Ranges and nullspaces. The importance of.

In the previous section we saw that the natural setting for linear algebra is where one has a *linear transformation* T mapping a *vector space* X into another vector space Y: in symbols, $T : X \mapsto Y$. The vector space Y contains the range of T but is often larger than the latter. The reason why Y is not simply $\mathrm{ran}(T)$ is that finding the range is not always so simple! Usually one can describe the general form of outputs as vectors of particular length or functions of so many variables. However, describing the range of a linear transformation exactly can be quite challenging. For instance, consider the transformation

$$T : f \mapsto \frac{df}{dx}.$$

Set X to be all differentiable functions on the unit interval which vanish at the endpoints:

$$X = \left\{ f \in C^1([0,1]) \mid f(0) = f(1) = 0 \right\}.$$

T is certainly linear and, as we know from previous homework, X is a vector space. Yet, is it obvious what $\mathrm{ran}(T)$ is?

Quite generally, the investigation of any linear transformation $T : X \mapsto Y$ begins as follows:

1. Find the nullspace $\mathrm{null}(T)$
2. Find the range $\mathrm{ran}(T)$

To see why these two questions supersede others, let us study a revealing example. Consider $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ whose matrix is:

$$\mathrm{mat}(T) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

The action of T on a generic input is given by the matrix-vector product:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1 + x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{1}$$

Since the sum $(x_1 + x_2)$ in Equation (1) can be absolutely arbitrary, we can replace it with C and write

$$\mathrm{ran}(T) = \left\{ C \begin{bmatrix} 1 \\ 2 \end{bmatrix} \middle| C \in \mathbb{R} \right\}.$$

To find the nullspace of T we set the generic output given by eq1 to zero. This leads to the condition: $x_1 + x_2 = 0$ or, equivalently, $x_2 = -x_1$. The latter shows that any input of the form:

$$\left[\begin{array}{c} x_1 \\ -x_1 \end{array} \right] = x_1 \left[\begin{array}{c} 1 \\ -1 \end{array} \right]$$

produces the zero-vector as the output. Since x_1 can be completely arbitrary

$$\text{null}(T) = \left\{ C \left[\begin{array}{c} 1 \\ -1 \end{array} \right] \middle| C \in \mathbb{R} \right\}.$$

Notice that both the range and the nullspace are one-dimensional vector subspaces of $\mathbb{R}^2$. To understand the importance of these subspaces, let us consider the task of inverting T. That is, we are given an output

$$\mathbf{y} = \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right]$$

and asked to produce an input

$$\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]$$

such that $T(\mathbf{x}) = \mathbf{y}$.

As the first order of business, let us determine whether the problem is feasible. By definition, $\text{ran}(T)$ is the set of all possible outputs. Therefore, if $\mathbf{y}$ is not in the range, there is no $\mathbf{x}$ in the domain that can possibly produce it. Our explicit formula for $\text{ran}(T)$ gives us a simple feasibility criterion: The problem is solvable if and only if the second component of $\mathbf{y}$ is twice the first. This shows, for instance, that we cannot possibly solve (in the usual sense) the system

$$\left[\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \left[\begin{array}{c} 1 \\ 3 \end{array} \right] \tag{2}$$

because the right-hand side is *not in the range* of T. This does not mean that we cannot investigate equations such as (2): in fact, we will do so later. Yet, in the meantime, let us follow the principle:

The data must be in the range

and consider a solvable problem:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \tag{3}$$

Notice that the formula for the inverse of a two-by-two matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{a\,d - b\,c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

does not apply to mat(T) on account of division by zero—the matrix mat(T) is singular. Yet Equation (3) has solutions such as

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

The problem is that there are multiple solutions which suggests that we modify our question. Instead of looking for "the solution" of Equation (3) let us attempt to describe the set of *all* of its solutions. The key observation here is that

$$\mathbf{x}_1 - \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \text{null}(T)$$

In words: the two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ differ by an element of the nullspace. Is it always the case? The following theorem shows that the answer is "Yes".

Theorem 1. *Let* $\mathbf{x_1}$ *and* $\mathbf{x}_2$ *be any two solutions of* $T(\mathbf{x}) = \mathbf{y}$. *Then*

$$\mathbf{x}_1 - \mathbf{x}_2 \in \text{null}(T).$$

Proof. Let $\mathbf{n} = \mathbf{x}_1 - \mathbf{x}_2$. To show that $\mathbf{n} \in \text{null}(T)$ we must demonstrate that $T(\mathbf{n}) = \mathbf{0}$. Using linearity of T and the fact that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of $T(\mathbf{x}) = \mathbf{y}$ we can write

$$T(\mathbf{n}) = T(\mathbf{x}_1 - \mathbf{x}_2) = T(\mathbf{x}_1) - T(\mathbf{x}_2) = \mathbf{y} - \mathbf{y} = 0.$$

This completes the proof. $\square$

As an immediate corollary, we can describe the set of all solutions of (3) using a particular solution $\mathbf{x}_1$ and the nullspace as follows:

$$\mathbf{x} = \mathbf{x}_1 + \text{null}(T) = \left\{ \left[\begin{array}{c} 3 \\ 1 \end{array} \right] + C \left[\begin{array}{c} 1 \\ -1 \end{array} \right] \middle| C \in \mathbb{R} \right\}.$$

This is called the *general solution.* Let us check:

$$\begin{aligned} T(\mathbf{x}) &= T\left(\left[\begin{array}{c} 3 \\ 1 \end{array} \right] + C \left[\begin{array}{c} 1 \\ -1 \end{array} \right] \right) \\ &= T\left(\left[\begin{array}{c} 3 \\ 1 \end{array} \right] \right) + C\,T\left(\left[\begin{array}{c} 1 \\ -1 \end{array} \right] \right) \\ &= \left[\begin{array}{c} 4 \\ 8 \end{array} \right] + C \left[\begin{array}{c} 0 \\ 0 \end{array} \right] = \left[\begin{array}{c} 4 \\ 8 \end{array} \right] \end{aligned}$$

This shows that for any choice of C the vector $\mathbf{x}$ is a solution; conversely, if $\mathbf{z}$ is any solution, Theorem 1 shows that we can choose C so that

$$\mathbf{z} = \mathbf{x}_1 + C \left[\begin{array}{c} 1 \\ -1 \end{array} \right]$$

for *that* C.

Exercises

1. Find the ranges and nullspaces of the following transformations:

   (a) $T_1 : \mathbb{R}^2 \mapsto \mathbb{R}^2$; the action is rotation by θ degrees (counterclockwise)

   (b) $T_2 : f \mapsto \int_0^1 t\, f(t)\, dt$; the domain of T_2 is all cubic polynomials.

   (c) $T^3 : \mathbb{R}^3 \mapsto \mathbb{R}^3$; the action is matrix-vector multiplication

   $$\text{mat}(T_3) = \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 3 & 3 \\ -2 & 5 & 9 \end{array} \right]$$

   (d) $T_4 : f \mapsto \frac{d^2 f}{dx^2} - 9\, f$; the domain of T_4 is all possible linear combinations of exponential functions.

2. Find the general solutions of the following systems:

   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$

   (b) $\dfrac{dy}{dx} + y = \sin(x)$.
   Note: Use the Linear Algebra principles explained in this handout rather than outside knowledge.

3. Give an example of a linear transformation from $\mathbb{R}^4$ into $\mathbb{R}^3$ whose nullspace is two-dimensional. What is the dimension of the range?

4. Let $S : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear operator. Is it possible to have

$$\text{null}(S) = \left\{ C \begin{bmatrix} 1 \\ 3 \end{bmatrix} \middle| C \in \mathbb{R}^1 \right\}$$

   and

$$\text{ran}(S) = \left\{ C \begin{bmatrix} 3 \\ -1 \end{bmatrix} \middle| C \in \mathbb{R}^1 \right\}?$$

   If so, what can you say about the matrix of S? Explain your answers.