A Dynamic Framework for Identification and Estimation of Nonseparable Production Functions

Job Market Presentation

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Introduction

- This paper identifies and estimates a **nonseparable** production function with **unobserved heterogeneity**
- Two important contributions to the literature
  1. New estimates of heterogeneous production functions
     - Nonseparable model allows interactions between unobserved production shocks and inputs
     - Captures **factor-specific** productivity changes
  2. New framework to capture heterogeneous productivity dynamics and uncertainty
     - Incorporates **asymmetric persistence** in productivity history
     - Driven by size and sign of productivity shocks

Production Functions

- Production functions are important in many economic models
- They link outputs to inputs (e.g. capital, labor) and represent firm technology
- Estimates can be used in the following applications:
  1. Industry productivity
  2. Trade liberalization
  3. Capital misallocation
  4. Market power
- A correctly specified production function is crucial for correct inference in these applications

Simultaneity Bias

- Researchers don't observe productivity
- Firm chooses inputs depending on their productivity
- A more productive firm may hire more/less workers
- In this case, labor estimates will be biased
- **Proxy variable** approaches of OP, LP, ACF remain a popular tool
- Basic idea: A policy function (e.g. material demand) is inverted as a function of productivity
- Substitute inverted function into production function and estimate in two-step approach ACF Estimator

Limitations of Proxy Variables

- No unobserved errors in the policy function
  - No measurement error
  - No unobserved demand shocks
- Productivity and an unobserved production shock are additive
  - Implies productivity is factor-neutral
  - Technology is fixed across firms
- Productivity process and additive shocks
  - Dispersion and Skewness in productivity dynamics
  - How do these shocks affect different firms?

Why Nonseparable Models (Production)

- Model misspecification and heterogeneous production
- Labor augmenting productivity and biased markups
- Decreasing labor share of income
- Recent papers use structural approach to estimating factor-specific productivity
  1. Doraszelski and Jaumandreu, 2018
  2. Dermirer, 2020
- Structural unobservables beyond labor productivity?

Why Nonseparable Models (Productivity)

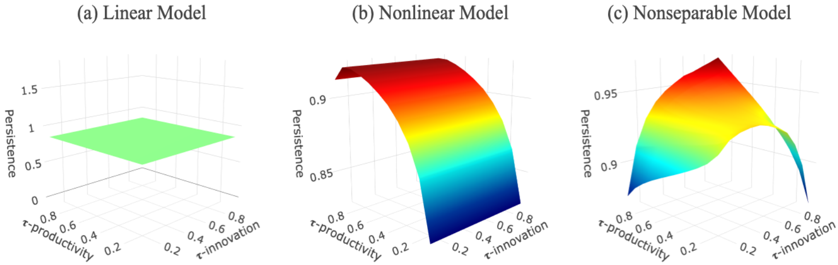
- Business cycles, microeconomic uncertainty and asymmetric adjustment paths
- Role of uncertainty in decisions for investment and hiring (Bloom et al., 2018)
- Previous papers examine role of higher moments in productiity
- My framework allows for parsimonious modelling of higher moments in productivity shocks
- Impact of productivity shocks on history of productivity and production inputs

Why Nonseparable Models (Productivity)

1. Impact of productivity skewness (Salgado, Guvenen, and Bloom, 2019)
    - Pro-cyclical skewness of productivity shocks
    - More flexible process may better explain business cycle phenomenon
2. Misallocation of Inputs
    - Role of uncertainty in dispersion of marginal product of inputs (David and Venkateswaran, 2019), (Asker, Collard-Wexler, and Loecker, 2014)
    - Asymmetric productivity persistence induces heterogeneous responses of long-run decisions for investment and labor

Preview of Results

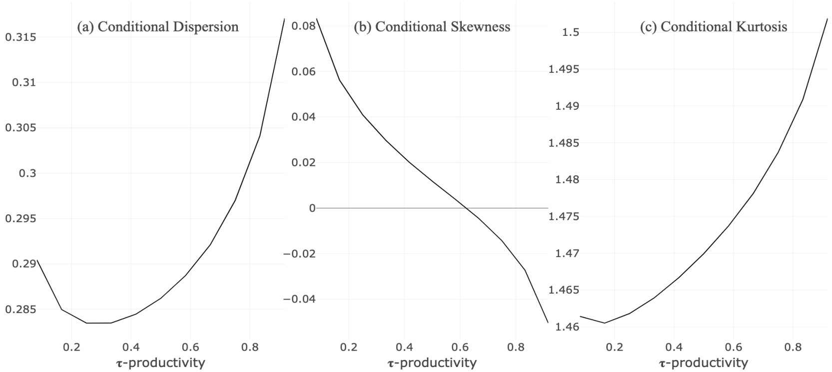
Figure 1: Productivity Persistence Comparison



*Panel (a): Productivity persistence from a linear model. Panel (b): Productivity persistence for a nonlinear model that is separable in unobserved shocks. Panel (c): Productivity persistence estimated in the nonseparable model.

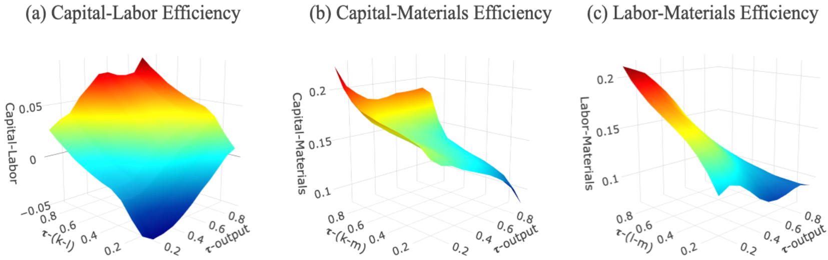
Preview of Results

Figure 2: Higher Moments of the Conditional Productivity Distribution



Preview of Results

Figure 3: Effects of Input Composition on Productivity



*Panel (a): Capital-labor effect evaluated at τ_η and percentiles of capital-labor τ-(k-l). Panel (b): Capital-materials effect evaluated at τ_η and percentiles of capital-materials τ-(k-m). Panel (c): Labor-materials effect evaluated at τ_η and percentiles of labor-materials τ-(l-m).

Summary of Findings

Data: U.S. Compustat public manufacturing firms

- Asymmetric persistence in productivity
  1. Positive shocks for low productivity firms
  2. Negative shocks for high productivity firms

  have lower persistence of productivity
- Asymmetric adjustments of inputs with respect to productivity and its shocks
- Non-Hicks neutrality of productivity
  1. Positive and negative capital intensity effects
  2. Positive capital-materials intensity effects
  3. Positive labor-materials intensity effects

Outline for the Rest of Talk

1. ~~Introduction~~
2. Economic Model
3. Econometric Identification
4. Econometric Procedure and Quantile Modelling
5. Results
6. Conclusions

The Production Function

- Nonseparable model for a firm's gross-output production function (in logs)

$$y_{it} = Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \eta_{it}), \quad \eta_{it} \sim Uniform(0,1) \tag{1}$$

Skorohod Representation

Assumption 1 (Production Function)

1. η_{it} *is independent of* η_{is} *for all* $t \neq s$ *conditional on* $(k_{it}, l_{it}, m_{it}, \omega_{it})$
2. η_{it} *follows a standard uniform distribution independent of* $(k_{it}, l_{it}, m_{it}, \omega_{it})$.
3. $\tau \to Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau)$ *is strictly increasing on* $(0,1)$.

Quantile Function

- For a given τ, the conditional quantile function for the random variable $Y|X$ is defined as

$$Q_\tau(Y|X) = \inf\{y \in \mathbb{R} : \tau \leq F_{Y|X}(y|x)\}, \quad \tau \in (0,1)$$

where $F_{Y|X}$ is continuous and strictly increasing

- Quantiles are a way to flexibly model the entire distribution of response variable
- Estimates recover structural response functions for specific firms
- The standard production function and quantiles is an unexplored area
- Doty and Song, 2021 propose a simple model and discuss implications Markups

Productivity

- Productivity evolves according to an exogenous first-order Markov process

$$\omega_{it} = Q_t^{\omega}(\omega_{it-1}, \xi_{it}), \quad \xi_{it} \sim Uniform(0,1), \tag{2}$$

where $\xi_{i1}, \ldots, \xi_{iT}$ are innovation shocks to productivity

- The function Q^{ω} is a function that allows the persistence in productivity in firms to be nonlinear across different quantiles

Assumption 2 (Productivity)

1. ξ_{it} *independent of* ξ_{is} *for all* $t \neq s$ *conditional on* ω_{it-1}
2. ξ_{it} *follows a standard uniform distribution independent of* ω_{it-1}.
3. $\tau \to Q_t^{\omega}(\omega_{it-1}, \tau)$ *is strictly increasing on* $(0,1)$.

Flexible Inputs

- Labor and material inputs are chosen to maximize current period profits

$$l_{it} = Q_t^\ell(k_{it}, \omega_{it}, \epsilon_{\ell,it}), \quad \epsilon_{\ell,it} \sim Uniform(0,1), \tag{3}$$

$$m_{it} = Q_t^m(k_{it}, l_{it}, \omega_{it}, \epsilon_{m,it}), \quad \epsilon_{m,it} \sim Uniform(0,1), \tag{4}$$

Assumption 3 (Flexible Inputs)

1. *$\epsilon_{\ell,it}$ and $\epsilon_{m,it}$ are mutually independent over time conditional on state variables.*
2. *$\epsilon_{\ell,it}$ and $\epsilon_{m,it}$ follow a standard uniform distribution independent of state variables.*
3. *$\tau \to Q_t^\ell(k_{it}, \omega_{it}, \tau)$ and $\tau \to Q_t^m(k_{it}, l_{it}, \omega_{it}, \tau)$ are strictly increasing on $(0,1)$.*

Flexible Inputs

- Shocks may be multi-dimensional or non-monotone
- For example, the labor demand function can be written

$$l_{it} = \tilde{\ell}_t(k_{it}, \omega_{it}, \varepsilon_{\ell,it})$$

- Consistently estimate objects like average derivative of inputs with respect to productivity using quantiles
- Under my assumptions

$$\mathbb{E}_{\varepsilon^{\ell}_{it}}\left[\frac{\partial \tilde{\ell}_t(k_{it}, \omega_{it}, \varepsilon_{\ell,it})}{\partial \omega_{it}}\right] = \mathbb{E}_{\epsilon^{\ell}_{it}}\left[\frac{\partial Q^{\ell}_t(k_{it}, \omega_{it}, \epsilon_{\ell,it})}{\partial \omega_{it}}\right]$$

Capital and Investment

- Capital accumulates to the following generalized law of motion

$$K_{it} = \kappa_t(K_{it-1}, I_{it-1}, \upsilon_{it-1}) \tag{5}$$

where I_{it-1} denotes firm investment in the prior period

- In each period, a firm chooses investment to maximize its discounted future profits:

$$I_{it} = \iota_t(K_{it}, \omega_{it}) = \underset{I_t \geq 0}{\text{argmax}} \left[\Pi_t(K_{it}, \omega_{it}) - c(I_{it}, \omega_{it}) + \beta \mathbb{E}\left[V_{t+1}(K_{it+1}, \omega_{it+1}) | \mathcal{I}_t \right] \right], \tag{6}$$

- $\pi_t(\cdot)$ is current period profits as a function of the state variables $c(\cdot)$ is the cost function, $\mathcal{I}_t$ is information set

Capital and Investment

- Empirical investment rule is

$$i_{it} = Q_t^{\iota}(k_{it}, \omega_{it}, \zeta_{it}), \quad \zeta_{it} \sim Uniform(0,1). \tag{7}$$

Assumption 4 (Capital Accumulation and Investment)

1. *ζ_{it} independent of ζ_{is} for all $t \neq s$ conditional on (k_{it}, ω_{it})*
2. *ζ_{it} follows a standard uniform distribution independent of (k_{it}, ω_{it}).*
3. *η_{it} and ζ_{it} are independent conditional on $(k_{it}, l_{it}, m_{it}, \omega_{it})$. In addition, υ_{it} is independent of η_{it} conditional on $(k_{it}, l_{it}, m_{it}, \omega_{it})$*
4. *$\tau \to Q_t^i(k_{it}, \omega_{it}, \tau)$ is strictly increasing on $(0,1)$*

Nonparametric Identification

- Let $Z_t = (l_t, k_t, m_t, k_{t+1})$ denote conditioning variables

Assumption 5 (Conditional Independence)

1. $f(y_t | y_{t+1}, I_t, \omega_t, Z_t) = f(y_t | \omega_t, Z_t)$
2. $f(y_{t+1} | I_t, \omega_t, Z_t) = f(y_{t+1} | \omega_t, Z_t)$

- Conditional on ω_t and Z_t, y_{t+1} and I_t do not provide any additional information about y_t
- Conditional on ω_t and Z_t, I_t does not provide any additional information about y_{t+1}
- Satisfied by mutual independence assumptions on η_t and ζ_t conditional on $(\omega_t, k_t, l_t, m_t)$

Nonparametric Identification

- Conditional density as a function of observable to densities containing unobserved productivity
- Using the conditional independence assumption, I can write

$$f_{y_t,I_t|y_{t+1},Z_t} = \int f_{y_t|Z_t,\omega_t} f_{I_t|Z_t,\omega_t} f_{\omega_t|y_{t+1},Z_t} d\omega_t \tag{8}$$

- Goal is identification of $f_{y_t|Z_t,\omega_t}$, $f_{I_t|Z_t,\omega_t}$, and $f_{\omega_t|y_{t+1},Z_t}$
- Proof following nonclassical, nonlinear error-in-variable set-up of Hu and Schennach, 2008
- y_t is the outcome variable, I_t is the mismeasured regressor, y_{t+1} is the IV, ω_t is the unobserved variable

Nonparametric Identification

Assumption 6 (Identification)

(a) *The joint density* $f_{y_t, I_t, y_{t+1}|Z_t}$ *is bounded as well as its joint and marginal densities*

(b) $\forall \bar{\omega}_t \neq \tilde{\omega}_t \in \Omega_t, Pr[f_{I_t|\omega_t, Z_t}(I_t|\bar{\omega}_t, Z_t) \neq f_{I_t|\omega_t, Z_t}(I_t|\tilde{\omega}_t)|Z_t] > 0$

(c) *There exists a functional* Γ *such that* $\Gamma[f_{y_t|\omega_t, Z_t}(y_t|\omega_t, Z_t)] = \omega_t$

(d) *The linear operators* $L_{y_t|Z_t,\omega_t}$ *and* $L_{y_{t+1}|Z_t,\omega_t}$ *corresponding to* $f_{y_t|Z_t,\omega_t}$ *and* $f_{y_{t+1}|Z_t,\omega_t}$ *are injective*

Linear Operators

Nonparametric Identification

- Assumption 6(a) requires bounded densities
- Assumption 6(b) is a uniqueness condition
- I assume $E[I_t|\omega_t, Z_t]$ is strictly increasing in ω_t
- Assumption 6(c) normalizes a measure of location on $f_{y_t|Z_t,\omega_t}$
  - Achieved by the normalization $E[y_t|\omega_t, 0] = \omega_t$
  - Requires centering some coefficients in my model (shown later)
- Assumption 6(d) is similar to a nonparametric IV rank condition which rules out a “weak” instrument

Nonparametric Identification

Theorem 1 (Identification)

Under Assumption 6, given the observed density $f_{y_t, I_t | y_{t+1}, Z_t}$, the equation

$$f_{y_t, I_t | y_{t+1}, Z_t} = \int f_{y_t | \omega_t, Z_t} f_{I_t | \omega_t, Z_t} f_{\omega_t | y_{t+1}, Z_t} d\omega_t \tag{9}$$

admits a unique solution for $f_{y_t | \omega_t, Z_t}$, $f_{I_t | \omega_t, Z_t}$, and $f_{\omega_t | y_{t+1}, Z_t}$

- The proof follows using Hu and Schennach, 2008
- However it does not directly identify the Markov transition function for productivity $f_{\omega_{it+1} | \omega_{it}}(\omega_{it+1} | \omega_{it})$

Identification of Productivity Process

Corollary 1 (Stationary)

Suppose that the production function is stationary, $f_{y_t|\omega_t,Z_t} = f_{y_1|\omega_1,Z_1}, \forall t \in \{1,\cdots,T\}$. Then, the observed density $f_{y_t,I_t|y_{t+1},Z_t}$, uniquely determines the density $f_{\omega_{t+1}|\omega_t}, \forall t \in \{1,\ldots,T-1\}$

Corollary 2 (Non-Stationary)

The observed density $f_{y_{t+1},I_{t+1}|y_{t+2},Z_{t+1}}$, uniquely determines the density $f_{\omega_{t+1}|\omega_t}, \forall t \in \{1,\ldots,T-2\}$

Econometric Procedure: Production

- The production function is specified as

$$Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) = \sum_{r_k=0}^{R_k} \sum_{r_l=0}^{R_l} \sum_{r_m=0}^{R_m} \sum_{r_\omega=0}^{R_\omega} \beta_{r_k,r_l,r_m,r_\omega}(\tau) k_{it}^{r_k} l_{it}^{r_l} m_{it}^{r_m} \omega_{it}^{r_\omega}. \tag{10}$$

or

$$Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) = \sum_{s_k=0}^{S_k} \sum_{s_l=0}^{S_l} \sum_{s_m=0}^{S_m} \gamma_{s_k,s_l,s_m,s_\omega}(\tau) k_{it}^{s_k} l_{it}^{s_l} m_{it}^{s_m} + \sum_{p_k=0}^{P_k} \sum_{p_l=0}^{P_l} \sum_{p_m=0}^{P_m} \sum_{p_\omega=1}^{P_\omega} \sigma_{p_k,p_l,p_m,p_\omega}(\tau) k_{it}^{p_k} l_{it}^{p_l} m_{it}^{p_m} \omega_{it}^{p_\omega}. \tag{11}$$

- I take $S_k = S_l = S_m = 2$, $P_k = P_l = P_m = 2$, and $P_\omega = 1$

Econometric Procedure: Production

$$
\begin{aligned}
Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}, \tau) = & \sum_{s_k=0}^{S_k=2} \sum_{s_l=0}^{S_l=2} \sum_{s_m=0}^{S_m=2} \gamma_{s_k, s_l, s_m, s_\omega}(\tau) k_{it}^{s_k} l_{it}^{s_l} m_{it}^{s_m} \\
& + \omega_{it} \Bigg[\underbrace{\sigma_\omega(\tau)}_{\text{Hicks-neutral effect}} + \underbrace{\frac{(\sigma_k(\tau) + \sigma_l(\tau) + \sigma_m(\tau)) s_{it}}{3} + \frac{(\sigma_{kk}(\tau) + \sigma_{ll}(\tau) + \sigma_{mm}(\tau)) s_{it}^2}{3}}_{\text{scale effect}} \\
& + \underbrace{\frac{(\sigma_k(\tau) - \sigma_l(\tau))(k_{it} - l_{it})}{3} + \frac{(\sigma_{kk}(\tau) - 3/2\sigma_{kl}(\tau) + \sigma_{ll}(\tau))(k_{it} - l_{it})^2}{3}}_{\text{capital-labor effect}} \\
& + \underbrace{\frac{(\sigma_k(\tau) - \sigma_m(\tau))(k_{it} - m_{it})}{3} + \frac{(\sigma_{kk}(\tau) - 3/2\sigma_{km}(\tau) + \sigma_{mm}(\tau))(k_{it} - m_{it})^2}{3}}_{\text{capital-materials effect}} \\
& + \underbrace{\frac{(\sigma_l(\tau) - \sigma_m(\tau))(l_{it} - m_{it})}{3} + \frac{(\sigma_{ll}(\tau) - 3/2\sigma_{lm}(\tau) + \sigma_{mm}(\tau))(l_{it} - m_{it})^2}{3}}_{\text{labor-materials effect}} \Bigg].
\end{aligned}
\tag{12}
$$

Econometric Procedure: Productivity

- I specify productivity using a polynomial series

$$Q_t^{\omega}(\omega_{it-1}, \tau) = \sum_{j=0}^{J_\omega} \rho_j(\tau)\omega_{it-1}^j. \tag{13}$$

Selection Bias

- Initial productivity

$$Q^{\omega_1}(k_{i1}, \tau) = \sum_{j=0}^{J_{\omega_1}} \rho_{\omega_1,j}(\tau) k_{i1}^j. \tag{14}$$

- Models entry in the Compustat sample
- Productivity and R&D: Doraszelski and Jaumandreu, 2013

$$Q^{\omega}(\omega_{it-1}, r_{it-1}, \tau) = \mathbb{1}\{R_{it-1} = 0\}Q^{\omega}(\omega_{it-1}, \tau) + \mathbb{1}\{R_{it-1} > 0\}Q^{\omega,r}(\omega_{it-1}, r_{it-1}, \tau). \tag{15}$$

R&D Firms

Econometric Procedure: Flexible Inputs

- I specify the labor input demand equation as follows:

$$Q_t^{\ell}(k_{it}, \omega_{it}, \tau) = \sum_{j=0}^{J_\ell} \alpha_{\ell,j}(\tau)\phi_{\ell,j}(k_{it}, \omega_{it}), \qquad (16)$$

  where ϕ_ℓ is modeled using a tensor product Hermite polynomial of degree 3

- If labor is dynamic, can include l_{it-1} as an additional state variable Labor Dynamics

- Material inputs are specified as

$$Q_t^{m}(k_{it}, l_{it}, \omega_{it}, \tau) = \sum_{j=0}^{J_m} \alpha_{m,j}(\tau)\phi_{m,j}(k_{it}, l_{it}, \omega_{it}). \qquad (17)$$

Econometric Procedure: Investment

- The investment demand function is specified as

$$i_{it} = Q_t^i(k_{it}, \omega_{it}, \tau) = \sum_{j=0}^{J_\iota} \delta_j(\tau)\phi_{i,j}(k_{it}, \omega_{it}). \tag{18}$$

where $\phi_{\iota,j}$ is specified similarly as the labor and material input decision rule.

- In the case where investment is censored, I can write

$$Q_t^{\iota^*}(k_{it}, \omega_{it}, \tau) = \max\{0, \sum_{j=1}^{J_\iota} \delta_j(\tau)\phi_{\iota,j}(k_{it}, \omega_{it})\}, \tag{19}$$

due to the equivariance properties of quantiles

Econometric Procedure

The independence assumptions in Assumptions 1, 2, 3, and 4 imply:

1. **Production:** $\Pr[y_{it} \leq Q_t(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau)|k_{it}, l_{it}, m_{it}, \omega_{it}] = \tau$
2. **Productivity:** $\Pr[\omega_{it} \leq \sum_{j=0}^{J_\omega} \rho_j(\tau)\omega_{it-1}^j|\omega_{it-1}] = \tau$
3. **Initial Productivity:** $\Pr[\omega_{i1} \leq \sum_{j=0}^{J_{\omega_1}} \rho_{\omega_1,j}(\tau)k_{i1}^j|k_{i1}] = \tau$
4. **Labor:** $\Pr[l_{it} \leq \sum_{j=1}^{J_\ell} \alpha_{l,j}(\tau)\phi_{l,j}(k_{it}, \omega_{it}))|k_{it}, \omega_{it}] = \tau$
5. **Materials:** $\Pr[m_{it} \leq \sum_{j=1}^{J_m} \delta_j(\tau)\phi_{m,j}(k_{it}, l_{it}, \omega_{it}))|k_{it}, l_{it}, \omega_{it}] = \tau$
6. **Investment:** $\Pr[i_{it} \leq \sum_{j=1}^{J_\iota} \delta_j(\tau)\phi_{\iota,j}(k_{it}, \omega_{it}))|k_{it}, \omega_{it}] = \tau$

Econometric Restrictions

Which imply the following conditional moment restrictions

1. **Production:**
   $\mathbb{E}[\Psi_\tau(y_{it} - Q_t(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau)))|k_{it}, l_{it}, m_{it}] = 0$
2. **Productivity:** $\mathbb{E}[\Psi_\tau(\omega_{it} - \sum_{j=0}^{J_\omega} \rho_j(\tau)\omega_{it-1}^j)|\omega_{it-1}] = 0$
3. **Initial Productivity:** $\mathbb{E}[\Psi_\tau(\omega_{i1} - \sum_{j=0}^{J_{\omega_1}} \rho_{\omega_1,j}(\tau)k_{i1}^j)|k_{i1}] = 0$
4. **Labor:** $\mathbb{E}[\Psi_\tau(l_{it} - \sum_{j=1}^{J_\ell} \alpha_{l,j}(\tau)\phi_{l,j}(k_{it}, \omega_{it}))|k_{it}, \omega_{it}] = 0$
5. **Materials:**
   $\mathbb{E}[\Psi_\tau(m_{it} - \sum_{j=1}^{J_m} \alpha_{m,j}(\tau)\phi_{m,j}(k_{it}, l_{it}, \omega_{it}))|k_{it}, l_{it}, \omega_{it}] = 0$
6. **Investment:** $\mathbb{E}[\Psi_\tau(i_{it} - \sum_{j=1}^{J_\iota} \delta_j(\tau)\phi_{\iota,j}(k_{it}, \omega_{it}))|k_{it}, \omega_{it}] = 0$

Econometric Restrictions

- The function $\Psi_\tau(u) = \tau - \mathbb{1}\{u < 0\}$
- Consider the moment condition corresponding to production
- The law of iterated expectations gives the integrated moment condition

$$\mathbb{E}\left[\int \Psi_\tau\left(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}; \beta(\tau))\right) \otimes \begin{pmatrix} k_{it} \\ l_{it} \\ m_{it} \\ \omega_{it} \end{pmatrix} g_i(\omega_i^T; \theta(\cdot)) d\omega_i^T\right] = 0, \tag{20}$$

- The posterior density $g_i(\omega_i^T; \theta(\cdot)) = f(\omega_i^T | y_i^T, k_i^T, l_i^T, m_i^T, i_i^T, \theta(\cdot))$ involves the entire set of model parameters

Implementation

- Impossible to estimate the model parameters in a τ-by-τ procedure
- Continuous model parameters are approximated by piecewise linear splines
- θ is a piecewise-polynomial interpolating splines with Q knots constant on $[0, \tau_1]$ and $[\tau_Q, 1)$ Splines
- Posterior density has a closed form expression Likelihood
- Integration is done using Metropolis Hastings algorithm
- Estimates are updated using these draws in a sequential algorithm

Implementation

1. *Stochastic E-Step*: Draw M values $\omega_i^{(m)} = (\omega_{i1}^{(m)}, \omega_{i2}^{(m)}, \ldots, \omega_{iT}^{(m)})$ from

$$g_i(\omega_i^T; \hat{\theta}^{(s)}) = f(\omega_i^T | y_i^T, k_i^T, l_i^T, m_i^T, i_i^T, ; \hat{\theta}^{(s)}) \propto$$

$$\prod_{t=1}^{T} f(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}; \hat{\beta}^{(s)}) f(l_{it}|k_{it}, \omega_{it}; \hat{\alpha}_l{}^{(s)}) f(m_{it}|k_{it}, l_{it}, \omega_{it}; \hat{\alpha}_m^{(s)})$$

$$\times f(i_{it}|k_{it}, \omega_{it}; \hat{\delta}^{(s)}) \prod_{t=2}^{T} f(\omega_{it}|\omega_{it-1}; \hat{\rho}^{(s)}) f(\omega_{i1}|k_{i1}; \hat{\rho}_{\omega_1}^{(s)})$$

2. *Maximization Step*: For $q = 1, \ldots, Q$, solve (e.g for production function)

$$\hat{\boldsymbol{\beta}}(\tau_q)^{(s+1)} = \underset{\boldsymbol{\beta}(\tau_q)}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_q} \left(y_{it} - Q_t^y(k_{it}, l_{it}, m_{it}, \omega_{it}^{(m)}; \beta(\tau_q)) \right)$$

Empirical Implementation

- Repeat Step 2 for estimating the productivity process, input decision rules and investment
- Take $M = 1$ in the MCEM algorithm and the report estimates as the average of the last $\tilde{S} = S/2$ draws
- A similar estimator was proposed by Wei and Carroll, 2009 for error-in-variables models
- Arellano and Bonhomme, 2016 extend this methodology to nonlinear quantile panel data models
- Arellano, Blundell, and Bonhomme, 2017 use this to document nonlinear persistence in income and consumption dynamics

Empirical Implementation

- $J_y = 20. J_\omega = J_{\omega_1} = 3, J_\ell = J_\iota = 16, J_m = 27$ so total parameters, $J = 85$
- 500 random walk Metropolis-Hastings
- 200 EM steps (half used for burn-in)
- $Q = 11$ for grid size for the interpolating spline
- Initial values for productivity are simulated from TFP estimated from the LP model
- Replication code is available on Github

Asymptotic Properties

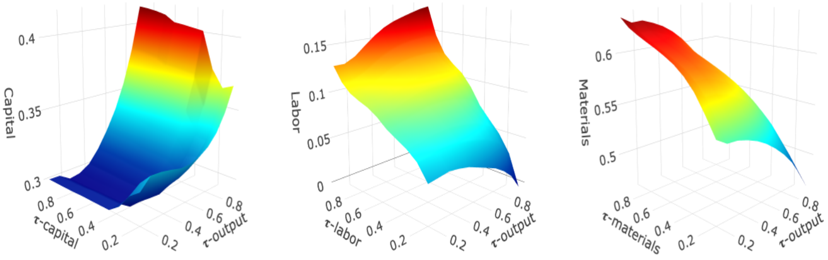
- Nielsen, 2000 provides a.normality conditions when M-step is MLE
- Arellano and Bonhomme, 2016 provide conditions when M-step is QR
- Assuming correctly specified model, they show consistency and asymptotic normality
- Future work: nonparametric model with time-varying unobservable
- Instead of confidence intervals, I report posterior intervals from the Markov chain in the paper

Application

- Standard and Poors Compustat database 1997 – 2016
- Productivity is simulated from its estimated parameters and used to construct investment, inputs, and output using their estimated parameters
- Capital is simulated from a linear accumulation process with depreciation rate 0.02
- Production function elasticities estimated as marginal quantile effects (derivative wrt to inputs)
- Factor-specific effects estimated as marginal quantile effects (derivative of elasticities wrt productivity)

Production Elasticities

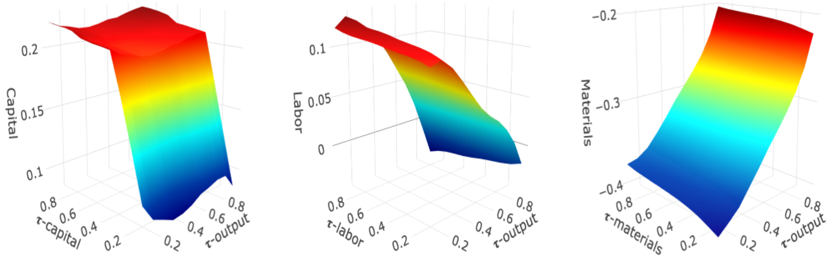
Figure 4: Output Elasticities



*Output elasticities evaluated at percentiles of the conditional output distribution τ_η and percentiles of capital, labor, and materials averaged over values of the other inputs.

Production Elasticities

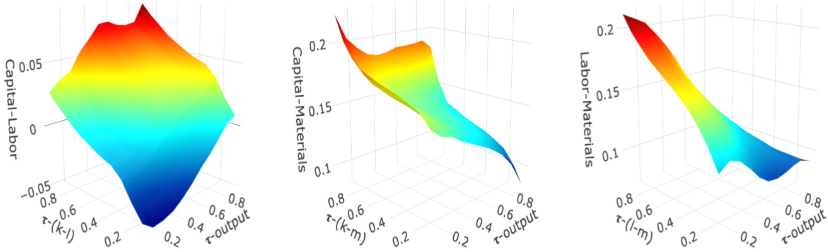
Figure 5: Effect of Productivity on Output Elasticities



*Input efficiencies evaluated at percentiles of the conditional output distribution τ_η and percentiles of capital, labor, and materials averaged over values of the other inputs.

Production Elasticities

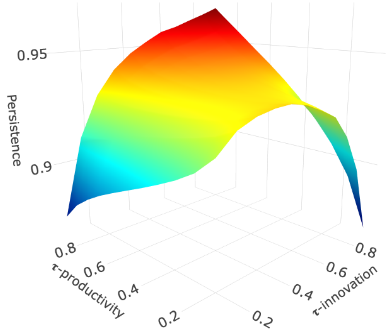
Figure 6: Effects of Input Composition on Productivity



*Panel (a): Capital-labor effect evaluated at τ_η and percentiles of capital-labor τ-(k-l). Panel (b): Capital-materials effect evaluated at τ_η and percentiles of capital-materials τ-(k-m). Panel (c): Labor-materials effect evaluated at τ_η and percentiles of labor-materials τ-(l-m).

Productivity Persistence

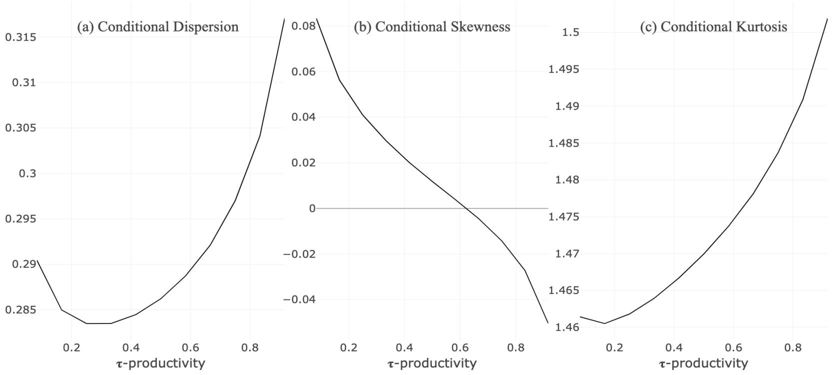
Figure 7: Productivity Persistence



*Estimates of average productivity persistence evaluated at τ_{ξ} and percentiles of previous productivity.

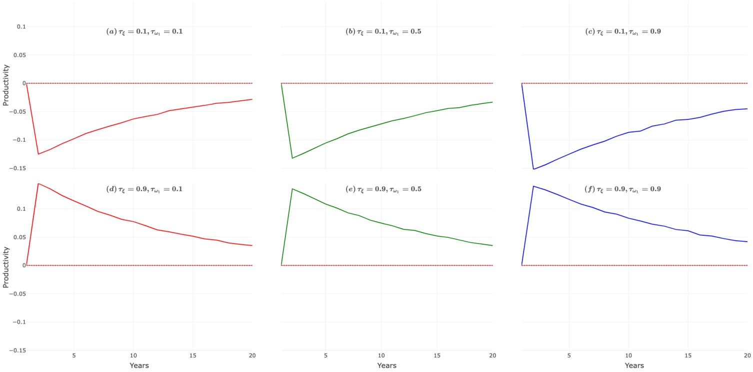
Preview of Results

Figure 8: Higher Moments of the Conditional Productivity Distribution



Productivity Innovation Shocks

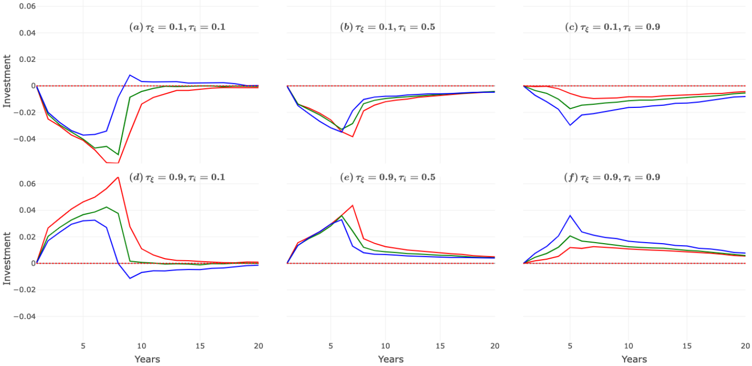
Figure 9: Shocks to Productivity



*Differences between firms hit with medium productivity shock $\tau_\xi = 0.5$ and firms hit by low shock $\tau_\xi = 0.1$ (top row) and high shock $\tau_\xi = 0.9$ (bottom row) at different percentiles of initial productivity.

Productivity Innovation Shocks to Investment

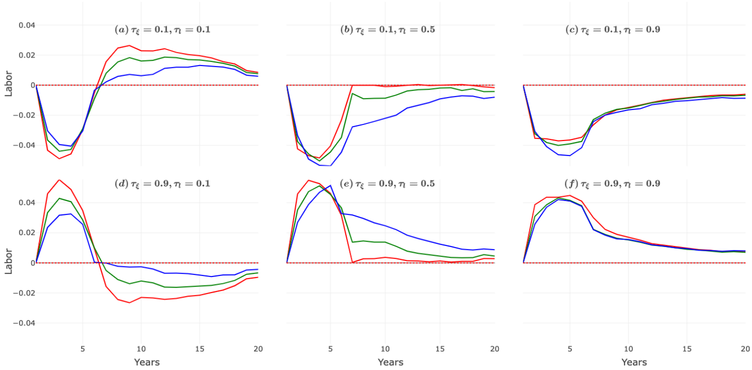
Figure 10: Shocks to Investment



*Differences between firms hit with medium productivity shock $\tau_\xi = 0.5$ and firms hit by low shock $\tau_\xi = 0.1$ (top row) and high shock $\tau_\xi = 0.9$ (bottom row) at different percentiles of investment demand.

Productivity Innovation Shocks to Labor

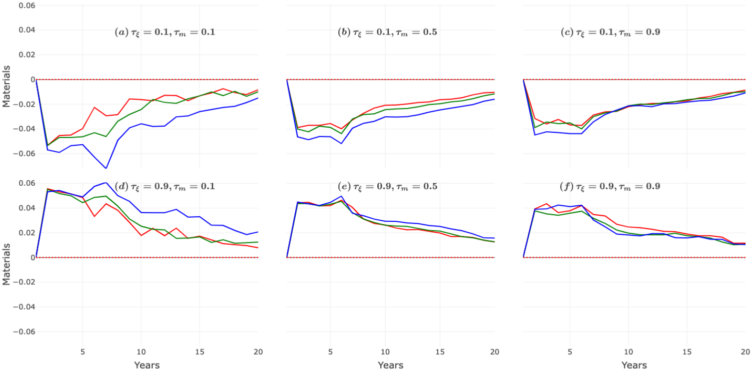
Figure 11: Shocks to Labor



*Differences between firms hit with medium productivity shock $\tau_\xi = 0.5$ and firms hit by low shock $\tau_\xi = 0.1$ (top row) and high shock $\tau_\xi = 0.9$ (bottom row) at different percentiles of labor demand.

Productivity Innovation Shocks to Materials

Figure 12: Shocks to Materials



*Differences between firms hit with medium productivity shock $\tau_\xi = 0.5$ and firms hit by low shock $\tau_\xi = 0.1$ (top row) and high shock $\tau_\xi = 0.9$ (bottom row) at different percentiles of materials demand.

Conclusion

- Nonparametric identification of production function, input demand, and productivity
- Proposed an estimation framework to document firm heterogeneity in production, input decisions, and productivity
- Variation in factor-correlated productivity shocks
- Asymmetric productivity persistence depends on size of innovation shock and levels of previous productivity
- Asymmetric impact of innovation shocks to inputs after bad/good shocks

Conclusion

- Extension to more structural models
- Incorporating shape constraints in estimation
- Structural production function with labor-augmenting productivity
- Other multi-dimensional unobservables such as fixed effects
- Implications for TFP estimation and markups

ACF Estimator

- ACF procedure for estimating a *value-added* production function (in logs):

$$y_{it} = \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \varepsilon_{it}, \tag{21}$$

- y_{it} is value-added output for firm i and time t
- l_{it} denotes labor input
- k_{it} denotes capital input
- ω_{it} is unobserved productivity
- ε_{it} denotes an independent and identically distributed (i.i.d) shock to production
- The constant β_0 is omitted since it is not separately identified from the mean of productivity.

ACF Estimator

- ACF introduces an intermediate input demand function defined as
$$m_{it} = m_t(k_{it}, l_{it}, \omega_{it}) \tag{22}$$
- The function m is assumed to be strictly increasing in ω_{it} for all k_{it} and l_{it}.
- Productivity can then be expressed as
$$\omega_{it} = m_t^{-1}(k_{it}, l_{it}, m_{it}) \tag{23}$$
- Substituting this equation into the production function
$$y_{it} = \beta_k k_{it} + \beta_l l_{it} + m_t^{-1}(k_{it}, l_{it}, m_{it}) + \varepsilon_{it} = \Phi_t(k_{it}, l_{it}, m_{it}) + \varepsilon_{it}. \tag{24}$$
- The function, $\Phi_t(k_{it}, l_{it}, m_{it})$, is identified by the following first stage moment restriction
$$\mathbb{E}[\varepsilon_{it} | \mathcal{I}_{it}] = 0 \tag{25}$$

ACF Estimator

- For the second stage, assume that productivity follows an AR(1) process given by

$$\omega_{it} = \mathbb{E}[\omega_{it}|\omega_{it-1}] + \xi_{it} = \rho\omega_{it-1} + \xi_{it}, \tag{26}$$

- ξ_{it} denotes an innovation to productivity which satisfies $\mathbb{E}[\xi_{it}|\mathcal{I}_{it-1}] = 0$.
- Plugging into the production function gives

$$\begin{aligned} y_{it} &= \beta_k k_{it} + \beta_l l_{it} + \rho\omega_{it-1} + \xi_{it} + \varepsilon_{it} \\ &= \beta_k k_{it} + \beta_l l_{it} + \rho(\Phi_{t-1}(k_{it-1}, l_{it-1}, m_{it-1}) - \beta_k k_{it-1} - \beta_l l_{it-1}) + \end{aligned}$$

- The production function parameters β_k, β_l and ρ are identified from the moment restrictions given by

$$\mathbb{E}[\xi_{it} + \varepsilon_{it}|\mathcal{I}_{it-1}] = 0. \tag{27}$$

- Estimation using Generalized Method of Moments (GMM)

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Skorohod Representation

- This representation comes from the fact that η_{it} can be defined as

$$\eta_{it} = F(y_{it}|k_{it}, l_{it}, m_{it}, \omega_{it}) \tag{28}$$

- η_{it} is then uniformly distributed independently of $(k_{it}, l_{it}, m_{it}, \omega_{it})$ on $(0, 1)$

$$y_{it} = F^{-1}(k_{it}, l_{it}, m_{it}, \omega_{it}, \eta_{it}), \quad \eta_{it} \sim Uniform(0, 1) \tag{29}$$

which is the quantile function

Go Back

Quantiles and Markups

- Markups can be obtained from a static cost minimization problem

$$\min_{L_t, M_t} P_{l_t} L_t + P_{m_t} M_t, \quad s.t.\ Q_t^y(K_{it}, L_{it}, M_{it}, \omega_{it}, \eta_{it}) = Y_t$$

- From FOC for inputs $X_t = \{L_t, M_t\}$ and rearranging

$$\mu_t = \frac{P_t Y_t}{P_{X_t} X_t} \frac{\frac{\partial F(K_t, L_t, M_t, \omega_t, \eta_t)}{\partial X_t}}{F(K_t, L_t, M_t, \omega_t, \eta_t)} = \frac{\beta_X(K_t, L_t, M_t, \omega_t, \eta_t)}{S_{X_t}}$$

where S_{X_t} is the expenditure share of the input

- Firm-specific markups are recovered from

$$\hat{\mu}_t(\tau) = \frac{\hat{\beta}_X(K_t, L_t, M_t, \omega_t, \tau)}{S_{X_t}}$$

◂ Go Back

Linear Operators

Definition 1

(Integral Operator) *Let a and b denote random variables with supports $\mathcal{A}$ and $\mathcal{B}$. Given two corresponding spaces $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ of functions with domains $\mathcal{A}$ and $\mathcal{B}$, let $L_{b|a}$ denote the operator mapping $g \in \mathcal{G}(\mathcal{A})$ to $L_{b|a}g \in L_{b|a}\mathcal{G}(\mathcal{B})$ defined by*

$$[L_{b|a}g](b) \equiv \int_{\mathcal{A}} f_{b|a}(b|a)g(a)da,$$

where $f_{b|a}$ denotes the conditional density of b given a.

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Piecewise Linear Splines

- The functional coefficients are approximated by piecewise linear splines on a grid with Q knots

$$\theta(\tau) = \theta(\tau_q) + \frac{\tau - \tau_q}{\tau_{q+1} - \tau_q}(\theta(\tau_{q+1}) - \theta(\tau_q))), \quad \tau_q < \tau \leq \tau_{q+1}$$

- I specify exponential distributions in the tail intervals
- Intercept coefficients are quantiles of an exponential distribution

$$\theta_0(\tau) = \theta_0(\tau_1) + \frac{\ln(\tau/\tau_1)}{\lambda_\theta^-}, \quad \tau \leq \tau_1$$

and

$$\theta_0(\tau) = \theta_0(\tau_Q) + \frac{\ln((1-\tau)/(1-\tau_Q))}{\lambda_\theta^+}, \quad \tau > \tau_Q$$

λ_θ^- and λ_θ^+ are parameters of exponential distributions

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Likelihood Function

- With piecewise linear splines in quantiles, the likelihood function is available in closed-form
- Density of a random variable is inverse of derivative of quantile function
- Consider the conditional density of $Y|X$ indexed by $\theta(\cdot)$

$$
\begin{aligned}
&\mathbf{f}_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x};\theta) = \\
&\sum_{q=1}^{Q-1} \frac{\tau_{q+1}-\tau_q}{Q^y(x;\theta(\tau_{q+1})) - Q^y(x;\theta(\tau_q))} \mathbb{1}\{Q^y(x;\theta(\tau_q)) < y \leq Q^y(x;\theta(\tau_{q+1}))\} \\
&+ \tau_1 \lambda_\theta^- \exp(\lambda_\theta^-(y - Q^y(x;\theta(\tau_1)))) \mathbb{1}\{y \leq Q^y(x;\theta(\tau_1))\} \\
&+ (1-\tau_Q)\lambda_\theta^+ \exp(-\lambda_\theta^+(y - Q^y(x;\theta(\tau_Q)))) \mathbb{1}\{y > Q^y(x;\theta(\tau_Q))\}
\end{aligned}
$$

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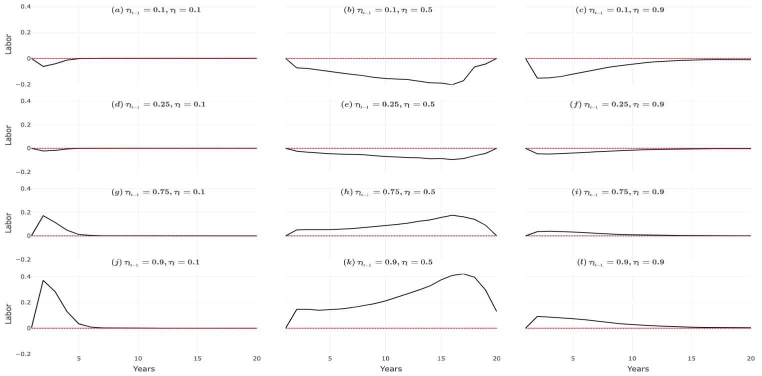
Labor Adjustments



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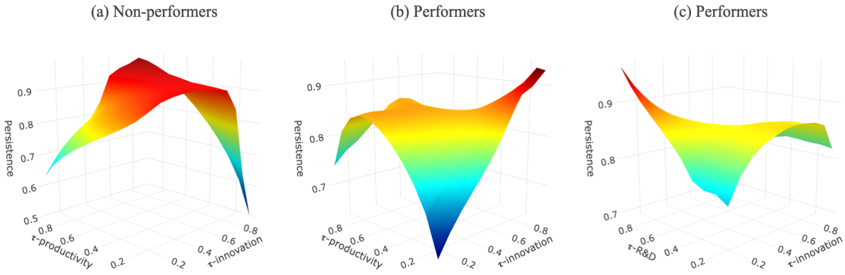
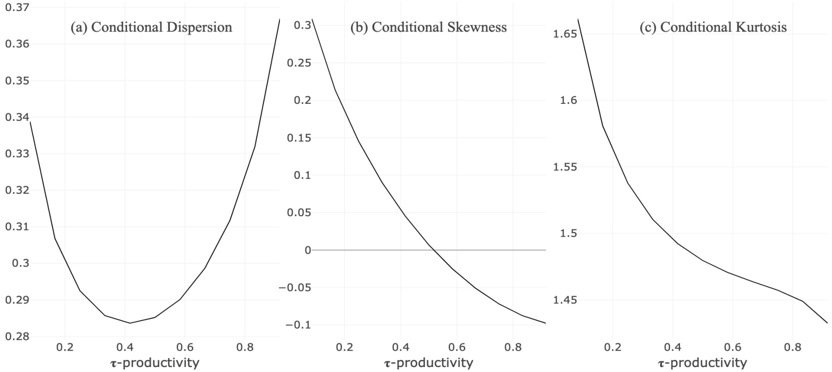
Productivity Persistence

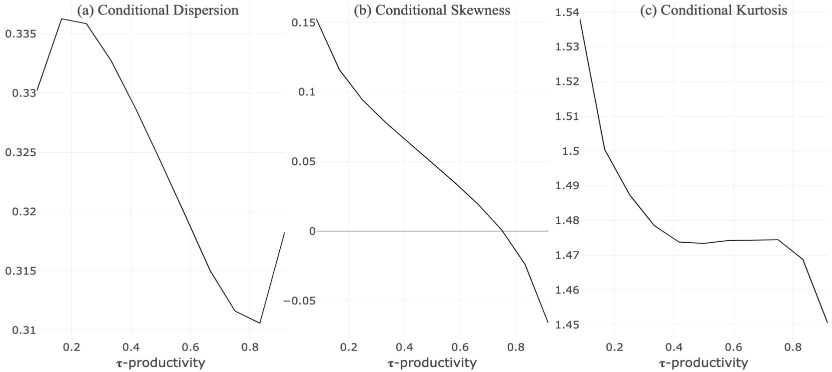


*Panel (a): Estimates of average productivity persistence for non R&D firms evaluated at τ_ξ and percentiles of previous productivity. Panel (b): Estimates of productivity persistence for R&D firms evaluated at τ_ξ and percentiles of previous productivity averaged over R&D. Panel (c): Estimates of productivity persistence for R&D firms evaluated at τ_ξ and percentiles of R&D averaged over productivity.

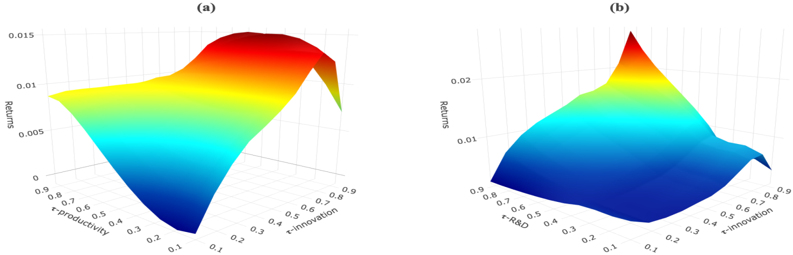
Higher Moments of the Conditional Productivity Distribution (Non R&D)



Higher Moments of the Conditional Productivity Distribution (R&D)



Returns to R&D



*Panel (a): Returns to R&D for firms evaluated at τ_ξ and percentiles of previous productivity averaged over R&D. Panel (b): Returns to R&D for firms evaluated at τ_ξ and percentiles of R&D averaged over productivity.

Correction for Selection Bias

- The exit rule can be written as

$$\begin{aligned} h_t(\omega_{it-1},\xi_{it}) \geq \underline{\omega}_t(k_{it}), \\ \xi_{it} \geq h_t^{-1}(\omega_{it-1}, \underline{\omega}_t(k_{it})), \\ \xi_{it} \geq \underline{\omega}_t(\omega_{it-1}, k_{it}) \end{aligned} \quad (30)$$

- If ξ_{it} is independent of (k_{it}, ω_{it-1}), $\xi_{it}|(k_{it}, \omega_{it-1}) \sim U(0,1)$
- The cutoff for which firms stay in operation can be estimated from

$$\underline{\omega}_t(\omega_{it-1}, k_{it}) = \text{Prob}(\chi_{it} = 1|\omega_{it-1}, k_{it}) \equiv p(\omega_{it-1}, k_{it}) \quad (31)$$

- Firms that receive an innovation shock greater than $p(\omega_{it-1}, k_{it})$ continue to operate
- Distribution of $\xi_{it}|(k_{it}, \omega_{it-1}, \chi_{it} = 1)$ is

$$\xi_{it}|(k_{it}, \omega_{it-1}, \chi_{it} = 1) \sim U(p(\omega_{it-1}, k_{it}), 1) \quad (32)$$

Correction for Selection Bias

- Consider a simple process for productivity $\omega_{it} = \rho(\xi_{it})\omega_{it-1}$

$$
\begin{aligned}
\text{Prob}(\omega_{it} &\leq \rho(\tau)\omega_{it-1} | \omega_{it-1}, k_{it}, \chi_{it} = 1) \\
&= \text{Prob}(\xi_{it} \leq \tau | \omega_{it-1}, k_{it}, \chi_{it} = 1) \\
&= \frac{\tau - p(\omega_{it-1}, k_{it})}{1 - p(\omega_{it-1}, k_{it})} \equiv G(\tau, p)
\end{aligned} \tag{33}
$$

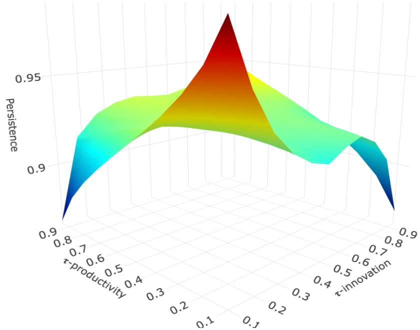
- If ω_{it} is known, then $\rho(\tau)$ is the solution to a rotated quantile regression problem

$$
\hat{\rho}(\tau) = \underset{\rho}{\text{argmin}} \sum_{i=1}^{N} \sum_{t=2}^{T} \chi_{it} \Big[G(\tau, \hat{p})(\omega_{it} - \rho\omega_{it-1})^{+} + (1 - G(\tau, \hat{p}))(\omega_{it} - \rho\omega_{it-1})^{-} \Big] \tag{34}
$$

where $a^{+} = max(a, 0)$ and $a^{-} = max(-a, 0)$

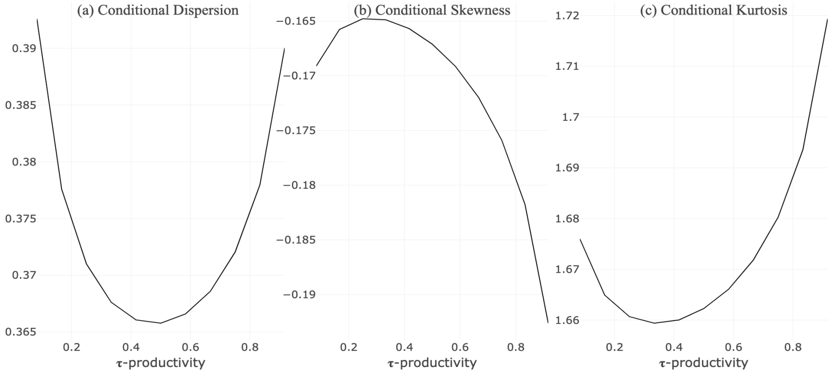
- $p(\omega_{it-1}, k_{it})$ can be estimated using a probit model

Productivity Persistence (Selection Corrected)

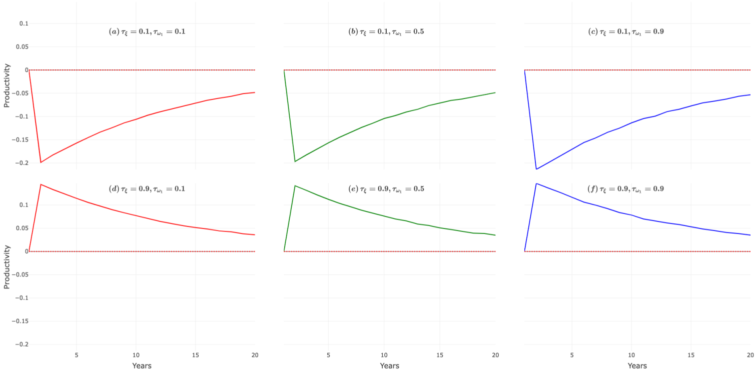


*Estimates of average productivity persistence evaluated at τ_ξ and percentiles of previous productivity.

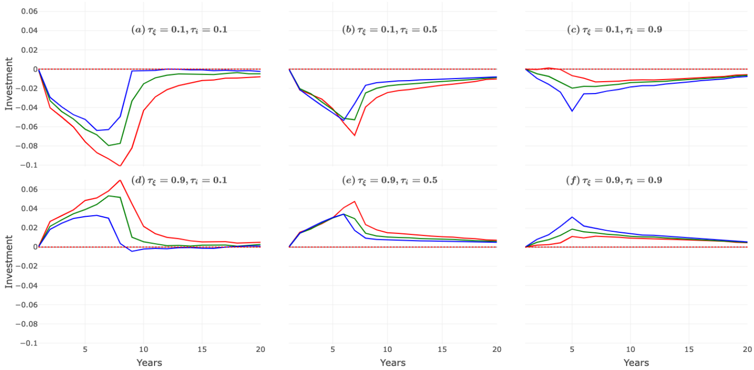
Higher Moments of the Conditional Productivity Distribution (Selection Corrected)



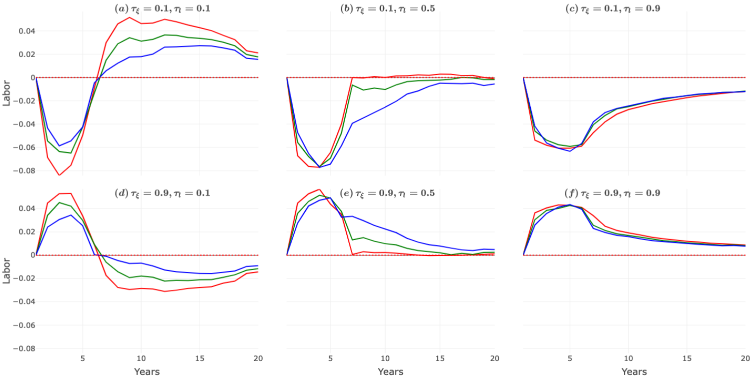
Productivity Innovation Shocks



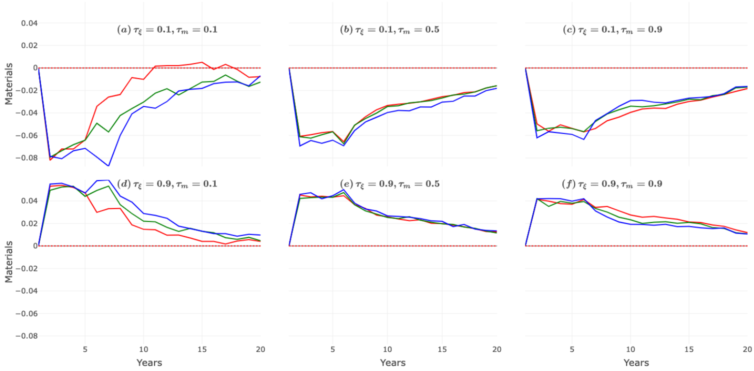
Productivity Innovation Shocks to Investment



Productivity Innovation Shocks to Labor



Productivity Innovation Shocks to Materials



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