This Notebook uses Exponential Generating Functions (EGFs) to compute the table in Figure 6 of "Invariant Synchrony and Anti-Synchrony Subspaces of Weighted Networks" by Nijholt, Sieben and Swift.

The first part of a sequence, for example $(e_n) = (e_0, e_1, e_2, \ldots)$, is represented as a list with the name of the sequence. Thus, $(e_0, e_1, e_2, \ldots, e_n) \longleftrightarrow$ e = {e $_0, e_1, e_2, \ldots, e_n$}. Note that in Mathematica the first element of the list is e[[1]] = e_0, and the last element of the list is e[[n+1]] = e_n.

The exponential generating function of the constant sequence {1,1,1, ..} is Exp[x]. The Range[0,10]! multiplies the coefficient c_n of the Taylor Series by n!

```
In[•]:= nMax = 20; (* A global variable used in
 all the cells.  So this cell must be run first *)

In[•]:= CoefficientList[Series[Exp[x], {x, 0, nMax}], x] * Range[0, nMax] !

Out[•]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

The number of polydiagonal subspaces of R^n is p_n (the Dowling numbers)

```
In[•]:= (*  The sequence (p_n) is obtained by Proposition 7.5, using the Exponential
 Generating Function (EGF) of the Dowling numbers found at the OEIS *)
Print["The sequence (pn) of the number of polydiagonal subspaces of R^n is ", p =
   CoefficientList[Series[Exp[x + (Exp[2 * x] - 1) / 2], {x, 0, nMax}], x] * Range[0, nMax] ! ];

The sequence (pn) of the number of polydiagonal subspaces of R^n is
 {1, 2, 6, 24, 116, 648, 4088, 28 640, 219 920, 1 832 224, 16 430 176, 157 554 048,
  1 606 879 040, 17 350 255 744, 197 553 645 440, 2 363 935 624 704, 29 638 547 505 408,
  388 328 781 668 864, 5 304 452 565 517 824, 75 381 218 537 805 824, 1 112 348 880 749 130 752}
```

The number of synchrony subspaces of R^n is s_n. (the Bell numbers.)

```
In[•]:= (*  The sequence (s_n) is obtained by Proposition 7.13,
using the Exponential Generating Function (EGF) *)
Print["The sequence (sn) of the number of polydiagonal subspaces of R^n is ",
  s = CoefficientList[Series[Exp[Exp[x] - 1], {x, 0, nMax}], x] * Range[0, nMax] ! ];

The sequence (sn) of the number of polydiagonal subspaces of R^n is
 {1, 1, 2, 5, 15, 52, 203, 877, 4140, 21 147, 115 975, 678 570, 4 213 597, 27 644 437, 190 899 322,
  1 382 958 545, 10 480 142 147, 82 864 869 804, 682 076 806 159, 5 832 742 205 057, 51 724 158 235 372}
```

The number of anti-synchrony subspaces of R^n is $a_n = p_n - s_n$. (This does not appear to be in the OEIS as of May 12, 2022). We can subtract either the sequences or the EGFs.

```
In[•]:= (*  The sequence (a_n) is obtained by subtracting the EFGs of (pn) and (sn) *)
Print["The sequence (an) of the number of anti-synchrony subspaces of R^n is ",
  a = CoefficientList[Series[Exp[x + (Exp[2 * x] - 1) / 2] -
       Exp[Exp[x] - 1], {x, 0, nMax}], x] * Range[0, nMax] ! ];

The sequence (an) of the number of anti-synchrony subspaces of R^n is
 {0, 1, 4, 19, 101, 596, 3885, 27 763, 215 780, 1 811 077, 16 314 201, 156 875 478,
  1 602 665 443, 17 322 611 307, 197 362 746 118, 2 362 552 666 159, 29 628 067 363 261,
  388 245 916 799 060, 5 303 770 488 711 665, 75 375 385 795 600 767, 1 112 297 156 590 895 380}
```

In[∘]:=
```
a == p - s
```
Out[∘]=
```
True
```

The number of minimally tagged anti-synchrony subspaces of R^n is $m_n = s_{n+1} - s_n$. The sequence for s_{n+1} is kluged by making a longer list and dropping the first element.

In[∘]:=
```
Print["The sequence (m_n) of the number of
    minimally tagged anti-synchrony subspaces of R^n is ", m =
   Drop[CoefficientList[Series[Exp[Exp[x] - 1], {x, 0, nMax + 1}], x] * Range[0, nMax + 1] !,
     1] - CoefficientList[Series[Exp[Exp[x] - 1], {x, 0, nMax}], x] * Range[0, nMax] ! ];
```

```
The sequence (m_n) of the number of minimally tagged anti-synchrony subspaces of R^n is
 {0, 1, 3, 10, 37, 151, 674, 3263, 17007, 94828, 562595, 3535027, 23430840, 163254885, 1192059223,
  9097183602, 72384727657, 599211936355, 5150665398898, 45891416030315, 423145657921379}
```

The number of fully tagged polydiagonal subspaces of R^n is f_n. The sequence for s_{n+1} is kluged by making a longer list and dropping the first element.

In[∘]:=
```
(*  The sequence (f_n) is obtained by the EFG of Proposition 7.18 *)
Print["The sequence (f_n) of the number of anti-synchrony subspaces of R^n is ",
  f = CoefficientList[Series[Exp[(Exp[2 * x] - 2 Exp[x] + 2 x + 1) / 2], {x, 0, nMax}], x] *
    Range[0, nMax] ! ];
```

```
The sequence (f_n) of the number of anti-synchrony subspaces of R^n is
 {1, 1, 2, 7, 29, 136, 737, 4537, 30914, 229831, 1850717, 16036912,
  148573889, 1463520241, 15259826402, 167789512807, 1939125333629,
  23484982837816, 297289975208417, 3924325664733097, 53906145745657634}
```

The number of evenly tagged subspaces of R^n is e_n. (This does not appear to be in the OEIS as of May 12, 2022.)

In[∘]:=
```
(*  The sequence (e_n) is obtained by the EGF of Proposition 7.15 *)Print[
  "The sequence (e_n) of the number of evenly tagged polydiagonal subspaces of R^n is ",
  e = CoefficientList[Series[Exp[BesselI[0, 2 x] / 2 - 1 / 2 + x], {x, 0, nMax}], x] *
    Range[0, nMax] ! ];
```

```
The sequence (e_n) of the number of evenly tagged polydiagonal subspaces of R^n is
 {1, 1, 2, 4, 13, 41, 176, 722, 3774, 18958, 116302, 687182, 4812226, 32541874, 255274826,
  1938568634, 16798483589, 141220228149, 1337121257864, 12305678519102, 126208299343263}
```

In[∘]:=
```
tableEGF = {Range[0, nMax], p, s, a, m, f, e};
TableForm[tableEGF, TableHeadings → {{"n", "p_n", "s_n", "a_n", "m_n", "f_n", "e_n"}, None}]
```

Out[∘]//TableForm=

n	0	1	2	3	4	5	6	7	8	9	10
p_n	1	2	6	24	116	648	4088	28640	219920	1832224	16430176
s_n	1	1	2	5	15	52	203	877	4140	21147	115975
a_n	0	1	4	19	101	596	3885	27763	215780	1811077	16314201
m_n	0	1	3	10	37	151	674	3263	17007	94828	562595
f_n	1	1	2	7	29	136	737	4537	30914	229831	1850717
e_n	1	1	2	4	13	41	176	722	3774	18958	116302

Note that $s_n = e_n$ for n = 0, 1, 2, and $s_n > e_n$ for $3 \le n \le 9$, and $s_n < e_n$ for $n \ge 10$. (We have not proved this last observation.)

That was easy to program, but it hides a computation done by the computer algebra system. See the notebook sequences-recurrence.nb for a more elementary method of computing these sequences.