#### PROBLEM 1A

*Claim.* The set of rational numbers (Q) is countable.

*Proof.* Let  $x \in \mathbb{Q}$ . Then, by definition, we can write  $x = \frac{a}{b}$  for some  $a, b \in \mathbb{Z}, b \neq 0$ . This fraction can be uniquely mapped to a tuple  $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ . This means that there exists an injective mapping  $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$  that maps each fraction to a tuple. Thus, we have:

$$|\mathbb{Q}| \le |\mathbb{Z} \times \mathbb{Z}| \le |\mathbb{N}|$$

By the countability of  $\mathbb{N}$ , we have shown that  $\mathbb{Q}$  is countable.

### PROBLEM 1B

Let *E* be an event, and let  $1_E : \Omega \to \{0,1\}$  be an indicator random variable such that:

$$1_E(\omega) = \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E \end{cases}$$

*Claim.* The expectation of  $1_E$  is equal to the probability that E occurs. In other words:

$$\mathbb{E}\left[1_{E}\right] = \mathbb{P}\left[E\right]$$

Proof.

$$\mathbb{E} [1_E] = \mathbb{P} [1_E = 1] \cdot 1 + \mathbb{P} [1_E = 0] \cdot 0$$

$$= \mathbb{P} [1_E = 1]$$

$$= \mathbb{P} [\{\omega \in \Omega : 1_E(\omega) = 1\}]$$

$$= \mathbb{P} [E]$$

### PROBLEM 2

*Claim.*  $e^{iy} = \cos y + i \sin y$  for  $y \in \mathbb{R}$ 

*Proof.* We begin by noting that  $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ . Plugging in x = yi for the exponent in  $e^x$  gives:

$$\begin{split} e^{yi} &= \sum_{n=0}^{\infty} \frac{(yi)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{y^{4n}}{(4n)!} + \sum_{n=0}^{\infty} i \cdot \frac{y^{4n+1}}{(4n+1)!} + \sum_{n=0}^{\infty} -1 \cdot \frac{y^{4n+2}}{(4n+2)!} + \sum_{n=0}^{\infty} -i \cdot \frac{y^{4n+3}}{(4n+3)!} \\ &= \sum_{n=0}^{\infty} \frac{y^{4n}}{(4n)!} - \frac{y^{4n+2}}{(4n+2)!} + i \left( \sum_{n=0}^{\infty} \frac{y^{4n+1}}{(4n+1)!} - \frac{y^{4n+3}}{(4n+3)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \left( \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \right) \\ &= \cos y + i \sin y \end{split}$$

This equality holds because the series representations of sin and cos are convergent. Thus,  $e^{iy} = \cos y + i \sin y$  for  $y \in \mathbb{R}$ , as required.

# Problem 3

Claim.  $P \neq NP$ 

*Proof.* We have already shown that  $P \subseteq NP$ , so to prove that  $NP \neq P$  it suffices to show that  $NP \not\subseteq P$ . We begin by fixing a language  $L \in NP$ .

TO-DO

## PROBLEM 4

*Claim.* Two identical decks of *n* cards have a *k*-matching with probability:

$$\pi_k = \frac{1}{k!} \left( 1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!} \right)$$

*Proof.* First, we pick and order k cards to be matched. The probability of the selected card orders matching is  $\frac{1}{k!}$ .

Now, we consider the probability that the remaining n-k cards do not match. Similar to the examples from class and the textbook, this is an instance of an indexed union of sets  $A_i \in \mathcal{F}$  such that  $A_i$  is the set of permutations in which f(i) = i. But since we are looking for the probability of this not happening, we consider the probability of the complement, computed as 1 minus the probability of this union of events.

By the Inclusion-Exclusion principle, based on the notes from class, we have that:

$$\mathbb{P}\left[\bigcup_{i=1}^{n-k} A_i\right] = \sum_{i=1}^{n-k} \frac{(-1)^i}{i!}$$

Putting everything together, we get the following, and we're done.

$$\pi_k = \frac{1}{k!} \left( 1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!} \right)$$