Converting Tensors from Cartesian to Cylindrical

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From `http://solidmechanics.org/text/AppendixD/AppendixD.htm` we get:

$$\begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{\theta z} & S_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{\theta z} & S_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Take the first two matrices and multiply them:
Note: to ease space constraints, the following translations have been defined:

$$\cos\theta = \mathbf{c}_\theta \tag{2}$$

$$\sin\theta = \mathbf{s}_\theta \tag{3}$$

$$\begin{aligned} &\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{\theta z} & S_{zz} \end{bmatrix} \\ &\implies \begin{bmatrix} \mathbf{c}_\theta S_{xx} + \mathbf{s}_\theta S_{yx} + 0 & \mathbf{c}_\theta S_{xy} + \mathbf{s}_\theta S_{yy} + 0 & \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} + 0 \\ -\mathbf{s}_\theta S_{xx} + \mathbf{c}_\theta S_{yx} + 0 & -\mathbf{s}_\theta S_{xy} + \mathbf{c}_\theta S_{yy} + 0 & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} + 0 \\ 0 + 0 + S_{zx} & 0 + 0 + S_{zy} & 0 + 0 + S_{zz} \end{bmatrix} \\ &\implies \begin{bmatrix} \mathbf{c}_\theta S_{xx} + \mathbf{s}_\theta S_{yx} & \mathbf{c}_\theta S_{xy} + \mathbf{s}_\theta S_{yy} & \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} \\ -\mathbf{s}_\theta S_{xx} + \mathbf{c}_\theta S_{yx} & -\mathbf{s}_\theta S_{xy} + \mathbf{c}_\theta S_{yy} & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} \end{aligned} \tag{4}$$

Multiply the result by the third matrix:

$$\begin{aligned} &\begin{bmatrix} \mathbf{c}_\theta S_{xx} + \mathbf{s}_\theta S_{yx} & \mathbf{c}_\theta S_{xy} + \mathbf{s}_\theta S_{yy} & \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} \\ -\mathbf{s}_\theta S_{xx} + \mathbf{c}_\theta S_{yx} & -\mathbf{s}_\theta S_{xy} + \mathbf{c}_\theta S_{yy} & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &\begin{bmatrix} \mathbf{c}_\theta^2 S_{xx} + \mathbf{s}_\theta \mathbf{c}_\theta S_{yx} + \mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{s}_\theta^2 S_{yy} & -\mathbf{c}_\theta \mathbf{s}_\theta S_{xx} - \mathbf{s}_\theta^2 S_{yx} + \mathbf{c}_\theta^2 S_{xy} + \mathbf{c}_\theta \mathbf{s}_\theta S_{yy} & \mathbf{c}_\theta S_{xy} + \mathbf{s}_\theta S_{yz} \\ -\mathbf{s}_\theta \mathbf{c}_\theta S_{xx} + \mathbf{c}_\theta^2 S_{yx} - \mathbf{s}_\theta^2 S_{xy} + \mathbf{c}_\theta \mathbf{s}_\theta S_{yy} & \mathbf{s}_\theta^2 S_{xx} - \mathbf{c}_\theta \mathbf{s}_\theta S_{yx} - \mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{c}_\theta^2 S_{yy} & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} \\ \mathbf{c}_\theta S_{zx} + \mathbf{s}_\theta S_{zy} & -\mathbf{s}_\theta S_{zx} + \mathbf{c}_\theta S_{zy} & S_{zz} \end{bmatrix} \end{aligned} \tag{5}$$

Combine like terms, assuming that the tensor is symmetric (ie. $S_{ij} = S_{ji}$):

$$(5) \implies \begin{bmatrix} \mathbf{c}_\theta^2 S_{xx} + 2\mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{s}_\theta^2 S_{yy} & -\mathbf{c}_\theta \mathbf{s}_\theta S_{xx} + (\mathbf{c}_\theta^2 - \mathbf{s}_\theta^2) S_{xy} + \mathbf{c}_\theta \mathbf{s}_\theta S_{yy} & \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} \\ -\mathbf{c}_\theta \mathbf{s}_\theta S_{xx} + (\mathbf{c}_\theta^2 - \mathbf{s}_\theta^2) S_{xy} + \mathbf{c}_\theta \mathbf{s}_\theta S_{yy} & \mathbf{s}_\theta^2 S_{xx} - 2\mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{c}_\theta^2 S_{yy} & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} \\ \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} & S_{zz} \end{bmatrix} \tag{6}$$

This can be further simplified by combinging the $\mathbf{c}_\theta \mathbf{s}_\theta$ groups:

$$\begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{\theta z} & S_{zz} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_\theta^2 S_{xx} + 2\mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{s}_\theta^2 S_{yy} & \mathbf{c}_\theta \mathbf{s}_\theta (S_{yy} - S_{xx}) + (\mathbf{c}_\theta^2 - \mathbf{s}_\theta^2) S_{xy} & \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} \\ \mathbf{c}_\theta \mathbf{s}_\theta (S_{yy} - S_{xx}) + (\mathbf{c}_\theta^2 - \mathbf{s}_\theta^2) S_{xy} & \mathbf{s}_\theta^2 S_{xx} - 2\mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{c}_\theta^2 S_{yy} & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} \\ \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} & -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} & S_{zz} \end{bmatrix} \tag{7}$$

To list off the unique components:

$$S_{rr} = \mathbf{c}_\theta^2 S_{xx} + 2\mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{s}_\theta^2 S_{yy} \tag{8}$$

$$S_{r\theta} = \mathbf{c}_\theta \mathbf{s}_\theta (S_{yy} - S_{xx}) + (\mathbf{c}_\theta^2 - \mathbf{s}_\theta^2) S_{xy} \tag{9}$$

$$S_{rz} = \mathbf{c}_\theta S_{xz} + \mathbf{s}_\theta S_{yz} \tag{10}$$

$$S_{\theta\theta} = \mathbf{s}_\theta^2 S_{xx} - 2\mathbf{c}_\theta \mathbf{s}_\theta S_{xy} + \mathbf{c}_\theta^2 S_{yy} \tag{11}$$

$$S_{\theta z} = -\mathbf{s}_\theta S_{xz} + \mathbf{c}_\theta S_{yz} \tag{12}$$

$$S_{zz} = S_{zz} \tag{13}$$