Probabilistic Bisimulation

Seminar *Probabilistic Models of Concurrency*

July 2020

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Outline

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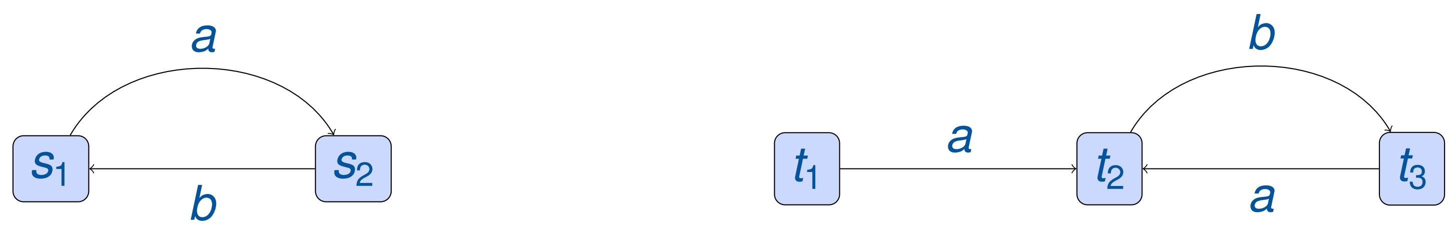
Bisimulation

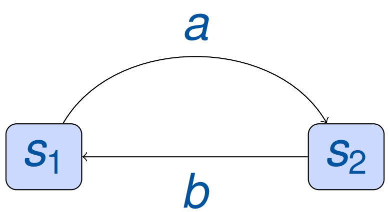
Motivation

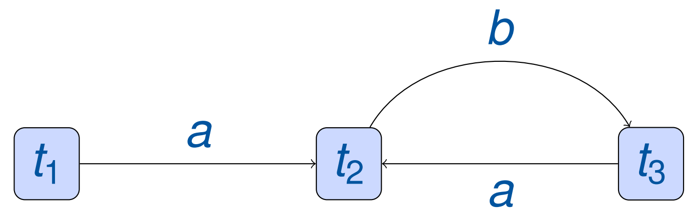
- Determine equivalence of transition systems (TS) under a certain logic
- Reducing size of TS while maintaining logical properties
- Optimize model checking performance



Two bisimilar transition systems

Introduction





Definition - Bisimulation for an ordinary TS

Let $TS = (S, Act, \rightarrow, I, AP, L)$ be a transition system. A bisimulation for TS is a binary relation $\mathcal{R}$ such that for all $(s_1, s_2) \in \mathcal{R}$:

- $L(s_1) = L(s_2)$
- If $s_1' \in \text{Post}(s_1)$, then there exists an $s_2' \in \text{Post}(s_2)$ with $(s_1', s_2') \in \mathcal{R}$
- If $s_2' \in \text{Post}(s_2)$, then there exists an $s_1' \in \text{Post}(s_1)$ with $(s_1', s_2') \in \mathcal{R}$

States s_1 and s_2 are bisimulation-equivalent (or bisimilar), denoted $s_1 \sim_{TS} s_2$, if there exists a bisimulation $\mathcal{R}$ for TS with $(s_1, s_2) \in \mathcal{R}$.

Introduction

Aim of this presentation

1. Introduce a variant of probabilistic automata (PLTS)
2. Define probabilistic bisimulation for these automata
3. Present an algorithm (Groote, Veduzsco, de Vink) to efficiently compute probabilistic bisimulation

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Why is this relevant?

Probabilistic bisimulation is constructed to maintain equivalence under the logic PCTL (covered in another presentation)
$\longrightarrow$ Use the quotient TS to reduce the amount of states
$\longrightarrow$ Useful for Model Checking

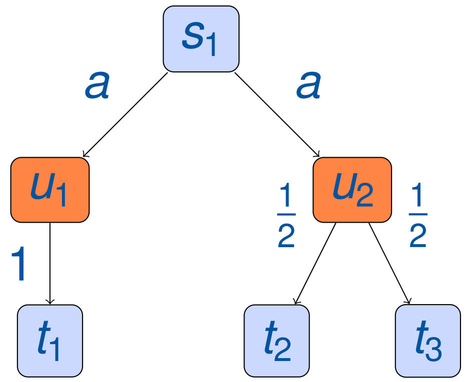
Outline

Introduction

Probabilistically labeled transition system (PLTS)

A probabilistically labeled transition system (PLTS) with a set of actions *Act* is a tuple $\mathcal{A} = (S, \rightarrow)$ such that

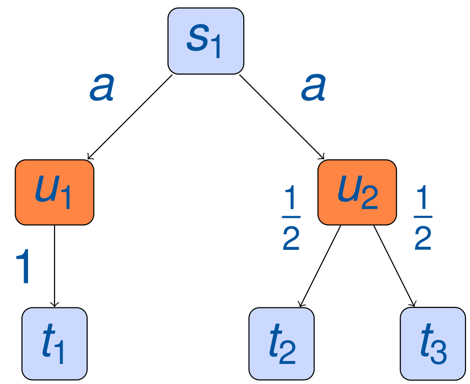
- S is a finite set, containing the states of the system
- $\mathcal{D}(S)$ is the set of probabilistic distributions over S
- $\rightarrow \subseteq S \times Act \times \mathcal{D}(S)$ is a finite relation, containing the transitions of the system



Introduction

Example

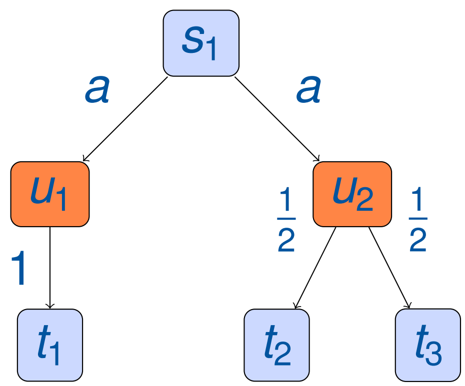
- $S = \{s_1, t_1, t_2, t_3\}$
- $Act = \{a\}$
- $\rightarrow = \{(s_1, a, u_1), (s_1, a, u_2)\}$
- u_1 denotes the distribution $u_1 : S \rightarrow [0, 1]$ with $u_1(t_1) = 1$
- u_2 denotes the distribution $u_2 : S \rightarrow [0, 1]$ with $u_2(t_2) = 0.5$ and $u_2(t_3) = 0.5$



Introduction

Notation

- S is called the set of action states
- For $f \in \mathcal{D}(S)$, we call u_f the associated probabilistic state
- The set of prob. states is called U
- We define $f[T] = \sum_{s \in T} f(s)$ for subsets $T \subseteq S$



Example of a PLTS

Definition - Probabilistic bisimulation

Let $\mathcal{A} = (S, \rightarrow)$ be a PLTS.

An equivalence relation $\sim$ on S is called a **probabilistic bisimulation** if and only if:

- For all states $s, t \in S$ such that $s \sim t$ and $s \xrightarrow{a} f$ for some $a \in \mathrm{Act}$ and $f \in \mathcal{D}(S)$ there is a $g \in \mathcal{D}(S)$ such that $t \xrightarrow{a} g$ and $f[B] = g[B] \quad \forall B \in S/\!\sim$

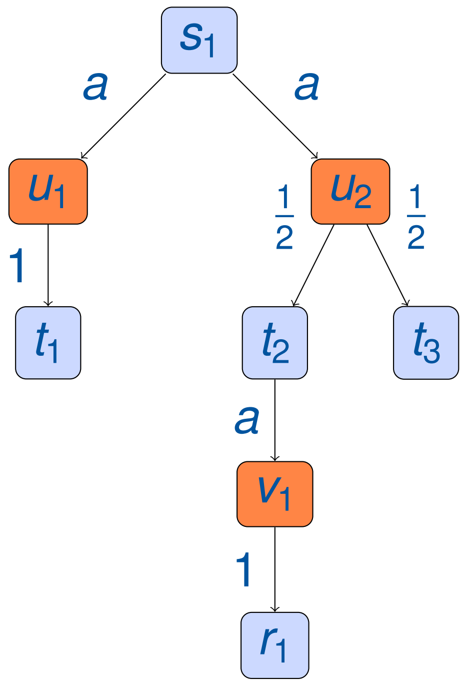
Two **action states** $s, t \in S$ are called probabilistically bisimilar if and only if:

- There is a probabilistic bisimulation $\sim$ such that $s \sim t$

Two **probabilistic states** u_f, u_g are called probabilistically bisimilar if and only if:

- There is a probabilistic bisimulation $\sim$ such that for all sets $B \in S/\!\sim$ it holds that $f[B] = g[B]$

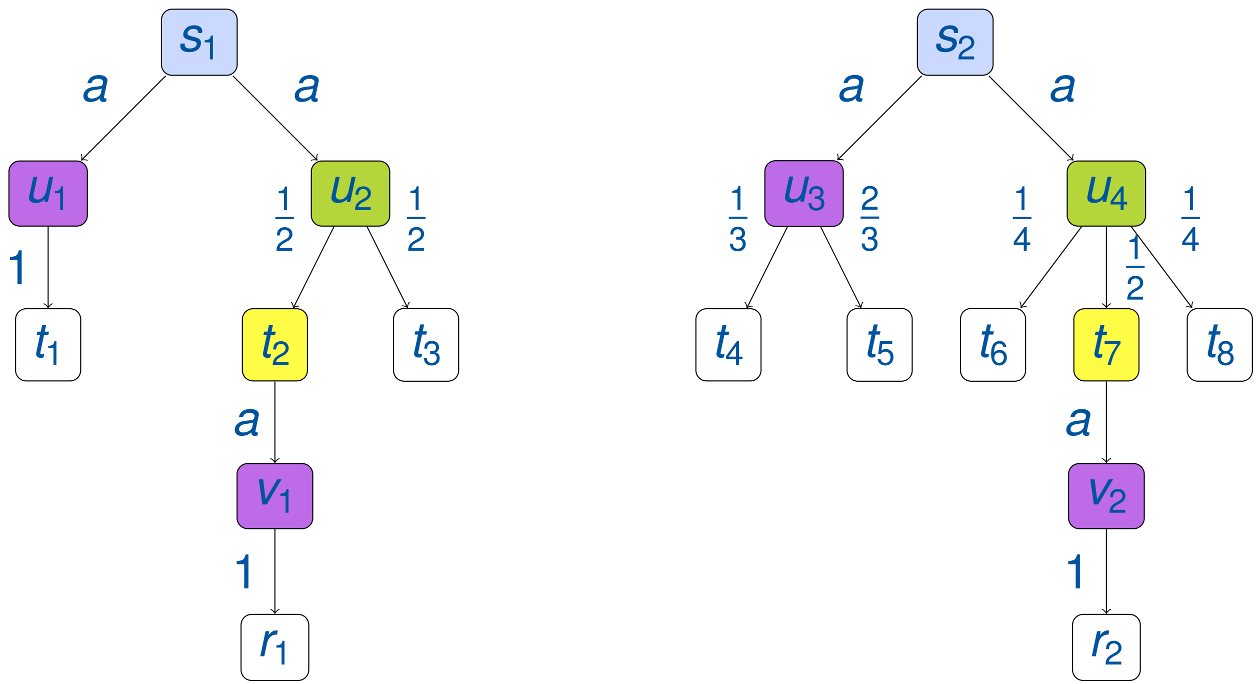
Probabilistic bisimulation



Two probabilistically bisimilar nondeterministic transition systems

Introduction

Probabilistic bisimulation



Two probabilistically bisimilar nondeterministic transition systems

Outline

Outline

Algorithm

General remarks

- View bisimilarity as a partition on the set $S \cup U$
- Partition refinement algorithm
- Keep two partitions:
  - $\mathcal{C}$ - The set of constellations
  - $\mathcal{B}$ - The set of blocks which will later be our bisimulation
  - $\mathcal{B}$ will always be a refinement of $\mathcal{C}$

Algorithm

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Structure of the algorithm

1. Initialize $\mathcal{B}$ and $\mathcal{C}$ with trivial partitions
2. While $\mathcal{B}$ is not equal to $\mathcal{C}$
   - Refine the partitions $\mathcal{B}$ and $\mathcal{C}$ according to certain criteria
   - Both partitions $\mathcal{B}$ and $\mathcal{C}$ will be more fine-grained
   - $\mathcal{B}$ will always be a refinement of $\mathcal{C}$

Algorithm

Preliminaries

- A set of action states B is called **stable** under a set of probabilistic states $C \subseteq U$ iff for all actions $a \in Act$ it holds that:

$$s \xrightarrow{a} C \iff t \xrightarrow{a} C \quad \forall s, t \in B$$

- A set of probabilistic states $B \subseteq U$ is called **stable** under a set of action states $C \subseteq S$ iff

$$u[C] = v[C] \quad \forall u, v \in B$$

- A partition $\mathcal{B}$ is called stable under another partition $\mathcal{C}$, if each block $B \in \mathcal{B}$ is stable under all sets of action states or probabilistic states, respectively.

Algorithm

Algorithm 1 Abstract partition refinement algorithm for probabilistic bisimulation

function PARTITION REFINEMENT
- $\mathcal{C} := \{S, U\}$
- $\mathcal{B} := \{U\} \cup \{S_A \mid A \subseteq Act\}$
- where $S_A = \left\{ s \in S \mid \forall a \in Act \left(\exists u \in U : s \xrightarrow{a} u \iff a \in A \right) \right\}$

Initialization

- **while** $\mathcal{C}$ contains a non-trivial constellation C **do**

Termination condition

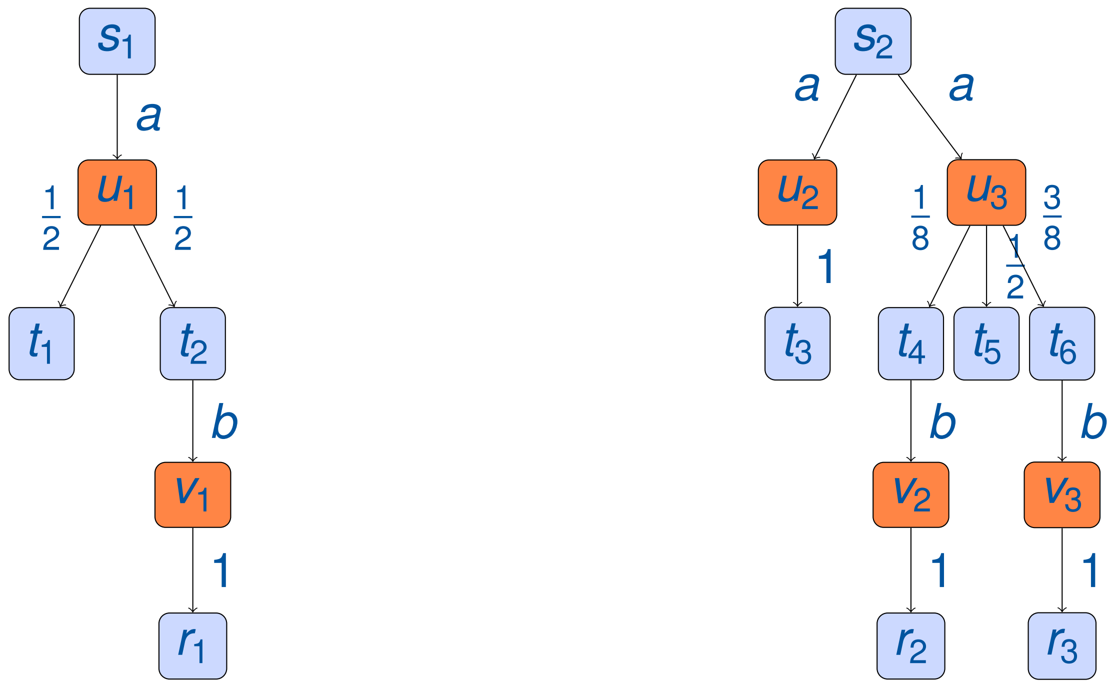
- choose block B_C from $\mathcal{B}$ in C
- replace in $\mathcal{C}$ constellation C by B_C and $C \backslash B_C$
- **if** C contains probabilistic states **then**
  - **for all** blocks B of action states in $\mathcal{B}$ unstable under B_C or $C \backslash B_C$ **do**
    - refine $\mathcal{B}$ by splitting B
    - into blocks of states with the same actions into B_C and $C \backslash B_C$
- **else**
  - **for all** blocks B of probabilistic states in $\mathcal{B}$ unstable under B_C **do**
    - refine $\mathcal{B}$ by splitting B
    - into blocks of states with equal probabilities into B_C

Refinement by splitting unstable sets

- **return** $\mathcal{B}$

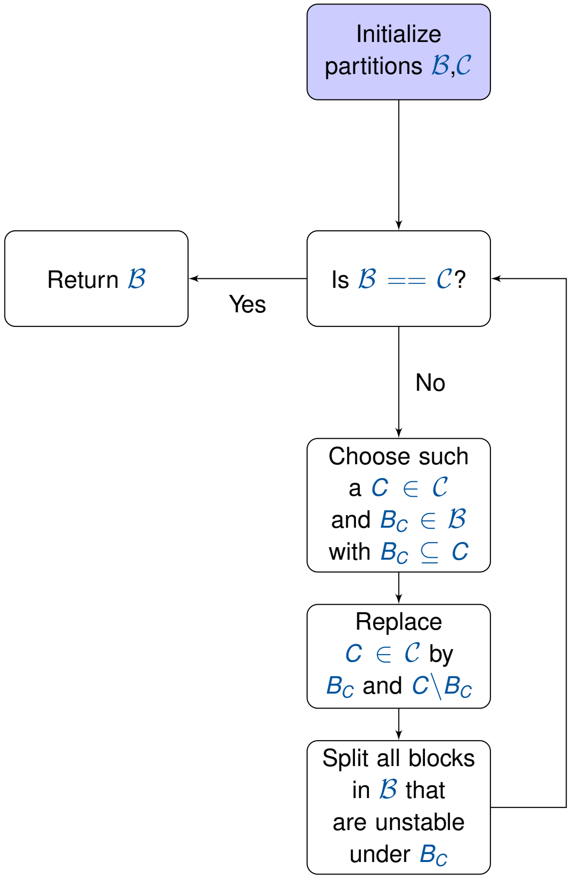
Outline

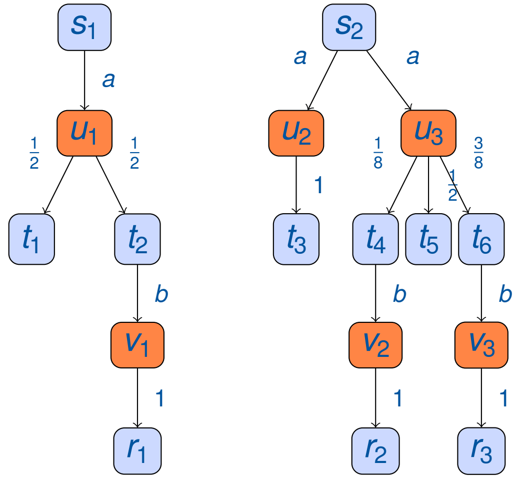
Example computation



Example PLTS for algorithm 1

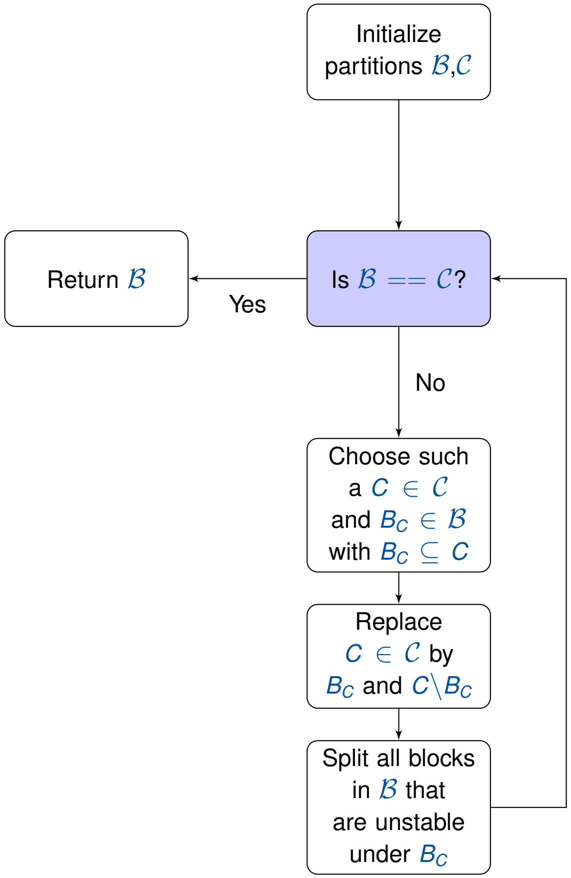
Algorithm

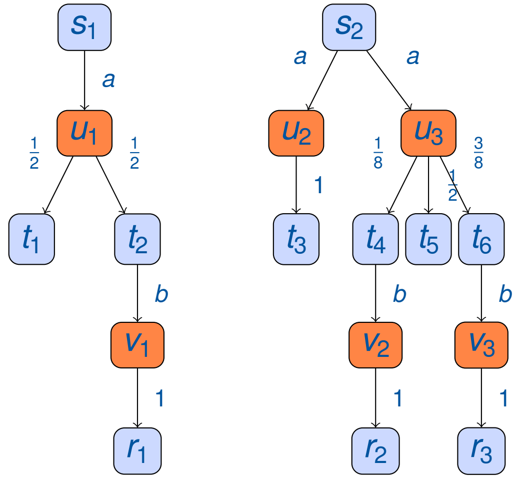




$$\mathcal{C} = \{U, S\}$$
$$= \{\{u_{1-3}, v_{1-3}\}, \{s_1, s_2, t_{1-6}, r_{1-3}\}\}$$
$$\mathcal{B} = \{U, S_{\{a\}}, S_{\{b\}}, S_{\varnothing}\}$$
$$= \{\{u_{1-3}, v_{1-3}\}, \{s_1, s_2\}, \{t_2, t_4, t_6\}, \{t_1, t_3, t_5, r_{1-3}\}\}$$

Algorithm





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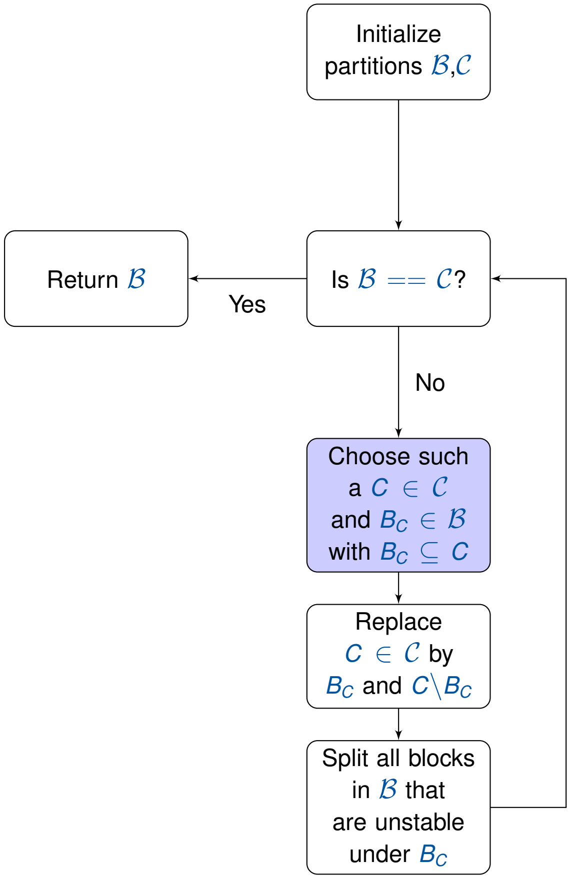
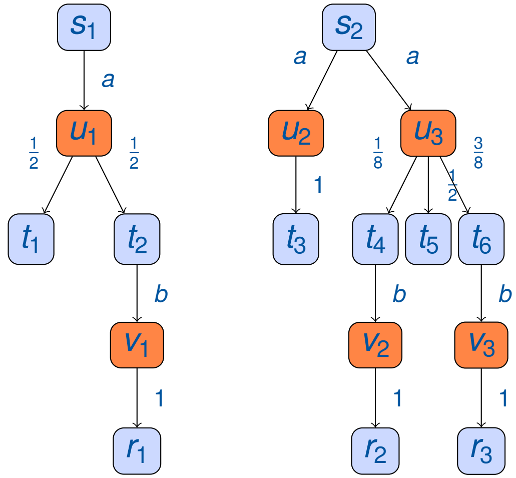
Algorithm





$$
\begin{aligned}
C &= S \\
B_C &= S_{\{a\}} \\
\mathcal{C} &= \{U, S\} \\
&= \{\{u_{1-3}, v_{1-3}\}, \{s_1, s_2, t_{1-6}, r_{1-3}\}\} \\
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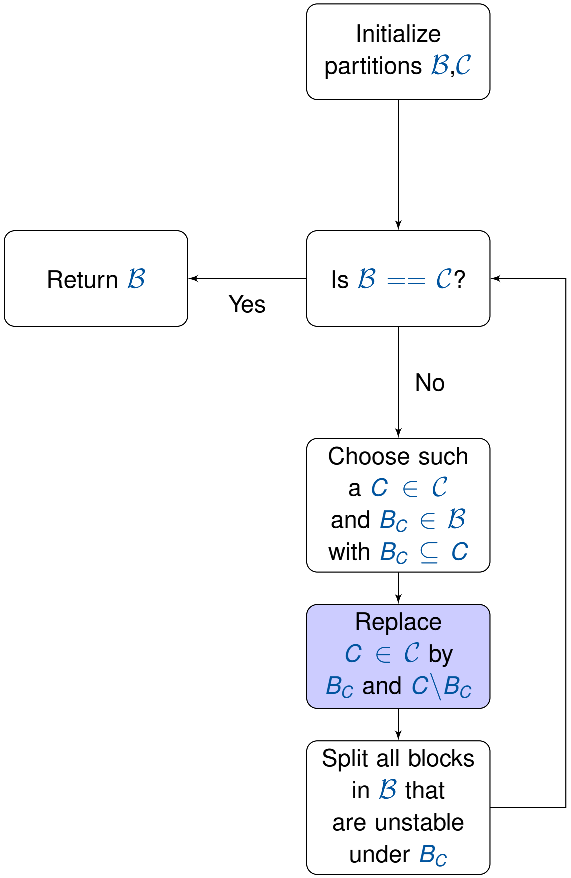
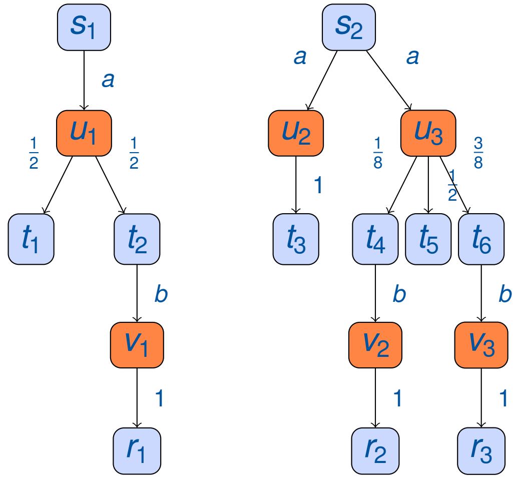
$$

Algorithm





$$C = S$$

$$B_C = S_{\{a\}}$$

$$\begin{aligned}\mathcal{C} &= \left\{U, S_{\{a\}}, S \setminus S_{\{a\}}\right\} \\ &= \left\{U, \{s_1, s_2\}, \{t_{1-6}, r_{1-3}\}\right\}\end{aligned}$$

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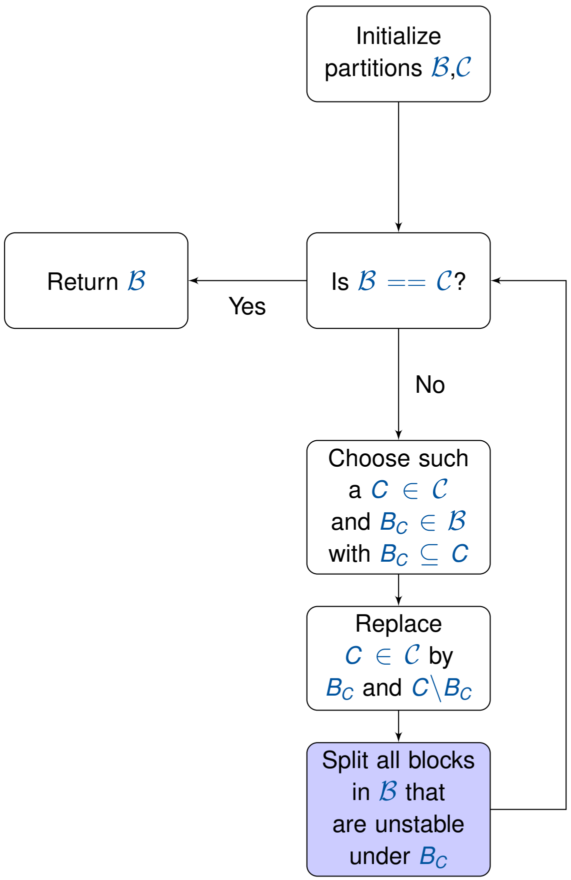
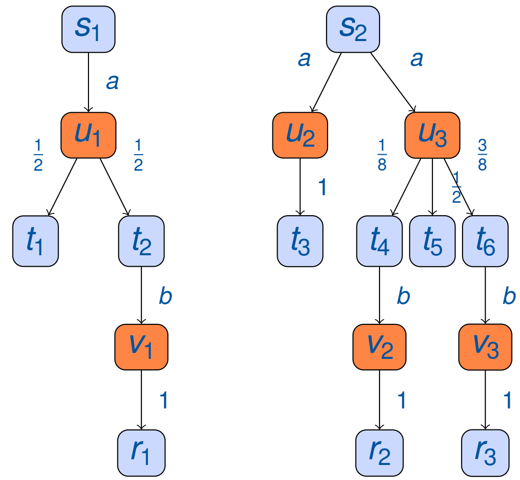
Algorithm





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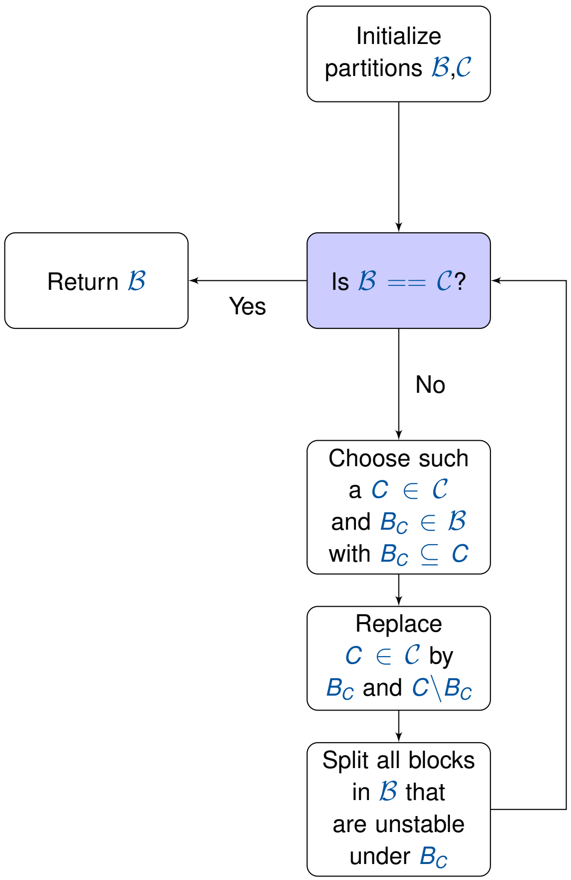

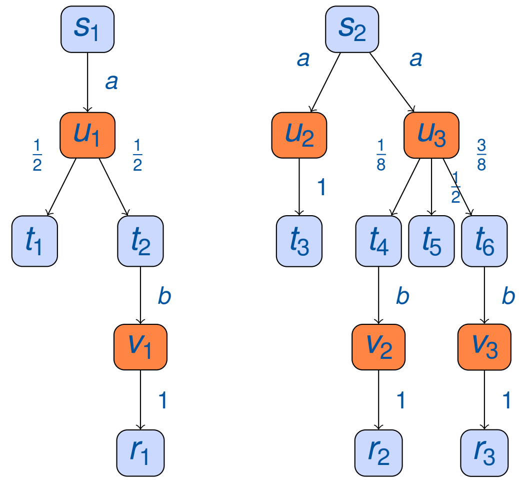

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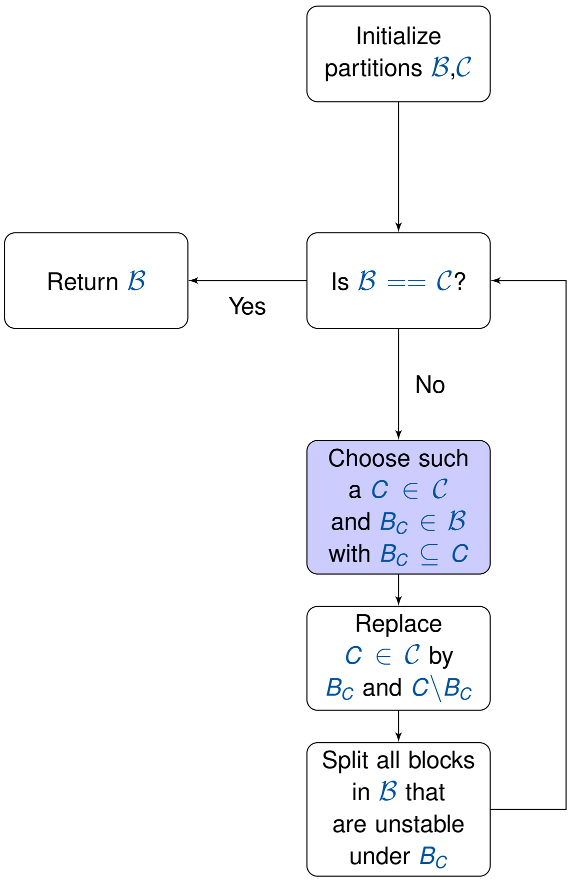

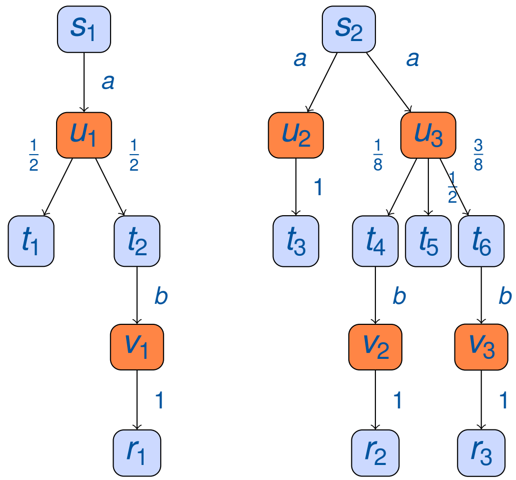

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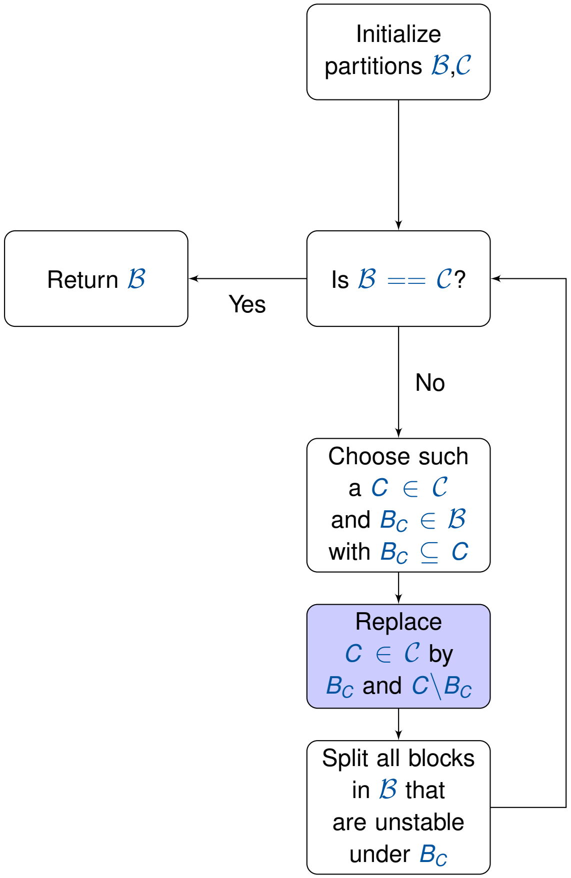

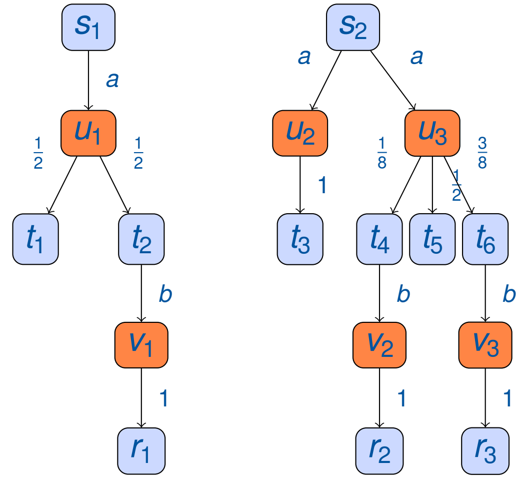

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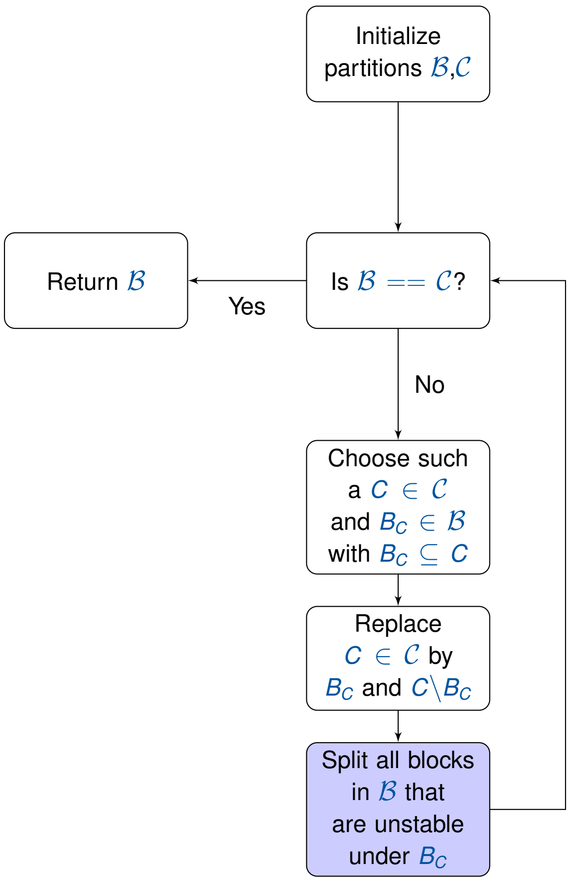

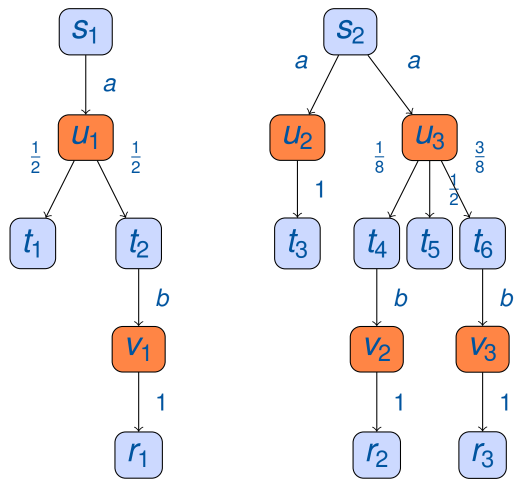

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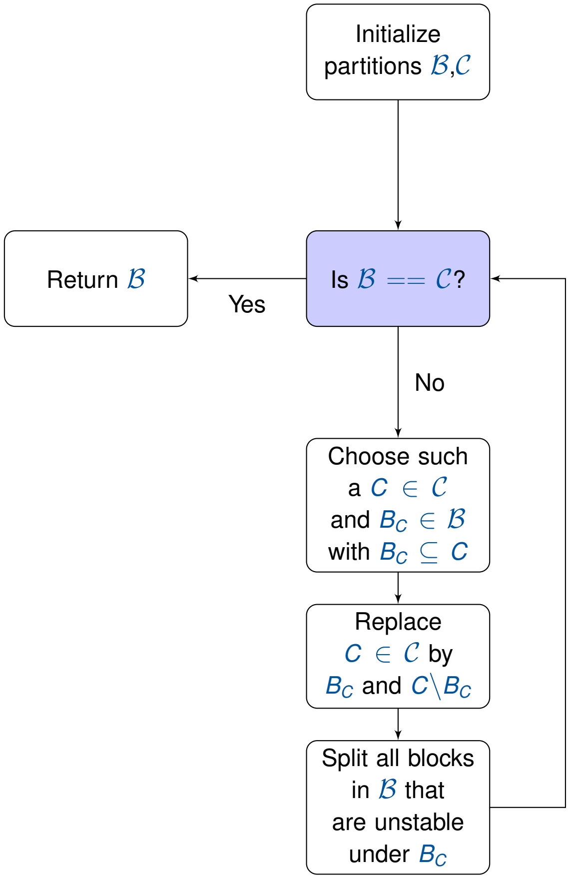

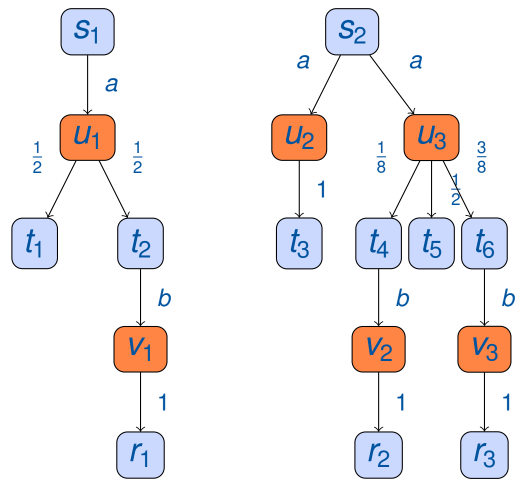

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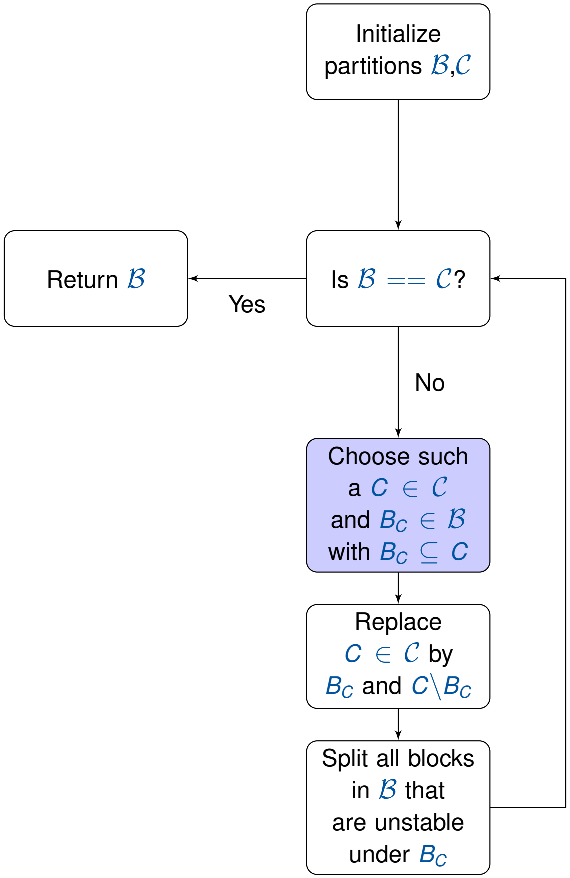

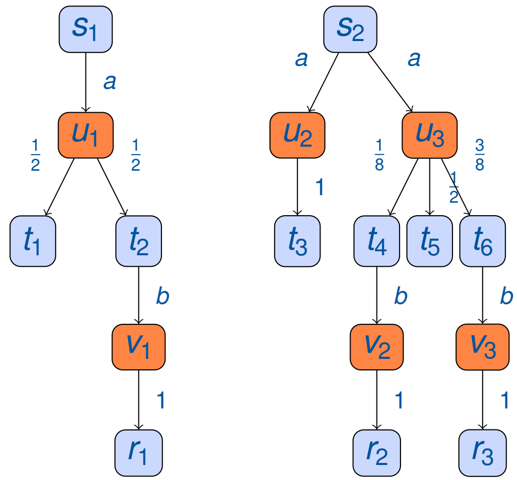

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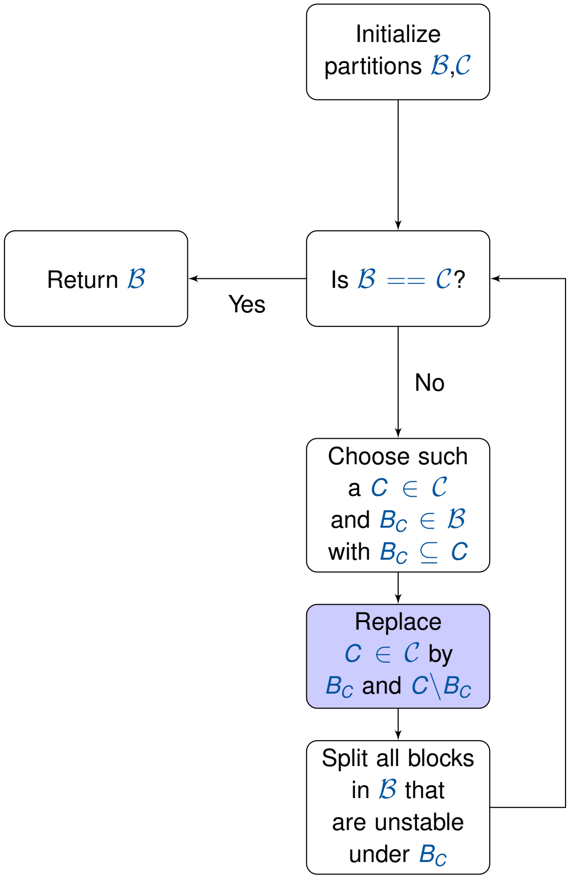

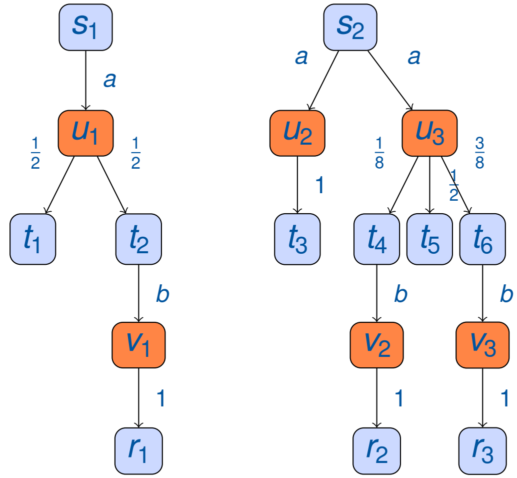

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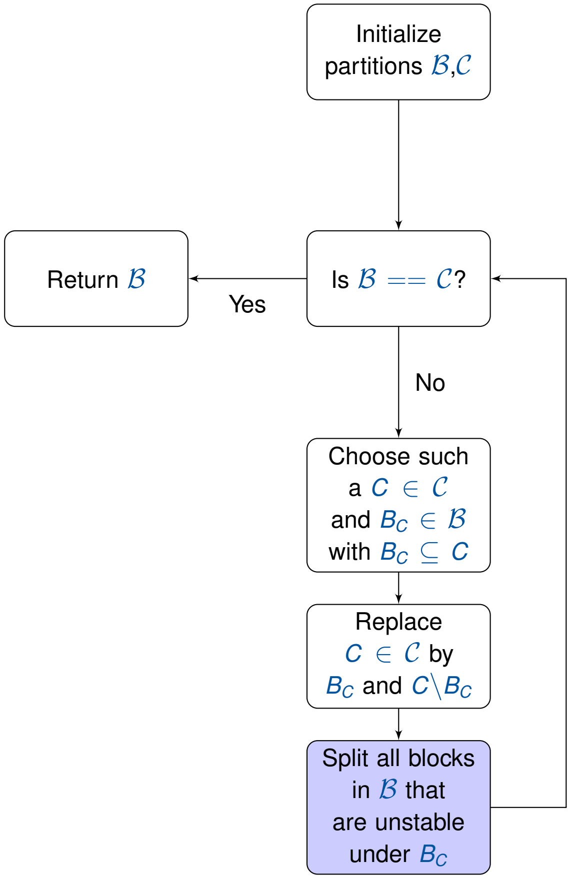

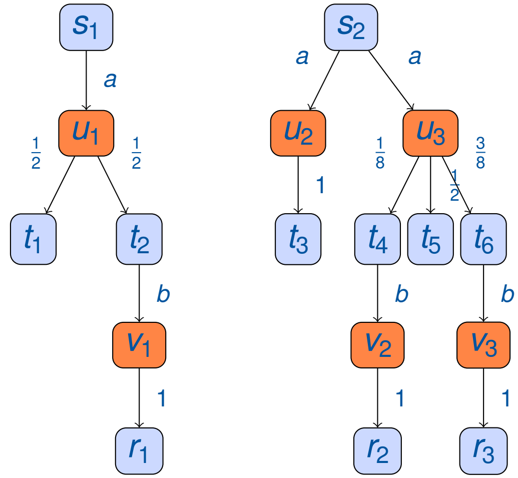

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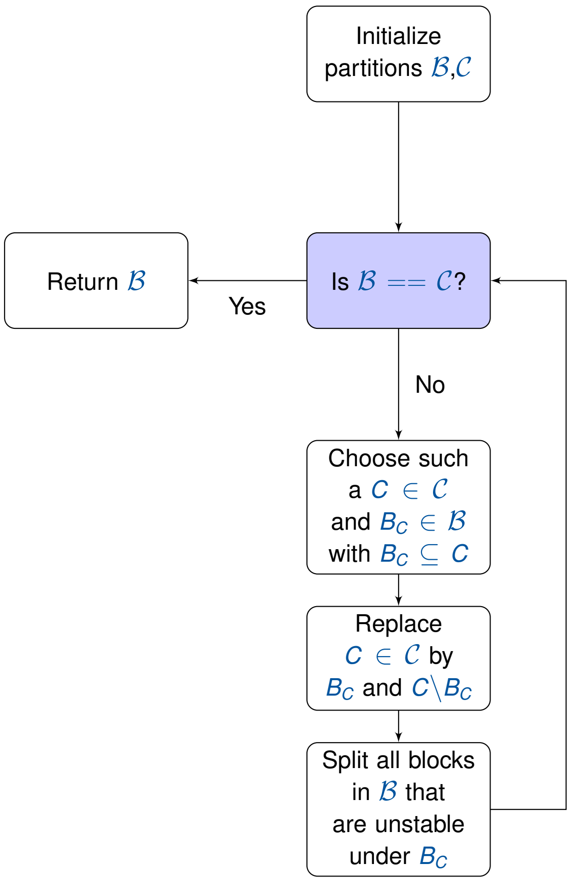
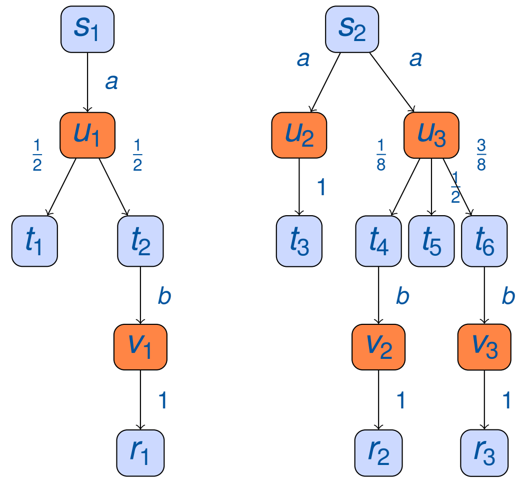
Algorithm





$$\mathcal{C} = \{\{u_1, u_3\}, \{u_2, v_{1-3}\}, \{s_1, s_2\}, \{t_2, t_4, t_6\}, \{t_1, t_3, t_5, r_{1-3}\}\}$$

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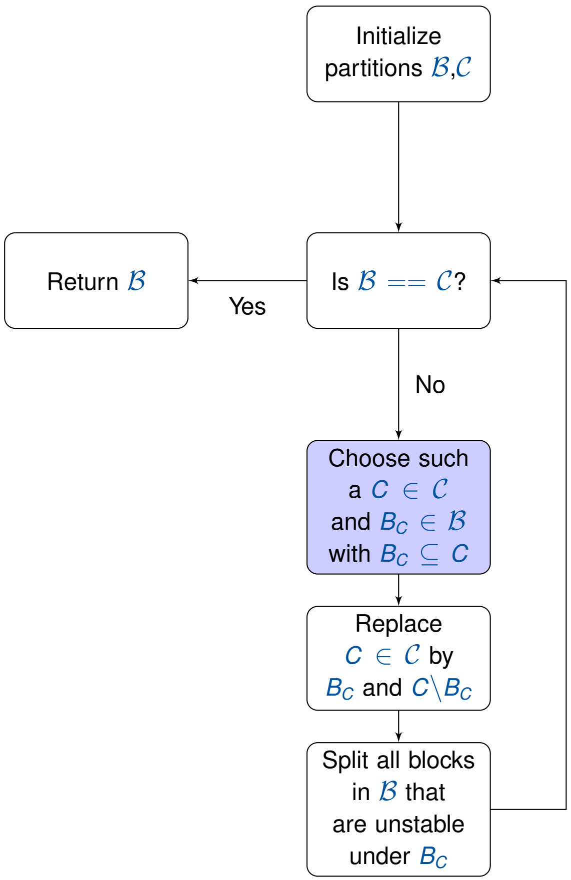
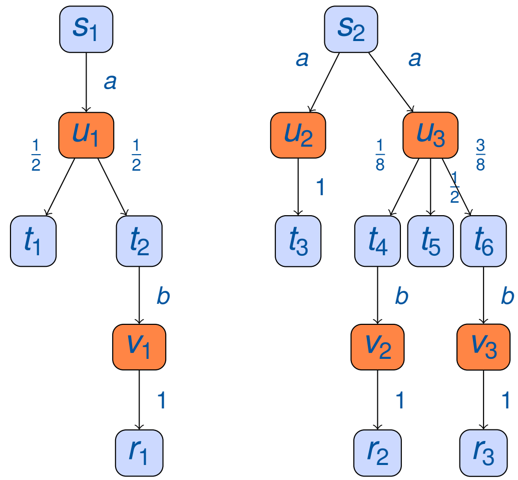
Algorithm





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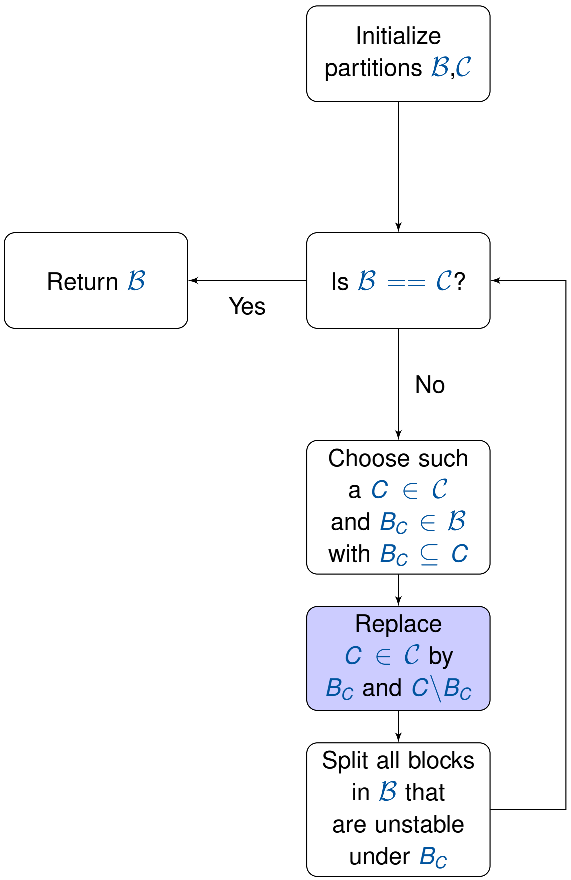

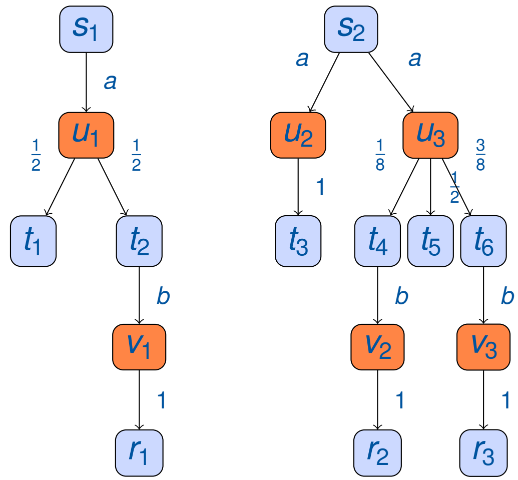

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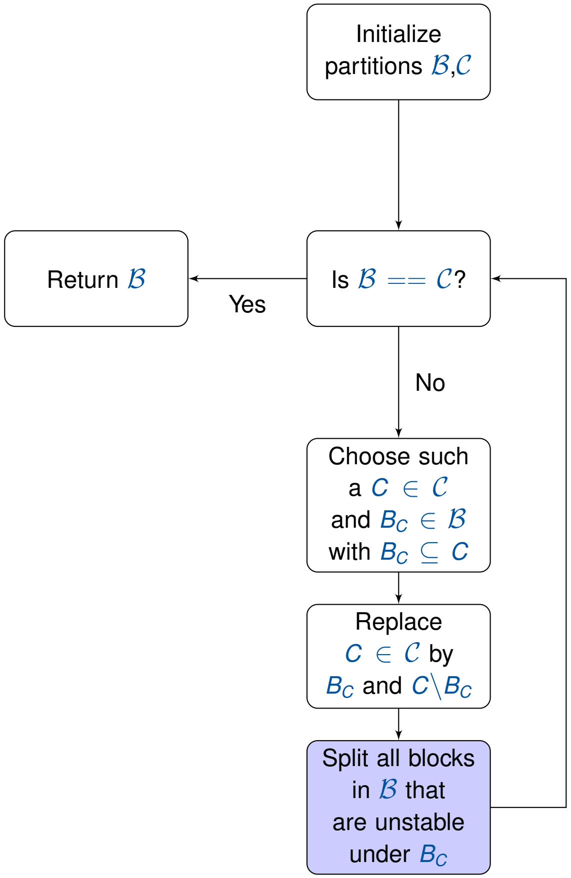

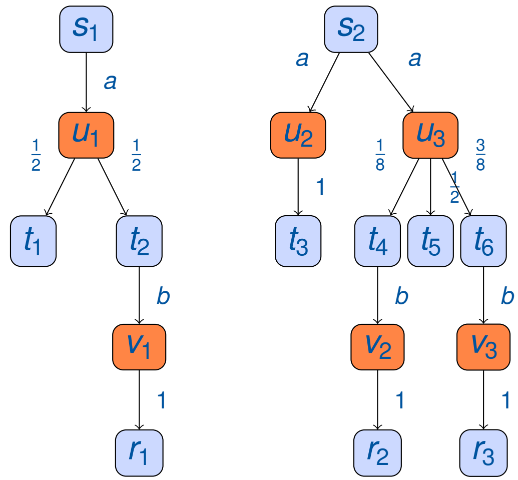

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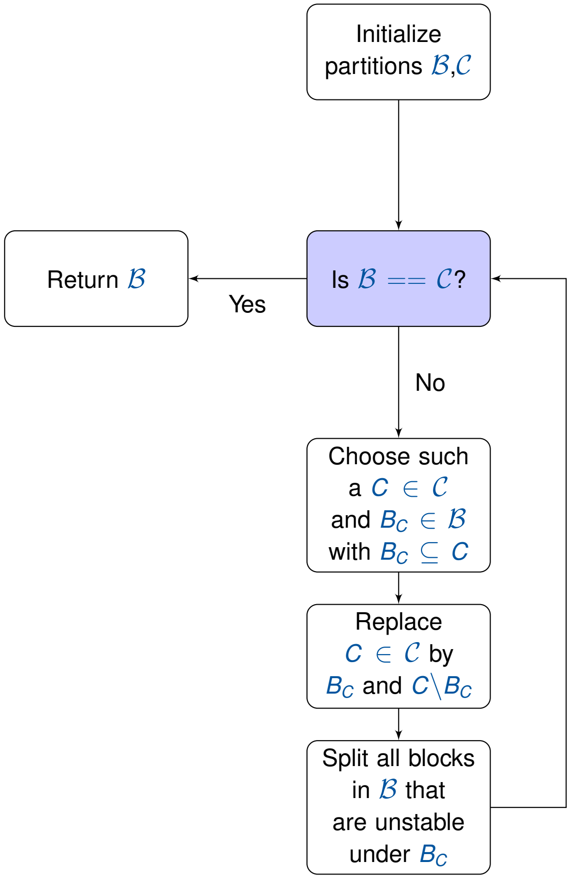

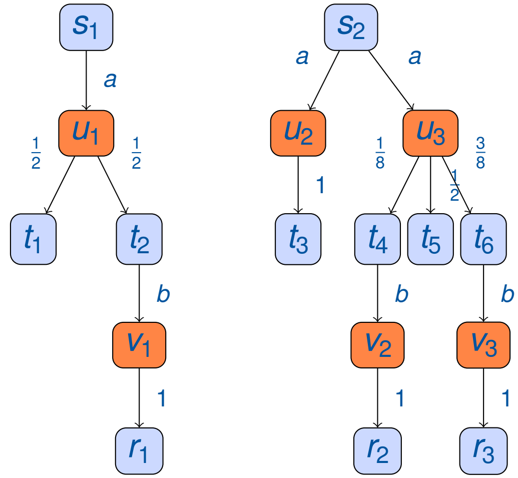

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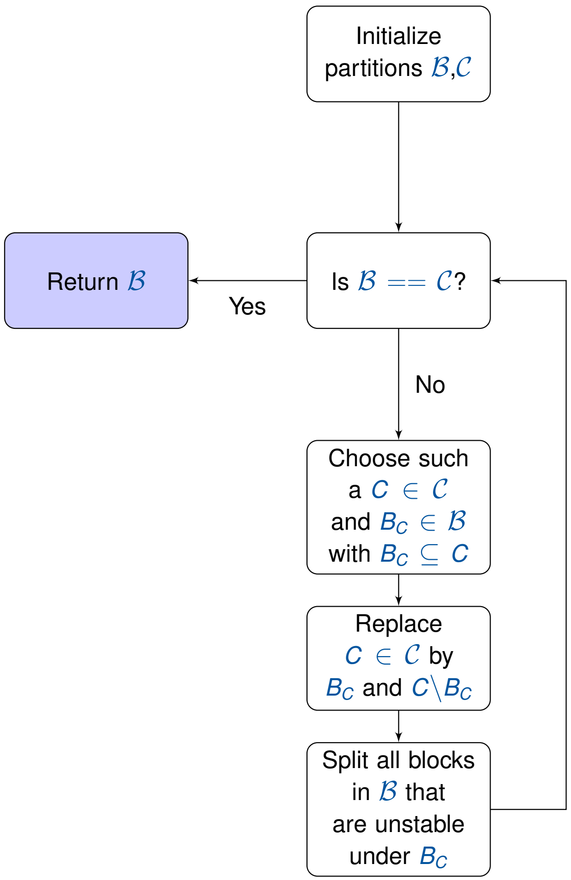

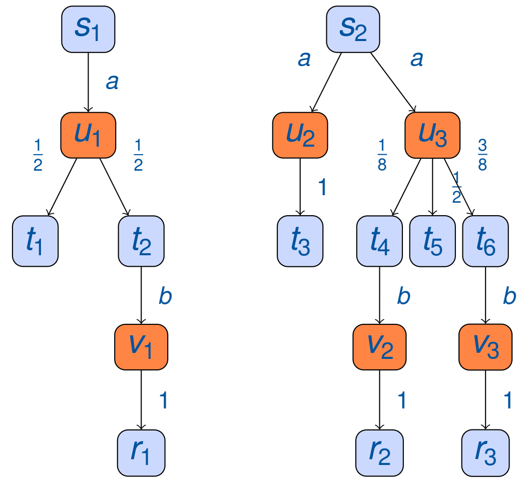

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Outline

Software Modeling and Verification Chair

RWTH AACHEN UNIVERSITY

Outline

Correctness

Invariant 1

Probabilistic bisimilarity $\simeq_p$ is a refinement of $\mathcal{B}$.

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Invariant 3

Partition $\mathcal{B}$ is stable under the set of constellations $\mathcal{C}$.

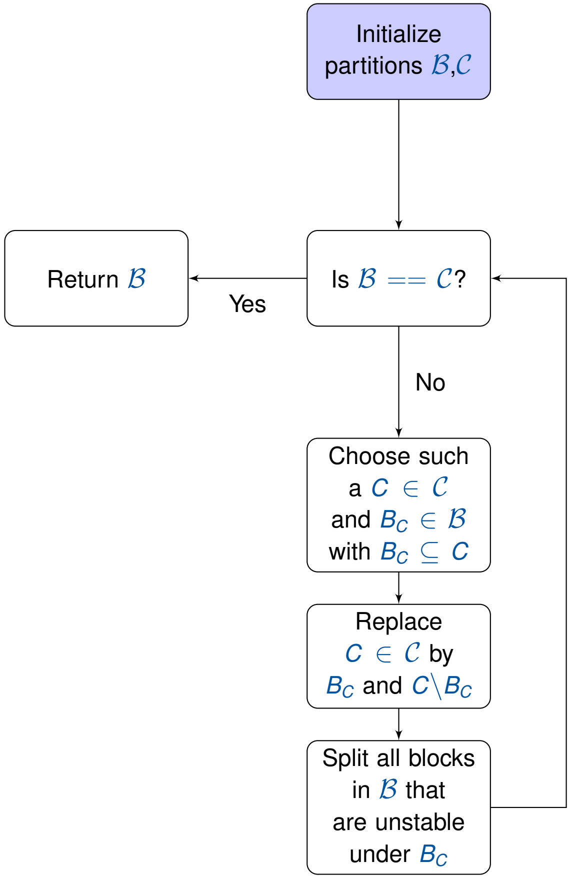
Lemma

If $\mathcal{B}$ is stable under itself, then $\sim_{\mathcal{B}}$ is a probabilistic bisimulation.

Outline

Software Modeling and Verification Chair
RWTH AACHEN UNIVERSITY

Complexity



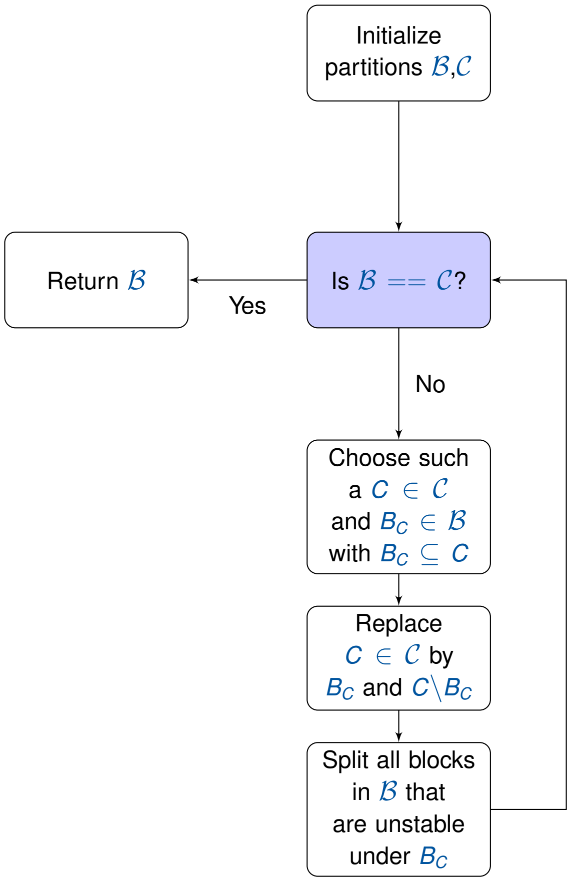
Notation

$n_a :=$ number of action states
$n_p :=$ number of probabilistic states
$m_a :=$ number of action transitions
$m_p :=$ number of probabilistic transitions

- Initialization: $O(1)$

Analysis

Complexity



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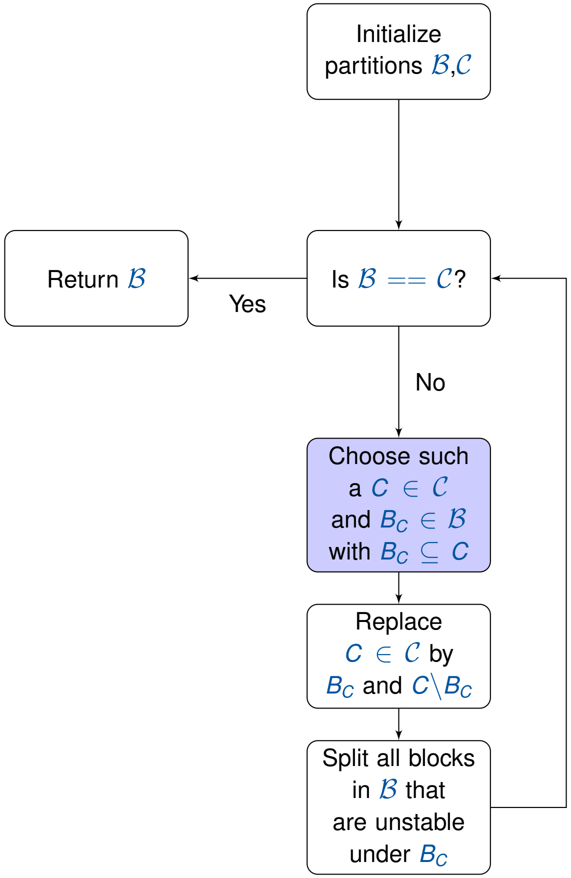
- Initialization: $O(1)$
- Checking invariant: $O(1)$

Complexity



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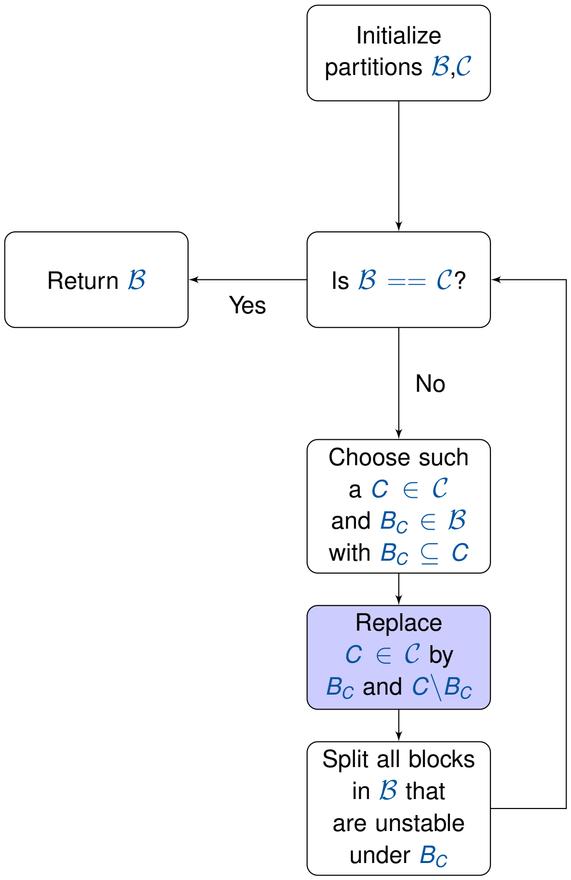
- Checking invariant: $O(1)$
- Choosing appropriate subsets: $O(1)$
  1. Keep a stack of constellations that need to be checked
  2. Choose $|B_C| \leq \frac{1}{2}|C|$
  3. Number of iterations $\leq \log(n_a) + \log(n_p)$

Complexity



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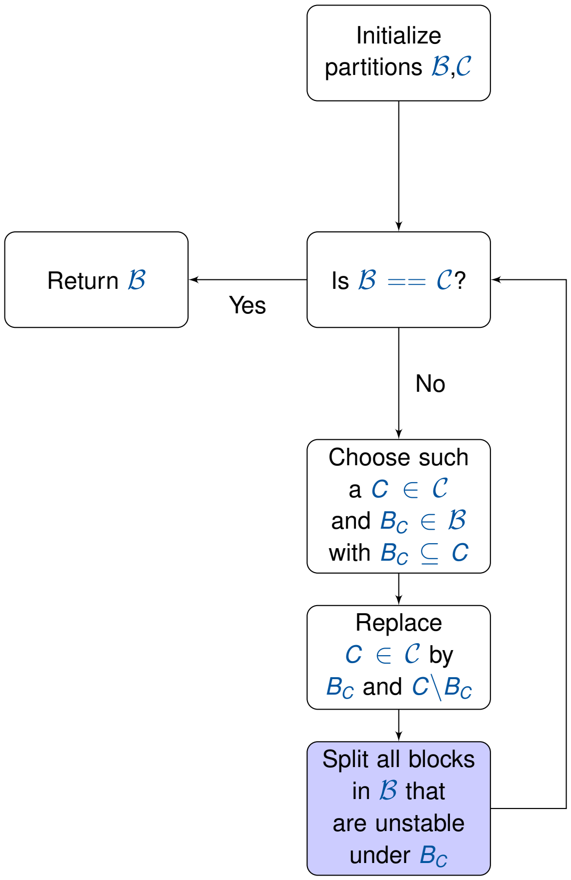
  3. Number of iterations $\leq \log(n_a) + \log(n_p)$
- Refining $\mathcal{C}$: $O(1)$

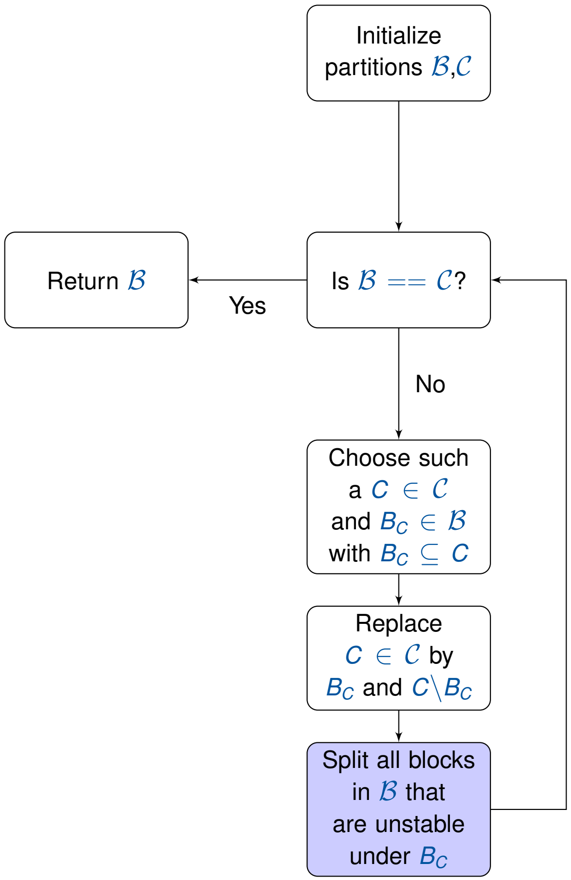
Complexity



- Refining $\mathcal{B}$:
  1. Mark unstable blocks B and determine the subsets B' that it has to be split into
  2. Create a new block for each B' for which $|B'| \leq \frac{1}{2}|B|$
  3. Each action state will only be moved $O(\log(n_a))$ times
  4. Each prob. state will only be moved $O(\log(n_p))$ times

Analysis

Complexity



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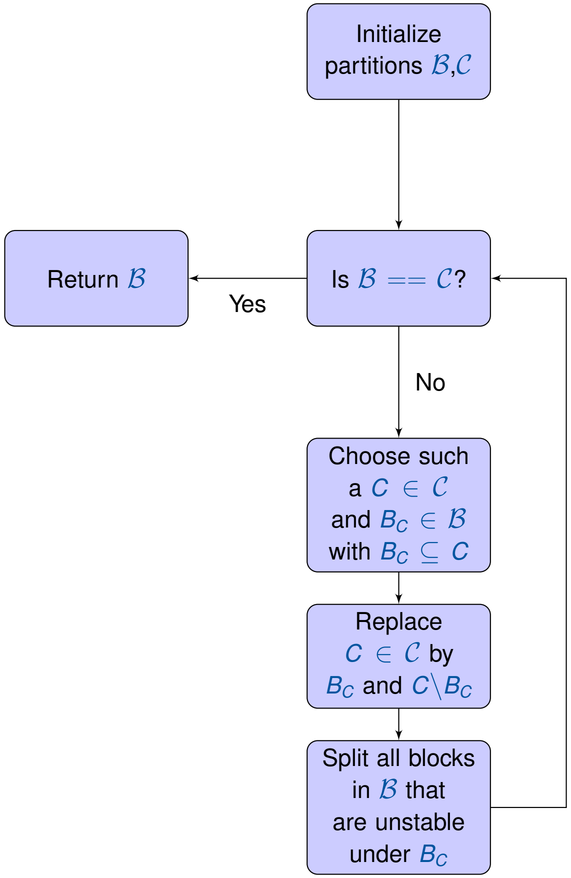
  3. Each action state will only be moved $O(\log(n_a))$ times
  4. Each prob. state will only be moved $O(\log(n_p))$ times
- Marking unstable blocks:
  1. This mainly works by iterating over incoming transitions to a block
  2. Marking procedures have complexity $O(\text{number of incoming transitions into } B_C)$
  3. Requires sorting that accumulates to $O(m_p \log(n_p))$

Complexity



Accumulated time complexity

In case all action states are reachable, the time complexity of the algorithm is $O((m_a + m_p) \log n_p + m_p \log n_a))$

Space complexity

All data structures are linear in the number of transitions and states. Thus, space complexity is $O\left(m_a + m_p\right)$

Outline

Outlook

Conclusion

The algorithm from Groote, Verduzco and de Vink is currently an efficient algorithm for computing probabilistic bisimilarity. It outperforms all previously known approaches.

Open research questions

- What is the best strategy of choosing blocks B_C? Right now, they are chosen non-deterministically
- Can this partition refinement approach be generalized for notions of equivalence other than bisimulation?

Outlook

References I

- C. Baier, B. Engelen, and M. Majster-Cederbaum.
  Deciding bisimilarity and similarity for probabilistic processes.
  Journal of Computer and System Sciences, 60(1):187 – 231, 2000.

- C. Baier and J.-P. Katoen.
  Principles of Model Checking.
  The MIT Press, 2008.

- J. Groote, H. Rivera Verduzco, and E. de Vink.
  An efficient algorithm to determine probabilistic bisimulation.
  Algorithms, 11(9):131, Sep 2018.

- R. Paige and R. E. Tarjan.
  Three partition refinement algorithms.
  SIAM Journal on Computing, 16(6):973–989, 1987.

Outlook

References II

- R. Segala and N. Lynch.
  Probabilistic simulations for probabilistic processes.
  In B. Jonsson and J. Parrow, editors, *CONCUR '94: Concurrency Theory*, pages 481–496, Berlin, Heidelberg, 1994. Springer Berlin Heidelberg.
- A. Turrini and H. Hermanns.
  Polynomial time decision algorithms for probabilistic automata.
  Information and Computation, 244:134 – 171, 2015.
- A. Valmari and G. Franceschinis.
  Simple o(m logn) time markov chain lumping.
  In J. Esparza and R. Majumdar, editors, *Tools and Algorithms for the Construction and Analysis of Systems*, pages 38–52, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.

Outline

Backup slides

Quotient transition system

For transition system $TS = (S, Act, \rightarrow, I, AP, L)$ and bisimulation $\sim_{TS}$, the quotient transition system $TS/\sim_{TS}$ is defined by

$$TS/\sim_{TS} = (S/\sim_{TS}, \tau, \longrightarrow', I', AP, L')$$

where:

- $I' = \{[s]_{\sim_{TS}} | \in I\}$
- $\rightarrow'$ is defined by

$$\frac{s \xrightarrow{\alpha} s'}{[s]_{\sim_{TS}} \xrightarrow{\tau} [s']_{\sim_{TS}}}$$

- $L'([s]_{\sim_{TS}}) = L(s)$

Bijection between equivalence relations and partitions

Let X be a finite set, $P \subseteq \mathcal{P}(X)$ a partition of X and $\sim$ an equivalence relation on X.

1. P induces an equivalence relation $\sim_P$ defined by $x \sim_P y \iff \exists A \in P : x \in A \land y \in A$
2. $\sim$ induces a partition $P_\sim := \{[x]_\sim \mid x \in X\}$ where $[x]_\sim$ denotes the equivalence class of x
3. There is a bijection between the set of equivalence relations on X and the set of partitions of X

Stability under itself

Lemma

Let $A = (S, \rightarrow)$ be a PLTS and $\mathcal{B}$ a partition of S.
If $\mathcal{B}$ is stable under itself, then the corresponding equivalence relation $\sim_B$ is a probabilistic bisimulation.

Proof

We check the definition of probabilistic bisimilarity:
Suppose two action states $s, t \in S$ are in the same equivalence class $B \in \mathcal{B}$. Let there be a transition $s \xrightarrow{a} u_f$ with $f \in \mathcal{D}(S)$. Then there is a block $B' \in P$ which contains u_f, so that $s \xrightarrow{a} B'$. Since $\mathcal{B}$ is stable under itself, B is stable under B'. Consequently, there is $g \in B'$ such that $t \xrightarrow{a} u_g$.
Now let $B'' \in P$ be an arbitrary block. Since B' is stable under B'', it holds that

$$u_f[B''] = u_g[B'']$$

Complete algorithm

Algorithm 2 Partition refinement algorithm for probabilistic bisimulation

function Partition Refinement $(S, U, \rightarrow)$	
$\mathcal{C} := \{S, U\}$	$O(n_a + n_p)$
$\mathcal{B} := \{U\} \cup \{S_A \mid A \subseteq Act\}$ where $S_A = \left\{ s \in S \mid \forall a \in Act \left(\exists u \in U : s \xrightarrow{a} u \iff a \in A \right) \right\}$	$O(n_p + n_a + m_a)$
group the incoming action transitions in each block per label	$O(m_a)$
initialise `state_to_constellation_cht` for each transition	$O(m_a)$
while $\mathcal{C}$ contains a non-trivial constellation C **do**	$\leq n$ iterations
choose a block B_C from $\mathcal{B}$ in C such that $\lvert B_C \rvert \leq \frac{1}{2}\lvert C \rvert$	
split constellation C into B_C and $C \backslash B_C$ in $\mathcal{C}$	$O(1)$
if C contains probabilistic states **then**	
for all incoming actions a of states in B_C **do**	$\leq \lvert Act \rvert$ iterations
$\langle \mathbf{B}_a, \text{left}_a, \text{mid}_a, \text{right}_a, \text{large}_a \rangle :=$aMark$(\mathcal{B}, C, B_C, a)$	$O($ nr of incoming a transitions in $B_C)$
for all blocks $B \in \mathbf{B}_a$ **do**	
for all non-empty subsets $B' \subseteq B$, different from $\text{large}_a(B)$ in $\{\text{left}(B)_a, \text{mid}_a(B), \text{right}_a(B)\}$ **do**	$O($ nr of incoming a transitions in $B_C)$
move B' out of B and add B' as new block to $\mathcal{B}$	$O($ nr of incoming transitions in $B')$
else	$O(\text{nr of incoming prob. transitions in } B_C)$ plus
$\langle \mathbf{B}_p, \text{left}_p, \text{mid}_p, \text{right}_p, \text{large}_p \rangle :=$ pMark $(\mathcal{B}, C, B_C)$	a sorting penalty
for all blocks $B \in \mathbf{B}_p$ **do**	
for all non-empty sets of states $B' \subseteq B$ not equal to $\text{large}_p(B)$ in $\{\text{left}_p(B)\} \cup \text{mid}_p(B) \cup \{\text{right}(B)_p\}$ **do**	$O($ nr of incoming prob. transitions in $B_C)$
move B' out of B and add B' as a new block to $\mathcal{B}$	$O($ nr of incoming transitions in $B')$
return $\mathcal{B}$	