A Dynamic Panel Data Framework for Identification and Estimation of Nonlinear Production Functions

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Abstract

This paper studies identification and estimation of a non-linear model for production functions with unobserved heterogeneity. Non-parametric identification results are established for the production function under stationarity assumptions. This paper then estimates the conditional quantiles of firm production using non-linear quantile regression. This paper finds that objects of interest, such as output elasticities with respect to inputs vary considerably with respect to the rank of unobserved technology shocks. In the application to US firm-level manufacturing data, this paper considers a Translog production function with non-Hicksian neutral productivity shocks and a productivity process that features non-linear persistence.

1 Introduction

2 Literature Review

3 The Model of Firm Production

Output

Consider a nonlinear model for a firm's gross-output production function (in logs)

$$y_{it} = f_t(k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\eta_t)) \tag{1}$$

where y_{it} is firm i's' output at time t and l_{it}, m_{it}, k_{it} denotes optimal input choices for labor, materials, and capital respectively. The unobserved productivity is denoted by ω_{it} which is correlated to input choices of the firm at time t. I let the output elasticities β be functionally dependent on unobserved production shocks $\eta_{i1}, \ldots, \eta_{iT}$ which are independent of input choices and productivity at time t.

Without loss of generality I normalize $\eta_{it} \sim U[0,1]$. This model corresponds to a nonlinear random coefficient model where the outcome y_{it} is assumed to be strictly monotonic in η_{it}. In practice I can allow for nonlinear interactions between inputs and unobserved productivity at different quantiles so that marginal effects can be modeled as non-Hick's neutral. For empirical simplicity I can model separability in the unobserved productivity to calculate total factor productivity (TFP). The function f_t is an unknown nonlinear function.

In this model, heterogeneity in production technology across firms is driven by the rank of the unobserved production shocks η_{it}. I specify the following functional forms for the input demand functions.

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Labor

Labor inputs are chosen to maximize current period profits and therefore are a function of current period state variables

$$l_{it} = \ell_t(k_{it}, \omega_{it}; \alpha_l(\epsilon_{it})) \tag{2}$$

where ϵ_{it} is iid and independent of current period state variables. I assume the labor demand function ℓ is strictly increasing in ϵ_{it} which is normalized to be uniformly distributed on the interval $[0,1]$

Materials

Material inputs are chosen to maximize current period profits and therefore are a function of current period state variables

$$m_{it} = \mu_t(k_{it}, \omega_{it}, \alpha_m(\varepsilon_{it})) \tag{3}$$

where ε_{it} is iid and independent of current period state variables. I assume the labor demand function μ is strictly increasing in ε_{it} which is normalized to be uniformly distributed on the interval $[0,1]$. I can extend this to the case where labor is chosen prior to choosing material inputs in which case I would include l_{it} as a state variable in equation (3).

Capital Accumulation

Capital accumulates to the following generalized law of motion

$$K_{it} = \kappa_t(K_{it-1}, I_{it-1}, \upsilon_{it-1}) \tag{4}$$

where I_{it-1} denotes firm investment in the prior period. Under this specification, capital is determined in the period $t-1$. I introduce a random error term υ_{it-1} which eliminates the deterministic relationship of capital with respect to previous period state and choice variables. I will show later that this step is crucial for our nonparametric identification result which is used by Hu *et al.* (2019). I assume this error term is independent of the arguments in the capital accumulation law.

Productivity

Productivity evolves according to the exogenous first order Markov process:

$$\omega_{it} = g(\omega_{it-1}; \rho(\xi_{it})) \tag{5}$$

where $\xi_{i1}, \ldots, \xi_{iT}$ are independent uniform random variables which represent innovation shocks to productivity. I assume ω_{it} is monotonic in ξ_{it} I let g be another unknown nonlinear function that allows the persistence in productivity in firms to be nonlinear across different quantiles. The exogeneity of the productivity process can be relaxed when I consider productivity enhancing activities such as R&D similar to Doraszelski and Jaumandreu (2013).

Investment

I introduce a dynamic model of firm investment that is a slight modification of Ericson and Pakes (1995) which can also be found in Hu and Shum (2013) and Ackerberg *et al.* (2007). In each period, a firm chooses investment to maximize its discounted future profits:

$$I_{it} = I^*(K_{it}, \omega_{it}; \delta(\zeta_{it})) = \underset{I_t \geq 0}{\operatorname{argmax}} \left[\Pi_t(K_{it}, \omega_{it}, \zeta_{it}) - c(I_{it}) + \beta \mathbb{E}\left[V_{t+1}(K_{it+1}, \omega_{it+1}, \zeta_{it+1}) | \mathcal{I}_t \right] \right], \tag{6}$$

Here, $\pi_t(\cdot)$ is current period profits as a function of the state variables and an unobservable demand shock ζ_{it}. These are shocks to a firm's product demand which are privately observed by each firm and i.i.d across i and t. I assume these shocks are independent from the firm's state variables. Current costs to investment are given by $c(I_t)$ and β is the firm's discount factor. Pakes (1994) provides specific conditions for which the investment policy function is strictly increasing in its unobservable components. I can interpret conditions for strict monotonicity here and leave a detailed explanation in the appendix. Without loss of generality, I normalize $\zeta_t \sim U[0,1]$

4 Identification

I begin by listing preliminary assumptions regarding the main econometric restrictions of the model, some of which were informally stated in the previous section.

Assumption 4.1.

(i) η_{it} follows a standard uniform distribution independent $\mathcal{I}_{it}$, where $\mathcal{I}_{it}$ denotes the firm $i's$ information set at time t

(ii) The production function $y_{it} = f_t(k_{it}, l_{it}, m_{it}, \omega_{it}, \beta(\eta_t))$ is strictly increasing in η_{it}

(iii) η_{it} is independent and identically distributed across firm's and time

Then, according to Assumption 4.1 the following moment restriction holds:

$$\mathbb{E}_{nT}[\Psi_\tau(y_{it} - f_t(k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau)))|\mathcal{I}_{it}] = 0, \tag{7}$$

where $\Psi_\tau(u) = \mathbb{1}\{u \leq 0\} - \tau$. Using the law of iterated expectations, the following unconditional integrated moment restriction can be written as

$$\mathbb{E}_{nT}\left[\int [k_{it}, l_{it}, m_{it}, \omega_{it}]\Psi_\tau(y_{it} - f_t(k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau)))g_i(\omega_i^T; \theta(\cdot))d\omega_i^T\right] = 0, \tag{8}$$

where $g_i(\omega_i^T; \theta(\cdot)) = f(\omega_i^T|y_i^T, k_i^T, l_i^T, m_i^T, I_i^T; \theta(\cdot))$ denotes the posterior density of $\omega_i^T = (\omega_{it}, \ldots, \omega_{iT})$ which depends on the continuum of parameters $\theta(\cdot) = (\beta(\cdot), \alpha_l(\cdot), \alpha_m(\cdot), \rho(\cdot), \delta(\cdot))$. I only consider exact identification for the above moment restriction, but I include investment in the conditioning set of the posterior density to capture the capital accumulation process which depends on last periods investment decisions, here $I_{it} \in \mathcal{I}_{it}$. In my model the posterior density is:

$$
\begin{aligned}
g_i(\omega_i^T; \theta(\cdot)) &\propto \prod_{t=1}^{T} f(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\cdot))f(l_{it}|k_{it}, \omega_{it}; \alpha_l(\cdot))f(m_{it}|k_{it}, \omega_{it}; \alpha_m(\cdot)) \\
&\quad \times f(I_{it}|k_{it}, \omega_{it}; \delta(\cdot))\prod_{t=2}^{T} f(\omega_{it}|\omega_{it-1}; \rho(\cdot))f_{i1}(\omega_{i1}; \rho_1(\cdot)),
\end{aligned} \tag{9}
$$

Therefore, the identification of the production function in (8) relies on identification of the conditional densities of production, labor, materials, investment, and the productivity process[1]. I show that these densities are nonparametrically identified using Hu and Schennach (2008). To show this, I introduce some new notation. Let $Z_t = (l_t, k_t, m_t, k_{t+1})$ denote conditioning variables where I have dropped the i subscript for convenience. Assume the following:

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[1]Note that since the capital accumulation process does not directly depend on productivity, it drops out of the conditioning density. Identifying the investment rule should be sufficient for identifying capital dependence on last period's productivity

Assumption 4.2. *(Conditional Independence):*
$$f(y_t|y_{t+1}, I_t, \omega_t, Z_t) = f(y_t|\omega_t, Z_t) \ and \ f(I_t|y_{t+1}, \omega_t, Z_t) = f(I_t|\omega_t, Z_t)$$

The first equality of Assumption 4.2 states that conditional on ω_t and Z_t, y_{t+1} and I_t do not provide any additional information about y_t. The second equality states that conditional on ω_t and Z_t, y_{t+1} does not provide any additional information about I_t. These are satisfied by mutual independence assumptions on η_t and ζ_t conditional on $(\omega_t, k_t, l_t, m_t)$ and the fact that η_{it} is assumed to be conditionally independent over time. The next assumption is more technical.

Assumption 4.3. *(Injectivity): The operators $L_{y_t|Z_t, \omega_t}$ and $L_{y_{t+1}|Z_t, \omega_t}$ are injective*

The above assumption allows us to take inverses of the operators. Consider the operator $L_{y_t|Z_t, \omega_t}$, following Hu and Schennach (2008), injectivity of this operator can be interpreted as its corresponding density $f_{y_t|Z_t, \omega_t}(y_t|Z_t, \omega_t)$ having sufficient variation in ω_t given Z_t. This assumption is often phrased as completeness condition in the nonparametric IV literature on the density $f_{y_t|k_t, \omega_t}(y_t|Z_t, \omega_t)$. More formally, for a given $Z_t \in Supp(Z_t)$

$$\int f_{y_t|Z_t, \omega_t}(y_t|Z_t, \omega_t) g(\omega_t) d\omega_t = 0 \tag{10}$$

for all y_t implies $g(\omega_t) = 0$ for all ω_t

For injectivity of the second operator $L_{y_{t+1}|Z_t, \omega_t}$ it is intuitive to consider y_{t+1} having sufficient variation for different values of ω_t given Z_t. Since productivity is specified as an AR(1) process and is highly persistent over time, this assumption is intuitive.

This assumption is more restrictive than that of Hu *et al.* (2019). Since their model is separable in ω_t they are able utilize convolution type arguments which require mild conditional independence assumptions as well as assumptions regarding characteristic functions. Injectivity is the price we pay for assuming a more general production function specification. I also require two additional assumptions for uniqueness of the identification strategy.

Assumption 4.4. *(Uniqueness): For any $\bar{\omega}_t, \tilde{\omega}_t \in \Omega$, the set $\{f_{I|\omega,Z}(I_t|\bar{\omega}_t, Z_t) \neq f_{I|\omega,Z}(I_t|\tilde{\omega}_t, Z_t)\}$ has positive probability whenever $\bar{\omega}_t \neq \tilde{\omega}_t$*

This assumption is relatively weak and is satisfied if there is conditional heteroskedasticity in $f_{I|\omega,Z}$ or if any functional of its distribution is strictly increasing in ω_t. For example, this assumption is satisfied if $E[I_t|\omega_t, Z_t]$ is strictly increasing in ω_t which is similar to the invertibility conditions required in Olley and Pakes (1996). This assumption is problematic if investment is censored, in which case $E[I_t|\omega_t, Z_t]$ can only be inverted for positive values of investment. In this case we can consider for some τ, $Q_\tau(I_t|\bar{\omega}, Z_t) \neq Q_\tau(I_t|\tilde{\omega}, Z_t)$. Since I will estimate input decision rules in my model using quantile regression, correcting for censoring is straightforward as I will show later. However, in my application censoring is not an issue since I use publicly listed firms, most of which who report positive capital expenditure in their income statements.

Assumption 4.5. *(Normalization): There exists a functional Γ such that $\Gamma[f_{y|\omega,Z}(y_t|\omega_t, Z_t)] = \omega_t$*

This functional does not need to be known, it is sufficient to consider a known function of the data distribution as shown by Arellano and Bonhomme (2016). For a general nonseparable panel model this assumption is satisfied if $E[y_t|\omega_t, Z_t]$ is strictly increasing in ω_{it}, then one could normalize $\omega_t = E[y_t|\omega_t, Z_t]$. In our empirical application, we use a non-separable translog production function. In this case our normalization can be achieved by $E[y_t|\omega_t, 0] = \omega_t$. With these assumptions we can now state the first part of our identification results.

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Theorem 4.1. *Under Assumptions 4.2, 4.3, 4.4, and 4.5, given the observed density $f_{y_t,I_t|y_{t+1},Z_t}$, the equation*

$$f_{y_t,I_t|y_{t+1},Z_t}(y_t,I_t|y_{t+1},Z_t) = \int f_{y_t|\omega_t,Z_t}(y_t|\omega_t,Z_t) f_{I_t|\omega_t,Z_t}(I_t|\omega_t,Z_t) f_{\omega_t|y_{t+1},Z_t}(\omega_t|y_{t+1},Z_t) d\omega_t \quad (11)$$

admits a unique solution for $f_{y_t|\omega_t,Z_t}$, $f_{I_t|\omega_t,Z_t}$ and $f_{\omega_t|y_{t+1},Z_t}$

   Proof: See Appendix.

This result identifies the conditional density of output and investment. It also identifies the marginal distribution for productivity since $f_{\omega_t} = \int f_{\omega_t|y_{t+1},Z_t} f_{y_{t+1},Z_t} d(y_{t+1}, Z_t)$ where f_{y_{t+1},Z_t} is known from the observed data. It also identifies the input decision rules for the labor and material inputs. We need additional results to identify the productivity process $f_{\omega_t|\omega_{t-1}}(\omega_t|\omega_{t-1})$ and the initial condition $f_{\omega_{t-1}}(\omega_{t-1})$. The requirements for identifying these distributions are different if the production function is stationary or non-stationary. Therefore I claim two separate results.

Corollary 4.1. *(Stationarity): Suppose that the production function is stationary i.e. $f_{y_t|\omega_t,Z_t} = f_{y_1|\omega_1,Z_1} \forall t \in \{1, \cdots, T\}$. Then, under Assumptions 4.2, 4.3, 4.4, and 4.5, the observed density $f_{y_t,I_t|y_{t+1},Z_t}$ uniquely determines the density $f_{\omega_{t+1}|\omega_t}$ and the initial condition f_{ω_t} for any $t \in \{1, \ldots, T-1\}$*

   Proof: See Appendix.

Corollary 4.2. *(Non-Stationary): Under Assumptions 4.2, 4.3, 4.4, and 4.5, the observed density $f_{y_{t+1},I_{t+1}|y_{t+2},y_t Z_{t+1}}$ uniquely determines the density $f_{\omega_{t+1}|\omega_t}$ and the initial condition f_{ω_t} for any $t \in \{1, \ldots, T-2\}$*

   Proof: See Appendix.

The main conclusion of these two corollaries state is that under the condition of stationarity, the productivity process can be identified with $T = 2$ observations per firms whereas under non-stationarity, the productivity process is identified with $T = 3$ observations per firm. The number of time periods required for identification increases with the length of the auto-regressive process. It is standard to estimate an AR(1) process for productivity. The data requirements are similar to the control function approach where the instrument set often includes secondary lags of inputs. The main identification assumptions are stronger than those in Hu *et al.* (2019), but necessary for the generality of the model I present in this paper.

5   Estimation Strategy

I specify the following functional forms for estimation of the model in Section 3.

Output:

$$Q_t(y_{it}|k_{it},l_{it},m_{it},\omega_{it},\tau) = Q(y_{it}|k_{it},l_{it},m_{it},\omega_{it},t,\tau)$$
$$= \beta_0(\tau) + \beta_t(\tau)t + \sum_{0<n_x\leq n} \beta_{n_k,n_l,n_m}(\tau) k_{it}^{n_k} l_{it}^{n_l} m_{it}^{n_m} \omega_{it} \quad (12)$$

where t denotes the age of firm i at time t, $n_x = n_k + n_l + n_m$. In my application I take $n_k = n_l = n_m = 2$. In this case the production function is specified as Translog with first-order interactions of productivity, in a sense, allowing for productivity to affect input usage beyond a Hicks-neutral effect. Therefore, I have a total of 21 parameters to estimate.

Labor Input:

I specify the labor input demand equation as follows:

$$Q_t(l_{it}|k_{it}, \omega_{it}, \tau) = Q(l_{it}|k_{it}, \omega_{it}, t, \tau)$$
$$= \sum_{j=1}^{J} \alpha_{l,j}(\tau)\psi_{l,j}(k_{it}, \omega_{it}, t) \tag{13}$$

where $\psi_{l,j}$ is a tensor product linear sieve. One could use basis functions as Fourier, B-splines, or Hermite polynomials. In practice I approximate this function by a third-order polynomial with interactions in ω_{it} and k_{it} and additive in firm age.

Materials Input:

I specify the material input demand equation as follows:

$$Q_t(m_{it}|k_{it}, \omega_{it}, \tau) = Q(m_{it}|k_{it}, \omega_{it}, t, \tau)$$
$$= \sum_{j=1}^{J} \alpha_{m,j}(\tau)\psi_{m,j}(k_{it}, \omega_{it}, t) \tag{14}$$

where $\psi_{m,j}$ is specified similarly as the labor decision rule. In the case where labor inputs are chosen prior to material inputs, one could include l_{it} as an additional state variable in the material input rule specified above.

Investment Demand:

I specify the investment demand equation as:

$$I_{it}^* = \iota_t(k_{it}, \omega_{it}, \tau) = \iota(k_{it}, \omega_{it}, t, \tau)$$
$$= \sum_{j=1}^{J} \alpha_{\delta,j}(\tau)\psi_{\iota,j}(k_{it}, \omega_{it}, t) \tag{15}$$

where $\psi_{\iota,j}$ is specified similarly as the labor and material input decision rule. In the case where investment is censored, I can write

$$I_{it} = \max\{0, I_{it}^*\} = \max\{0, I^*(K_{it}, \omega_{it}; \delta(\zeta_{it}))\} \tag{16}$$

in which case the conditional quantiles (in logs) can be written

$$Q_\tau(i_{it}|\mathcal{I}_{it}) = \max\{0, \sum_{j=1}^{J} \delta_j(\tau)\psi_{\iota,j}(k_{it}, \omega_{it}, t)\}, \tag{17}$$

due to the equivariance property of quantiles. The censored quantile regression model avoids distributional assumptions in estimation at the cost of computational complexity.

Persistent Productivity:

I specify productivity to transition according to:

$$\omega_{it} = g_t(\omega_{it-1}, \tau) = g(\omega_{it-1}, t, \tau) = \rho_0(\tau) + \rho_1(\tau)\omega_{it-1} + \rho_2(\tau)\omega_{it-1}^2 + \rho_3(\tau)\omega_{it-1}^3. \tag{18}$$

Later I consider the case where productivity may evolve endogenously. In this example, firms engage in activities hoping to increase future productivity. Activities such as R&D, exporting activity, or even spill-over effects from other firms are omitted variables in ξ_{it}. Consider a productivity enhancing activity such as R&D, in logs as

$$r_{it} = R_t(k_{it}, \omega_{it}, \varrho_{it}), \tag{19}$$

where ϱ_{it} is a shock to R&D that is independent from the state variables and normalized to be $U[0,1]$. It is straightforward to include this process in the identification argument. In this case we could rewrite 18 as

$$\omega_{it} = g_t(\omega_{it-1}, \tau) = g(\omega_{it-1}, t, \tau) = \sum_{j=1}^{J} \rho_j(\tau)\psi_{\omega,j}(\omega_{it-1}, r_{it-1}, t) \tag{20}$$

where $\psi_{\omega,j}$ can capture interaction effects between productivity and R&D expenditure. Another extension for the productivity process allows can also control for endogenous exit of firms from the sample. This insight is built from the Olley and Pakes (1996) model which provides an optimal exit rule for firms whose productivity falls below some threshold. In the Appendix, I show how a propensity score can be estimated in my framework and used in the productivity process as an additional control variable in $\psi_{\omega,j}$. I also need to specify initial conditions for productivity in the next equation

$$\omega_{i1} = g_{\omega_1}(t, \tau) = \sum_{j=1}^{J} \rho_{1,j}(\tau)\psi_{\omega_1,j}(t), \tag{21}$$

where $\psi_{\omega_1,j}$ is another flexible function, such as a univariate polynomial in age.

6 Implementation

I model the functional parameters $\theta(\cdot)$ using Wei and Carroll (2009) and Arellano and Bonhomme (2016). For example, the function $\beta_j(\tau_q)$ is modeled as a piecewise-polynomial interpolating splines on a grid $[\tau_1, \tau_2], [\tau_3, \tau_4], \ldots, [\tau_{Q-1}, \tau_Q]$, contained in the unit interval and is constant on $[0, \tau_1]$ and $[\tau_Q, 1)$ The intercept coefficient β_0 is specified as the quantile of an exponential distribution on $(0, \tau_1]$ (indexed by λ^-) and $[\tau_{Q-1}, 1)$ (indexed by λ^+). The remaining functional parameters are modeled similarly. I take $Q = 11$ and $\tau_q = \frac{q}{Q+1}$. In the following section I outline the model's restrictions and a feasible estimation strategy.

6.1 Model Restrictions

The following conditional moment restrictions hold as an implication of the conditional independence restrictions on the production function, input decision rules, and productivity process.

$$\mathbb{E}\left[\Psi_{\tau_q}\left(y_{it} - \beta_0(\tau) - \beta_t(\tau)t - \sum_{0 < n_x \leq n} \beta_{n_k, n_l, n_m}(\tau)k_{it}^{n_k} l_{it}^{n_l} m_{it}^{n_m} \omega_{it}\right)\middle| k_{it}, l_{it}, m_{it}, t\right] = 0 \tag{22}$$

$$\mathbb{E}\left[\Psi_{\tau_q}\left(l_{it} - \sum_{j=1}^{J} \alpha_{l,j}(\tau)\psi_{l,j}(k_{it}, \omega_{it}, t)\right)\middle| k_{it}, \omega_{it}, t\right] = 0 \tag{23}$$

$$\mathbb{E}\left[\Psi_{\tau_q}\left(m_{it} - \sum_{j=1}^{J} \alpha_{m,j}(\tau)\psi_{m,j}(k_{it}, \omega_{it}, t)\right)\middle| k_{it}, \omega_{it}, age_{it}\right] = 0 \tag{24}$$

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$$\mathbb{E}\left[\Psi_{\tau_q}\left(i_{it} - \sum_{j=1}^{J} \alpha_{\iota,j}(\tau)\psi_{l,j}(k_{it}, \omega_{it}, t)\right)\middle| k_{it}, \omega_{it}, t\right] = 0 \tag{25}$$

For $t \geq 2$,

$$\mathbb{E}\left[\Psi_{\tau_q}\left(\omega_{it} - \rho_0(\tau) - \rho_1(\tau)\omega_{it-1} - \rho_2(\tau)\omega_{it-1}^2 - \rho_3(\tau)\omega_{it-1}^3\right)\middle| \omega_{it-1}, t\right] = 0 \tag{26}$$

For initial productivity $(t = 1)$,

$$\mathbb{E}\left[\Psi_{\tau_q}\left(\omega_{i1} - \sum_{j=1}^{J} \rho_{1,j}(\tau)\psi_{\omega_1,j}(t)\right)\middle| t\right] = 0 \tag{27}$$

Clearly, estimating the above conditional moment restrictions are infeasible due to the unobserved productivity component. Therefore, I use the following unconditional moment restrictions and posterior distributions for ω_{it} to integrate out the unobserved productivity. Consider the following unconditional moment restriction of output:

$$\mathbb{E}\left[\int \Psi_{\tau_q}\left(y_{it} - \beta_0(\tau) - \beta_t(\tau)t - \sum_{0 < n_x \leq n} \beta_{n_k,n_l,n_m}(\tau)k_{it}^{n_k}l_{it}^{n_l}m_{it}^{n_m}\omega_{it}\right) \otimes \begin{pmatrix} k_{it} \\ l_{it} \\ m_{it} \\ \omega_{it} \\ age_{it} \end{pmatrix} g_i(\omega_{it};\theta)d\omega_{it}\right] = 0 \tag{28}$$

The posterior density in equation (28) is a closed-form expression when using piecewise linear splines for $\theta(\cdot)$. The estimation is an Expectation Maximization (EM) algorithm. In Arellano and Bonhomme (2016) and Arellano *et al.* (2017), the "M-step" is performed using quantile regression. Given an initial parameter value $\hat{\theta}^0$. Iterate on $s = 0, 1, 2, \ldots$ in the following two-step procedure until converge to a stationary distribution:

1. *Stochastic E-Step*: Draw M values $\omega_i^{(m)} = (\omega_{i1}^{(m)}, \omega_{i2}^{(m)}, \ldots, \omega_{iT}^{(m)})$ from

$$g_i(\omega_{it};\hat{\theta}^{(s)}) = f(\omega_{it}|y_{it}, k_{it}, l_{it}, m_{it}, I_t, ;\hat{\theta}^{(s)}) \propto$$

$$\prod_{t=1}^{T} f(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it};\hat{\theta}^{(s)})f(l_{it}|k_{it}, \omega_{it};\hat{\theta}^{(s)})f(m_{it}|k_{it}, \omega_{it};\hat{\theta}^{(s)})$$

$$\times f(i_{it}|k_{it}, \omega_{it};\hat{\theta}^{(s)})\prod_{t=2}^{T} f(\omega_{it}|\omega_{it-1};\hat{\theta}^{(s)})f(\omega_{i1};\hat{\theta}^{(s)})$$

2. *Maximization Step*: For $q = 1, \ldots, Q$, solve

$$\hat{\boldsymbol{\beta}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\beta}(\tau_q)}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \Psi_{\tau_q} \left(y_{it} - \beta_0(\tau_q) - \beta_t(\tau_q)t - \sum_{0 < n_x \leq n} \beta_{n_k,n_l,n_m}(\tau_q) k_{it}^{n_k} l_{it}^{n_l} m_{it}^{n_m} \omega_{it}^{(m)} \right)$$

$$\hat{\boldsymbol{\alpha}}_l(\tau_q)^{(s+1)} = \underset{\boldsymbol{\alpha}_l(\tau_q)}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \Psi_{\tau_q} \left(l_{it} - \sum_{j=1}^{J} \alpha_{l,j}(\tau_q) \psi_{l,j}(k_{it}, \omega_{it}^{(m)}, t) \right)$$

$$\hat{\boldsymbol{\alpha}}_m(\tau_q)^{(s+1)} = \underset{\boldsymbol{\alpha}_m(\tau_q)}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \Psi_{\tau_q} \left(m_{it} - \sum_{j=1}^{J} \alpha_{m,j}(\tau_q) \psi_{m,j}(k_{it}, \omega_{it}^{(m)}, t) \right)$$

$$\hat{\boldsymbol{\iota}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\iota}(\tau_q)}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \Psi_{\tau_q} \left(i_{it} - \sum_{j=1}^{J} \iota_j(\tau_q) \psi_{\iota,j}(k_{it}, \omega_{it}^{(m)}, t) \right)$$

$$\hat{\boldsymbol{\rho}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\rho}(\tau_q)}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \Psi_{\tau_q} \left(\omega_{it}^{(m)} - \rho_0(\tau_q) - \rho_1(\tau_q)\omega_{it-1}^{(m)} - \rho_2(\tau_q)\omega_{it-1}^{(m)2} - \rho_3(\tau_q)\omega_{it-1}^{(m)3} \right)$$

$$\hat{\boldsymbol{\rho}}_1(\tau_q)^{(s+1)} = \underset{\boldsymbol{\rho}_1(\tau_q)}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \Psi_{\tau_q} \left(\omega_{i1}^{(m)} - \sum_{j=1}^{J} \rho_{1,j}(\tau_q) \psi_{\omega_1,j}(t) \right)$$

7 Application

I apply the estimator to US manufacturers from the Standard and Poors Compustat database. The sample covers publicly traded firms and contains data from their financial statements. I collect a sample between 1990 and 2016 on sales, capital expenditures, property, plant, and equipment, employees, and other expenses to construct output, investment, capital stock, labor, and material inputs. The financial data is deflated using 3-digit deflators from the NBER-CES Manufacturing Industry Database. For firm's age I use the initial date of a firm's IPO. The sample is unbalanced as I do not wish to artificially reduce the sample size by truncating observations for firms who exit.

I use the stochastic EM algorithm presented in this paper. I run the estimation procedure with 500 random walk Metropolis-Hastings steps and 200 EM steps with $M = 1$. I target a 10% acceptance rate by setting the variance of the proposal distribution to $2.38^2/n_t$ where n_t is the number of time-period observations per firm. Once estimates are obtained I simulate productivity from its initial conditions and the decision rules for labor and materials which are aggregated to final output. Using the simulated data, I compute a variety of estimates.

Consider the average quantile marginal effect with respect to capital:

$$\hat{\bar{\beta}}_k(\tau) = \mathbb{E} \left[\frac{\partial Q_\tau(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it})}{\partial k_{it}} \right]$$

I compute this estimator for each input as well as the non-Hicks neutral effects. I use the same formulation to compute estimates of average quantile productivity persistence in addition to computing *individual quantile effects* of the production function with respect to inputs (capital) by

$$\hat{\beta}(\tau_\eta, \tau_k) = \mathbb{E} \left[\frac{\partial Q_{\tau_\eta}(y_{it}|Q_{\tau_k}(k_{it}), l_{it}, m_{it}, \omega_{it})}{\partial k_{it}} \right]$$

where $\hat{\beta}(\tau_\eta, \tau_k)$ is computed by the average derivative with respect to capital, averaged over $(l_{it}, m_{it}, \omega_{it})$ evaluated at percentiles of capital $Q_{\tau_k}(k_{it})$. I also calculate the individual quantile effects of productivity

given by

$$\hat{\rho}(\omega_{it-1}, \tau) = \hat{\rho}_1(\tau) + 2\hat{\rho}_2(\tau)\omega_{it-1} + 3\hat{\rho}_3(\tau)\omega_{it-1}^2$$

Also of interest are the dynamic effects of productivity innovation shocks on inputs. Similar to Hu *et al.* (2019) I estimate how quickly firms respond to shocks to current productivity. This analysis would show whether input decision rules such as labor and materials are subject to adjustment frictions. For example, if I find that labor responds positively to increases in productivity, then policies designed to increase productivity will have a faster effect depending on how quickly the firm is able to adjust how many workers it can hire. My model allows me to examine this effect on two different dimensions: the size of the shock to labor demand and the size of the innovation shock. The estimator is given by

$$\hat{l}(\tau_\epsilon, \tau_\xi) = \mathbb{E}\left[\frac{\partial Q_{\tau_\epsilon}\left(l_{it}|k_{it}, Q_{\tau_\xi}(\omega_{it})\right)}{\omega_{it}} \times \left(\frac{\partial Q_{\tau_\xi}(\omega_{it}|\omega_{it-1})}{\xi_{it}} \right) \right]$$

where $\partial Q_{\tau_\xi}(\omega_{it}|\omega_{it-1})/\partial \xi_{it}$ can be approximated by finite differences.

7.1 Results (Preliminary)
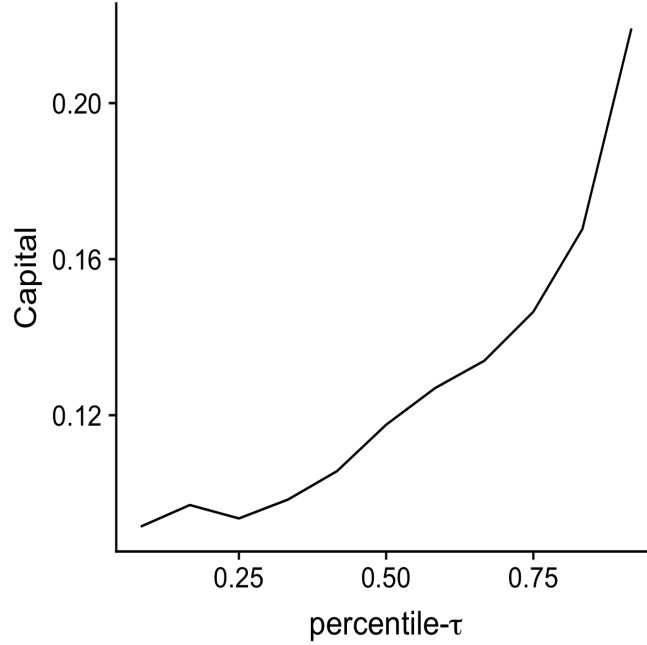
Figure 1: Average Capital Elasticity

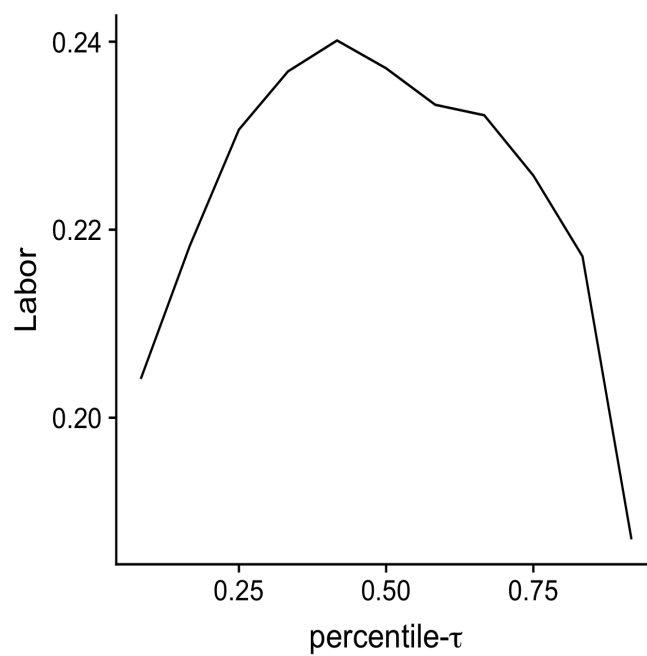Figure 2: Average Labor Elasticity

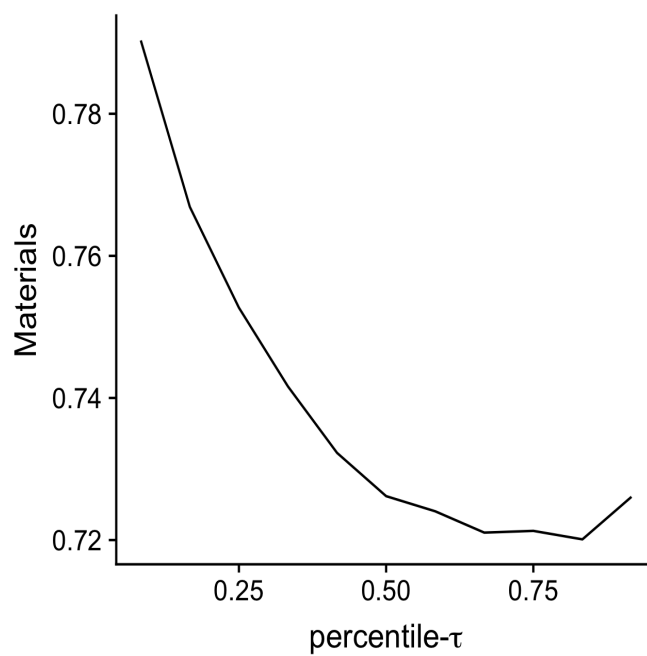Figure 3: Average Materials Elasticity

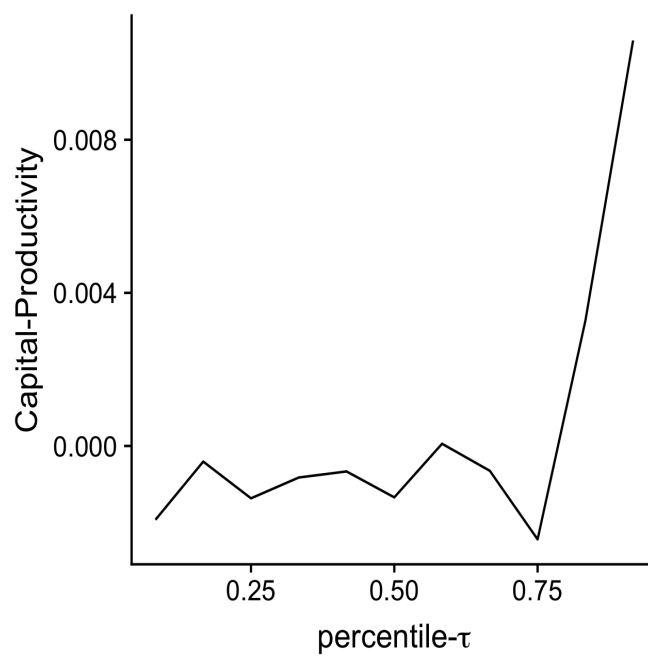Figure 4: Average Capital-Productivity Effect
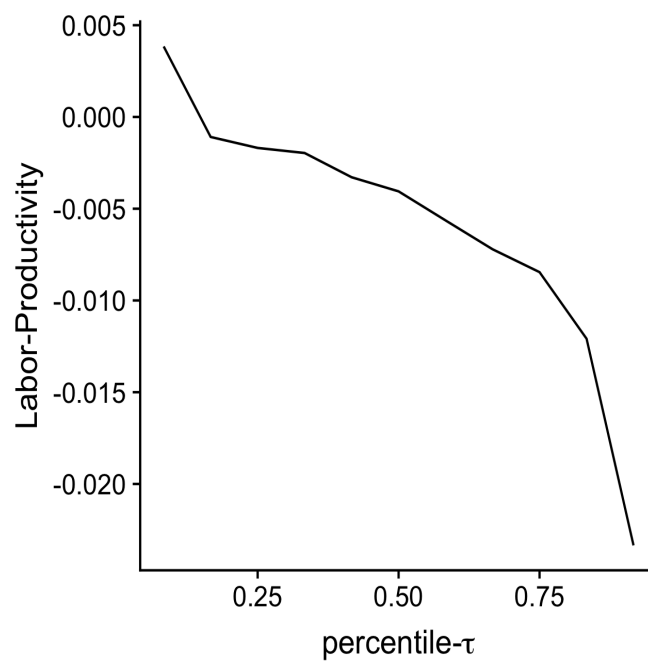

Figure 5: Average Capital-Labor Effect
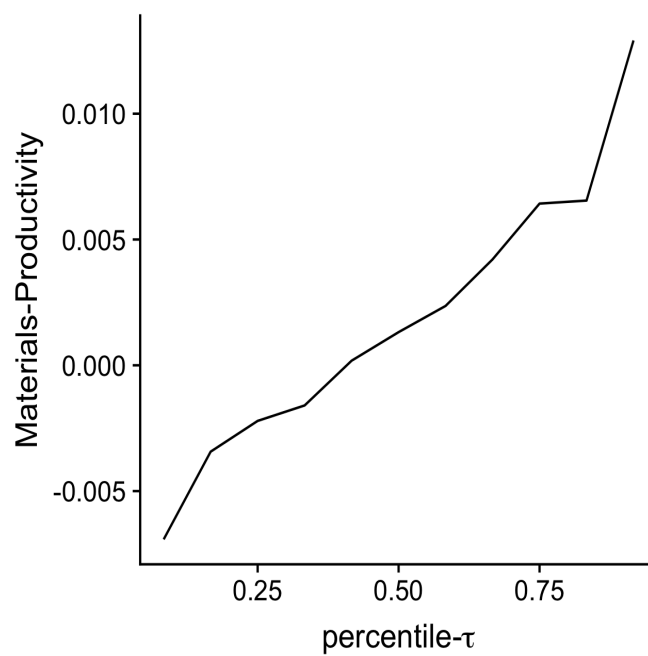
Figure 6: Average Capital-Materials Effect
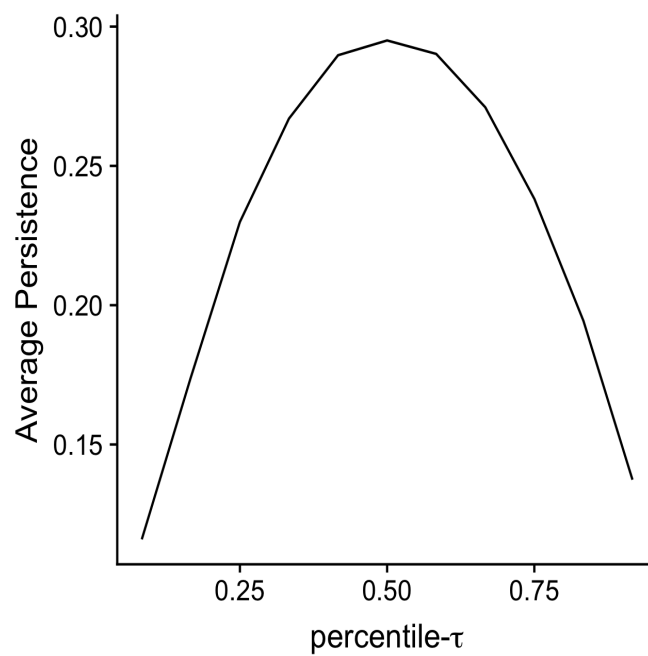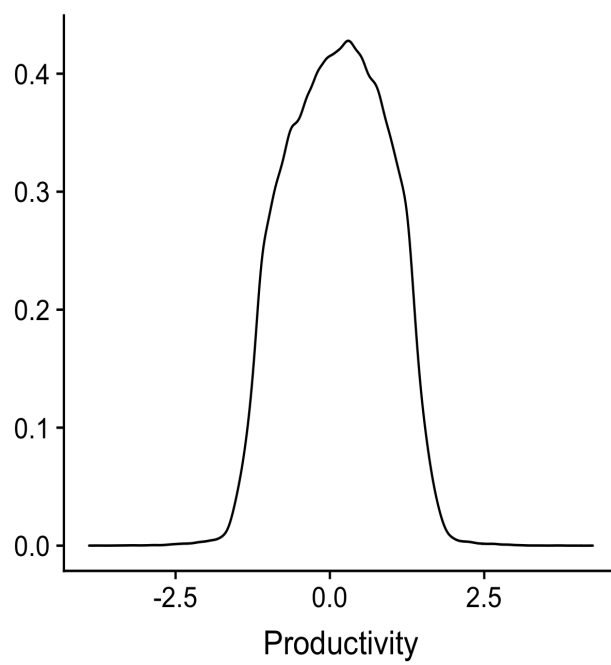

Figure 7: Average Persistence of Productivity

Figure 8: Distribution of Productivity

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