EE339 – Chapter 7 – Problem 4

7.4 Show that the coherent state defined through

$$a|\alpha >= \alpha|\alpha >$$

can be written $$|\alpha > = \sum_{n=0}^{\infty} exp\left(-\frac{|\alpha|^2}{2}\right)\frac{\alpha^n}{\sqrt{n!}}\ |n >.$$

Alright so to start this problem, we need to know that when a state forms a complete basis set (which the coherent state does), we can expand the state as follows:

$$|\phi_n\rangle = \sum_{n=0}^{\infty} C_n|\phi_n\rangle$$

Now, in our problem here we are given that a coherent state is **defined** as:

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

Where a is the lowering operator and α is the eigenvalue associated with the lowering operator. **Note:** the lowering operator is <u>not</u> a Hermitian operator, thus α is not necessarily real.

The lowering operating acting on a state is defined as follows:

$$a_-|\phi_n\rangle = \sqrt{n}\ |\phi_{n-1}\rangle$$

When $n > 0$, otherwise it is 0.

Alright, with that out of the way let's write:

$$a_-|\phi_n\rangle = \sum_{n=0}^{\infty} a_-\ C_n|\phi_n\rangle$$

$$a_-|\phi_n\rangle = \sum_{n=0}^{\infty} C_n\ \ a_-|\phi_n\rangle$$

Let's expand the first 3 (significant) terms:

$$a_-|\phi_n\rangle = \sum_{n=0}^{\infty} C_n\ \ a_-|\phi_n\rangle = C_1\sqrt{1}|\phi_0\rangle + C_2\sqrt{2}|\phi_1\rangle + C_3\sqrt{3}\ |\phi_2\rangle + \cdots$$

Now remember that the coherent state is defined also as $a_-|\phi_n\rangle = \alpha|\phi_n\rangle$

It just so happens that:

$$\alpha|\phi_n\rangle = \sum_{n=0}^{\infty} \alpha C_n|\phi_n\rangle$$

Where α is some constant (real or complex). Remember that this α is the eigenfunction of the lowering operator.

Let's expand this like we did before (first 3 significant terms) !

$$\alpha|\phi_n\rangle = \sum_{n=0}^{\infty} \alpha C_n|\phi_n\rangle = \alpha C_0|\phi_0\rangle + \alpha C_1|\phi_1\rangle + \alpha C_2|\phi_2\rangle + \cdots$$

So looking back (again) at the definition of the coherent state:

$$a_-|\phi_n\rangle = \alpha|\phi_n\rangle$$

$$\sum_{n=0}^{\infty} a_- \, C_n|\phi_n\rangle = \sum_{n=0}^{\infty} \alpha C_n|\phi_n\rangle$$

Then now, we can look at their expansions:

$$C_1\sqrt{1}|\phi_0\rangle + C_2\sqrt{2}|\phi_1\rangle + C_3\sqrt{3}\,|\phi_2\rangle + \cdots = \alpha C_0|\phi_0\rangle + \alpha C_1|\phi_1\rangle + \alpha C_2|\phi_2\rangle + \cdots$$

Not too pretty to look at, but the next thing we have to do is solve for the coefficients (C).

We can match up the state numbers and their coefficients and we can see that

$$C_1 = \alpha C_0$$

$$\sqrt{2}\, C_2 = \alpha C_1$$

Solving for C_2, we get the following:

$$C_2 = \frac{\alpha(\alpha C_0)}{\sqrt{2}} = \frac{\alpha^2 C_0}{\sqrt{2}}$$

Now, we can look at the third term and see that

$$\sqrt{3}\, C_3 = \alpha C_2$$

Solving for C_3, we can see that

$$C_3 = \frac{\alpha(\alpha^2 C_0)}{\sqrt{2}\,\sqrt{3}} = \frac{\alpha^3 C_0}{\sqrt{3!}}$$

If we keep expanding the terms, this pattern will emerge:

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0$$

Now remember our original expansion of $|\phi_n\rangle$? Hint- it's the very first equation of this document

$$|\phi_n\rangle = \sum_{n=0}^{\infty} C_n|\phi_n\rangle$$

If we plug in C_n into that equation, we have

$$|\phi_n\rangle = \sum_{n=0}^{\infty} C_n|\phi_n\rangle = |\phi_n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_0|\phi_n\rangle$$

Cool we're getting somewhere with this problem. We just have to find out what C_0 is!

So assuming that the wave function is normalized (which it has to be) we know that:

$$\langle \phi_n | \phi_n \rangle = 1$$

I mean technically… I should write:

$$\langle \phi_m | \phi_n \rangle = 1 \ if \ m = n, 0 \ otherwise$$

But we're going to just be using $\langle \phi_n | \phi_n \rangle = 1$ for now. Anyway, from this we know that

$$\langle \phi_n | \phi_n \rangle = \sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} C_0^* \frac{\alpha^n}{\sqrt{n!}} C_0 \ \langle \phi_n | \phi_n \rangle = 1$$

Since we know that $\langle \phi_n | \phi_n \rangle = 1$, we can write the following

$$\langle \phi_n | \phi_n \rangle = \sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} C_0^* \frac{\alpha^n}{\sqrt{n!}} C_0 \ (1) = 1$$

Now let's make this a little neater

$$\langle \phi_n | \phi_n \rangle = \left| \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_0 \frac{\alpha^n}{\sqrt{n!}} C_0 \ (1) \right| = \sum_{n=\,0}^{\infty} |C_0|^2 \frac{|\alpha|^{2n}}{n!} = 1$$

We can take the $|C_0|^2$ term out of the summation since it isn't dependent on n

$$\langle \phi_n | \phi_n \rangle = |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = 1$$

Remember back a few years ago, when you were studying calculus… you did the Taylor series expansion? Well here it is again! It turns out that the summation that you see (between $|C_0|^2$ and the = sign) is the Taylor series expansion of the exponential function.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Then we can generalize this to say:

$$e^{x^k} = \sum_{n=0}^{\infty} \frac{x^{kn}}{n!}$$

In our case, $k = 2$. So let's rewrite out equation a little bit:

$$\langle \phi_n | \phi_n \rangle = |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{\alpha^2} = \ 1$$

If you recall, we're doing this so that we can find what C_0 is. So let's find it

$$|C_0|^2 = e^{-\alpha^2}$$

$$C_0 = \sqrt{e^{-\alpha^2}} = \exp(-\frac{\alpha^2}{2})$$

So now we can rewrite our original equation (first equation on this document) as such:

$$|\phi_n\rangle = \sum_{n=0}^{\infty} C_n|\phi_n\rangle = |\phi_n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_0|\phi_n\rangle = \sum_{n=0}^{\infty} \exp\left(-\frac{\alpha^2}{2}\right)\frac{\alpha^n}{\sqrt{n!}}|\phi_n\rangle$$

Boom! It's exactly what we're trying to prove!

This is the answer to the original question 😊