Variational Models and Numerical Methods for Image Processing

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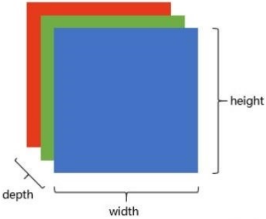
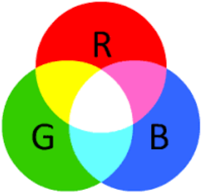
Outline

1. Introduction to image processing
2. Image denoising
3. Image contrast enhancement
4. Image stitching

Let Ω be an open set in $\mathbb{R}^2$ and $u : \bar{\Omega} \to \mathbb{R}$

- $u(x) \in [0, 255]\,, \forall x \in \bar{\Omega}$
- grayscale image: $0 \to$ black, $255 \to$ white
- color image: RGB channels



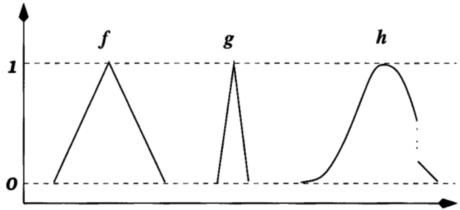


Definition (Total variation of one variable function)

Let $\Omega = (a,b) \subseteq \mathbb{R}$ and $\mathcal{P}_n = \{a = x_0, x_1, \cdots, x_{n-1}, x_n = b\}$, be an arbitrary partition of $\bar{\Omega}$. The total variation of a real-valued function $u : \Omega \to \mathbb{R}$ is defined as the quantity,

$$\|u\|_{TV(\Omega)} = \sup_{\mathcal{P}_n} \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| .$$

For example, the total variation of the following are the value 2.



Definition (Total variation of one variable function)

Let $\Omega = (a, b) \subseteq \mathbb{R}$ and $\mathcal{P}_n = \{a = x_0, x_1, \cdots, x_{n-1}, x_n = b\}$, be an arbitrary partition of $\bar{\Omega}$. The total variation of a real-valued function $u : \Omega \to \mathbb{R}$ is defined as the quantity,

$$\|u\|_{TV(\Omega)} = \sup_{\mathcal{P}_n} \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})|.$$

Theorem

If u is a smooth function, then

$$\|u\|_{TV(\Omega)} = \int_{\Omega} |u'(x)| dx.$$

Definition (Total variation of two variable function)

Let Ω be an open set of $\mathbb{R}^2$ and $u \in L^1(\mathbb{R})$. The total variation of u in Ω is defined as

$$\|u\|_{TV(\Omega)} = \sup\left\{\int_\Omega u \operatorname{div}\varphi\, dx : \varphi \in C_c^1\left(\Omega, \mathbb{R}^2\right), \|\varphi\|_{L^\infty(\Omega)} \leq 1\right\},$$

where $C_c^1(\Omega, \mathbb{R}^n)$ is the set of continuously differentiable vector functions of compact support contained in Ω, and $\|\cdot\|_{L^\infty(\Omega)}$ is the essential supremum norm.

Theorem

If u is a smooth function, then

$$\|u\|_{TV(\Omega)} = \int_\Omega |\nabla u| dx.$$

Definition (Bounded variation)

If $\|u\|_{TV(\Omega)} < \infty$, then we say that u is a function of bounded variation. Moreover, the space of functions of bounded variation $BV(\Omega)$ is defined as $u \in L^1(\Omega)$ such that the total variation is finite, i.e.,

$$BV(\Omega) = \left\{ u \in L^1(\Omega) : \|u\|_{TV(\Omega)} < \infty \right\}.$$

Remark

$BV(\Omega)$ is a Banach space with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|u\|_{TV(\Omega)}.$$

Let $[a, b] \subseteq \mathbb{R}$. We consider the functional,

$$E[y] = \int_a^b L\left(x, y, y'\right) dx,$$

where we assume that $y \in C^2([a, b])$ and $L \in C^2$ with respect to its arguments x, y and y'.

Euler-Lagrange equation (1-dimension)

A necessary condition for a local minimum y of E is

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

Let $\Omega \subseteq \mathbb{R}^2$ be an open set. We consider the functional,

$$E[u] = \int_\Omega L\left(x, y, u, u_x, u_y\right) d(x, y),$$

where we assume that $u \in C^2(\bar{\Omega})$ and $L \in C^2$ with respect to its arguments x, y, u, u_x and u_y.

Euler-Lagrange equation (2-dimension)

A necessary condition for a local minimum u of E is

$$\frac{\partial L}{\partial u} - \nabla \cdot \left(\frac{\partial L}{\partial u_x}, \frac{\partial L}{\partial u_y} \right) = 0.$$

$\xrightarrow{\text{denoising}}$

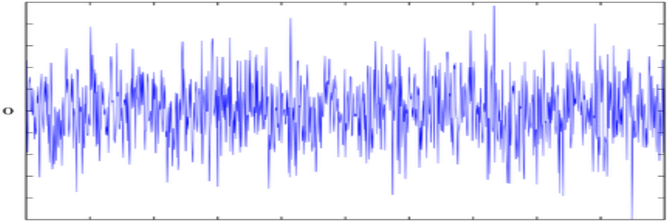
Mathematics method of image processing

1. Fourier transform
2. Heat-type equation
3. Machine learning
4. **Variational method (energy functional)**

Denoising (1-dimension)

Let u be a one-dimension signal. Our goal is to find u such that

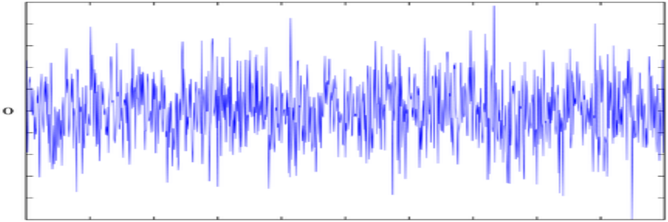
$$\min_u \left(\int_\Omega \left| u'(x) \right| dx + \qquad ? \qquad \right)$$



Denoising (1-dimension)

Let u be a one-dimension signal. Our goal is to find u such that

$$\min_u \left(\int_\Omega |u'(x)|\, dx + \text{ (some data fidelity term) } \right)$$



ROF model

ROF model (Physica D, 1992)

Let $f : \bar{\Omega} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a given noisy image. Rudin, Osher, and Fatemi proposed the model for image denoising:

$$\min_{u \in BV(\Omega)} \Bigg(\underbrace{\|u\|_{TV(\Omega)}}_{\text{regularizer}} + \frac{\lambda}{2} \underbrace{\int_\Omega (u-f)^2 d\boldsymbol{x}}_{\text{data fidelity}} \Bigg),$$

where $\lambda > 0$ is a tuning parameter which controls the regularization strengt.

Remark

1. A smaller value of λ will lead to a more regular solution.
2. The space of functions with bounded variation help remove spurious oscillations (noise) and preserve sharp signals (edges).
3. The *TV* term allows the solution to have discontinuities.

ROF model

ROF model (Physica D, 1992)

Let $f : \bar{\Omega} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a given noisy image. Rudin, Osher, and Fatemi proposed the model for image denoising:

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where $\lambda > 0$ is a tuning parameter which controls the regularization strengt.

Remark

The solution of this minimization problem is existence and uniqueness. (Use triangle inequality)

Discretization of the ROF model

- **ROF Model:**

$$\min_{u \in BV(\Omega)} \left(\|u\|_{TV(\Omega)} + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 d\boldsymbol{x} \right)$$

- **Discretization:**

$$\min_{u} \left(\sum_{i,j} \left| (\nabla u)_{i,j} \right| + \frac{\lambda}{2} \sum_{i,j} \left(u_{i,j} - f_{i,j} \right)^2 \right)$$

- **Constraint:**

$$\min_{d,u} \left(\sum_{i,j} \left| d_{i,j} \right| + \frac{\lambda}{2} \sum_{i,j} \left(u_{i,j} - f_{i,j} \right)^2 \right)$$

subject to $d_{i,j} = \nabla u_{i,j}$

Discretization of the ROF model

- **Constraint:**

$$\min_{d,u}\left(\sum_{i,j}|d_{i,j}|+\frac{\lambda}{2}\sum_{i,j}(u_{i,j}-f_{i,j})^2\right)$$

subject to $d_{i,j}=\nabla u_{i,j}$

- **Bregman iteration:**

$$\min_{d,u}\left(\sum_{i,j}|d_{i,j}|+\frac{\lambda}{2}\sum_{i,j}(u_{i,j}-f_{i,j})^2+\frac{\gamma}{2}\sum_{i,j}|d_{i,j}-\nabla u_{i,j}-b_{i,j}|^2\right)$$

Split Bregman algorithm

Bregman iteration

$$\min_{d,u}\left(\sum_{i,j}|d_{i,j}|+\frac{\lambda}{2}\sum_{i,j}(u_{i,j}-f_{i,j})^2+\frac{\gamma}{2}\sum_{i,j}|d_{i,j}-\nabla u_{i,j}-b_{i,j}|^2\right)$$

- **u-subproblem:**

  With d fixed, we solve

$$u^{(k+1)}=\arg\min_{u}\left(\frac{\lambda}{2}\sum_{i,j}(u_{i,j}-f_{i,j})^2+\frac{\gamma}{2}\sum_{i,j}\left|d_{i,j}^{(k)}-\nabla u_{i,j}-b_{i,j}^{(k)}\right|^2\right).$$

  Then consider the minimization problem

$$\min_{u}\int_{\Omega}\left(\frac{\lambda}{2}(u-f)^2+\frac{\gamma}{2}|d-\nabla u-b|^2\right)d\boldsymbol{x}.$$

  By Euler-Lagrange equation, we have

$$\lambda(u-f)-\gamma\left[\nabla\cdot(\nabla u-d+b)\right]=0,$$

- **u-subproblem (continue):**

or equivalently,

$$\lambda u - \gamma \Delta u = \lambda f - \gamma \nabla \cdot (d - b).$$

Notice that:

$$\Delta u_{i,j} = u_{i-1,j} + u_{i,j-1} + u_{i,j+1} + u_{i+1,j} - 4u_{i,j}.$$

So, we have

$$(\lambda + 4\gamma) u_{i,j} = c_{i,j} + \gamma \left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} \right),$$

where $c_{i,j} = (\lambda f - \gamma \nabla \cdot (d - b))_{i,j}$.

- **u-subproblem (continue):**

$$(\lambda + 4\gamma)u_{i,j} = c_{i,j} + \gamma\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right),$$

which is a symmetric and strictly diagonally dominant linear system, by the Jacobi iterative method:

$$u_{i,j}^{(k+1)} = \left[c_{i,j}^{(k)} + \gamma\left(u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\right)\right] / (\lambda + 4\gamma).$$

Split Bregman algorithm

Bregman iteration

$$\min_{d,u}\left(\sum_{i,j}|d_{i,j}|+\frac{\lambda}{2}\sum_{i,j}(u_{i,j}-f_{i,j})^2+\frac{\gamma}{2}\sum_{i,j}|d_{i,j}-\nabla u_{i,j}-b_{i,j}|^2\right)$$

- **d-subproblem:**

  With u fixed, we solve

$$d^{(k+1)}=\arg\min_{d}\left(\sum_{i,j}|d_{i,j}|+\frac{\gamma}{2}\sum_{i,j}\left|d_{i,j}-\nabla u_{i,j}^{(k+1)}-b_{i,j}^{(k)}\right|^2\right).$$

Soft Thresholding

Let $f(x)=\tau|x|+\frac{\rho}{2}(x-y)^2$. The optimal solution of $f(x)$ is

$$\arg\min_{x} f(x)=\frac{y}{|y|}\max\left\{|y|-\tau/\rho,0\right\}=:\mathcal{S}_{\tau/\rho}(y).$$

Split Bregman algorithm

- **d-subproblem (continue):**
  The soft thresholding give the closed form of d-subproblem. So we have

$$d_{i,j}^{(k+1)} = \mathcal{S}_{1/\gamma}\left(\nabla u_{i,j}^{(k+1)} + b_{i,j}^{(k)}\right).$$

- **Updating b:** $b_{i,j}^{(k+1)} = (b^{(k)} + \nabla u^{(k+1)} - d^{(k+1)})_{i,j}$.

Algorithm Split Bregman

Initialize $u = f, b = 0, d = 0$
while $\frac{\|u - u_{prev}\|}{\|u_{prev}\|}$ >tolerance **do**
  for $n = 1$ to maxstep **do**
    Solve the ***u*-subproblem**
    Solve the ***d*-subproblem**
    $b \leftarrow b + \nabla u - d$
  end for
end while
return u

Numerical experiments

ROF Model

$$\min_{u \in BV(\Omega)} \left(\|u\|_{TV(\Omega)} + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 dx \right)$$

Some Indices

Let $\tilde{u}$ be the clean image, $\bar{u}$ be the mean intensity of the clean image, and u be the produced image.

1. **Mean square error:** $\text{MSE} = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} (\tilde{u}_{i,j} - u_{i,j})^2$
2. **Peak signal to noise ratio:** $\text{PSNR} = 10 \log \left(\frac{(\max u)^2}{\text{MSE}(\tilde{u}, u)} \right)$
3. **Signal to noise ratio:** $\text{SNR} = 10 \log \left(\frac{\text{MSE}(\tilde{u}, \bar{u})}{\text{MSE}(\tilde{u}, u)} \right)$

Numerical experiments: color image

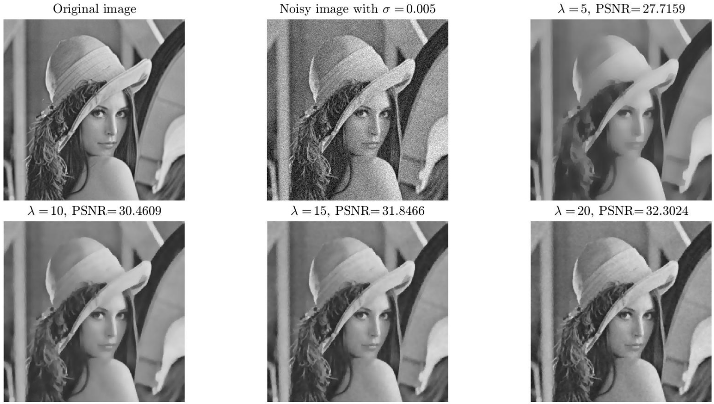


Figure: Lenna

Numerical experiments: color image

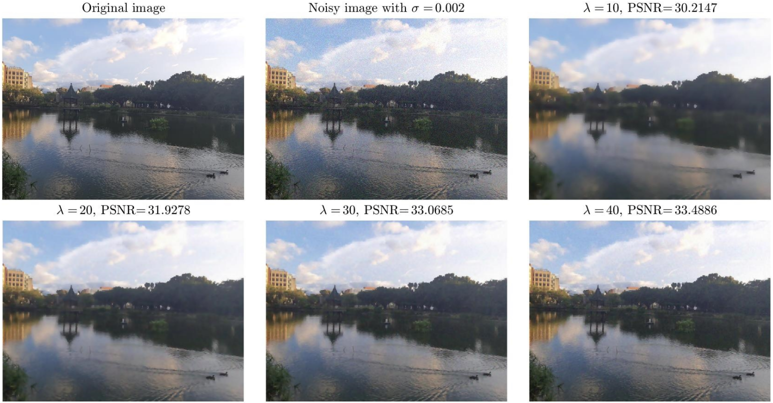


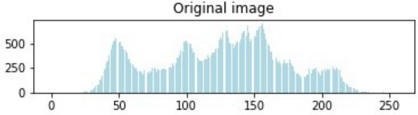
Figure: Drunken moon lake at NTU

Introduction to image contrast enhancement

$\xrightarrow{\text{contrast enhancement}}$

Histogram equalization (HE)

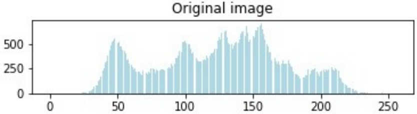
Origin



Histogram

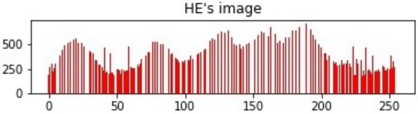
Histogram equalization (HE)

Origin image



Origin histogram

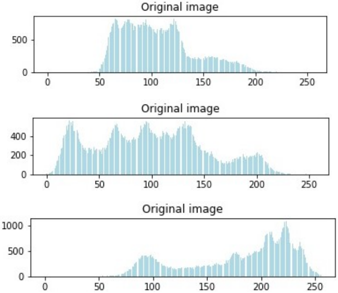
HE's image



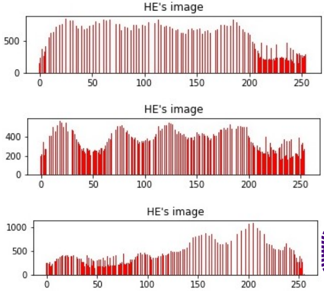
HE's histogram

Histogram equalization (HE)

Origin image



HE's image



Contrast enhancement

Morel-Petro-Sbert model (IPOL 2014)

Let $f : \overline{\Omega} \to \mathbb{R}$ be a given grayscale image. The Morel-Petro-Sbert oposed the model for image contrast enhancementis:

$$\min_u \Bigg(\underbrace{\frac{1}{2} \int_\Omega |\nabla u - \nabla f|^2 \, d\boldsymbol{x}}_{\text{data fidelity}} + \underbrace{\frac{\lambda}{2} \int_\Omega (u - \overline{u})^2 \, d\boldsymbol{x}}_{\text{regularizer}} \Bigg),$$

where $\overline{u} = \frac{1}{|\Omega|} \int_\Omega u \, dx$ is the mean value of u over Ω and $\lambda > 0$ balances between detail preservation and variance reduction.

Remark

The data fidelity term preserves image details presented in f and the regularizer reduces the variance of u to eliminate the effect of nonuniform illumination.

Contrast enhancement

The original model is simple but difficult to solve due to the $\overline{u}$ term. So, we assuming that $\overline{u} \approx \overline{f}$.

Petro-Sbert-Morel model (MAA 2014)

Petro-Sbert-Morel further improved their model by using the L^1 norm to obtain sharper edges:

$$\min_u \left(\int_\Omega |\nabla u - \nabla f| \, d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega (u - \overline{f})^2 \, d\boldsymbol{x} \right).$$

Remark

Requiring the desired image u being close to a pixel-independent constant $\overline{f}$ highly contradicts the requirement of ∇u being close to ∇f and restrains the parameter λ to be very small.

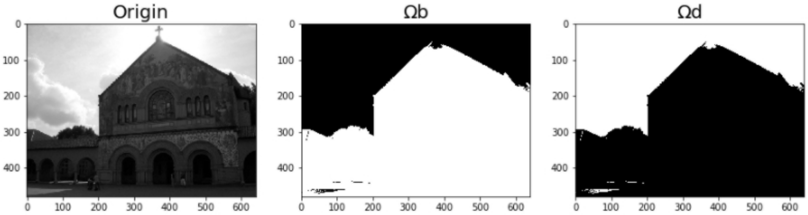
Contrast enhancement

First, We define

$$\Omega_d = \{\boldsymbol{x} \in \bar{\Omega} : f(\boldsymbol{x}) \leq \bar{f}\}, \text{ and } \Omega_b = \{\boldsymbol{x} \in \bar{\Omega} : f(\boldsymbol{x}) > \bar{f}\}$$

as the dark part and the bright part of the image Ω.
Second, define the adaptive functions

$$g(\boldsymbol{x}) = \begin{cases} \alpha \bar{f}, & \boldsymbol{x} \in \Omega_d \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b \end{cases}, \quad h(\boldsymbol{x}) = \begin{cases} \beta f(x), & \boldsymbol{x} \in \Omega_d \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b \end{cases}.$$



Contrast enhancement

Hsieh-Shao-Yang model (SIIMS 2020)

Hsieh-Shao-Yang proposed two adaptive functions g and h to replace $\overline{f}$ and the original input image f

$$\min_u \left(\int_\Omega |\nabla u - \nabla h| d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega (u-g)^2 d\boldsymbol{x} + \chi_{[0,255]}(u) \right),$$

where g and h are devised respectively as

$$g(\boldsymbol{x}) = \begin{cases} \alpha \bar{f}, & \boldsymbol{x} \in \Omega_d \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b \end{cases}, \quad h(\boldsymbol{x}) = \begin{cases} \beta f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_d \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b \end{cases},$$

with $\alpha > 0$ and $\beta > 1$ and the characteristic function is defined as

$$\chi_{[0,255]}(u) = \begin{cases} 0, & \text{range}(u) \subseteq [0,255] \\ \infty, & \text{otherwise} \end{cases}.$$

Discretization of model

- **Model:**

$$\min_u \left(\int_\Omega |\nabla u - \nabla h| d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega (u-g)^2 d\boldsymbol{x} + \chi_{[0,255]}(u) \right)$$

- **Discretization:**

$$\min_u \sum_{i,j} \left(\left| (\nabla u)_{i,j} - (\nabla h)_{i,j} \right| + \frac{\lambda}{2} \left(u_{i,j} - g_{i,j} \right)^2 \right) + \chi_{[0,255]}(u)$$

- **Constraint:**

$$\min_u \sum_{i,j} \left(\left| d_{i,j} \right| + \frac{\lambda}{2} \left(u_{i,j} - g_{i,j} \right)^2 \right) + \chi_{[0,255]}(v)$$

subject to $d = \nabla u - \nabla h$ and $v = u$

Discretization of model

- **Constraint:**

$$\min_{u} \sum_{i,j} \left(|d_{i,j}| + \frac{\lambda}{2} \left(u_{i,j} - g_{i,j}\right)^2 \right) + \chi_{[0,255]}(v)$$

subject to $d = \nabla u - \nabla h$ and $v = u$

- **Bregman iteration:**

$$\min_{u,d,v} \sum_{i,j} \left(|d_{i,j}| + \frac{\lambda}{2}(u_{i,j} - g_{i,j})^2 + \frac{\gamma}{2}|d_{i,j} - (\nabla u)_{i,j} + (\nabla h)_{i,j} - b_{i,j}|^2 \right.$$
$$\left. + \frac{\delta}{2} \left(v_{i,j} - u_{i,j} - c_{i,j}\right)^2 \right) + \chi_{[0,255]}(v)$$

Bregman iteration

$$\min_{u,d,v} \sum_{i,j} \Big(|d_{i,j}| + \frac{\lambda}{2}\left(u_{i,j} - g_{i,j}\right)^2 + \frac{\gamma}{2}\left|d_{i,j} - (\nabla u)_{i,j} + (\nabla h)_{i,j} - b_{i,j}\right|^2 + \frac{\delta}{2}\left(v_{i,j} - u_{i,j} - c_{i,j}\right)^2 \Big) + \chi_{[0,255]}(v)$$

- **u-subproblem:**

  With d and v fixed, we solve

$$u^{(k+1)} = \arg\min_u \sum_{i,j} \Big(\frac{\lambda}{2}\left(u_{i,j} - g_{i,j}\right)^2 + \frac{\gamma}{2}\left|d^{(k)}_{i,j} - (\nabla u)_{i,j} + (\nabla h)_{i,j} - b^{(k)}_{i,j}\right|^2 + \frac{\delta}{2}\left(v^{(k)}_{i,j} - u_{i,j} - c^{(k)}_{i,j}\right)^2 \Big).$$

- **u-subproblem (continue):**

  Consider the minimization problem

$$\min_u \int_\Omega \left(\frac{\lambda}{2}(u-g)^2 + \frac{\gamma}{2}|d - \nabla u + \nabla h - b|^2 + \frac{\delta}{2}(v-u-c)^2 \right) d\boldsymbol{x}.$$

  By Euler-Lagrange equation, we have

$$\lambda(u-g) - \delta(v-u-c) - \gamma\left[\nabla \cdot (\nabla u - d - \nabla h + b)\right] = 0,$$

  or equivalently,

$$(\lambda+\delta)u - \gamma\Delta u = \lambda g - \gamma\nabla\cdot(d + \nabla h - b) + \delta(v-c).$$

- **u-subproblem (continue):**

  Since $\Delta u_{i,j} = u_{i,j-1} + u_{i,j+1} + u_{i-1,j} + u_{i+1,j} - 4u_{i,j}$, we have

  $$(\lambda + \delta + 4\gamma)u_{i,j} = c_{i,j} + \gamma\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right),$$

  where $c_{i,j} = (\lambda g - \gamma \nabla \cdot (d + \nabla h - b) + \delta(v - c))_{i,j}$.

  which is a symmetric and strictly diagonally dominant linear system, by the Jacobi iterative method:

  $$u_{i,j}^{(k+1)} = \left[c_{i,j}^{(k)} + \gamma\left(u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\right)\right] / (\lambda + \delta + 4\gamma).$$

Bregman iteration

$$\min_{u,d,v} \sum_{i,j} \left(|d_{i,j}| + \frac{\lambda}{2} \left(u_{i,j} - g_{i,j}\right)^2 + \frac{\gamma}{2} \left|d_{i,j} - (\nabla u)_{i,j} + (\nabla h)_{i,j} - b_{i,j}\right|^2 + \frac{\delta}{2} \left(v_{i,j} - u_{i,j} - c_{i,j}\right)^2 \right) + \chi_{[0,255]}(v)$$

- **d-subproblem:**

  With u and v fixed, we solve

  $$d^{(k+1)} = \arg\min_{d} \left(\sum_{i,j} |d_{i,j}| + \frac{\gamma}{2} \sum_{i,j} \left| d_{i,j} - \nabla u_{i,j}^{(k+1)} + (\nabla h)_{i,j} - b_{i,j}^{(k)} \right|^2 \right).$$

  By soft-thresholding, we have

  $$d_{i,j}^{(k+1)} = \mathcal{S}_{1/\gamma} \left(\nabla u_{i,j}^{(k+1)} - (\nabla h)_{i,j} + b_{i,j}^{(k)} \right).$$

Bregman iteration

$$\min_{u,d,v} \sum_{i,j} \left(|d_{i,j}| + \frac{\lambda}{2} \left(u_{i,j} - g_{i,j}\right)^2 + \frac{\gamma}{2} \left|d_{i,j} - (\nabla u)_{i,j} + (\nabla h)_{i,j} - b_{i,j}\right|^2 + \frac{\delta}{2} \left(v_{i,j} - u_{i,j} - c_{i,j}\right)^2 \right) + \chi_{[0,255]}(v)$$

- **v-subproblem:**

With u and d fixed, we solve

$$v^{(k+1)} = \arg\min_{v} \left(\sum_{i,j} \frac{\delta}{2} \left(v_{i,j} - u_{i,j}^{(k+1)} - c_{i,j}^{(k)} \right)^2 \right) + \chi_{[0,255]}(v).$$

Then we can be solved by pixelwise:

$$v_{i,j} = \min\{\max\{u_{i,j} + c_{i,j}, 0\}, 255\}.$$

- **Updating b:** $b_{i,j}^{(k+1)} = (b^{(k)} + \nabla u^{(k+1)} - \nabla h - d^{(k+1)})_{i,j}.$

- **Updating c:** $c_{i,j}^{(k+1)} = (c^{(k)} + u^{(k+1)} - v^{(k+1)})_{i,j}.$

Split Bregman algorithm

Algorithm Split Bregman

Initialize $u = h, v = h, d = 0, b = 0, c = 0$
while $\frac{\|u - u_{prev}\|_2}{\|u_{prev}\|_2} >$tolerance **do**
  for $n = 1$ to maxstep **do**
    Solve the $\boldsymbol{u}$-**subproblem**
    Solve the $\boldsymbol{d}$-**subproblem**
    Solve the $\boldsymbol{v}$-**subproblem**
    $b \leftarrow b + \nabla u - \nabla h - d$
    $c \leftarrow c + u - v$
  end for
end while
return u

Numerical experiments: grayscale image

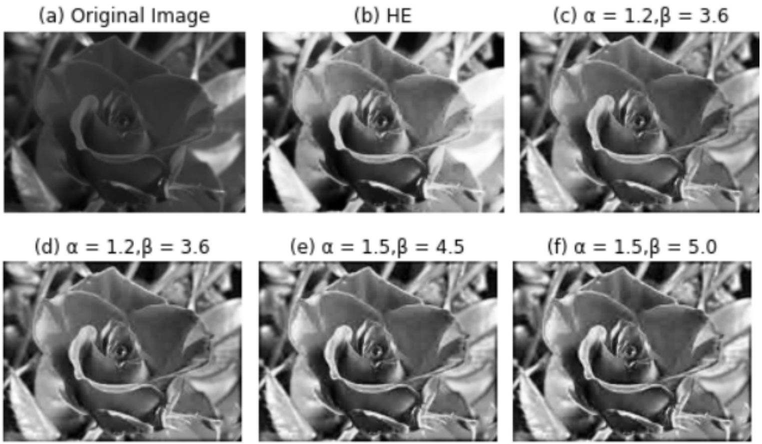


Figure: Rose ($\lambda = 0.0005$)

Color RGB images

- The domain division for color RGB images denoted by (f_R, f_G, f_B) is conducted as follows. First, we define the maximum image as

$$f_{\max}(\boldsymbol{x}) = \max\{f_R(\boldsymbol{x}), f_G(\boldsymbol{x}), f_B(\boldsymbol{x})\}, \ \forall \boldsymbol{x} \in \overline{\Omega}.$$

- For example,

65	27	100
22	31	47
112	54	78

58	21	10
145	213	48
132	2	9

15	122	200
189	32	45
12	52	79

$\downarrow$

65	122	200
189	213	48
132	54	79

Color RGB images

Let $\overline{f}_{\max} = \frac{1}{|\Omega|}\int_\Omega f_{\max}\,d\boldsymbol{x}$. Then we divide the image domain Ω into two parts

$$\Omega_d = \{\boldsymbol{x} \in \overline{\Omega} : f_{\max}(\boldsymbol{x}) \leq \overline{f}_{\max}\},$$
$$\Omega_b = \{\boldsymbol{x} \in \overline{\Omega} : f_{\max}(\boldsymbol{x}) > \overline{f}_{\max}\}.$$

Numerical experiments: color image

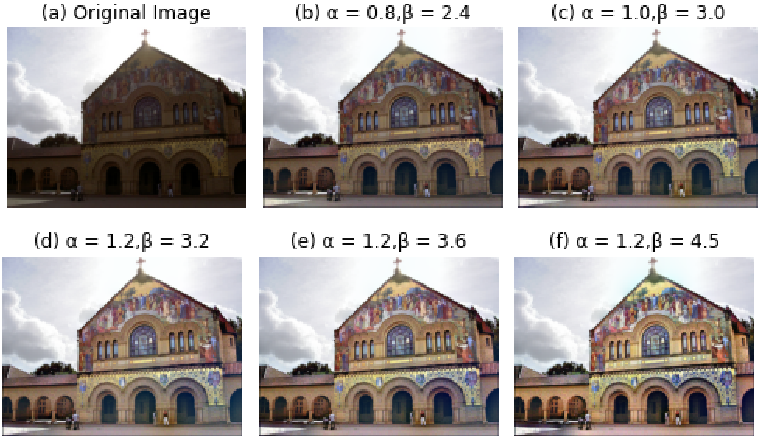
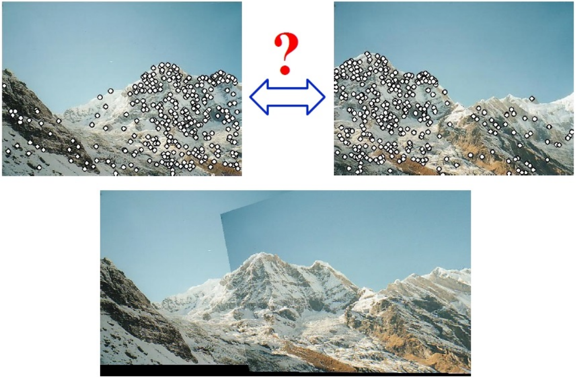


Figure: House ($\lambda = 0.0005$)

Introduction to image stitching

1. Image alignment
   - Scale-invariant feature transform (SIFT): find features
   - Homography: match features
2. Image blending: linear blending



Scale-invariant feature transform (SIFT)

David Lowe first proposed the SIFT algorithm. SIFT's main purpose is to extract feature points from an image, which can also be said to be local feature blocks of the image and have the scale, rotation, luminosity invariant, and stability to viewing angle changes and noise.

SIFT steps

The SIFT algorithm can be divided into the following four major steps.

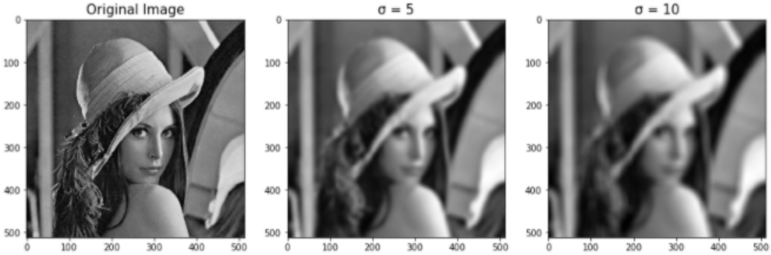
1. detect the scale space extreme value
2. key point positioning
3. determine the direction
4. describe the key points

SIFT: Gaussian blur

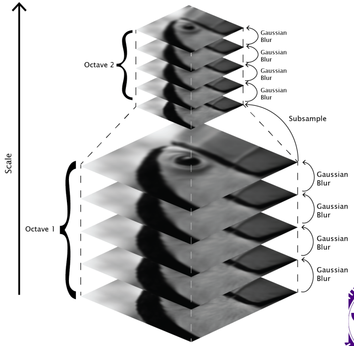
Let $L(x,y,\sigma)$ be the convolution of a Gaussian function $G(x,y,\sigma)$ with different scale and the original image $I(x,y)$. That is,

$$L(x,y,\sigma) = G(x,y,\sigma) * I(x,y),$$

where $G(x,y,\sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\frac{m}{2})^2+(y-\frac{n}{2})^2}{2\sigma^2}}$, $m = n = 6\sigma + 1$, (x,y) is the pixel position, and σ is the scale factor.

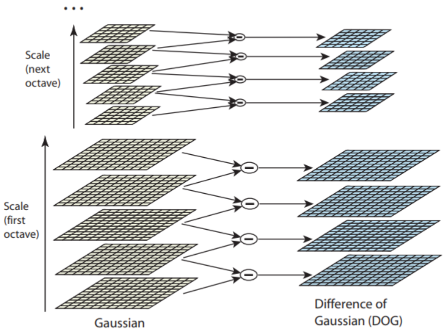


We use the Gaussian pyramid to realize the scale-space, which requires two steps: first, do Gaussian blur of different scales on the image. Second, down-sampling the picture.



SIFT: difference of Gaussian pyramid (DOG)

To find the image's feature points, we first subtracted the images of the same octave of Gaussian pyramids to obtain another pyramid, which we call the differential of Gaussian pyramid.

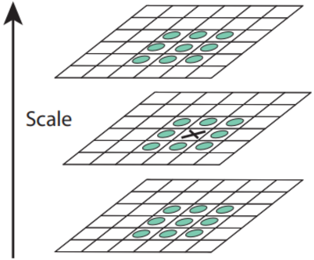


DOG can also be presented in the form

$$D(x,y,\sigma) = L(x,y,k\sigma) - L(x,y,\sigma).$$

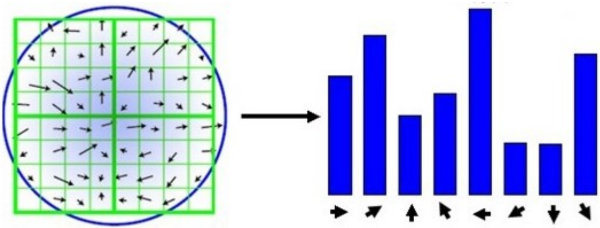
SIFT: find extrema

When we look for extreme points in the difference pyramid, we will use the information at different intervals in the same octave. In the other words, the upper and lower layers will also be considered, that is, 26 points surrounding the candidate, as shown in the figure below:

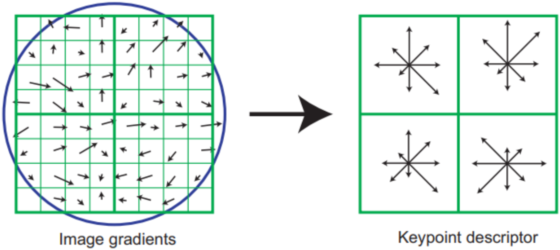


When facing two images in different directions, we can find the same key point and keep them in the same direction. Define the direction and magnitude of a pixel (x,y).

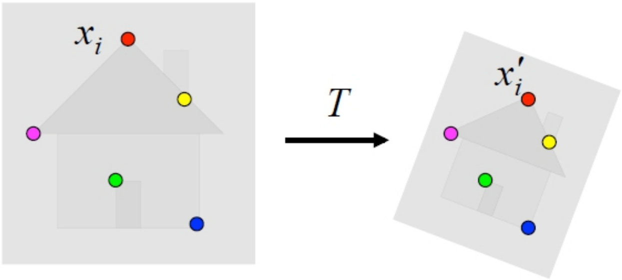
$$m(x,y)=\sqrt{(L(x+1,y)-L(x-1,y))^2+(L(x,y+1)-L(x,y-1))^2},$$
$$\theta(x,y)=\tan^{-1}\left(\frac{L(x,y+1)-L(x,y-1)}{L(x+1,y)-L(x-1,y)}\right).$$



The neighborhood of the key point is divided into 16 sub-regions. Each sub-region is a seed point. Let each sub-point have eight directions (and their magnitudes). Simply calculating, a key point will be described by a 128 (16×8)-dimensional vector. The eight directions are obtained by weighting the information around each seed point.



Alignment as fitting



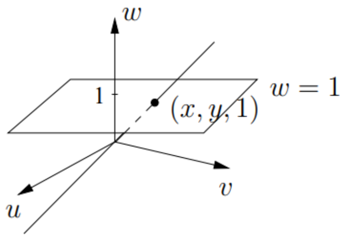
Homogeneous coordinates

- Converting to homogeneous image coordinates:

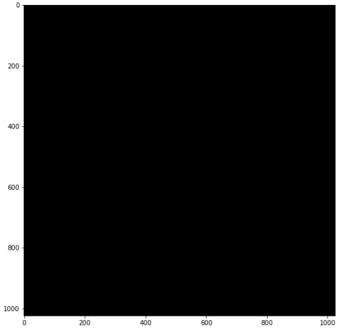
$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Converting from homogeneous image coordinates:

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix}, \text{ where } w \neq 0 \longrightarrow \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \end{bmatrix}$$



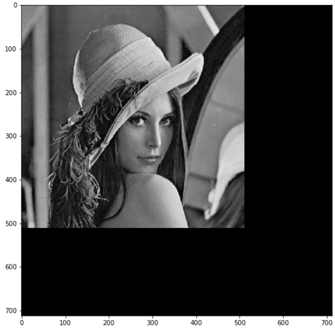
Transformation: scale



$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix},$$

where $a, b > 0$.

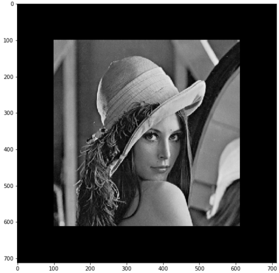
Transformation: translation



$$\begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix},$$

where $e, f \in \mathbb{R}$.

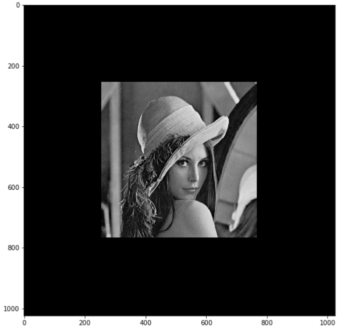
Transformation: rotation



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix},$$

where $\theta \in (0, 2\pi)$.

Transformation: shear



$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

or

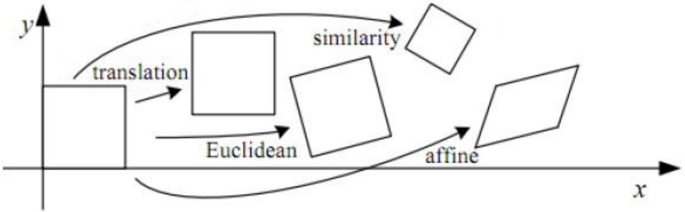
$$\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix},$$

where $a, b > 0$.

Affine transformation

Remark

Scale + Translation + Rotation + Shear = Affine transform



Affine transformation

Remark

Scale + Translation + Rotation + Shear = Affine transform

Affine transformation

A 2D affine transformation is composed of a linear transformation by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and a translation by a vector $\begin{bmatrix} e \\ f \end{bmatrix} \in \mathbb{R}^2$, given as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

Affine transformation

Homogeneous expression of 2D affine transformation

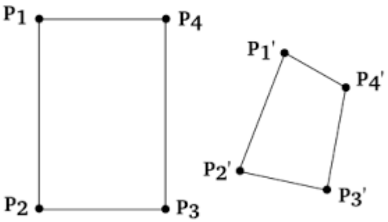
The homogeneous expression of the affine transformation is given as

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

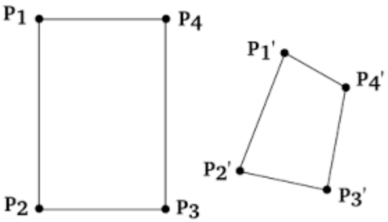
Remark

The affine transformation has 6 degree of freedom.

Homography



Homography



Homogeneous expression of homography

The homogeneous expression of the homography is given as

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

Homogeneous expression of homography

The homogeneous expression of the homography is given as

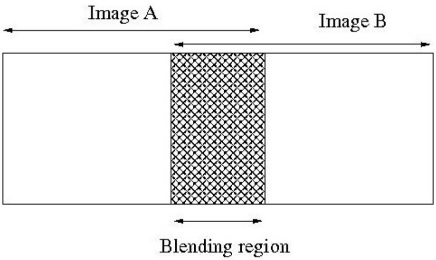
$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

Remark

The homogeneous expression of the homography has 8 degree of freedom (9 parameters, but scale is arbitrary).

Why do we need to do image blending?

Image blending



$$\text{Pano}(i,j) = (1 - w(i,j)) * A(i,j) + w(i,j) * B(i,j),$$

where $\text{Pano}(i,j)$ is the i,j^{th} pixel of the panoramic image, $A(i,j)$ is the i,j^{th} pixel of Image A, $B(i,j)$ is the i,j^{th} pixel in the panoramic image corresponds to which in Image B after transformed by homography, and the closer the weight w is to Image B, the larger w is, $0 \leq w \leq 1$.

Image blending

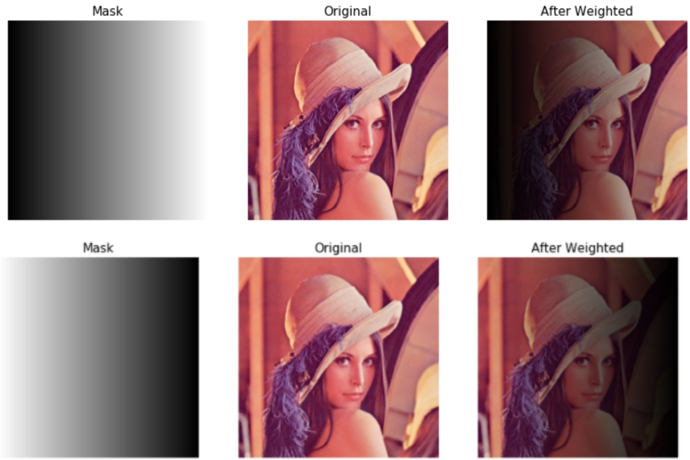


Figure: Linear blending

Experiments: draw matches

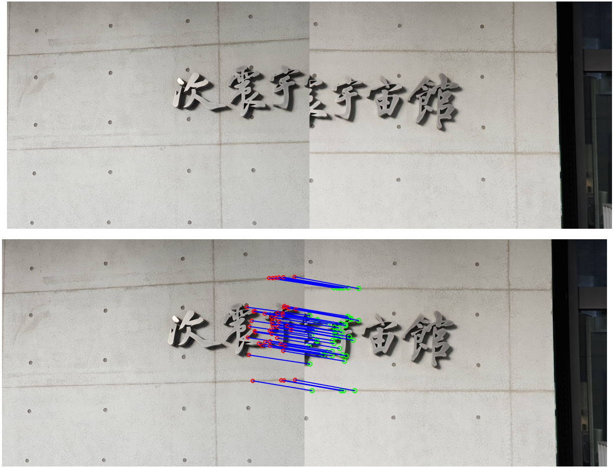


Figure: Draw matches

Experiments: homography

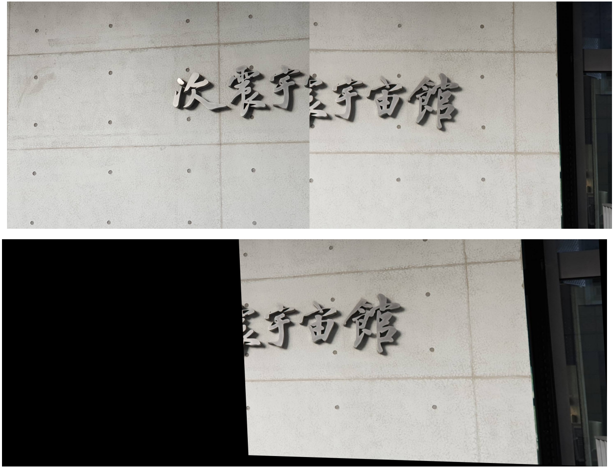


Figure: Warp perspective

Experiments: blending

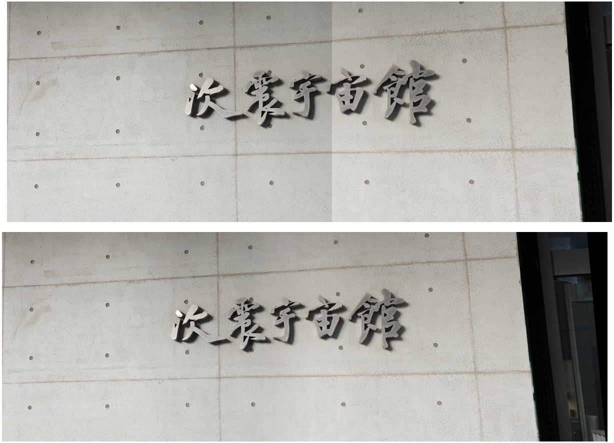


Figure: No blending versus linear blending

Experiments: panorama

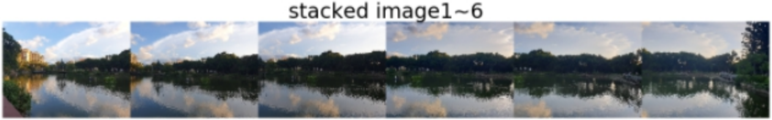


Figure: Drunken moon lake at NTU

Experiments: contrast enhancement

Figure: Drunken moon lake at NTU

Experiments: contrast enhancement

Figure: Street tree at NTU

Conclusion

In this research program, we have learned some topics for mathematical image processing, including the bounded variation function space BV, the basics of calculus of variation and the Euler-Lagrange equation, the Rudin-Osher-Fatemi model for image denoising, an adaptive contrast enhancement model, some knowledge for image alignment, and the split Bregman iterative scheme for solving the associated minimization problems. We have also studied the famous feature detection algorithm, called the scale-invariant feature transform (SIFT). Moreover, we have successfully realized some image stitching problems using MATLAB.

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Thanks for listening!