Exercise Set 6

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1

Show S = span $(v_1, v_2, ..., v_n)$ is a n dimensional subspace of X.
If X is an M-dimensional subspace then $v_1 \in$ X, $v_2 \in$ X, and $v_n \in$ X. So all vectors $v_1 \rightarrow v_n$ are in X. The vector space X can be identified as a linear combination of its basis vectors. One common basis to choose which we can identify all vector is X with is its standard basis. Therefore:

$$X \equiv x_1 \begin{bmatrix} 1 \\ 0 \\ ... \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ ... \\ 0 \end{bmatrix} + ... + x_m \begin{bmatrix} 0 \\ ... \\ 0 \\ 1 \end{bmatrix}$$

All vectors that lie in the vector space X can be rewritten as a particular linear combination of the basis of X. So vector v_1 can be rewritten as:

$$v_1 \equiv C_1 \begin{bmatrix} 1 \\ 0 \\ ... \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \\ ... \\ 0 \end{bmatrix} + ... + C_m \begin{bmatrix} 0 \\ ... \\ 0 \\ 1 \end{bmatrix}$$

Also, two vectors added or scaled in any way could be rewritten in this form as another vector of the vector space, X. Such that:

$$v_1 + v_2 \equiv (C_{1_1} + C_{1_2}) \begin{bmatrix} 1 \\ 0 \\ ... \\ 0 \end{bmatrix} + (C_{2_1} + C_{2_2}) \begin{bmatrix} 0 \\ 1 \\ ... \\ 0 \end{bmatrix} + ... + (C_{m_1} + C_{m_2}) \begin{bmatrix} 0 \\ ... \\ 0 \\ 1 \end{bmatrix}$$

And since $(C_{1_1} + C_{1_2}) \in \mathbb{R}$ we can rewrite this as another constant say C_1. There for any linear combination of vectors in the vector space can be added and scaled to be rewritten as a vector lying in the space using its standard basis. Thus any linear combination of vectors in X produces vectors in X, and since our definition of a vector space is a linear combination of vectors in the space will produce another vector in the space we can see that for any subspace of X such that S = span$(v_1, v_2, ..., v_n)$ will be a subspace of X. Since a span in the linear combination of vectors and the basis of all the vectors in S can be written using the same standard basis as X.

If the dimension of X is exactly equal to the number of vectors in S than we know that S is one possible basis for X as long as no vectors in S are scalar multiples of each other. Also, If the dimension of X is greater than S than we know that vectors in S are not linearly independent and therefore cannot be a basis of X. If the number of vectors in S are less than the dimensions of X than we know S is a subspace of X and is not a basis for X.

2

We may write a vector in X using the basis $v_1,\ v_2,\ v_3,\ ...,\ v_n$ and we call that vector v. If we wish too write v as:

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n$$

But we also wish to write v as:

$$v = b_1v_1 + b_2v_2 + ... + b_nv_n$$

than we are actually writing the same vector. Since $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ and the basis vectors used for both are the same we are essentially writing the same vector twice and all of a_i will equal b_i for every i.

This can be shown by first looking at the vector v. We know that v must equal v and v-v = 0. Using this knowledge we can write out the vector v in both of its forms.

$$\begin{aligned} v &= v \\ a_1v_1 + a_2v_2 + ... + a_nv_n &= b_1v_1 + b_2v_2 + ... + b_nv_n \\ (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + ...(a_n - b_n)v_n &= 0; \\ \therefore a_1 - b_1\ AND\ a_2 - b_2\ AND\ all\ a_n - b_n &= 0 \end{aligned}$$

$$\therefore a_1 = b_1 \ AND \ a_2 = b_2 \ AND \ a_n = b_n$$

And we have shown that $a_i = b_i$ for all i when these vectors are compared in the same basis.

3

If we take P^2 which is the space of all quadratic polynomials we can represent it in standard basis as:

$$P^2 \equiv a_0 + a_1 x + a_2 x^2 \equiv a_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where the standard basis of P^2 is:

$$basis(P^2) = (1, x, x^2) \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}$$

We can choose to write out P^2 in terms of a different basis of our choice. One possible choice would be to pick a non-standard basis such as:

$$basis = (1 - x, x^2 + x, 1 + x + x^2)$$

or even:

$$basis = (1 - x - x^2, x - 1 - x^2, 1 + x^2)$$

These are but two out of an infinite possibility of ways to describe the space made by P^2. If we look to the standard basis where P^2 is identified with vectors of only numbers we can represent the space of P^2 with an infinite number of 3, 3x1 vectors which are linearly independent, since P^2 is isomorphic to the vector space $\mathbb{R}^3$.

4

Given a linear operator L:$\mathbb{R}^2 \to \mathbb{R}^2$ which can be represented in the standard basis as:

$$mat(L) = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

If we wish, we are able to rewrite L in terms of a non-standard basis v_1, v_2 where:

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ AND \ v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We start by writing taking the basis of L and finding a way to rewrite then in terms of the new basis v_1, v_2. If we start with the basis of L to be standard than they are:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \ AND \ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If we write v_1 and v_2 in terms of these basis we will have a way to convert between the two. So:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

If we now expand the v basis out in terms of the standard basis we see that:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now combining like terms we see:

$$1 = C_1 - 2C_2$$

AND

$$0 = C_1 + C_2$$

We can now solve this system of equations to find that: $C_1 = \frac{1}{3}$ and $C_2 = \frac{-1}{3}$. Doing this same process again for the other basis vector we get to this system of equations:

$$0 = C_3 - 2C_2$$

AND

$$1 = C_3 + C_4$$

Which give us solutions for C_3 and C_4 to be: $C_3 = \frac{2}{3}$ and $C_4 = \frac{1}{3}$. From this we need only put together out coefficients in terms of a matrix to see that the linear operator L written in terms of the v bases described above is:

$$mat(L) = \begin{bmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Where now each component of the matrix is relative to the new basis v_1 and v_2

5

The trace of a 2x2 matrix is such that a matrix:

$$mat() = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the Sum of the diagonals a and d: a+d. A traceless matrix is one than where a+d = 0. which could also be defined as a = -d. So a traceless matrix would be any matrix that looked like:

$$mat() = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

If X is the set of all traceless matrices than X is a vector space. We know X is a vector space because we can write it as the span of its basis. X's basis are:

$$basis = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

such that the span of the basis will produce the vector space X:

$$X = span(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})$$

meaning that:

$$X = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Just to check this result with our definition of a vector space we can see that X will have a zero vector on the all zero input. Also, We can write a linear combination of Vectors in X as another vector in X such that:

$$C_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + C_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + C_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + C_5 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + C_6 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} =$$

$$(C_1 + C_4) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (C_2 + C_5) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (C_3 + C_6) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Which since $C_n \in \mathbb{R}$ and combination of C's will be $\in \mathbb{R}$. So this linear combination of vectors in X is:

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$a, b, c \in \mathbb{R}$$

Thus X is a 3 dimensional vector space