A Dynamic Framework for Identification and Estimation of Nonseparable Production Functions

Job Market Presentation

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Introduction

- This paper identifies and estimates a **nonseparable** production function with **unobserved heterogeneity**
- Two important contributions to the literature
  1. New nonparametric estimates of firm-specific production functions
     - Nonseparable model allows interactions between unobserved production shocks and inputs
     - Captures sources of unobserved heterogeneity arising from factor-specific productivity changes
     - Important implications for models that use production function estimates
  2. New framework to capture heterogeneous productivity dynamics
     - Incorporates **asymmetric persistence** in productivity history
     - Driven by size and sign of productivity shocks
     - Can explain variation in productivity rankings across time

Production Functions

- Production functions are important in many economic models
- They link outputs to inputs (e.g. capital, labor) and represents firm technology
- Estimates can be used in the following applications:
  1. Measuring returns to scale
  2. Gains from trade
  3. Capital misallocation
  4. Estimates of market power (e.g. markups)
- A correctly specified production function is crucial for correct inference in these areas
- For example, biased flexible input elasticities are transmitted to markup estimates

Simultaneity Bias

- Significant progress in solving one type of bias in production function estimation: **simultaneity bias**
- Researchers don't observe productivity
- Firm chooses inputs depending on their productivity
- A more productive firm may hire more/less workers
- In this case, labor estimates will be biased
- **Proxy variable** approaches of OP, LP, ACF remain a popular tool
- Basic idea: A policy function (e.g. material demand) is inverted as a function of productivity
- Substitute inverted function into production function and estimate in two-step approach

Limitations to Proxy Variables

- This approach relies on many crucial assumptions
- No unobserved errors in the policy function
  - No measurement error
  - No unobserved demand shocks
- Productivity and an unobserved production shock are additive (e.g. log-Cobb-Douglas)
  - Implies productivity is "factor-neutral"
  - Technology is fixed across firms
  - Estimates only capture average firm production
- Productivity process is subject to additive shocks
  - Empirical evidence is in favor of more flexible productivity dynamics
  - Since productivity is transmitted to inputs it is important to understand its dynamics to test model validity

Why Nonseparable Models

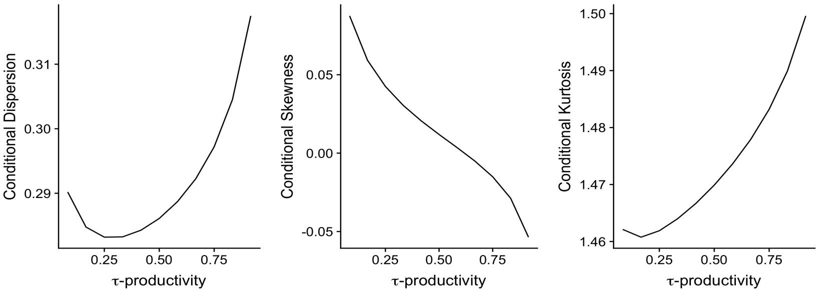
- Empirical research points to missing heterogeneity from technological change, which favors inputs such as labor
- Labor augmenting productivity explains much of the variation in markups over time (Dermirer, 2020)
- It is also the primary driver of the fall in labor shares across many advanced economies (Doraszelski and Jaumandreu, 2018)
- Recent advancements have used a structural approach to estimating factor-specific productivity
- These rely on either a parametric or nonparametric inversion of policy functions
- These techniques are still invalid in the presence of unobservables in the policy functions

Why Nonseparable Models

- Common finding is that productivity is significantly heterogeneous across firms even in narrowly defined industries
- The cross-sectional distribution of productivity varies over time
- These dynamic effects can change the production function at different points in time
- A more flexible productivity process may provide insight to how firms asymmetrically adjust inputs and output in accordance to shocks
- Large degree of persistence in productivity rankings in the U.S. (Bartelsman and Dhrymes, 1998)
- High and lower productivity firms may change productivity rankings depending on the size of sign of shocks

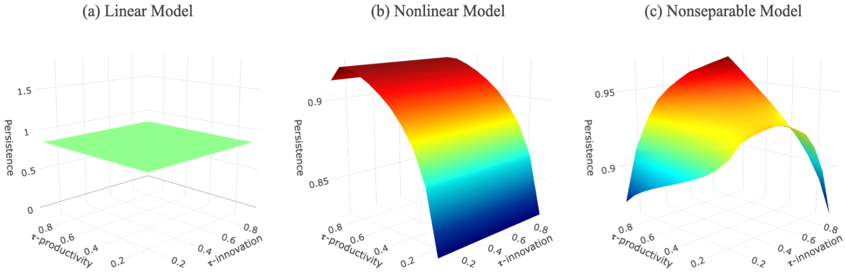
Preview of Results

Figure 1: Summary Statistics of Productivity Dynamics



Preview of Results

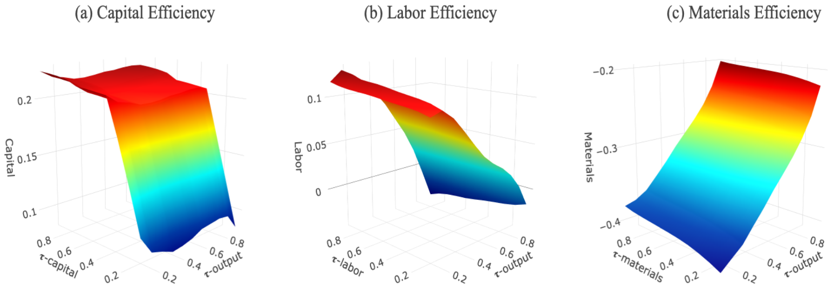
Figure 2: Productivity Persistence



*Panel (a): Productivity persistence from a linear model. Panel (b): Productivity persistence for a nonlinear model that is separable in unobserved shocks. Panel (c): Productivity persistence estimated in the nonseparable model.

Preview of Results

Figure 3: Non-Hicks Neutral Elasticities



*Panel (a): Capital efficiency evaluated at τ_η and percentiles of capital τ_k averaged over values of (l_{it}, m_{it}) that correspond to τ_k. Panel (b): Labor efficiency evaluated at τ_η and percentiles of labor τ_l averaged over values of (k_{it}, m_{it}). Panel (c): Materials efficiency evaluated at τ_η and percentiles of materials τ_m averaged over values of (k_{it}, l_{it}).

Summary of Findings

Data: U.S. Compustat public manufacturing firms

- Asymmetric persistence in productivity
  1. Positive shocks for low productivity firms
  2. Negative shocks for high productivity firms

  have lower persistence of productivity
- Asymmetric adjustments of inputs with respect to productivity and shocks
- Length of time until recovery from bad productivity shocks varies by firm
- Nonlinearities in production function estimates
  1. Positive capital productivity effects
  2. Positive and negative labor productivity effects
  3. Negative material productivity effects

Outline for the Rest of Talk

1. ~~Introduction~~
2. Economic Model
3. Econometric Identification
4. Econometric Procedure and Quantile Modelling
5. Results
6. Conclusions

The Production Function

- Consider a nonseparable model for a firm's gross-output production function (in logs)

$$y_{it} = f_t(k_{it}, l_{it}, m_{it}, \omega_{it}, \eta_{it}) \tag{1}$$

- Allows for non-linear interactions between inputs and unobserved productivity
- Assume the following

Assumption 1 (Production Function)

1. *The unanticipated production shocks η_{it} are i.i.d. over firms and time.*
2. *The unanticipated production shock η_{it} follows a standard uniform distribution independent of $(k_{it}, l_{it}, m_{it}, \omega_{it},)$.*
3. $\tau \rightarrow Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau)$ *is strictly increasing on* $(0, 1)$.

Quantile Function

- For a given τ, the conditional quantile function for the random variable $Y|X$ is defined as

$$Q_\tau(Y|X) = \inf\{y \in \mathbb{R} : \tau \leq F_{Y|X}(y|x)\}, \quad \tau \in (0,1)$$

  where $F_{Y|X}$ is continuous and strictly increasing
- This quantile assumption will be important later on as it allows me to recover estimates of firm specific production functions, input demands, and productivity
- The standard production function and quantiles is an unexplored area
- Doty and Song, 2021 consider the case for a simple additive-in-productivity model and discuss some of its implications

Productivity

- Productivity evolves according to an exogenous first-order Markov process given by

$$\omega_{it} = Q_t^{\omega}(\omega_{it-1}, \xi_{it}), \quad \xi_{it} \sim \mathit{Uniform}(0,1), \tag{2}$$

  where $\xi_{i1}, \ldots, \xi_{iT}$ are independent uniform random variables which represent innovation shocks to productivity
- The function Q^{ω} is a nonlinear function that allows the persistence in productivity in firms to be nonlinear across different quantiles

Assumption 2 (Productivity)

1. *The productivity innovation shocks ξ_{it} are i.i.d. across firms and time.*
2. *ξ_{it} follows a standard uniform distribution independent of previous period productivity ω_{it-1}.*
3. *$\tau \to Q_t^{\omega}(\omega_{it-1}, \tau)$ is strictly increasing on $(0,1)$.*

Flexible Inputs

- Labor and Material inputs are chosen to maximize current period profits
- Therefore they are a function of current period state variables

$$l_{it} = Q_t^{\ell}(k_{it}, \omega_{it}, \epsilon_{\ell,it}), \quad \epsilon_{\ell,it} \sim Uniform(0,1), \tag{3}$$

$$m_{it} = Q_t^{m}(k_{it}, l_{it}, \omega_{it}, \epsilon_{m,it}), \quad \epsilon_{m,it} \sim Uniform(0,1), \tag{4}$$

- $\epsilon_{\ell,it}$ and $\epsilon_{m,it}$ are i.i.d. unobservable input demand shocks that are assumed to be independent of current period state variables
- In the control function approach, with material inputs as a proxy, this function could not be inverted as an expression of productivity only
- This can also be extended to the case where labor has adjustment frictions

Flexible Inputs

Assumption 3 (Flexible Inputs)

1. *The unobserved input demand shocks* $\epsilon_{\ell,it}$ *and* $\epsilon_{m,it}$ *are i.i.d. across firms and time.*
2. $\epsilon_{\ell,it}$ *and* $\epsilon_{m,it}$ *follow a standard uniform distribution independent of* (k_{it}, ω_{it}) *and* $(k_{it}, l_{it}, \omega_{it})$*, respectively.*
3. $\tau \to Q_t^{\ell}(k_{it}, \omega_{it}, \tau)$ *and* $\tau \to Q_t^{m}(k_{it}, l_{it}, \omega_{it}, \tau)$ *are strictly increasing on* $(0, 1)$*.*

Capital and Investment

- Capital accumulates to the following generalized law of motion

$$K_{it} = \kappa_t(K_{it-1}, I_{it-1}, \upsilon_{it-1}) \tag{5}$$

where I_{it-1} denotes firm investment in the prior period

- Eliminates the deterministic relationship of capital with respect to previous period state and choice variables
- Assume this error term is independent of the arguments in the capital accumulation law
- In each period, a firm chooses investment to maximize its discounted future profits:

$$I_{it} = \iota_t(K_{it}, \omega_{it}) = \underset{I_t \geq 0}{\text{argmax}} \left[\Pi_t(K_{it}, \omega_{it}) - c(I_{it}, \omega_{it}) + \beta \mathbb{E}\left[V_{t+1}(K_{it+1}, \omega_{it+1}) | \mathcal{I}_t\right] \right], \tag{6}$$

Capital and Investment

- $\pi_t(\cdot)$ is current period profits as a function of the state variables $c(\cdot)$ is the cost function, $\mathcal{I}_t$ is information set
- Empirical investment rule is

$$i_{it} = Q_t^i(k_{it}, \omega_{it}, \zeta_{it}), \quad \zeta_{it} \sim Uniform(0,1). \tag{7}$$

Assumption 4 (Capital Accumulation and Investment)

1. *The unobserved investment demand shocks ζ_{it} is i.i.d. across firms and time.*
2. *ζ_{it} follows a standard uniform distribution independent of (k_{it}, ω_{it}).*
3. *The production shock η_{it} and ζ_{it} are independent conditional on $(k_{it}, l_{it}, m_{it}, \omega_{it})$. In addition, υ_{it} is independent of η_{it} conditional on $(k_{it}, l_{it}, m_{it}, \omega_{it})$*
4. *$\tau \to Q_t^i(k_{it}, \omega_{it}, \tau)$ is strictly increasing on $(0,1)$*

Nonparametric Identification

- I show that the conditional densities corresponding to production, inputs, and productivity are nonparametrically identified
- Let $Z_t = (l_t, k_t, m_t, k_{t+1})$ denote conditioning variables

Assumption 5 (Conditional Independence)

1. $f(y_t|y_{t+1}, I_t, \omega_t, Z_t) = f(y_t|\omega_t, Z_t)$
2. $f(y_{t+1}|I_t, \omega_t, Z_t) = f(y_{t+1}|\omega_t, Z_t)$

- First equality states that conditional on ω_t and Z_t, y_{t+1} and I_t do not provide any additional information about y_t
- Second equality states that conditional on ω_t and Z_t, I_t does not provide any additional information about y_{t+1}
- Satisfied by mutual independence assumptions on η_t and ζ_t conditional on $(\omega_t, k_t, l_t, m_t)$

Nonparametric Identification

- Begin by relating a conditional density as a function of observable to densities containing unobserved productivity
- Using the conditional independence assumption, I can write

$$f_{y_t, I_t | y_{t+1}, Z_t} = \int f_{y_t | Z_t, \omega_t} f_{I_t | Z_t, \omega_t} f_{\omega_t | y_{t+1}, Z_t} d\omega_t \tag{8}$$

- The identification strategy follows HS by using a eigenvalue-eigenfunction decomposition of integral operators of (8)

Definition 1

(Integral Operator) *Let a and b denote random variables with supports $\mathcal{A}$ and $\mathcal{B}$. Given two corresponding spaces $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ of functions with domains $\mathcal{A}$ and $\mathcal{B}$, let $L_{b|a}$ denote the operator mapping $g \in \mathcal{G}(\mathcal{A})$ to $L_{b|a} g \in L_{b|a} \mathcal{G}(\mathcal{B})$ defined by*

$$[L_{b|a} g](b) \equiv \int_{\mathcal{A}} f_{b|a}(b|a) g(a) da,$$

where $f_{b|a}$ denotes the conditional density of b given a.

Nonparametric Identification

- The observed density in (8) can be written in operator notation

$$L_{y_t,I_t|y_{t+1},Z_t} = L_{y_t|Z_t,\omega_t}\Delta_{I_t|Z_t,\omega_t}L_{\omega_t|y_{t+1},Z_t} \tag{9}$$

- Will show that under a set of assumptions, the conditional density is identified from an eigenvalue-eigenfunction decomposition of (9)

Assumption 6 (Injectivity)

The operators $L_{y_t|Z_t,\omega_t}$ *and* $L_{y_{t+1}|Z_t,\omega_t}$ *are injective*

- The above assumption allows us to take inverses of the operators.
- Consider the operator $L_{y_t|Z_t,\omega_t}$, injectivity of this operator can be interpreted as its corresponding density $f_{y_t|Z_t,\omega_t}(I_t|K_t,\omega_t)$ having sufficient variation in ω_t given Z_t.
- This assumption is often phrased as completeness condition in the nonparametric IV literature on the density $f_{y_t|Z_t,\omega_t}(y_t|Z_t,\omega_t)$.

Nonparametric Identification

Assumption 7 (Uniqueness)

For any $\bar{\omega}_t, \tilde{\omega}_t \in \Omega$, the set $\{f_{I_t|\omega_t,Z_t}(I_t|\bar{\omega}_t, Z_t) \neq f_{I_t|\omega_t,Z_t}(I_t|\tilde{\omega}_t, Z_t)\}$ has positive probability whenever $\bar{\omega}_t \neq \tilde{\omega}_t$.

- This assumption is relatively weak
- Satisfied if there is conditional heteroskedasticity in $f_{I|\omega,Z}$
- Satisfied if any functional of its distribution is strictly increasing in ω_t
- I assume $E[I_t|\omega_t, Z_t]$ is strictly increasing in ω_t
- Similar to the invertibility condition in Olley and Pakes, 1996

Nonparametric Identification

Assumption 8 (Normalization)

There exists a functional Γ *such that* $\Gamma[f_{y_t|\omega_t,Z_t}(y_t|\omega_t, Z_t)] = \omega_t$.

- This functional does not need to be known
- Sufficient to consider a known function of the data distribution as shown by Arellano and Bonhomme, 2016
- In my empirical application, I consider a nonseparable translog production function
- The assumption can be satisfied by the normalization $E[y_t|\omega_t, 0] = \omega_t$
- For more generalized production functions, if $E[y_t|\omega_t, Z_t]$ is strictly increasing in ω_t, then one could normalize $\omega_t = E[y_t|\omega_t, Z_t]$
- These restrictions are easily adaptable in estimation as it amounts to centering the coefficients in the model

Nonparametric Identification

Theorem 1 (Identification)

Under Assumptions 5, 6, 7, and 8, given the observed density $f_{y_t, I_t | y_{t+1}, Z_t}$*, the equation*

$$f_{y_t, I_t | y_{t+1}, Z_t} = \int f_{y_t | \omega_t, Z_t} f_{I_t | \omega_t, Z_t} f_{\omega_t | y_{t+1}, Z_t} d\omega_t \qquad (10)$$

admits a unique solution for $f_{y_t | \omega_t, Z_t}$, $f_{I_t | \omega_t, Z_t}$*, and* $f_{\omega_t | y_{t+1}, Z_t}$

- The proof follows using Hu and Schennach, 2008
- However it does not directly identify the Markov transition function for productivity $f_{\omega_{it+1} | \omega_{it}}(\omega_{it+1} | \omega_{it})$

Nonparametric Identification

Corollary 1 (Identification of Markov Process: Stationarity Case)

Suppose that the production function is stationary, $f_{y_t|\omega_t,Z_t} = f_{y_1|\omega_1,Z_1}, \forall t \in \{1, \cdots, T\}$*. Then, under Assumptions 5, 6, 7, and 8, the observed density* $f_{y_t,I_t|y_{t+1},Z_t}$*, uniquely determines the density* $f_{\omega_{t+1}|\omega_t}, \forall t \in \{1, \ldots, T-1\}$

Corollary 2 (Identification of Markov Process: Non-Stationarity Case)

Under Assumptions 5, 6, 7, and 8, the observed density $f_{y_{t+1},I_{t+1}|y_{t+2},Z_{t+1}}$*, uniquely determines the density* $f_{\omega_{t+1}|\omega_t}, \forall t \in \{1, \ldots, T-2\}$

Econometric Procedure: Production

- The production function is specified as Translog with non-Hicks neutral effects

$$
\begin{aligned}
&Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) = \\
&\quad \gamma_0(\tau) + (\gamma_k(\tau) + \sigma_k(\tau)\omega_{it})k_{it} + (\gamma_l(\tau) + \sigma_l(\tau)\omega_{it})l_{it} + (\gamma_m(\tau) + \sigma_m(\tau)\omega_{it})m_{it} \\
&\quad + (\gamma_{kl}(\tau) + \sigma_{kl}(\tau)\omega_{it})k_{it}l_{it} + (\gamma_{lm}(\tau) + \sigma_{lm}(\tau)\omega_{it})l_{it}m_{it} + (\gamma_{km}(\tau) + \sigma_{km}(\tau)\omega_{it})k_{it}m_{i} \\
&\quad + (\gamma_{kk}(\tau) + \sigma_{kk}(\tau)\omega_{it})k_{it}^2 + (\gamma_{ll}(\tau) + \sigma_{ll}(\tau)\omega_{it})l_{it}^2 + (\gamma_{mm}(\tau) + \sigma_{mm}(\tau)\omega_{it})m_{it}^2 + \sigma_\omega(
\end{aligned}
\tag{11}
$$

- Similar model was estimated by Ackerberg and Chen (2015)
- In my approach I can simulate productivity from estimated intital conditions and Markov process to compute average derivative effects
- Provides a better picture of heterogeneity instead of reporting individual coefficients

Econometric Procedure: Productivity

- I specify productivity using 3rd order polynomial

$$Q^{\omega}(\omega_{it-1}, \tau) = \rho_0(\tau) + \rho_1(\tau)\omega_{it-1} + \rho_2(\tau)\omega_{it-1}^2 + \rho_3(\tau)\omega_{it-1}^3. \tag{12}$$

- Initial productivity

$$Q^{\omega_1}(k_{i1}, \tau) = \sum_{j=1}^{J} \rho_{\omega_1,j}(\tau)\phi_{\omega_1,j}(k_{i1}), \tag{13}$$

- I can also consider the case where productivity may evolve endogenously as Doraszelski and Jaumandreu, 2013

$$Q^{\omega}(\omega_{it-1}, r_{it-1}, \tau) = \mathbb{1}\{R_{it-1} = 0\}Q^{\omega}(\omega_{it-1}, \tau) + \mathbb{1}\{R_{it-1} > 0\}Q^{\omega,r}(\omega_{it-1}, r_{it-1}, \tau). \tag{14}$$

Econometric Procedure: Flexible Inputs

- I specify the labor input demand equation as follows:

$$Q_t^{\ell}(k_{it}, \omega_{it}, \tau) = \sum_{j=1}^{J} \alpha_{\ell,j}(\tau)\phi_{\ell,j}(k_{it}, \omega_{it}), \tag{15}$$

where $\phi_{\ell,j}$ can be another non-linear function

- Material inputs are specified as

$$Q_t^{m}(k_{it}, l_{it}, \omega_{it}, \tau) = \sum_{j=1}^{J} \alpha_{m,j}(\tau)\phi_{m,j}(k_{it}, l_{it}, \omega_{it}), \tag{16}$$

- Again, $\phi_{m,j}$ can be a non-linear function

Econometric Procedure: Investment

- The investment demand function is specified as

$$i_{it} = Q_t^i(k_{it}, \omega_{it}, \tau) = \sum_{j=1}^{J} \delta_j(\tau)\phi_{i,j}(k_{it}, \omega_{it}), \tag{17}$$

  where $\phi_{\iota,j}$ is specified similarly as the labor and material input decision rule.

- In the case where investment is censored, I can write

$$Q_t^{i^*}(k_{it}, \omega_{it}, \tau) = \max\{0, \sum_{j=1}^{J} \delta_j(\tau)\phi_{i,j}(k_{it}, \omega_{it})\}, \tag{18}$$

  due to the equivariance properties of quantiles

- The censored quantile regression model avoids distributional assumptions at the cost of computational complexity
- Censored investment levels are not an issue in Compustat

Econometric Restrictions

The following conditional moment restrictions hold:

$$\mathbb{E}\left[\Psi_\tau\left(y_{it} - Q_t(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau))\right)\middle| k_{it}, l_{it}, m_{it}\right] = 0 \tag{19}$$

$$\mathbb{E}\left[\Psi_\tau\left(l_{it} - \sum_{j=1}^{J} \alpha_{l,j}(\tau)\phi_{l,j}(k_{it}, \omega_{it})\right)\middle| k_{it}, \omega_{it}\right] = 0 \tag{20}$$

$$\mathbb{E}\left[\Psi_\tau\left(m_{it} - \sum_{j=1}^{J} \alpha_{m,j}(\tau)\phi_{m,j}(k_{it}, \omega_{it})\right)\middle| k_{it}, \omega_{it}\right] = 0 \tag{21}$$

$$\mathbb{E}\left[\Psi_\tau\left(i_{it} - \sum_{j=1}^{J} \delta_j(\tau)\phi_{\iota,j}(k_{it}, \omega_{it})\right)\middle| k_{it}, \omega_{it}\right] = 0 \tag{22}$$

For $t \geq 2$,

$$\mathbb{E}\left[\Psi_\tau\left(\omega_{it} - \rho_0(\tau) - \rho_1(\tau)\omega_{it-1} - \rho_2(\tau)\omega_{it-1}^2 - \rho_3(\tau)\omega_{it-1}^3\right)\middle| \omega_{it-1}\right] = 0, \tag{23}$$

$$\mathbb{E}\left[\Psi_\tau\left(\omega_{i1} - \sum_{j=1}^{J} \rho_{\omega_1,j}(\tau)\phi_{\omega_1,j}(k_{i1})\right)\middle| k_{i1}\right] = 0, \tag{24}$$

Econometric Restrictions

- The function $\Psi_\tau(u) = \tau - \mathbb{1}\{u < 0\}$
- Estimating the above conditional moment restrictions is infeasible due to the unobserved productivity component
- Use the unconditional moment restriction and integrate out productivity
- Let the finite and functional parameters be indexed by a finite dimensional parameter vector $\theta(\cdot)$.
- To fix ideas, consider the unconditional moment restriction corresponding to the production function

$$\mathbb{E}\left[\int \Psi_\tau\left(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau))\right) \otimes \begin{pmatrix} k_{it} \\ l_{it} \\ m_{it} \\ \omega_{it} \end{pmatrix} g_i(\omega_i^T; \theta(\cdot)) d\omega_i^T\right] = 0, \tag{25}$$

- The posterior density $g_i(\omega_i^T; \theta(\cdot)) = f(\omega_i^T | y_i^T, k_i^T, l_i^T, m_i^T, i_i^T, \theta(\cdot))$ involves the entire set of model parameters

Implementation

- Therefore, it is impossible to estimate the model parameters in a τ-by-τ procedure
- To eliminate the intractability of this problem, the continuous model parameters are approximated by piece-wise linear splines
- θ is a piecewise-polynomial interpolating splines on a grid $[\tau_1, \tau_2], [\tau_3, \tau_4], \dots, [\tau_{Q-1}, \tau_Q]$, contained in the unit interval and is constant on $[0, \tau_1]$ and $[\tau_Q, 1)$
- The intercept coefficient β_0 is specified as the quantile of an exponential distribution on $(0, \tau_1]$ (indexed by λ^-) and $[\tau_{Q-1}, 1)$ (indexed by λ^+).
- The remaining functional parameters are modeled similarly.
- With piece-wise linear splines, the posterior density has a closed form expression without relying on strong distributional assumptions for estimation

Implementation

- In order to estimate the model, the integral inside the expectation of Equation (25) needs to be approximated
- This can be done using quadrature methods or Monte Carlo integration by converting the problem into a weighted quantile regression
- Due to the high-dimensionality of my application I choose to use a random-walk Metropolis Hastings algorithm to compute the integral
- This is known as a Monte Carlo Expectation Maximization (MCEM) procedure where the maximization step is performed using quantile regression
- This type of estimator is used by Arellano and Bonhomme, 2016 and Arellano, Blundell, and Bonhomme, 2017

Implementation

Given an initial parameter value $\hat{\theta}^0$. Iterate on $s = 0, 1, 2, \ldots$ in the following two-step procedure until converge to a stationary distribution:

1. *Stochastic E-Step*: Draw M values $\omega_i^{(m)} = (\omega_{i1}^{(m)}, \omega_{i2}^{(m)}, \ldots, \omega_{iT}^{(m)})$ from

$$g_i(\omega_i^T; \hat{\theta}^{(s)}) = f(\omega_i^T | y_i^T, k_i^T, l_i^T, m_i^T, i_i^T, ; \hat{\theta}^{(s)}) \propto$$

$$\prod_{t=1}^{T} f(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}; \hat{\beta}^{(s)}) f(l_{it}|k_{it}, \omega_{it}; \hat{\alpha}_l{}^{(s)}) f(m_{it}|k_{it}, l_{it}, \omega_{it}; \hat{\alpha}$$

$$\times f(i_{it}|k_{it}, \omega_{it}; \hat{\delta}^{(s)}) \prod_{t=2}^{T} f(\omega_{it}|\omega_{it-1}; \hat{\rho}^{(s)}) f(\omega_{i1}|k_{i1}; \hat{\rho}_{\omega_1}^{(s)})$$

2. *Maximization Step*: For $q = 1, \ldots, Q$, solve (e.g for production function)

$$\hat{\boldsymbol{\beta}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\beta}(\tau_q)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q}\Big(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}^{(m)}; \beta(\tau_q)) \Big)$$

Empirical Implementation

- $\psi_\tau(u) = (\tau - \mathbb{1}\{u < 0\})u$ is the "check" function from quantile regression
- Repeat Step 2 for estimating the productivity process, input decision rules and investment
- Take $M = 1$ in the MCEM algorithm and the report estimates as the average of the last $\tilde{S} = S/2$ draws
- This is known as the stochastic EM algorithm (stEM) of Celeux and Diebolt, 1985
- The sequence of maximizers $\hat{\theta}^{(s)}$ is a time-homogeneous Markov chain which, if ergodic, will converge to its stationary distribution
- Nielsen, 2000 provides sufficient conditions for ergodicity and asymptotic properties of the estimator when the "M-step" is solved using maximum likelihood
- Arellano and Bonhomme, 2016 discuss the asymptotic properties of the estimator when the M-step is solved using quantile regression

Empirical Implementation

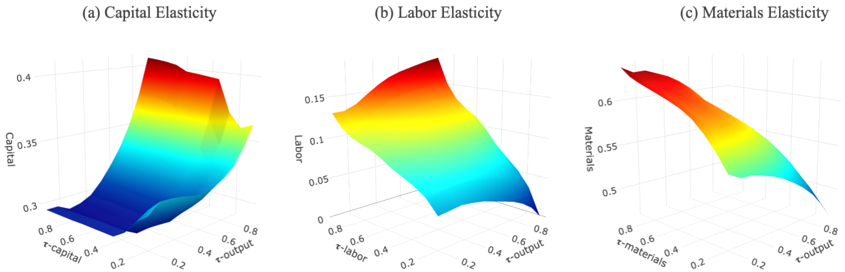
- Estimation procedure is ran with 500 random walk Metropolis-Hastings steps keeping the last draw for estimation
- 200 EM steps where the average is taken over half the draws
- $Q = 11$ for grid size for the interpolating spline
- Experimented with many different proposal distributions and initial values
- Normal distribution centered at the current draw of productivity with variance equal 0.01
- Acceptance rate $\approx 10\%$
- Initial values for productivity are simulated from TFP estimated from the LP model
- Replication code is available on author's Github

Application

- Standard and Poors Compustat database 1997 – 2016
- Productivity is simulated from its estimated parameters and used to construct investment, inputs, and output using their estimated parameters
- Capital is simulated from a linear accumulation process with constant depreciation rate 0.02
- Results are not too different from reasonable specifications for the capital accumulation process
- Interested in a variety of average and individual marginal quantile effects
- Using these estimates, can analyze how firms react to latest shocks to production, inputs, and productivity
- How long does it take for firms to recover from bad shocks to productivity?

Production Elasticities

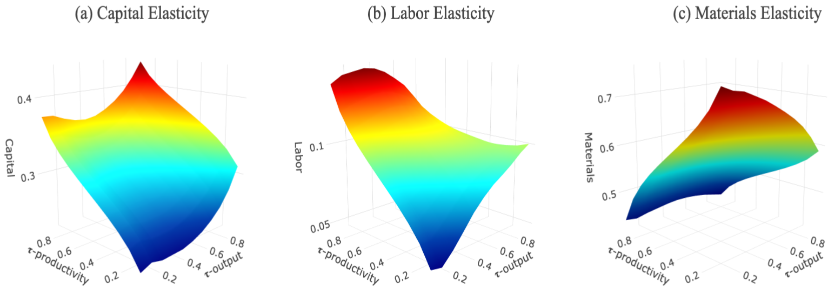
Figure 4: Output Elasticities



*Panel (a): Capital elasticity evaluated at τ_η and percentiles of capital τ_k averaged over values of (l_{it}, m_{it}) that correspond to τ_k. Panel (b): Labor elasticity evaluated at τ_η and percentiles of labor τ_l averaged over values of (k_{it}, m_{it}). Panel (c): Materials elasticity evaluated at τ_η and percentiles of materials τ_m averaged over values of (k_{it}, l_{it}).

Production Elasticities

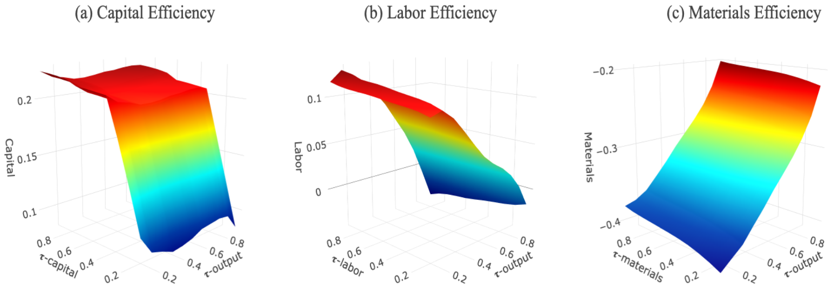
Figure 5: Output Elasticities



*Panel (a): Capital elasticity evaluated at τ_η and τ-productivity averaged over values of (k_{it}, l_{it}, m_{it}) that correspond to τ-productivity. Panel (b): Labor elasticity evaluated at evaluated at τ_η and τ-productivity averaged over values of (k_{it}, l_{it}, m_{it}) that correspond to τ-productivity. Panel (c): Materials elasticity evaluated at τ_η and τ-productivity averaged over values of (k_{it}, l_{it}, m_{it}) that correspond to τ-productivity.

Production Elasticities

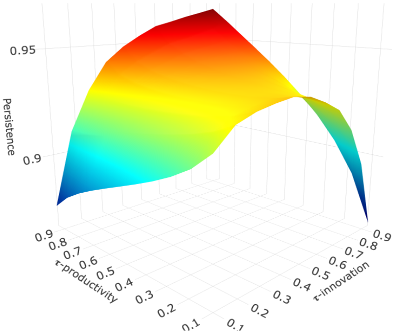
Figure 6: Non-Hicks Neutral Elasticities



*Panel (a): Capital efficiency evaluated at τ_η and percentiles of capital τ_k averaged over values of (l_{it}, m_{it}) that correspond to τ_k. Panel (b): Labor efficiency evaluated at τ_η and percentiles of labor τ_l averaged over values of (k_{it}, m_{it}). Panel (c): Materials efficiency evaluated at τ_η and percentiles of materials τ_m averaged over values of (k_{it}, l_{it}).

Productivity Persistence

Figure 7: Productivity Persistence

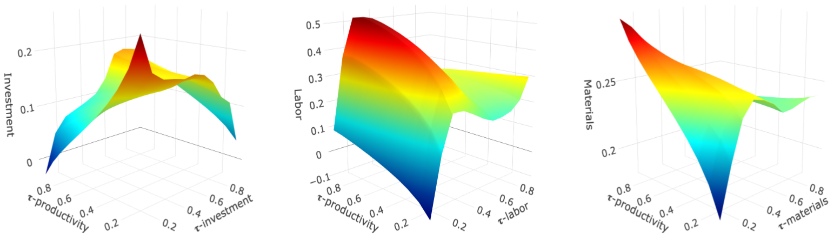


*Estimates of average productivity persistence evaluated at τ_ξ and percentiles of previous productivity.

Marginal Productivities

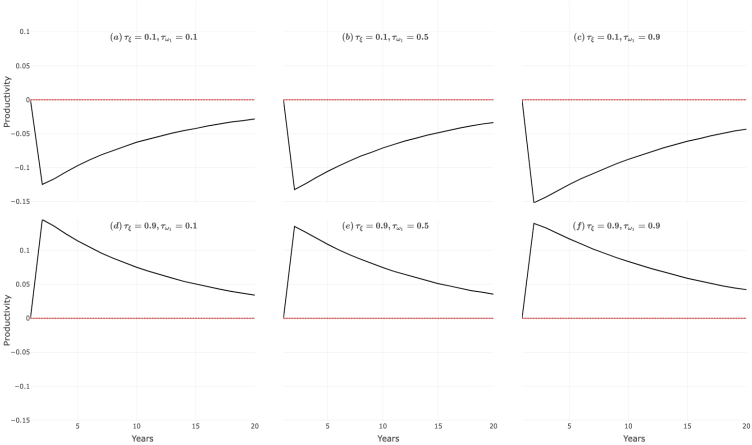
Figure 8: Input Demand Responses to Productivity

(a) Investment Response (b) Labor Response (c) Materials Response



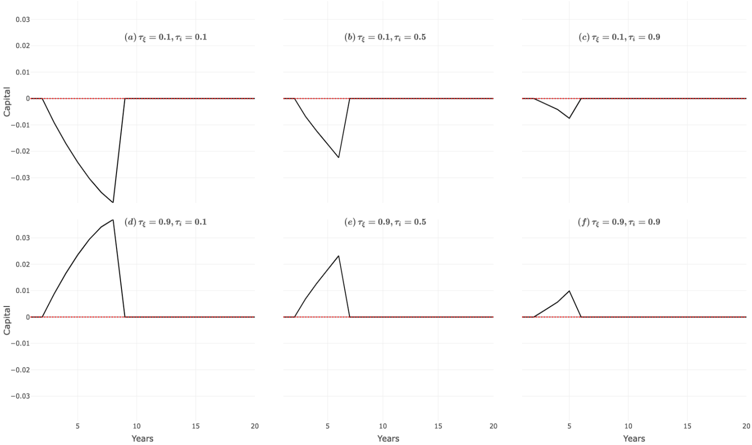
*Panel (a): Investment demand evaluated at τ_ζ and percentiles of productivity τ_ω averaged over values of k_{it}. Panel (b): Labor demand evaluated at $\tau_{\epsilon\ell}$ and percentiles of productivity τ_ω averaged over values of k_{it}. Panel (c): Material demand evaluated at τ_{ϵ_m} and percentiles of productivity τ_ω averaged over values of k_{it} and l_{it}

Productivity Innovation Shocks



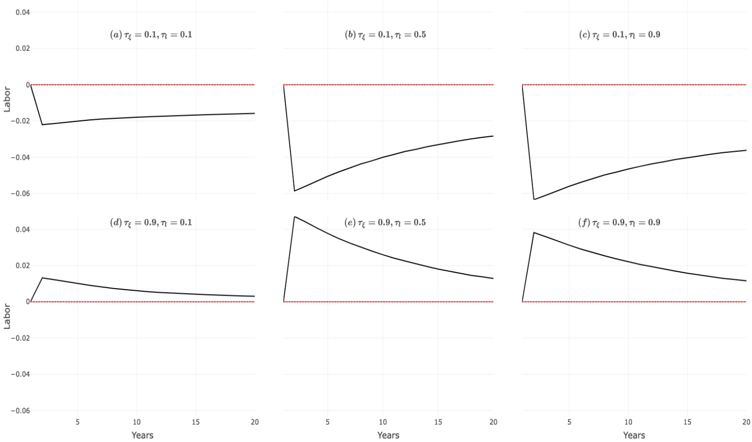
*Top row: Differences between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity. Bottom row: Differences between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of initial productivity.

Productivity Innovation Shocks to Capital



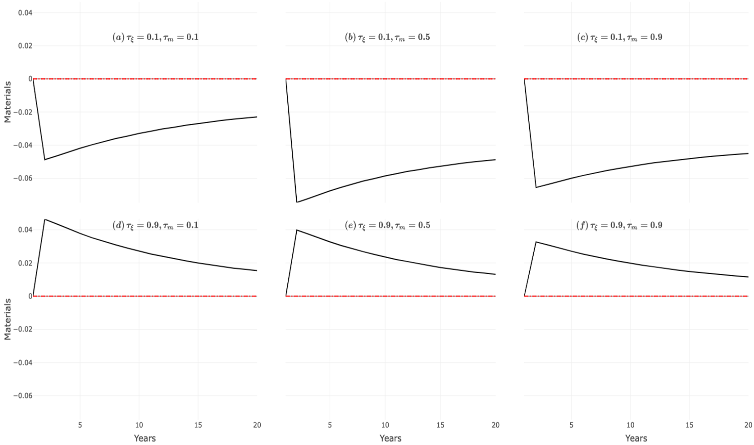
*Top row: Differences between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of investment demand. Bottom row: Differences between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of investment demand

Productivity Innovation Shocks to Labor



*Top row: Differences between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of labor demand. Bottom row: Differences between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of labor demand.

Productivity Innovation Shocks to Materials



*Top row: Differences between firms hit with low productivity shock $\tau_\xi = 0.1$ and medium shock $\tau_\xi = 0.5$ at different levels of materials demand. Bottom row: Differences between firms hit with high productivity shock $\tau_\xi = 0.9$ and medium shock $\tau_\xi = 0.5$ at different levels of materials demand.

Conclusion

- Nonparametric identification of production function, input demand, and productivity
- Proposed an estimation framework to document firm heterogeneity in production, input decisions, and productivity
- Asymmetric productivity persistence depends on size of innovation shock and levels of previous productivity
- Some firms respond to productivity shocks by using more inputs, but this affect is asymmetric for different productivity levels input demand size
- Asymmetric impact of innovation shocks to inputs after bad/good shocks
- Extension to multi-dimensional unobservables: fixed effects, labor-augmenting productivity
- Implications for TFP estimates? Market power?