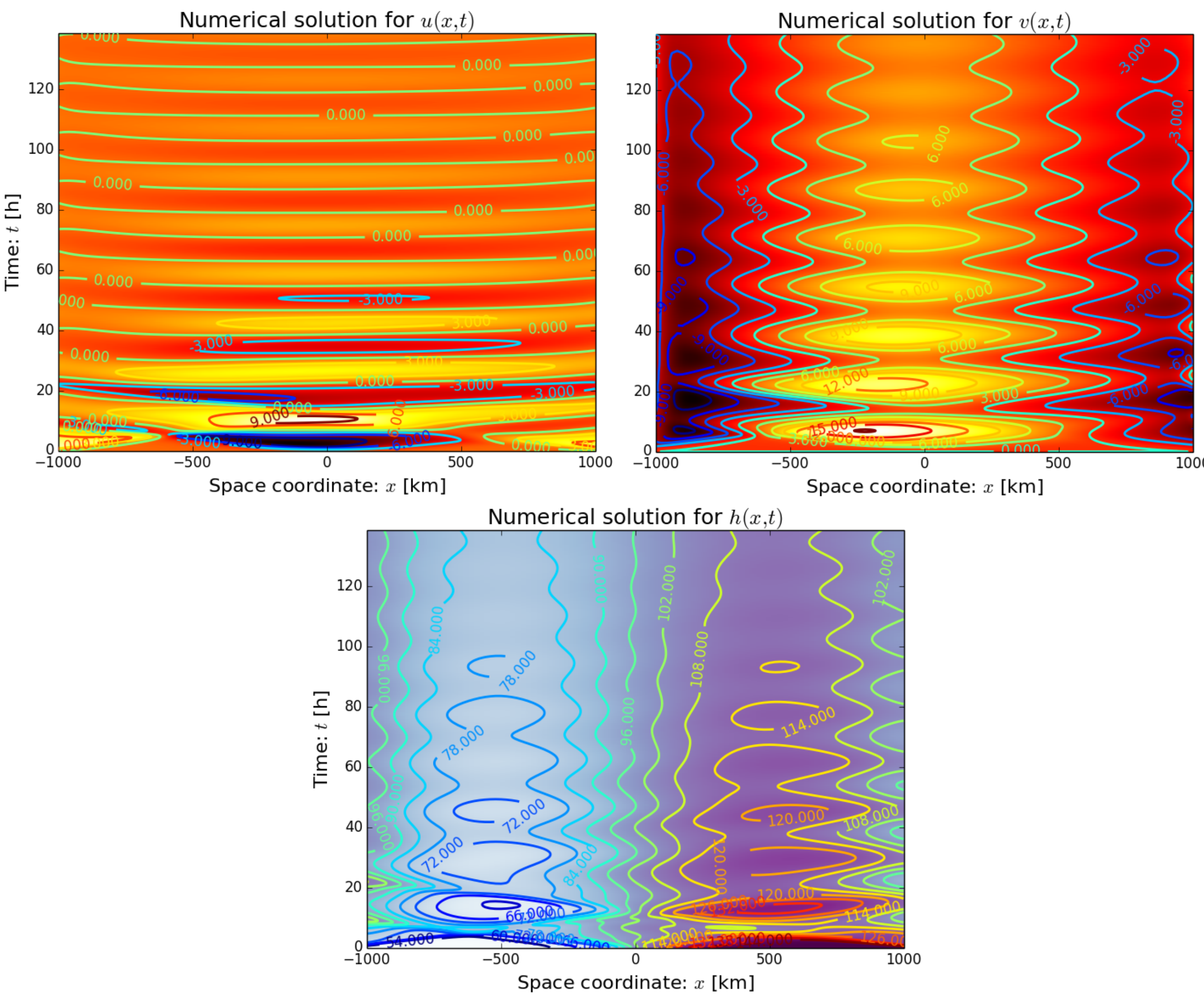

Figure 1: Numerical solution to the nonlinear rotating shallow water equations (11)-(13) (with initial condition $h|_{t=0} = H + \Delta H \tanh(\kappa x)$ where $H = 100\text{m}, \Delta H = H/2, \kappa = 10/L, L = 2000\text{km}$ and open boundaries where the exterior solution is set to $u = v = 0, h = H$) using the method of characteristics with $\Delta x = L/100, \Delta t = 10\text{s}$ and 10 points in the FRS-zone.