A Dynamic Panel Data Framework for Identification and Estimation of Nonlinear Production Functions

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1   Introduction

Consider a nonlinear model for a firm's gross-output production function (in logs)

$$y_{it} = \sum_{j=1}^{J_1} \beta_j(\eta_{it}) g_j(x_{it}) + \omega_{it}(\eta_{it}) \tag{1}$$

where y_{it} is firm i's' output at time t and x_{it} denotes optimal input choices of the firm which include predetermined inputs such as capital and flexible inputs such as labor and other intermediate inputs. The unobserved productivity is denoted by ω_{it} which is correlated to input choices of the firm at time t. We let the output elasticities β and productivity ω_{it} to be functionally dependent on unobserved production shocks $\eta_{i1}, \ldots, \eta_{iT}$ that are uncorrelated to input choices and productivity at time t.

Without loss of generality we normalize η_{it} to be uniformly distributed on the interval $[0, 1]$. This model corresponds to a nonlinear random coefficient model where the outcome y_{it} is monotonic in η_{it}. In practice we can allow for nonlinear interactions between inputs and unobserved productivity at different quantiles so that marginal effects can be modeled as non-Hick's neutral. For empirical simplicity we assume separability in the unobserved productivity. The function g is an unknown nonlinear function.

In this model, heterogeneity in production technology across firms is driven by the rank of the unobserved production shocks η_{it}. In the following section we discuss how in certain specification, this is a valid approach to modeling heterogeneous input choices by connecting to the literature on production uncertainty and the firm's profit maximization problem.

We specify the productivity process as

$$\omega_{it} = \sum_{j=1}^{J_2} \rho_j(\xi_{it}) h_j(\omega_{it-1}) \tag{2}$$

where $\xi_{i1}, \ldots, \xi_{iT}$ are independent uniform random variables which represent innovation shocks to productivity. We assume ω_{it} is monotonic in ξ_{it} We let h be another unknown nonlinear function that allows the persistence in productivity in firms to be nonlinear across different quantiles.

2   Identification

To ease notation, let $X_t = \{Y_t, L_t, \iota_t\}$ and $W_t = \{K_{it}, I_t\}$. Our goal is identification of the Markov law of motion $f_{X_t, W_t, \omega_t | X_{t-1}, W_{t-1}, \omega_{t-1}}$ which we assume to be stationary. We assume the researcher observes

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a panel dataset consisting of i.i.d observations of firm output and input choices with the number of time periods $T \geq 4$ for a large number of firms. We introduce the following assumptions that simplify the expression for the law of motion.

Assumption 2.1. *(Production Dynamics)*

a) *Non-dynamic output and inputs:* $f_{X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1}} = f_{X_t,W_t,\omega_t|W_{t-1},\omega_{t-1}}$

b) *First-order Markov:* $f_{X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1},\mathcal{I}_{<t-1}} = f_{X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1}}$, *where* $\mathcal{I}_{<t-1}$ *is the firm's information set up to time* $t-1$

c) *Limited Feedback:* $f_{W_t|W_{t-1},\omega_t,\omega_{t-1}} = f_{W_t|W_{t-1},\omega_t}$

Assumption (2.1)(a) states that input decisions for labor and material inputs maximize profits in the current period and do not have any dynamic implications. This can be relaxed to include cases where labor has dynamic effects due to hiring/firing costs. This assumption also does not allow for dynamic effects of output, which is standard in the production function literature.

Using Assumption (2.1) the Markov law of motion can be factored into:

$$
\begin{aligned}
f(X_t,W_t,\omega_t|X_{t-1},W_{t-1},\omega_{t-1},\mathcal{I}_{<t-1}) &= f(X_t,W_t,\omega_t|W_{t-1},\omega_{t-1}) \\
&= f(X_t|W_t,\omega_t,W_{t-1},\omega_{t-1})f(W_t|\omega_t,W_{t-1},\omega_{t-1})f(\omega_t|W_{t-1},\omega_{t-1})
\end{aligned}
\tag{3}
$$

We can simplify the first density on the last line of the equation (3) as

$$
\begin{aligned}
f(X_t|W_t,\omega_t,W_{t-1},\omega_{t-1}) &= f(Y_t|W_t,\omega_t,W_{t-1},\omega_{t-1})f(L_t|W_t,\omega_t,W_{t-1},\omega_{t-1})(\iota_t|W_t,\omega_t,W_{t-1},\omega_{t-1}) \\
&= f(Y_t|L_t,\iota_t,K_t,\omega_t)f(L_t|K_t,\omega_t)f(\iota_t|K_t,\omega_t)
\end{aligned}
\tag{4}
$$

Furthermore, the second density on the last line of the equation (3) becomes

$$
\begin{aligned}
f(W_t|\omega_t,W_{t-1},\omega_{t-1}) &= f(W_t|W_{t-1},\omega_t) = f(I_t,K_t|I_{t-1},K_{t-1},\omega_t) \\
&= f(I_t|K_t,I_{t-1},K_{t-1},\omega_t)f(K_t|I_{t-1},K_{t-1},\omega_t) \\
&= f(I_t|K_t,\omega_t)f(K_t|I_{t-1},K_{t-1},\omega_t)
\end{aligned}
\tag{5}
$$

To show that both Assumption (2.1)(b-c) are satisfied, we introduce a dynamic model of firm investment that is a slight modification of Ericson and Pakes (1995). We begin by providing the evolution process for $\omega_t \in \mathbb{R}$ which is given by:

$$
\omega_t = g(\omega_{t-1},\xi_t),
\tag{6}
$$

where the function $g(\cdot,\xi_t)$ is strictly increasing in the iid innovation shock, $\xi_t \in \mathbb{R}$. Note that equation (6) implies $f_{\omega_t|W_{t-1},\omega_{t-1}} = f_{\omega_t|\omega_{t-1}}$ which implies that productivity evolves exogenously. This can be relaxed when we consider productivity enhancing activities such as R&D similar to Doraszelski and Jaumandreu (2013).

Capital accumulates according to the following process:

$$
K_t = \kappa_t(K_{t-1},I_{t-1},\upsilon_t),
\tag{7}
$$

where the function κ is strictly increasing in its last argument and υ_t denotes an iid shock independent of the other arguments. A special case of equation (7) is the usual capital accumulation law $K_t = (1-\delta)K_{t-1}+I_{t-1}+\upsilon_t$. Here υ_t are other factors that affect the capital accumulation process need better examples of these.

In each period, a firm chooses investment to maximize its discounted future profits:

$$I_t = I^*(K_t, \omega_t, \zeta_t) = \underset{I_t \geq 0}{\operatorname{argmax}} \left[\Pi_t(K_t, \omega_t) - c(I_t, \zeta_t) + \beta \mathbb{E} \left[V_{t+1}(K_{t+1}, \omega_{t+1}, \zeta_{t+1}) | \mathcal{I}_t \right] \right], \qquad (8)$$

where $\pi_t(\cdot)$ is current period profits as a function of the state variables, $c(I_t)$ is the cost of current investment and β is the firm's discount factor. We introduce an additional state variable ζ_t which could represent other factors that shift firm's investment costs. We assume the cost function $c(\cdot, \zeta_t)$ is decreasing in ζ_t. Under certain conditions, the investment policy function is monotonic increasing in ζ_t. Without loss of generality, we normalize $\zeta_t \sim U[0, 1]$. In a later section we show how to modify identification and estimation strategies to cases when investment is censored or there is selection bias due to endogenous entry/exit which can be the case in certain industries. Note that the restrictions on the capital accumulation process in (7) and the investment problem in (8) satisfy the Limited Feedback condition in Assumption (2.1)(c).

The specifications for the static inputs, labor and intermediate inputs such as materials, fuels, and electricity, are much easier to state. We let optimal labor demand be given by:

$$l_t = \ell(k_t, \omega_t, \epsilon_t) \qquad (9)$$

where the function, $\ell(\cdot, \cdot, \epsilon_t)$ is strictly increasing in ϵ_t which is assumed to be independent of the other arguments. We normalize this to be standard uniform each period. We follow a similar model for the intermediate inputs:

$$\iota_t = \iota_t(k_t, \omega_t, \varepsilon_t) \qquad (10)$$

where the function, $\iota(\cdot, \cdot, \varepsilon_t)$ is strictly increasing in ε_t which is assumed to be independent of the other arguments. We normalize this to be standard uniform each period.

In order to factorize the density in equation (4) we need the following assumptions:

Assumption 2.2. *(Conditional Independence)*

   a) *The production shock η_t, labor shock ϵ_t and intermediate input shock ε_t are mutually independent conditional on $(W_t, \omega_t, W_{t-1}, \omega_{t-1})$*

   b) *The production shock η_t is independent of ζ_t conditional on $(L_t, \iota_t, K_t, \omega_t)$*

   c) *ϵ_t and ε_t are independent of ζ_t conditional on (K_t, ω_t)*

Assumption (2.2) is similar to the mutual independence assumptions made by Hu *et al.* (2019). In their paper, they provide interpretations of these errors that are likely to satisfy the conditional independence restrictions. For example, this assumption is likely to be valid if we interpret each of these shocks as optimization errors. Unlike Hu *et al.* (2019) we cannot interpret these shocks as unobserved idiosyncratic cost shocks since this would imply that input demand functions, are decreasing in cost shocks, which violates the monotonically increasing assumption we need for representing these functions by their conditional quantiles.

We outline the identification procedure similar to Hu and Shum (2012). First we define a function

$$V_t \equiv h_t(\{X_t, W_t\}), \qquad (11)$$

where the function $h_t : \mathbb{R}^{d_X + d_W} \to \mathbb{R}$ is known with d_X being the dimension of X_t and d_W being the dimension of W_t. Under assumption (2.1), we can write:

$$f_{X_{t+1}, W_{t+1}, X_t, W_t, X_{t-1}, W_{t-1}, X_{t-2}, W_{t-2}} = \int f_{X_{t+1}, W_{t+1} | W_t, \omega_t} f_{X_t, W_t | W_{t-1}, \omega_t} f_{X_{t-1}, W_{t-1}, X_{t-2}, W_{t-2}, \omega_t} d\omega_t$$

$$(12)$$

The linear operator $L_{V_{t-2}, \bar{x}_{t-1}, \bar{w}_{t-1}, \bar{x}_t, \bar{w}_t, V_{t+1}}$ as a mapping from the $\mathcal{L}^p$ space of functions of V_{t+1} to the $\mathcal{L}^p$ space of functions of V_{t-2} is defined as

$$
\begin{aligned}
& (L_{V_{t-2}, \bar{x}_{t-1}, \bar{w}_{t-1}, \bar{x}_t, \bar{w}_t, V_{t+1}} h)(v_{t-2}) \\
& = \int f_{V_{t-2}, X_{t-1}, W_{t-1}, X_t, W_t, V_{t+1}}(v_{t-2}, \bar{x}_{t-1}, \bar{w}_{t-1}, \bar{x}_t, \bar{w}_t, v_{t+1}) h(v_{t+1}) dv_{t+1} : \\
& h \in \mathcal{L}^p(\mathcal{V}_{t+1}), \bar{x}_{t-1} \in \mathcal{X}_{t-1}, \bar{x}_t \in \mathcal{X}_t, \bar{w}_{t-1} \in \mathcal{W}_{t-1}, \bar{w}_t \in \mathcal{W}_t,
\end{aligned}
\tag{13}
$$

where $\mathcal{V}_t, \mathcal{X}_{t-1}, \mathcal{X}_t, \mathcal{W}_{t-1}$ and $\mathcal{W}_t$ are the supports of $V_t, X_{t-1}, X_t, W_{t-1}$ and W_t respectively. We also define the diagonal operator as:

$$
\begin{aligned}
(D_{\bar{x}_t, \bar{w}_t | \bar{w}_{t-1}, \omega_t} h)(\omega_t) &= f_{X_t, W_t |, W_{t-1}, \omega_t}(\bar{x}_t, \bar{w}_t | \bar{w}_{t-1}, \omega_t) h(\omega_t) \\
h &\in \mathcal{L}^p(\Omega_t), \bar{x}_t \in \mathcal{X}_t, \bar{w}_t \in \mathcal{W}_t, \bar{w}_{t-1} \in \mathcal{W}_{t-1}
\end{aligned}
\tag{14}
$$

We provide a few lemmas that allow us to represent the observed density $f_{X_{t+1}, W_{t+1}, X_t, W_t, X_{t-1}, W_{t-1}, X_{t-2}, W_{t-2}}$ and the Markov law of motion $f_{X_t, W_t, \omega_t | X_{t-1}, W_{t-1}, \omega_{t-1}}$ using operator notation.

Lemma 2.1. *(Representation of the Observed Density)* *For any $t \in \{3, \ldots, T-1\}$. Assumption (2.1) implies that, for any $(x_t, w_t, x_{t-1}, w_{t-1}) \in \mathcal{X}_t \times \mathcal{W}_t \times \mathcal{X}_{t-1} \times \mathcal{W}_{t-1}$,*

$$
L_{V_{t+1}, x_t, w_t, x_{t-1}, w_{t-1}} = L_{V_{t+1} | w_t, \omega_t} D_{x_t, w_t | w_{t-1}, \omega_t} L_{x_{t-1}, w_{t-1}, V_{t-2}, \omega_t}
\tag{15}
$$

Lemma 2.2. *(Representation of the Markov Law of Motion)* *For any $t \in \{3, \ldots, T-1\}$ and under certain invertibility conditions provided below give, for any $(x_t, w_t, x_{t-1}, w_{t-1}) \in \mathcal{X}_t \times \mathcal{W}_t \times \mathcal{X}_{t-1} \times \mathcal{W}_{t-1}$,*

$$
L_{x_t, w_t, \omega_t | w_{t-1}, \omega_{t-1}} = L_{V_{t+1} | w_t, \omega_t}^{-1} L_{V_{t+1}, x_t, w_t, x_{t-1}, w_{t-1}, V_{t-2}} L_{V_t, x_{t-1}, w_{t-1}, V_{t-2}}^{-1} L_{V_t | w_{t-1}, \omega_t}
\tag{16}
$$

The identification argument then depends on the existence and uniqueness a spectral decomposition of the linear operator $L_{V_{t+1}, x_t, w_t, x_{t-1}, w_{t-1}}$ which corresponds to the observed density $f_{X_{t+1}, W_{t+1}, X_t, W_t, X_{t-1}, W_{t-1}, X_{t-2}, W_{t-2}}$. We state these assumptions and their interpretation in the production function literature and in context of our model.

3   Estimation Strategy

Our estimation strategy does not follow directly from the identification conditions discussed earlier in this paper, but on a set of restrictions discussed in the appendix. Our empirical specification for the Markovian transitions of productivity, output, and capital evolution closely resemble Arellano *et al.* (2017).

3.1   Persistent Productivity

Let age_{it} denote the age of firm i at time t, we specify productivity to transition according to:

$$
Q_t(\omega_{it-1}, \tau) = Q(\omega_{it-1}, age_{it}, \tau) = \rho_0(\tau) + \rho_\omega(\tau)\omega_{it-1} + \rho_a(\tau)age_{it}
\tag{17}
$$

The quantile function for ω_{i1} is specified in a similar way

$$
Q_{\omega_1}(age_{i1}, \tau) = \rho_{01}(\tau) + \rho_{a1}(\tau)age_{i1}
\tag{18}
$$

3.2 Output

We specify the output equation as follows:

$$Q_t(x_{it}, \omega_{it}, \tau) = Q(x_{it}, age_{it}, \omega_{it}, \tau) =$$
$$\beta_0(\tau) + \beta_\omega(\tau)\omega_{it} + \beta_a(\tau)age_{it} + \beta_l(\tau)l_{it} + \beta_k(\tau)k_{it} + \beta_m(\tau)m_{it} + \beta_u(\tau)u_{it} \tag{19}$$

The above representation of the production function is a simple case. One can also consider more flexible specifications that allow for interactions of productivity and input choices thus allowing for non-Hicks neutral technology changes.

3.3 Investment Demand

We specify the investment process as the censored variable, $I_t = \max\{0, I_t^*\}$ where $I_t^* = \iota_t(\omega_{it}, k_{it}, \zeta_{it})$. We specify the investment demand equation as:

$$\iota_t(\omega_{it}, k_{it}, \tau) = \iota(\omega_{it}, k_{it}, age_{it}, \tau) = \iota_\omega(\tau)\omega_{it} + \iota_k(\tau)k_{it} + \iota_a(\tau)age_{it} + \iota_0(\tau) \tag{20}$$

Due to the equivariance of quantiles to monotone transformations we can write the censored investment variable as:

$$Q(\omega_{it}, k_{it}, \tau) = \max\{0, \iota_0(\tau) + \iota_\omega(\tau)\omega_{it} + \iota_k(\tau)k_{it} + \iota_a(\tau)age_{it}\} \tag{21}$$

3.4 Static Inputs

Finally, we specify a reduced-form demand function for the static inputs of $z = m$ and u (in logs) as:

$$\mu_{zt+1}(k_{it+1}, \omega_{it+1}, \tau) = \mu_z(k_{it+1}, \omega_{it+1}, age_{it+1}, \tau) = \mu_{zk}k_{it+1} + \mu_{z\omega}\omega_{it+1} + \mu_{za}age_{it} + \mu_{z0}(\tau) \tag{22}$$

4 Implementation

To ease notation, let δ denote a finite dimensional parameter vector that contains the functions $\rho_0, \rho_\omega, \rho_a, \rho_{01}, \rho_{a_1}$ and θ denote the finite dimensional parameter vector that contains the functions $\beta_0, \beta_\omega, \beta_a, \beta_l, \beta_k, \beta_m, \beta_u, \iota_0, \gamma_0, \gamma_\omega, \gamma_a, \kappa_0$ and μ_{x0}. The vector θ also contains the finite dimensional parameters $\iota_\omega, \iota_k, \iota_a, \kappa_K, \kappa_I, \kappa_a, \mu_{xk}$ and μ_{xa}. We model the functional parameters using Wei and Carroll (2009) andArellano and Bonhomme (2016). For example, the function ρ_ω is modeled as a piecewise-polynomial interpolating splines on a grid $[\tau_1, \tau_2], [\tau_3, \tau_4], \ldots, [\tau_{J-1}, \tau_J]$, contained in the unit interval and is constant on $[0, \tau_1]$ and $[\tau_J, 1)$ The intercept coefficient ρ_0 is specified as the quantile of an exponential distribution on $(0, \tau_1]$ (indexed by λ_ω^-) and $[\tau_{J-1}, 1)$ (indexed by λ_ω^+).

The function ρ_ω is modeled as piecewise linear on $[\tau_1, \tau_J]$. The remaining functional parameters are modeled similarly. We take $J = 11$ and $\tau_j = \frac{j}{J+1}$. Following Arellano *et al.* (2017) we parameterize the distribution of ϵ to be log-normal so we set, for example, $\mu_{z0}(\tau) = \alpha_{z0} + \sigma_z \Phi^{-1}(\tau)$ although this can be relaxed. In the following section we outline the model's restrictions and a feasible estimation strategy.

4.1 Model Restrictions

Let $\Psi_\tau(u) = \tau - \mathbb{1}\{u < 0\}$ denote the first derivative of the quantile check function $\psi_\tau(u) = (\tau - \mathbb{1}\{u < 0\})u$. The following conditional moment restrictions hold as an implication of the conditional independence restrictions in section 2. Therefore, we estimate the parameters of interest from the following conditional moment restrictions.

$$\mathbb{E}\left[\Psi_{\tau_j}\begin{pmatrix}\xi_{it+1}\\\eta_{it}\end{pmatrix}\Big|\omega_{it},age_{it},\tilde{z}_{td}\right]=0 \tag{23}$$

For investment,

$$\mathbb{E}\left[\Psi_{\tau_j}(\zeta_{it})|\omega_{it},k_{it},age_{it}\right]=0, \tag{24}$$

and lastly for the static input:

$$\mathbb{E}\left[\epsilon_{it+1}|\omega_{it},age_{it},\tilde{z}_{td}\right]=0, \tag{25}$$

Why do we need a moment restriction for ζ_{it}? Current productivity enters the investment equation so we need a model for investment. See footnote 23 page 709 in Arellano *et al.* (2017)

where $\tilde{z}_{td}=I_t,l_t,k_t,m_t,$ and u_t. To fix ideas, we first specify how to estimate the conditional moments for persistent productivity and production.

For $t\geq 2$

$$\mathbb{E}\left[\Psi_{\tau_j}(\xi_{it+1})\Big|\omega_{it},age_{it},\tilde{z}_{td}\right]=$$
$$\mathbb{E}\left[\Psi_{\tau_j}(\omega_{it+1}-\bar{\rho}_0(\tau_j)-\bar{\rho}_\omega(\tau_j)\omega_{it}-\bar{\rho}_a(\tau_j)age_{it})\Big|\omega_{it},age_{it},\tilde{z}_{td}\right]=0 \tag{26}$$

$$\mathbb{E}\left[\Psi_{\tau_j}(\eta_{it})\Big|\omega_{it},age_{it},\tilde{z}_{td}\right]=$$
$$\mathbb{E}\Big[\Psi_{\tau_j}(y_{it}-\bar{\beta}_0(\tau_j)-\bar{\beta}_\omega(\tau_j)\omega_{it}-\bar{\beta}_a(\tau_j)age_{it}-\bar{\beta}_l(\tau_j)l_{it}-\bar{\beta}_k(\tau_j)k_{it}$$
$$-\bar{\beta}_m(\tau_j)m_{it}-\bar{\beta}_u(\tau_j)u_{it})\Big|\omega_{it},age_{it},\tilde{z}_{td}\Big]=0 \tag{27}$$

Here $\bar{\rho}(\tau_j)$ and $\bar{\beta}(\tau_j)$ denote the true values of $\rho(\tau_j)$ and $\beta(\tau_j)$ for $j\in\{1,\ldots,J\}$. Clearly, estimating the above conditional moment restrictions are infeasible due to the unobserved productivity component. Therefore, we use the following unconditional moment restrictions and posterior distributions for ω_{it} to integrate out the unobserved productivity. Due to the law of iterated expectations we now have the following integrated moment conditions:

$$\mathbb{E}\left[\int\left(\Psi_{\tau_j}(\omega_{it+1}-\bar{\rho}_0(\tau_j)+\bar{\rho}_\omega(\tau_j)\omega_{it}-\bar{\rho}_a(\tau_j)age_{it})\otimes\begin{pmatrix}1\\\omega_{it}\\age_{it}\\\tilde{z}_{td}\end{pmatrix}\right)g_i(\omega_{it};\bar{\delta},\bar{\theta})d\omega_{it}\right]=0 \tag{28}$$

and

$$\mathbb{E}\left[\int\left(\Psi_{\tau_j}(y_{it} - \bar{\beta}_0(\tau_j) + \bar{\beta}_\omega(\tau_j)\omega_{it} + \bar{\beta}_a(\tau_j)age_{it} + \bar{\beta}_l(\tau_j)l_{it} + \bar{\beta}_k(\tau_j)k_{it}\right.\right.$$

$$\left.\left. + \bar{\beta}_m(\tau_j)m_{it} + \bar{\beta}_u(\tau_j)u_{it} \otimes \begin{pmatrix} 1 \\ \omega_{it} \\ age_{it} \\ \tilde{z}_{td} \end{pmatrix}\right)g_i(\omega_{it}; \bar{\delta}, \bar{\theta})d\omega_{it}\right] = 0, \tag{29}$$

where $\bar{\theta}$ and $\bar{\delta}$ denote the true values of θ and δ. The posterior distribution is specified as (age omitted for ease of notation):

$$g_i(\omega_{it}; \bar{\delta}, \bar{\theta}) = g(\omega_{it}|m_{it+1}, y_{it}, \tilde{z}_{td}; \bar{\delta}, \bar{\theta}) \propto$$

$$\prod_{t=1}^{T} g(y_{it}|\omega_{it}, l_{it}, k_{it}, m_{it}, u_{it}; \bar{\delta}, \bar{\theta}) \times \prod_{t=1}^{T} g(I_{it}|\omega_{it}, l_{it}, k_{it}, m_{it}, u_{it}; \bar{\delta}, \bar{\theta}) \times$$

$$\prod_{t=1}^{T} g(m_{it+1}|\omega_{it}, l_{it}, k_{it}, m_{it}, u_{it}; \bar{\delta}, \bar{\theta}) \times \prod_{t=2}^{T} g(\omega_{it}|\omega_{it-1}; \bar{\delta}) \times g(\omega_{i1}; \bar{\delta}) \tag{30}$$

The unconditional moment restrictions implied by (23) are over-identified. The posterior density in equation (30) is a closed-form expression when using piece-wise linear splines for $(\delta(\cdot), \theta(\cdot))$. Therefore, we use a Generalized Method of Moments (GMM) objective function to estimate (δ, θ) The estimation is an Expectation Maximization (EM) algorithm. In Arellano and Bonhomme (2016) and Arellano *et al.* (2017), the "M-step" is performed using quantile regression. In our M-step, we replace quantile regression with GMM since our moment conditions are smooth due to the presence of the integral. Given an initial parameter value $(\hat{\delta}^0, \hat{\theta}^0)$. Iterate on $s = 0, 1, 2, \ldots$ in the following two-step procedure until converge to a stationary distribution:

1. *Stochastic E-Step*: Draw M values $\omega_i^{(m)} = (\omega_{i1}^{(m)}, \omega_{i2}^{(m)}, \ldots, \omega_{iT}^{(m)})$ from

$$g(\omega_{it}|m_{it+1}, y_{it}, \tilde{z}_{td}; \hat{\delta}^{(s)}, \hat{\theta}^{(s)}) \propto$$

$$\prod_{t=1}^{T} g(y_{it}|\omega_{it}, l_{it}, k_{it}, m_{it}, u_{it}; \hat{\delta}^{(s)}, \hat{\theta}^{(s)}) \times \prod_{t=1}^{T} g(I_{it}|\omega_{it}, l_{it}, k_{it}, m_{it}, u_{it}; \hat{\delta}^{(s)}, \hat{\theta}^{(s)}) \times$$

$$\prod_{t=1}^{T} g(m_{it+1}|\omega_{it}, l_{it}, k_{it}, m_{it}, u_{it}; \hat{\delta}^{(s)}, \hat{\theta}^{(s)}) \times \prod_{t=2}^{T} g(\omega_{it}|\omega_{it-1}; \hat{\delta}^{(s)}) \times g(\omega_{i1}; \hat{\delta}^{(s)})$$

2. *Maximization Step*: For $j = 1, \ldots, J$, solve the following GMM objective functions

$$\hat{\boldsymbol{\rho}}(\tau_j)^{(s+1)} = \underset{\boldsymbol{\rho}(\tau_j)}{\operatorname{argmin}} \hat{M}_{n,t,m}(\boldsymbol{\rho}, \tau_j)' \hat{W}_{\boldsymbol{\rho}} \hat{M}_{n,t,m}(\boldsymbol{\rho}, \tau_j)$$

$$\hat{\boldsymbol{\rho}}_1(\tau_j)^{(s+1)} = \underset{\boldsymbol{\rho}_1(\tau_j)}{\operatorname{argmin}} \hat{M}_{n,1,m}(\boldsymbol{\rho}_1, \tau_j)' \hat{W}_{\boldsymbol{\rho}_1} \hat{M}_{n,1,m}(\boldsymbol{\rho}_1, \tau_j)$$

$$\hat{\boldsymbol{\beta}}(\tau_j)^{(s+1)} = \underset{\boldsymbol{\beta}(\tau_j)}{\operatorname{argmin}} \hat{M}_{n,t,m}(\boldsymbol{\beta}, \tau_j)' \hat{W}_{\boldsymbol{\beta}} \hat{M}_{n,t,m}(\boldsymbol{\beta}, \tau_j)$$

where, for example

$$\boldsymbol{\rho}(\tau_j) = (\rho_0(\tau_j), \rho_\omega(\tau_j), \rho_a(\tau_j))$$

$$\hat{M}_{n,t,m}(\boldsymbol{\rho}, \tau_j) = \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \Psi_{\tau_j}(\xi_{it+1}(\boldsymbol{\rho}(\tau_j))) Z_{it}^{(m)}$$

$$Z_{it}^{(m)} = (1, \omega_{it}^{(m)}, age_{it}, \tilde{z}_{td})$$

$$\xi_{t+1}(\boldsymbol{\rho}(\tau_j)) = \omega_{it+1}^{(m)} - \rho_0(\tau_j) + \rho_\omega(\tau_j)\omega_{it}^{(m)} - \rho_a(\tau_j)age_{it}$$

and $\hat{W}_{\boldsymbol{\rho}}$ is a weighting matrix. We estimate each moment restriction separately as computation is less prone to error compared to estimating jointly.

We estimate the investment equation without over-identifying conditions because I do not know how to estimate a censored quantile regression model using GMM

$$\mathbb{E}\left[\int \left(\Psi_{\tau_j}(I_{it} - \max\{0, \iota_0(\tau) + \iota_\omega(\tau)\omega_{it} + \iota_k(\tau)k_{it} + \iota_a(\tau)age_{it}\}) \otimes \begin{pmatrix} 1 \\ \omega_{it} \\ k_{it} \\ age_{it} \end{pmatrix}\right) g_i(\omega_{it}; \bar{\delta}, \bar{\theta})d\omega_{it}\right] = 0$$

(31)

which can be estimated using

$$\hat{\boldsymbol{\iota}}(\tau_j)^{(s+1)} = \underset{\boldsymbol{\iota}(\tau_j)}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{m=1}^{M} \psi_{\tau_j}\left(I_{it} - \max\{0, \iota_0(\tau) + \iota_\omega(\tau)\omega_{it}^{(m)} + \iota_k(\tau)k_{it} + \iota_a(\tau)age_{it}\right)$$

(32)

this step can be performed using any existing censored quantile regression using the package *quantreg*.

The moment restrictions involving the static input can be written as:

$$\mathbb{E}\left[\int (z_{it+1} - \alpha_{z0} - \mu_{zk}k_{it+1} + \mu_{z\omega}\omega_{it+1} - \mu_{za}age_{it}) \otimes \begin{pmatrix} 1 \\ \omega_{it} \\ age_{it} \\ \tilde{z}_{td} \end{pmatrix} g_i(\omega_{it}; \bar{\delta}, \bar{\theta})d\omega_{it}\right] = 0,$$

(33)

which can be estimated using GMM. With estimates $(\hat{\alpha}_{z0}, \hat{\mu}_{k\omega}, \hat{\mu}_{z\omega}, \hat{\mu}_{za})$, the variance of the static input demand shock can be found by the following equation:

$$\hat{\sigma}_z^2 = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[\int \left(z_{it+1} - \hat{\alpha}_{z0} - \hat{\mu}_{zk}k_{it+1} + \hat{\mu}_{z\omega}\omega_{it+1} - \hat{\mu}_{za}age_{it}\right)^2 g_i(\omega_{it}; \bar{\delta}, \bar{\theta})\right)d\omega_{it}\right]$$

(34)

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