A Dynamic Panel Data Framework for Identification and Estimation of Nonlinear Production Functions

Justin Doty

May 15, 2020

- Consider a firm's gross-output production function

$$Y_{it} = F_t(K_{it}, L_{it}, \iota_t, \omega_{it}, \eta_{it}), \quad (1)$$

where Y_{it} denotes a firm's final output, K_{it} denotes capital stock, L_{it} denotes amount of labor used and ι_t is an intermediate input. ω_{it} is a firm's productivity, a state variable which is unobserved to the econometrician. η_{it} is an iid shock, independent of the firms input choices at time t

- Assume F_t is monotonic increasing in η_{it}
- Identification issue: Firms optimal input choices are a function of the unobserved ω_{it} which leads to simultaneity bias when estimating the production function
- Many papers have proposed solutions to the simultaneity problem such as firm fixed effects, IV, control functions, etc.

Introduction

- Many solutions focus on Cobb-Douglas production function (the simple linear case)
- Estimates dont account for other unobservable firm-specific heterogeneity
- We focus on how to estimate the conditional quantiles of (1)
- Therefore we need a general identification strategy as well as a flexible estimation strategy for the nonlinear model
- This paper would be the first to directly apply identification results using the dynamic nature of firm's input decision problems
- Then, it will apply an EM algorithm type approach to estimate the production function by integrating over the unobservable productivity term

Identification

- For simplicity, let $X_t = \{Y_t, L_t, \iota_t\}$ and $W_t = \{K_{it}, I_t\}$
- Our goal is identification of the Markov law of motion $f_{X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1}}$ which we assume to be stationary
- We assume the researcher observes a panel dataset consisting of i.i.d observations of firm output and input choices with the number of time periods $T \geq 4$
- We introduce the following assumptions that simplify the expression for the law of motion.

Assumption 1

1. *Non-dynamic output and inputs:* $f_{X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1}} = f_{X_t,W_t,\omega_t|W_{t-1},\omega_{t-1}}$
2. *First-order Markov:* $f_{X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1},\mathcal{I}_{<t-1}} = f_{X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1}}$, where $\mathcal{I}_{<t-1}$ is the firm's information set up to time $t-1$
3. *Limited Feedback:* $f_{W_t|W_{t-1},\omega_t,\omega_{t-1}} = f_{W_t|W_{t-1},\omega_t}$

Identification

- Using Assumption 1 the Markov law of motion can be factored into:

$$\begin{aligned} f(X_t, W_t, \omega_t | X_{t-1}, W_{t-1}, \omega_{t-1}, \mathcal{I}_{<t-1}) &= f(X_t, W_t, \omega_t | W_{t-1}, \omega_{t-1}) \\ = f(X_t | W_t, \omega_t, W_{t-1}, \omega_{t-1}) & f(W_t | \omega_t, W_{t-1}, \omega_{t-1}) f(\omega_t | W_{t-1}, \omega_{t-1}) \end{aligned} \tag{2}$$

- We can simplify the first density on the last line of the equation (2) as

$$\begin{aligned} & f(X_t | W_t, \omega_t, W_{t-1}, \omega_{t-1}) = f(Y_t | W_t, \omega_t, W_{t-1}, \omega_{t-1}) \\ & \times f(L_t | W_t, \omega_t, W_{t-1}, \omega_{t-1}) f(\iota_t | W_t, \omega_t, W_{t-1}, \omega_{t-1}) \\ & = f(Y_t | L_t, \iota_t, K_t, \omega_t) f(L_t | K_t, \omega_t) f(\iota_t | K_t, \omega_t) \end{aligned} \tag{3}$$

- Furthermore, the second density on the last line of the equation (2) becomes

$$\begin{aligned} f(W_t | \omega_t, W_{t-1}, \omega_{t-1}) = f(W_t | W_{t-1}, \omega_t) &= f(I_t, K_t | I_{t-1}, K_{t-1}, \omega_t) \\ &= f(I_t | K_t, I_{t-1}, K_{t-1}, \omega_t) f(K_t | I_{t-1}, K_{t-1}, \omega_t) \\ &= f(I_t | K_t, \omega_t) f(K_t | I_{t-1}, K_{t-1}, \omega_t) \end{aligned} \tag{4}$$

Identification

- The previous derivations relied on conditional independence assumptions I will mention later
- Assumption 1 part 2 and 3 are satisfied from the our dynamic model of firm investment
- The evolution process for $\omega_t \in \mathbb{R}$ is given by

$$\omega_t = g(\omega_{t-1}, \xi_t), \tag{5}$$

  where the function $g(\cdot, \xi_t)$ is strictly increasing in the iid innovation shock, $\xi_t \in \mathbb{R}$
- Equation (5) implies $f_{\omega_t | W_{t-1}, \omega_{t-1}} = f_{\omega_t | \omega_{t-1}}$ which implies that productivity evolves exogenously
- This can be relaxed when we consider productivity enhancing activities such as R&D

- Capital accumulates according to the following process:

$$K_t = \kappa_t(K_{t-1}, I_{t-1}, \upsilon_t), \tag{6}$$

where the function κ is strictly increasing in its last argument and υ_t denotes an iid shock independent of the other arguments

- A special case of equation (6) is the usual capital accumulation law $K_t = (1-\delta)K_{t-1} + I_{t-1}$
- Here υ_t are other factors that affect the capital accumulation process
- In each period, a firm chooses investment to maximize its discounted future profits:

$$I_t = I^*(K_t, \omega_t, \zeta_t) = \underset{I_t \geq 0}{\operatorname{argmax}} \left[\Pi_t(K_t, \omega_t) - c(I_t, \zeta_t) + \beta \mathbb{E}\left[V_{t+1}(K_{t+1}, \omega_{t+1}, \zeta_{t+1}) | \mathcal{I}_t \right] \right], \tag{7}$$

where $\pi_t(\cdot)$ is current period profits as a function of the state variables, $c(I_t)$ is the cost of current investment and β is the firm's discount factor

Identification

- We introduce an additional state variable ζ_t which could represent other factors that shift firm's investment costs
- We assume the cost function $c(\cdot, \zeta_t)$ is decreasing in ζ_t. Under certain conditions, the investment policy function is monotonic increasing in ζ_t
- Without loss of generality, we normalize $\zeta_t \sim U[0, 1]$
- Also considering how to modify identification and estimation strategies to cases when investment is censored or there is selection bias due to endogenous entry/exit
- The restrictions on the capital accumulation process in (6) and the investment problem in (7) satisfy the Limited Feedback condition in Assumption 1.

Identification

- The specifications for the static inputs, labor and intermediate inputs such as materials, fuels, and electricity, are much easier to state. We let optimal labor demand be given by:

$$l_t = \ell(k_t, \omega_t, \epsilon_t) \tag{8}$$

  where the function, $\ell(\cdot, \cdot, \epsilon_t)$ is strictly increasing in ϵ_t which is assumed to be independent of the other arguments.

- We normalize this to be standard uniform each period. We follow a similar model for the intermediate inputs:

$$\iota_t = \iota_t(k_t, \omega_t, \varepsilon_t) \tag{9}$$

  where the function, $\iota(\cdot, \cdot, \varepsilon_t)$ is strictly increasing in ε_t which is assumed to be independent of the other arguments.

- We normalize this to be standard uniform each period.

Identification

- In order to factorize the density in equation (3) we need the following assumptions:
  Assumption 2
  1. The production shock η_t, labor shock ϵ_t and intermediate input shock ε_t are mutually independent conditional on $(W_t, \omega_t, W_{t-1}, \omega_{t-1})$
  2. The production shock η_t is independent of ζ_t conditional on $(L_t, \iota_t, K_t, \omega_t)$
  3. ϵ_t and ε_t are independent of ζ_t conditional on (K_t, ω_t)
- Assumption 2 is similar to the mutual independence assumptions made by [**?**].
- In their paper, they provide interpretations of these errors that are likely to satisfy the conditional independence restrictions.
- For example: optimization errors or measurement error or factor specific innovation shocks

Identification

- We outline the identification procedure similar to [**?**]. First, let $V_t = \{X_t, W_t\}$. Under Assumption 1, we can write:

$$f_{X_{t+1},W_{t+1},X_t,W_t,X_{t-1},W_{t-1},X_{t-2},W_{t-2}} = \int f_{X_{t+1},W_{t+1}|W_t,\omega_t} f_{X_t,W_t|W_{t-1},\omega_t} \times f_{X_{t-1},W_{t-1},X_{t-2},W_{t-2},\omega_t} d\omega_t \tag{10}$$

- The identification argument then depends on the existence and uniqueness a spectral decomposition of the linear operators associated with the observed density.
- If we can show the linear operator associated with the Markov Law of Motion can be factored into linear operators corresponding to identified densities we can then say our model is identified

Identification

- We define a linear operator $L_{V_{t-2},\bar{x}_{t-1},\bar{w}_{t-1},\bar{x}_t,\bar{w}_t,V_{t+1}}$ as a mapping from the $\mathcal{L}^p$ space of functions of V_{t+1} to the $\mathcal{L}^p$ space of functions of V_{t-2} as follows:

$$
\begin{aligned}
&(L_{V_{t-2},\bar{x}_{t-1},\bar{w}_{t-1},\bar{x}_t,\bar{w}_t,V_{t+1}}h)(v_{t-2}) \\
&\quad = \int f_{V_{t-2},X_{t-1},W_{t-1},X_t,W_t,V_{t+1}}(v_{t-2},\bar{x}_{t-1},\bar{w}_{t-1},\bar{x}_t,\bar{w}_t,v_{t+1})h(v_{t+1})dv_{t+1}, \\
&h \in \mathcal{L}^p(\mathcal{V}_{t+1}), \bar{x}_{t-1} \in \mathcal{X}_{t-1}, \bar{x}_t \in \mathcal{X}_t, \bar{w}_{t-1} \in \mathcal{W}_{t-1}, \bar{w}_t \in \mathcal{W}_t,
\end{aligned} \tag{11}
$$

where $\mathcal{V}_t, \mathcal{X}_{t-1}, \mathcal{X}_t, \mathcal{W}_{t-1}$ and $\mathcal{W}_t$ are the supports of $V_t, X_{t-1}, X_t, W_{t-1}$ and W_t respectively

- We also define the diagonal operator as:

$$
\begin{aligned}
(D_{\bar{x}_t,\bar{w}_t|\bar{w}_{t-1},\omega_t}h)(\omega_t) &= f_{X_t,W_t|,W_{t-1},\omega_t}(\bar{x}_t,\bar{w}_t|\bar{w}_{t-1},\omega_t)h(\omega_t) \\
&h \in \mathcal{L}^p(\Omega_t), \bar{x}_t \in \mathcal{X}_t, \bar{w}_t \in \mathcal{W}_t, \bar{w}_{t-1} \in \mathcal{W}_{t-1}
\end{aligned} \tag{12}
$$

Identification

- We rely on a few lemmas that allow us to represent the observed density and the Markov Law of Motion via linear operators
  Lemma 1 For any $t \in \{3, \ldots, T-1\}$. Assumption 1 implies that, for any $(x_t, w_t, x_{t-1}, w_{t-1}) \in \mathcal{X}_t \times \mathcal{W}_t \times \mathcal{X}_{t-1} \times \mathcal{W}_{t-1}$,

$$L_{V_{t+1},x_t,w_t,x_{t-1},w_{t-1}} = L_{V_{t+1}|w_t,\omega_t} D_{x_t,w_t|w_{t-1},\omega_t} L_{x_{t-1},w_{t-1},V_{t-2},\omega_t} \quad (13)$$

  Lemma 2

$$L_{x_t,w_t,\omega_t|w_{t-1},\omega_{t-1}} = L^{-1}_{V_{t+1}|w_t,\omega_t} L_{V_{t+1},x_t,w_t,x_{t-1},w_{t-1},V_{t-2}} L^{-1}_{V_t,x_{t-1},w_{t-1},V_{t-2}} L_{V_t|w_{t-1},\omega_t} \quad (14)$$

- In what sense are the linear operators invertible?
- I will talk about this on the next slide
- Intuitively, the Markov Law of Motion is identified if we can $L_{V_t|w_{t-1},\omega_t}$ using $L_{V_{t+1},x_t,w_t,x_{t-1},w_{t-1}}$
- The rest of the operators in (14) correspond to densities of observed data so those are already identified

Identification

Assumption 3

There exists variable(s) V_t such that

1. For any $w_t \in \mathcal{W}_t$, $L_{V_{t+1}|w_t,\omega_t}$ is one-to-one
2. For any $(x_{t-1}, w_{t-1}) \in \mathcal{X}_{t-1} \times \mathcal{W}_{t-1}$, $L_{V_t,x_{t-1},w_{t-1},V_{t-2}}$ is one-to-one
3. For any $(x_t, w_t) \in \mathcal{X}_t \times \mathcal{W}_t$, there exists a $(x_{t-1}, w_{t-1}) \in \mathcal{X}_{t-1} \times \mathcal{W}_{t-1}$ and a neighborhood $\mathcal{N}^r$ around $(x_t, w_t, x_{t-1}, w_{t-1})$ such that, for any $(\bar{x}_t, \bar{w}_t, \bar{x}_{t-1}, \bar{w}_{t-1}) \in \mathcal{N}^r$, $L_{V_{t-2},\bar{x}_t,\bar{w}_t,\bar{x}_{t-1},\bar{w}_{t-1},V_{t-2}}$ is one-to-one

- This is a difficult assumption to interpret
- I am trying to find sufficient conditions in context of our model that satisfy injectivity via bounded completeness of the corresponding densities
- Other assumptions include uniqueness of eigenvalues in the decomposition of $L_{V_{t+1},x_t,w_t,x_{t-1},w_{t-1},V_{t-2}}$ and a normalization assumption
- Also need conditions to identify the initial distributions