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NEW TABLES OF COEFFICIENTS AND PERCENTAGE POINTS FOR THE *W* TEST FOR NORMALITY

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Recent work has made available new tables of expected values, variances, and covariances of normal order statistics to higher precision and greater accuracy than existing tables. The coefficients associated with calculation of the Shapiro-Wilk W statistic used in testing departure from normality are based upon expressions involving expected values of normal order statistics and the corresponding variance-covariance matrix. These coefficients have been calculated on the basis of the new tables for sample sizes up to 50. Percentage points of the distribution of W are evaluated via simulation using the new coefficients, and an approximation is provided. The first and half moments of W also are given.

KEY WORDS: normal order statistics, Shapiro-Wilk W statistic, test of normality, W test.

1. INTRODUCTION

Since the introduction of the W test of normality by Shapiro and Wilk (1965), many authors have applied and extended the technique. The computation of W usually has been based on tables of coefficients provided originally by Shapiro and Wilk which, in turn, were based on variances and covariances of normal order statistics as provided by Sarhan and Greenberg (1956) and also appearing in Sarhan and Greenberg (1962). The moments were given to ten decimal places (d.p.) for sample sizes n up to 20. For larger sample sizes, approximations were used for the coefficient calculations. Recently, Parrish (1992a, 1992b) computed high-precision values for the expected values, variances, and covariances of normal order statistics. Expected values and standard deviations were computed to 25 d.p. for n up to 500. Variances and covariances were computed to 25 d.p. for $n = 2(1)20$, to 20 d.p. for $n = 21(1)30$, to 15 d.p. for $n = 31(1)40$, and to 10 d.p. for $n = 41(1)50$. A brief summary of the computational techniques is provided in the Appendix. These values can be utilized to calculate with higher accuracy the coefficients used for computing W for sample sizes up to 50. In addition, percentage points of the distribution of W and low-order moments also can be computed with better accuracy. The percentage points presented here differ somewhat from those reported by Shapiro and Wilk and also from those based on code by Royston (1982b).

2. COEFFICIENTS FOR CALCULATION OF W

For a random sample of n observations, the W test statistic is defined by

$$W = \frac{(\boldsymbol{a}'\mathbf{y})^2}{\sum_{i=1}^{n} (y_i - \bar{y})^2}$$

where $\mathbf{y}$ represents the vector of ordered observations with elements y_i ($i = 1, \ldots, n$) having mean value $\bar{y}$, and where $\boldsymbol{a}'$ represents a set of corresponding coefficients. Of course, the numerator for W may be written as $(\Sigma a_i y_i)^2$, where the summation extends over $i = 1, \ldots, n$. Note also that the bounds on W are from $na_1^2/(n - 1)$ to 1.

The coefficients are determined on the basis of the expected values, variances, and covariances of order statistics for a sample of size n from the unit normal distribution. Using $\boldsymbol{\mu}$ to denote the vector of expected values and $\boldsymbol{\Sigma}$ to denote the variance-covariance matrix, the coefficients are given by

$$\boldsymbol{a}' = \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{-1/2}}$$

The values of a_i for $n = 2(1)50$ were calculated using the values given by Parrish; these are presented in Table 1. The table contains values for $-a_i$, noting that $a_{n-i+1} = -a_i$.

By definition, $\mathbf{a'a} = 1$ for each sample size. As a numerical check, this identity holds to at least 31 d.p. for all cases reported. The identity $\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{1} = 0$ holds also to at least 31 d.p. In comparison to the 4-d.p. table of coefficients of Shapiro and Wilk (1965), there is uniform agreement up to sample size 20. Thereafter, however, there are several instances of relatively large differences between the two sets of values.

Royston (1982a) described an algorithm for approximating these coefficients with emphasis on higher sample sizes. Royston's computed values agree very well with those of Shapiro and Wilk for n between 21 and 50, being based partly on the same technique of approximation. In comparison to current values, however, maximum absolute (relative) differences observed for a_i values were 0.0087 (2.1%), 0.0176 (4.6%), and 0.0243 (6.9%) for n equal to 30, 40, and 50, respectively. The maximum differences all occur for a_1, which is the weight applied to the most extreme observations in the sample.

3. LOW-ORDER MOMENTS OF W

The expected value of W is given by

$$E(W) = \frac{(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + 1)}{(n - 1)(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})}$$

and the first half-moment is

$$E(W^{1/2}) = \frac{\Gamma[(n-1)/2](\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{2^{1/2}\Gamma(n/2)(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^{1/2}}$$

(see Shapiro and Wilk, 1965). Computed values of these moments for $n = 2(1)50$ are given in Table 2 to 20 d.p. In comparison to the values given by Shapiro and Wilk for $n = 3(1)20$, there are some discrepancies in the fourth decimal digit.

4. PERCENTAGE POINTS OF W

Simulation techniques were employed to obtain empirical percentage points for the distribution of W. For each size n, 100,000 random normal samples of size n were generated using an inverse CDF technique; see routine RNNOR of IMSL (1987). For each sample, the value of W was computed. Percentage points of the resulting empirical distributions were calculated according to the quantile estimator given by

$$\hat{\mu}_p = (0.5 + i - np)W_{(i)} + (0.5 - i + np)W_{(i+1)}$$

where μ_p represents the $100p$-th percentile, $n = 100{,}000$, $i = [np + 0.5]$ (an integer), and where $W_{(i)}$ is the ith order statistic of the set of W values; see Parrish (1990). Percentage points were computed for p = 0.005, 0.01, 0.02, 0.025, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.40, 0.50, 0.60, 0.70, 0.75, 0.80, 0.85, 0,90, 0.95, 0.975, 0.98, 0.99, and 0.995. These results are presented in Table 3. The reported values were convergent to within one digit in the third decimal place.

The Q-Q plot of cumulative distribution function values in Figure 1 shows the relationship between the empirical percentage points W_p calculated on the basis of the high-precision coefficient values and those produced by Royston's approximate technique, for the case of $n = 50$. This indicates that the empirical values are consistently greater than those of Royston. For values of n below 20, there is very little difference between the values. For values of n = 30, 40, and 50, however, the maximum differences are 3.1%, 5.3%, and 9.2%, respectively, when expressed relative to the 99% empirical quasi-range ($W_{0.995} - W_{0.005}$).

Royston (1982) described an effective technique for approximating percentiles of the distribution of W, based on a power transformation of the form $Y = (1 - W)^\tau$. For appropriate values of τ, Y has approximately a normal distribution. Thus, an approximation of a percentage point of W can be obtained by applying the inverse transformation using the corresponding percentage point of the normal distribution, provided the mean and strandard deviation are known. By using simulation, Royston determined a value of τ for each n that maximized the coefficient of determination for a weighted regression of empirical percentage points of W on corresponding values of the normal distribution. Polynomials then were used to obtain an approximation for τ as a function of n. Using high-precision

Table 1 Coefficients for the *W* test of normality

	Sample size n				
i		*2*	*3*	*4*	*5*
1		0.7071067812	0.7071067812	0.6871551280	0.6646474775
2			0.0000000000	0.1667867802	0.2413373793
3					0.0000000000
	6	*7*	*8*	*9*	*10*
1	0.6431054172	0.6232895333	0.6052472633	0.5888321091	0.5738570793
2	0.2806345383	0.3031014983	0.3164469157	0.3244228324	0.3290504445
3	0.0875195881	0.1401414979	0.1743201616	0.1976490410	0.2141165910
4		0.0000000000	0.0561211289	0.0946648285	0.1223707080
5				0.0000000000	0.0399168226
	11	*12*	*13*	*14*	*15*
1	0.5601422499	0.5475281705	0.5358777533	0.5250744090	0.5150192099
2	0.3315011888	0.3324925825	0.3324834790	0.3317775940	0.3305816528
3	0.2260112813	0.2347342583	0.2411873679	0.2459746345	0.2495139501
4	0.1429441989	0.1585949356	0.1707215330	0.1802509537	0.1878205734
5	0.0695232230	0.0921573545	0.1098628215	0.1239635841	0.1353555195
6	0.0000000000	0.0302668380	0.0538770994	0.0726976454	0.0879586645
7			0.0000000000	0.0239718197	0.0433516067
8					0.0000000000
	16	*17*	*18*	*19*	*20*
1	0.5056280678	0.4968292406	0.4885612340	0.4807710708	0.4734128767
2	0.3290396258	0.3272536929	0.3252975150	0.3232248788	0.3210754680
3	0.2521023330	0.2539557503	0.2552343204	0.2560587664	0.2565214513
4	0.1938820587	0.1987636435	0.2027090203	0.2059024776	0.2084856299
5	0.1446662435	0.1523481263	0.1587350801	0.1640786216	0.1685715416
6	0.1005055905	0.1109391506	0.1196979973	0.1271098197	0.1334242040
7	0.0592758746	0.0725350058	0.0836968501	0.0931803327	0.1013014106
8	0.0195963678	0.0358670219	0.0495487472	0.0611755362	0.0711446023
9		0.0000000000	0.0164092927	0.0303185106	0.0422290548
10				0.0000000000	0.0140026086
	21	*22*	*23*	*24*	*25*
1	0.4664467268	0.4598377036	0.4535551262	0.4475719153	0.4418640698
2	0.3188787999	0.3166569623	0.3144265488	0.3122000466	0.3099868465
3	0.2566939530	0.2566323737	0.2563811303	0.2559757039	0.2554446647
4	0.2105688392	0.2122391882	0.2135661558	0.2146057276	0.2154034217
5	0.1723638838	0.1755739987	0.1782963546	0.1806071621	0.1825684929
6	0.1388344530	0.1434924768	0.1475191999	0.1510119860	0.1540500328
7	0.1083028893	0.1143743953	0.1196661221	0.1242985139	0.1283692276
8	0.0797578651	0.0872490460	0.0938019389	0.0995630671	0.1046506560
9	0.0525164046	0.0614677911	0.0693069211	0.0762107318	0.0823210553
10	0.0260695452	0.0365554708	0.0457331335	0.0538160456	0.0609739073
11	0.0000000000	0.0121324451	0.0227298981	0.0320512598	0.0403000747
12			0.0000000000	0.0106449407	0.0200482635
13					0.0000000000

Table 1 (continued)

	26	*27*	*28*	*29*	*30*
1	0.4364102318	0.4311913235	0.4261902430	0.4213916082	0.4167815395
2	0.3077939859	0.3056267018	0.3034888455	0.3013831975	0.2993117080
3	0.2548111830	0.2540941704	0.2533091512	0.2524689344	0.2515841371
4	0.2159965457	0.2164159026	0.2166870959	0.2168315372	0.2168672338
5	0.1842313407	0.1856379285	0.1868234674	0.1878175140	0.1886450282
6	0.1566983562	0.1590107760	0.1610321820	0.1628002757	0.1643469233
7	0.1319582284	0.1351315786	0.1379442958	0.1404425356	0.1426652801
8	0.1091611237	0.1131738590	0.1167547950	0.1199591189	0.1228333526
9	0.0877529291	0.0926006375	0.0969421793	0.1008426256	0.1043566768
10	0.0673433859	0.0730358257	0.0781428732	0.0827406551	0.0868929286
11	0.0476387943	0.0541987726	0.0600874378	0.0653935408	0.0701910651
12	0.0284039853	0.0358675038	0.0425649038	0.0485996488	0.0540574905
13	0.0094386991	0.0178561866	0.0254011137	0.0321944514	0.0383357562
14		0.0000000000	0.0084444544	0.0160369307	0.0228935039
15				0.0000000000	0.0076134408

	31	*32*	*33*	*34*	*35*
1	0.4123474753	0.4080780148	0.4039627823	0.3999923112	0.3961579430
2	0.2972756810	0.2952759175	0.2933128271	0.2913865147	0.2894968487
3	0.2506635954	0.2497146895	0.2487436027	0.2477555297	0.2467548445
4	0.2168094072	0.2166709859	0.2164629985	0.2161948911	0.2158747843
5	0.1893272049	0.1898821345	0.1903253306	0.1906701564	0.1909281695
6	0.1656992159	0.1668803105	0.1679100996	0.1688057516	0.1695821477
7	0.1446456552	0.1464119710	0.1479885486	0.1493963843	0.1506536862
8	0.1254169698	0.1277436640	0.1298423546	0.1317379895	0.1334521932
9	0.1075306355	0.1104039444	0.1130103996	0.1153791146	0.1175352940
10	0.0906534904	0.0940680413	0.0971756462	0.1000098881	0.1025997896
11	0.0745421847	0.0784995325	0.0821079619	0.0854059298	0.0884265934
12	0.0590101353	0.0635180246	0.0676324729	0.0713973324	0.0748503029
13	0.0439077750	0.0489798916	0.0536107452	0.0578502486	0.0617411619
14	0.0291099437	0.0347660481	0.0399288904	0.0446552907	0.0489937246
15	0.0145072955	0.0207737038	0.0264896074	0.0317198175	0.0365193539
16	0.0000000000	0.0069104320	0.0132065399	0.0189625234	0.0242410031
17			0.0000000000	0.0063094121	0.0120893874
18					0.0000000000

	36	*37*	*38*	*39*	*40*
1	0.3924517404	0.3888664111	0.3853952412	0.3820320373	0.3787710751
2	0.2876435153	0.2858260608	0.2840439256	0.2822964709	0.2805830004
3	0.2457452368	0.2447298236	0.2437112409	0.2426917189	0.2416731446
4	0.2155096836	0.2151056517	0.2146679507	0.2142011605	0.2137092772
5	0.1911094042	0.1912226036	0.1912754110	0.1912745293	0.1912258537
6	0.1702522385	0.1708273376	0.1713173630	0.1717310383	0.1720760607
7	0.1517763111	0.1527781229	0.1536712875	0.1544665180	0.1551732792
8	0.1350037920	0.1364092438	0.1376829911	0.1388377541	0.1398847738
9	0.1195008598	0.1212949605	0.1229343895	0.1244339305	0.1258066434
10	0.1049705543	0.1071441683	0.1091398918	0.1109746636	0.1126634352

Table 1 (continued)

11	0.0911986879	0.0937472355	0.0960941216	0.0982585671	0.1002575176
12	0.0780239734	0.0809466579	0.0836430716	0.0861348820	0.0884411623
13	0.0653203350	0.0686196961	0.0716670478	0.0744867118	0.0771000561
14	0.0529858135	0.0566675032	0.0600700041	0.0632205491	0.0661430081
15	0.0409352519	0.0450079774	0.0487725460	0.0522594182	0.0554952212
16	0.0290952390	0.0335708457	0.0377071373	0.0415381925	0.0450937076
17	0.0174004411	0.0222941283	0.0268146765	0.0310002662	0.0348840459
18	0.0057907930	0.0111214682	0.0160419397	0.0205950910	0.0248179962
19		0.0000000000	0.0053395736	0.0102762695	0.0148516753
20				0.0000000000	0.0049440918

	41	*42*	*43*	*44*	*45*
1	0.3756070541	0.3725350582	0.3695505198	0.3666491887	0.3638271034
2	0.2789027767	0.2772550356	0.2756389962	0.2740538696	0.2724988661
3	0.2406571144	0.2396449769	0.2386378700	0.2376367512	0.2366424238
4	0.2131957957	0.2126637789	0.2121159166	0.2115545751	0.2109818394
5	0.1911345836	0.1910053165	0.1908421272	0.1906486361	0.1904280663
6	0.1723592414	0.1725866244	0.1727635870	0.1728949253	0.1729849265
7	0.1557999582	0.1563540092	0.1568420751	0.1572700910	0.1576433729
8	0.1408340158	0.1416943414	0.1424736528	0.1431790153	0.1438167624
9	0.1270641038	0.1282166039	0.1292733215	0.1302424635	0.1311313880
10	0.1142194504	0.1156544773	0.1169790060	0.1182024131	0.1193331038
11	0.1021059670	0.1038172270	0.1054031535	0.1068743374	0.1082402661
12	0.0905787649	0.0925626332	0.0944060615	0.0961209143	0.0977178112
13	0.0795259282	0.0817810149	0.0838801407	0.0858365195	0.0876619648
14	0.0688583927	0.0713852704	0.0737401095	0.0759375664	0.0779907276
15	0.0585033361	0.0613043797	0.0639166017	0.0663562150	0.0686376715
16	0.0483996851	0.0514789947	0.0543518343	0.0570361104	0.0595477552
17	0.0384949462	0.0418583377	0.0449965676	0.0479294009	0.0506743847
18	0.0287428900	0.0323979447	0.0358079007	0.0389945792	0.0419773060
19	0.0191017927	0.0230579646	0.0267476195	0.0301948747	0.0334210285
20	0.0095330456	0.0138018277	0.0177809160	0.0214970133	0.0249736258
21	0.0000000000	0.0045951585	0.0088753544	0.0128702296	0.0166057823
22			0.0000000000	0.0042854453	0.0082900138
23					0.0000000000

	46	*47*	*48*	*49*	*50*
1	0.3610805671	0.3584061243	0.3558005410	0.3532607867	0.3507840182
2	0.2709731999	0.2694760935	0.2680067810	0.2665645105	0.2651485459
3	0.2356555587	0.2346767133	0.2337063471	0.2327448354	0.2317924810
4	0.2103995493	0.2098093308	0.2092126222	0.2086106971	0.2080046843
5	0.1901832932	0.1899168868	0.1896311485	0.1893281424	0.1890097226
6	0.1730374321	0.1730558909	0.1730434054	0.1730027722	0.1729365164
7	0.1579666933	0.1582443460	0.1584802025	0.1586777604	0.1588401854
8	0.1443925851	0.1449116092	0.1453784606	0.1457973238	0.1461719907
9	0.1319467089	0.1326943841	0.1333797949	0.1340078086	0.1345828412
10	0.1203786312	0.1213458006	0.1222407543	0.1230690568	0.1238357434
11	0.1095094625	0.1106895990	0.1117876072	0.1128097444	0.1137617138
12	0.0992062826	0.1005949091	0.1018914211	0.1031028353	0.1042354485

Table 1 (continued)

13	0.0893670710	0.0909613579	0.0924534230	0.0938509919	0.0951611725
14	0.0799113106	0.0817098458	0.0833957951	0.0849777624	0.0864633642
15	0.0707738969	0.0727764767	0.0746558557	0.0764213781	0.0780817119
16	0.0619009901	0.0641085570	0.0661818771	0.0681313152	0.0699659895
17	0.0532471542	0.0556616808	0.0579305124	0.0600648517	0.0620750784
18	0.0447732600	0.0473977700	0.0498645416	0.0521859361	0.0543728771
19	0.0364449666	0.0392834958	0.0419516359	0.0444628060	0.0468292242
20	0.0282315360	0.0312891934	0.0341630315	0.0368677677	0.0394164990
21	0.0201049228	0.0233879277	0.0264728170	0.0293756506	0.0321108620
22	0.0120390223	0.0155548020	0.0188571357	0.0219636245	0.0248899521
23	0.0040090437	0.0077663449	0.0112936741	0.0146103332	0.0177335170
24		0.0000000000	0.0037611431	0.0072956146	0.0106222495
25				0.0000000000	0.0035377912

coefficient values, the present author followed a similar approach and obtained values of τ for $n = 7(1)50$; these are plotted in Figure 2. Royston's results exhibited a discontinuity at $n = 20$, which apparently was due to the use of approximate coefficients above this level, and the values tended to increase above $n = 20$. In contrast, current values showed no discontinuity and showed a consistently declining exponential trend up to $n = 50$. Implementation of this approximation also requires calculation of the mean and standard deviation of the approximating normal distribution. Royston obtained these by conducting further simulations using τ values calculated from the polynomial function, and, similarly, the mean and standard deviation were fitted using polynomials. For this study, the mean and standard deviation were computed for $n = 7(1)50$ on the basis of the regression of empirical percentage points W_p versus standard normal percentage points for p ranging from 0.005 to 0.995. R^2 values for all cases were at least 0.999. Plots of the mean and standard deviation so obtained are given in Figures 3 and 4.

Approximations for the mean μ, the standard deviation σ, and the exponent τ, as functions of n, were obtained by using the following model forms:

$$\mu = A_0 - A_1 e^{-c_1 n} - A_2 e^{-c_2 n},$$

$$\sigma = A_3 e^{-c_3 n},$$

$$\tau = A_4 e^{-c_4 n} + A_5 e^{-c_5 n}.$$

The parameters A_i and C_i were estimated by using nonlinear least squares to minimize the function

$$\sum_n \sum_p \left(\frac{(1 - W_p)^\tau - \mu}{\sigma} - Z_p \right)^2$$

Table 2 Low-order moments of the Shapiro-Wilk W statistic

	Expected value	*Half-moment*
2	1.00000000000000000000	1.00000000000000000000
3	0.91349667156634403713	0.95492965855137201461
4	0.90134519044754513483	0.94860607100455176481
5	0.90260843946125172617	0.94940601433948222235
6	0.90714614811641694356	0.95190950844072190868
7	0.91234611623600380971	0.95472848337660872056
8	0.91741539192094758982	0.95745045124918655928
9	0.92211342347220650495	0.95995593518890414679
10	0.92638807705246033976	0.96222346577841627740
11	0.93025382416783535182	0.96426509595203587480
12	0.93374706814312365772	0.96610316450704737430
13	0.93690901753580717613	0.96776161652006365754
14	0.93977917406865121558	0.96926284465681634435
15	0.94239310007495775322	0.97062670877930646601
16	0.94478193531821190766	0.97187041788235817517
17	0.94697260747540038439	0.97300873535868117156
18	0.94898828567435564682	0.97405428331057649864
19	0.95084888532062159215	0.97501785408068273388
20	0.95257154612940791482	0.97590869412744424108
21	0.95417105567720966067	0.97673474988580386552
22	0.95566021282936844758	0.97750287549381897308
23	0.95705013453551454870	0.97821900616304097490
24	0.95835051274254200149	0.97888830207528633362
25	0.95956982881124831504	0.97951526761339007590
26	0.96071553235951415086	0.98010385021201581895
27	0.96179419059883489611	0.98065752247435142336
28	0.96281161330195789883	0.98117935058079124809
29	0.96377295767263980247	0.98167205146856415872
30	0.96468281663238855126	0.98213804079942762298
31	0.96554529340195746272	0.98257947335199262952
32	0.96636406472883727551	0.98299827716580251738
33	0.96714243468154719357	0.98339618251445152654
34	0.96788338058167134825	0.98377474658398625977
35	0.96858959236100004900	0.98413537457120792105
36	0.96926350640141707580	0.98447933778647664352
37	0.96990733472896596327	0.98480778924085681398
38	0.97052309028234917024	0.98512177711283064956
39	0.97111260885310429550	0.98542225642127860286
40	0.97167756819435038516	0.98571009917574988482
41	0.97221950471289455497	0.98598610322966457887
42	0.97273982809210959919	0.98625100002497102641
43	0.97323983413752123150	0.98650546138631832536
44	0.97372071609123614432	0.98675010549771871360
45	0.97418357462334380940	0.98698550217392305224
46	0.97462942667689210948	0.98721217752155076934
47	0.97505921331670195411	0.98743061807070342071
48	0.97547380670972191675	0.98764127444557188837
49	0.97587401634656281830	0.98784456463277569408
50	0.97626059459539153246	0.98804087689639025830

Table 3 Empirical percentage points of the Shapiro-Wilk *W* statistic

N	*0.005*	*0.010*	*0.020*	*0.025*	*0.050*	*0.100*	*0.150*	*0.200*	*0.250*	*0.300*	*0.400*	*0.500*
3	0.752	0.755	0.759	0.761	0.772	0.794	0.815	0.835	0.854	0.872	0.905	0.933
4	0.674	0.692	0.716	0.726	0.761	0.800	0.826	0.846	0.860	0.872	0.893	0.914
5	0.671	0.700	0.732	0.743	0.777	0.813	0.835	0.853	0.867	0.879	0.899	0.915
6	0.691	0.720	0.750	0.760	0.793	0.828	0.849	0.864	0.877	0.887	0.905	0.919
7	0.709	0.737	0.768	0.778	0.809	0.840	0.860	0.874	0.885	0.894	0.910	0.923
8	0.726	0.754	0.784	0.793	0.821	0.851	0.869	0.882	0.892	0.901	0.915	0.927
9	0.745	0.772	0.799	0.808	0.835	0.863	0.879	0.890	0.899	0.907	0.920	0.932
10	0.760	0.786	0.812	0.821	0.845	0.871	0.886	0.897	0.906	0.913	0.925	0.935
11	0.773	0.798	0.823	0.830	0.854	0.879	0.893	0.903	0.911	0.918	0.929	0.939
12	0.786	0.809	0.832	0.840	0.862	0.885	0.898	0.908	0.916	0.922	0.933	0.942
13	0.797	0.819	0.840	0.847	0.869	0.891	0.904	0.913	0.920	0.926	0.936	0.944
14	0.805	0.827	0.848	0.854	0.875	0.896	0.908	0.917	0.924	0.930	0.939	0.947
15	0.813	0.836	0.856	0.863	0.882	0.902	0.913	0.921	0.928	0.933	0.942	0.949
16	0.823	0.843	0.863	0.869	0.887	0.905	0.916	0.924	0.930	0.935	0.944	0.951
17	0.831	0.850	0.869	0.875	0.892	0.910	0.920	0.928	0.934	0.938	0.946	0.953
18	0.840	0.857	0.874	0.880	0.897	0.913	0.924	0.931	0.936	0.941	0.948	0.955
19	0.844	0.861	0.879	0.884	0.900	0.917	0.926	0.933	0.939	0.943	0.950	0.956
20	0.850	0.867	0.883	0.888	0.905	0.920	0.929	0.936	0.941	0.945	0.952	0.958
21	0.857	0.873	0.888	0.893	0.908	0.923	0.932	0.938	0.943	0.947	0.954	0.959
22	0.861	0.876	0.891	0.896	0.911	0.926	0.934	0.940	0.945	0.949	0.955	0.961
23	0.865	0.880	0.894	0.900	0.914	0.928	0.936	0.942	0.947	0.950	0.957	0.962
24	0.870	0.885	0.899	0.903	0.917	0.930	0.938	0.944	0.948	0.952	0.958	0.963
25	0.875	0.888	0.902	0.906	0.919	0.932	0.940	0.945	0.950	0.953	0.959	0.964
26	0.877	0.892	0.905	0.909	0.922	0.934	0.942	0.947	0.951	0.955	0.960	0.965
27	0.881	0.895	0.908	0.912	0.924	0.936	0.944	0.949	0.953	0.956	0.961	0.966
28	0.885	0.898	0.910	0.914	0.926	0.938	0.945	0.950	0.954	0.957	0.962	0.967
29	0.888	0.900	0.912	0.916	0.928	0.939	0.946	0.951	0.955	0.958	0.963	0.968
30	0.892	0.904	0.916	0.919	0.930	0.942	0.948	0.953	0.956	0.959	0.964	0.969
31	0.894	0.906	0.917	0.921	0.932	0.943	0.949	0.954	0.957	0.960	0.965	0.969
32	0.897	0.908	0.919	0.923	0.934	0.944	0.951	0.955	0.958	0.961	0.966	0.970
33	0.899	0.910	0.921	0.925	0.935	0.945	0.952	0.956	0.959	0.962	0.967	0.971
34	0.901	0.912	0.923	0.926	0.937	0.947	0.953	0.957	0.960	0.963	0.968	0.971
35	0.904	0.915	0.925	0.928	0.938	0.948	0.954	0.958	0.961	0.964	0.968	0.972
36	0.908	0.917	0.927	0.930	0.940	0.949	0.955	0.959	0.962	0.965	0.969	0.973
37	0.909	0.918	0.928	0.932	0.941	0.950	0.956	0.960	0.963	0.965	0.970	0.973
38	0.911	0.920	0.930	0.933	0.942	0.951	0.957	0.960	0.964	0.966	0.970	0.974
39	0.912	0.922	0.931	0.934	0.943	0.952	0.958	0.961	0.964	0.967	0.971	0.974
40	0.913	0.924	0.932	0.935	0.944	0.953	0.958	0.962	0.965	0.967	0.971	0.975
41	0.916	0.925	0.935	0.938	0.946	0.954	0.959	0.963	0.966	0.968	0.972	0.975
42	0.917	0.926	0.935	0.938	0.947	0.955	0.960	0.964	0.967	0.969	0.973	0.976
43	0.919	0.928	0.936	0.939	0.948	0.956	0.961	0.964	0.967	0.969	0.973	0.976
44	0.920	0.929	0.938	0.940	0.949	0.957	0.962	0.965	0.968	0.970	0.973	0.976
45	0.922	0.930	0.939	0.942	0.950	0.958	0.962	0.966	0.968	0.970	0.974	0.977
46	0.924	0.932	0.940	0.943	0.951	0.958	0.963	0.966	0.969	0.971	0.974	0.977
47	0.925	0.933	0.941	0.944	0.951	0.959	0.963	0.967	0.969	0.971	0.975	0.978
48	0.927	0.935	0.942	0.945	0.953	0.960	0.964	0.967	0.970	0.972	0.975	0.978
49	0.928	0.936	0.943	0.946	0.953	0.961	0.965	0.968	0.970	0.972	0.976	0.978
50	0.930	0.937	0.944	0.947	0.954	0.961	0.965	0.968	0.971	0.973	0.976	0.979

Table 3 (continued)

N	*0.600*	*0.700*	*0.750*	*0.800*	*0.850*	*0.900*	*0.950*	*0.975*	*0.980*	*0.990*	*0.995*
3	0.957	0.976	0.983	0.989	0.994	0.997	0.999	1.000	1.000	1.000	1.000
4	0.933	0.950	0.958	0.967	0.975	0.983	0.992	0.996	0.997	0.998	0.999
5	0.930	0.945	0.953	0.960	0.968	0.975	0.984	0.990	0.992	0.995	0.997
6	0.932	0.945	0.951	0.958	0.965	0.972	0.981	0.986	0.988	0.991	0.994
7	0.935	0.946	0.952	0.958	0.964	0.971	0.979	0.985	0.986	0.989	0.992
8	0.938	0.948	0.953	0.959	0.964	0.971	0.979	0.984	0.985	0.988	0.991
9	0.941	0.951	0.955	0.960	0.965	0.971	0.978	0.983	0.984	0.987	0.990
10	0.944	0.953	0.957	0.961	0.966	0.972	0.978	0.983	0.984	0.987	0.989
11	0.947	0.955	0.959	0.963	0.968	0.972	0.979	0.983	0.984	0.987	0.989
12	0.949	0.957	0.961	0.964	0.968	0.973	0.979	0.983	0.984	0.987	0.989
13	0.951	0.958	0.962	0.966	0.969	0.974	0.979	9.983	0.984	0.987	0.989
14	0.954	0.960	0.964	0.967	0.971	0.975	0.980	0.984	0.985	0.987	0.989
15	0.956	0.962	0.965	0.968	0.972	0.976	0.980	0.984	0.985	0.988	0.989
16	0.957	0.963	0.966	0.969	0.973	0.976	0.981	0.984	0.985	0.988	0.990
17	0.959	0.965	0.967	0.970	0.973	0.977	0.982	0.985	0.986	0.988	0.990
18	0.960	0.966	0.969	0.971	0.974	0.978	0.982	0.985	0.986	0.988	0.990
19	0.962	0.967	0.969	0.972	0.975	0.978	0.982	0.985	0.986	0.988	0.990
20	0.963	0.968	0.970	0.973	0.976	0.979	0.983	0.986	0.986	0.989	0.990
21	0.964	0.969	0.971	0.974	0.976	0.979	0.983	0.986	0.987	0.989	0.990
22	0.965	0.970	0.972	0.975	0.977	0.980	0.983	0.986	0.987	0.989	0.990
23	0.966	0.971	0.973	0.975	0.978	0.981	0.984	0.987	0.987	0.989	0.990
24	0.967	0.972	0.974	0.976	0.978	0.981	0.984	0.987	0.988	0.989	0.991
25	0.968	0.972	0.975	0.977	0.979	0.981	0.985	0.987	0.988	0.990	0.991
26	0.969	0.973	0.975	0.977	0.979	0.982	0.985	0.987	0.988	0.990	0.991
27	0.970	0.974	0.976	0.978	0.980	0.982	0.985	0.988	0.988	0.990	0.991
28	0.971	0.974	0.976	0.978	0.980	0.983	0.986	0.988	0.989	0.990	0.991
29	0.971	0.975	0.977	0.979	0.981	0.983	0.986	0.988	0.989	0.990	0.992
30	0.972	0.976	0.977	0.979	0.981	0.983	0.986	0.988	0.989	0.990	0.992
31	0.973	0.976	0.978	0.980	0.982	0.984	0.987	0.989	0.989	0.991	0.992
32	0.973	0.977	0.978	0.980	0.982	0.984	0.987	0.989	0.989	0.991	0.992
33	0.974	0.977	0.979	9.981	0.982	0.984	0.987	0.989	0.990	0.991	0.992
34	0.975	0.978	0.979	0.981	0.983	0.985	0.987	0.989	0.990	0.991	0.992
35	0.975	0.978	9.980	0.981	0.983	0.985	0.988	0.989	0.990	0.991	0.992
36	0.976	0.979	0.980	0.982	0.983	0.985	0.988	0.990	0.990	0.991	0.992
37	0.976	0.979	0.981	0.982	0.984	0.986	0.988	0.990	0.990	0.992	0.993
38	0.977	0.979	9.981	0.982	0.984	0.986	0.988	0.990	0.991	0.992	0.993
39	0.977	0.980	0.981	0.983	0.984	0.986	0.988	0.990	0.990	0.992	0.993
40	0.978	0.980	0.982	0.983	0.985	0.986	0.989	0.990	0.991	0.992	0.993
41	0.978	0.981	0.982	0.983	0.985	0.987	0.989	0.990	0.991	0.992	0.993
42	0.978	0.981	0.982	0.984	0.985	0.987	0.989	0.991	0.991	0.992	0.993
43	0.979	0.981	0.983	0.984	0.985	0.987	0.989	0.991	0.991	0.992	0.993
44	0.979	0.982	0.983	0.984	0.986	0.987	0.989	0.991	0.991	0.992	0.993
45	0.979	0.982	0.983	0.984	0.986	0.987	0.989	0.991	0.991	0.993	0.993
46	0.980	0.982	0.984	0.985	0.986	0.988	0.990	0.991	0.991	0.993	0.993
47	0.980	0.983	0.984	0.985	0.986	0.988	0.990	0.991	0.992	0.993	0.994
48	0.980	0.983	0.984	0.985	0.987	0.988	0.990	0.991	0.992	0.993	0.994
49	0.981	0.983	0.984	0.985	0.987	0.988	0.990	0.991	0.992	0.993	0.994
50	0.981	0.983	0.984	0.986	0.987	0.988	0.990	0.992	0.992	0.993	0.994

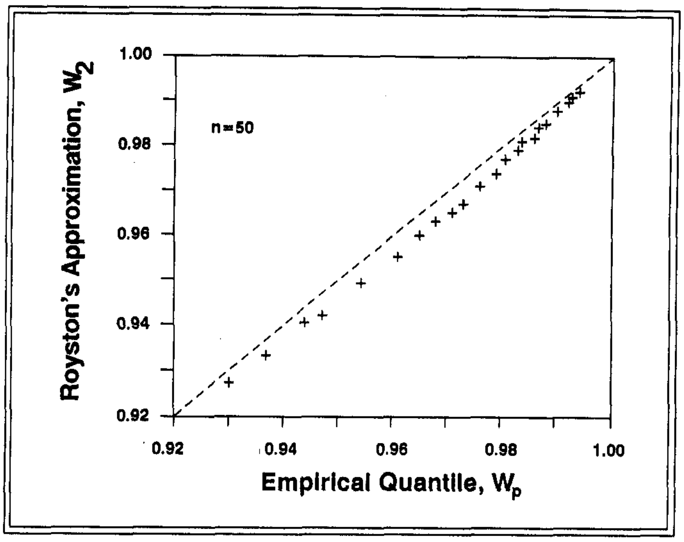

Figure 1 Quantile-quantile plot for Shapiro-Wilk W statistic based on empirical values (W_p) and on Royston's approximation (W_2), for sample size $n = 50$.

where W_p represents an empirical percentage point and Z_p represents that of the standard normal. The summations extend over $n = 7(1)50$ and $p = 0.005, 0.01, 0.02, 0.025, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.40, 0.50, 0.60, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 0.975, 0.98, 0.99$, and 0.995. The final estimates are given in Table 4. Using the estimated values, simulations were conducted to evaluate the first four moments of the the function $[(1 - W_p)^\tau - \mu]/\sigma$; results are given in Table 5. The moment values are in close agreement with those of the standard normal (i.e., 0,1,0,3) in all cases. Thus, percentage points of W are given approximately by

$$W_p \approx 1 - (\mu + \sigma Z_p)^{1/\tau}$$

for $7 \leq n \leq 50$.

For n greater than 20, percentage points developed using the present technique generally are closer to empirical values than are those based on Royston's method. In Table 6, percentage points calculated using the two approximation methods are summarized for $n = 10(10)50$. The table contains empirical percentage points (W_p), approximations based on the present method (W_1), and approximations based on Royston's method (W_2). Also, given in the table is the percent absolute difference (RE_1 and RE_2) between each approximation and the empirical values, expressed relative to the 99% empirical quasi-range. Small values of RE indicate closer agreement with empirical values. The values in Table 6 show that the two methods perform about equally well for smaller values of n. For larger values of n, the present method is uniformly better. In several cases, the difference in relative errors is more than 8%.

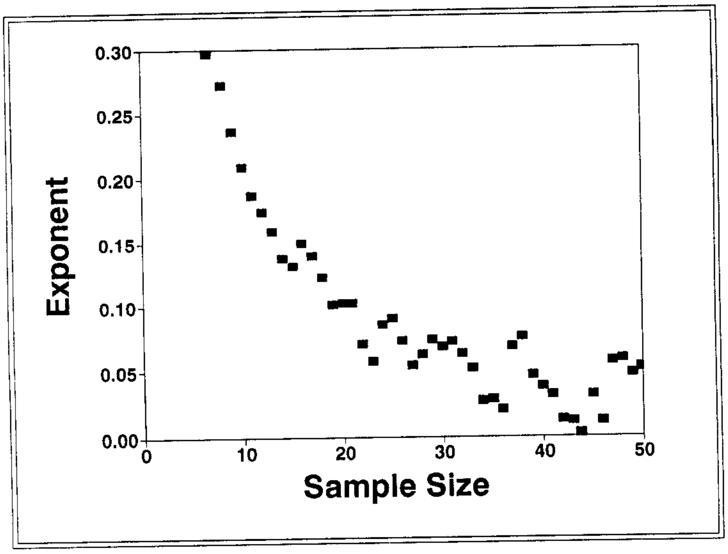

Figure 2 Plot of exponent versus sample size used in the normalizing power transformation $(1\text{-}W)^{\tau}$ derived from regression of empirical quantiles versus standard normal quantiles.

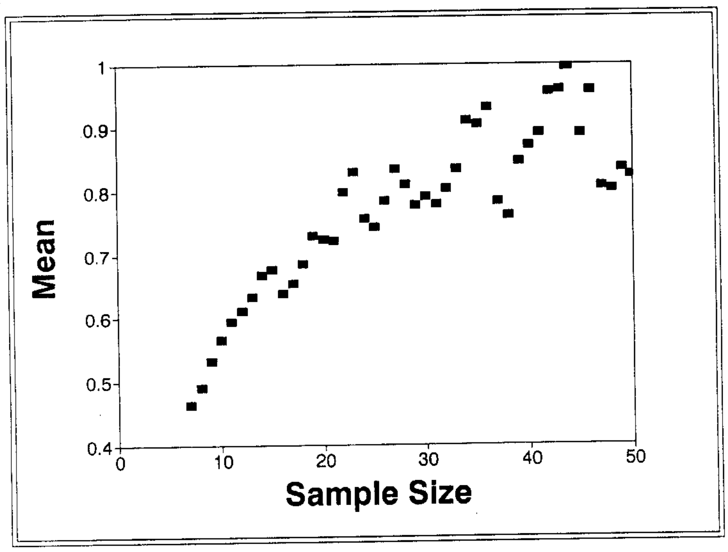

Figure 3 Plot of mean versus sample size used in the transformation $[(1 - W)^{\tau} - \mu]/\sigma$ derived from regression of empirical quantiles versus standard normal quantiles.

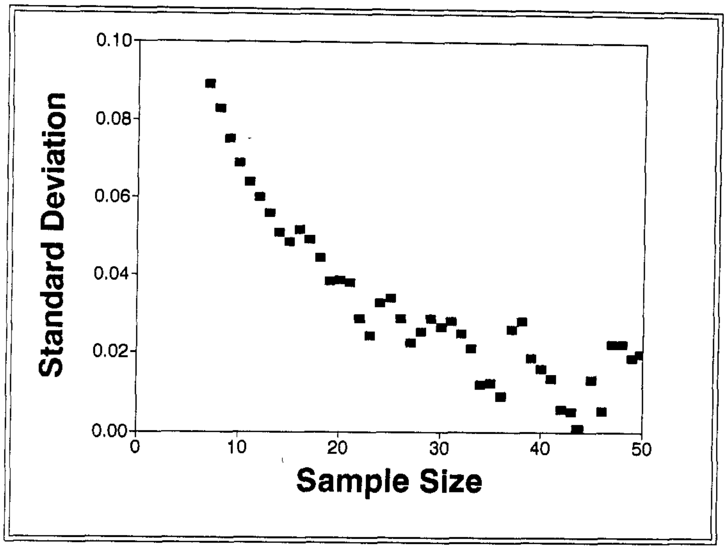

Figure 4 Plot of standard deviation versus sample size used in the transformation $[(1 - W)^{\gamma} - \mu]/\sigma$ derived from regression of empirical quantiles versus standard normal quantiles.

Table 4 Coefficients used in approximations based on normalizing transformations

$A_0 = 1.00317$	
$A_1 = 0.25318$	$C_1 = 0.05687$
$A_2 = 0.64930$	$C_2 = 0.06740$
$A_3 = 0.14761$	$C_3 = 0.07159$
$A_4 = 0.34797$	$C_4 = 0.07322$
$A_5 = 0.50182$	$C_5 = 0.20230$

Table 5 Moments of transformed *W* values obtained in a simulation of 10,000 samples from a standard normal distribution

n	μ	σ^2	$\beta_1^{1/2}$	β_2
10	0.004	1.006	−0.005	2.943
15	−0.004	0.992	0.002	3.020
20	0.000	0.992	−0.007	3.058
25	0.003	0.995	0.020	3.086
30	0.001	1.000	−0.021	3.122
35	0.007	0.998	−0.008	3.073
40	−0.003	0.994	−0.002	3.110
45	0.010	0.991	−0.011	3.006
50	0.003	0.985	0.013	3.082

Table 6 Selected empirical quantiles and approximations based on two methods

$n = 10$																						
p	0.005	0.010	0.020	0.025	0.050	0.100	0.150	0.200	0.250	0.300	0.400	0.500	0.600	0.700	0.750	0.800	0.850	0.900	0.950	0.975	0.980	0.990	0.995
W	0.760	0.786	0.812	0.821	0.845	0.871	0.886	0.897	0.906	0.913	0.925	0.935	0.944	0.953	0.957	0.961	0.966	0.972	0.978	0.983	0.984	0.987	0.989
W_1	0.762	0.787	0.812	0.819	0.844	0.869	0.885	0.896	0.905	0.912	0.924	0.935	0.944	0.952	0.957	0.961	0.966	0.971	0.978	0.983	0.984	0.987	0.990
W_2	0.759	0.785	0.812	0.820	0.846	0.872	0.887	0.898	0.907	0.914	0.927	0.937	0.946	0.954	0.958	0.962	0.967	0.972	0.978	0.983	0.984	0.987	0.989
RE_1	1.1	0.4	0.2	0.7	0.3	0.7	0.6	0.6	0.6	0.4	0.3	0.2	0.1	0.3	0.1	0.1	0.0	0.3	0.0	0.1	0.0	0.1	0.3
RE_2	0.4	0.2	0.2	0.5	0.3	0.3	0.5	0.6	0.5	0.7	0.7	0.7	0.7	0.4	0.5	0.6	0.4	0.1	0.1	0.1	0.1	0.0	0.1
$n = 20$																						
p	0.005	0.010	0.020	0.025	0.050	0.100	0.150	0.200	0.250	0.300	0.400	0.500	0.600	0.700	0.750	0.800	0.850	0.900	0.950	0.975	0.980	0.990	0.995
W	0.850	0.867	0.883	0.888	0.905	0.920	0.929	0.936	0.941	0.945	0.952	0.958	0.963	0.968	0.970	0.973	0.976	0.979	0.983	0.986	0.986	0.989	0.990
W_1	0.850	0.867	0.883	0.888	0.904	0.920	0.929	0.936	0.941	0.945	0.952	0.958	0.963	0.968	0.971	0.973	0.976	0.979	0.983	0.986	0.987	0.989	0.990
W_2	0.852	0.868	0.883	0.888	0.904	0.919	0.928	0.935	0.940	0.944	0.951	0.957	0.963	0.968	0.970	0.973	0.976	0.979	0.983	0.986	0.986	0.988	0.990
RE_1	0.2	0.1	0.0	0.1	0.7	0.2	0.1	0.3	0.2	0.1	0.2	0.1	0.3	0.2	0.6	0.2	0.0	0.1	0.0	0.2	0.4	0.4	0.1
RE_2	1.5	0.6	0.3	0.2	1.0	0.7	0.7	1.0	0.8	0.6	0.4	0.5	0.2	0.2	0.1	0.1	0.3	0.2	0.2	0.3	0.3	0.4	0.1
$n = 30$																						
p	0.005	0.010	0.020	0.025	0.050	0.100	0.150	0.200	0.250	0.300	0.400	0.500	0.600	0.700	0.750	0.800	0.850	0.900	0.950	0.975	0.980	0.990	0.995
W	0.892	0.904	0.916	0.919	0.930	0.942	0.948	0.953	0.956	0.959	0.964	0.969	0.972	0.976	0.977	0.979	0.981	0.983	0.986	0.988	0.989	0.990	0.992
W_1	0.891	0.903	0.915	0.918	0.930	0.941	0.948	0.952	0.956	0.959	0.964	0.969	0.972	0.976	0.978	0.979	0.981	0.983	0.986	0.988	0.989	0.990	0.992
W_2	0.891	0.902	0.913	0.917	0.928	0.939	0.945	0.950	0.954	0.957	0.962	0.967	0.971	0.974	0.976	0.978	0.980	0.983	0.986	0.988	0.988	0.990	0.991
RE_1	1.5	1.3	1.4	0.7	0.2	1.0	0.3	0.6	0.1	0.2	0.3	0.5	0.3	0.2	0.5	0.4	0.3	0.5	0.3	0.3	0.1	0.4	0.5
RE_2	0.9	1.7	2.6	2.1	2.1	3.1	2.5	2.8	2.0	1.9	1.6	2.3	1.4	1.7	0.8	0.9	0.8	0.5	0.5	0.2	0.6	0.0	0.7
$n = 40$																						
p	0.005	0.010	0.020	0.25	0.050	0.100	0.150	0.200	0.250	0.300	0.400	0.500	0.600	0.700	0.750	0.800	0.850	0.900	0.950	0.975	0.980	0.990	0.995
W	0.913	0.924	0.932	0.935	0.944	0.953	0.958	0.962	0.965	0.967	0.971	0.975	0.978	0.980	0.982	0.983	0.985	0.986	0.989	0.990	0.991	0.992	0.993
W_1	0.914	0.924	0.933	0.936	0.945	0.953	0.959	0.962	0.965	0.968	0.971	0.975	0.978	0.980	0.982	0.983	0.985	0.986	0.989	0.990	0.991	0.992	0.993
W_2	0.913	0.922	0.930	0.932	0.941	0.949	0.954	0.958	0.961	0.963	0.967	0.971	0.974	0.977	0.979	0.980	0.982	0.984	0.987	0.989	0.989	0.991	0.992
RE_1	1.3	0.6	1.0	0.9	0.8	0.5	0.7	0.3	0.1	0.7	0.6	0.3	0.4	0.5	0.3	0.2	0.4	0.5	0.5	0.3	0.4	0.2	0.3
RE_2	0.5	3.0	2.8	3.3	4.2	4.9	4.9	5.2	5.3	4.6	4.4	5.0	4.7	3.4	3.9	3.2	3.5	2.2	2.7	1.5	2.0	1.5	1.3
$n = 50$																						
p	0.005	0.010	0.020	0.025	0.050	0.100	0.150	0.200	0.250	0.300	0.400	0.500	0.600	0.700	0.750	0.800	0.850	0.900	0.950	0.975	0.980	0.990	0.995
W	0.930	0.937	0.944	0.947	0.954	0.961	0.965	0.968	0.971	0.973	0.976	0.979	0.981	0.983	0.984	0.986	0.987	0.988	0.990	0.992	0.992	0.993	0.994
W_1	0.928	0.936	0.944	0.946	0.953	0.961	0.965	0.968	0.971	0.973	0.976	0.979	0.981	0.983	0.985	0.986	0.987	0.988	0.990	0.992	0.992	0.993	0.994
W_2	0.927	0.933	0.940	0.942	0.949	0.955	0.960	0.963	0.965	0.967	0.971	0.974	0.977	0.979	0.981	0.982	0.984	0.985	0.988	0.990	0.990	0.991	0.992
RE_1	3.3	1.8	0.6	1.5	0.8	0.3	0.2	0.3	0.6	0.6	0.1	0.5	0.1	0.6	0.8	0.5	0.0	0.6	0.4	0.6	0.0	0.0	0.4
RE_2	4.6	5.6	6.4	7.9	8.5	8.8	8.5	8.4	9.2	9.0	8.2	8.1	7.0	5.9	5.3	6.2	5.3	4.1	3.5	3.8	3.0	2.5	2.4

Cumulative probabilities for W can be approximated according to

$$P(W \leq w) \approx P\left(Z \geq \frac{(1 - w)^{\tau} - \mu}{\sigma}\right)$$

for $7 \leq n \leq 50$, where Z is a standard normal variable.

5. SUMMARY

The tables of coefficients presented here were developed on the basis of expected values, variances, and covariances of normal order statistics that recently were computed to high precision. The availability of these values made it possible to compute to high precision the coefficients used in the Shapiro-Wilk W statistic. Previous values were of limited precision, and those for higher values of n had been based on approximations. Similarly, the first and half moments of W were computed. Percentage points were developed using simulation methodology with calculations of W being based on the higher precision coefficients. Simple functional approximations were developed for the power transformation coefficients and the corresponding mean and standard deviation that are used to compute percentage points of W. The resulting approximations for the percentage points were shown to be more precise than previously available values for sample sizes between 25 and 50. The present technique can be implimented as easily as the method based on polynomial approximations. Inasmuch as the general relationships for τ, μ, and σ (as functions of n) differ from those reported by Royston (1982), it would be of interest to investigate further these approximations for sample sizes above $n = 50$.

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APPENDIX

Computation of expected values, variances, and covariances of normal order statistics

Parrish (1992a) used the following approximations for computing the first two noncentral moments of the ith order statistic in a normal sample of size n. Numerical computations are based on Gauss-Legendre quadrature.

$$\begin{aligned} E[X_{i|n}^r] &= K_{in} \int_{-\infty}^{\infty} x^r f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i}\,dx \\ &= K_{in} \int_{-12.2}^{12.2} x^r f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i}\,dx + e_{in} \\ &\approx K_{in} \sum_{j=1}^{N} w_j x_j^r f(x_j)[F(x_j)]^{i-1}[1 - F(x_j)]^{n-i} \end{aligned}$$

for $r = 1,2$, where N is a predetermined number of integration points, x_i and w_i are Gaussian points and weights, respectively,

$$K_{in} = \frac{n!}{(n-i)!(i-1)!},$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2x^2},$$

and

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

This method provided more than 25 decimal places of precision for expected values and variances.

Covariance computations were based on a similar approach. Noncentral product moments for the ith and jth order statistics from a normal sample of size n may be written as follows.

$$\begin{aligned} E[X_{i|n}X_{j|n}] &= K_{ijn} \int_{-\infty}^{\infty} \int_{-\infty}^{y} xyf(x)f(y)[F(x)]^{i-1} \\ &\quad \times [1 - F(y)]^{n-j}[F(y) - F(x)]^{j-i-1}\,dx\,dy \\ &= K_{ijn} \sum_{r=0}^{j-i-1} \sum_{s=0}^{j-i-1-r} \frac{(-1)^{r+s}(j-i-1)!}{r!s!(j-i-1-r-s)!} \gamma_{i+r,n-j+s+1} \end{aligned}$$

where

$$K_{ijn} = \frac{n!}{(i-1)!(n-j)!(j-i-1)!},$$

$$\gamma_{i,j} = \frac{1}{ij}(\alpha_{i,j} + i\beta_{i-1,j} - \psi_{i,j}),$$

$$\alpha_{i,j} = \int_{-\infty}^{\infty} x[F(x)]^i[1 - F(x)]^j \, dx,$$

$$\beta_{i,j} = \int_{-\infty}^{\infty} x^2 f(x)[F(x)]^i[1 - F(x)]^j \, dx,$$

and

$$\Psi_{i,j} = \int_{-\infty}^{\infty} [F(x)]^i \int_x^{\infty} [1 - F(y)]^j \, dy \, dx;$$

see Godwin (1949). By using finite limits of -12.2 to 12.2 in each integral and applying Gauss-Legendre integration techniques, the following approximations were obtained.

$$\alpha_{i,j} \approx \sum_{k=1}^{N} w_k x_k [F(x_k)]^i[1 - F(x_k)]^j,$$

$$\beta_{i,j} \approx \sum_{k=1}^{N} w_k x_k^2 f(x_k)[F(x_k)]^i[1 - F(x_k)]^j,$$

$$\Psi_{i,j} \approx \sum_{k=1}^{N} \sum_{m=1}^{N} w_{km}[F(x_k)]^i[F(x_m)]^j.$$

These approximations enable the computation of product moments to high precision. Using these results, high-precision values for covariances can be obtained.