A Dynamic Panel Data Framework for Identification and Estimation of Nonlinear Production Functions

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Introduction: Random Coefficient Model

- Consider the random coefficient gross output production function (in logs)

$$y_{it} = \beta_0(\eta_{it}) + \beta_k(\eta_{it})k_{it} + \beta_l(\eta_{it})l_{it} + \beta_m(\eta_{it})m_{it} + \omega_{it} \qquad (1)$$

  where y_{it} denotes output, l_{it} denotes labor input for firm i at time t, k_{it} denotes capital input, m_{it} denotes material input, ω_{it} is unobserved productivity and η_{it} denotes an iid shock to production.
- If the production function is strictly increasing in η_{it} then the conditional quantile of (1) are

$$Q_\tau(y_{it}|\mathcal{I}_{it}) = \beta_0(\tau) + \beta_k(\tau)k_{it} + \beta_l(\tau)l_{it} + \beta_m(\tau)m_{it} + \omega_{it} \qquad (2)$$

  where $\mathcal{I}_{it}$ is the firm's information set at time t which is independent of η_{it}.
- In this specification, ω_{it} is a location-shifter of the conditional output distribution

Random Coefficient Model

- Doty and Song (2020) show how to identify and estimate this simple linear random coefficient model by extending the control variable approach of Levinsohn and Petrin, 2003 of a value-added production function
- I will briefly review our approach here. They assume the following
  Assumption 1
  1. The production function $y_{it} = f_t(k_{it}, l_{it}, \omega_{it}, \eta_{it})$ is strictly increasing in η_{it}
  2. The firm's information set at time t includes current and past productivity shocks $\{\omega_{it}\}_{t=0}^{t}$, but does not include past productivity shocks $\{\omega_{it}\}_{t=t+1}^{\infty}$. η_{it} is independent of $\mathcal{I}_{it}$
  3. Firm's productivity shocks evolve according to a first-order Markov process

$$\omega_{it} = \rho\omega_{it-1} + \xi_{it} \tag{3}$$

where the iid productivity innovations ξ_{it} are independent of $\mathcal{I}_{it-1}$. We require that $P[\xi_{it} \leq F_{\xi}^{-1}(\tau)|\mathcal{I}_{it-1}] = P[\xi_{it} \leq F_{\xi}^{-1}(\tau)] = \tau$

Random Coefficient Model

Assumption 1 continued

1. Firms accumulate capital according to

$$K_{it} = \kappa_t(I_{it-1}, K_{it-1}). \tag{4}$$

   where K_{it-1} and I_{it-1} denote previous period capital and investment
2. Firm's intermediate input demand function is given by $m_{it} = m_t(k_{it}, \omega_{it})$
3. The intermediate input demand function $m_t(k_{it}, \omega_{it})$ is strictly increasing in ω_{it}

- We invert intermediate input demand $\omega_{it} = m^{-1}(k_{it}, m_{it})$ and substitute into the production function. We treat m_t^{-1} as a nonparametric function (k_{it}, m_{it}). We then have:

$$y_{it} = \beta_k(\eta_{it})k_{it} + \beta_l(\eta_{it})l_{it} + m_t^{-1}(k_{it}, m_{it}) = \beta_l(\eta_{it})l_{it} + \Phi(k_{it}, m_{it}, \eta_{it}) \tag{5}$$

Random Coefficient Model

- Using Assumption 1 we have the following identification condition for the first stage:

$$P\left(y_{it} \leq \beta_l(\tau) l_{it} + \Phi(k_{it}, m_{it}; \tau) \middle| \mathcal{I}_{it}\right) = \tau \tag{6}$$

- We use a linear approximation $\Phi(k_{it}, m_{it}, \eta_{it}) = \beta_k(\eta_{it}) k_{it} + \beta_m(\eta_{it}) m_{it}$ so we can estimate $\beta_l(\tau)$ and $\Phi(k_{it}, m_{it}, \tau)$ using linear quantile regression.
- We write a second stage identification condition in only the ξ_{it} component, similar to Ackerberg, Caves, and Frazer, 2015 by concentrating out the constant $\beta_0(\tau)$ and ρ. For a hypothetical guess of $\beta_k(\tau)$ we can write

$$\beta_0(\tau) + \omega_{it} = y_{it} - \hat{\beta}_l(\tau) l_{it} - \beta_k(\tau) k_{it} = \hat{\Phi}(k_{it}, m_{it}; \tau) - \beta_k(\tau) k_{it} \tag{7}$$

Random Coefficient Model

- We can rewrite the AR(1) productivity process as

$$\hat{\Phi}(k_{it}, m_{it}; \tau) - \beta_k(\tau) k_{it} = \beta_0(\tau) + \rho(\hat{\Phi}(k_{it-1}, m_{it-1}; \tau) - \beta_k(\tau) k_{it-1}) + \xi_{it} \tag{8}$$

and note that

$$Q_\tau(\omega_{it} | \mathcal{I}_{it-1}) = \beta_0(\tau) + \rho(\hat{\Phi}(k_{it-1}, m_{it-1}; \tau) - \beta_k(\tau) k_{it-1}) + F_\xi^{-1}(\tau) \tag{9}$$

- We estimate $(\beta_0(\tau) + F_\xi^{-1}(\tau), \rho)$ using the procedure of He, 1997
  1. First, a median regression of $\hat{\Phi}(k_{it}, m_{it}; \tau) - \beta_k(\tau) k_{it}$ on $\hat{\Phi}(k_{it-1}, m_{it-1}; \tau) - \beta_k(\tau) k_{it-1}$ to obtain estimates of $(\beta_0(\tau), \rho)$
  2. Second, let $\hat{\xi}_{it} = \hat{\Phi}(k_{it}, m_{it}; \tau) - \beta_k(\tau) k_{it} - \hat{\beta}_0(\tau) - \hat{\rho}(\hat{\Phi}(k_{it-1}, m_{it-1}; \tau) - \beta_k(\tau) k_{it-1})$. Then take the τth quantile of $\hat{\xi}_{it}$ as an estimate for $\hat{F}_\xi^{-1}(\tau)$

Random Coefficient Model

- Therefore, we have the following identification condition

$$P[\hat{\xi}_{it}(\beta_k(\tau)) \leq \hat{F}_{\xi}^{-1}(\tau)|\mathcal{I}_{it-1}] = P[\hat{\xi}_{it}(\beta_k(\tau)) \leq \hat{F}_{\xi}^{-1}(\tau)] = \tau \quad (10)$$

- This can be represented by conditional moment restrictions

$$\mathbb{E}[\mathbb{1}\{\hat{\xi}_{it}(\beta_k(\tau)) - \hat{F}_{\xi}^{-1}(\tau) \leq 0\} - \tau|\mathcal{I}_{it-1}] = 0 \quad (11)$$

  where $\mathbb{1}\{\cdot\}$ is the indicator function. To estimate the production function parameters we use the unconditional moments implied by (11)

$$\mathbb{E}[Z_{it-1}(\mathbb{1}\{\hat{\xi}_{it}(\beta_k(\tau)) - \hat{F}_{\xi}^{-1}(\tau) \leq 0\} - \tau)] = 0 \quad (12)$$

- We propose smoothing the indicator function using the methodology proposed by Kaplan and Sun, 2016 and Castro, Galvao, Kaplan, and Liu, 2018 for nonlinear conditional quantile models.

Limitations and Extensions

- Proxy variable approach only works for linear production function models
- Nonlinear production functions would allow for non-Hicks neutral technology shocks
- A more general model could allow for location-scale effects of productivity or even more general distributional effects (e.g $\omega_{it} = \omega_{it}(\tau)$)
- The productivity process could be generalized to a quantile auto-regressive process, allowing for rich distributional effects of innovation shocks on future productivity
- As a consequence, we could document the different effects an innovation shock has on the firm-size distribution which we cannot do in the linear model.
- In order to do this, we apply Arellano and Bonhomme, 2016 and use nonlinear IV identification results from Hu and Schennach, 2008 and Hu and Shum, 2012 for dynamic models

Dynamic Panel Data Framework

- Consider a nonlinear model for a firm's gross-output production function

$$Y_{it} = F_t(K_{it}, L_{it}, M_{it}, \omega_{it}, \eta_{it}) \tag{13}$$

- Without loss of generality we normalize η_{it} to be uniformly distributed on the interval $[0, 1]$
- We assume F is strictly increasing in η_{it}
- Labor inputs are chosen to maximize current period profits and therefore are a function of current period state variables

$$L_{it} = \ell_t(K_{it}, \omega_{it}, \epsilon_{it}) \tag{14}$$

  where ϵ_{it} is iid and independent of current period state variables.
- We assume the labor demand function ℓ is strictly increasing in ϵ_{it} which is normalized to be uniformly distributed on the interval $U[0, 1]$

- Material inputs are chosen to maximize current period profits and therefore are a function of current period state variables

$$M_{it} = \mu_t(K_{it}, \omega_{it}, \varepsilon_{it}) \tag{15}$$

where ε_{it} is iid and independent of current period state variables.

- We assume the material demand function μ is strictly increasing in ε_{it} which is normalized to be uniformly distributed on the interval $U[0, 1]$.
- We can extend this to the case where labor is chosen prior to choosing material inputs in which case we would include L_{it} as a state variable in equation (15).
- Capital accumulates to the following generalized law of motion

$$K_{it} = \kappa_t(K_{it-1}, I_{it-1}, \upsilon_{it-1}) \tag{16}$$

where I_{it-1} denotes firm investment in the prior period.

Dynamic Panel Data Framework

- We introduce a random error term υ_{it-1} which eliminates the deterministic relationship of capital with respect to previous period state and choice variables.
- We assume this error term is independent of the arguments in the capital accumulation law and that the function $\kappa$is strictly increasing in this term and is normalized to be uniformly distributed on the interval $U[0,1]$.
- Productivity evolves according to the exogenous first order Markov process:

$$\omega_{it} = g(\omega_{it-1}, \xi_{it}) \tag{17}$$

- where $\xi_{i1}, \ldots, \xi_{iT}$ are independent uniform random variables which represent innovation shocks to productivity.
- We assume ω_{it} is monotonic in ξ_{it}
- The exogeneity of the productivity process can be relaxed when we consider productivity enhancing activities such as R&D similar to Doraszelski and Jaumandreu, 2013.

Dynamic Panel Data Framework

- We introduce a dynamic model of firm investment that is a slight modification of Ericson and Pakes, 1995
- In each period, a firm chooses investment to maximize its discounted future profits:

$$I_{it} = I^*(K_{it}, \omega_{it}, \zeta_{it}) = \underset{I_t \geq 0}{\text{argmax}} \left[\Pi_t(K_{it}, \omega_{it}, \zeta_{it}) - c(I_{it}) + \beta \mathbb{E}\left[V_{t+1}(K_{it+1}, \omega_{it+1}, \zeta_{it+1}) | \mathcal{I}_t \right] \right], \tag{18}$$

where $\pi_t(\cdot)$ is current period profits as a function of the state variables and an unobservable demand shock ζ_{it}.

- These are shocks to a firm's product demand which are privately observed by each firm and i.i.d across i and t.
- We assume these shocks are independent from the firm's state variables.
- Pakes, 1994 provides specific conditions for which the investment policy function is strictly increasing in its unobservable components.
- Without loss of generality, we normalize $\zeta_t \sim U[0, 1]$

Dynamic Panel Data Framework: Identification

- Our goal is identification of the Markov law of motion $f_{Y_t,L_t,M_t,K_t,I_t\omega_t|Y_{t-1},L_{t-1},M_{t-1},K_{t-1},I_{t-1}\omega_{t-1}}$ which we assume to be stationary.
- We assume the researcher observes a panel dataset consisting of i.i.d observations of firm output and input choices with the number of time periods $T \geq 3$ for a large number of firms.
- We formalize the independence conditions stated in the earlier section as well as conditional independence assumptions for identification.
- For ease of notation we drop the i subscript and let $X_t = (Y_t, K_t, L_t, M_t, I_t)$ denote all the observable data and $V_t = (Y_t.L_t, M_t, I_t)$ denote the variables that depend only on the current state variables (K_t, ω_t)

Dynamic Panel Data Framework: Identification

Assumption: Production Dynamics

1. *First-Order Markov* $f_{X_t,\omega_t|X_{t-1},\omega_{t-1},\mathcal{I}_{t<t-1}} = f_{X_t,\omega_t|X_{t-1},\omega_{t-1}}$
2. *Markov Decision Rules:* $f_{V_t|K_t,X_{t-1},\omega_t} = f_{V_t|K_t,\omega_t}$
3. *Limited Feedback:* $f_{X_t|X_{t-1},\omega_t,\omega_{t-1}} = f_{X_t|X_{t-1},\omega_t}$
4. *Independence*
    1. η_t, ϵ_t and ε_t are mutually independent of ζ_t conditional on $(K_t, L_t, M_t, \omega_t)$
    2. η_t is mutually independent of $\upsilon_{t-2}, \epsilon_{t-1}, \varepsilon_{t-1}$, and ζ_{t-1} conditional on $(K_t, L_t, M_t, \omega_t)$
    3. The error terms $(\eta_t, \epsilon_t, \varepsilon_t, \upsilon_t, \xi_t, \zeta_t)$ are independent of their respective functional arguments

We will proceed in steps in factoring the Markov law of motion into the densities we are interested in identifying. Using Assumption *Production Dynamics:*

$$f_{X_t,\omega_t|X_{t-1},\omega_{t-1}} = f_{Y_t|K_t,L_t,M_t,\omega_t} f_{K_t|K_{t-1},I_{t-1}} f_{I_t|K_t,\omega_t} \times f_{L_t|K_t,\omega_t} f_{M_t|K_t,\omega_t} f_{\omega_t|\omega_{t-1}} \tag{19}$$

Dynamic Panel Data Framework: Identification

- We use the identification results of Hu and Shum, 2012 to identify the Markov law of motion in equation (19) using the fact that investment depends only on current state variables this allows us to use three repeated measures of investment (I_t, I_{t-1}, I_{t-2}) for identification.
- We begin by relating a joint density of observables to unobserved densities. Let $Z_t = (Y_t, L_t, M_t)$ denote the variables that are functions of the current state variables excluding investment.

$$f_{X_t|X_{t-1},X_{t-2}} = \int f_{I_t|K_t,\omega_t} f_{Z_t|K_t,\omega_t} f_{K_t|K_{t-1},I_{t-1}} f_{\omega_t|X_{t-1},X_{t-2}} d\omega_t \tag{20}$$

- The observed density in (20) can be written in operator notation

$$L_{X_t|X_{t-1},X_{t-2}} = L_{I_t|K_t,\omega_t} \Delta_{Z_t|K_t,\omega_t} \Delta_{K_t|K_{t-1},I_{t-1}} L_{\omega_t|X_{t-1},X_{t-2}} \tag{21}$$

We will show that under a set of assumptions, the Markov law of motion in (19) is identified from a eigenvalue-eigenfunction decomposition of (21) using the arguments of Hu and Schennach, 2008.

Dynamic Panel Data Framework: Identification

Assumption: Boundedness The joint density of X_{t+1}, X_t and X_{t-1} is bounded. All conditional and marginal densities are also bounded.

Assumption: Injectivity The operators $L_{I_t|K_t,\omega_t}$, $L_{\omega_t|X_{t-1},X_{t-2}}$, $\Delta_{Z_t|K_t,\omega_t}$ and $\Delta_{K_t|K_{t-1},I_{t-1}}$ are injective

- The above assumption allows us to take inverses of the operators.
- Consider the operator $L_{I_t|K_t,\omega_t}$, following Hu and Schennach, 2008, injectivity of this operator can be interpreted as its corresponding density $f_{I_t|K_t,\omega_t}(I_t|K_t,\omega_t)$ having sufficient variation in ω_t given K_t.
- This assumption is often phrased as completeness condition in the nonparametric IV literature on the density $f_{I_t|K_t,\omega_t}(I_t|K_t,\omega_t)$.
- Sufficient conditions for injectivity can be found in the convolution literature.
- In our econometric model discussed in the next section, we consider specifications that necessary for injectivity without placing extensive restrictions on the economic primitives governing the investment process and productivity evolution.

Dynamic Panel Data Framework: Identification

- Injectivity of the operator $L_{\omega_t|X_{t-1},X_{t-2}}$ is equivalent to showing the injectivity of $L_{\omega_t|X_{t-1},X_{t-2}} = L_{\omega_t|,\omega_{t-1}} L_{\omega_{t-1},X_{t-1},X_{t-2}}$.
- Therefore we require injectivity of both $L_{\omega_t|,\omega_{t-1}}$ and $L_{\omega_{t-1},X_{t-1},X_{t-2}}$.
- The first operator corresponds to the Markov process for productivity $f_{\omega_t|\omega_{t-1}}$. For the second operator, it is injective when productivity evolves exogenously as specified in our model.
- Invertibility of the diagonal operators $\Delta_{Z_t|K_t,\omega_t}$ and $\Delta_{K_t|K_{t-1},I_{t-1}}$ requires the kernels of these operators to be nonzero along its support.
- This is satisfied in our model since we assume Y_t, L_t, M_t, K_t to be strictly increasing in $\eta_t, \epsilon_t, \varepsilon_t, \upsilon_{t-1}$ respectively.
- It assumed that the densities of these error terms are nonzero so that each density is nonzero and furthermore bounded above.

Dynamic Panel Data Framework: Identification

- Our next assumption places mild restrictions on the operator $\Delta_{Z_t|K_t,\omega_t}$ so that the eigenvalues of our decomposition argument are unique.

  Assumption: Uniqueness

  For any Z_t, K_t and any $\bar{\omega}_t \neq \tilde{\omega}_t$: $k(Z_t, K_t, \bar{\omega}_t) \neq k(Z_t, K_t, \tilde{\omega}_t)$ where,

$$k(Z_t, K_t, \omega_t) = f_{Z_t|K_t,\omega_t}(Z_t|K_t, \omega_t) = f_{Y_t|K_t,L_t,M_t,\omega_t}(Y_t|K_t, L_t, M_t, \omega_t) \\ \times f_{L_t|K_t,\omega_t}(L_t|K_t, \omega_t) f_{M_t|K_t,\omega_t}(M_t|K_t, \omega_t) \tag{22}$$

- This requires the density $f_{Z_t|K_t,\omega_t}$ be nonidentical and different values of ω_t.
- This corresponds to the densities $f_{Y_t|K_t,L_t,M_t,\omega_t}$, $f_{L_t|K_t,\omega_t}$ and $f_{M_t|K_t,\omega_t}$ being nonidentical and different values of ω_t.
- It is satisfied when either Y_t, L_t, M_t are strictly increasing in ω_t or if their respective error terms are conditionally heteroskedastic.

Assumption: Monotonicity and Normalization

For any $K_t \in Supp(K_t)$, there exists a known functional M such that $M[f_{I_t|K_t,\omega_t}(I_t|K_t,\omega_t)]$ is monotonic in ω_t. This functional is normalized such that $M[f_{I_t|K_t,\omega_t}(I_t|K_t,\omega_t)] = \omega_t$

- The above assumption is used to pin down the eigenfunctions to each unobserved ω_t.
- In practice, this functional can be the mean, median, mode, or any quantile of the distribution $f_{I_t|K_t,\omega_t}(I_t|K_t,\omega_t)$.
- In either case, this places restrictions on the parameters of the investment process which we discuss in the next later.

Integrating (20) with respect to (Y_t, L_t, M_t) gives us in operator notation:

$$L_{I_t,K_t|X_{t-1},X_{t-2}} = L_{I_t|K_t,\omega_t} \Delta_{K_t|K_{t-1},I_{t-1}} L_{\omega_t|X_{t-1},X_{t-2}} \tag{23}$$

Dynamic Panel Data Framework: Identification

- Combining equation (21) and (23) and using injectivity gives

$$L_{I_t,K_t|X_{t-1},X_{t-2}} L^{-1}_{X_t|X_{t-1},X_{t-2}} = L_{I_t|K_t,\omega_t} \Delta_{Z_t|K_t,\omega_t} L^{-1}_{I_t|K_t,\omega_t} \tag{24}$$

- The left hand side of equation (24) is a function of observable data whereas the right-hand side are the unobservable densities of interest indexed by the unobservable ω_t.
- The following theorem of Hu and Schennach, 2008 gives us our main identification result.

Theorem: Identification Under assumptions *production dynamics, boundedness, injectivity, uniqueness, monotonicity and normalization*, the density $f_{X_t|X_{t-1},X_{t-2}}$ uniquely determines the Markov law of Motion in equation (19).

Dynamic Panel Data Framework: Estimation

- Our empirical specification for the Markovian transitions of productivity, output, and capital evolution closely resemble Arellano, Blundell, and Bonhomme, 2017. We let $(y_{it}, k_{it}, l_{it}, m_{it}, i_{it})$ denote the logarithms of $(Y_{it}, K_{it}, L_{it}, M_{it}, I_{it})$ respectively.

  Output: Let age_{it} denote the age of firm i at time t. We specify the output equation as follows:

$$\begin{aligned} Q_t(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) &= Q(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}, age_{it}, \tau) \\ &= \sum_{j=1}^{J} \beta_j(\tau)\psi_j(k_{it}, l_{it}, m_{it}, \omega_{it}, age_{it}) \end{aligned} \tag{25}$$

  Labor Input: We specify the labor input demand equation as follows:

$$\begin{aligned} Q_t(l_{it}|k_{it}, \omega_{it}, \tau) &= Q(l_{it}|k_{it}, \omega_{it}, age_{it}, \tau) \\ &= \sum_{j=1}^{J} \gamma_j(\tau)\psi_j(k_{it}, \omega_{it}, age_{it}) \end{aligned} \tag{26}$$

Material Input: We specify the material input demand equation as follows

$$\begin{aligned} Q_t(m_{it}|k_{it},\omega_{it},\tau) &= Q(m_{it}|k_{it},\omega_{it},age_{it},\tau) \\ &= \sum_{j=1}^{J}\delta_j(\tau)\psi_j(k_{it},\omega_{it},age_{it}) \end{aligned} \tag{27}$$

Investment Demand: We specify the investment demand equation as

$$\begin{aligned} i_t = \iota_t(k_{it},\omega_{it},\zeta_{it}) &= \iota(k_{it},\omega_{it},age_{it},\zeta_{it}) \\ &= \iota_0 + \sum_{j=1}^{J}\iota_j\psi_j(k_{it},\omega_{it},age_{it}) + \zeta_{it}, \end{aligned} \tag{28}$$

where $E[\zeta_{it}|k_{it},\omega_{it}] = 0$

- The above specification is a nonlinear regression model.
- The conditional mean zero assumption provides the normalization assumption required

Dynamic Panel Data Framework: Estimation

Persistent Productivity: We specify productivity to transition according to:

$$Q_t(\omega_{it-1}, \tau) = Q(\omega_{it-1}, age_{it}, \tau) = \sum_{j=1}^{J} \rho_j(\tau)\psi_j(\omega_{it-1}, age_{it}) \tag{29}$$

The quantile function for ω_{i1} is specified in a similar way

$$Q(\omega_{i1}, age_{i1}, \tau) = \sum_{j=1}^{J} \rho_j^1(\tau)\psi_j(\omega_{i1}, age_{i1}) \tag{30}$$

- A specification for the capital accumulation process may not be needed since it does not depend on ω_{it}
- Given these specifications, we discuss how the model restrictions yield a viable estimation strategy

Dynamic Panel Data Framework: Estimation

- To ease notation, we let the finite and functional parameters be indexed by a finite dimensional parameter vector θ.
- We model the functional parameters using Wei and Carroll, 2009 and Arellano and Bonhomme, 2016. For example, the function $\beta_j(\tau_q)$ is modeled as a piecewise-polynomial interpolating splines on a grid $[\tau_1, \tau_2], [\tau_3, \tau_4], \ldots, [\tau_{Q-1}, \tau_Q]$, contained in the unit interval and is constant on $[0, \tau_1]$ and $[\tau_Q, 1)$
- The intercept coefficient β_0 is specified as the quantile of an exponential distribution on $(0, \tau_1]$ (indexed by λ^-) and $[\tau_{Q-1}, 1)$ (indexed by λ^+).
- The remaining functional parameters are modeled similarly. We take $Q = 11$ and $\tau_q = \frac{q}{Q+1}$.
- Following Arellano, Blundell, and Bonhomme, 2017 we parameterize the distribution of ζ_{it} to be log-normal so we set, for example, $\iota_0(\tau_q) = \iota_0 + \sigma_\zeta \Phi^{-1}(\tau_q)$. In the following section we outline the model's restrictions and a feasible estimation strategy.

Dynamic Panel Data Framework: Estimation

- Let $\Psi_\tau(u) = \tau - \mathbb{1}\{u < 0\}$ denote the first derivative of the quantile check function $\psi_\tau(u) = (\tau - \mathbb{1}\{u < 0\})u$.
- The following conditional moment restrictions hold as an implication of the conditional independence restrictions. Therefore, we estimate the parameters of interest from the following conditional moment restrictions.
- To fix ideas, we focus on how to estimate the production function and investment equation

$$\mathbb{E}\left[\Psi_{\tau_q}(\eta_{it})\middle| k_{it}, l_{it}, m_{it}, age_{it}\right] = 0 \tag{31}$$

$$\mathbb{E}\left[\zeta_{it}\middle|\omega_{it}, k_{it}, age_{it}\right] = 0 \tag{32}$$

Dynamic Panel Data Framework: Estimation

Rewriting these moment conditions as:

$$\mathbb{E}\left[\Psi_{\tau_q}(\eta_{it})\middle| k_{it}, l_{it}, m_{it}, \omega_{it}, age_{it}\right] =$$
$$\mathbb{E}\left[\Psi_{\tau_q}(y_{it} - \sum_{j=1}^{J} \bar{\beta}_j(\tau_q)\psi_j(k_{it}, l_{it}, m_{it}, \omega_{it}, age_{it}))\middle| k_{it}, l_{it}, m_{it}, \omega_{it}, age_{it}\right] = 0 \tag{33}$$

and

$$\begin{aligned}\mathbb{E}\left[\zeta_{it}\middle| k_{it}, \omega_{it}, age_{it}\right] =& \\ \mathbb{E}\left[i_{it} - \bar{\iota_0} - \sum_{j=1}^{J} \bar{\iota}_j\psi_j(k_{it}, \omega_{it}, age_{it})\middle| k_{it}, \omega_{it}, age_{it}\right] &= 0\end{aligned} \tag{34}$$

Dynamic Panel Data Framework: Estimation

- Here $\bar{\beta}_j(\tau_q), \bar{\iota_0}$ and $\bar{\iota_j}$ denote the true values of $\beta_j(\tau_q), \iota_0$ and ι_j for $j \in \{1, \ldots, J\}$ and $q \in \{1, \ldots, Q\}$.
- Clearly, estimating the above conditional moment restrictions are infeasible due to the unobserved productivity component.
- Therefore, we use the following unconditional moment restrictions and posterior distributions for ω_{it} to integrate out the unobserved productivity.
- Due to the law of iterated expectations we now have the following integrated moment conditions:

$$\mathbb{E}\left[\int \left(\Psi_{\tau_q}(y_{it} - \sum_{j=1}^{J} \bar{\beta}_j(\tau_q)\psi_j(k_{it}, l_{it}, m_{it}, \omega_{it}, age_{it})) \otimes \begin{pmatrix} k_{it} \\ l_{it} \\ m_{it} \\ \omega_{it} \\ age_{it} \end{pmatrix} \right) f_i(\omega_{it}; \bar{\theta}) d\omega_{it}\right] = 0 \tag{35}$$

and for investment

$$\mathbb{E}\left[\int\left(\left(i_{it}-\bar{\iota_0}-\sum_{j=1}^{J}\bar{\iota}_j\psi_j(k_{it},\omega_{it},age_{it})\right)\otimes\begin{pmatrix}1\\k_{it}\\\omega_{it}\\age_{it}\end{pmatrix}\right)f_i(\omega_{it};,\bar{\theta})d\omega_{it}\right]=0, \tag{36}$$

where $\bar{\theta}$ denotes the true values of θ. The posterior distribution is specified as (age omitted for ease of notation):

$$\begin{aligned}f_i(\omega_{it};\bar{\theta}) &= f(\omega_{it}|y_{it},k_{it},l_{it},m_{it},i_{it};\bar{\theta}) \propto \\ &\prod_{t=1}^{T} f(y_{it}|k_{it},l_{it},m_{it},\omega_{it};\bar{\theta})f(l_{it}|k_{it},\omega_{it};\bar{\theta})f(m_{it}|k_{it},\omega_{it};\bar{\theta}) \\ &\times f(i_{it}|k_{it},\omega_{it};\bar{\theta})f(\omega_{i1};\bar{\theta})\prod_{t=2}^{T} f(\omega_{it}|\omega_{it-1};\bar{\theta})\end{aligned} \tag{37}$$

Dynamic Panel Data Framework: Estimation

- The posterior density in equation (37) is a closed-form expression when using piecewise linear splines for $\theta(\cdot)$.
- The estimation is an Expectation Maximization (EM) algorithm. In Arellano and Bonhomme, 2016 and Arellano, Blundell, and Bonhomme, 2017, the "M-step" is performed using quantile regression.
- Given an initial parameter value $\hat{\theta}^0$. Iterate on $s = 0, 1, 2, \ldots$ in the following two-step procedure until converge to a stationary distribution:

1. *Stochastic E-Step*: Draw M values $\omega_i^{(m)} = (\omega_{i1}^{(m)}, \omega_{i2}^{(m)}, \ldots, \omega_{iT}^{(m)})$ from

$$
\begin{aligned}
f_i(\omega_{it}; \hat{\theta}^{(s)}) = f(\omega_{it} | y_{it}, k_{it}, l_{it}, m_{it}, i_t, ; \hat{\theta}^{(s)}) \propto \\
\prod_{t=1}^{T} f(y_{it} | k_{it}, l_{it}, m_{it}, \omega_{it}; \hat{\theta}^{(s)}) f(l_{it} | k_{it}, \omega_{it}; \hat{\theta}^{(s)}) f(m_{it} | k_{it}, \omega_{it}; \hat{\theta}^{(s)}) \\
\times f(i_{it} | k_{it}, \omega_{it}; \hat{\theta}^{(s)}) f(\omega_{i1}; \hat{\theta}^{(s)}) \prod_{t=2}^{T} f(\omega_{it} | \omega_{it-1}; \hat{\theta}^{(s)})
\end{aligned}
$$

2. *Maximization Step*: For $q = 1, \ldots, Q$, solve

$$\hat{\boldsymbol{\beta}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\beta}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \Psi_{\tau_q} \left(y_{it} - \sum_{j=1}^{J} \beta_j(\tau_q) \psi_j(k_{it}, l_{it}, m_{it}, \omega_{it}^{(m)}, age_{it}) \right)$$

$$\hat{\boldsymbol{\iota}}^{(s+1)} = \underset{\boldsymbol{\iota}}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \left(i_{it} - \iota_0 - \sum_{j=1}^{J} \iota_j \psi_j(k_{it}, \omega_{it}^{(m)}, age_{it}) \right)^2$$

- The parameters of the production function equation in (36) can be estimated using a nonlinear regression for a given draw of $\omega_{it}^{(m)}$.
- Then, the variance of the shock ζ_{it} can be estimated using

$$\hat{\sigma_\zeta}^2 = \frac{1}{NTM} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \left[\left(i_{it} - \hat{\iota_0} - \sum_{j=1}^{J} \hat{\iota}_j \psi_j(k_{it}, \omega_{it}^{(m)}, age_{it}) \right)^2 \right] \tag{38}$$

so that $\hat{\iota_0}(\tau_q) = \hat{\iota_0} + \hat{\sigma_\zeta} \Phi^{-1}(\tau_q)$

Conclusion

- Identification strategy is not new, most economists have avoided it due to injectivity assumptions and panel time length requirements
- These data requirements become even more stringent if other unobservables (firm fixed effects) are added
- Implementation in progress
- The estimation procedure is novel and allows us to document a variety of heterogeneous quantile marginal effects, for example, the average quantile marginal effect of labor would be

$$\phi_t(k_t, l_t, m_t, \tau) = \mathbb{E}\left[\frac{\partial Q(y_t | k_t, l_t, m_t, \omega_t, \tau)}{\partial l_t}\right] \tag{39}$$

- Distributional effects of productivity

$$\phi_t(k_t, l_t, m_t, \omega_t, \tau) = \frac{\partial Q(y_t | k_t, l_t, m_t, \omega_t, \tau)}{\partial \omega_t} \tag{40}$$

Conclusion

- Dynamic effects of innovation shocks on production

$$\frac{\partial}{\partial \xi_{it}} \left[\frac{\partial Q(y_t | k_t, l_t, m_t, Q_t(\omega_{t-1}, \xi_t), \tau)}{\partial \omega_t} \right] \Bigg|_{\xi = \tau_\xi} \\ = \phi_t(k_t, l_t, m_t, Q_t(\omega_{t-1}, \tau_\xi), \tau) \frac{\partial Q_t(\omega_{t-1}, \tau_\xi)}{\partial \xi} \tag{41}$$

- "Impulse response" functions can be written as

$$\frac{\partial}{\partial \omega_{t-1}} \left[\frac{\partial Q_t(\omega_{t-1}, \tau_\xi)}{\partial \xi} \right] \tag{42}$$

  and can be approximated by finite differences across τ_ξ to study the asymmetric impacts of innovation shocks at different points on the output distribution over firm age

- Many more interesting effects can be formalized and calculated