ELSEVIER

Topology and its Applications 89 (1998) 75–120

TOPOLOGY AND ITS APPLICATIONS

Properly injective spaces and function spaces

Martín Hötzel Escardó [1]

Department of Computing, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK

Received 1 March 1997; received in revised form 7 July 1997

Abstract

Given an injective space D (a continuous lattice endowed with the Scott topology) and a subspace embedding $j : X \to Y$, Dana Scott asked whether the higher-order function $[X \to D] \to [Y \to D]$ which takes a continuous map $f : X \to D$ to its greatest continuous extension $\bar{f} : Y \to D$ along j is Scott continuous. In this case the extension map is a subspace embedding. We show that the extension map is Scott continuous iff D is the trivial one-point space or j is a proper map in the sense of Hofmann and Lawson.

In order to avoid the ambiguous expression "proper subspace embedding", we refer to proper maps as finitary maps. We show that the finitary sober subspaces of the injective spaces are exactly the stably locally compact spaces. Moreover, the injective spaces over finitary embeddings are the algebras of the upper power space monad on the category of sober spaces. These coincide with the retracts of upper power spaces of sober spaces. In the full subcategory of locally compact sober spaces, these are known to be the continuous meet-semilattices. In the full subcategory of stably locally compact spaces these are again the continuous lattices.

The above characterization of the injective spaces over finitary embeddings is an instance of a general result on injective objects in poset-enriched categories with the structure of a KZ-monad established in this paper, which we also apply to various full subcategories closed under the upper power space construction and to the upper and lower power locale monads.

The above results also hold for the injective spaces over *dense* subspace embeddings (continuous Scott domains). Moreover, we show that every sober space has a smallest finitary *dense* sober subspace (its *support*). The support always contains the subspace of maximal points, and in the stably locally compact case (which includes densely injective spaces) it *is* the subspace of maximal points iff that subspace is compact.

Keywords: Injective spaces; Function spaces; Proper maps; Lower and upper power spaces; Stably locally compact spaces; Subspaces of maximal points; Locales; Continuous lattices; Continuous Scott domains; Domain theory; KZ-monads; Kan extensions

AMS classification: 06B35; 54C20; 54C35; 18C20

[1] E-mail: mhe@dcs.ed.ac.uk, URL: http://theory.doc.ic.ac.uk/people/Escardo.

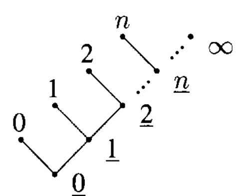
0166-8641/98/$19.00
PII: S0166-8641(97)00225-3

1. Introduction

Although the fundamental rôle of injective spaces in the mathematical theory of computation was emphasized by Dana Scott in his seminal papers [33] and [35], injective spaces have been neglected in the subsequent development of the theory (but see [30,15]). In this section we recall their rôle and raise questions related to function spaces and higher-order functions which are answered in the technical development that follows. We also briefly discuss applications, introduce preliminary background and give a summary of the main results of this paper.

1.1. Embedding spaces into domains

In applications of domain theory [2] to denotational semantics [16,13] and integration [8,11], one starts by implicitly or explicitly embedding given spaces X, Y, Z, ... into appropriate domains C, D, E, ... endowed with the Scott topology. One of the simplest examples is given by the embedding of the discrete space of natural numbers into the so-called flat domain $\mathbb{N}_\perp$ of natural numbers [36,29,31]. A slightly more elaborate example is given by the embedding of (the one-point compactification of) the discrete space of natural numbers into the so-called domain of lazy natural numbers [16,2]:



The choice of domains depends, among other things, on the model of computation on the space. For example, the flat and lazy domains of natural numbers, respectively capture call-by-value and call-by-name evaluation of the successor map [16].

More sophisticated examples of such embeddings include: the Euclidean real line into the domain of compact real intervals ordered by reverse inclusion [34,13,10], the same space into the ideal completion of the rational basis of the interval domain, or a similar algebraic domain [14], Cantor space 2^ω into the domain $2^\infty = 2^* \cup 2^\omega$ of finite and infinite sequences ordered by prefix [31,43,39] (similarly, Baire space $\mathbb{N}^\omega$ into the domain $\mathbb{N}^\infty$), the space of total functions $\mathbb{N} \to \mathbb{N}$ endowed with the compact-open topology (another version of Baire space) into the domain of partial functions ordered by graph inclusion [36,31], any second countable T_0 space into the domain $\mathcal{P}\omega$ of subsets of natural numbers ordered by inclusion [35], any second countable T_0 space as an isochordal subspace (see below for the definition) of $\mathcal{T}^\omega$, where $\mathcal{T}$ is the flat domain $\mathbb{T}_\perp$ of truth values [30], any locally compact Hausdorff space into its upper power space [9], any Polish space onto the subspace of maximal points of a continuous dcpo (directed complete poset) [25,12].

1.2. Injective spaces

Given embeddings of spaces X and Y into computational models C and D, we model continuous maps $X \to Y$ by Scott continuous functions $C \to D$. Therefore it is natural to demand that the continuous functions $C \to D$ capture the continuous maps $X \to Y$, in the sense that every continuous map of the latter kind (co)extends to a continuous function of the former kind. Since every continuous map $f : X \to Y$ trivially coextends to a continuous map $f : X \to D$, we only need to consider extensions of continuous maps $f : X \to D$ to continuous functions $\bar{f} : C \to D$. This brings us to the subject of injective spaces.

A space D is *injective* in the ambient category of T_0 spaces if every continuous map $f : X \to D$ extends to a continuous map $\bar{f} : Y \to D$, for any space Y containing X as a subspace [33, p. 99], as illustrated in the diagram below:

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "\bar{f}"] \\ & D & \end{tikzcd}$$

where $j : X \hookrightarrow Y$ is the inclusion. (Notice that in principle there is nothing canonical about $\bar{f}$.)

A main result in loc. cit. is the characterization of the injective spaces as the continuous lattices endowed with the Scott topology. For example, by the previous discussion, the continuous endomaps of the continuous lattice $\mathcal{P}\omega$ capture the continuous maps between any two second countable T_0 spaces [35, p. 527].

If one restricts *subspace* to *dense subspace* in the definition, one speaks of *densely injective spaces*, which are characterized as the *continuous Scott domains* [15, p. 127] (that is, continuous dcpos with least upper bounds of bounded subsets). A continuous Scott domain fails to be a continuous lattice only by lacking a (compact) top element [15, pp. 52, 53] and the continuous Scott domains coincide with the closed subspaces of the continuous lattices [15, p. 109]. The examples of embeddings given above, except for the ones into the ideal completion of the rational intervals, $\mathcal{P}\omega$ and $\mathcal{T}^{\omega}$, and the ones about Polish spaces, are instances of *dense* embeddings into *continuous Scott domains*. In fact, these are examples of dense embeddings of *Hausdorff* spaces *onto* subspaces of maximal points of continuous Scott domains.

Gordon Plotkin [30, p. 233] calls a subspace inclusion $X \subseteq Y$ *isochordal* if for any two disjoint open sets $U, U' \subseteq X$ there are disjoint open sets $V, V' \subseteq Y$ with $V \cap X = U$ and $V' \cap X = U'$. For example, every dense embedding is isochordal.

If one restricts *subspace* to *isochordal subspace* in the definition, one speaks of *isochordally injective spaces*. Plotkin [30, pp. 233, 234] characterized the second countable isochordally injective spaces as the countably based coherently complete domains (that is, countably based continuous dcpos in which every pairwise bounded subset has a least upper bound). A result by Paul Taylor [40, 2.5.4 and 2.6.4b] implies that the countability hypothesis is not necessary. An example of isochordal embedding into a coherently complete domain which is not a dense embedding is given by the embedding of the real line into the ideal completion of the compact rational intervals ordered by reverse inclusion, defined by $x \mapsto \{[p,q] \mid p < x < q\}$.

More generally, if in the definition j ranges over a given class J of embeddings, then one speaks of *injective spaces over* J [15, p. 121].

1.3. Injective spaces and function spaces

By the above discussion, if $j : X \to C$ and $k : Y \to D$ are subspace embeddings into injective spaces, then the *continuous functions* $C \to D$ capture the *continuous maps* $X \to Y$ in the sense of the (co)extension property. When one considers *higher-order* maps $[X \to Y] \to Z$, such as the integration and supremum operators discussed in [11], one is led to consider the case in which the *function space* $[C \to D]$ captures the the *function space* $[X \to Y]$, in the stronger sense of having it embedded as a subspace, via some continuous (co)extension map $[X \to Y] \to [C \to D]$. By the above remarks, it suffices to consider continuous extension maps $[X \to D] \to [C \to D]$.

Although in principle there is nothing canonical about the extended map $\bar{f}$ in the definition of injective space, this turns out to be the case. In fact, Scott [33, p. 116] showed that if D is injective and $j : X \to Y$ is a subspace embedding, then every continuous map $f : X \to D$ has a *greatest* continuous extension along j, which will be convenient to denote by $f/j : Y \to D$, given by

$$f/j(y) = \bigvee^{\uparrow}_{y \in V \in \Omega Y} \bigwedge f(j^{-1}(V)).$$

Here hom-sets are ordered by the pointwise ordering induced by the specialization order of the target space and ΩY is the frame of open sets of Y.

Having established this result, Scott asked whether the greatest-extension map $f \mapsto f/j$ is Scott continuous (very much doubting that this would be the case in general). Here $[X \to D]$ and $[Y \to D]$ are endowed with the Scott topology, which coincide with the Isbell topology if X and Y are exponentiable [26, p. 154] (which further coincide with the compact-open topology if X and Y are locally compact [23, p. 61]). Since the greatest-extension map is a right inverse of the restriction map $g \mapsto g \circ j$, which is always Scott continuous, the greatest-extension map is a subspace embedding iff it is Scott continuous, and in this case $[X \to D]$ is a retract of $[Y \to D]$.

We show that the greatest-extension map $f \mapsto f/j$ is Scott continuous iff D is the trivial one-point space or j is a proper map in the sense of Hofmann and Lawson [18, p. 154]. Briefly, a continuous map $j : X \to Y$ is proper if the right adjoint $\forall_j : \Omega X \to \Omega Y$ of its associated frame map $\Omega j : \Omega Y \to \Omega X$ defined by $\Omega j(V) = j^{-1}(V)$ is Scott continuous.

The terminology "proper" has been used in several slightly distinct senses in the literature [2]—see, e.g., [4, pp. 97–107]; [20, p. 104]; [41] and the remarks by Johnstone [20,

[2] If X and Y are Hausdorff spaces then all definitions are equivalent.

p. 121] and Vickers [42, Section 5]. To make things worse, in our case we have the unfortunate ambiguity of the expression "proper subspace embedding", which can mean either an embedding onto a proper subspace or an embedding which is a proper map in the sense just defined. We have therefore decided to refer to the proper maps in the sense of Hofmann and Lawson as *finitary maps* and to the subspaces whose inclusion map is finitary as *finitary subspaces*. The terminology "finitary" is borrowed from Banaschewski [3, p. 649], who calls a nucleus on a locale finitary if it is Scott continuous, and it is justified by the fact that a subspace embedding is finitary iff its induced nucleus is finitary.

In view of the above result on injective spaces and finitary embeddings, we are led to investigate the finitary subspaces of injective spaces. More concrete characterizations of the notions of finitary map and finitary subspace are given in the technical development that follows this introduction. For the time being, we remark that the finitary sober subspaces of a (densely) injective space D are the sober subspaces X such that $Q \cap X$ is compact for every compact saturated set $Q \subseteq D$. Also, the finitary sober subspaces of the (densely) injective spaces are exactly the stably locally compact spaces. This is a consequence of more general results, including the following.

In the ambient category of sober spaces, the full subcategories of respectively compact, locally compact, spectral and stably locally compact spaces are closed under the formation of finitary subspaces. Here we do not assume the Hausdorff separation axiom in the definition of compactness; a space is compact iff it satisfies the Heine–Borel property. Stably locally compact spaces are considered in [20, p. 313]; [17,38]. Such spaces are called coherent in [2], but Johnstone [20, p. 63] calls coherent the spaces which Vickers [43, p. 120] and Smyth [39, p. 649] call spectral.

Also, in this ambient category, for every subspace X of a space Y there is a smallest finitary subspace $\overline{X}$ of Y containing X as a subspace, which we refer to as the *finitary hull* of X. Moreover, every space has a smallest finitary *dense* subspace (its *support*), which is the finitary hull of its subspace of maximal points. In the stably locally compact case, the support *is* the subspace of maximal points iff that subspace is compact.

The above results hold in the more general category of locales. Moreover, the full subcategory of locales with enough points is closed under the formation of finitary sublocales. Also, the smallest finitary dense sublocale of a locale is the finitary hull of its smallest dense sublocale.

1.4. Injective spaces and upper power spaces

Having established that the "good denominators" are the finitary subspace embeddings, one wonders what the finitarily injective spaces are. We show that they are precisely the algebras of the upper power space monad in the category of sober spaces (considering the empty set as a point of the upper power space construction), which coincide with the retracts of upper power spaces of sober spaces. Moreover, greatest extensions exist and are given by

$$f/j(y) = \bigwedge f\left(j^{-1}(\uparrow y)\right).$$

The full subcategory of locally compact sober spaces is closed under the upper power space construction. The finitarily injective spaces in this subcategory are the continuous meet-semilattices with unit, by virtue of the characterization of the algebras given in [32, pp. 135, 140]. The full subcategory of stably compact spaces is also closed under the upper power space construction. In this subcategory, the finitarily injective spaces are again the continuous lattices.

1.5. Injective spaces and KZ-monads

The above characterization of the finitarily injective spaces is a particular case of a more general result on KZ-monads [24] on poset-enriched categories established in the present paper. Moreover, this result is applied to the lower and upper power locale monads discussed in [32] (see below). If the underlying functor of the monad is Scott continuous on hom-posets (which is the case in our applications), the extension map is also Scott continuous, so that we do not lose the continuity of the extension map along finitary embeddings when we consider the larger class of finitarily injective spaces.

1.6. Injective spaces and lower power spaces

In applications of domain theory (see, e.g., Scott [35, p. 528] and Plotkin [31, p. 14]), one sometimes wishes to consider *least* extensions. We show that the least continuous extension along a subspace embedding $j:X \to Y$ of an injective-valued map $f:X \to D$ exists iff D is trivial or j is semi-open, in the sense that its associated frame map $\Omega j:\Omega Y \to \Omega X$ has a *left* adjoint $\exists_j:\Omega X \to \Omega Y$. The process of taking least continuous extensions is always Scott continuous, essentially because $\exists_j$, being a left adjoint, preserves all joins.

By an application of the general result on KZ-monads and a result by Steve Vickers [42, Proposition 4.6], we conclude that the injective locales over semi-open embeddings are the algebras of the *lower* power locale monad. It is plausible the same result holds for the injective *spaces* over semi-open embeddings, but we do not pause to check whether this is the case.

1.7. Injective spaces and Kan extensions

By definition, f/j is the greatest continuous map $g:Y \to D$ such that $g \circ j = f$. Scott [33, p. 116] remarked that f/j is in fact the greatest continuous map g such that $g \circ j \leqslant f$. This shows that f/j is the right Kan extension of f along j. The general definition of a Kan extension of a functor can be found in [7, p. 39] and [27, p. 232], and its specialization to a monotone map can be found in [1, p. 22]. In this paper we consider Kan extensions of arrows of poset-enriched categories.

By virtue of Scott's remark, D is injective iff for every subspace embedding $j : X \to Y$ the restriction map $g \mapsto g \circ j : \hom(Y, D) \to \hom(X, D)$ has an *injective* right adjoint $f \mapsto f/j : \hom(X, D) \to \hom(Y, D)$. We make this characterization into a definition of *right injective* object in a poset-enriched category. By omitting the injectivity condition on the right adjoint, we obtain a definition of *right Kan object*; this means that in general we only have $(f/j) \circ j \leqslant f$. Similarly, we obtain definitions of left Kan and left injective objects.

We show that every injective space over subspace embeddings is a right Kan space over *arbitrary continuous maps*. Similarly, every right injective space over finitary embeddings is a right Kan space over arbitrary finitary maps, and every left injective locale over semi-open embeddings is a left Kan locale over arbitrary semi-open maps.

1.8. Injective spaces and dense embeddings

The above results on greatest extensions and continuity of the extension map generalize to densely injective spaces. Also, the injective spaces over finitary dense embeddings are characterized via an application of the above result on KZ-monads to the upper power space monad without the empty set as a point of the upper power space construction.

1.9. Injective spaces and isochordal embeddings

Unfortunately, the isochordally injective spaces fail to enjoy both the least- and greatest-extension properties, as simple counter-examples which can be safely left to the reader show.

2. Kan objects in poset-enriched categories

2.1. Poset-enriched categories

A *poset-enriched category* is a category whose hom-sets are posets and whose composition operation is monotone. A *poset-functor* between poset-enriched categories is a functor which is monotone on hom-posets. A poset-functor $U : \mathcal{X} \to \mathcal{A}$ is *poset-faithful* if $Uf \leqslant Ug$ in $\mathcal{A}$ implies $f \leqslant g$ in $\mathcal{X}$.

The main example of a poset-enriched category is Poset, the category of posets and monotone maps with hom-sets ordered pointwise. If $\mathcal{X}$ is any category and $U : \mathcal{X} \to \text{Poset}$ is a faithful functor, then there is a unique way of making $\mathcal{X}$ into a poset-enriched category so as to also make U into a poset-faithful functor, given by the definition

$$f \leqslant g \text{ in } \mathcal{X} \quad \text{iff} \quad Uf \leqslant Ug \text{ in Poset},$$

because one direction of the definition is equivalent to saying that U is a poset-functor and the other is equivalent to saying that U is poset-faithful.

Our main example of such a situation is given by the category Sp_0 of T_0 topological spaces and continuous maps with U the specialization-order functor. The specialization order [15, p. 123]; [20, p. 45] on the points of a space X is defined by $x \leqslant y$ iff every neighborhood of x is a neighborhood of y iff x belongs to the closure of $\{y\}$. This definition makes $\leqslant$ into a reflexive and transitive relation, which is antisymmetric iff X is T_0. Also, it is clear from the definition that any continuous map preserves this preorder. Then the functor in question sends a T_0 space to its set of points ordered by the specialization order, and a continuous map to itself. Thus, the induced poset-enrichment in Sp_0 is given simply by $f \leqslant g$ in $\hom(X, Y)$ iff $f(x) \leqslant g(x)$ for all $x \in X$.

Definition 2.1.1. Let $\mathcal{X}$ be a poset-enriched category, $\mathcal{I}$ be any category, $\mathcal{F}: \mathcal{I} \to \mathcal{X}$ be a functor, and $P \in \mathcal{X}$ be a limit of $\mathcal{F}$ with projections $\{\pi_i : P \to \mathcal{F}i\}_{i \in \mathcal{I}}$. Then there is a bijection between the set of cones under X and the set $\hom(X, P)$, which sends the cone $\{g_i : X \to \mathcal{F}i\}_{i \in \mathcal{I}}$ to the unique $g : X \to P$ such that $g_i = \pi_i \circ g$ for all $i \in \mathcal{I}$. We say that the given limit is a *poset-limit* if the above bijection is an *order-isomorphism* for each X, where cones under X are ordered by $\{g_i : X \to \mathcal{F}i\}_{i \in \mathcal{I}} \leqslant \{h_i : X \to \mathcal{F}i\}_{i \in \mathcal{I}}$ iff $g_i \leqslant h_i$ for all $i \in \mathcal{I}$.

This notion is a particular case of the general notion of $\mathcal{V}$-limit in a $\mathcal{V}$-enriched category, which can be found in, e.g., [7, p. 7].

All limits in Poset are poset-products, and the same holds for Sp_0, poset-enriched as above, because the specialization-order functor $U : \mathrm{Sp}_0 \to \mathrm{Poset}$ preserves limits.

2.2. *Adjunctions between objects of poset-enriched categories*

For a complete account to adjunctions between posets the reader is referred to [15] or [2]. In this subsection we fix terminology and basic facts about adjunctions between objects of poset-enriched categories.

An *adjunction* between objects X and Y of a poset-enriched category is a pair of arrows $l : X \to Y$ and $r : Y \to X$ such that $l \circ r \leqslant \mathrm{id}_Y$ and $\mathrm{id}_X \leqslant r \circ l$. Such an adjunction is denoted by $l \dashv r$, and l and r are said to be, respectively *left* and *right adjoint* to each other. In an adjunction $l \dashv r$, each adjunct l and r is uniquely determined by the other.

An adjunction $l \dashv r$ is *reflective* if $l \circ r = \mathrm{id}_Y$, and it is *coreflective* if $\mathrm{id}_X = r \circ l$. A poset-functor $\mathcal{F}$ preserves adjunctions in the sense that $l \dashv r$ implies $\mathcal{F}l \dashv \mathcal{F}r$; moreover, if $\mathcal{F}$ is poset-faithful then it reflects adjunctions in the sense that $l \dashv r$ whenever $\mathcal{F}l \dashv \mathcal{F}r$.

Adjunctions compose in the sense that if $l : X \to Y$, $l' : Y \to Z$, $r : Y \to X$ and $r' : Z \to Y$ are arrows with $l \dashv r$ and $l' \dashv r'$, then $l' \circ l \dashv r \circ r'$.

2.3. *Kan extensions of arrows of poset-enriched categories*

We first briefly specialize the definition of Kan extension [7, p. 39]; [27, p. 232] from functors to monotone maps (cf. [1, p. 22]), and then we consider its (immediate) generalization to arrows of poset-enriched categories.

Let $j : X \to Y$ and $f : X \to D$ be monotone maps between posets. A *right Kan extension* of f along j is a monotone map $f/j : Y \to D$ such that

$$(f/j) \circ j \leqslant f, \tag{Kan$_1$}$$

$$g \circ j \leqslant f \text{ for } g : Y \to D \quad \text{implies} \quad g \leqslant f/j. \tag{Kan$_2$}$$

Inequality (Kan$_1$) is illustrated in the following diagram:

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & \arrow[d, phantom, "\geqslant"] & Y \arrow[dl, "f/j"] \\ & D & \end{tikzcd}$$

Dually, we obtain the definition of left Kan extension by reversing the inequalities. We denote the left Kan extension of f along j by $f \backslash j$ whenever it exists.

By definition, the right Kan extension of f along j, if it exists, is the greatest map $g : Y \to D$ such that $g \circ j \leqslant f$. This is equivalent to saying that for all $g : Y \to D$,

$$g \circ j \leqslant f \quad \text{iff} \quad g \leqslant f/j,$$

which shows that right Kan extensions along a fixed $j : X \to Y$ exist for all $f : X \to D$ iff the composition map

$$g \mapsto g \circ j : \mathrm{hom}(Y, D) \to \mathrm{hom}(X, D)$$

has a right adjoint

$$f \mapsto f/j : \mathrm{hom}(X, D) \to \mathrm{hom}(Y, D).$$

In [7,27], f/j and $f \backslash j$ are denoted by $\mathrm{Ran}_j f$ and $\mathrm{Lan}_j f$, respectively. Our notation makes the basic properties of Kan extensions easier to remember, because they resemble the properties of quotients (cf. Theorem 2.3.3 below).

The above definition formally applies to arrows between objects of poset-enriched categories. Before considering this generalization, we consider some basic properties which make sense only for monotone maps between posets. Propositions 2.3.1 and 2.3.2 are obtained by specializing the corresponding results in loc. cit. from functors to monotone maps.

Kan extensions generalize adjunctions, as Proposition 2.3.1, Theorem 2.3.4 and Corollary 2.3.5 below show:

Proposition 2.3.1. *A monotone map $f : X \to D$ has a right Kan extension along a monotone map $j : X \to Y$ if the set $f(j^{-1}(\uparrow y))$ has a meet in D for each $y \in Y$, and in this case it is given by*

$$f/j(y) = \bigwedge f\left(j^{-1}(\uparrow y)\right).$$

In particular, a subset $X \subseteq D$ has a meet iff the inclusion map has a right Kan extension along the unique map $X \to \mathbf{1}$, where $\mathbf{1}$ is the one-point poset, which has to be the meet of X considered as a map $\mathbf{1} \to D$. Therefore every monotone map $f : X \to D$

has a right Kan extension along any monotone map $j : X \to Y$ iff D has all meets and hence is a complete lattice.

An *order-embedding* [5, p. 10] is a monotone map $j : X \to Y$ which reflects order in the sense that $j(x) \leqslant j(x')$ implies $x \leqslant x'$. The following proposition shows that the right Kan extension f/j is an *actual* extension if j is an order-embedding:

Proposition 2.3.2. *If $j : X \to Y$ is an* order-embedding *and $f : X \to D$ is a monotone map with a right Kan extension along j, then $(f/j) \circ j = f$.*

We now prove the facts that generalize to Kan extensions of arrows of poset-enriched categories.

Theorem 2.3.3. *Let $j : X \to Y$, $k : Y \to Z$, $f : X \to D$, and $r : D \to E$ be arrows of a poset-enriched category. Then the following properties hold whenever the right Kan extensions f/j, $(f/j)/k$ and $f/(k \circ j)$, f/f and $(r \circ f)/j$ exist:*

(1) $f/\mathrm{id}_X = f$,
(2) $(f/j)/k = f/(k \circ j)$,
(3) $\mathrm{id}_D \leqslant f/f = (f/f) \circ (f/f)$
  (*if* $\mathrm{id}_D = f/f$ *then f is said to be codense* [27, p. 242]),
(4) $r \circ (f/j) \leqslant (r \circ f)/j$, *equality holding if r has a left adjoint.*

These (in)equations are illustrated in the following diagrams:

$$
\begin{tikzcd}
X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "="] & X \arrow[dl, "f"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "j"] \arrow[dr, "f"'] \arrow[dr, phantom, "\geqslant", near start, shift left=3] & Y \arrow[r, "k"] \arrow[d, "f/j"] & Z \arrow[dl, "(f/j)/k=f/(k\circ j)"] \arrow[dl, phantom, "\geqslant", near start, shift right=3] \\
& D &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\geqslant"] & D \arrow[dl, "f/f\geqslant \mathrm{id}_D"] \\
& D &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "j"] \arrow[dr, "f"] \arrow[ddr, bend right, "r\circ f"'] & {} \arrow[d, phantom, "\geqslant"] & Y \arrow[dl, "f/j"'] \arrow[ddl, bend left, "(r\circ f)/j"] \\
\arrow[r, phantom, "="] & D \arrow[d, "r"] \arrow[r, phantom, "\leqslant"] & {} \\
& E &
\end{tikzcd}
$$

Proof. (1) Trivial.

(2) By two applications of (Kan_1), $((f/j)/k) \circ k \circ j \leqslant (f/j) \circ j \leqslant f$. By (Kan_2), $(f/j)/k \leqslant f/(k \circ j)$. In the other direction, $(f/(k \circ j)) \circ k \circ j \leqslant f$ by (Kan_1). By two applications of (Kan_2), $f/(k \circ j) \leqslant (f/j)/k$.

(3) Since $\mathrm{id}_D \circ f \leqslant f$, we conclude that $\mathrm{id}_D \leqslant f/f$ by (Kan$_2$). By monotonicity of composition, $f/f \leqslant (f/f) \circ (f/f)$. In the other direction, $(f/f) \circ (f/f) \circ f \leqslant (f/f) \circ f \leqslant f$ by two applications of (Kan$_1$). Therefore $(f/f) \circ (f/f) \leqslant f/f$ by (Kan$_2$).

(4) $r \circ (f/j) \circ j \leqslant r \circ f$ by (Kan$_1$). Therefore $r \circ (f/j) \leqslant (r \circ f)/j$ by (Kan$_2$). In order to establish the inequality in the other direction, assume that r has a left adjoint l. By (Kan$_1$), $((r \circ f)/j) \circ j \leqslant r \circ f$. By composing with l on the left and using the fact that $l \circ r \leqslant \mathrm{id}_Y$,

$$l \circ \big((r \circ f)/j\big) \circ j \leqslant l \circ r \circ f \leqslant f.$$

Hence $l \circ ((r \circ f)/j) \leqslant f/j$ by (Kan$_2$). By composing with r on the left,

$$r \circ l \circ \big((r \circ f)/j\big) \leqslant r \circ (f/j).$$

But $\mathrm{id}_X \leqslant r \circ l$. Therefore $(r \circ f)/j \leqslant r \circ (f/j)$. □

Item (4) generalizes [33, Lemma 3.9], whose statement amounts to the equation $j \circ (g/e) = (j \circ g)/e$, from greatest extensions to right Kan extensions (and from coreflective adjunctions to adjunctions).

Theorem 2.3.4. *Let $l : Y \to X$ and $r : X \to Y$ be arrows of a poset-enriched category.*

(1) *If $l \dashv r$ then every arrow $f : X \to D$ has a right Kan extension along r given by $f/r = f \circ l$. In particular, $l = \mathrm{id}_X/r$.*

(2) *If id_X and r have right Kan extensions along r and $r \circ (\mathrm{id}_X/r) = r/r$, then $\mathrm{id}_X/r \dashv r$.*

Proof. (1) $f \circ l \circ r \leqslant f$ because $l \circ r \leqslant \mathrm{id}_X$ by definition of adjunction. This establishes (Kan$_1$). Assume that $g \circ r \leqslant f$ for $g : Y \to D$. Since $g \circ r \circ l \leqslant f \circ l$ and since $\mathrm{id}_Y \leqslant r \circ l$ by definition of adjunction, we conclude that $g \leqslant f \circ l$. This establishes (Kan$_2$). The particular case follows by taking $D = X$ and $f = \mathrm{id}_X$.

(2) By (Kan$_1$), $(\mathrm{id}_X/r) \circ r \leqslant \mathrm{id}_X$. This establishes one half of the adjunction. Since $r/r \geqslant \mathrm{id}_Y$ by Theorem 2.3.3(3), we conclude that $r \circ (\mathrm{id}_X/r) \geqslant \mathrm{id}_Y$ by the hypothesis and transitivity, which establishes the other half. Therefore $\mathrm{id}_X/r \dashv r$. □

Corollary 2.3.5. *The arrow $r : X \to Y$ has a left adjoint iff id_X and r have a right Kan extension along r and the condition $r \circ (\mathrm{id}_X/r) = r/r$ holds, and in this case it is id_X/r.*

Proof. If $l \dashv r$ then $r \circ (\mathrm{id}_X/r) = r \circ (\mathrm{id}_X \circ l) = r \circ l = r/r$ by Theorem 2.3.4(1). The converse is Theorem 2.3.4(2). □

2.4. *Kan objects in poset-enriched categories*

Let $\mathcal{X}$ be a poset-enriched category and J be any subcategory of $\mathcal{X}$.

Definition 2.4.1. Let D be an object of $\mathcal{X}$.

(1) D is a *right Kan object* over J if for each $j : X \to Y$ in J the composition map

$$\_\circ j : \hom(Y, D) \to \hom(X, D)$$

has a right adjoint

$$\_/j : \hom(X, D) \to \hom(Y, D),$$

also denoted by Ran_j^D.

(2) D is a *right injective object* over J if in addition the right adjoint is an *injective* function, which means that the right Kan extension f/j is an actual extension, in the sense that $(f/j) \circ j = f$.

Left Kan objects and left injective objects are defined dually (at the level of hom-posets), and left-Kan-extension maps are denoted by Lan_j^D.

Let $\mathcal{X}/J$ denote the subcategory of $\mathcal{X}$ consisting of right Kan objects over J and right adjoints between them. The following is a corollary of Theorem 2.3.3:

Proposition 2.4.2. *The equations*

$$\mathrm{Ran}(X, D) = \hom(X, D),$$

$$\mathrm{Ran}(j : X \to Y,\ r : D \to E) = f \mapsto r \circ f/j : \hom(X, D) \to \hom(Y, E)$$

define a functor $\mathrm{Ran} : J \times \mathcal{X}/J \to \mathrm{Poset}$.

Notice that $\mathrm{Ran}_j^D = \mathrm{Ran}(j, \mathrm{id}_D)$.

2.4.1. Kan objects and retractions

A *retraction* of an arrow $s : Y \to X$ is an arrow $r : X \to Y$ with $r \circ s = \mathrm{id}_Y$ and a *section* of an arrow $r : X \to Y$ is an arrow $s : Y \to X$ such that $r \circ s = \mathrm{id}_Y$. An object Y is a *retract* of an object X iff there is a section $s : Y \to X$ iff there is a retraction $r : X \to Y$.

Lemma 2.4.3. *If an object is right Kan (respectively right injective) over J, so is any of its retracts. Moreover, if $r : D \to E$ has a section $s : E \to D$ with D right Kan and $j : X \to Y$ is a member of J, then for every $f : X \to E$ the right Kan extension $f/j : Y \to E$ is given by*

$$f/j = r \circ \big((s \circ f)/j\big).$$

The construction of f/j is illustrated in the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "j"] \arrow[dr, "f"] \arrow[ddr, "s\circ f"'] & & Y \arrow[dl, "f/j"'] \arrow[ddl, "(s\circ f)/j"] \\
& E \arrow[d, shift right, "s"'] & \\
& D \arrow[u, shift right, "r"'] &
\end{tikzcd}$$

Proof. Let f/j be defined by the above equation. By monotonicity of composition,

$$\begin{aligned}(f/j)\circ j &= r\circ\big((s\circ f)/j\big)\circ j\\ &\leqslant r\circ s\circ f \qquad \text{because } ((s\circ f)/j)\circ j\leqslant s\circ f \text{ by } (\mathrm{Kan}_1)\\ &= \mathrm{id}_E\circ f = f,\end{aligned}$$

equality holding if D is right injective. This establishes (Kan_1). Let $g:Y\to E$ with $g\circ j\leqslant f$. By monotonicity of composition, $s\circ g\circ j\leqslant s\circ f$. By ($\mathrm{Kan}_2$), $s\circ g\leqslant (s\circ f)/j$. By monotonicity of composition, $r\circ s\circ g\leqslant r\circ((s\circ f)/j)$. But $r\circ s=\mathrm{id}_E$. Hence $g\leqslant f/j$, which establishes (Kan_2). Therefore E is right Kan, and right injective if D is. □

Notice that the following lemma does *not* hold for *right Kan* objects:

Lemma 2.4.4. *If D is injective over J and $j:D\to Y$ in J then D is a retract of Y.*

Proof. By definition of injectivity, there is some extension of id_D along j, which means that j is a section. □

2.4.2. Kan objects and products

Lemma 2.4.5. *If each component of a poset-product is a right Kan (respectively right injective) object over J, so is the product. Moreover, if $E=\prod_{i\in I}D_i$ is such a product with projections $p_i:E\to D_i$ and $j:X\to Y$ is a member of J, then for every arrow $f:X\to E$, the right Kan extension $f/j:Y\to E$ is given by*

$$f/j=\big\langle (p_i\circ f)/j\big\rangle_{i\in I}.$$

Proof. Let f/j be defined by the above equation. By Definition 2.1.1,

$$\begin{aligned}(f/j)\circ j &= \big\langle (p_i\circ f)/j\big\rangle_{i\in I}\circ j\\ &= \big\langle (p_i\circ f)/j\circ j\big\rangle_{i\in I}\\ &\leqslant \langle p_i\circ f\rangle_{i\in I} \qquad \text{because } (p_i\circ f)/j\circ j\leqslant p_i\circ f \text{ by } (\mathrm{Kan}_1)\\ &= f,\end{aligned}$$

equality holding if each D_i is right injective. This establishes (Kan_1). Let $g:Y\to E$ with $g\circ j\leqslant f$. By monotonicity of composition, $p_i\circ g\circ j\leqslant p_i\circ f$. By ($\mathrm{Kan}_2$) $p_i\circ g\leqslant (p_i\circ f)/j$. By Definition 2.1.1,

$$\langle p_i\circ g\rangle_{i\in I}\leqslant \big\langle (p_i\circ f)/j\big\rangle_{i\in I}.$$

But $\langle p_i\circ g\rangle_{i\in I}=g$. This means that $g\leqslant f/j$, which establishes (Kan_2). Therefore E is right Kan, and right injective if each D_i is. □

2.4.3. Kan objects and inverse limits

Let Δ be a directed set considered as a category in the usual way [27, p. 11] and $\mathcal{F} : \Delta^{\mathrm{op}} \to \mathcal{X}$ be a functor onto *right Kan objects over* J and *right adjoints*. For all $m \leqslant n$ in Δ, define

$$D_m = F(m), \qquad r_{mn} = F(m \to n) : D_n \to D_m,$$

where $m \to n$ is the unique arrow from m to n. Then for all $m \leqslant n \leqslant p \in \Delta$,

$$r_{mm} = \mathrm{id}_{D_m}, \qquad r_{mn} \circ r_{np} = r_{mp}.$$

The following proposition and its proof generalize [33, Proposition 4.1]:

Proposition 2.4.6. *If* $\mathcal{F} : \Delta^{\mathrm{op}} \to \mathcal{X}$ *has a poset-limiting cone*

$$\begin{tikzcd} & D_\infty \arrow[dl, "p_m"'] \arrow[dr, "p_n"] & \\ D_m & & D_n \arrow[ll, "r_{mn}"] \end{tikzcd} \qquad m \leqslant n \in \Delta$$

then D_∞ *is a right Kan object over* J, *and it is a right injective object over* J *if each* D_m *is.*

Proof. Let $j : X \to Y$ in J and $f : X \to D_\infty$ be any arrow. In order to show that f has a right Kan extension along j, we first check that

$$\begin{tikzcd} & Y \arrow[dl, "(p_m \circ f)/j"'] \arrow[dr, "(p_n \circ f)/j"] & \\ D_m & & D_n \arrow[ll, "r_{mn}"] \end{tikzcd} \qquad m \leqslant n \in \Delta$$

is a cone:

$$\begin{aligned} r_{mn} \circ \big((p_n \circ f)/j\big) &= (r_{mn} \circ p_n \circ f)/j && \text{by Theorem 2.3.3(4)} \\ &= (p_m \circ f)/j && \text{by the given limiting cone.} \end{aligned}$$

We then use the same argument as in Lemma 2.4.5 to show that f has a right Kan extension along j given by

$$f/j = \big\langle (p_m \circ f)/j \big\rangle_{m \in \Delta},$$

which is an actual extension if each D_m is right injective. Therefore D_∞ is right Kan, and right injective if each D_m is. $\square$

2.4.4. Kan objects and adjunctions between poset-enriched categories

Let $\mathcal{A}$ be a poset-enriched category and K be any class of arrows of $\mathcal{A}$.

Lemma 2.4.7. *Let* $\mathcal{F} : \mathcal{X} \to \mathcal{A}$ *and* $\mathcal{G} : \mathcal{A} \to \mathcal{X}$ *be poset-functors with* $\mathcal{F} \dashv \mathcal{G}$. *If* $\mathcal{F}(J) \subseteq K$ *then* $\mathcal{G}$ *maps right Kan (respectively right injective) objects over* K *to right Kan* *(respectively right injective) objects over J. Moreover, if D is a right Kan object over K, $g : X \to \mathcal{G}D$ is any arrow, and $j : X \to Y$ is a member of J, then*

$$g/j = \phi(\phi^{-1}(g)/\mathcal{F}j),$$

where

$$\phi = \phi_{X,D} : \mathrm{hom}(\mathcal{F}X, D) \to \mathrm{hom}(X, \mathcal{G}D)$$

is the natural isomorphism which specifies the adjunction $\mathcal{F} \dashv \mathcal{G}$.

Proof. First, notice that ϕ is an order-isomorphism, because if $\eta_X : X \to \mathcal{G}\mathcal{F}X$ is the unit of the adjunction then $\phi(f) = \mathcal{G}f \circ \eta_X$ [27, p. 80, Eq. (5)], which shows that ϕ is a monotone map as $\mathcal{G}$ is a poset-functor and composition is monotone. Let g/j be defined by the above equation. Then

$$\begin{aligned}(g/j) \circ j &= \phi\big(\phi^{-1}(g)/\mathcal{F}j\big) \circ j \\ &= \phi\big(\big(\phi^{-1}(g)/\mathcal{F}j\big) \circ \mathcal{F}j\big) && \text{by [27, p. 79, Eq. (3)]} \\ &\leqslant \phi\big(\phi^{-1}(g)\big) && \text{by } (\mathrm{Kan}_1) \\ &= g,\end{aligned}$$

equality holding if D is right injective. This establishes (Kan_1). Now assume that $h \circ j \leqslant g$ for $h : X \to \mathcal{G}D$. Then $\phi^{-1}(h \circ j) \leqslant \phi^{-1}(g)$. But $\phi^{-1}(h \circ j) = \phi^{-1}(h) \circ \mathcal{F}j$ by [27, Eq. (4)]. Hence $\phi^{-1}(h) \leqslant \phi^{-1}(g)/\mathcal{F}j$ by (Kan_2), and $h \leqslant \phi(\phi^{-1}(g)/\mathcal{F}j) = g/j$. This establishes ($\mathrm{Kan}_2$), which shows that g/j is a right Kan extension of f along j, being an actual extension if D is injective. □

We shall apply both this lemma and its symmetric version, which states that if $\mathcal{G}(K) \subseteq J$ then $\mathcal{F}$ maps right Kan (respectively right injective) objects over J to right Kan (respectively right injective) objects over K.

2.4.5. Kan objects and Cartesian closed categories

Here we assume that $\mathcal{X}$ has finite products. An object X of $\mathcal{X}$ is *exponentiable* if the functor $\_ \times X : \mathcal{X} \to \mathcal{X}$ has a right adjoint, which we shall denote by $[X \to \_]$. The following lemma generalizes [20, Lemma 4.10]:

Lemma 2.4.8. *Let X be an object of $\mathcal{X}$ and assume that $j \times \mathrm{id}_X : Y \times X \to Z \times X$ in J for each $j : Y \to Z$ in J. If X is exponentiable then the functor $[X \to \_]$ preserves right Kan (respectively right injective) objects over J.*

Proof. This is a particular case of Lemma 2.4.7. □

A category is *Cartesian closed* if it has finite products and every object is exponentiable [27, p. 95].

Corollary 2.4.9. *Assume that $j \times \mathrm{id}_X : Y \times X \to Z \times X$ in J for all $j : Y \to Z$ in J and $X \in \mathcal{X}$. If the right Kan (respectively right injective) objects over J are exponentiable, then they form a Cartesian closed full subcategory.*

2.5. Right injective spaces

Scott [33] proved that the injective T_0 spaces (over subspace embeddings) are the continuous lattices endowed with the Scott topology by the following chain of deductions: (1) Sierpinski space is injective. (Sierpinski space is the space $\mathbb{S}$ with points $\perp$ and $\top$ such that $\{\top\}$ is open but $\{\perp\}$ is not.) (2) The Cartesian product of any number of injective spaces is injective. (3) A retract of an injective space is injective. (4) Every T_0 space can be embedded into an injective space; in fact, into a Cartesian power of Sierpinski space. (We can let I be the set of open sets of X and define an embedding $j : X \to \mathbb{S}^I$ by $j(x)(U) = \top$ iff $x \in U$.) (5) An injective space is a retract of every space of which it is a subspace—cf. Lemma 2.4.4. (6) Therefore the injective spaces are exactly the retracts of the Cartesian powers of Sierpinski space. (7) A finite lattice is a continuous lattice. (8) The Cartesian product of any number of continuous lattices is a continuous lattice with Scott topology agreeing with the product topology. (9) A retract of a continuous lattice is a continuous lattice with the subspace topology agreeing with the Scott topology. (10) Every continuous lattice is an injective space under the Scott topology. (11) Therefore the injective spaces are exactly the continuous lattices.

In fact, Scott [33, p. 116] proved more, namely that every continuous lattice is what we call a right injective space (over subspace embeddings), which shows that the injective spaces coincide with the right injective spaces. But notice that Lemmas 2.4.3–2.4.5, together with Lemma 2.5.2 below also give a proof of this fact, which does not refer to the definition of continuous lattice. Notice also that the proof of the fact that a Cartesian power of injective spaces is injective depends on the Axiom of Choice, which is not necessary to prove that a Cartesian power of right injective spaces is right injective (Lemma 2.4.5), because in the latter case we have canonical extensions available.

Definition 2.5.1. If X is a space then ΩX denotes its lattice of open sets ordered by inclusion. Let $f : X \to Y$ be a continuous map. Then the equation

$$\Omega f(V) = f^{-1}(V)$$

gives rise to a well-defined function $\Omega f : \Omega Y \to \Omega X$, by continuity of f. Since Ωf preserves all joins, it has a right adjoint, denoted by $\forall_f : \Omega X \to \Omega Y$, which has to be given by

$$\forall_f(U) = \bigcup \{V \in \Omega Y \mid \Omega f(V) \subseteq U\}.$$

There is a bijection between $\hom(X, \mathbb{S})$ and ΩX, given by $f \mapsto \Omega f(\top)$. Let $U \mapsto \chi_U$ denote its inverse. Since $f \leqslant g$ in $\hom(X, \mathbb{S})$ iff $\Omega f(\top) \subseteq \Omega g(\top)$ in ΩX, this bijection is an order-isomorphism.

Lemma 2.5.2. *Sierpinski space is a right Kan space over arbitrary continuous maps and a right injective space over subspace embeddings. Moreover, for every $U \in \Omega X$ and every continuous $j : X \to Y$, the right Kan extension of $\chi_U : X \to \mathbb{S}$ along j is given by*

$$\chi_U / j = \chi_{\forall_j(U)}.$$

Proof. $\chi_V \circ j = \chi_{j^{-1}(V)} = \chi_{\Omega j(V)}$ for all $V \in \Omega Y$. Hence $g \mapsto g \circ j$ has a right adjoint iff Ωj has a right adjoint, which is always the case. This shows that $\mathbb{S}$ is right Kan over arbitrary continuous maps and establishes the above equation. Moreover, $\forall_j$ is injective iff Ωj is surjective iff j is a subspace embedding. Therefore $\mathbb{S}$ is right injective over subspace embeddings (and only over subspace embeddings). □

The following proposition shows that the injective and the right injective spaces coincide:

Proposition 2.5.3. *The right injective spaces over subspace embeddings are the retracts of Cartesian powers of Sierpinski space.*

Proof. Such a retract is right injective by Lemma 2.4.3. If D is right injective then, being a T_0 space, it can be embedded into a Cartesian power of Sierpinski space, which is right injective by Lemmas 2.5.2 and 2.4.5. But then it is a retract of the Cartesian power, by Lemma 2.4.4. □

Proposition 2.5.4. *Every (right) injective space over subspace embeddings is a right Kan space over arbitrary continuous maps.*

Proof. By Lemmas 2.5.2, 2.4.5, and 2.4.3, the retracts of Cartesian powers of Sierpinski space are right Kan spaces over arbitrary continuous maps. □

Is the converse true? Thomas Erker (private communication) has shown that it is not. He has also obtained an internal characterization of the right Kan spaces.

2.6. A remark on Scott's extension process

Scott's formula

$$f/j(y) = \bigvee^{\uparrow}_{y \in V \in \Omega Y} \bigwedge f(j^{-1}(V))$$

discussed in the introduction not only produces the greatest extension of $f : X \to D$ along a subspace embedding $j : X \to Y$, where D is a continuous lattice, but also produces the *right Kan extension* of f along *any* continuous map $j : X \to Y$.

In fact, by turning some equalities into inequalities, the proof given in [33, pp. 109, 110] covers this generalized situation. Moreover, essentially the same proof establishes the following proposition, which has the claim as a corollary:

Proposition 2.6.1. *Let Y and D be T_0 spaces with D injective and let $g : Y \to D$ be a* monotone *function with respect to the specialization orders of Y and D. Then there is a greatest* continuous *map $\underline{g} : Y \to D$ pointwise below g, given by*

$$\underline{g}(y) = \bigvee^{\uparrow}_{y \in V \in \Omega Y} \bigwedge g(V).$$

Moreover, $\underline{g}$ agrees with g at every point of continuity of g.

We refer to $\underline{g}$ as the *continuous coreflection* of g.

Corollary 2.6.2. *For every injective-valued continuous map $f : X \to D$ and every continuous map $j : X \to Y$, the right Kan extension of f along j is the continuous coreflection of the right Kan monotone extension of f along j.*

2.7. Continuity of the right-Kan-extension map for right injective spaces

Definition 2.7.1. A continuous map $f : X \to Y$ is *finitary* if the function $\forall_f : \Omega X \to \Omega Y$ is Scott continuous (cf. Definition 2.5.1).

We say that a T_0 space is *nontrivial* if it contains at least two distinct points.

Theorem 2.7.2. *The following are equivalent for any continuous map $j : X \to Y$:*

(1) $\mathrm{Ran}_j^D : [X \to D] \to [Y \to D]$ *is Scott continuous for each injective space D.*

(2) $\mathrm{Ran}_j^{\mathbb{S}} : [X \to \mathbb{S}] \to [Y \to \mathbb{S}]$ *is Scott continuous.*

(3) $\mathrm{Ran}_j^D : [X \to D] \to [Y \to D]$ *is Scott continuous for some nontrivial injective space D.*

(4) *j is a finitary map.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(2) $\Leftrightarrow$ (4) It is clear from Lemma 2.5.2 that $\mathrm{Ran}_j^{\mathbb{S}}$ is Scott continuous iff $\forall_j$ is Scott continuous.

(3) $\Rightarrow$ (2) Since D is nontrivial, $\mathbb{S}$ can be embedded into D by $\bot_{\mathbb{S}} \mapsto \bot_D$ and $\top_{\mathbb{S}} \mapsto \top_D$. Since $\mathbb{S}$ is right injective by Lemma 2.5.2, it is a retract of D by Lemma 2.4.4. Therefore $\mathrm{Ran}_j^{\mathbb{S}}$ is Scott continuous by Lemma 2.4.3.

(2) $\Rightarrow$ (1) Being a T_0 space, D can be embedded into a Cartesian power of $\mathbb{S}$, say $\mathbb{S}^I$, and hence it is a retract of $\mathbb{S}^I$ by Lemma 2.4.4. By Lemma 2.4.5, $\mathrm{Ran}_j^{\mathbb{S}^I}$ is Scott continuous. But then so is Ran_j^D, by Lemma 2.4.3. $\Box$

Corollary 2.7.3. *For all continuous maps $j : X \to Y$ and all injective spaces D, the map $\mathrm{Ran}_j^D : [X \to D] \to [Y \to D]$ is Scott continuous iff D is trivial or j is finitary.*

Proof. This is an immediate consequence of Theorem 2.7.2, because if D is trivial then D is a one-point space and hence Ran_j^D is a constant map. $\Box$

2.8. Existence of left Kan extensions for injective spaces

Definition 2.8.1. A continuous map $f : X \to Y$ is *semi-open* if the function $\Omega f : \Omega Y \to \Omega X$ has a left adjoint, denoted by $\exists_j : \Omega X \to \Omega Y$ (cf. Definition 2.5.1).

Theorem 2.8.2. *The following are equivalent for any continuous map $j : X \to Y$:*

(1) $\mathrm{Lan}_j^D : [X \to D] \to [Y \to D]$ *exists for each injective space D.*

(2) $\mathrm{Lan}_j^{\mathbb{S}} : [X \to \mathbb{S}] \to [Y \to \mathbb{S}]$ *exists.*

(3) $\mathrm{Lan}_j^D : [X \to D] \to [Y \to D]$ *exists for some nontrivial injective space* D.
(4) j *is a semi-open map.*
Moreover, in this case the left-Kan-extension maps are Scott continuous, and they produce actual extensions if j *is a subspace embedding.*

Proof. Similar to the proof of Theorem 2.7.2. First, with a proof similar to that of Lemma 2.5.2, we show that the following holds: A continuous map $\chi_U : X \to \mathbb{S}$ has a left Kan extension along a continuous map $j : X \to Y$ iff j is semi-open. Moreover, in this case the left Kan extension $\chi_U \backslash j : Y \to \mathbb{S}$ is given by

$$\chi_U \backslash j = \chi_{\exists_j(U)},$$

and it is an actual extension iff j is a subspace embedding. $\square$

2.9. Right injective spaces over dense embeddings

We omit the details of the (routine) proofs of the following facts:

(1) The right injective spaces over dense embeddings are the closed subspaces of the right injective spaces. Therefore the right injective spaces over dense embeddings are the continuous Scott domains endowed with the Scott topology.
(2) The right injective spaces over dense embeddings are right Kan spaces over continuous maps with dense image.
(Use Proposition 2.5.4 and (1) above.)
(3) The results of Section 2.6 hold for densely injective spaces.
(Scott's formula gives rise to a well-defined function f/j. In fact, by density, $j^{-1}(V)$ is nonempty for every nonempty $V \in \Omega Y$, and hence the meet is well-defined.)
(4) Theorem 2.7.2 and Corollary 2.7.3 hold for densely injective spaces.
(Use (1) above.)
(5) Theorem 2.8.2 holds for densely injective spaces.
(Every continuous Scott domain D is semi-openly embedded in the continuous lattice $D^\top$ which results from adding a (compact) top element to D. In fact, let $j : D \to D^\top$ be the inclusion. Then $\Omega j(V) = V \backslash \{\top\}$, and it is easy to see that Ωj has a left adjoint $\exists_j : \Omega D \to \Omega D^\top$ given by $\exists_j(U) = \uparrow_{D^\top} U$, or, equivalently, $\exists_j(U) = U \cup \{\top\}$ if U is nonempty and $\exists_j(\emptyset) = \emptyset$.)

3. Finitary sublocales and subspaces

3.1. Finitary sublocales

The results on finitary sublocales discussed in this subsection are transferred to finitary sober subspaces in Section 3.2 below, via the usual adjunction between the category of locales and the category of spaces.

We assume some familiarity with the terminology and basic results on frames and locales as described in [20] or [43]. Recall that a frame is a complete lattice in which finite meets distribute over arbitrary joins, and that a frame homomorphism preserves finite meets and arbitrary joins. The category of spaces and continuous maps is denoted by Sp, the category of frames and frame homomorphisms is denoted by Frm, and the opposite of Frm is denoted by Loc. The objects of Loc are referred to as locales, and its arrows of are referred to as *continuous maps*.

A *nucleus* [20, p. 49] on a locale A is a map $j : A \to A$ such that

(N_1) $a \leqslant j(a)$,

(N_2) $j(a \wedge b) = j(a) \wedge j(b)$,

(N_3) $j(j(a)) \leqslant j(a)$.

By (N_2), a nucleus is monotone, and hence by (N_1) and (N_3) it is idempotent. Therefore a nucleus is a closure operator (an idempotent monotone map above the identity) which preserves binary meets. A nucleus is *finitary* if it preserves directed joins [3, p. 649].

A *sublocale* [20, p. 50] of a locale A is a locale of the form $A_j = j(A) = \{j(a) \mid a \in A\}$ for some nucleus $j : A \to A$. A *finitary sublocale* is a sublocale induced by a finitary nucleus.

The following two lemmas are applied in Section 3.2. We include them at this point for the sake of motivation:

Lemma 3.1.1. *A subspace embedding $j : X \to Y$ is finitary iff the induced nucleus*

$$j_X \stackrel{\text{def}}{=} \forall_j \circ \Omega j : \Omega Y \to \Omega Y$$

is finitary.

Proof. If $\forall_j$ is finitary so is j_X because Ωj preserves all joins. Conversely, if j_X is finitary then $\forall_j$ preserves directed joins because j_X is a Scott continuous idempotent and Ωj is a Scott continuous surjection [2, Proposition 3.17]. □

A space is *sober* iff every completely prime filter of open sets is the open neighborhood filter of a unique point [20, p. 43]. A locale A is called *spatial* (or said to have enough points) [20, p. 43] if for all a and b in A with $a \not\leqslant b$ there is a completely prime filter $\mathcal{F} \subseteq A$ with $a \in \mathcal{F}$ but $b \notin \mathcal{F}$. The category of sober spaces and continuous maps is denoted by Sob. The categories of spatial locales and sober spaces are equivalent.

Lemma 3.1.2. *For any sober space Y there is an inclusion-preserving bijection between sober subspaces of Y and spatial sublocales of ΩY. Moreover, the above bijection can be given by the map which sends a sober subspace X to the nucleus j_X defined by*

$$j_X = \forall_j \circ \Omega j,$$

where $j : X \to Y$ is the inclusion. Its inverse sends a nucleus j to the subspace X_j defined by

$$X_j = \{y \in Y \mid \forall U \in \Omega Y\colon\ y \in j(U) \Rightarrow y \in U\} = \{y \in Y \mid y \notin j(Y \setminus {\downarrow} y)\}.$$

Proof. See [28, pp. 504, 505]. □

3.1.1. Spatial locales

Lemma 3.1.3. *Let A_j be a finitary sublocale of a locale A. If $\mathcal{F}$ is a Scott open filter in A then $j(\mathcal{F})$ is a Scott open filter in A_j.*

Proof. First, notice that $j(\mathcal{F}) \subseteq \mathcal{F}$, because id $\leqslant j$ and $\mathcal{F}$ is an upper set in A. Let $a \in j(\mathcal{F})$ and $b \in A_j$ with $a \leqslant b$. Since $a \in \mathcal{F}$ and $\mathcal{F}$ is a filter, $b \in \mathcal{F}$. It follows that $b \in j(\mathcal{F})$, because $b = j(b)$. Hence $j(\mathcal{F})$ is an upper set in A_j. Now let $a, b \in j(\mathcal{F})$ and $a', b' \in \mathcal{F}$ with $a = j(a')$ and $b = j(b')$. Since $a' \wedge b' \in \mathcal{F}$ as $\mathcal{F}$ is a filter and since $a \wedge b = j(a') \wedge j(b') = j(a' \wedge b')$, it follows that $a \wedge b \in j(\mathcal{F})$. Therefore $j(\mathcal{F})$ is a filter. Finally, let $\bigvee$ and $\bigvee_j$ denote the join operations of A and A_j, respectively, and let $\Delta \subseteq A_j$ be a directed set with $\bigvee_j \Delta \in j(\mathcal{F})$. We know that $\bigvee_j \Delta = j(\bigvee \Delta)$[20, p. 49]. Since j is finitary, $j(\bigvee \Delta) = \bigvee j(\Delta)$. Therefore $\bigvee_j \Delta = \bigvee \Delta$, because $j(\Delta) = \Delta$ as $\Delta \subseteq A_j$. Since $j(\mathcal{F}) \subseteq \mathcal{F}$ and $\mathcal{F}$ is Scott open in A, there is some $b \in \Delta$ with $b \in \mathcal{F}$. Hence $j(b) \in j(\mathcal{F})$. But $j(b) = b$, because $b \in \Delta \subseteq A_j$. Therefore $j(\mathcal{F})$ is Scott open in A_j. □

The following is the only result in Section 3.1 that makes use of the Axiom of Choice (in the form of Zorn's Lemma):

Theorem 3.1.4. *Spatial locales are closed under the formation of finitary sublocales.*

Proof. Let $a \nleqslant b$ in A_j for some finitary nucleus $j : A \to A$ on a spatial locale A. Then $a \nleqslant b$ in A. Since A is spatial, there is a completely prime filter $\mathcal{F} \subseteq A$ with $a \in \mathcal{F}$ but $b \notin \mathcal{F}$. Since $\mathcal{F}$ is Scott open, so is $j(\mathcal{F})$ by Lemma 3.1.3. Since $j(\mathcal{F}) \subseteq \mathcal{F}$ and $b = j(b)$, we have that $b \notin j(\mathcal{F})$. Hence we may use Zorn's Lemma to enlarge $j(\mathcal{F})$ to a Scott open filter $\mathcal{G}$ maximal amongst those not containing b, because a directed union of Scott open filters is a Scott open filter. By Lemma VII-4.3 of [20, pp. 310, 311], which states that every Scott open filter $\mathcal{G}$ maximal amongst Scott open filters not containing b is prime, recalling that a filter is completely prime iff it is prime and Scott open [15, p. 257], we conclude that $\mathcal{G}$ is a completely prime filter not containing b. But $a \in j(\mathcal{F}) \subseteq \mathcal{G}$. Therefore A_j is spatial. □

3.1.2. Stably locally compact locales

A locale A is *compact* [20, p. 80] if its top element 1 is compact in the sense that $1 \ll 1$, and it is *locally compact* if it is a continuous lattice [20, p. 310]; [15, p. 270].

Remark 3.1.5. The classes of compact locales and locally compact locales are closed under the formation of finitary sublocales.

Proof. (1) A nucleus is a closure operator, and the corestriction of a Scott continuous closure operator to its image preserves compact elements [15, p. 87].

(2) The image of a Scott continuous idempotent defined on a continuous lattice is a continuous lattice [15, p. 63]. □

A meet-semilattice is *stably continuous* [20, p. 296] if it is continuous, $1 \ll 1$, and its way-below relation is *multiplicative*, in the sense that $x \ll y$ and $x \ll z$ together imply $x \ll y \wedge z$. A locale is *stably locally compact* if it is stably continuous [20, p. 313]; [38, p. 321]. A locale is *spectral* [43, pp. 119, 120] if it is algebraic and the compact elements form a sublattice. Since it suffices that the compact elements form a meet-semilattice, we see that every spectral locale is stably locally compact.

Lemma 3.1.6.

(1) *Let D and E be dcpos with binary meets and $l : E \to D$ be a map which preserves binary meets and has a Scott continuous injective right adjoint $r : D \to E$. If E is a continuous poset with multiplicative way-below relation, so is D.*

(2) *Let E be a poset with binary meets and $j : E \to E$ be a Scott continuous, meet-preserving closure operator. If E is a continuous poset with multiplicative way-below relation, so is $j(E)$ with the inherited order.*

Proof. (1) D is a retract of E because r, being a right adjoint, is injective iff $l \circ r = \mathrm{id}_D$. Hence D is a continuous dcpo with binary meets [2, Proposition 3.1.3, Theorem 3.1.4].

Assume that $x \ll y$ and $x \ll z$ in D. Since $l \circ r = \mathrm{id}_D$, we have that $x \ll l(r(y))$ and $x \ll l(r(y))$. Since D and E are continuous and l is Scott continuous (it preserves all joins), by the so-called ε-δ characterization of Scott continuity [2, Proposition 2.2.11]; [15, pp. 112, 119], there are $y' \ll r(y)$ and $z' \ll r(z)$ in E such that already $x \ll l(y')$ and $x \ll l(z')$ in D. By multiplicativity of the way-below relation of E and the fact that r is a right adjoint and hence preserves meets,

$$y' \wedge z' \ll r(y) \wedge r(z) = r(y \wedge z).$$

Since l preserves the way-below relation as it has a Scott continuous right adjoint [2, Proposition 3.1.14],

$$l(y' \wedge z') \ll l\bigl(r(y \wedge z)\bigr) = y \wedge z.$$

But $x \leqslant l(y') \wedge l(z') = l(y' \wedge z')$. Hence $x \ll y \wedge z$ in D. Therefore the way-below relation of D is multiplicative.

(2) The map j factors through its image $D \stackrel{\text{def}}{=} j(E)$ as $j = r \circ l$ with $r : D \to E$ the inclusion, $l : E \to D$ the corestriction of j to its image, and l left adjoint to r [15, p. 22]. Since j is idempotent and monotone, its image E is a continuous dcpo with binary meets [2, Proposition 3.1.2, Theorem 3.1.4]. Since j is Scott continuous so is r [2, Proposition 3.1.7]. Also, it is clear that l preserves binary meets and r is injective. Therefore the result follows from (1). □

Theorem 3.1.7. *The classes of stably locally compact locales and spectral locales are closed under the formation of finitary sublocales.*

Proof. Immediate consequence of Lemma 3.1.6 and Remark 3.1.5. For the spectral case, it suffices to note that Scott continuous images of algebraic lattices are algebraic. □

3.1.3. Finitary hulls of sublocales

The set $N(A)$ of nuclei on a locale A is ordered by $j \leqslant k$ iff $j(a) \leqslant k(a)$ for all $a \in A$. By [20, p. 51], we know that $N(A)$ is a frame dual to the set of sublocales of A ordered by inclusion, in the sense that $j \leqslant k$ iff $A_j \supseteq A_k$. Meets in $N(A)$ are given pointwise, in the sense that for all $J \subseteq N(A)$ one has that $\left(\bigwedge J\right)(a) = \bigwedge_{j \in J} j(a)$. Joins are harder to describe explicitly, because a pointwise join is not necessarily idempotent, though we may note that the sublocale $A_{\bigvee J}$ is simply the set-theoretic intersection $\bigcap_{j \in J} A_j$. But (arbitrary) joins of *finitary* nuclei are easy to describe explicitly. Let $F(A)$ denote the set of finitary nuclei on a locale A.

We first remark that *directed* joins of *finitary* nuclei *are* computed pointwise. In fact, let $J \subseteq F(A)$ be directed, and define $i(a) = \bigvee^{\uparrow}_{j \in J} j(a)$. In order to establish the claim, it is enough to show that i is a nucleus. It is clear that i is above the identity. It preserves binary meets because each $j \in J$ does and because binary meets distribute over joins. Finally, it is idempotent by virtue of the following calculation:

$$\begin{aligned} i\big(i(a)\big) &= \bigvee_{j \in J}^{\uparrow} j\big(i(a)\big) = \bigvee_{j \in J}^{\uparrow} j\Big(\bigvee_{k \in J}^{\uparrow} k(a)\Big) = \bigvee_{j \in J}^{\uparrow} \bigvee_{k \in J}^{\uparrow} j\big(k(a)\big) \\ &= \bigvee_{j \in J}^{\uparrow} j\big(j(a)\big) = \bigvee_{j \in J}^{\uparrow} j(a) = i(a). \end{aligned}$$

Lemma 3.1.8. *$F(A)$ is a subframe of $N(A)$. Moreover, for all $J \subseteq F(A)$,*

$$\Big(\bigvee J\Big)(a) = \bigvee_{\alpha \in J^*}^{\uparrow} \alpha(a),$$

where J^* is the set of finite compositions of members of J.

Proof. The empty meet is finitary because it is the identity, and it is clear from frame-distributivity that finitary nuclei are closed under binary meets. Each member of J^* is a Scott continuous map above the identity and preserves binary meets, but is not necessarily idempotent. The set J^* is nonempty because it contains the identity map, and if $\alpha, \beta \in J^*$ then $\beta \circ \alpha \in J^*$ is above α and β because α and β are above the identity. Hence J^* is a directed collection of Scott continuous maps, and the function $i : A \to A$ defined by

$$i(a) = \bigvee_{\alpha \in J^*}^{\uparrow} \alpha(a)$$

is a Scott continuous map above the identity. It is idempotent because

$$\begin{aligned} i\big(i(a)\big) = i\Big(\bigvee_{\alpha \in J^*}^{\uparrow} \alpha(a)\Big) &= \bigvee_{\alpha \in J^*}^{\uparrow} i\big(\alpha(a)\big) = \bigvee_{\alpha \in J^*}^{\uparrow} \bigvee_{\beta \in J^*}^{\uparrow} \beta\big(\alpha(a)\big) \\ &= \bigvee_{\alpha, \beta \in J^*}^{\uparrow} \beta\big(\alpha(a)\big) = \bigvee_{\gamma \in J^*}^{\uparrow} \gamma(a) = i(a), \end{aligned}$$

where the equations of the second row follow from the fact that $J^* = \{\beta \circ \alpha \mid \alpha, \beta \in J^*\}$. Since each member of J^* preserves binary meets,

$$
\begin{aligned}
i(a) \wedge i(b) &= \bigvee^{\uparrow}_{\alpha \in J^*} \alpha(a) \wedge \bigvee^{\uparrow}_{\beta \in J^*} \beta(b) = \bigvee^{\uparrow}_{\alpha \in J^*} \Big(\alpha(a) \wedge \bigvee^{\uparrow}_{\beta \in J^*} \beta(b) \Big) \\
&= \bigvee^{\uparrow}_{\alpha \in J^*} \bigvee^{\uparrow}_{\beta \in J^*} \alpha(a) \wedge \beta(b) = \bigvee^{\uparrow}_{\alpha, \beta \in J^*} \alpha(a) \wedge \beta(b) \\
&= \bigvee^{\uparrow}_{\gamma \in J^*} \gamma(a) \wedge \gamma(b) = \bigvee^{\uparrow}_{\gamma \in J^*} \gamma(a \wedge b) = i(a \wedge b),
\end{aligned}
$$

because for all $\alpha, \beta \in J^*$ there is $\gamma \in J^*$ above α and β and hence $\alpha(a) \wedge \beta(b) \leqslant \gamma(a) \wedge \gamma(b)$. This shows that i is a finitary nucleus. It remains to show that i is the least upper bound of J in $N(A)$. Since $J \subseteq J^*$, we have that i is an upper bound of J. Let k be another upper bound. For finitely many $j_1, \ldots, j_n \in J$, we have that $j_1 \circ \cdots \circ j_n \leqslant k^n \leqslant k$ by monotonicity of composition and the fact that k is an idempotent above the identity. Since $j_1 \circ \cdots \circ j_n$ is an arbitrary member of J^*, this means that k is an upper bound of J^* in the set of monotone endomaps of A ordered pointwise. Therefore $i \leqslant k$. □

Corollary 3.1.9. *Finitary sublocales are closed under the formation of arbitrary intersections.*

A sublocale is *dense* [20, p. 50] if it is induced by a nucleus j with $j(0) = 0$. Every locale has a smallest dense sublocale, induced by the double Heyting complement nucleus $a \mapsto \neg\neg a$ [20, pp. 50, 51].

Theorem 3.1.10. *For every sublocale B of a locale A there is a smallest finitary sublocale $\overline{B}$ of A containing B as a sublocale, called the* finitary hull *of B (relative to A). In particular, every locale has a smallest finitary dense sublocale, called its* support *and denoted by* Supp(A).

Proof. $\overline{B}$ is the intersection of the finitary sublocales containing B (which include A). The particular case follows from the fact that a sublocale larger than a dense sublocale is itself dense, and hence we can take the finitary hull of the smallest dense sublocale. □

Remark 3.1.11.

(1) In general, the smallest dense sublocale of a spatial locale is *not* spatial again. But, by Theorem 3.1.4, if a locale is spatial so is its smallest finitary dense sublocale.

(2) We can construct the smallest finitary dense sublocale directly, without appealing to the smallest dense sublocale. In fact, let J be the set of finitary dense nuclei on a locale A. Then Corollary 3.1.8 shows that $\bigvee J$ preserves 0 and is finitary. Hence $\bigvee J \in J$ and $\bigvee J$ has to be the greatest finitary dense nucleus on A. Therefore $A_{\bigvee J}$ is the smallest finitary dense sublocale of A.

3.1.4. Finitary hulls in the stably locally compact case

The following lemma is well known [15, p. 63]; [21, p. 21]; [26, p. 146]:

Lemma 3.1.12. *Let D be a continuous dcpo, E be a directed complete poset, and let $f : D \to E$ be a* monotone *function. Then there is a greatest Scott* continuous *map $\underline{f}$ below f, given by*

$$\underline{f}(x) = \bigvee_{y \ll x}^{\uparrow} f(y).$$

We refer to $\underline{f}$ as the *continuous coreflection* of f (cf. Proposition 2.6.1).

Lemma 3.1.13. *The continuous coreflection of a closure operator defined on a continuous dcpo is itself a closure operator.*

Proof. Let D be a continuous dcpo and $j : D \to D$ be a closure operator. Since j is above the identity of D, so is $\underline{j}$, because the identity is Scott continuous and $\underline{j}$ is the greatest Scott continuous function below j. Since $\underline{j} \leqslant j$, by monotonicity of the composition operator we have that $\underline{j} \circ \underline{j} \leqslant j \circ j \leqslant j$. Since $\underline{j}$ is the greatest Scott continuous map below j and $\underline{j} \circ \underline{j}$ is Scott continuous, we have that $\underline{j} \circ \underline{j} \leqslant \underline{j}$. Therefore $\underline{j}$ is a closure operator. □

Lemma 3.1.14. *Let D be a continuous poset with binary meets and multiplicative way-below relation. If a monotone map $j : D \to D$ preserves binary meets, so does its continuous coreflection.*

Proof. In any continuous poset with binary meets, the binary meet operation is Scott continuous [33, p. 106]. Hence

$$\begin{aligned}
\underline{j}(x \wedge y) &= \bigvee_{c \ll x \wedge y}^{\uparrow} j(c) = \bigvee_{c \ll x, c \ll y}^{\uparrow} j(c) \quad \text{by multiplicativity} \\
&= \bigvee_{a \ll x, b \ll y}^{\uparrow} j(a \wedge b) = \bigvee_{a \ll x, b \ll y}^{\uparrow} j(a) \wedge j(b) \\
&= \bigvee_{a \ll x}^{\uparrow} j(a) \wedge \bigvee_{b \ll y}^{\uparrow} j(b) \quad \text{by Scott continuity of the meet operation} \\
&= \underline{j}(x) \wedge \underline{j}(y). \qquad \square
\end{aligned}$$

Theorem 3.1.15. *For every sublocale A_j of a stably locally compact locale A, the finitary hull of A_j is induced by the continuous coreflection of j. In particular,*

$$\mathrm{Supp}(A) = A_{\underline{\neg\neg}}.$$

Proof. Immediate consequence of Lemmas 3.1.13 and 3.1.14. □

3.1.5. Finitary maps and upper power locales

For a definition of the upper power locale monad $\mathcal{U} = (\mathcal{U}, \eta, \mu)$ and see [32] or [42] (see also [43]).

Definition 3.1.16. Let $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ be a poset-functor on a poset-enriched category $\mathcal{X}$. A *right $\mathcal{F}$-arrow* is a map $f : X \to Y$ in $\mathcal{X}$ such that $\mathcal{F}f : \mathcal{F}X \to \mathcal{F}Y$ has a left adjoint, denoted by $f^{-1} : \mathcal{F}Y \to \mathcal{F}X$. If the adjunction is reflective (i.e., $f^{-1} \circ \mathcal{F}f = \mathrm{id}_{\mathcal{F}X}$), we say that f is a *right $\mathcal{F}$-embedding*. Left $\mathcal{F}$-arrows and left $\mathcal{F}$-embeddings are defined by reversing the adjunction and changing reflectiveness to coreflectiveness (which amounts to the same equation).

We shall prove a topological version of the following result (Proposition 3.2.17):

Proposition 3.1.17 (Vickers). *Let $f : X \to Y$ be a continuous map of locales.*
(1) *f is a finitary map iff it is a right $\mathcal{U}$-arrow.*
(2) *f is a finitary sublocale embedding iff it is a right $\mathcal{U}$-embedding.*

Proof. For (1) see [42, Proposition 5.6]. Item (2) is not stated in [42], but it immediately follows from the construction given in the proof. □

3.2. Finitary subspaces

A space is called *locally compact* if every point has a neighborhood base of compact sets. A sober space is locally compact iff its locale of open sets is locally compact [15, p. 259]. A sober space is called *spectral* [43, p. 120] if it has a base of compact open sets. A sober space is spectral iff its locale of open sets is spectral [43]. A sober space is called *stably locally compact* [38, p. 321] if it is compact and locally compact, and the intersection of any two compact saturated sets is compact. Recall that a compact set is *saturated* iff it is the intersection of its neighborhoods iff it is an upper set with respect to the specialization order. A sober space is stably locally compact iff its locale of open sets is stably locally compact [38, p. 323]. Stably locally compact spaces include compact Hausdorff spaces, continuous lattices and continuous Scott domains endowed with the Scott topology (and more generally FS-domains) [22,2].

Theorem 3.2.1. *The classes of sober spaces consisting of*
(1) *compact spaces,*
(2) *locally compact spaces,*
(3) *spectral spaces,*
(4) *stably locally compact spaces*
are closed under the formation of finitary sober subspaces.

Proof. Immediate consequence of Theorems 3.1.4 and 3.1.7, Remark 3.1.5, and Lemmas 3.1.1 and 3.1.2. □

Theorem 3.2.2. *For every subspace X of a sober space Y there is a smallest finitary sober subspace $\overline{X}$ of Y containing X as subspace, called the* finitary sober hull *of X. In particular, every sober space has a smallest finitary dense sober subspace, called its* support *and denoted by* $\mathrm{Supp}(X)$.

Proof. Immediate consequences of Theorem 3.1.4 and Lemmas 3.1.1 and 3.1.2. □

3.2.1. A point-set characterization of finitary sober subspaces

Theorem 3.2.3 (Hofmann and Lawson). *Let X and Y be sober spaces. Then a continuous map $f : X \to Y$ is finitary iff the following conditions hold:*

(1) *If $Q \subseteq Y$ is compact saturated then $f^{-1}(Q) \subseteq X$ is compact.*

(2) *If $C \subseteq X$ is closed then $\downarrow f(C) \subseteq Y$ is closed.*

Moreover, if X and Y are locally compact, then condition (2) *follows from condition* (1).

Proof. See [18, Proposition 3.3]; [17, Remark 1.3]. □

Corollary 3.2.4. *A subspace inclusion $X \subseteq Y$ of sober spaces is finitary iff the following conditions hold:*

(1) *$Q \cap X$ is compact for every compact saturated set $Q \subseteq Y$.*

(2) *$\downarrow C$ is closed in Y for every closed set $C \subseteq X$.*

Moreover, if X and Y are locally compact, then condition (2) *follows from condition* (1).

Proof. If $j : X \to Y$ is the subspace inclusion then $j^{-1}(Q) = Q \cap X$ and $j(C) = C$. □

In particular, a subspace of a compact Hausdorff space is finitary iff it is closed.

Remark 3.2.5. Condition (2) of Theorem 3.2.3 is equivalent to

(2′) For every open set $U \subseteq X$, the set $U_f \stackrel{\text{def}}{=} \{y \in Y \mid f^{-1}(\uparrow y) \subseteq U\}$ is open.

If $j : X \to Y$ is a subspace inclusion then $U_j = \{y \in Y \mid \uparrow y \cap X \subseteq U\}$.

Proof. Condition (2) is clearly equivalent to:

(2″) For every open set $U \subseteq X$, the set $Y \setminus \downarrow f(X \setminus U)$ is open.

But

$$\begin{aligned} y \in Y\setminus\downarrow f(X\setminus U) \quad &\text{iff} \quad y \notin \downarrow f(X\setminus U)\\ &\text{iff} \quad \forall x \in X : x \notin U \Rightarrow y \not\leqslant f(x)\\ &\text{iff} \quad \forall x \in X : y \leqslant f(x) \Rightarrow x \in U\\ &\text{iff} \quad \forall x \in X : x \in f^{-1}(\uparrow y) \Rightarrow x \in U\\ &\text{iff} \quad f^{-1}(\uparrow y) \subseteq U. \end{aligned}$$

Therefore $U_f = Y\setminus\downarrow f(X\setminus U)$, which shows that (2′) is equivalent to (2″). □

Lemma 3.2.6. *If $f : X \to Y$ is a finitary map of sober spaces then*

$$\forall_f(U) = U_f.$$

Proof. By Theorem 3.2.3 and Remark 3.2.5, the set U_f is open. Also, we know that

$$\forall_f(U) = \bigcup \{V \in \Omega Y \mid f^{-1}(V) \subseteq U\}.$$

In order to conclude that $U_f \subseteq \forall_f(U)$, we show that $f^{-1}(U_f) \subseteq U$. Let $x \in f^{-1}(U_f) \subseteq U$. Then $f(x) \in U_f$ and $f^{-1}(\uparrow f(x)) \in U$. Therefore $x \in U$, which establishes $f^{-1}(U_f) \subseteq U$. Conversely, let $y \in \forall_f(U)$. Then there is some $V \in \Omega Y$ such that $y \in V$ and $f^{-1}(V) \subseteq U$. But $\uparrow y \subseteq V$ and hence $f^{-1}(\uparrow y) \subseteq f^{-1}(V)$, which shows that $y \in U_f$. Therefore $\forall_f(U) \subseteq U_f$. □

Remark 3.2.7. Any subspace embedding $j : X \to Y$ is an order-embedding (with respect to the specialization order).

Proof. The fact that j is an embedding can be expressed by saying that Ωj is surjective. By a general property of adjunctions, $\Omega j \circ \forall_j = \mathrm{id}_{\Omega X}$. Assume that $j(x) \leqslant j(y)$ and let U be an open neighborhood of x. Then $x \in \Omega j(\forall_j(U))$ and hence $j(x) \in \forall_j(U)$. It follows that $j(y) \in \forall_j(U)$, because open sets are upper closed. But this means that $y \in \Omega j(\forall_j(U)) = U$. Therefore $x \leqslant y$. □

Proposition 3.2.8. *A finitary map of sober spaces is a subspace embedding iff it is an order-embedding.*

Proof. ($\Rightarrow$) Remark 3.2.7.

($\Leftarrow$) Let $j : X \to Y$ be a finitary order-embedding of sober spaces. We first show that $j^{-1}(\uparrow j(x)) = \uparrow x$. Let $y \in j^{-1}(\uparrow j(x))$. This means that $j(x) \leqslant j(y)$. Hence $x \leqslant y$ and $y \in \uparrow x$. Conversely, let $y \in \uparrow x$. Then $x \leqslant y$ and $j(x) \leqslant j(y)$, which means $y \in j^{-1}(\uparrow j(x))$. Therefore $j^{-1}(\uparrow j(x)) = \uparrow x$ as desired. Let $U \in \Omega X$. Then

$$\begin{aligned}\Omega j \circ \forall_j(U) &= j^{-1}\big(\{y \in Y \mid j^{-1}(\uparrow y) \subseteq U\}\big) \qquad \text{by Lemma 3.2.6}\\ &= \big\{x \in X \mid j^{-1}\big(\uparrow j(x)\big) \subseteq U\big\} = \{x \in X \mid \uparrow x \subseteq U\} = U.\end{aligned}$$

Hence Ωj is surjective. Therefore j is a subspace embedding. □

Lemma 3.2.9. *Finitary maps of locally compact sober spaces are closed under finite products.*

Proof. Let $f : X \to A$ and $g : Y \to Z$ be finitary maps of locally compact sober spaces and let $Q \subseteq A \times B$ be a compact saturated set. Then

$$(f \times g)^{-1}(Q) = f^{-1}\big(p(Q)\big) \times g^{-1}\big(q(Q)\big)$$

is a compact set, where $p : A \times B \to A$ and $q : A \times B \to B$ are the projections, because the continuous maps p and q preserve compactness and saturatedness, f and g are finitary and hence reflect compact saturated sets by Theorem 3.2.3, and the product of two compact sets is compact (Tychonoff Theorem). Therefore $f \times g$ is finitary by Theorem 3.2.3, because $X \times Y$ and $A \times B$ are locally compact sober. □

3.2.2. Supports and subspaces of maximal points

The subspace of maximal points of a space X with respect to the specialization order is denoted by $\mathrm{Max}(X)$. Every *finite* T_0 space X has $\mathrm{Max}(X)$ as its smallest dense subspace. We now consider a more general situation.

Remark 3.2.10. $\mathrm{Max}(X)$ is dense in X for any sober space X.

Proof. By sobriety, the specialization order of X is directed complete [20, p. 46], and thus every point is below a maximal point by Zorn's Lemma. Hence $\downarrow\mathrm{Max}(X) = X$. Therefore $\mathrm{Max}(X)$ is dense in X, because closed sets are lower sets. □

Lemma 3.2.11. *Let $X \subseteq Y$ be a finitary inclusion of sober spaces. If X is dense in Y then* $\mathrm{Max}(Y) \subseteq X$.

In particular,

(1) $\mathrm{Max}(Y) \subseteq \mathrm{Supp}(Y)$.

(2) $\mathrm{Supp}(Y)$ *is the finitary sober hull of* $\mathrm{Max}(Y)$.

(3) *If Y is T_1 then* $\mathrm{Supp}(Y) = Y$.

Proof. Since X is closed in X, $\downarrow X$ is closed in Y. But then $\downarrow X$ is the closure of X in Y by Corollary 3.2.4, because closed sets are lower sets. By density, $\downarrow X = Y$. This means that every element of Y is below some element of X. □

Lemma 3.2.12. *Let X be a stably locally compact space. Then any two disjoint compact saturated subsets of X can be separated by disjoint neighborhoods.*

In particular,

(1) $\mathrm{Max}(X)$ *is Hausdorff.*

(2) *X is T_1 iff it is compact Hausdorff.*

(3) *$\downarrow Q$ is closed for every compact saturated set $Q \subseteq X$.*

Proof. We show that if $Q, R \subseteq X$ are compact saturated sets such that every neighborhood of Q meets every neighborhood of R, then Q meets R. Let $\mathcal{F}$ and $\mathcal{G}$ be the open neighborhood filters of Q and R, respectively, and put

$$\mathcal{H} = \{U \cap V \mid U \in \mathcal{F} \text{ and } V \in \mathcal{G}\}.$$

Then $\mathcal{H}$ is a proper filter. Let $U \in \mathcal{F}$ and $V \in \mathcal{G}$. By compactness of Q and R, there are $U' \ll U$ and $V' \ll V$ in $\mathcal{F}$ and $\mathcal{G}$, respectively. Since $U' \cap V'$ is way-below U and way-below V, it follows that $U' \cap V' \ll U \cap V$, by stability. This shows that $\mathcal{H}$ is Scott open. By the Hofmann–Mislove Theorem [2,19,39], $\mathcal{H}$ is the open neighborhood filter of a (unique) nonempty compact saturated set, which by construction of $\mathcal{H}$ is contained in Q and R.

Of the conclusions (1)–(3), only (3) is not immediate. Let p be a limit point of Q. Then, by definition, every neighborhood of q meets Q. Hence every neighborhood of $\uparrow p$ meets every neighborhood of Q. It follows that $\uparrow p$ meets Q, because $\uparrow p$ is compact saturated. Therefore $p \in \downarrow Q$. □

Theorem 3.2.13. *For any stably locally compact space X,* $\mathrm{Supp}(X) = \mathrm{Max}(X)$ *iff* $\mathrm{Max}(X)$ *is compact.*

Proof. ($\Rightarrow$) $\mathrm{Supp}(X)$ is compact by Corollary 3.2.1.

($\Leftarrow$) $\mathrm{Max}(X)$ is sober because it is Hausdorff by Lemma 3.2.12. By stability of X, $\mathrm{Max}(X) \cap Q$ is compact saturated for every compact saturated set $Q \subseteq X$. Hence $\mathrm{Max}(X)$ is a finitary sober subspace of X by Corollary 3.2.4. It follows that $\mathrm{Supp}(X) \subseteq \mathrm{Max}(X)$, because $\mathrm{Max}(X)$ is dense by Remark 3.2.10. Therefore $\mathrm{Supp}(X) = \mathrm{Max}(X)$ by Lemma 3.2.11. □

The interior and closure of a subset A of a space X are denoted by A° and A^-, respectively.

Proposition 3.2.14. *For any stably locally compact space X,*

$$\mathrm{Supp}(X) = \bigcap_{V \ll U} U \cup X \setminus (V^-)^\circ = \bigcap_{Q \subseteq U} U \cup X \setminus ({\downarrow} Q)^\circ,$$

where U and V range over ΩX, and Q ranges over the compact saturated subsets of X.

If X is a continuous poset endowed with the Scott topology then Q can taken as the upper set of a finite set.

Proof. In the following calculation, step (†) follows from the fact that for any locally compact space X, one has that $V \ll U$ in ΩX iff $V \subseteq Q \subseteq U$ for some compact saturated set $Q \subseteq X$ [15, pp. 40, 259]:

$$\begin{aligned}
\mathrm{Supp}(X) &= \{x \in X \mid \forall U \in \Omega X : x \in \underline{\neg\neg}(U) \Rightarrow x \in U\} \\
&\quad \text{by Theorem 3.1.15 and Lemma 3.1.2} \\
&= \Big\{x \in X \mid \forall U \in \Omega X : x \in \bigcup \{\neg\neg V \mid V \ll U\} \Rightarrow x \in U\Big\} \\
&= \Big\{x \in X \mid \forall U \in \Omega X : x \in \bigcup \{(V^-)^\circ \mid V \ll U\} \Rightarrow x \in U\Big\} \\
&\quad \text{because } \neg\neg V = (V^-)^\circ \\
&= \{x \in X \mid \forall U \in \Omega X : (\exists V \ll U : x \in (V^-)^\circ) \Rightarrow x \in U\} \\
&= \{x \in X \mid \forall U \in \Omega X : \forall V \ll U : x \in (V^-)^\circ \Rightarrow x \in U\} \\
&= \{x \in X \mid \forall U \in \Omega X : \forall Q \subseteq U : x \in (Q^-)^\circ \Rightarrow x \in U\} \qquad (\dagger) \\
&= \{x \in X \mid \forall U \in \Omega X : \forall Q \subseteq U : x \in ({\downarrow} Q)^\circ \Rightarrow x \in U\} \\
&\quad \text{by Lemma 3.2.12.} \qquad \square
\end{aligned}$$

It follows that the following conditions are equivalent:

(1) $x \notin \mathrm{Supp}(X)$.

(2) There are $V \ll U$ with $x \in (V^-)^\circ$ but $x \notin U$.

(3) There are $Q \subseteq U$ with $x \in ({\downarrow} Q)^\circ$ but $x \notin U$.

(4) There is $V \ll X \setminus {\downarrow} x$ with $x \in (V^-)^\circ$.

We now briefly consider some examples of supports from applications of domain theory to denotational semantics. Recall that continuous Scott domains endowed with the Scott topology are stably locally compact spaces [2].

(1) For any flat domain $A_\perp$ one has that $\mathrm{Supp}(A_\perp) = A$ iff A is finite.

(2) Let $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ be the domain of finite and infinite sequences over a set Σ, with the Scott topology induced by the prefix order. Then
   (a) If Σ is finite then $\mathrm{Supp}(\Sigma^\infty) = \Sigma^\omega$, because Σ^ω is compact.
   (b) If Σ is infinite then $\mathrm{Supp}(\Sigma^\infty) = \Sigma^\infty$, because there is no other compact set Q containing $\mathrm{Max}(\Sigma^\infty) = \Sigma^\omega$ such that $\uparrow d \cap Q$ is compact for every finite $d \in \Sigma^\infty$, as the only compact sets are the upper sets of finite sets.

(3) Let L be the domain of lazy natural numbers. Then $\mathrm{Supp}(L)$ is the one-point compactification of the discrete space of natural numbers, because this is the subspace of maximal points.

(4) Let $\mathcal{I}$ be the domain consisting of the closed intervals of the unit interval with the Scott topology induced by reverse inclusion [13,11]. Then $\mathrm{Supp}(\mathcal{I}) \cong [0, 1]$.

(5) Let $\mathcal{R}$ be the space consisting of the compact intervals of the real line with the Scott topology induced by reverse inclusion [34]. Then $\mathcal{R}$ fails to be a continuous Scott domain because it lacks a bottom element. The support of its lifting is given by $\mathrm{Supp}(\mathcal{R}_\perp) \cong \mathbb{R}_\perp$, where $\mathbb{R}_\perp$ is the Euclidean real line with a bottom element in its specialization order.

3.2.3. Finitary subspaces of injective spaces

The following concept is due to Smyth [37] (see also [32,43]). Let X be a T_0 space. The *upper power space* of X is the space $\mathcal{U}X$ whose points are the compact saturated sets of X and whose topology is generated by the base $\{\Box U \mid U \in \Omega X\}$, where

$$\Box U = \{Q \in \mathcal{U}X \mid Q \subseteq U\}.$$

Then $\mathcal{U}X$ is a T_0 space with specialization order given by

$$Q \leqslant Q' \quad \text{iff} \quad Q \supseteq Q'.$$

The upper power space construction preserves sobriety, compactness and local compactness [32, pp. 127, 132]. If X is locally compact and sober, then the specialization order of $\mathcal{U}X$ makes the set of points of $\mathcal{U}X$ into a continuous dcpo, and the topology of $\mathcal{U}X$ coincides with the Scott topology induced by the specialization order [32, pp. 132, 133]. If the empty compact set is omitted from the set of points of the upper power construction, the above definitions and facts are valid [32, p. 140]. This variant of the upper power space of X is denoted by $\mathcal{U}^+X$.

Lemma 3.2.15. *Let X be a T_0 space. The map $\eta_X : X \to \mathcal{U}X$ defined by*

$$\eta_X(x) = \uparrow x$$

is a finitary subspace embedding. Moreover, for all $U \in \Omega X$,

$$\forall_{\eta_X}(U) = \Box U.$$

The same results hold if the empty compact set is omitted from the set of points of $\mathcal{U}X$. *Moreover, in this case* $\eta_X : X \to \mathcal{U}^+X$ *is a* dense *subspace embedding.*

Proof. Let $U \in \Omega X$. Then $\eta_X^{-1}(\Box U) = \{x \in X \mid \uparrow x \subseteq U\} = U$. Hence η_X is an embedding. In order to conclude that $\forall_{\eta_X}(U) = \Box U$ it suffices to show that $U \mapsto \Box U$ is right adjoint to $\Omega\eta_X$. The above argument also shows that $\Omega\eta_X(\Box U) = U$ for all $U \in \Omega$. It remains to show that $V \subseteq \Box(\Omega\eta_X(V))$ for all $V \in \Omega\mathcal{U}X$. We have that

$$\Box\big(\Omega\eta_X(V)\big) = \Box\{x \in X \mid \uparrow x \in V\} = \big\{Q \in \mathcal{U}X \mid Q \subseteq \{x \in X \mid \uparrow x \in V\}\big\}.$$

Let $V \in \Omega\mathcal{U}X$ and $Q \in V$. Then $\uparrow q \in V$ for all $q \in Q$, because Q is saturated and hence $Q \leqslant \uparrow q$. Therefore $Q \in \Box(\Omega\eta_X(V))$, which establishes the adjunction. In order to see that η_X is finitary, let $\Delta \subseteq \Omega X$ be directed. Then

$$\begin{aligned}\Box \cup \Delta &= \Big\{Q \in \mathcal{U}X \mid Q \subseteq \bigcup \Delta\Big\}\\ &= \{Q \in \mathcal{U}X \mid \exists U \in \Delta\colon\ Q \subseteq U\} \qquad \text{by compactness of } Q\\ &= \bigcup_{U \in \Delta}\{Q \in \mathcal{U}X \mid Q \subseteq U\} = \bigcup_{U \in \Delta} \Box U.\end{aligned}$$

Finally, in order to conclude that $\eta_X : X \to \mathcal{U}^+X$ is dense, it suffices to show that $\forall_{\eta_X}(\emptyset) = \emptyset$. But this follows from the fact that $\forall_{\eta_X}(\emptyset) = \Box\emptyset = \emptyset$ (without omitting the empty compact set one would have $\Box\emptyset = \{\emptyset\}$). $\Box$

Theorem 3.2.16. *The finitary sober (dense) subspaces of the (densely) injective spaces are the stably locally compact spaces, up to homeomorphism.*

Proof. Every continuous lattice and every continuous Scott domain is stably locally compact and hence so is any sober subspace by Theorem 3.2.1. Conversely, for any stably locally compact space X, the map $\eta_X : X \to \mathcal{U}X$ is an embedding into a continuous lattice. In fact, $\mathcal{U}X$ is a always meet-semilattice under the specialization order, it is a continuous poset with Scott topology coinciding with the intrinsic topology by sobriety and local compactness, and it is a join-semilattice by stability. Therefore $\mathcal{U}X$ is a continuous lattice by directed completeness of the specialization order. Similarly, the map $\eta_X : X \to \mathcal{U}^+X$ is a dense embedding into a continuous Scott domain. $\Box$

Notice that any compact Hausdorff space X is homeomorphic to $\mathrm{Supp}\,\mathcal{U}^+X$. More generally, for any sober space X, $\mathrm{Supp}\,X$ is homeomorphic to $\mathrm{Supp}\,\mathcal{U}^+X$.

3.2.4. Finitary maps and upper power spaces

The upper power space constructor $\mathcal{U}$ defined in Theorem 3.2.3 becomes a poset-functor $\mathcal{U} : \mathrm{Sp}_0 \to \mathrm{Sp}_0$ if we define

$$\mathcal{U}f(Q) = \uparrow f(Q)$$

for all $f : X \to Y$ in Sp_0 [32, p. 127]. Continuity of $\mathcal{U}f$ follows from the fact that

$$(\mathcal{U}f)^{-1}(\Box V) = \Box f^{-1}(V),$$

which, by virtue of Lemma 3.2.15, can be expressed as

$$\Omega\mathcal{U}f \circ \forall_{\eta_X} = \forall_{\eta_Y} \circ \Omega f.$$

Moreover, with this definition, η becomes a natural transformation $\mathrm{Id} \to \mathcal{U}$.

Proposition 3.2.17. *Let $f : X \to Y$ be a continuous map of sober spaces.*
(1) *f is finitary iff it is a right $\mathcal{U}$-arrow.*
(2) *f is a finitary subspace embedding iff it is a right $\mathcal{U}$-embedding.*

Cf. Definition 3.1.16 and Proposition 3.1.17.

Proof. ($\Rightarrow$) Define $g : \mathcal{U}Y \to \mathcal{U}X$ by $g(Q) = f^{-1}(Q)$. Since f is finitary, g is a well-defined set-theoretical function by Theorem 3.2.3. It is also continuous, because for all $U \in \Omega X$,

$$\begin{aligned} g^{-1}(\Box U) &= \{Q \in \mathcal{U}X \mid g(Q) \in \Box U\} = \{Q \in \mathcal{U}X \mid f^{-1}(Q) \subseteq U\} \\ &= \{Q \in \mathcal{U}X \mid \forall q \in Q\colon\ f^{-1}(\uparrow q) \subseteq U\} && \text{because } Q \text{ is an upper set} \\ &= \{Q \in \mathcal{U}X \mid \forall q \in Q\colon\ q \in \forall_f(U)\} && \text{by Lemma 3.2.6} \\ &= \{Q \in \mathcal{U}X \mid Q \subseteq \forall_f(U)\} = \Box\forall_f(U). \end{aligned}$$

In order to show that $g \dashv \mathcal{U}f$ (reflectively if f is an embedding), we have to show that (i) $g(\mathcal{U}f(P)) \supseteq P$ for all $P \in \mathcal{U}X$ (equality holding if f is an embedding), and (ii) $\mathcal{U}f(g(Q)) \subseteq Q$ for all $Q \in \mathcal{U}Y$.

(i) Let $p \in P$. Then $f(p) \in \uparrow f(P)$, which means $p \in f^{-1}(\uparrow f(P)) = g(\mathcal{U}f(P))$. Conversely, assume that f is a subspace embedding and let $x \in g(\mathcal{U}f(P)) = f^{-1}(\uparrow f(P))$. This means that $f(x) \in \uparrow f(P)$. Hence there is some $p \in P$ such that $f(p) \leqslant f(x)$. It follows that $p \leqslant x$, because f is an order-embedding by Remark 3.2.7. Therefore $x \in P$ because P is saturated.

(ii) Let $q \in \mathcal{U}f(g(Q)) = \uparrow f(f^{-1}(Q))$. Then $y \leqslant q$ for some $y \in f(f^{-1}(Q))$. But $y = f(x)$ for some $x \in f^{-1}(Q)$, i.e., for some x with $f(x) \in Q$. Since $f(x) \leqslant q$ and Q is saturated, $q \in Q$.

($\Leftarrow$) (In this part—which is adapted and expanded from [42, Proposition 4.6]—we do not need the fact that X and Y are sober.) Let $g : \mathcal{U}Y \to \mathcal{U}X$ be a left adjoint of $\mathcal{U}f$ and define $G : \Omega X \to \Omega Y$ by $G = \Omega\eta_Y \circ \Omega g \circ \forall_{\eta_X}$. Then

$$\begin{aligned} &\Omega f \circ G \\ &\quad= \Omega f \circ \Omega\eta_Y \circ \Omega g \circ \forall_{\eta_X} \\ &\quad= \Omega\eta_X \circ \Omega\mathcal{U}f \circ \Omega g \circ \forall_{\eta_X} && \text{by naturality of } \eta \text{ and contravariance of } \Omega \\ &\quad\leqslant \Omega\eta_X \circ \forall_{\eta_X} && \text{because } g \circ \mathcal{U}f \leqslant \mathrm{id}_{\Omega\mathcal{U}X} \text{ and } \Omega \text{ is contravariant} \\ &\quad= \mathrm{id}_{\Omega X} && \text{by Lemma 3.2.15,} \end{aligned}$$

equality holding if f is a right $\mathcal{U}$-embedding, and

$$\begin{aligned} G \circ \Omega f &= \Omega\eta_Y \circ \Omega g \circ \forall_{\eta_X} \circ \Omega f \\ &= \Omega\eta_Y \circ \Omega g \circ \Omega\mathcal{U}f \circ \forall_{\eta_Y} \\ &\geqslant \Omega\eta_Y \circ \forall_{\eta_Y} \quad \text{because } \mathcal{U}f \circ g \geqslant \mathrm{id}_{\Omega Y} \text{ and } \Omega \text{ is contravariant} \\ &= \mathrm{id}_{\Omega Y} \qquad \text{by Lemma 3.2.15.} \end{aligned}$$

Hence $\Omega f \dashv G$, which shows that $G = \forall_f$. But G preserves directed joins because $\Omega\eta_Y$ and Ωg preserve all joins and η_X is finitary by Lemma 3.2.15. Therefore f is finitary. Now assume that f is a right $\mathcal{U}$-embedding. Then $\Omega f \circ \forall_f = \Omega f \circ G = \mathrm{id}_{\Omega X}$, which means that Ωf is surjective. Therefore f is a subspace embedding. $\square$

We say that a map is *dense* if its image is dense.

Proposition 3.2.18. *Let $f : X \to Y$ be a continuous map of sober spaces.*
(1) *f is a finitary* dense *map iff it is a right $\mathcal{U}^+$-arrow.*
(2) *f is a finitary* dense *embedding iff it is a right $\mathcal{U}^+$-embedding.*

Proof. ($\Rightarrow$) As in the proof of Proposition 3.2.17, noting that $f^{-1}(Q)$ is nonempty for every nonempty compact saturated Q. In fact, f is dense iff $\forall_f(\emptyset) = \emptyset$. But

$$\forall_f(\emptyset) = \{y \in Y \mid f^{-1}(\uparrow y) \subseteq \emptyset\}$$

by Lemma 3.2.6. Hence $f^{-1}(\uparrow y)$ is nonempty for every $y \in Y$.

($\Leftarrow$) As in the proof of Proposition 3.2.17, noting that $\forall_f = G = \Omega\eta_Y \circ \Omega g \circ \forall_{\eta_X}$, that $\forall_{\eta_X}(\emptyset) = \emptyset$ by Lemma 3.2.15, and that trivially $\Omega g(\emptyset) = \emptyset$ and $\Omega\eta_Y(\emptyset) = \emptyset$. $\square$

For each T_0 space X, define $\mu_X : \mathcal{U}\mathcal{U}X \to \mathcal{U}X$ by

$$\mu_X(\mathcal{Q}) = \bigcup \mathcal{Q}.$$

Then μ_X is well-defined, and it is continuous because

$$\mu_X^{-1}(\Box V) = \Box\Box V.$$

Moreover, $\mu : \mathcal{U}\mathcal{U} \to \mathcal{U}$ is a natural transformation and $\mathcal{U} = (\mathcal{U}, \eta, \mu)$ is a monad [27, p. 133] on Sp_0 [32, pp. 128, 129]. We refer to it as the *upper power space monad.* Similarly, we have an upper power space monad $\mathcal{U}^+ = (\mathcal{U}^+, \eta, \mu)$.

The following lemma shows that $\mathcal{U}$ and $\mathcal{U}^+$ are right KZ-monads in the sense of Definition 4.1.2 below:

Lemma 3.2.19. *For every T_0 space X, $\eta_{\mathcal{U}X} \leqslant \mathcal{U}\eta_X$ and $\eta_{\mathcal{U}^+X} \leqslant \mathcal{U}^+\eta_X$.*

Proof. Let $Q \in \mathcal{U}X$ and $P \in \mathcal{U}\eta_X(Q)$. This means that $P \subseteq \uparrow q$ for some $q \in Q$. Hence $P \subseteq Q$, because Q is saturated. This means that $P \in \eta_{\mathcal{U}X}(Q)$. Therefore $\eta_{\mathcal{U}X}(Q) \supseteq \mathcal{U}\eta_X(Q)$. The second inequality is proved in the same way. $\square$

4. Injective objects which are the algebras of KZ-monads

4.1. KZ-monads in poset-enriched categories

We now specialize Kock's notion of KZ-doctrine in a 2-category [24] obtaining the notion of KZ-monad in a poset-enriched category (cf. [24, Theorem 5.4]). We first summarize the (poset-duals of the) main results of Kock's paper specialized to poset-enriched categories:

Lemma 4.1.1 (Kock). *Let $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ be a monad in a poset-enriched category $\mathcal{X}$, and assume that $\mathcal{T}$ is a poset-functor. Then the following conditions are equivalent:*

(KZ$_0$) $\eta_{\mathcal{T}X} \leqslant \mathcal{T}\eta_X$ *for all* $X \in \mathcal{X}$.

(KZ$_1$) *For all $X \in \mathcal{X}$, an arrow $\alpha : \mathcal{T}X \to X$ is a structure map iff $\eta_X \dashv \alpha$ is a coreflective adjunction (i.e., an adjunction with $\alpha \circ \eta_X = \mathrm{id}_X$).*

(KZ$_2$) $\eta_{\mathcal{T}X} \dashv \mu_X$ *for all* $X \in \mathcal{X}$.

(KZ$_3$) $\mu_X \dashv \mathcal{T}\eta_X$ *for all* $X \in \mathcal{X}$.

Proof. (KZ$_0$) $\Rightarrow$ (KZ$_1$) $\Rightarrow$ (KZ$_2$) We show that (KZ$_0$) $\Rightarrow$ (KZ$_1$)($\Rightarrow$) $\Rightarrow$ (KZ$_2$) $\Rightarrow$ (KZ$_1$) ($\Leftarrow$).

(KZ$_0$) $\Rightarrow$ (KZ$_1$) ($\Rightarrow$) The unit law for structure maps says that $\alpha \circ \eta_X = \mathrm{id}_X$. But also

$$\begin{aligned}
\eta_X \circ \alpha &= \mathcal{T}\alpha \circ \eta_{\mathcal{T}X} && \text{by naturality of } \eta\\
&\leqslant \mathcal{T}\alpha \circ \mathcal{T}\eta_X && \text{by (KZ}_0)\\
&= \mathcal{T}(\alpha \circ \eta_X)\\
&= \mathcal{T}\mathrm{id}_X = \mathrm{id}_{\mathcal{T}X} && \text{again by the unit law.}
\end{aligned}$$

Therefore $\eta_X \dashv \alpha$ coreflectively.

(KZ$_1$)($\Rightarrow$) $\Rightarrow$ (KZ$_2$) We have that $\eta_{\mathcal{T}X} \dashv \mu_X$ because μ_X is a structure map for $\mathcal{T}X$.

(KZ$_2$) $\Rightarrow$ (KZ$_1$)($\Leftarrow$) By coreflectiveness we obtain the unit law $\alpha \circ \eta_X = \mathrm{id}_X$ for structure maps. By combining the adjunction $\eta_X \dashv \alpha$ of the hypothesis of (KZ$_1$)($\Leftarrow$) with the adjunction $\eta_{\mathcal{T}X} \dashv \mu_X$ of the assumption (KZ$_2$), we get $\eta_{\mathcal{T}X} \circ \eta_X \dashv \alpha \circ \mu_X$. But we also have that $\mathcal{T}\eta_X \circ \eta_X \dashv \alpha \circ \mathcal{T}\alpha$, because

$$\begin{aligned}
&\alpha \circ \mathcal{T}\alpha \circ \mathcal{T}\eta_X \circ \eta_X\\
&\quad = \alpha \circ \mathcal{T}(\alpha \circ \eta_X) \circ \eta_X\\
&\quad = \alpha \circ \eta_X \quad \text{because } \eta_X \dashv \alpha \text{ is coreflective and hence } \alpha \circ \eta_X = \mathrm{id}_{\mathcal{T}X}\\
&\quad = \mathrm{id}_X,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{T}\eta_X \circ \eta_X \circ \alpha \circ \mathcal{T}\alpha\\
&\quad \leqslant \mathcal{T}\eta_X \circ \mathrm{id}_{\mathcal{T}X} \circ \mathcal{T}\alpha && \text{because } \eta_X \dashv \alpha \text{ and hence } \eta_X \circ \alpha \leqslant \mathrm{id}_{\mathcal{T}X}\\
&\quad = \mathcal{T}(\eta_X \circ \alpha)\\
&\quad \leqslant \mathcal{T}\mathrm{id}_{\mathcal{T}X} && \text{because } \eta_X \dashv \alpha = \mathrm{id}_{\mathcal{T}\mathcal{T}X}.
\end{aligned}$$

But $\eta_{\mathcal{T}X} \circ \eta_X = \mathcal{T}\eta_X \circ \eta_X$ by naturality of η. Hence $\alpha \circ \mathcal{T}\alpha = \alpha \circ \mu_X$, because each side of the equation has the same left adjoint. This establishes the associativity law. Therefore α is a structure map.

(KZ_2) $\Rightarrow$ (KZ_0) By assumption, $\eta_{\mathcal{T}X} \circ \mu_X \leqslant \mathrm{id}_{\mathcal{T}\mathcal{T}X}$. Hence $\eta_{\mathcal{T}X} \circ \mu_X \circ \mathcal{T}\eta_X \leqslant \mathcal{T}\eta_X$. Therefore $\eta_{\mathcal{T}X} \leqslant \mathcal{T}\eta_X$ by the unit laws.

(KZ_0) $\Rightarrow$ (KZ_3) One of the unit laws is $\mu_X \circ \mathcal{T}\eta_X = \mathrm{id}_{\mathcal{T}X}$. But also

$$\begin{aligned}
\mathcal{T}\eta_X \circ \mu_X &= \mu_{\mathcal{T}X} \circ \mathcal{T}\mathcal{T}\eta_X && \text{by naturality of } \mu \\
&\geqslant \mu_{\mathcal{T}X} \circ \mathcal{T}\eta_{\mathcal{T}X} && \text{by } (\mathrm{KZ}_0) \\
&\geqslant \mu_{\mathcal{T}X} \circ \eta_{\mathcal{T}\mathcal{T}X} && \text{by } (\mathrm{KZ}_0) \text{ again} \\
&= \mathrm{id}_{\mathcal{T}\mathcal{T}X} && \text{by the other unit law.}
\end{aligned}$$

Therefore $\mu_X \dashv \mathcal{T}\eta_X$.

(KZ_3) $\Rightarrow$ (KZ_0) By assumption, $\mathcal{T}\eta_X \circ \mu_X \geqslant \mathrm{id}_{\mathcal{T}\mathcal{T}X}$. Hence $\mathcal{T}\eta_X \circ \mu_X \circ \eta_{\mathcal{T}X} \geqslant \eta_{\mathcal{T}X}$. Therefore $\mathcal{T}\eta_X \geqslant \eta_{\mathcal{T}X}$ by the unit laws. □

Definition 4.1.2. Let $\mathcal{X}$ be a poset-enriched category. A *right KZ-monad* in $\mathcal{X}$ is a monad $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ in $\mathcal{X}$ with $\mathcal{T}$ a poset-functor, subject to the equivalent conditions of Lemma 4.1.1. Left KZ-monads are defined poset-dually, by reversing the inequalities and the adjunctions between arrows.

By Lemma 3.2.19, the upper power space monads $\mathcal{U}$ and $\mathcal{U}^+$ are right KZ.

Notation 4.1.3. By (KZ_1), every object has at most one structure map. The unique structure map of a $\mathcal{T}$-algebra A of a KZ-monad $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ is denoted by m_A.

4.2. Injective objects over right $\mathcal{T}$-embeddings

Let $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ be a right KZ-monad in a poset-enriched category $\mathcal{X}$.

Remark 4.2.1. $\eta_X : X \to \mathcal{T}X$ is a right $\mathcal{T}$-embedding with $\eta_X^{-1} = \mu_X$ (cf. Definition 3.1.16).

Proof. Axiom (KZ_3) says that $\mu_X \dashv \mathcal{T}\eta_X$. This adjunction is reflective by virtue of the unit law $\mu_X \circ \mathcal{T}\eta_X = \mathrm{id}_{\mathcal{T}X}$. □

Theorem 4.2.2. *The following statements are equivalent for any object $A \in \mathcal{X}$:*

(1) *A is right injective over right $\mathcal{T}$-embeddings.*

(2) *A is injective over right $\mathcal{T}$-embeddings.*

(3) *A is a $\mathcal{T}$-algebra.*

These conditions imply

(4) *A is a right Kan object over right $\mathcal{T}$-arrows.*

Moreover, assuming that the equivalent conditions (1)–(3) *hold, if $j : X \to Y$ is a right $\mathcal{T}$-arrow and $f : X \to A$ is any arrow, then*

$$f/j = m_A \circ \mathcal{T}f \circ j^{-1} \circ \eta_Y.$$

The construction of f/j is illustrated in the following diagrams:

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "f/j"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}X \arrow[dr, "\mathcal{T}f"'] & & \mathcal{T}Y \arrow[ll, "j^{-1}"'] & & Y \arrow[ll, "\eta_Y"'] \arrow[ddll, dotted, "f/j"] \\ & \mathcal{T}A \arrow[dr, "m_A"'] & & & \\ & & A & & \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) By Remark 4.2.1, η_A is a $\mathcal{T}$-embedding. Hence there is an extension $m: \mathcal{T}A \to A$ of the identity of A along η_A. This means that $m \circ \eta_A = \mathrm{id}_A$. Hence

$$\begin{aligned} \eta_A \circ m &= \mathcal{T}m \circ \eta_{\mathcal{T}A} && \text{by naturality of } \eta \\ &\leqslant \mathcal{T}m \circ \mathcal{T}\eta_A && \text{by axiom } \mathrm{KZ}_0 \\ &= \mathcal{T}(m \circ \eta_A) = \mathcal{T}\mathrm{id}_A = \mathrm{id}_{\mathcal{T}A}. \end{aligned}$$

Thus $\eta_A \dashv m$ is a coreflective adjunction. By axiom (KZ_1), m is a structure map and therefore A is an algebra.

(3) $\Rightarrow$ (1) and (4) Let $f/j = m_A \circ \mathcal{T}f \circ j^{-1} \circ \eta_Y$. Then

$$\begin{aligned} (f/j) \circ j &= m_A \circ \mathcal{T}f \circ j^{-1} \circ \eta_Y \circ j \\ &= m_A \circ \mathcal{T}f \circ j^{-1} \circ \mathcal{T}j \circ \eta_X && \text{by naturality of } \eta \\ &\leqslant m_A \circ \mathcal{T}f \circ \eta_X && \text{because } j^{-1} \circ \mathcal{T}j \leqslant \mathrm{id}_{\mathcal{T}X} \\ &= m_A \circ \eta_A \circ f && \text{by naturality of } \eta \\ &= \mathrm{id}_A \circ f = f && \text{by the unit law for structure maps,} \end{aligned}$$

equality holding if $j^{-1} \circ \mathcal{T}j = \mathrm{id}_{\mathcal{T}X}$ iff j is a $\mathcal{T}$-embedding. If $g \circ j \leqslant f$ for $g: Y \to A$ then

$$\begin{aligned} f/j &= m_A \circ \mathcal{T}f \circ j^{-1} \circ \eta_Y && \text{by definition of } f/j \\ &\geqslant m_A \circ \mathcal{T}(g \circ j) \circ j^{-1} \circ \eta_Y && \text{because } g \circ j \leqslant f \\ &= m_A \circ \mathcal{T}g \circ \mathcal{T}j \circ j^{-1} \circ \eta_Y \\ &\geqslant m_A \circ \mathcal{T}g \circ \eta_Y && \text{because } \mathcal{T}j \circ j^{-1} \geqslant \mathrm{id}_{\mathcal{T}Y} \\ &= m_A \circ \eta_A \circ g && \text{by naturality of } \eta \\ &= \mathrm{id}_A \circ g = g && \text{by the unit law for structure maps.} \end{aligned}$$

This shows that f/j is the right Kan extension of f along j, being an actual extension if j is a $\mathcal{T}$-embedding. Therefore A is a right Kan object over $\mathcal{T}$-arrows and a right injective object over $\mathcal{T}$-embeddings. □

Corollary 4.2.3. *The following are equivalent for any object $X \in \mathcal{X}$:*

(1) *X is a $\mathcal{T}$-algebra.*

(2) *X is a retract of a free $\mathcal{T}$-algebra.*
(3) *X is a retract of a $\mathcal{T}$-algebra.*

Proof. (1) $\Rightarrow$ (2) The unit law for structure maps says that $m_X \circ \eta_X = \mathrm{id}_X$. This means that X is a retract of $\mathcal{T}X$.

(2) $\Rightarrow$ (3) Immediate.

(3) $\Rightarrow$ (1) Since $\mathcal{T}$-algebras are injective over $\mathcal{T}$-embeddings by Theorem 4.2.2, X is injective over the same maps by Lemma 2.4.3, and hence it is a $\mathcal{T}$-algebra by Theorem 4.2.2. $\square$

4.3. Right $\mathcal{T}$-arrows between $\mathcal{T}$-algebras

Let $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ be a right KZ-monad in a poset-enriched category $\mathcal{X}$.

Proposition 4.3.1 (Kock). *Let A and B be $\mathcal{T}$-algebras. If an arrow $f: A \to B$ is a right adjoint then it is a $\mathcal{T}$-algebra homomorphism.*

Proof. Let $g: B \to A$ with $g \dashv f$. Since $\eta_A \dashv m_A$ by (KZ$_2$), we have that $\eta_A \circ g \dashv f \circ m_A$. Since $\eta_B \dashv m_B$ and $\mathcal{T}g \dashv \mathcal{T}f$, we have that $\mathcal{T}g \circ \eta_B \dashv m_B \circ \mathcal{T}f$. But $\mathcal{T}g \circ \eta_B = \eta_A \circ g$ by naturality of η. Hence $f \circ m_A = m_B \circ \mathcal{T}f$ by uniqueness of right adjoints. This means that f is a $\mathcal{T}$-algebra homomorphism. $\square$

Proposition 4.3.2. *Let A and B be $\mathcal{T}$-algebras. If for an arrow $f: A \to B$ the arrow $\mathcal{T}f$ has a (reflective) left adjoint, so does f.*

Proof. Given $f^{-1}: \mathcal{T}B \to \mathcal{T}A$ with $f^{-1} \dashv \mathcal{T}f$, define $g: B \to A$ by $g = m_A \circ f^{-1} \circ \eta_B$. Then

$$\begin{aligned} g \circ f &= m_A \circ f^{-1} \circ \eta_B \circ f \\ &= m_A \circ f^{-1} \circ \mathcal{T}f \circ \eta_A && \text{by naturality of } \eta \\ &\leqslant m_A \circ \mathrm{id}_{\mathcal{T}A} \circ \eta_A && \text{because } f^{-1} \dashv \mathcal{T}f \text{ and hence } f^{-1} \circ \mathcal{T}f \leqslant \mathrm{id}_{\mathcal{T}A} \\ &= m_A \circ \eta_A = \mathrm{id}_A && \text{by the unit law for algebras,} \end{aligned}$$

equality holding if $f^{-1} \dashv \mathcal{T}f$ is reflective, and

$$\begin{aligned} f \circ g &= f \circ m_A \circ f^{-1} \circ \eta_B \\ &= m_B \circ \mathcal{T}f \circ f^{-1} \circ \eta_B && \text{because } f \text{ is a } \mathcal{T}\text{-algebra homomorphism} \\ &\geqslant m_B \circ \mathrm{id}_{\mathcal{T}B} \circ \eta_B && \text{because } f^{-1} \dashv \mathcal{T}f \text{ and hence } \mathcal{T}f \circ f^{-1} \geqslant \mathrm{id}_{\mathcal{T}B} \\ &= m_B \circ \eta_B = \mathrm{id}_B && \text{by the unit law for algebras.} \quad \square \end{aligned}$$

Corollary 4.3.3. *A map $f: A \to B$ is a right $\mathcal{T}$-arrow iff it has a left adjoint. Moreover, in this case it is a $\mathcal{T}$-algebra homomorphism.*

But notice that there is no reason why a $\mathcal{T}$-algebra homomorphism should be a right adjoint. The following proposition generalizes Theorem 2.3.3(4) from right adjoints to $\mathcal{T}$-algebra homomorphisms:

Proposition 4.3.4. *Let $j : X \to Y$ be a right $\mathcal{T}$-arrow, $h : A \to B$ be a $\mathcal{T}$-algebra homomorphism, and $f : X \to A$ be any arrow. Then*

$$h \circ (f/j) = (h \circ f)/j.$$

Proof. This follows from the routine calculation

$$\begin{aligned} h \circ (f/j) &= h \circ m_A \circ \mathcal{T}f \circ j^{-1} \circ \eta_Y && \text{by Theorem 4.2.2} \\ &= m_B \circ \mathcal{T}h \circ \mathcal{T}f \circ j^{-1} \circ \eta_Y && \text{by the homomorphism law} \\ &= m_B \circ \mathcal{T}(h \circ f) \circ j^{-1} \circ \eta_Y && \\ &= (h \circ f)/j && \text{by Theorem 4.2.2.} \qquad \square \end{aligned}$$

Let $\mathcal{T}$-right denote the lluf subcategory of right $\mathcal{T}$-arrows and $\mathcal{T}$-alg denote the category of $\mathcal{T}$-algebras and $\mathcal{T}$-homomorphisms.

Corollary 4.3.5. *The equations*

$$\mathrm{Ran}(X, A) = \mathrm{hom}(X, A),$$

$$\mathrm{Ran}(j : X \to Y,\ h : A \to B)(f : X \to A) = h \circ f/j : Y \to B$$

define a functor $\mathrm{Ran} : \mathcal{T}\text{-right} \times \mathcal{T}\text{-alg} \to \mathrm{Poset}$.

4.4. Injective locales over finitary and semi-open embeddings

Theorem 4.4.1. *The* (*right*) *injective locales over finitary sublocale embeddings are the algebras of the upper power locale monad.*

Proof. This follows from Proposition 3.1.17 and Theorem 4.2.2, using the fact that the upper power locale monad is right KZ [42]. $\square$

For a definition of the lower power locale monad $\mathcal{L}$ see [32] or [42] (see also [43]).

Proposition 4.4.2 (Vickers). *Let $f : X \to Y$ be a continuous map of locales.*

(1) *f is a semi-open map iff it is a left $\mathcal{L}$-arrow.*

(2) *f is a semi-open sublocale embedding iff it is a left $\mathcal{L}$-embedding.*

Proof. For item (1) see [42, Proposition 4.6]. Item (2) is not stated in [42], but it immediately follows from the construction given in the proof. $\square$

Theorem 4.4.3. *The* (*left*) *injective locales over semi-open sublocale embeddings are the algebras of the lower power locale monad.*

Proof. Immediate consequence of Proposition 4.4.2 and Theorem 4.2.2, using the fact that the lower power locale monad is left KZ [42]. □

For concrete characterizations of the algebras of the lower and upper power locale monads see [32]. It is plausible that Theorem 4.4.3 also holds for injective *spaces* over semi-open embeddings, but we do not pause to check whether this is the case.

4.5. Finitarily injective spaces

We say that a space is *finitarily injective* if it is injective over finitary subspace embeddings.

Lemma 4.5.1. *The finitarily injective T_0 spaces are sober. Moreover, the finitarily injective spaces in the category of T_0 spaces coincide with the finitarily injective spaces in the category of sober spaces.*

Proof. Let $\mathcal{S} : \mathrm{Sp}_0 \to \mathrm{Sob}$ denote the sobrification functor and $s_X : X \to \mathcal{S}X$ denote the natural embedding of a T_0 space X into its sobrification. If D is finitarily injective in Sp_0 then $s_D : D \to \mathcal{S}D$ is a finitary embedding, because D is T_0 and Ωs_D is a frame isomorphism. Hence D is a retract of $\mathcal{S}D$ by Lemma 2.4.4. But retracts of sober spaces are sober [32, p. 23]. Hence D is sober. Therefore D is finitarily injective in Sob, because there are fewer finitary embeddings in Sob than in Sp_0. The converse follows from the symmetric version of Lemma 2.4.7, because $\mathcal{S}$ is left adjoint to the inclusion functor $\mathrm{Sob} \to \mathrm{Sp}_0$ [20, p. 44] and $\mathcal{S}$ clearly preserves finitary embeddings. □

Proposition 4.5.2 (Schalk). *A T_0 space X is a $\mathcal{U}$-algebra iff it has meets of compact sets and $\bigwedge : \mathcal{U}X \to X$ is a continuous map, and in this case $\bigwedge : \mathcal{U}X \to X$ is the structure map of X. A continuous function between $\mathcal{U}$-algebras is a $\mathcal{U}$-algebra homomorphism iff it preserves meets of compact sets.*

Proof. See [32, pp. 130, 140]. □

Theorem 4.5.3. *The following statements are equivalent for any T_0 space D:*

(1) *D is right injective over finitary embeddings.*

(2) *D is injective over finitary embeddings.*

(3) *D is a sober $\mathcal{U}$-algebra.*

(4) *D is a retract of the upper power space of a sober space.*

These conditions imply

(5) *D is a right Kan space over arbitrary finitary maps.*

Moreover, assuming that the equivalent conditions (1)–(4) *hold, if $j : X \to Y$ is a finitary map of sober spaces and $f : X \to D$ is any continuous map then*

$$f/j(y) = \bigwedge f\left(j^{-1}(\uparrow y)\right).$$

Proof. By Lemma 4.5.1, we can consider the upper power space monad restricted to the category of sober spaces and apply the characterization of finitary maps and embeddings established in Proposition 3.2.17 to obtain a characterization of the finitarily injective spaces via Theorem 4.2.2 and Corollary 4.2.3. For condition (5) we observe that

$$\begin{aligned} f/j(y) &= m_D \circ \mathcal{U}f \circ j^{-1} \circ \eta_Y(y) && \text{by Theorem 4.2.2} \\ &= \bigwedge \uparrow f(j^{-1}(\uparrow y)) && \text{by Propositions 4.5.2 and 3.2.17, and definition of } \eta_X \\ &= \bigwedge f(j^{-1}(\uparrow y)). \quad \square \end{aligned}$$

By Proposition 2.3.1, this means that the greatest *monotone* extension is continuous:

Corollary 4.5.4. *The specialization-order functor* $\boldsymbol{U} : \mathrm{Sp}_0 \to$ Poset *preserves right Kan extensions of maps* $f : X \to D$ *with values on finitarily injective spaces along finitary maps* $j : X \to Y$*, in the sense that* $\boldsymbol{U}(f/j) = \boldsymbol{U}f/\boldsymbol{U}j$.

Recall that if X is a locally compact sober space, so is $\mathcal{U}X$.

Proposition 4.5.5 (Schalk). *The algebras of the upper power space monad restricted to sober locally compact spaces are the continuous meet-semilattices with unit (endowed with the Scott topology). The homomorphisms are the continuous maps which preserve finite meets.*

Proof. See [32, p. 133]. $\square$

Theorem 4.5.6. *The (right) injective spaces over finitary embeddings in the category of locally compact sober spaces are the continuous meet-semilattices with unit. Moreover, if* D *is such an injective space and* $j : X \to Y$ *is a finitary embedding of locally compact sober spaces then*

$$f \mapsto f/j : [X \to D] \to [Y \to D]$$

is a subspace embedding.

Proof. We know that $f/j = m_D \circ \mathcal{U}f \circ j^{-1} \circ \eta_Y$. The maps m_D, j^{-1} and η_Y have already been shown to be continuous. Also, $\mathcal{U}$ is locally continuous [2] and the composition operation is Scott continuous. Hence the map $f \mapsto f/j$ is continuous. It follows that it is a subspace embedding, because it has $g \mapsto g \circ j$ as a retraction (in fact, as a reflective left adjoint). $\square$

Corollary 4.5.7. *If* $j : X \to D$ *and* $k : Y \to E$ *are embeddings of locally compact sober spaces with* D *and* E *finitarily injective and* j *finitary, then the map*

$$f \mapsto (k \circ f)/j : [X \to Y] \to [D \to E]$$

is a subspace embedding.

Theorem 4.5.8. *The full subcategory of finitarily injective spaces in the category of locally compact spaces is Cartesian closed.*

Proof. Since locally compact spaces are exponentiable [26, p. 149], the result follows from Lemmas 2.4.8 and 3.2.9. □

Theorem 4.5.9. *The (right) injective spaces over finitary embeddings in the category of stably locally compact spaces are the continuous lattices.*

Proof. Let X be stably locally compact. Then $\mathcal{U}X$ is a continuous lattice as it was shown in the proof of Theorem 3.2.16. Hence the result follows from the fact that the class of continuous lattices is closed under the formation of retracts. □

This result completes a circle. The injective spaces over subspace embeddings are the continuous lattices (endowed with the Scott topology). However, we have seen that for a nontrivial continuous lattice D and subspace embedding $j: X \to Y$, the right Kan extension map $\mathrm{Ran}_j^D : [X \to D] \to [Y \to D]$ is a subspace embedding iff $j: X \to Y$ is a finitary map. Then two natural questions have arisen: (1) What are the finitary subspaces of the continuous lattices? (2) Given that the finitary subspace embeddings are well-behaved, what are the finitarily injective spaces? The answer to (1) is that the finitary subspaces of the continuous lattices are the T_0 spaces whose frames of open sets are stably continuous. In particular, the finitary *sober* subspaces of the continuous lattices are the stably locally compact spaces, and the finitary T_1 subspaces of the continuous lattices are the compact Hausdorff spaces. The answer to (2) is that the finitarily injective spaces are the algebras of the upper power space monad in the category of sober spaces. Since there are fewer finitary embeddings than arbitrary embeddings, there are more finitarily injective spaces than injective spaces. In particular, the finitarily injective spaces in the full subcategory of locally compact sober spaces are the continuous meet-semilattices with unit. Now, given (1) and (2), we are led to ask (3): What are the finitarily injective spaces in the full subcategory of stably locally compact spaces? Perhaps surprisingly, and quite satisfyingly, the answer to (3) is that the finitarily injectives in this subcategory are again the continuous lattices.

Similar results are obtained for injectivity over finitary dense embeddings via Proposition 3.2.18 and another version of Proposition 4.5.5 [32, p. 140]. We state some of them:

Theorem 4.5.10. *The (right) injective spaces over finitary* dense *embeddings in the category of locally compact sober spaces are the continuous meet-semilattices without unit. Moreover, if D is such an injective space and $j: X \to Y$ is a finitary dense embedding of locally compact sober spaces then*

$$f \mapsto f/j : [X \to D] \to [Y \to D]$$

is a subspace embedding.

Theorem 4.5.11. *The (right) injective spaces over finitary dense embeddings in the category of stably locally compact spaces are the continuous Scott domains.*

And a similar circle is completed, whose analogous discussion need not be included. But another discussion is in order. In practice, one often works with embeddings of spaces onto maximal points of continuous Scott domains. Such embeddings are necessarily dense embeddings of Hausdorff spaces. Since the finitary subspaces of the continuous Scott domains are those whose frames of open sets are stably continuous, such embeddings are finitary for, and only for, compact Hausdorff spaces. This has connections with the idea of support, for the support of a continuous Scott domain always contains the subspace of maximal points, and it is the subspace of maximal points iff that subspace is compact. If $j : X \to D$ is an infinitary embedding of a Hausdorff space X onto the subspace of maximal points of D, then the support of D is not T_1 and it is strictly larger than X. For example, for X the Euclidean real line and D the interval domain, the support of D is the real line with a bottom element in the specialization order.

A general result by Jimmie Lawson [25] implies that the subspaces of maximal points of the continuous Scott domains are Polish spaces (spaces whose topology is induced by a complete metric). It follows at once that the subspaces of maximal points of the stably locally compact spaces are also Polish spaces. However, it does not follow that all Polish spaces arise in this way.

4.6. Added in proof

Alan Day [6] showed that the algebras of the filter monad on Sp_0 are the continuous lattices endowed with the Scott topology, and that the algebra homomorphisms are the meet-preserving Scott continuous maps. It turns out that the filter monad is right KZ, as it can be routinely checked. Moreover, all continuous maps are right $\mathcal{T}$-arrows, and the right $\mathcal{T}$-embeddings are the subspace embeddings. From this and Theorem 4.2.2 we obtain an alternative proof of the fact that the injective spaces over subspace embeddings are the continuous lattices, and also of Proposition 2.5.4, which states that every injective space over subspace embeddings is a right Kan space over arbitrary continuous maps. In fact, via an application of [6, Theorem 4.3] [3], which characterizes the structure maps of the algebras, our Theorem 4.2.2 produces Scott's extension formula discussed in Section 2.6.

In unpublished joint work with Bob Flagg, we have established other injectivity results in topology via the general injectivity result for KZ-monads. These include: the injective spaces over flat embeddings are the stably locally compact spaces (similarly for locales), and the injective spaces over locally dense embeddings are the L-domains endowed with the Scott topology.

[3] Notice that the statement of the cited theorem has a misprint. See item (7) of the proof for the correct formulation.

Acknowledgements

I benefited from many long discussions with Steve Vickers. Samson Abramsky pointed out that my right extensions are particular cases of right Kan extensions. Bob Flagg, Till Plewe and Philipp Sünderhauf let me know of some imprecisions in preliminary versions of this paper. I also had interesting discussions with Abbas Edalat, Achim Jung, Klaus Keimel and Mike Smyth.

The author was supported by the Brazilian agency CNPq and an ARC project "A Computational Approach to Measure and Integration Theory". The diagrams were drawn with Paul Taylor's `diagrams` package. Many mathematical symbols were produced with Jeremy Gibbons and Alan Jefrey's `stmaryrd` symbols package.

References

[1] S. Abramsky, Abstract interpretation, logical relations and Kan extensions, J. Logic Comput. 1 (1) (1990) 5–40.

[2] S. Abramsky and A. Jung, Domain theory, in: S. Abramsky, D.M. Gabbay and T.S.E. Maibaum, eds., Handbook of Logic in Computer Science, Vol. 3 (Clarendon Press, Oxford, 1994) 1–168.

[3] B. Banaschewski, Another look at the localic Tychonoff Theorem, Comm. Math. Univ. Carolin. 29 (4) (1988) 647–655.

[4] N. Bourbaki, General Topology, Vol. 1 (Addison-Wesley, London, 1966).

[5] B.A. Davey and H.A. Priestley, Introduction to Lattices and Order (Cambridge University Press, Cambridge, 1990).

[6] A. Day, Filter monads, continuous lattices and closure systems, Canad. J. Math. XXVII (1) (1975) 50–59.

[7] E.J. Dubuc, Kan Extensions in Enriched Category Theory, Lecture Notes in Math. 145 (Springer, Berlin, 1970).

[8] A. Edalat, Domain theory and integration, Theoret. Comput. Sci. 151 (1995) 163–193.

[9] A. Edalat, Dynamical systems, measures and fractals via domain theory, Inform. and Comput. 120 (1) (1995) 32–48.

[10] A. Edalat, The Scott topology induces the weak topology, Math. Structures Comput. Sci. (1996).

[11] A. Edalat and M.H. Escardó, Integration in Real PCF (extended abstract), in: Proceedings Eleventh Annual IEEE Symposium on Logic in Computer Science, New Brunswick, NJ (1996) 382–393.

[12] A. Edalat and R. Heckmann, A computational model for metric spaces, Theoret. Comput. Sci. 193 (1998) 53–73.

[13] M.H. Escardó, PCF extended with real numbers, Theoret. Comput. Sci. 162 (1) (1996) 79–115.

[14] P. Di Gianantonio, Real number computability and domain theory, Inform. and Comput. 127 (1) (1996) 11–25.

[15] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, A Compendium of Continuous Lattices (Springer, New York, 1980).

[16] C.A. Gunter, Semantics of Programming Languages—Structures and Techniques (The MIT Press, London, 1992).

[17] K.H. Hofmann, Stably continuous frames, and their topological manifestations, in: Categorical Topology, Proc. Toledo, Ohio, 1983 (Heldermann, 1984) 282–307.

[18] K.H. Hofmann and J.D. Lawson, On the order theoretical foundation of a theory of quasicompactly generated spaces without separation axiom, in: R.-E. Hoffmann, ed., Continuous Lattices and Related Topics, Mathematik-Arbeitspapiere 27, University of Bremen (1982) 143–160.
[19] K.H. Hofmann and M. Mislove, Local compactness and continuous lattices, in: B. Banaschewski and R.-E. Hoffmann, eds., Continuous Lattices, Lecture Notes in Math. 871 (Springer, Berlin, 1981) 209–248.
[20] P.T. Johnstone, Stone Spaces (Cambridge University Press, Cambridge, 1982).
[21] A. Jung, Cartesian Closed Categories of Domains, CWI Tract 66 (Centre for Mathematics and Computer Science, Amsterdam, The Netherlands, 1989).
[22] A. Jung, The classification of continuous domains, in: Logic in Computer Science (IEEE Computer Society Press, 1990) 35–40.
[23] K. Keimel and G. Gierz, Halbstetige Funktionen und stetige Verbände, in: R.-E. Hoffmann, ed., Continuous Lattices and Related Topics, Mathematik-Arbeitspapiere 27, Universität Bremen (1982) 59–67.
[24] A. Kock, Monads for which structures are adjoint to units (version 3), J. Pure Appl. Algebra 104 (1995) 41–59.
[25] J. Lawson, Spaces of maximal points, Math. Structures Comput. Sci. 7 (1997) 543–555.
[26] J.D. Lawson, The versatile continuous order, in: M. Main, A. Melton, M. Mislove and D. Schimidt, eds., Mathematical Foundations of Programming Languages, Lecture Notes in Comput. Sci. 298 (Springer, Berlin, 1987) 134–160.
[27] S. MacLane, Categories for the Working Mathematician (Springer, New York, 1971).
[28] S. MacLane and I. Moerdijk, Shieves in Geometry and Logic: A First Introduction to Topos Theory (Springer, New York, 1st ed, 1992).
[29] G. Plotkin, LCF considered as a programming language, Theoret. Comput. Sci. 5 (1) (1977) 223–255.
[30] G. Plotkin, $\mathcal{T}^{\omega}$ as a universal domain, J. Comput. System Sci. 17 (1978) 209–236.
[31] G. Plotkin, Domains, Post-graduate lectures in advanced domain theory, University of Edinburgh, Department of Computer Science (1980). http://ida.dcs.qmw.ac.uk/sites/other/domain.notes.other.
[32] A. Schalk, Algebras for generalized power constructions, PhD Thesis, Technische Hochschule Darmstadt (July 1993). ftp://ftp.cl.cam.ac.uk/papers/as213/diss.dvi.gz.
[33] D.S. Scott, Continuous lattices, in: F.W. Lawvere, ed., Toposes, Algebraic Geometry and Logic, Lectures Notes in Math. 274 (Springer, Berlin, 1972) 97–136.
[34] D.S. Scott, Lattice theory, data types and semantics, in: Formal Semantics of Programming Languages (Prentice-Hall, Englewood Cliffs, NJ, 1972) 66–106.
[35] D.S. Scott, Data types as lattices, SIAM J. Comput. 5 (September 1976) 522–587.
[36] D.S. Scott, A type-theoretical alternative to CUCH, ISWIM and OWHY, Theoret. Comput. Sci. 121 (1993) 411–440; Reprint of a manuscript produced in 1969.
[37] M.B. Smyth, Power domains and predicate transformers: a topological view, in: J. Diaz, ed., Automata, Languages and Programming, Lecture Notes in Comput. Sci. 154 (Springer, Berlin, 1983) 662–675.
[38] M.B. Smyth, Stable compactification I, J. London Math. Soc. 45 (2) (1992) 321–340.
[39] M.B. Smyth, Topology, in: S. Abramsky, D.M. Gabbay and T.S.E. Maibaum, eds., Handbook of Logic in Computer Science, Vol. 1 (Clarendon Press, Oxford, 1992) 641–761.
[40] P. Taylor, Recursive domains, indexed category theory and polymorphism, PhD Thesis, Cambridge University (1986).
[41] J.J.C. Vermeulen, Proper maps of locales, J. Pure Appl. Algebra 92 (1994) 79–107.
[42] S. Vickers, Locales are not pointless, in: C. Hankin, I. Mackie and R. Nagarajan, eds., Theory and Formal Methods 1994: Proceedings of the Second Imperial College Department

of Computing Workshop on Theory and Formal Methods, Møller Centre, Cambridge, 11–14 September 1994 (IC Press, 1995).
[43] S. Vickers, Topology via Logic (Cambridge University Press, Cambridge, 1989).