# Exercise Set 2

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## Operators

There are three operators that will will consider.

1. $T_1$ : f $\rightarrow x\frac{df}{dx}$ The domain of the operator is quadratic polynomials: $f = a_0 + a_1x + a_2x^2$
2. $T_2$ takes points in the plane. The action is rotation by angle $\theta$ about the origin.
3. $T_3$ takes as its input the position, velocity, and acceleration of a free falling object; the output is the position, velocity, and acceleration 1 second later. (Note that the equation of motion of the object can be stated as $\frac{d^3x}{dt^3} = 0$ where x(t) is the position.)

# 1 Problem 1

Represent each operator with a matrix.

## 1.1 $T_1$

In order to write $T_1$ in matrix form, first we will define our function as a vector, secondly preform the operation once on the function, and thirdly we will write the solution as a vector using the same definition from the function space.

Starting with the function space:

$$f = a_0 + a_1 x + a_2 x^2 \equiv \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Now we must preform the operation $T_1$ on f

$$\begin{aligned} T[f] &= x\left(\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1 x) + \frac{d}{dx}(a_2 x^2)\right) \\ &= 0 + a_1 x + 2a_2 x^2 \end{aligned}$$

$$T[f] = \begin{bmatrix} 0 \\ a_1 \\ 2a_2 \end{bmatrix}$$

Now working backward we can form the matrix that would perform this operation.

$$\begin{bmatrix} & & \\ & T & \\ & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ 2a_2 \end{bmatrix}$$

filling in the values required to turn the initial vector into the final vector we find that the operator $T_1$ is

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## 1.2 $T_2$

In order to derive a rotation matrix for the coordinates (x,y) we will start first with a rotation of zero degrees. In order to get this result from an x, y vector we can write:

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is the case when the rotation $(\theta) = 0$. In order to write a formula that emulates this we may write the zero's as $\sin\theta$ and the one's as $\cos\theta$. However in order to achieve the proper rotation, the sin term that is on the y component must be multiplied by -1. This was derived through trial and error process. This means that in order to transform the vector:

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

into a rotated vector of the form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix}$$

we would need a matrix of the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

## 1.3 $T_3$

Operator $T_3$ is an action upon the vector:

$$\begin{bmatrix} x_0 \\ v_0 \\ a_0 \end{bmatrix}$$

Moving from the definition

$$\frac{d^3x}{dt^3} = 0$$

we will integrate in order to find expressions for a, v, and x. From our basic physics knowledge it is known that

$$\frac{d^2x}{dt^2} = a$$
$$\frac{dx}{dt} = v$$

Also, with the knowledge that a is constant we are able to derive a value for v and then a value for x. To find v will will integrate a.

$$\begin{aligned} v &= \int a dt \\ &= at + c \\ v(0) &= v_0 \\ v(t) &= at + v_0 \end{aligned}$$

Now we can integrate our solution for v in order to find x:

$$\begin{aligned} x &= \int v(t) dt \\ &= \int at + v_0 \\ &= \frac{1}{2}at^2 + v_0 t + c \\ x(0) &= x_0 \\ x(t) &= \frac{1}{2}at^2 + v_0 t + x_0 \end{aligned}$$

now we have an equation representing a, v and x.

If we apply the transformation now to a,v,x than we will be able find a matrix representing $T_3$. Transformation $T_3$ applied to the vector

$$\begin{bmatrix} x_0 \\ v_0 \\ a_0 \end{bmatrix}$$

applying the operation one step at a time:
$a_0$ is transformed into $a_0$
$v_0$ is transformed into $a_0 + v_0$
$x_0$ is transformed into $\frac{1}{2}a_0 + v_0 + x_0$
giving us a vector:

$$\begin{bmatrix} x_0 + v_0 + \frac{1}{2}a_0 \\ v_0 + a_0 \\ a_0 \end{bmatrix}$$

Using the same process to work backwards and find the matrix we see that:

$$\begin{bmatrix} & & \\ & T & \\ & & \end{bmatrix} * \begin{bmatrix} x_0 \\ v_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} x_0 + v_0 + \frac{1}{2}a_0 \\ v_0 + a_0 \\ a_0 \end{bmatrix}$$

and this gives us

$$\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x_0 \\ v_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} x_0 + v_0 + \frac{1}{2}a_0 \\ v_0 + a_0 \\ a_0 \end{bmatrix}$$

# 2 Problem 2

Using Matrix representation find compute $T_1^2(x^2)$

Now if we write out the operator in matrix form and multiply it by itself then we can find an operator for $T_1^2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Applying the $T_1^2$ on $x^2$, first we must write $x^2$ as a vector. Using the same form as we defined for this domain.

$$x^2 \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and now applying the $T_1^2$ operator:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

# 3 Problem 3

An inverse of the rotation matrix can be found through a trial and error process or geometric reasoning. In order for the coordinate (x,y)to be rotated clockwise instead of counter clock wise than the sine term on the y coordinate must be positive and the sine term on the x coordinate must be negative. This produces this matrix:

$$T_2^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

When the operator $T_3$ is multiplied with $T_3^{-1}$ than it should produce a net effect of no rotation. As should the other multiplication $T_3^{-1}$ times $T_3$. We can show this by doing the matrix multiplication as seeing the resultant matrix:

$T_3^{-1} * T_3$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T_3 * T_3^{-1}$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# 4 Problem 4

Computing $T_3^n$

$$T_3 = \begin{bmatrix} 1 & 1 & .5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3^3 = \begin{bmatrix} 1 & 3 & 4.5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3^4 = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

From this we see that the general formula for $T_3^n$ is:

$$T_3^n = \begin{bmatrix} 1 & n & \frac{n^2}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

The physical meaning of the $T_3^2$ operator is that instead of giving the acceleration, velocity, and displacement one second ahead of the given vector it will give it two seconds a head (t+2 instead of t+1).