The many faces of a linear operator

Recall that the symbol P_n in our linear algebra language stands for polynomials of degree n (in whatever variable). Consider the operator $T : P_2 \mapsto P_2$ defined by

$$T(f) = \frac{d}{dx}(x\, f).$$

In the previous section we discovered that the identification

$$a_0 + a_1\, x + a_2\, x^2 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \tag{1}$$

allows us to represent T with a three-by-three matrix of the form:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \tag{2}$$

What if we change the way we represent quadratics by 3-vectors? For instance, we could write

$$a_0 + a_1\, x + a_2\, x^2 = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}. \tag{3}$$

Notice the difference between Equations (1) and (3). In the first case we sort the powers in ascending order: first component of the vector stores the constant, second—the linear coefficient, third—the quadratic coefficient. In the second case the situation is reversed: we sort the powers in descending order. It is easy to see that identification (3) leads to the matrix representation

$$T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

which is similar to, yet different from Equation (4). Thus the matrix representation of a linear transformation (or operator) is not unique: it depends on how we represent the inputs (and outputs!) as vectors.

As the exercises at the end of this section show, *any* matrix representation of a linear transformation can be used in computations. For this reason, the matrices representing a given transformation are termed *similar*. However, one quickly discovers that some matrix representations are much more convenient than others. In particular, linear transformations become especially simple when their matrices are *diagonal* as in Equations (2) and (4). To see why diagonal representation are nice, let us compute the matrix of T^n using the first convention (1). The result is, simply:

$$T^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}.$$

The matrix of the inverse operator is also very simple, namely:

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

(Verify that $T^{-1}T = TT^{-1} = I$). Now try it with a *dense* matrix, say

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 2 & -7 & 4 \end{bmatrix}.$$

We can still compute A^n and A^{-1} using matrix multiplication. However, the computations become strenuous and, if we increase the dimensions of A, even MATLAB becomes inadequate. This raises a key question:

> Given a linear transformation, can we represent its inputs and outputs vectorially in such a manner that the matrix of the transformation becomes diagonal?

Exercises

1. Consider the following operator acting on P_2:

$$T : f \mapsto f + 2\,\frac{df}{dx} + \frac{d^2 f}{dx^2}.$$

   (a) Find the matrix representation of T using the convention (1) from this section; call the matrix A.

(b) Find the matrix representation of T using the convention (3) from this section; call the matrix B.

(c) Use the matrices A and B to compute $T(x + x^2)$ and show that the result is the same.

(d) Let $\mathcal{P}$ denote the operator which permutes the elements of a 3-vector as follows:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

Find the matrix of $\mathcal{P}$ and call it C.

(e) Two of the four matrices $A\,C$, $C\,A$, $C\,B$, $B\,C$ are the same. Which ones? Support your answer with a verbal explanation.

2. In this exercise we will need Euler's formula for the complex exponential: $e^{i\,\theta} = \cos(\theta) + i\,\sin(\theta)$. (Here $i = \sqrt{-1}$). Let $D = \frac{d}{dt}$.

(a) Consider the operator D acting on inputs of the form:

$$a\,\cos(t) + b\,\sin(t) \equiv \begin{bmatrix} a \\ b \end{bmatrix}.$$

Find the matrix of D and call it A.

(b) Now consider the same operator D acting on inputs of the form:

$$c_1\,e^{i\,t} + c_2\,e^{-i\,t} \equiv \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

where c_1 and c_2 are complex numbers. Find the matrix of D and call it B.

(c) Write a general formula for A^n and B^n where n is a positive integer. Which matrix representation of D is easier to use and why?

3. Let K be the following operator acting on P_1:

$$K : f \mapsto \int_0^1 (2 + 2\,x - 6\,t)\;f(t)\,dt.$$

(a) Find the matrix of K using the identification:

$$a + b\,t = \begin{bmatrix} a \\ b \end{bmatrix}$$

(b) Use the matrix you found in the previous part to compute $K(2 + 3\,t)$. Confirm the result of matrix-vector multiplication using Calculus.

(c) Consider the action of K on inputs $c_1\,f_1 + c_2\,f_2$ where

$$f_1 = 1 - (1 + i)\,t, \quad f_2 = 1 - (1 - i)\,t, \quad (i = \sqrt{-1}),$$

and c's are arbitrary complex numbers. Find the matrix of K using the standard identification

$$c_1\,f_1 + c_2\,f_2 \equiv \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

(d) Show that any linear function in P_1 can be written as a linear combination $c_1\,f_1 + c_2\,f_2$ with f_1 and f_2 as above. What is your conclusion? Write a paragraph summarizing your observations in this problem.

4. Consider the matrices

$$A = \begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Are these matrices similar (in the sense explained in this section)? Whatever answer you choose, provide a clear verbal explanation. If you find this problem impossibly difficult, explain what makes it so.