Exact solution of the statistical mechanics of the classical two-dimensional Coulomb gas

L. Šamaj
Institute of Physics, Slovak Academy of Sciences,
Dúbravská cesta 9, 84511 Bratislava, Slovak Republic

The symmetric Coulomb gas, defined in an infinite two-dimensional (2D) space of points $\mathbf{r} \in \mathrm{R}^2$, consists of particles $\{i\}$ of charge $\{q_i = \pm q\}$. The interaction energy of particles is $\sum_{i<j} q_i q_j v(\mathbf{r}_i - \mathbf{r}_j)$, where the Coulomb potential v is the solution of the 2D Poisson equation $\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r})$. Explicitly, $v(\mathbf{r}) = -\ln(|\mathbf{r}|/L)$ where L is a length scale. This definition of the Coulomb potential in 2D maintains many generic properties (e.g. sum rules) of "real" 3D Coulomb fluids with the interaction potential $v(\mathbf{r}) = 1/|\mathbf{r}|$, $\mathbf{r} \in \mathrm{R}^3$. The Coulomb gas is treated here as classical (i.e. non-quantum), in thermodynamic equilibrium at the inverse temperature $\beta = 1/(k_{\mathrm{B}}T)$. For the considered case of *pointlike* particles, the system is stable against the collapse of positive-negative pairs of charges provided that the corresponding Boltzmann factor $r^{-\beta q^2}$ can be integrated at short distances in 2D, i.e. in the high-temperature region $\beta q^2 < 2$; in what follows, we shall restrict ourselves to this stability region.

Through a simple scaling argument, the exact equation of state for the pressure P, $\beta P = n(1-\beta/4)$ (n is the total particle density), has been known for a long time [1]. The complete bulk thermodynamics (free energy, internal energy, specific heat, etc.) was derived only quite recently [2], based on an equivalence between the 2D Coulomb gas and the 2D Euclidean sine-Gordon field theory with a conformal normalization of the cosine field and due to a progress in the method of thermodynamic Bethe ansatz for the latter model. From a gnoseological point of view, the 2D Coulomb gas is the only exactly solvable case of a continuous fluid in more than one dimension. Later on, the form-factor approach was applied to calculate the large-distance asymptotic behavior of the charge and number density pair-correlation functions [3].

The surface thermodynamic properties (surface tension) of the 2D Coulomb gas in contact with an ideal conductor wall and an ideal dielectric wall were obtained in the whole stability range of $\beta q^2 < 2$ [4] through the mappings onto the boundary sine-Gordon field theory with integrable Dirichlet and Neumann boundary conditions, respectively.

The exact solvability of the 2D Coulomb gas in specific model situations permits one to confirm or to disprove generally used hypothesis.

The infinite dilution limit of a colloidal mixture is equivalent to the problem of one guest arbitrarily Q-charged particle immersed in the bulk of an electrolyte. Plausible arguments were given to conjecture that the induced electric potential far from the guest charge exhibits the form predicted by the Debye-Hückel (DH) theory, with a renormalized-charge prefactor Q_{ren} [5]. Moreover, one expects that the renormalized charge Q_{ren} saturates monotonically at some finite value when the bare charge $Q \to \infty$. When the electrolyte is modeled by the 2D Coulomb gas, the explicit results for the renormalized charge through the sine-Gordon format were obtained in Refs. [6, 7]. While these results confirm the adequacy of the concept of renormalized charge, in contrast to the saturation hypothesis the renormalized charge does not saturate at a specific finite value as $Q \to \infty$, but oscillates between two extreme values.

The DH theory describes rigorously a Coulomb fluid in the high-temperature $\beta \to 0$ regime. It is generally believed that the DH theory and the systematic high-temperature series expansion provide an adequate description also in the region of small *strictly positive* values of $\beta > 0$. This hypothesis was tested in Refs. [8, 9] on the asymptotic large-distance behavior of the charge and density correlation functions of the 2D Coulomb gas. In the case of the charge correlation function, the leading asymptotic term at strictly positive $\beta > 0$ was shown to be also the leading one in the high-temperature $\beta \to 0$ limit. This is no longer true for the density correlation function because its large-distance and high-temperature limits do not commute.

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[1] A. Salzberg and S. Prager, J. Chem. Phys. **38**, 2587 (1963).
[2] L. Šamaj and I. Travěnec, J. Stat. Phys. **101**, 713 (2000).
[3] L. Šamaj and B. Jancovici, J. Stat. Phys. **106**, 301 (2002); **106**, 323 (2002).
[4] L. Šamaj and B. Jancovici, J. Stat. Phys. **103**, 717 (2001); L. Šamaj, *ibid.* **103**, 737 (2001).
[5] Y. Levin, Rep. Prog. Phys. **65**, 1577 (2002).
[6] L. Šamaj, J. Stat. Phys. **120**, 125 (2005).
[7] L. Šamaj, J. Stat. Phys. **124**, 1179 (2006).
[8] L. Šamaj and Z. Bajnok, Phys. Rev. E **72**, 061503 (2005).

[9] L. Šamaj, A trickiness of the high-temperature limit for number density correlation functions in classical Coulomb fluids, J. Stat. Phys. (2007), in press.