Baeten, J., and B. Boerboom. "Ω can be anything it should not be." Indagationes Mathematicae, vol. 82 (1979), pp. 111-120.

MATHEMATICS

# Ω Can be anything it should not be

by J. Baeten and B. Boerboom

*Mathematical Institute, University of Utrecht*

Communicated by Prof. A. S. Troelstra at the meeting of September 30, 1978

ABSTRACT

Let $\Omega$ be the $\lambda$-term $(\lambda x.xx)\,(\lambda x.xx)$. Correcting and supplementing Jacopini [1975], it will be shown by a prooftheoretical argument that Con($\lambda\eta+\Omega=M$) for an arbitrary closed $\lambda$-term $M$.

By changing the pairingfunction in the graphmodel $P\omega$, cf. Scott [1975], it will be shown that for arbitrary closed $M$ one may have $P\omega \models \Omega = M$, giving a modeltheoretic proof of Con($\lambda+\Omega=M$).

## § 1. PROOFTHEORETICAL PROOF OF Con ($\lambda\eta+\Omega=M$)

In this paper we use the notation as in Barendregt [1977]. In this section we follow the line of argument as given in Jacopini [1975] to show that for an arbitrary closed $\lambda$-term $M$ Con ($\lambda\eta+\Omega=M$).

In 1.1–1.7 we give a criterion for Con ($\lambda\eta+M=N$), and in the rest of this section we use this to show that Con ($\lambda\eta+\Omega=M$) for $M \in \Lambda^0$.

1.0 NOTE: Throughout this section all our reasoning takes place in the theory $\lambda\eta$, so when writing $M=N$, we mean $\lambda\eta \vdash M=N$.

1.1 DEF.: Let $X, Y, U, V \in \Lambda^0$, then $X \overset{UV}{\longleftrightarrow} Y$ iff $\exists Q \in \Lambda^0\; QUV=X$ and $QVU=Y$.

1.2 LEMMA: For all $X, Y, X', Y' \in \Lambda^0$:

1. $X \overset{UV}{\longleftrightarrow} X$

2. $X \overset{UV}{\longleftrightarrow} Y$, then $Y \overset{UV}{\longleftrightarrow} X$
3. $X \overset{UV}{\longleftrightarrow} Y$ and $X' \overset{UV}{\longleftrightarrow} Y'$, then $XX' \overset{UV}{\longleftrightarrow} YY'$
4. $U \overset{UV}{\longleftrightarrow} V$

PROOF:

1. $K(KX)UV = KXV = KXU = K(KX)VU$
2. If $QUV = X$ and $QVU = Y$, then $(\lambda xyz \cdot xzy)QUV = QVU = Y$ and $(\lambda xyz \cdot xzy)QVU = QUV = X$.
3. and 4. are equally simple.

1.3 DEF.:

1. $\overset{UV}{\sim}$ is the transitive closure of $\overset{UV}{\longleftrightarrow}$, so $X \overset{UV}{\sim} Y$ iff $\exists Z_1, \ldots, Z_n \in \Lambda^0$, $n \geqslant 0$ such that $X \leftrightarrow Z_1 \leftrightarrow \ldots \leftrightarrow Z_n \leftrightarrow Y$.
2. $l$th $(X \sim Y) = n$, if in 1. $n$ is chosen minimal.

Notation: $X \underset{n}{\sim} Y$, e.g. $X \underset{0}{\sim} Y$ means $X \leftrightarrow Y$.

1.4 LEMMA: For all $X, Y, Z, X', Y' \in \Lambda^0$:

1. $X \overset{UV}{\sim} X$
2. $X \overset{UV}{\sim} Y$, then $Y \overset{UV}{\sim} X$
3. $X \overset{UV}{\sim} Y$ and $X' \overset{UV}{\sim} Y'$, then $XX' \overset{UV}{\sim} YY'$
4. $X \overset{UV}{\sim} Y$ and $Y \overset{UV}{\sim} Z$, then $X \overset{UV}{\sim} Z$
5. $X \overset{UV}{\sim} Y$, then $\lambda x \cdot X \overset{UV}{\sim} \lambda x \cdot Y$
6. $U \overset{UV}{\sim} V$

PROOF: 1, 2, 4, 5, 6 trivial.

3. By induction on max ($l$th $(X \sim Y)$, $l$th $(X' \sim Y')$).

CONCLUSION: $\overset{UV}{\sim}$ is a congruence relation on $\Lambda^0 \times \Lambda^0$.

1.5 LEMMA: For all $X, Y, U, V \in \Lambda^0$: $X \overset{UV}{\sim} Y \Leftrightarrow \lambda\eta + U = V \vdash X = Y$.

PROOF: Trivial.

1.6 DEF.: $U$ is separable from $V$, $U$ sep $V$ iff $T \overset{UV}{\sim} F$.

1.7 THM.: For $M, N \in \Lambda^0$: $M$ sep $N$ iff $\neg$ Con $(\lambda\eta + M = N)$.

PROOF:

$M$ sep $N \Leftrightarrow T \overset{MN}{\sim} F \Leftrightarrow \lambda\eta + M = N \vdash T = F \Leftrightarrow \neg$ Con $(\lambda\eta + M = N)$.

1.8 DEF.:

1. $\underline{\Lambda}$ is a set of words over the following alphabet:
   $x_0, x_1, \ldots; \underline{\Omega}, \lambda, (\,,\,)$.
   Note: $\underline{\Omega}$ is a special symbol, but $\Omega \equiv (\lambda x \cdot xx)(\lambda x \cdot xx)$.
2. $\underline{\Lambda}$ is inductively defined by: for $i \in \omega$ $x_i \in \underline{\Lambda}$; $\underline{\Omega} \in \underline{\Lambda}$; if $M, N \in \underline{\Lambda}$, then $MN \in \underline{\Lambda}$; if $M \in \underline{\Lambda}$, then $\lambda x M \in \underline{\Lambda}$.

3. $M \underset{\underline{\beta\eta}}{\longrightarrow} N$ if 1. $M \equiv C[(\lambda xP)Q]$ and $N \equiv C[P[x:=Q]]$

2. $M \equiv C[\lambda xPx]$ and $N \equiv C[P]$ $(x \notin FV(P))$

3. $M \equiv C[\underline{\Omega}]$ and $N \equiv C[\underline{\Omega}]$

$\underset{\underline{\beta\eta}}{\twoheadrightarrow}$ is defined as the transitive and reflexive closure of $\underset{\underline{\beta\eta}}{\longrightarrow}$.

4. For a $\underline{\Lambda}$-term we define $|M|$ to be $M$ without any underlining, and $M \simeq M' \Leftrightarrow |M| \equiv |M'|$.
In particular $\underline{\Omega} \simeq \Omega$, for $|\underline{\Omega}| \equiv \Omega$.

5. For $M \in \underline{\Lambda}$ we define $\phi_x: \underline{\Lambda} \to \Lambda$ for $x \notin FV(M)$ as follows:
$M \equiv x_i$, then $\phi_x(x_i) \equiv x_i$
$M \equiv \underline{\Omega}$, then $\phi_x(\underline{\Omega}) \equiv x$
$M \equiv PQ$, then $\phi_x(PQ) \equiv \phi_x(P)\phi_x(Q)$
$M \equiv \lambda x_i P$, then $\phi_x(\lambda x_i P) \equiv \lambda x_i \phi_x(P)$

With these definitions we can keep track of $\Omega$'s in a $\Lambda$-term $M$, while reducing this term.

First we will obtain a version of the "Genericity Lemma" in 1.11.

1.9 LEMMA: Let $M, N \in \Lambda$ and $M' \in \underline{\Lambda}$, with $M \simeq M'$ and $M \underset{\beta\eta}{\twoheadrightarrow} N$, then there is a $N' \in \underline{\Lambda}$ such that $N \simeq N'$ and $M' \underset{\underline{\beta\eta}}{\twoheadrightarrow} N'$.

PROOF: It is sufficient to show this for $M \underset{\beta\eta}{\longrightarrow} N$.
Then the proof is easy by simulating the reduction of the redex in $M$ in $M'$.

1.10 LEMMA: Let $M, N \in \underline{\Lambda}$, and $M \underset{\underline{\beta\eta}}{\twoheadrightarrow} N$, then $\phi_x(M) \underset{\beta\eta}{\twoheadrightarrow} \phi_x(N)$ (for $x \notin FV(MN)$)

PROOF: By induction on $\underset{\underline{\beta\eta}}{\twoheadrightarrow}$: in the reduction $M \underset{\underline{\beta\eta}}{\twoheadrightarrow} N$ replace the constant $\underline{\Omega}$ everywhere by the fresh variable $x$.

1.11 THM.: (Version of the Genericity Lemma).
Let $G\Omega \underset{\beta\eta}{\twoheadrightarrow} N$ and $N$ in $\beta\eta$-normal form, then for all $M \in \Lambda$ $GM \underset{\beta\eta}{\twoheadrightarrow} N$.

PROOF: By lemma 1.9 we have: there is a $N' \in \underline{\Lambda}$, such that $G\underline{\Omega} \underset{\underline{\beta\eta}}{\twoheadrightarrow} N'$ and $N \simeq N'$.
By lemma 1.10: $\phi_x(G\underline{\Omega}) \underset{\beta\eta}{\twoheadrightarrow} \phi_x(N')$. Now $\Omega \not\subset N$ hence $N \equiv N'$ and $N' \equiv \phi_x(N')$. Therefore $Gx \underset{\beta\eta}{\twoheadrightarrow} N$ for $x \notin FV(G)$, hence $GM \underset{\beta\eta}{\twoheadrightarrow} N$, for all $M$.

1.12 LEMMA: Let $M, N \in \underline{\Lambda}$, then $M \simeq N \Rightarrow \exists C(x_0, x_1, x_2)$ $(x_0, x_1, x_2 \in FV(C))$
such that $C[x_0:=x, x_1:=x, x_2:=\underline{\Omega}] \equiv \phi_x M$
and $C[x_0:=x, x_1:=\underline{\Omega}, x_2:=x] \equiv \phi_x N$.

PROOF: It is obvious that $M$ is identical with $N$, except for the occurrences of $\Omega$ and $\underline{\Omega}$. Now we will consider $M$ and $N$ as contexts of $\Omega$ and $\underline{\Omega}$, where every occurrence of $\Omega$ and $\underline{\Omega}$ is mentioned.

We have e.g. $M \equiv C'[\Omega, \Omega, \underline{\Omega}, \underline{\Omega}, \ldots]$ and $N \equiv C'[\Omega, \underline{\Omega}, \Omega, \underline{\Omega}, \ldots]$ (with $x \notin FV(MN)$), giving us: $\phi_x M \equiv C'[\Omega, \Omega, x, x, \ldots]$ and $\phi_x N \equiv C'[\Omega, x, \Omega, x, \ldots]$.

We can now distinguish between four cases:

1. an occurrence of $x$ in $\phi_x M$, where in the corresponding place in $\phi_x N$ occurs a $x$.
2. an occurrence of $x$ in $\phi_x M$, where in the corresponding place in $\phi_x N$ occurs an $\Omega$.
3. an occurrence of $\Omega$ in $\phi_x M$, where in the corresponding place in $\phi_x N$ occurs a $x$.
4. an occurrence of $\Omega$ in $\phi_x M$, where in the corresponding place in $\phi_x N$ occurs an $\Omega$.

Now substitute "fresh" variables $x_0, x_1, x_2$ in $\phi_x M$ and $\phi_x N$ respectivily for occurrences of 1, 2 and 3.

This gives the desired $C$. (In the example: $C(x_0, x_1, x_2) \equiv C'[\Omega, x_2, x_1, x_0, \ldots]$.)

1.13 LEMMA: If $C_0[\Omega] = C_1[\Omega]$, then there is a $C$ such that

$$C_0[x] \twoheadrightarrow_{\beta\eta} C[x_0 := x, x_1 := x, x_2 := \Omega], \text{ and}$$
$$C_1[x] \twoheadrightarrow_{\beta\eta} C[x_0 := x, x_1 := \Omega, x_2 := x].$$

PROOF: By the Church-Rosser theorem we obtain for some $Z$

$$C_0[\Omega] \twoheadrightarrow_{\beta\eta} Z \text{ and } C_1[\Omega] \twoheadrightarrow_{\beta\eta} Z.$$

By lemma 1.9:

$$\begin{array}{ccccccc}
C_0[\Omega] & \twoheadrightarrow_{\beta\eta} & Z & \qquad & Z & \twoheadleftarrow_{\beta\eta} & C_1[\Omega] \\
\mid & & \mid & & \mid & & \mid \\
\simeq & & \simeq & & \simeq & & \simeq \\
\mid & & \mid & & \mid & & \mid \\
C_0[\underline{\Omega}] & \twoheadrightarrow_{\underline{\beta\eta}} & M & & N & \twoheadleftarrow_{\underline{\beta\eta}} & C_1[\underline{\Omega}]
\end{array}$$

$\simeq$ being an equivalencerelation on $\underline{\Lambda} \times \underline{\Lambda}$, $\phi_x(C_0[\underline{\Omega}]) \twoheadrightarrow_{\beta\eta} \phi_x M$, $\phi_x(C_1[\underline{\Omega}]) \twoheadrightarrow_{\beta\eta} \phi_x N$ and $M \simeq N$ hold.

Then lemma 1.12 gives us the desired result.

1.14 DEF.: Let $X, Y, U, V \in \Lambda^0$, then

$$X \xrightarrow{UV} Y \Leftrightarrow Y \xleftarrow{UV} X \Leftrightarrow \exists P \in \Lambda^0 \ (PU = X \text{ and } PV = Y).$$

1.15 LEMMA:

1. If $T \xleftarrow{U\Omega} Z$, then $T = Z$ (for $U, Z \in \Lambda^0$)
2. If $Z \xrightarrow{U\Omega} F$, then $Z = F$.

PROOF:

1. Suppose $T \xleftarrow{U\Omega} Z$, then $\exists P\ PU = Z$ and $P\Omega = T$. Now apply theorem 1.11, $T$ in $\beta\eta$-normal form. Then $PM \twoheadrightarrow_{\beta\eta} T$ for all $M$, hence $PU = T$.
2. Analogous.

1.16 LEMMA:

1. $X \xleftrightarrow{UV} Y \Rightarrow \exists Z(X \xleftarrow{UV} Z \xrightarrow{UV} Y)$ (for $X, Y, Z \in \Lambda^0$)
2. $X \xrightarrow{U\Omega} Z \xleftarrow{U\Omega} Y \Rightarrow X \xleftrightarrow{U\Omega} Y$.

PROOF:

1. Suppose there is a $Q$ such that $QUV = X$ and $QVU = Y$, then define $Z \equiv QUU$, $P \equiv QU$ and $P' \equiv \lambda x \cdot QxU$. Then $PU \equiv QUU \equiv Z$ and $PV \equiv QUV = X$, so $X \xleftarrow{UV} Z$, and also $P'U \equiv (\lambda x \cdot QxU)U = QUU \equiv Z$ and $P'V \equiv (\lambda x \cdot QxU)V = QVU = Y$, so $Z \xrightarrow{UV} Y$.
2. Suppose there are $P, P'$ such that $PU = X$, $P\Omega = Z$ and $P'U = Y$, $P'\Omega = Z$, so $P\Omega = P'\Omega$. By lemma 1.13

$$Px \twoheadrightarrow_{\beta\eta} C[x_0 := x,\ x_1 := x,\ x_2 := \Omega]$$

and

$$P'x \twoheadrightarrow_{\beta\eta} C[x_0 := x,\ x_1 := \Omega,\ x_2 := x]$$

for certain $C$.

Therefore

$$PU = C[x_0 := U,\ x_1 := U,\ x_2 := \Omega]$$

and

$$P'U = C[x_0 := U,\ x_1 := \Omega,\ x_2 := U].$$

Now define $Q \equiv \lambda xy C[x_0 := U,\ x_1 := x,\ x_2 := y]$.

Then

$$QU\Omega \equiv (\lambda xy C[x_0 := U,\ x_1 := x,\ x_2 := y])U\Omega = \\ = C[x_0 := U,\ x_1 := U,\ x_2 := \Omega] = PU = X$$

and

$$Q\Omega U = C[x_0 := U,\ x_1 := \Omega,\ x_2 := U] = P'U = Y,$$

hence $X \xleftrightarrow{U\Omega} Y$.

1.17 LEMMA: $\neg\, T \overset{U\Omega}{\sim} F$ (for any $U \in \Lambda^0$).

PROOF: We show by induction on $n$: $\neg\, T \underset{n}{\overset{U\Omega}{\sim}} F$.

1. $n = 0$. $T \xleftrightarrow{U\Omega} F$, then by 1.16.1 for some $Z$ $T \xleftarrow{U\Omega} Z \xrightarrow{U\Omega} F$, hence by 1.15 $T = Z$ and $Z = F$, so $T = F$, and that is a contradiction. Hence $\neg\, T \leftrightarrow F$.

2. $n>0$. Suppose $T \underset{n}{\overset{U\Omega}{\sim}} F$, then $\exists Z_1, \ldots, Z_n\ T \leftrightarrow Z_1 \leftrightarrow \ldots \leftrightarrow Z_n \leftrightarrow F$, and by 1.16.1 $\exists W_1, \ldots, W_{n+1}\ T \leftarrow W_1 \rightarrow Z_1 \leftarrow \ldots \rightarrow Z_n \leftarrow W_{n+1} \rightarrow F$. By 1.15 $T = W_1$ and $W_{n+1} = F$, so this reduces the chain to:

$$T \rightarrow Z_1 \leftarrow W_2 \rightarrow \ldots \leftarrow W_n \rightarrow Z_n \leftarrow F,$$

and by 1.16.2 $T \leftrightarrow W_2 \leftrightarrow \ldots \leftrightarrow W_n \leftrightarrow F$, so $T \underset{n-1}{\sim} F$, and this contradicts the induction hypothesis.

1.18 THM.: For all $M \in \Lambda^0$ Con $(\lambda\eta + \Omega = M)$.

PROOF: Let $M \in \Lambda^0$. Suppose $M$ sep $\Omega$. Then by 1.6 $T \overset{M\Omega}{\sim} F$ contradicting 1.17. Therefore $\neg\, M$ sep $\Omega$ and hence by 1.7 Con $(\lambda\eta + \Omega = M)$.

1.19 REMARK

1. It is not the case that, if $P$ is unsolvable, then Con $(\lambda + P = M)$ for all $M \in \Lambda^0$.

   Take e.g. $P \equiv YK$ (the fixed-point of $K$), then $Px = KPx = P$, hence $P = I \vdash x = Ix = Px = P$ for all $x$, contradiction.
2. Jacopini [1975] gives an example of an unsolvable $P$, which is of order 0 (a term $P$ is of *order 0* iff $P$ does not reduce to a term of the form $\lambda xQ$), such that $\neg$ Con $(\lambda + P = I)$: let $\omega_3 \equiv \lambda x \cdot xxx$, and $P \equiv \omega_3\omega_3 \equiv \Omega_3$, then: $I = \Omega_3 \vdash I = \omega_3\omega_3 = \omega_3\omega_3\omega_3 = \Omega_3\omega_3 = I\omega_3 = \omega_3$, and it is not difficult to derive a contradiction from $\lambda x \cdot x = \lambda x \cdot xxx$. (Cf. Böhm [1968]).

## § 2. MODELTHEORETICAL PROOF OF Con $(\lambda + \Omega = M)$

For all $M \in \Lambda^0$, we will define a bijective pairingfunction $C_M$, such that $P\omega,\ C_M \models \Omega = M$.

2.1 Some definitions and remarks in connection with $P\omega$:

1. $e_n = \{x_0, \ldots, x_k\}$ iff $n = \sum_0^k 2^{x_i}$ and $x_0 < x_1 < \ldots < x_k$.
2. The letter $C$ will be reserved for bijective pairing-functions: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. $C^*$ is the "ordinary" pairing: $C^*(n, m) = \frac{1}{2}(n+m)(n+m+1) + m$.
3. Abstraction and application in $P\omega$, given a pairing $C$: if $f\colon P\omega^{n+1} \rightarrow P\omega$ is continuous, then $\lambda^* x \cdot f(x, \vec{y}) = \{C(n, m) | m \in f(e_n, \vec{y})\}$; if $u, x \in P\omega$, then $u \cdot x = \{m | \exists n\ e_n \subseteq x \text{ and } C(n, m) \in u\} \in P\omega$.
4. Interpretation from $\lambda$ to $P\omega$, given $C$ and a valuation $\varrho$ (by induction): $[x]_C^\varrho = \varrho(x)$; $[MN]_C^\varrho = [M]_C^\varrho \cdot [N]_C^\varrho$; $[\lambda x \cdot M]_C^\varrho = \lambda^* d \cdot [M]_C^{\varrho(x/d)}$.
5. By definition of $e_n$ and $C^*$: if $e_q \subseteq e_k$, then $q \leqslant k$; if $m \in e_k$ then $m < k$; if $C^*(a, b) = c$, then $c \geqslant a$ and $c \geqslant b$. If $C^*(a, b) = c$, we call $a = Ic$ and $b = Jc$.

2.2 LEMMA:

1. If $a \in [\Omega]_C$, then $\exists k (C(k, a) \in e_k)$.
2. If $e_k = \{C(k, a)\}$, then $a \in [\Omega]_C$.

PROOF:

1. Suppose $a \in [\Omega]_C$. Let $\omega = [\lambda x \cdot xx]_C$. By 2.1.3 $\exists k(e_k \subseteq \omega$ and $C(k, a) \in \omega)$. Let $k_0 = \mu k[e_k \subseteq \omega$ and $C(k, a) \in \omega]$ ("the smallest $k$ such that . . ."). $C(k_0, a) \in \omega \Rightarrow a \in e_{k_0} \cdot e_{k_0} \Rightarrow \exists q(e_q \subset e_{k_0}$ and $C(q, a) \in e_{k_0})$. Hence $e_q \subseteq \omega$ and $C(q, a) \in \omega$, so $k_0 \leqslant q$ by the minimality of $k_0$; and by 2.1.5 $q \leqslant k_0$, i.e. $q = k_0$, therefore $C(k_0, a) \in e_{k_0}$.
2. Suppose $e_k = \{C(k, a)\}$, then $e_k \subseteq e_k$ and $C(k, a) \in e_k$, so $a \in e_k \cdot e_k$, i.e. $C(k, a) \in \omega$. Therefore $e_k \subseteq \omega$, and $a \in [\Omega]_C$.

2.3 COR.: $[\Omega]_{C*} = \emptyset$.

PROOF: By 2.2.1 and 2.1.5. Cf. Scott [1975].

2.4 LEMMA: If $A \subseteq \mathbb{N}$, then there is a pairing $C_A$ such that $[\Omega]_{C_A} = A$.

PROOF:

1. If $A = \emptyset$, then $[\Omega]_{C*} = \emptyset = A$. Define $C_A = C^*$.
2. If $A \neq \emptyset$, then let $\{a_i | i \in \mathbb{N}\}$ be an enumeration (possibly with repetitions) of the elements of $A$. We define $C_A$ in stages, by interchanging values of $C^*$.

Step 0: if $C^*(1, a_0) = p_0$, define $C_A(1, a_0) = 0$ and $C_A(0, 0) = p_0$; step $n$, for $n > 0$: let $k'$ be the smallest $k$ such that $C_A(2^k, a_n)$ or $C_A(Ik, Jk)$ is not yet defined in any of the previous steps. Then define $C_A(2^{k'}, a_n) = k'$ and $C_A(Ik', Jk') = C^*(2^{k'}, a_n)$. For all pairs $(p, q)$, such that $C_A(p, q)$ is not defined in any of the steps above, define $C_A(p, q) = C^*(p, q)$.

Clearly this definition makes $C_A$ a bijective pairing.

a. Let $m \in \mathbb{N}$. By step $m$, there is a $k \in \mathbb{N}$ such that $C_A(2^k, a_m) = k$. By 2.1.1 $e_{2^k} = \{k\}$, so by 2.2.2 $a_m \in [\Omega]_{C_A}$. Therefore $[\Omega]_{C_A} \supseteq A$.
b. If $a \in [\Omega]_{C_A}$, then by 2.2.1 $\exists k\ C_A(k, a) \in e_k$. By 2.1.5 $C_A(k, a) < k$, so $C_A(k, a) < C^*(k, a)$. Let $C_A(k, a)$ be defined in step $i$. It follows easily that $a = a_i$, so $[\Omega]_{C_A} \subseteq A$.

$a$ and $b$ give $[\Omega]_{C_A} = A$, in all cases.

In 2.5 and 2.6, we will formulate a finiteness condition, in order to be able to construct $C_M$ for each $M \in \Lambda^0$, such that $[\Omega]_{C_M} = [M]_{C_M}$.

2.5 DEF.: Let $M \in \Lambda$, $\varrho$ be a valuation, and $a \in \mathbb{N}$. Then:

1. $P: \mathbb{N} \times \mathbb{N} \dashrightarrow \mathbb{N}$, an injective partial function with a *finite* domain, is called a *forcing condition.*
2. When $P$ is a forcing condition, then $P \Vdash a \in [M]^\varrho$ ($P$ *forces* $a \in [M]^\varrho$), iff for all pairings $C$, if $C \supseteq P$, then $a \in [M]^\varrho_C$.

2.6 THM.: Let $M \in \Lambda$, $\varrho$ be a valuation, $C$ a pairing, and $a \in \mathbb{N}$. Then: if $a \in [M]^\varrho_C$, then there is a forcing condition $P$ such that $P \subseteq C$ and $P \Vdash a \in [M]^\varrho$.

PROOF: By induction on the structure of $M$:

1. $M \equiv x$. Then $[x]^\varrho_C = \varrho(x)$. Suppose $a \in \varrho(x)$. This is independent of $C$,

so if $a \in [M]^{\varrho}_{C}$, then for all pairing $C'$, $a \in [x]^{\varrho}_{C'}$. Then $\emptyset \Vdash a \in [x]^{\varrho}$ follows.

2. $M \equiv \lambda x \cdot R$, and suppose the theorem is proven for $R$. Then

$$[M]^{\varrho}_{C} = \lambda^* d \cdot [R]^{\varrho(x/d)}_{C} = \{C(n, m) | m \in [R]^{\varrho(x/e_n)}_{C}\}.$$

Suppose $a \in [M]$. $\exists n, m(a = C(n, m)$ and $m \in [R]^{\varrho(x/e_n)}_{C})$. By induction there is a forcing condition $P \subseteq C$ with $P \Vdash m \in [R]^{\varrho(x/e_n)}$. Define $P^* = P \cup \{(n, m, a)\}$. Clearly $P^*$ is a forcing condition, $P^* \subseteq C$, and $P^* \Vdash m \in [R]^{\varrho(x/e_n)}$. Now if $C' \supseteq P^*$ is a pairing, then $m \in [R]^{\varrho(x/e_n)}_{C'}$, and $C'(n, m) = a \in [M]^{\varrho}_{C'}$, so $P^* \Vdash a \in [M]^{\varrho}$.

3. $M \equiv RQ$, and suppose the theorem is proven for $R$ and $Q$. Now

$$[M]_C = [R]_C \cdot [Q]_C = \{m | \exists n \; e_n \subseteq [Q] \text{ and } C(n, m) \in [R]\}.$$

Suppose $a \in [M]$, then for some $n, q$ $e_n = \{x_0, \dots, x_k\} \subseteq [Q]$ and $C(n, a) = q \in [R]$. Then by induction there are forcing conditions $P_0, \dots, P_k, P_{k+1}$ such that $P_i \subseteq C$ $(0 \leqslant i \leqslant k+1)$ and $P_i \Vdash x_i \in [Q]$ $(0 \leqslant i \leqslant k)$ and $P_{k+1} \Vdash q \in [R]$. Define $P = \bigcup_{i=1}^{k+1} P_i \cup \{(n, a, q)\}$. Clearly $P$ is a forcing condition and $P \subseteq C$. Also $P \Vdash x_i \in [Q]$ $(0 \leqslant i \leqslant k)$ and $P \Vdash q \in [R]$. Now, if $C' \supseteq P$ is a pairing, then $e_n = \{x_0, \dots, x_k\} \subseteq [Q]_{C'}$ and $q \in [R]_{C'}$ and $C'(n, a) = q$, so $a \in [M]_{C'}$

2.7 REMARK: We want a pairing $C_M$ such that $[\Omega]_{C_M} = [M]_{C_M}$ (for $M \in \Lambda^0$). The construction of 2.4 is not sufficient now, because the set $[M]_C$ changes, when $C$ changes.

2.6 gives us the solution: $a \in [M]_C$ is forced by a finite $P \subseteq C$, so when we define $C'$ such that $C'(2^m, a) = m$ for a certain $m \in \Omega$ and still $C' \supseteq P$, we will have $a \in [M]_{C'}$ and $a \in [\Omega]_{C'}$. Then we can proceed with the "next" element in $[M]$. In detail, we do this in 2.8.

2.8 DEF.:

1. By induction we define $a_n \in \Omega \cup \{\frac{1}{2}\}$, pairings $C_n$ and forcing conditions $P_n$ for each $M \in \Lambda^0$, such that for $n \leqslant k$ $P_n \subseteq C_k$, for all $n$ $P_n \subseteq P_{n+1}$ and, if $a_n \neq \frac{1}{2}$, $P_n \Vdash a_n \in [M]$.

a. $n = 0$. Define $a_0 = \frac{1}{2}$, $C_0 = C^*$ and $P_0 = \emptyset$.

b. $n > 0$. Suppose $a_{n-1}$, $C_{n-1}$ and $P_{n-1}$ defined. Then:
case 1: $[M]_{C_{n-1}} \subset \{a_1, \dots, a_{n-1}\}$. Define $a_n = a_{n-1}$, $C_n = C_{n-1}$, $P_n = P_{n-1}$.
case 2: $[M]_{C_{n-1}} \not\subset \{a_1, \dots, a_{n-1}\}$. Define $a_n = \mu x[x \in [M]_{C_{n-1}} - \{a_1, \dots, a_{n-1}\}]$. By 2.6 there is a forcing condition $Q$ such that $Q \Vdash a_n \in [M]$ and $Q \subseteq C_{n-1}$. Let

$$q = \mu x[x \notin \text{Ran}\,(P_{n-1} \cup Q) \text{ and } (2^x, a_n) \notin \text{Dom}\,(P_{n-1} \cup Q)];$$

$q$ exists, because $P_{n-1} \cup Q$ is finite.

Define $C_n(x, y) = C_{n-1}(x, y)$ if $(x, y) \neq (2^q, a_n)$ and $(x, y) \neq (Iq, Jq)$; $C_n(2^q, a_n) = q$ and $C_n(Iq, Jq) = C^*(2^q, a_n)$.

Define $P_n = P_{n-1} \cup Q \cup \{(2^q, a_n, q), (Iq, Jq, C^*(2^q, a_n))\}$.

Note that in both cases $C_n$ is a bijective pairing, $P_n$ a forcing condition and the conditions mentioned above hold.
2. Define $C_M$ by $C_M(x, y) = \lim_{n\to\infty} C_n(x, y)$. Result: $P_n \subseteq C_M$ for all $n$.

2.9 LEMMA: $C_M$ is well-defined and a bijective pairing.

PROOF: Let $x, y \in \mathbf{\Pi}$. If $C_n(x, y) = C^*(x, y)$ for all $n \in \mathbf{\Pi}$, trivially $C_M(x, y) = C^*(x, y)$. If that is not the case, there is a $k>0$ such that $(x, y) \notin \mathrm{Dom}\, P_{k-1}$ but $(x, y) \in \mathrm{Dom}\, P_k$, by 2.8.

Hence $C_i(x, y) = C^*(x, y)$, if $0 \leqslant i \leqslant k-1$, and $C_i(x, y) = C_k(x, y)$ for $i > k$ by the condition for $q$ in 2.8. So $C_M(x, y) = C_k(x, y)$.

Therefore is $C_n(x, y)$ for all $x$ and $y$ eventually constant, and is $C_M$ well-defined. It is easily checked that $C_M$ is bijective.

2.10 LEMMA: For all $a, q \in \mathbf{\Pi}$ $\{(2^q, a, q)\} \Vdash a \in [\Omega]$.

PROOF: $e_{2^q} = \{q\}$, so if $C(2^q, a) = q$, or $C \supseteq \{(2^q, a, q)\}$ for a certain pairing $C$, then $a \in [\Omega]_C$ by 2.2.2.

2.11 THM.: For all $M \in \Lambda^0$ $[\Omega]_{C_M} = [M]_{C_M}$ ($C_M$ as defined in 2.8).

PROOF:

a. We claim $[M]_{C_M} = \{a_i | i \in \mathbf{\Pi}\} - \{\frac{1}{2}\}$ ($a_i$ as defined in 2.8).
1. Let $i \in \mathbf{\Pi}$ and $a_i \neq \frac{1}{2}$. Then $P_i \Vdash a_i \in [M]$ and $P_i \subseteq C_k$ for $k \geqslant i$ by 2.8, hence $P_i \subseteq C_M$, thus $a_i \in [M]_{C_M}$ by 2.9.
2. Suppose $a \in [M]_{C_M}$. By 2.6 there is a forcing condition $P \subseteq C_M$ such that $P \Vdash a \in [M]$. Since for all $x, y$ $C_n(x, y)$ is eventually constant and $P$ is finite, $\exists i_0 \in \mathbf{\Pi}\ \forall k > i_0$ ($C_k \supseteq P$ and $a \in [M]_{C_k}$).

   Now if for certain $k > i_0$ case 1 in 2.8 holds, i.e. $[M]_{C_k} \subseteq \{a_1, \ldots, a_k\}$, then $a = a_i$ for certain $i \in \mathbf{\Pi}$.

   Now, suppose on the contrary that for every $k > i_0$ case 2 holds. Then $a_{i_0+1}, a_{i_0+2}, \ldots$ are all distinct, so there is a $a_k$ in this sequence with $a_k > a$. Because we chose each $a_i$ minimal, $a$ itself must be in this sequence, so $a = a_i$ for certain $i \in \mathbf{\Pi}$. We conclude that in both cases $a \in \{a_i | i \in \mathbf{\Pi}\} - \{\frac{1}{2}\}$, i.e. $[M]_{C_M} \subseteq \{a_i | i \in \mathbf{\Pi}\}$. This proves claim a.

b. We claim $[\Omega]_{C_M} = \{a_i | i \in \mathbf{\Pi}\} - \{\frac{1}{2}\}$.
1. Let $i \in \mathbf{\Pi}$ and $a_i \neq \frac{1}{2}$. Then, for certain $q \in \mathbf{\Pi}$, $(2^q, a_i, q) \in P_i$, and by 2.10 $P_i \Vdash a_i \in [\Omega]$. Hence, since $P_i \subseteq C_M$, $a_i \in [\Omega]_{C_M}$ by 2.9.
2. Suppose $a \in [\Omega]_{C_M}$. By 2.2.1 $\exists k$ ($C_M(k, a) \in e_k$), and by 2.1.5 $C_M(k, a) < k \leqslant C^*(k, a)$. By construction of $C_n$ there is a $n \in \mathbf{\Pi}$ such that
   $$C^*(k, a) = C_n(k, a) > C_{n+1}(k, a) = C_M(k, a).$$
   It follows by 2.8 that $(k, a) = (2^q, a_n)$, so $a = a_n$, and $[\Omega]_{C_M} \subseteq \{a_i | i \in \mathbf{\Pi}\}$. This proves claim *b*.
   a and b together prove $[\Omega]_{C_M} = [M]_{C_M}$.

2.12 NOTE: The construction of $C_M$ is not recursive.

REFERENCES

Barendregt, H. P. – The type free Lambda calculus, Handbook of Mathematical Logic, Editor J. Barwise, North Holland, Amsterdam, pages 1092–1132 (1977).

Böhm, C. – Alcune proprietà delle forme $\beta$-$\eta$-normali nel $\lambda$-$K$ calcolo. Pubbl. IAC – CNR n. 696, Roma, 1968.

Jacopini, G. – $\lambda$-Calculus and Computer Science Theory, Editor C. Böhm, Lecture Notes in Computer Science, 37, pages 213–220 (1975).

Scott, D. S. – Lambda calculus and recursion theory, Proceedings Third Scandinavian Logic Symposium, North Holland, pages 154–193 (1975).