Chapter 3 Moments and Other Expected Values

3.1. Introduction

In the last chapter we gave the derivation of the distribution of an order statistic, the joint distribution of two order statistics, and also the distributions of some systematic statistics, such as the sample range, the sample quasi-range, etc. By making use of these results, we first present in this chapter some fundamental formulas that are necessary for the computation of single and product moments of order statistics. Next, we present some recurrence relations and identities satisfied by these moments of order statistics. Further, we consider some specific distributions such as the uniform, exponential, logistic, Weibull, gamma, normal, and half logistic, and discuss the evaluation of moments of order statistics in these cases. We also present some interesting distributional results involving order statistics from the uniform, exponential, and normal populations. Some recurrence relations for the single and product moments of order statistics are given for the exponential, logistic, gamma, normal, and half logistic cases and their uses are illustrated. Finally, we explain the David-Johnson (1954) approximation for the moments of order statistics from an arbitrary continuous distribution. This method of approximation, based on the probability integral transformation and the explicit expressions of the moments of uniform order statistics, provides a simple and practical method of approximating the single and the product moments of order statistics for large sample sizes. A detailed account of this and some other methods of approximation for the moments of order statistics may be found in the recent monograph on this topic by Arnold and Balakrishnan (1989).

3.2. Some Basic Formulas

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with pdf $f(x)$ and cdf $F(x)$, and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics obtained from the above sample. Let us denote the single moment $E(X_{i:n}^k)$ by $\alpha_{i:n}^{(k)}$. Then, from the density function of $X_{i:n}$ in (2.4.4), we have

$$\begin{aligned}\alpha_{i:n}^{(k)} = E(X_{i:n}^k) &= \int_{-\infty}^{\infty} x^k f_{i:n}(x)\, dx \\ &= \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} x^k \{F(x)\}^{i-1}\{1-F(x)\}^{n-i} f(x)\, dx, \\ &\qquad i = 1, 2, \ldots, n,\ k \geq 1. \qquad (3.2.1)\end{aligned}$$

For convenience, let us denote $\alpha_{i:n}^{(1)}$ simply by $\alpha_{i:n}$. Then from the first two single moments of $X_{i:n}$, we may compute the variance as

$$\beta_{i,i:n} = \operatorname{Var}(X_{i:n}) = \alpha_{i:n}^{(2)} - \alpha_{i:n}^2, \qquad 1 \leq i \leq n. \qquad (3.2.2)$$

Next, let us denote the product moment $E(X_{i:n}X_{j:n})$ by $\alpha_{i,j:n}$. Then, from the joint density function of $X_{i:n}$ and $X_{j:n}$ in (2.3.5), we have

$$\begin{aligned}\alpha_{i,j:n} &= E(X_{i:n}X_{j:n}) \\ &= \iint_{-\infty<x<y<\infty} xy f_{i,j:n}(x, y)\, dy\, dx \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \iint_{-\infty<x<y<\infty} xy\{F(x)\}^{i-1}\{F(y)-F(x)\}^{j-i-1} \\ &\quad \times \{1-F(y)\}^{n-j} f(x) f(y)\, dy\, dx, \qquad 1 \leq i < j \leq n. \qquad (3.2.3)\end{aligned}$$

For convenience, we may also sometimes denote $\alpha_{i:n}^{(2)}$ by $\alpha_{i,i:n}$. From the first two single moments and the above product moments, we may compute the covariance of $X_{i:n}$ and $X_{j:n}$ as

$$\beta_{i,j:n} = \operatorname{Cov}(X_{i:n}, X_{j:n}) = \alpha_{i,j:n} - \alpha_{i:n}\alpha_{j:n}, \qquad 1 \leq i < j \leq n. \qquad (3.2.4)$$

The formulas (3.2.1) and (3.2.3) will enable us to derive exact and explicit expressions for the single and product moments of order statistics, respectively, in most cases. When such an explicit algebraic integration is not possible, the formulas (3.2.1) and (3.2.3) may be used for purposes of numerical integration. For example, one may refer to Tietjen *et al.* (1977) for the case of the normal distribution.

3.3. Recurrence Relations and Identities

In this section we present some basic recurrence relations and identities satisfied by the single and the product moments, and also covariances of order statistics. All the results presented in this section for moments of order statistics from an arbitrary continuous distribution and many others have been listed and analyzed by Malik *et al.* (1988) in their recent expository review article on this topic, and similar results for some specific continuous distributions have been reviewed by Balakrishnan *et al.* (1988). Interested readers may also refer to Arnold and Balakrishnan (1989) for a compendium of all these results.

By using the identities

$$\sum_{i=1}^{n} X_{i:n}^k = \sum_{i=1}^{n} X_i^k \tag{3.3.1}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} X_{i:n}X_{j:n} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_iX_j \tag{3.3.2}$$

and taking expectation on both sides, we derive the identities

$$\sum_{i=1}^{n} \alpha_{i:n}^{(k)} = n\alpha_{1:1}^{(k)} = nE(X^k) \tag{3.3.3}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{i,j:n} = n\alpha_{1:1}^{(2)} + n(n-1)\alpha_{1:1}^2. \tag{3.3.4}$$

In particular, for $k=1$ and 2, (3.3.3) yields the identities

$$\sum_{i=1}^{n} \alpha_{i:n} = n\alpha_{1:1} = nE(X) \tag{3.3.5}$$

and

$$\sum_{i=1}^{n} \alpha_{i:n}^{(2)} = n\alpha_{1:1}^{(2)} = nE(X^2); \tag{3.3.6}$$

see also Hoeffding (1953). Upon using (3.3.6) in (3.3.4), we simply derive the identity

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \alpha_{i,j:n} = \binom{n}{2} \alpha_{1:1}^2. \tag{3.3.7}$$

Now, by making use of the fact that

$$\alpha_{1,2:2} = E(X_{1:2}X_{2:2}) = E(X_1X_2) = \alpha_{1:1}^2, \tag{3.3.8}$$

we may also rewrite the identity in (3.3.7) as (Govindarajulu, 1963)

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_{i,j:n} = \binom{n}{2} \alpha_{1,2:2}. \tag{3.3.9}$$

From (3.3.4) and (3.3.5), we also obtain the identity

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{i,j:n} &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j:n} - \left(\sum_{i=1}^{n} \alpha_{i:n}\right)\left(\sum_{j=1}^{n} \alpha_{j:n}\right)\\ &= n(\alpha_{1:1}^{(2)} - \alpha_{1:1}^2)\\ &= n\beta_{1,1:1} = n \operatorname{Var}(X).\end{aligned} \tag{3.3.10}$$

The identities in (3.3.3), (3.3.4), and (3.3.10) are very simple to use and, hence, can be applied effectively to check the accuracy of the computation of the single moments, product moments, and covariances of order statistics, respectively.

In addition to the above identities, the following recurrence relations are also satisfied by these quantities.

Relation 3.3.1. *For $i = 1, 2, \ldots, n-1$ and $k \geq 1$,*

$$i\alpha_{i+1:n}^{(k)} + (n-i)\alpha_{i:n}^{(k)} = n\alpha_{i:n-1}^{(k)}. \tag{3.3.11}$$

Proof. From Eq. (3.2.1) we have for $i = 1, 2, \ldots, n-1$

$$\begin{aligned} n\alpha_{i:n-1}^{(k)} &= \frac{n!}{(i-1)!(n-i-1)!} \int_{-\infty}^{\infty} x^k \{F(x)\}^{i-1}\{1-F(x)\}^{n-i-1} f(x)\, dx\\ &= \frac{n!}{(i-1)!(n-i-1)!} \int_{-\infty}^{\infty} x^k \{F(x)\}^{i-1}\{1-F(x)\}^{n-i-1}\\ &\quad \times [F(x) + 1 - F(x)] f(x)\, dx\\ &= \frac{n!}{(i-1)!(n-i-1)!} \left[\int_{-\infty}^{\infty} x^k \{F(x)\}^{i}\{1-F(x)\}^{n-i-1} f(x)\, dx\right.\\ &\quad \left. + \int_{-\infty}^{\infty} x^k \{F(x)\}^{i-1}\{1-F(x)\}^{n-i} f(x)\, dx\right]\\ &= i\alpha_{i+1:n}^{(k)} + (n-i)\alpha_{i:n}^{(k)}. \end{aligned}$$

It may be noted that Relation 3.3.1 just requires the value of the kth moment of a single order statistic in a sample of size n for the evaluation of the kth moment of all n order statistics, assuming that these moments in samples of size less than n are known. This relation was first derived by Cole (1951) and was proved for the case of discrete populations by Abdel-Aty (1954), Melnick (1964), and Balakrishnan (1986). Arnold and Meeden (1975) called Relation 3.3.1 "the triangle rule" and discussed some applications of it in the area of characterization of distributions. While Arnold (1977) has given extensions of this relation, Balakrishnan (1988b) has recently generalized it to the case in which the order statistics arise from n independent and nonidentically distributed random variables.

For even values of n, say $n = 2m$, by setting $i = m$ in Relation 3.3.1 we immediately obtain the relation

$$\tfrac{1}{2}\{\alpha^{(k)}_{m+1:2m} + \alpha^{(k)}_{m:2m}\} = \alpha^{(k)}_{m:2m-1}, \tag{3.3.12}$$

which, for the case $k = 1$, simply implies that the expected value of the median in a sample of even size $(2m)$ is exactly equal to the expected value of the median in a sample of odd size $(2m-1)$.

Relation 3.3.2. *For $i = 1, 2, \ldots, n-1$ and $k \geq 1$,*

$$\alpha^{(k)}_{i:n} = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{j} \binom{j-1}{i-1} \alpha^{(k)}_{j:j}. \tag{3.3.13}$$

Proof. By considering the expression of $\alpha^{(k)}_{i:n}$ in (3.2.1) and expanding the term $\{1 - F(x)\}^{n-i}$ in the integrand binomially in powers of $F(x)$, we get

$$\alpha^{(k)}_{i:n} = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \int_{-\infty}^{\infty} x^k \{F(x)\}^{i+j-1} f(x)\, dx,$$

which, when simplified, yields the relation in (3.3.13).

Relation 3.3.3. *For $i = 2, 3, \ldots, n$ and $k \geq 1$,*

$$\alpha^{(k)}_{i:n} = \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{n}{j} \binom{j-1}{n-i} \alpha^{(k)}_{1:j}. \tag{3.3.14}$$

Proof. By considering the expression of $\alpha^{(k)}_{i:n}$ in (3.2.1) and writing the term $\{F(x)\}^{i-1}$ in the integrand as $[1 - \{1 - F(x)\}]^{i-1}$, and then expanding it binomially in powers of $1 - F(x)$, we obtain

$$\alpha^{(k)}_{i:n} = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \int_{-\infty}^{\infty} x^k \{1 - F(x)\}^{n-i+j} f(x)\, dx,$$

which, when simplified, yields the relation in (3.3.14).

Relations 3.3.2 and 3.3.3 have been derived by Srikantan (1962) and Govindarajulu (1963), and both are quite useful as they express the kth moment of the ith order statistic in a sample of size n in terms of the kth moment of the largest and the smallest order statistics in samples of size n and less, respectively. These relations could have also been obtained by repeated application of Relation 3.3.1. Hence, as one would expect, we note from Relations 3.3.2 and 3.3.3 that we just require the value of the kth moment of a single order statistic (either the largest or the smallest) in a sample of size n for the evaluation of the kth moment of all n order statistics, given these moments in samples of size less than n. One needs to be somewhat careful, however, in applying Relations 3.3.2 and 3.3.3, as increasing values of n result in large combinatorial terms and hence in an error of large magnitude; Srikantan (1962) has investigated the cumulative rounding error propagated by the use of these two relations. Recently, Balakrishnan (1988b) has generalized Relations 3.3.2 and 3.3.3 to the case in which the order statistics arise from n independent and nonidentically distributed random variables.

In the case of certain specific distributions such as the gamma, Weibull, extreme value, and geometric, by noting that the moments $\alpha_{n:n}^{(k)}$ or $\alpha_{1:n}^{(k)}$ may be derived exactly, some authors have employed one of Relations 3.3.1–3.3.3 for computing the single moments of all the remaining order statistics; see, for example, Kimball (1947), Lieblein (1955), Gupta (1960), Margolin and Winokur (1967), and Saleh *et al.* (1975). These computations have often been checked by the identity in (3.3.3). This is not meaningful, since Balakrishnan and Malik (1986) have shown that the identity in (3.3.3) will be automatically satisfied if one of Relations 3.3.1–3.3.3 is used in the computational procedure. Let us demonstrate this point by considering Relation 3.3.2. By setting $i=1, i=2, \ldots, i=n-1$ in Relation 3.3.2 and then adding the resulting $n-1$ equations, we obtain

$$\sum_{i=1}^{n-1} \alpha_{i:n}^{(k)} = \binom{n}{1} \alpha_{1:1}^{(k)} + \sum_{r=2}^{n-1} (-1)^{r-1} \binom{n}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \alpha_{r:r}^{(k)}$$
$$+(-1)^{n-1} \sum_{r=0}^{n-2} (-1)^r \binom{n-1}{r} \alpha_{n:n}^{(k)}. \tag{3.3.15}$$

By making use of the combinatorial identities

$$\sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} = 0 \quad \text{and} \quad \sum_{r=0}^{n-2} (-1)^r \binom{n-1}{r} = (-1)^n$$

in (3.3.15), we get

$$\sum_{i=1}^{n-1} \alpha_{i:n}^{(k)} = n\alpha_{1:1}^{(k)} - \alpha_{n:n}^{(k)},$$

which simply reveals that the identity in (3.3.3) will be satisfied automatically. In other words, when we use the identity in (3.3.3) as a check, it will not discover any error in the starting calculations, i.e., in the value of $\alpha_{i:i}^{(k)}$ or $\alpha_{1:i}^{(k)}$ for $i=1,2,\ldots,n$. However, as rightly mentioned by Balakrishnan and Malik (1986), the identity in (3.3.3) is so simple to use that one might still apply it to check, at least partially, whether the other single moments have been computed correctly from the values of $\alpha_{i:i}^{(k)}$ or $\alpha_{1:i}^{(k)}$.

Relation 3.3.4. *For* $2\le i<j\le n$,

$$(i-1)\alpha_{i,j:n}+(j-i)\alpha_{i-1,j:n}+(n-j+1)\alpha_{i-1,j-1:n}=n\alpha_{i-1,j-1:n-1}. \tag{3.3.16}$$

Proof. From Eq. (3.2.3) we have for $2\le i<j\le n$

$$\begin{aligned}
n\alpha_{i-1,j-1:n-1}&\\
&=\frac{n!}{(i-2)!(j-i-1)!(n-j)!}\iint_{-\infty<x<y<\infty} xy\{F(x)\}^{i-2}\{F(y)-F(x)\}^{j-i-1}\\
&\quad\times\{1-F(y)\}^{n-j}f(x)f(y)\,dy\,dx\\
&=\frac{n!}{(i-2)!(j-i-1)!(n-j)!}\iint_{-\infty<x<y<\infty} xy\{F(x)\}^{i-2}\{F(y)-F(x)\}^{j-i-1}\\
&\quad\times\{1-F(y)\}^{n-j}[F(x)+\{F(y)-F(x)\}+\{1-F(y)\}]f(x)f(y)\,dy\,dx\\
&=\frac{n!}{(i-2)!(j-i-1)!(n-j)!}\Bigg[\iint_{-\infty<x<y<\infty} xy\{F(x)\}^{i-1}\\
&\quad\times\{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j}f(x)f(y)\,dy\,dx\\
&\quad+\iint_{-\infty<x<y<\infty} xy\{F(x)\}^{i-2}\{F(y)-F(x)\}^{j-i}\\
&\quad\times\{1-F(y)\}^{n-j}f(x)f(y)\,dy\,dx\\
&\quad+\iint_{-\infty<x<y<\infty} xy\{F(x)\}^{i-2}\{F(y)-F(x)\}^{j-i-1}\\
&\quad\times\{1-F(y)\}^{n-j+1}f(x)f(y)\,dy\,dx\Bigg]\\
&=(i-1)\alpha_{i,j:n}+(j-i)\alpha_{i-1,j:n}+(n-j+1)\alpha_{i-1,j-1:n}.
\end{aligned}$$

This relation was originally pointed out for the case of normal order statistics by Teichroew (1956) and was proved in general by Govindarajulu (1963). It may be noted that Relation 3.3.4 will enable us to compute all the product moments $\alpha_{i,j:n}$ $(1 \le i < j \le n)$ if we know $n-1$ suitably chosen moments; for example, the knowledge of the $n-1$ immediate upper-diagonal product moments, viz., $\alpha_{i,i+1:n}$ $(1 \le i \le n-1)$, will suffice. This important application of Relation 3.3.4 has been realized and utilized extensively for reducing the amount of numerical computation by several authors including Gupta (1960), Shah (1966), Saleh *et al.* (1975), Joshi (1982), Balakrishnan and Joshi (1984), Balakrishnan (1985), Balakrishnan and Puthenpura (1986), Ragab and Green (1984), and Balakrishnan *et al.* (1987). Relation 3.3.4 has been generalized by Balakrishnan (1988b) to the case in which the order statistics arise from n independent and nonidentically distributed random variables.

It is important to mention here that Balakrishnan and Malik (1986) have shown that the identity in (3.3.7) will be automatically satisfied if Relation 3.3.4 is used in the computational procedure. To see this, by setting $n=3$, $i=2$, and $j=3$ in (3.3.16) and then using (3.3.8), we obtain

$$\alpha_{1,2:3} + \alpha_{1,3:3} + \alpha_{2,3:3} = 3\alpha_{1,2:2} = 3\alpha_{1:1}^2,$$

which is precisely the identity in (3.3.7) for $n=3$. For a general n, by setting $i=2,3,\ldots,n-1$ and $j=i+1,\ i+2,\ldots,n$, and then adding the resulting $(n-2)(n-1)/2$ equations, we derive

$$(n-2)\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \alpha_{i,j:n} = n\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} \alpha_{i,j:n-1}, \tag{3.3.17}$$

which shows immediately that the identity in (3.3.7) will be automatically satisfied.

Relation 3.3.5. *For* $2 \le i < j \le n$,

$$\begin{aligned}(i-1)\beta_{i,j:n} &+ (j-i)\beta_{i-1,j:n} + (n-j+1)\beta_{i-1,j-1:n}\\ &= n\{\beta_{i-1,j-1:n-1} + (\mu_{i-1:n-1} - \mu_{i-1:n})(\mu_{j-1:n-1} - \mu_{j:n})\}.\end{aligned} \tag{3.3.18}$$

Proof. From Relation 3.3.4 we have for $2 \le i < j \le n$

$$\begin{aligned}(i-1)\beta_{i,j:n} &+ (j-i)\beta_{i-1,j:n} + (n-j+1)\beta_{i-1,j-1:n}\\ &= n\beta_{i-1,j-1:n-1} + n\alpha_{i-1:n-1}\alpha_{j-1:n-1} - (i-1)\alpha_{i:n}\alpha_{j:n}\\ &\quad -(j-i)\alpha_{i-1:n}\alpha_{j:n} - (n-j+1)\alpha_{i-1:n}\alpha_{j-1:n}.\end{aligned} \tag{3.3.19}$$

Now consider

$$
\begin{aligned}
(i-1)\alpha_{i:n}\alpha_{j:n} &+ (j-i)\alpha_{i-1:n}\alpha_{j:n} + (n-j+1)\alpha_{i-1:n}\alpha_{j-1:n} \\
&= \alpha_{j:n}\{(i-1)\alpha_{i:n} + (n-i+1)\alpha_{i-1:n}\} + (n-j+1)\alpha_{i-1:n}(\alpha_{j-1:n} - \alpha_{j:n}) \\
&= n\alpha_{i-1:n-1}\alpha_{j:n} + n\alpha_{i-1:n}(\alpha_{j-1:n-1} - \alpha_{j:n})
\end{aligned}
$$

by using (3.3.11). Upon substituting for this in the RHS of (3.3.19) and simplifying the resulting expression, we derive the relation in (3.3.18).

Relation 3.3.5 has been proved recently by Balakrishnan (1989c) and has also been generalized by him to the case in which the order statistics arise from n independent and nonidentically distributed random variables. In addition to the above results, several recurrence relations and identities are available in the literature for these moments of order statistics. A detailed description of all these results may be found in the recent monograph by Arnold and Balakrishnan (1989).

By a systematic application of several recurrence relations satisfied by the single and product moments of order statistics, Joshi and Balakrishnan (1982) have obtained upper bounds for the number of single and double integrals to be evaluated for the calculation of all means, variances, and covariances in a sample of size n, provided these are available in samples of sizes $n-1$ and less. Their result is summarized in the following theorem.

Theorem 3.3.1. *In order to determine the means, variances, and covariances of order statistics in a sample of size n drawn from an arbitrary distribution, given these quantities in samples of sizes $n-1$ and less, one has to evaluate at most two single integrals and $(n-2)/2$ double integrals if n is even, and two single integrals and $(n-1)/2$ double integrals if n is odd.*

For distributions that are symmetric about zero, it may be easily noted from Eq. (2.4.13) that

$$X_{i:n} \stackrel{d}{=} (-X)_{n-i+1:n}, \qquad 1 \le i \le n, \tag{3.3.20}$$

and

$$(X_{i:n}, X_{j:n}) \stackrel{d}{=} ((-X)_{n-j+1:n}, (-X)_{n-i+1:n}), \qquad 1 \le i < j \le n. \tag{3.3.21}$$

Therefore, for distributions which are symmetric about zero, we have the results

$$\alpha^{(k)}_{n-i+1:n} = (-1)^k \alpha^{(k)}_{i:n}, \qquad 1 \le i \le n, k \ge 1, \tag{3.3.22}$$

$$\alpha_{n-j+1,n-i+1:n} = \alpha_{i,j:n}, \qquad 1 \le i < j \le n, \tag{3.3.23}$$

and

$$\begin{aligned}\beta_{n-j+1,n-i+1:n} &= \alpha_{n-j+1,n-i+1:n} - \alpha_{n-j+1:n}\alpha_{n-i+1:n}\\ &= \alpha_{i,j:n} - \alpha_{j:n}\alpha_{i:n}\\ &= \beta_{i,j:n}, \qquad 1 \le i \le j \le n. \end{aligned} \tag{3.3.24}$$

The above results help reduce the number of computations to be done for the case in which the parent population distribution is symmetric about zero. These results also help us reduce the bounds given in Theorem 3.3.1 for the number of single and double integrals to be evaluated in order to compute the means, variances, and covariances of order statistics in a sample of size n. This result, due to Joshi (1971), is summarized in the following theorem.

Theorem 3.3.2. *In order to determine the means, variances, and covariances of order statistics in a sample of size n drawn from an arbitrary distribution symmetric about zero, given these quantities for all sample sizes less than n, one has to evaluate at most one single integral if n is even, and one single integral and $(n-1)/2$ double integrals if n is odd.*

The important point to be noted from Theorem 3.3.2 is that for even values of n there is no need to evaluate any double integral. This is quite apparent from a result of Joshi and Balakrishnan (1982), wherein they expressed the product moment $\alpha_{i,j:n}$, for the case in which the population distribution is symmetric about zero and n is even, explicitly in terms of the first single moments and the product moments of order statistics in samples of sizes $n-1$ and less.

3.4. Results for the Uniform Distribution

Let $U_{1:n} \le U_{2:n} \le \cdots \le U_{n:n}$ be the order statistics from a uniform $(0, 1)$ distribution. Then, from (3.2.1), we get for $1 \le i \le n$ and $k \ge 1$

$$\begin{aligned}\alpha_{i:n}^{(k)} &= E(U_{i:n}^k)\\ &= \frac{n!}{(i-1)!(n-i)!}\int_0^1 x^{k+i-1}(1-x)^{n-i}\,dx\\ &= \frac{n!}{(i-1)!}\,\frac{(k+i-1)!}{(n+k)!}\\ &= \frac{i(i+1)\cdots(i+k-1)}{(n+1)(n+2)\cdots(n+k)}. \end{aligned} \tag{3.4.1}$$

Thus, by setting $p_i = i/(n+1)$ and $q_i = 1 - p_i$, we find from (3.4.1) that

$$\alpha_{i:n} = E(U_{i:n}) = p_i \tag{3.4.2}$$

and

$$\begin{aligned}\beta_{i,i:n} &= \mathrm{Var}(U_{i:n}) \\ &= \frac{i(i+1)}{(n+1)(n+2)} - \frac{i^2}{(n+1)^2} \\ &= p_i q_i/(n+2). \end{aligned} \tag{3.4.3}$$

Similarly, from (3.2.3), we get for $1 \le i < j \le n$ and $k_1, k_2 \ge 1$

$$\begin{aligned}\alpha_{i,j:n}^{(k_1,k_2)} &= E(U_{i:n}^{k_1} U_{j:n}^{k_2}) \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \iint\limits_{0<x<y<1} x^{k_1+i-1}(y-x)^{j-i-1} \\ &\quad \times y^{k_2}(1-y)^{n-j}\, dx\, dy \\ &= \frac{n!}{(i-1)!} \frac{(k_1+i-1)!}{(k_1+j-1)!} \frac{(k_1+k_2+j-1)!}{(n+k_1+k_2)!} \\ &= \frac{i(i+1)\cdots(i+k_1-1)(j+k_1)(j+k_1+1)\cdots(j+k_1+k_2-1)}{(n+1)(n+2)\cdots(n+k_1+k_2)}.\end{aligned} \tag{3.4.4}$$

By setting $k_1 = k_2 = 1$ in (3.4.4), we immediately obtain

$$\alpha_{i,j:n} = \frac{i(j+1)}{(n+1)(n+2)}, \tag{3.4.5}$$

which, when used with (3.4.2), yields

$$\begin{aligned}\beta_{i,j:n} &= \mathrm{Cov}(U_{i:n}, U_{j:n}) \\ &= \frac{i(j+1)}{(n+1)(n+2)} - \frac{ij}{(n+1)^2} \\ &= p_i q_j/(n+2). \end{aligned} \tag{3.4.6}$$

Proceeding similarly, we derive for $1 \le i_1 < i_2 < i_3 < i_4 \le n$

$$\begin{aligned} E(U_{i_1:n}^{k_1} U_{i_2:n}^{k_2} U_{i_3:n}^{k_3} U_{i_4:n}^{k_4}) &= \frac{n!}{(i_1-1)!} \frac{(k_1+i_1-1)!}{(k_1+i_2-1)!} \frac{(k_1+k_2+i_2-1)!}{(k_1+k_2+i_3-1)!} \\ &\quad \times \frac{(k_1+k_2+k_3+i_3-1)!}{(k_1+k_2+k_3+i_4-1)!} \frac{(k_1+k_2+k_3+k_4+i_4-1)!}{(n+k_1+k_2+k_3+k_4)!}. \end{aligned} \tag{3.4.7}$$

The first four cumulants and cross-cumulants of uniform order statistics derived from (3.4.7) have been used by David and Johnson (1954) in developing some approximations for the first four cumulants and cross-cumulants of order statistics from an arbitrary continuous distribution. This method of approximation is discussed in detail in Section 3.10.

For $1 \le i_1 < i_2 < \cdots < i_\ell \le n$, David and Johnson (1954) have also derived the general expression

$$E\left(\prod_{j=1}^{\ell} U_{i_j:n}^{k_j}\right) = \frac{n!}{\left(n+\sum_{j=1}^{\ell} k_j\right)!} \prod_{j=1}^{\ell} \left\{\frac{(i_j+k_1+k_2+\cdots+k_j-1)!}{(i_j+k_1+k_2+\cdots+k_{j-1}-1)!}\right\}, \tag{3.4.8}$$

with k_0 taken as 0.

Instead of employing the above direct integration approach to evaluate the single moments and product moments of uniform order statistics, one may use the following theorem and simplify the process considerably.

Theorem 3.4.1. *For the uniform* $(0, 1)$ *distribution, the variables* $V_1 = U_{i:n}/U_{j:n}$ *and* $V_2 = U_{j:n}$, *for* $1 \le i < j \le n$, *are statistically independent, with* V_1 *and* V_2 *having* beta$(i, j-i)$ *and* beta$(j, n-j+1)$ *distributions, respectively.*

Proof. From Eq. (2.3.6), we have the joint density function of $U_{i:n}$ and $U_{j:n}$ $(1 \le i < j \le n)$ as

$$f_{i,j:n}(u_{i:n}, u_{j:n}) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} u_{i:n}^{i-1}(u_{j:n}-u_{i:n})^{j-i-1}(1-u_{j:n})^{n-j},$$
$$0 < u_{i:n} < u_{j:n} < 1. \tag{3.4.9}$$

By making the transformation

$$V_1 = \frac{U_{i:n}}{U_{j:n}}, \; V_2 = U_{j:n} \quad \Rightarrow \quad U_{i:n} = V_1 V_2, \qquad U_{j:n} = V_2$$

and realizing that the Jacobian of this transformation is V_2, we obtain the joint density function of V_1 and V_2 as

$$g(v_1, v_2) = \frac{(j-1)!}{(i-1)!(j-i-1)!} v_1^{i-1}(1-v_1)^{j-i-1} \frac{n!}{(j-1)!(n-j)!} v_2^{j-1}(1-v_2)^{n-j},$$
$$0 \le v_1, v_2 \le 1. \tag{3.4.10}$$

By a factorization theorem (see Rao, 1973), we observe from (3.4.10) that the variables V_1 and V_2 are statistically independent. Furthermore, we also immediately note that V_1 and V_2 have beta$(i, j-i)$ and beta$(j, n-j+1)$ distributions, respectively.

As mentioned earlier, we may use Theorem 3.4.1 to simplify the evaluation of product moments. For example, we may find

$$\begin{aligned}\alpha_{i,j:n}^{(k_1,k_2)} &= E(V_1^{k_1} V_2^{k_1+k_2})\\ &= E(V_1^{k_1})E(V_2^{k_1+k_2})\\ &= \frac{(j-1)!(k_1+i-1)!}{(i-1)!(k_1+j-1)!}\,\frac{n!(k_1+k_2+j-1)!}{(j-1)!(n+k_1+k_2)!},\end{aligned}$$

which is exactly the same as the expression derived in (3.4.4) by direct integration.

Theorem 3.4.1 can, in a straightforward manner, be generalized as follows.

Theorem 3.4.2. *For the uniform* $(0, 1)$ *distribution, the variables*

$$V_1 = \frac{U_{i_1:n}}{U_{i_2:n}},\ V_2 = \frac{U_{i_2:n}}{U_{i_3:n}}, \dots, V_{\ell-1} = \frac{U_{i_{\ell-1}:n}}{U_{i_\ell:n}},\ V_\ell = U_{i_\ell:n}$$

for $1 \le i_1 < i_2 < \cdots < i_\ell \le n$ *are all statistically independent, having* beta(i_1, i_2-i_1), beta$(i_2, i_3-i_2), \dots,$ beta$(i_{\ell-1}, i_\ell - i_{\ell-1})$, *and* beta$(i_\ell, n-i_\ell+1)$ *distributions, respectively.*

Now, by using Theorem 3.4.2 and writing

$$\begin{aligned}E\left(\prod_{j=1}^{\ell} U_{i_j:n}^{k_j}\right) &= E\left(\prod_{j=1}^{\ell} V_j^{k_1+k_2+\cdots+k_j}\right)\\ &= \prod_{j=1}^{\ell} E(V_j^{k_1+k_2+\cdots+k_j}),\end{aligned}$$

we derive the formula given in (3.4.8).

It is quite important to realize here that if $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ are the order statistics from an arbitrary continuous distribution with cdf $F(x)$, then all the formulas given in this section for the uniform order statistics $U_{i:n}$ continue to hold for the order statistics $F(X_{i:n})$. This is obvious from the fact that (see Section 2.4) $F(X_{i:n}) \stackrel{d}{=} U_{i:n}$ for $i = 1, 2, \dots, n$. Thus, for example, by combining this fact with the formula in (3.4.1), we simply obtain

$$E\{F(X_{i:n})\}^k = \frac{i(i+1)\cdots(i+k-1)}{(n+1)(n+2)\cdots(n+k)}. \tag{3.4.11}$$

In particular, by setting $i = 1$ in (3.4.11), we get

$$E\{F(X_{1:n})\}^k = k!/\{(n+1)(n+2)\cdots(n+k)\}. \tag{3.4.12}$$

3.5. Results for the Exponential Distribution

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics from the standard exponential distribution with pdf

$$f(x) = e^{-x}, \qquad 0 \leq x < \infty \tag{3.5.1}$$

and cdf

$$F(x) = 1 - e^{-x}, \qquad 0 \leq x < \infty. \tag{3.5.2}$$

Then, from (2.2.1) and (3.5.1), we have the joint density function of $X_{i:n}$ $(i = 1, 2, \ldots, n)$ as

$$\begin{aligned} &f_{1,2,\ldots,n:n}(x_{1:n}, x_{2:n}, \ldots, x_{n:n}) \\ &\qquad = n!\, e^{-\sum_{i=1}^{n} x_{i:n}}, \qquad 0 \leq x_{1:n} < x_{2:n} < \cdots < x_{n:n} < \infty. \end{aligned} \tag{3.5.3}$$

Let us now consider the transformation

$$Y_1 = nX_{1:n},\ Y_2 = (n-1)(X_{2:n} - X_{1:n}), \ldots,\ Y_n = X_{n:n} - X_{n-1:n},$$

or equivalently,

$$X_{1:n} = \frac{Y_1}{n},\ X_{2:n} = \frac{Y_1}{n} + \frac{Y_2}{n-1}, \ldots,\ X_{n:n} = \frac{Y_1}{n} + \frac{Y_2}{n-1} + \cdots + Y_n. \tag{3.5.4}$$

By noting that the Jacobian of the transformation in (3.5.4) is $1/n!$, we derive the joint density function of $Y_1, Y_2, \ldots, Y_n$ from (3.5.3) to be

$$g(y_1, y_2, \ldots, y_n) = e^{-\sum_{i=1}^{n} y_i}, \qquad 0 \leq y_1, y_2, \ldots, y_n < \infty. \tag{3.5.5}$$

By using the factorization theorem (Rao, 1973), we immediately observe that the variables $Y_1, Y_2, \ldots, Y_n$ are all statistically independent, and also that they all have standard exponential distributions. This result is due to Sukhatme (1937). (3.5.4) enables us, therefore, to write

$$X_{i:n} = \sum_{\ell=1}^{i} Y_\ell/(n - \ell + 1). \tag{3.5.6}$$

Thus, in (3.5.6), we have expressed the ith order statistic in a sample of size n from the standard exponential distribution in (3.5.1) as a linear function of i independent standard exponential random variables. As a result, we see immediately that the exponential order statistics form an additive Markov chain; for example, see Rényi (1953) or Karlin and Taylor (1975).

From the representation of the ith exponential order statistic given in (3.5.6), we obtain at once

$$\alpha_{i:n} = \sum_{\ell=1}^{i} E(Y_\ell)/(n-\ell+1) = \sum_{\ell=1}^{i} 1/(n-\ell+1), \qquad 1 \le i \le n, \quad (3.5.7)$$

$$\beta_{i,i:n} = \sum_{\ell=1}^{i} \operatorname{Var}(Y_\ell)/(n-\ell+1)^2 = \sum_{\ell=1}^{i} 1/(n-\ell+1)^2, \qquad 1 \le i \le n, \quad (3.5.8)$$

and for $1 \le i < j \le n$

$$\begin{aligned} \beta_{i,j:n} &= \sum_{\ell=1}^{i} \operatorname{Var}(Y_\ell)/(n-\ell+1)^2 \\ &= \sum_{\ell=1}^{i} 1/(n-\ell+1)^2 = \beta_{i,i:n}. \end{aligned} \qquad (3.5.9)$$

Higher order moments of $X_{i:n}$ may also be derived similarly from (3.5.6).

The following theorem, due to Malmquist (1950), gives a distributional result satisfied by the uniform order statistics by making use of the representation of the exponential order statistic given in (3.5.6).

Theorem 3.5.1. *Let $U_{1:n} \le U_{2:n} \le \cdots \le U_{n:n}$ be the order statistics from a uniform $(0, 1)$ distribution. Then, with $U_{n+1:n} \equiv 1$, the random variables*

$$V_i^* = \left\{ \frac{U_{i:n}}{U_{i+1:n}} \right\}^i, \qquad i = 1, 2, \ldots, n,$$

are all statistically independent, each having a uniform $(0, 1)$ distribution.

Proof. With $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denoting the exponential order statistics, since the random variable $X = -\ln U$ has a standard exponential distribution with density as in (3.5.1) and also that $-\ln u$ is a monotonically decreasing function in u, we have the relation

$$X_{i:n} = -\ln U_{n-i+1:n}, \qquad 1 \le i \le n. \qquad (3.5.10)$$

Hence, we obtain from (3.5.10) that for $1 \le i \le n$

$$\begin{aligned} V_i^* &= \left\{ \frac{U_{i:n}}{U_{i+1:n}} \right\}^i = \left\{ \frac{e^{-X_{n-i+1:n}}}{e^{-X_{n-i:n}}} \right\}^i \\ &= e^{-i(X_{n-i+1:n} - X_{n-i:n})} \\ &= e^{-Y_{n-i+1}} \end{aligned} \qquad (3.5.11)$$

upon using (3.5.6). As a result, we see that the variables V_i^*, $i = 1, 2, \ldots, n$, are all statistically independent. Further, since the Y_i's are standard exponential random variables, we also observe from (3.5.11) that the variables V_i^* all have a uniform $(0, 1)$ distribution. Hence, the theorem.

Several characterization results are available for the exponential distribution based on order statistics. Interested readers may refer to the monograph on this topic by Galambos and Kotz (1978).

In the following two theorems, we present some simple recurrence relations satisfied by the single and the product moments of order statistics from the exponential distribution. As pointed out by Joshi (1978), these recurrence relations are so easy to use that one can write a simple computer program to evaluate the first k single moments and the product moments of all order statistics without introducing serious rounding errors, at least up to moderately large sample sizes. Also, the method of derivation of these relations may be extended to the truncated exponential distributions; see, for example, Joshi (1979a), Joshi (1982), and Balakrishnan and Joshi (1984).

Theorem 3.5.2. *For the standard exponential distribution, we have*

$$\alpha_{1:n}^{(k)} = \frac{k}{n}\,\alpha_{1:n}^{(k-1)}, \qquad n \geq 1,\, k \geq 1 \tag{3.5.12}$$

and

$$\alpha_{i:n}^{(k)} = \alpha_{i-1:n-1}^{(k)} + \frac{k}{n}\,\alpha_{i:n}^{(k-1)}, \qquad 2 \leq i \leq n,\, k \geq 1, \tag{3.5.13}$$

with $\alpha_{i:n}^{(0)} \equiv 1$ *for* $1 \leq i \leq n$.

Proof. Let us consider the expression of $\alpha_{i:n}^{(k-1)}$ in (3.2.1). By noting that $f(x) = 1 - F(x)$ for the standard exponential distribution, we may write for $1 \leq i \leq n$ and $k \geq 1$

$$\alpha_{i:n}^{(k-1)} = \frac{n!}{(i-1)!(n-i)!} \int_0^\infty x^{k-1}\{F(x)\}^{i-1}\{1-F(x)\}^{n-i+1}\, dx. \tag{3.5.14}$$

Integrating the RHS of (3.5.14) by parts, treating x^{k-1} for integration and the rest of the integrand for differentiation, we get for $1 \leq i \leq n$ and $k \geq 1$

$$\alpha_{i:n}^{(k-1)} = \frac{n!}{(i-1)!(n-i)!k}\left[(n-i+1)\int_0^\infty x^k\{F(x)\}^{i-1}\{1-F(x)\}^{n-i} f(x)\, dx \right.$$
$$\left. -(i-1)\int_0^\infty x^k\{F(x)\}^{i-2}\{1-F(x)\}^{n-i+1} f(x)\, dx\right]. \tag{3.5.15}$$

The recurrence relation in (3.5.12) follows readily from (3.5.15) if i is set equal to 1. If we split the first integral on the RHS of (3.5.15) into two and combine with the second integral, (3.5.15) may be rewritten as

$$\alpha_{i:n}^{(k-1)} = \frac{n!}{(i-1)!(n-i)!k}\left[n\int_0^\infty x^k\{F(x)\}^{i-1}\{1-F(x)\}^{n-i}f(x)\,dx\right.$$

$$\left.-(i-1)\int_0^\infty x^k\{F(x)\}^{i-2}\{1-F(x)\}^{n-i}f(x)\,dx\right]. \qquad (3.5.16)$$

Eq. (3.5.16), when simplified, yields the relation in (3.5.13).

Theorem 3.5.3. *For the standard exponential distribution, we have*

$$\alpha_{i,i+1:n} = \alpha_{i:n}^{(2)} + \frac{1}{n-i}\alpha_{i:n}, \qquad 1 \le i \le n-1, \qquad (3.5.17)$$

and

$$\alpha_{i,j:n} = \alpha_{i,j-1:n} + \frac{1}{n-j+1}\alpha_{i:n}, \qquad 1 \le i < j \le n, j-i \ge 2. \qquad (3.5.18)$$

Proof. Let us first consider the expression of $f_{i,j:n}(x, y)$ in (2.3.5). As before, by using the fact that $f(y) = 1 - F(y)$ for the standard exponential distribution, we may write for $1 \le i < j \le n$

$$\alpha_{i:n} = E(X_{i:n}X_{j:n}^0)$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_0^\infty x\{F(x)\}^{i-1}I(x)f(x)\,dx, \qquad (3.5.19)$$

where

$$I(x) = \int_x^\infty \{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j+1}\,dy. \qquad (3.5.20)$$

Integrating the RHS of (3.5.20) by parts, treating dy for integration and the rest of the integrand for differentiation, we get for $j = i+1$

$$I(x) = (n-i)\int_x^\infty y\{1-F(y)\}^{n-i-1}f(y)\,dy - x\{1-F(x)\}^{n-i}, \qquad (3.5.21)$$

and for $1 \le i < j \le n$ and $j - i \ge 2$

$$I(x) = (n-j+1)\int_x^\infty y\{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j}f(y)\,dy$$

$$-(j-i-1)\int_x^\infty y\{F(y)-F(x)\}^{j-i-2}\{1-F(y)\}^{n-j+1}f(y)\,dy. \qquad (3.5.22)$$

By substituting the expressions of $I(x)$ in (3.5.21) and (3.5.22) in Eq. (3.5.19) and simplifying the resulting equation, we derive the recurrence relations given in (3.5.17) and (3.5.18).

It should be pointed out here that the relation in (3.5.17) itself is sufficient for the evaluation of all the product moments of order statistics, because the remaining product moments, viz., $\alpha_{i,j:n}$ for $1 \le i < j \le n$ and $j - i \ge 2$, can all be computed from the product moments $\alpha_{i,i+1:n}$ $(1 \le i \le n-1)$ by using Relation 3.3.4.

3.6. Results for the Logistic Distribution

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics from a logistic population with pdf

$$f(x) = e^{-x}/(1+e^{-x})^2, \qquad -\infty < x < \infty \tag{3.6.1}$$

and cdf

$$F(x) = 1/(1+e^{-x}), \qquad -\infty < x < \infty. \tag{3.6.2}$$

Then, from the expression of $f_{i:n}(x)$ in (2.4.4) we obtain the moment generating function of $X_{i:n}$ as

$$\begin{aligned} m_{i:n}(t) &= E(e^{tX_{i:n}}) \\ &= \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} e^{tx} \left\{\frac{1}{1+e^{-x}}\right\}^{i-1} \left\{\frac{e^{-x}}{1+e^{-x}}\right\}^{n-i} \frac{e^{-x}}{(1+e^{-x})^2}\, dx. \end{aligned} \tag{3.6.3}$$

By making the substitution $u = 1/(1+e^{-x})$ in the integral on the RHS of (3.6.3), we get

$$\begin{aligned} m_{i:n}(t) &= \frac{n!}{(i-1)!(n-i)!} \int_0^1 \left(\frac{u}{1-u}\right)^t u^{i-1}(1-u)^{n-i}\, du \\ &= \frac{n!}{(i-1)!(n-i)!} \int_0^1 u^{i+t-1}(1-u)^{n-i-t}\, du \\ &= \Gamma(i+t)\Gamma(n-i+1-t)/\{\Gamma(i)\Gamma(n-i+1)\}. \end{aligned} \tag{3.6.4}$$

From the moment generating function of $X_{i:n}$ in (3.6.4), we derive

$$\begin{aligned} \alpha_{i:n} &= \frac{d}{dt} m_{i:n}(t)\big|_{t=0} \\ &= \frac{\Gamma'(i)}{\Gamma(i)} - \frac{\Gamma'(n-i+1)}{\Gamma(n-i+1)} \\ &= \psi(i) - \psi(n-i+1), \end{aligned} \tag{3.6.5}$$

where $\psi(z)=d/dz \ln\Gamma(z)=\Gamma'(z)/\Gamma(z)$ is the psi (or digamma) function. Similarly, we derive

$$\begin{aligned}\alpha_{i:n}^{(2)}&=\frac{d^2}{dt^2}\,m_{i:n}(t)\big|_{t=0}\\&=\frac{\Gamma''(i)}{\Gamma(i)}-2\frac{\Gamma'(i)}{\Gamma(i)}\frac{\Gamma'(n-i+1)}{\Gamma(n-i+1)}+\frac{\Gamma''(n-i+1)}{\Gamma(n-i+1)}\\&=[\psi'(i)+\{\psi(i)\}^2]-2\psi(i)\psi(n-i+1)+[\psi'(n-i+1)+\{\psi(n-i+1)\}^2]\\&=\psi'(i)+\psi'(n-i+1)+[\psi(i)-\psi(n-i+1)]^2,\end{aligned}\tag{3.6.6}$$

where $\psi'(z)=d/dz\,\psi(z)=d^2/dz^2\ln\Gamma(z)$ is the derivative of the psi function and commonly termed as the trigamma function. From (3.6.6) and (3.6.5), we immediately derive the variance of $X_{i:n}$ as

$$\begin{aligned}\beta_{i,i:n}&=\alpha_{i:n}^{(2)}-\alpha_{i:n}^2\\&=\psi'(i)+\psi'(n-i+1).\end{aligned}\tag{3.6.7}$$

These expressions have been derived by Birnbaum and Dudman (1963), Gupta and Shah (1965), and Tarter and Clark (1965). From the formulas of the mean and the variance of $X_{i:n}$ in (3.6.5) and (3.6.7), respectively, we observe that

$$\alpha_{i:n}=-\alpha_{n-i+1:n}\quad\text{and}\quad\beta_{i,i:n}=\beta_{n-i+1,n-i+1:n}.$$

This is because of the symmetry of the logistic distribution (see Section 3.3).

Proceeding similarly, Gupta and Shah (1965) and Shah (1966) have derived an expression for the product moment of $X_{i:n}$ and $X_{j:n}$ as

$$\begin{aligned}\alpha_{i,j:n}=\alpha_{j:n}^{(2)}&+\sum_{r=i}^{j-1}\sum_{s=1}^{r-1}\left\{(-1)^{r+i}\binom{r-1}{i-1}\binom{n}{r}\binom{j-i-r+s}{s}B(s,n-r+1)\right.\\&\left.\times\alpha_{j+s-r:n+s-r}\right\}+\binom{n}{r}\sum_{r=0}^{j-i-1}\left[(-1)^r\binom{n-i}{r}\frac{1}{i+r}\{-\psi'(n-j+1)\right.\\&\left.+(\psi(n-j+1)-\psi(n-i-r+1))(\psi(j-i-r)-\psi(n-j+1))\}\right].\end{aligned}\tag{3.6.8}$$

The digamma and trigamma functions involved in the formulas in (3.6.5)–(3.6.8) have been computed rather extensively; for example, see Abramowitz and Stegun (1965) or Davis (1935). Reference may also be made to Bernardo (1976) and Schneider (1978) for Fortran programs for the computation of these functions. By following an exactly similar approach, Balakrishnan and Leung (1988a) have derived expressions for the means, variances, and covariances of order statistics from a Type I generalized logistic distribution.

In the following two theorems, we present several recurrence relations satisfied by the single and the product moments of logistic order statistics. These relations, established by Shah (1966, 1970), will enable one to compute the first k single moments and the product moments of all order statistics in a very simple recursive way without accumulating serious rounding errors. Recently, Balakrishnan and Malik (1990) have successfully employed this recursive procedure and tabulated the means, variances, and covariances of logistic order statistics for sample sizes up to 50.

Theorem 3.6.1. *For the logistic population with pdf as in* (3.6.1), *we have*

$$\alpha_{i+1:n+1}^{(k)} = \alpha_{i:n}^{(k)} + \frac{k}{i}\,\alpha_{i:n}^{(k-1)}, \qquad 1 \le i \le n,\ k \ge 1, \tag{3.6.9}$$

with $\alpha_{i:n}^{(0)} \equiv 1$ *for* $1 \le i \le n$.

Proof. Let us first consider the expression of $\alpha_{i:n}^{(k-1)}$ in (3.2.1). By realizing that $f(x) = F(x)\{1-F(x)\}$ for the logistic population, we may write for $1 \le i \le n$ and $k \ge 1$

$$\alpha_{i:n}^{(k-1)} = \frac{n!}{(i-1)!(n-i)!}\int_{-\infty}^{\infty} x^{k-1}\{F(x)\}^{i}\{1-F(x)\}^{n-i+1}\,dx. \tag{3.6.10}$$

Integrating the RHS of (3.6.10) by parts, treating x^{k-1} for integration and the rest of the integrand for differentiation, we obtain for $1 \le i \le n$ and $k \ge 1$

$$\alpha_{i:n}^{(k-1)} = \frac{n!}{(i-1)!(n-i)!k}\left[(n-i+1)\int_{-\infty}^{\infty} x^{k}\{F(x)\}^{i}\{1-F(x)\}^{n-i}f(x)\,dx \right.$$
$$\left. -\,i\int_{-\infty}^{\infty} x^{k}\{F(x)\}^{i-1}\{1-F(x)\}^{n-i+1}f(x)\,dx\right]. \tag{3.6.11}$$

If we split the first integral on the RHS of (3.6.11) into two and combine with the second integral, (3.6.11) may be rewritten as

$$\alpha_{i:n}^{(k-1)} = \frac{n!}{(i-1)!(n-i)!k}\left[(n+1)\int_{-\infty}^{\infty} x^{k}\{F(x)\}^{i}\{1-F(x)\}^{n-i}f(x)\,dx \right.$$
$$\left. -\,i\int_{-\infty}^{\infty} x^{k}\{F(x)\}^{i-1}\{1-F(x)\}^{n-i}f(x)\,dx\right]. \tag{3.6.12}$$

Eq. (3.6.12), when simplified, yields the relation in (3.6.9).

Theorem 3.6.1 will allow one to calculate all the moments $\alpha_{i:n+1}^{(k)}$ $(2 \le i \le n+1)$ starting from $\alpha_{1:1}^{(k)}$ in a very simple recursive manner. Note that because of the symmetry of the logistic distribution about zero, we simply have $\alpha_{1:n+1}^{(k)} = (-1)^k \alpha_{n+1:n+1}^{(k)}$ and hence $\alpha_{1:n+1}^{(k)}$ will also be known. Thus, for example, starting with $\alpha_{1:1} = 0$ and $\alpha_{1:1}^{(2)} = \pi^2/3$, one can employ Theorem 3.6.1 to compute the means and variances of all order statistics.

Theorem 3.6.2. *For the logistic population with pdf as in* (3.6.1), *we have*

$$\alpha_{i,i+1:n+1} = \frac{n+1}{n-i+1}\left[\alpha_{i,i+1:n} - \frac{i}{n+1}\alpha_{i+1:n+1}^{(2)} - \frac{1}{n-i}\alpha_{i:n}\right],$$

$$1 \le i \le n-1, \tag{3.6.13}$$

and

$$\alpha_{i,j:n+1} = \frac{n+1}{n-j+2}\left[\alpha_{i,j:n} - \alpha_{i,j-1:n} + \frac{n-j+2}{n+1}\alpha_{i,j-1:n+1} - \frac{1}{n-j+1}\alpha_{i:n}\right],$$

$$1 \le i < j \le n, j-i \ge 2. \tag{3.6.14}$$

Proof. By considering the expression of $f_{i,j:n}(x, y)$ in (2.3.5) and using the fact that $f(y) = F(y)\{1-F(y)\}$ for the logistic population with pdf as in (3.6.1), we may write for $1 \le i < j \le n$

$$\alpha_{i:n} = E(X_{i:n}X_{j:n}^0)$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_{-\infty}^{\infty} x\{F(x)\}^{i-1} I(x) f(x)\, dx, \tag{3.6.15}$$

where

$$I(x) = \int_x^{\infty} \{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j+1} F(y)\, dy. \tag{3.6.16}$$

By writing $F(y)$ as $1-\{1-F(y)\}$ and splitting the integral in (3.6.16) accordingly into two, we get

$$I(x) = \int_x^{\infty} \{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j+1}\, dy$$

$$-\int_x^{\infty} \{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j+2}\, dy. \tag{3.6.17}$$

Integrating the RHS of (3.6.17) by parts treating dy for integration, we obtain for $j=i+1$

$$I(x)=\left[(n-i)\int_x^\infty y\{1-F(y)\}^{n-i-1}f(y)\,dy-x\{1-F(x)\}^{n-i}\right]$$
$$-\left[(n-i+1)\int_x^\infty y\{1-F(y)\}^{n-i}f(y)\,dy-x\{1-F(x)\}^{n-i+1}\right], \tag{3.6.18}$$

and for $j-i\geq 2$

$$I(x)=\left[(n-j+1)\int_x^\infty y\{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j}f(y)\,dy\right.$$
$$\left.-(j-i-1)\int_x^\infty y\{F(y)-F(x)\}^{j-i-2}\{1-F(y)\}^{n-j+1}f(y)\,dy\right]$$
$$-\left[(n-j+2)\int_x^\infty y\{F(y)-F(x)\}^{j-i-1}\{1-F(y)\}^{n-j+1}f(y)\,dy\right.$$
$$\left.-(j-i-1)\int_x^\infty y\{F(y)-F(x)\}^{j-i-2}\{1-F(y)\}^{n-j+2}f(y)\,dy\right]. \tag{3.6.19}$$

Substitution of the expression for $I(x)$ from (3.6.18) in Eq. (3.6.15) gives for $1\leq i\leq n-1$

$$\alpha_{i:n}=(n-i)\left[\alpha_{i,i+1:n}-\alpha_{i:n}^{(2)}+\frac{(n-i+1)}{(n+1)}\alpha_{i:n+1}^{(2)}-\frac{(n-i+1)}{(n+1)}\alpha_{i,i+1:n+1}\right]. \tag{3.6.20}$$

Now, by using the result

$$(n-i+1)\alpha_{i:n+1}^{(2)}-(n+1)\alpha_{i:n}^{(2)}=-i\alpha_{i+1:n+1}^{(2)}$$

obtained from Relation 3.3.1 in Eq. (3.6.20) and simplifying the resulting expression, we derive the recurrence relation in (3.6.13). Similarly, by substituting the expression for $I(x)$ from (3.6.19) in Eq. (3.6.15) and simplifying the resulting expression, we derive the recurrence relation in (3.6.14).

It should be mentioned here that the relation in (3.6.13) itself is sufficient for the evaluation of all the product moments of order statistics. To see this, we first note from (3.3.8) that

$$\alpha_{1,2:2} = \alpha_{1:1}^2 = 0 \tag{3.6.21}$$

and that

$$\alpha_{n,n+1:n+1} = \alpha_{1,2:n+1} \tag{3.6.22}$$

because of the symmetry of the logistic distribution. Thus, the relation in (3.6.13), when used along with (3.6.21) and (3.6.22), will enable one to calculate the product moments $\alpha_{i,i+1:n+1}$ $(1 \leq i \leq n)$. This is sufficient for the evaluation of all the product moments, because the remaining product moments, viz., $\alpha_{i,j:n+1}$ for $1 \leq i < j \leq n+1$ and $j-i \geq 2$, can all be systematically computed by using Relation 3.3.4.

By proceeding on exactly similar lines, Balakrishnan and Joshi (1983) and Balakrishnan and Kocherlakota (1986) have derived several recurrence relations satisfied by the single and the product moments of order statistics from the truncated logistic distributions. Explicit finite series expressions for the moments of order statistics have been derived in this case by Tarter (1966).

3.7. Results for the Gamma Distribution

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics from a gamma population with pdf

$$f(x) = \frac{1}{\Gamma(\rho)} e^{-x} x^{\rho-1}, \qquad 0 \leq x < \infty, \rho > 0. \tag{3.7.1}$$

The standard exponential distribution considered in Section 3.5 is observed to be a particular case (when $\rho = 1$) of the gamma distribution in (3.7.1).

Let us now consider the case when ρ is an integer and present the formulas that have been derived by Gupta (1960, 1962) for the computation of the single and the product moments of order statistics. First of all, we note easily that when ρ is an integer, the cumulative distribution function $F(x)$ of the gamma population can be written as a partial sum of the probabilities in a Poisson distribution, viz.,

$$F(x) = \sum_{\ell=\rho}^{\infty} e^{-x} x^{\ell} / \ell!, \qquad x \geq 0. \tag{3.7.2}$$

Now, from (3.2.1) we have for $1 \leq i \leq n$ and $k \geq 1$

$$\alpha_{i:n}^{(k)}=\frac{n!}{(i-1)!(n-i)!}\int_0^\infty x^k\{F(x)\}^{i-1}\{1-F(x)\}^{n-i}f(x)\,dx$$
$$=\frac{n!}{(i-1)!(n-i)!}\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\int_0^\infty x^k\{1-F(x)\}^{n-i+r}f(x)\,dx. \tag{3.7.3}$$

By using the expressions of $f(x)$ and $F(x)$ in (3.7.1) and (3.7.2), respectively, in Eq. (3.7.3), we obtain for $1\le i\le n$ and $k\ge 1$

$$\alpha_{i:n}^{(k)}=\frac{n!}{(i-1)!(n-i)!}\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\int_0^\infty x^k\left\{\sum_{\ell=0}^{\rho-1}e^{-x}x^\ell/\ell!\right\}^{n-i+r}$$
$$\times\frac{1}{\Gamma(\rho)}e^{-x}x^{\rho-1}\,dx$$
$$=\frac{n!}{(i-1)!(n-i)!}\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\frac{1}{\Gamma(\rho)}\int_0^\infty e^{-(n-i+r+1)x}$$
$$\times x^{k+\rho-1}\left\{\sum_{\ell=0}^{\rho-1}x^\ell/\ell!\right\}^{n-i+r}dx$$
$$=\frac{n!}{(i-1)!(n-i)!\Gamma(\rho)}\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\sum_{s=0}^{(\rho-1)(n-i+r)}a_s(\rho,n-i+r)$$
$$\times\int_0^\infty e^{-(n-i+r+1)x}x^{k+\rho+s-1}\,dx$$
$$=\frac{n!}{(i-1)!(n-i)!\Gamma(\rho)}\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\sum_{s=0}^{(\rho-1)(n-i+r)}a_s(\rho,n-i+r)$$
$$\times\Gamma(k+\rho+s)/(n-i+r+1)^{k+\rho+s}, \tag{3.7.4}$$

where $a_s(\rho, n-i+r)$ is the coefficient of x^s in the expansion of

$$\left\{\sum_{\ell=0}^{\rho-1}x^\ell/\ell!\right\}^{n-i+r}.$$

The kth moment of the smallest order statistic may be obtained from (3.7.4) by setting $i=1$ to be

$$\alpha_{1:n}^{(k)}=\frac{n}{\Gamma(\rho)}\sum_{s=0}^{(\rho-1)(n-1)}a_s(\rho,n-1)\Gamma(k+\rho+s)n^{-(k+\rho+s)}. \tag{3.7.5}$$

The coefficients $a_s(\rho, n-1)$ can be generated very easily as follows. First of all, we see that

$$a_0(\rho,1)=1,\ a_1(\rho,1)=1,\ a_2(\rho,1)=\frac{1}{2!},\ \ldots,\ a_{\rho-1}(\rho,1)=\frac{1}{(\rho-1)!}. \tag{3.7.6}$$

Next, let us consider $a_s(\rho, m)$ for $m \geq 2$, given by

$$\begin{aligned}
a_s(\rho, m) &= \text{coefficient of } x^s \text{ in } \left\{\sum_{\ell=0}^{\rho-1} x^\ell/\ell!\right\}^m \\
&= \sum_{\ell=0}^{\rho-1} \left[\text{coefficient of } x^\ell \text{ in } \left\{\sum_{\ell=0}^{\rho-1} x^\ell/\ell!\right\}\right] \\
&\quad \times \left[\text{coefficient of } x^{s-\ell} \text{ in } \left\{\sum_{\ell=0}^{\rho-1} x^\ell/\ell!\right\}^{m-1}\right] \\
&= \sum_{\ell=0}^{\rho-1} \frac{1}{\ell!} a_{s-\ell}(\rho, m-1). \qquad (3.7.7)
\end{aligned}$$

Thus, by starting with the values of $a_s(\rho, 1)$ given in (3.7.6), we can compute the coefficients $a_s(\rho, m)$ for any value of m by repeated application of the recurrence relation in (3.7.7). After computing the coefficients $a_s(\rho, m)$ this way, one may either directly use the formula in (3.7.4) to compute the single moments $\alpha_{i:n}^{(k)}$, or simply compute the moments $\alpha_{1:n}^{(k)}$ from (3.7.5) and then employ Relation 3.3.3 to compute the single moments of other order statistics.

By using the above formulas, Gupta (1960, 1962) has tabulated the first four moments of all order statistics for sample sizes up to 10 when $\rho = 1(1)5$ and for the smallest order statistic up to sample size 15. Breiter and Krishnaiah (1968) have provided similar tables for sample sizes up to 9 when $\rho = 0.5(1)10.5$. Harter (1970b) has tabulated just the means of all order statistics for sample sizes 40 and less when $\rho = 0.5(0.5)4.0$.

From (3.2.3) we have for $1 \leq i < j \leq n$

$$\begin{aligned}
\alpha_{i,j:n} &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \int_0^\infty \int_x^\infty xy\{F(x)\}^{i-1}\{F(y)-F(x)\}^{j-i-1} \\
&\quad \times \{1-F(y)\}^{n-j} f(x) f(y)\, dy\, dx \\
&= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s} \\
&\quad \times \int_0^\infty \int_x^\infty xy\{1-F(x)\}^{j-i-1-s+r}\{1-F(y)\}^{n-j+s} f(x) f(y)\, dy\, dx.
\end{aligned} \qquad (3.7.8)$$

By using the expressions of the pdf and the cdf in (3.7.1) and (3.7.2), respectively, in Eq. (3.7.8), we obtain for integer values of ρ and $1 \le i < j \le n$

$$\alpha_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s}$$
$$\times \int_0^\infty x \left\{ \sum_{\ell=0}^{\rho-1} e^{-x} x^\ell / \ell! \right\}^{j-i-1-s+r} \frac{1}{\Gamma(\rho)} e^{-x} x^{\rho-1}$$
$$\times \left[\int_x^\infty y \left\{ \sum_{\ell=0}^{\rho-1} e^{-y} y^\ell / \ell! \right\}^{n-j+s} \frac{1}{\Gamma(\rho)} e^{-y} y^{\rho-1}\, dy \right] dx.$$
$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!\{\Gamma(\rho)\}^2} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s}$$
$$\times \sum_{t_1=0}^{(\rho-1)(j-i-1-s+r)} a_{t_1}(\rho, j-i-1-s+r) \sum_{t_2=0}^{(\rho-1)(n-j+s)} a_{t_2}(\rho, n-j+s)$$
$$\times \int_0^\infty e^{-(j-i-s+r)x} x^{t_1+\rho} \left[\int_x^\infty e^{-(n-j+s+1)y} y^{t_2+\rho}\, dy \right] dx. \tag{3.7.9}$$

Let us consider the integral

$$I(x) = \int_x^\infty e^{-(n-j+s+1)y} y^{t_2+\rho}\, dy.$$

By making the transformation $u = (n-j+s+1)y$, we get

$$I(x) = \frac{1}{(n-j+s+1)^{t_2+\rho+1}} \int_{(n-j+s+1)x}^\infty e^{-u} u^{(t_2+\rho+1)-1}\, du,$$

which, when used with (3.7.2), gives

$$I(x) = \frac{\Gamma(t_2+\rho+1)}{(n-j+s+1)^{t_2+\rho+1}} \sum_{\ell=0}^{t_2+\rho} e^{-(n-j+s+1)x} \frac{(n-j+s+1)^\ell x^\ell}{\ell!}$$
$$= \Gamma(t_2+\rho+1) \sum_{\ell=0}^{t_2+\rho} e^{-(n-j+s+1)x} x^\ell / \{(n-j+s+1)^{t_2+\rho+1-\ell} \ell!\}. \tag{3.7.10}$$

Upon substituting the above expression of $I(x)$ in (3.7.9), we derive for integer values of ρ and $1 \le i < j \le n$

$$\alpha_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!\{\Gamma(\rho)\}^2} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s}$$
$$\times \sum_{t_1=0}^{(\rho-1)(j-i-1-s+r)} a_{t_1}(\rho, j-i-1-s+r)$$
$$\times \sum_{t_2=0}^{(\rho-1)(n-j+s)} a_{t_2}(\rho, n-j+s) \Gamma(t_2+\rho+1)$$
$$\times \sum_{\ell=0}^{t_2+\rho} \int_0^\infty e^{-(n-i+r+1)x} x^{t_1+\rho+\ell}\, dx / \{(n-j+s+1)^{t_2+\rho+1-\ell} \ell!\}$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!\{\Gamma(\rho)\}^2} \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{r+s} \binom{i-1}{r} \binom{j-i-1}{s}$$
$$\times \sum_{t_1=0}^{(\rho-1)(j-i-1-s+r)} a_{t_1}(\rho, j-i-1-s+r)$$
$$\times \sum_{t_2=0}^{(\rho-1)(n-j+s)} a_{t_2}(\rho, n-j+s)\Gamma(t_2+\rho+1)$$
$$\times \sum_{\ell=0}^{t_2+\rho} \Gamma(t_1+\rho+\ell+1)/\{(n-i+r+1)^{t_1+\rho+\ell+1}(n-j+s+1)^{t_2+\rho-\ell+1}\ell!\}. \tag{3.7.11}$$

The formula in (3.7.11), derived by Gupta (1960, 1962), may be used to compute the covariances of gamma order statistics for integer values of ρ. Prescott (1974) has tabulated the covariances of order statistics for sample sizes up to 10 when $\rho = 2(1)5$. Also, Young (1971) has derived some expressions for the moments of gamma order statistics for the limiting case, i.e., when $\rho \to \infty$. In addition, for the case in which ρ is an integer, Joshi (1979b) has established the recurrence relations

$$\alpha_{1:n}^{(k)} = \frac{k}{n}\Gamma(\rho) \sum_{j=0}^{\rho-1} \alpha_{1:n}^{(k+j-\rho)}/j!, \qquad k \geq 1 \tag{3.7.12}$$

and

$$\alpha_{i:n}^{(k)} = \alpha_{i-1:n-1}^{(k)} + \frac{k}{n}\Gamma(\rho) \sum_{j=0}^{\rho-1} \alpha_{i:n}^{(k+j-\rho)}/j!, \qquad 2 \leq i \leq n, k \geq 1, \tag{3.7.13}$$

and has successfully applied them to compute the existing negative moments of all order statistics by using the table of first four positive moments prepared by Gupta (1960, 1962). Reference may also be made to Tadikamalla (1977), who has developed some approximation to the moments of gamma order statistics via a four-parameter Burr distribution approximation to the gamma distribution in (3.7.1), and to Tiku and Malik (1972) for a three-moment chi square and t approximations for the distributions of gamma order statistics.

3.8. Results for the Weibull Distribution

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics from a Weibull population with pdf

$$f(x) = e^{-x^{\delta}} \delta x^{\delta-1}, \qquad 0 \leq x < \infty, \delta > 0, \tag{3.8.1}$$

and cdf

$$F(x)=1-e^{-x^\delta}, \qquad 0\le x<\infty,\ \delta>0. \tag{3.8.2}$$

Then, from (3.2.1) we obtain for $1\le i\le n$ and $k\ge 1$

$$\begin{aligned}\alpha_{i:n}^{(k)}&=\frac{n!}{(i-1)!(n-i)!}\int_0^\infty x^k(1-e^{-x^\delta})^{i-1}(e^{-x^\delta})^{n-i}\,e^{-x^\delta}\,\delta x^{\delta-1}\,dx\\&=\frac{n!}{(i-1)!(n-i)!}\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\int_0^\infty e^{-(n-i+r+1)x^\delta}\,x^k\,\delta x^{\delta-1}\,dx.\end{aligned} \tag{3.8.3}$$

By setting $u=x^\delta$ in the integral in (3.8.3), we get for $1\le i\le n$ and $k\ge 1$

$$\begin{aligned}\alpha_{i:n}^{(k)}&=\frac{n!}{(i-1)!(n-i)!}\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\int_0^\infty e^{-(n-i+r+1)u}u^{k/\delta}\,du\\&=\frac{n!}{(i-1)!(n-i)!}\Gamma\left(1+\frac{k}{\delta}\right)\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\Big/(n-i+r+1)^{1+(k/\delta)}.\end{aligned} \tag{3.8.4}$$

By making use of the formula in (3.8.4), first derived by Lieblein (1955), Govindarajulu and Joshi (1968) have tabulated the means and variances of all order statistics for sample sizes up to 10 when $\delta=1, 2, 2.5, 3(1)10$. Harter (1970b) has prepared a more extensive table of means of order statistics for samples of sizes 40 and less when $\delta=0.5(0.5)4(1)8$.

Next, from the expression of the product moment in (3.2.3) we have for $1\le i<j\le n$

$$\begin{aligned}\alpha_{i,j:n}&=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_0^\infty\int_0^y xy(1-e^{-x^\delta})^{i-1}(e^{-x^\delta}-e^{-y^\delta})^{j-i-1}\\&\quad\times(e^{-y^\delta})^{n-j}\,e^{-x^\delta}\delta x^{\delta-1}\,e^{-y^\delta}\,\delta y^{\delta-1}\,dx\,dy\\&=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\delta^2\sum_{r=0}^{i-1}\sum_{s=0}^{j-i-1}(-1)^{j-i-1-s+r}\binom{i-1}{r}\binom{j-i-1}{s}\\&\quad\times\int_0^\infty\int_0^y e^{-(r+s+1)x^\delta}\,e^{-(n-i-s)y^\delta}x^\delta y^\delta\,dx\,dy\\&=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{r=0}^{i-1}\sum_{s=0}^{j-i-1}(-1)^{j-i-1-s+r}\binom{i-1}{r}\binom{j-i-1}{s}\\&\quad\times\phi_\delta(r+s+1,\,n-i-s),\end{aligned} \tag{3.8.5}$$

where $\phi_\delta(a, b)$ is Lieblein's ϕ-function defined by

$$\phi_\delta(a,b)=\delta^2\int_0^\infty\int_0^y e^{-ax^\delta-by^\delta}x^\delta y^\delta\,dx\,dy. \tag{3.8.6}$$

In the following we present an algebraic evaluation of the above ϕ-function, as derived by Lieblein (1955) and also explained in detail by Balakrishnan and Kocherlakota (1985), while studying the moments of order statistics from the double Weibull distribution.

By considering the definition of $\phi_\delta(a, b)$ in (3.8.6) and integrating x by parts, we get

$$\phi_\delta(a, b) = -\frac{\Gamma\left(1+\frac{2}{\delta}\right)}{a(a+b)^{1+(2/\delta)}} + \frac{1}{a}\int_0^\infty \int_0^y y\, e^{-ax^\delta}\, e^{-by^\delta}\, \delta y^{\delta-1}\, dx\, dy. \quad (3.8.7)$$

Differentiating Eq. (3.8.7) partially with respect to a, we obtain

$$\frac{\partial \phi_\delta(a, b)}{\partial a} = \frac{\Gamma\left(2+\frac{2}{\delta}\right)}{a(a+b)^{2+(2/\delta)}} - \frac{1}{a}\left(1+\frac{1}{\delta}\right)\phi_\delta(a, b),$$

which, when rewritten, yields the differential equation

$$\frac{\partial \phi_\delta(a, b)}{\partial a} + \frac{1}{a}\left(1+\frac{1}{\delta}\right)\phi_\delta(a, b) = \frac{\Gamma\left(2+\frac{2}{\delta}\right)}{a(a+b)^{2+(2/\delta)}}. \quad (3.8.8)$$

Now by treating (3.8.8) as a linear differential equation in $\phi_\delta(a, b)$ and solving it, we get for $a \geq b$

$$\phi_\delta(a, b) = \frac{1}{a^{1+(1/\delta)}}\left[\Gamma\left(2+\frac{2}{\delta}\right)\int_b^a \frac{x^{1/\delta}}{(b+x)^{2+(2/\delta)}}\, dx + K(b)\right], \quad (3.8.9)$$

where $K(b)$ is the constant of integration to be determined. By setting $w = x/(b+x)$ in the integral on the RHS of (3.8.9), we may rewrite (3.8.9) as

$$\phi_\delta(a, b) = \frac{1}{a^{1+(1/\delta)}}\left[\frac{\Gamma\left(2+\frac{2}{\delta}\right)}{b^{1+(1/\delta)}}\int_{1/2}^{a/(a+b)} w^{1/\delta}(1-w)^{1/\delta}\, dw + K(b)\right]. \quad (3.8.10)$$

In order to determine the constant $K(b)$, let us set $a = b$ in (3.8.10); then, we get

$$K(b) = b^{1+(1/\delta)}\phi_\delta(b, b). \quad (3.8.11)$$

Now, from (3.8.6) we have

$$\phi_\delta(b, b) = \delta^2 \int_0^\infty \int_0^y e^{-bx^\delta}\, e^{-by^\delta}\, x^\delta y^\delta\, dx\, dy,$$

which, because of the symmetry in x and y, may be written as

$$\phi_\delta(b, b) = \frac{1}{2}\,\delta^2\left\{\int_0^\infty e^{-bx^\delta} x^\delta\, dx\right\}^2$$
$$= \frac{1}{2}\left\{\Gamma\left(1+\frac{1}{\delta}\right)\Big/ b^{1+(1/\delta)}\right\}^2$$
$$= \left\{\Gamma\left(2+\frac{2}{\delta}\right)\Big/ b^{2+(2/\delta)}\right\}\frac{1}{2}\,B\left(1+\frac{1}{\delta}, 1+\frac{1}{\delta}\right). \qquad (3.8.12)$$

In (3.8.12), $B(1+(1/\delta), 1+(1/\delta))$ denotes the complete beta integral

$$\int_0^1 w^{1/\delta}(1-w)^{1/\delta}\, dw,$$

which, by its symmetry around $\frac{1}{2}$, may be written equivalently as

$$2\int_0^{1/2} w^{1/\delta}(1-w)^{1/\delta}\, dw.$$

By making use of this expression in (3.8.12), we obtain

$$\phi_\delta(b, b) = \frac{\Gamma\left(2+\frac{2}{\delta}\right)}{b^{2+(2/\delta)}}\int_0^{1/2} w^{1/\delta}(1-w)^{1/\delta}\, dw, \qquad (3.8.13)$$

which, when used in (3.8.11), immediately yields

$$K(b) = \frac{\Gamma\left(2+\frac{2}{\delta}\right)}{b^{1+(1/\delta)}}\int_0^{1/2} w^{1/\delta}(1-w)^{1/\delta}\, dw. \qquad (3.8.14)$$

Upon substituting the expression of $K(b)$ from (3.8.14) in Eq. (3.8.10), we derive an explicit algebraic formula for the ϕ-function to be

$$\phi_\delta(a, b) = \frac{\Gamma\left(2+\frac{2}{\delta}\right)}{(ab)^{1+(1/\delta)}}\int_0^{a/(a+b)} w^{1/\delta}(1-w)^{1/\delta}\, dw$$
$$= \frac{\left\{\Gamma\left(1+\frac{1}{\delta}\right)\right\}^2}{(ab)^{1+(1/\delta)}}\,\mathrm{IB}_{a/(a+b)}\left(1+\frac{1}{\delta}, 1+\frac{1}{\delta}\right), \qquad a \geq b, \qquad (3.8.15)$$

where $\mathrm{IB}_p(d_1, d_2)$ is Karl Pearson's (1934) incomplete beta function defined as

$$\mathrm{IB}_p(d_1, d_2) = \frac{1}{B(d_1, d_2)}\int_0^p w^{d_1-1}(1-w)^{d_2-1}\, dw, \qquad 0 \leq p \leq 1. \quad (3.8.16)$$

Furthermore, we also have from (3.8.6) that

$$\phi_\delta(a, b)+\phi_\delta(b, a)=\delta^2 \int_0^\infty e^{-ax^\delta} x^\delta \, dx \int_0^\infty e^{-by^\delta} y^\delta \, dy$$

$$=\left\{\Gamma\left(1+\frac{1}{\delta}\right)\right\}^2 \Big/ (ab)^{1+(1/\delta)}. \tag{3.8.17}$$

So, after computing the function $\phi_\delta(a, b)$ from (3.8.15) for $a \geq b$, one can use the relation in (3.8.17) to compute the function $\phi_\delta(a, b)$ for $a < b$. These formulas for the ϕ-function will then enable one to compute the product moments of Weibull order statistics from (3.8.5). Govindarajulu and Joshi (1968) have used this procedure and tabulated the covariances of order statistics for sample sizes up to 10 when $\delta = 1, 2, 2.5, 3(1)10$.

3.9. Results for the Normal Distribution

Let us consider the case when the order statistics $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are from the standard normal population with cdf $F(x)$ and pdf

$$f(x)=\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty. \tag{3.9.1}$$

Derivations of the single and the product moments of order statistics explicitly in terms of some elementary functions have been attempted by several authors, including Jones (1948), Godwin (1949), Ruben (1954, 1956), Watanabe *et al.* (1957, 1958), Bose and Gupta (1959), and David (1963). These authors have been successful in achieving this for small sample sizes at least. To explain the method of derivation of single moments, let us follow the lines of Bose and Gupta (1959) and denote

$$I_n(a)=\int_{-\infty}^\infty \{F(ax)\}^n \frac{1}{\sqrt{\pi}} e^{-x^2} \, dx, \qquad n=0, 1, 2, \ldots. \tag{3.9.2}$$

Setting $n=0$ in (3.9.2), we immediately obtain

$$I_0(a)=1. \tag{3.9.3}$$

Let us now consider the expression of $\alpha_{2:2}$ obtained from (3.2.1) as

$$\alpha_{2:2}=2\int_{-\infty}^\infty xF(x)f(x) \, dx. \tag{3.9.4}$$

By using the property that $f'(x)=-xf(x)$ for the standard normal distribution in the integral on the RHS of (3.9.4), and then integrating by parts treating $f'(x)$ for integration and the rest of the integrand for differentiation, we obtain

$$\alpha_{2:2}=\int_{-\infty}^{\infty}\frac{1}{\pi}e^{-x^2}\,dx=\frac{1}{\sqrt{\pi}}I_0(a)=\frac{1}{\sqrt{\pi}}=0.5641895835$$

from (3.9.3). Also, because of the symmetry of $f(x)$, we have $\alpha_{1:2}=-\alpha_{2:2}=-0.5641895835$.

Next, let us consider the integral

$$\int_{-\infty}^{\infty}\left\{F(ax)-\frac{1}{2}\right\}^{2m+1}\frac{1}{\sqrt{\pi}}e^{-x^2}\,dx \tag{3.9.5}$$

for $m=0,1,2,\ldots$. Since the integrand in (3.9.5) is an odd function of x, we immediately have

$$\int_{-\infty}^{\infty}\left\{F(ax)-\frac{1}{2}\right\}^{2m+1}\frac{1}{\sqrt{\pi}}e^{-x^2}\,dx=0, \qquad m\geq 0. \tag{3.9.6}$$

Upon expanding the term $\{F(ax)-\frac{1}{2}\}^{2m+1}$ binomially in the above integral and using (3.9.2) to express the integrals in terms of $I_n(a)$, we get

$$\sum_{i=0}^{2m+1}(-1)^i\binom{2m+1}{i}\frac{1}{2^i}I_{2m+1-i}(a)=0, \qquad m\geq 0,$$

which yields the relation

$$I_{2m+1}(a)=\sum_{i=0}^{2m+1}(-1)^{i+1}\binom{2m+1}{i}\frac{1}{2^i}I_{2m+1-i}(a), \qquad m\geq 0. \tag{3.9.7}$$

Thus, for example, setting $m=0$ in (3.9.7), we obtain

$$I_1(a)=\frac{1}{2}I_0(a)=\frac{1}{2}. \tag{3.9.8}$$

Now, from (3.2.1) let us consider the expression of $\alpha_{3:3}$ and integrate by parts as before by using $f'(x)=-xf(x)$. We then obtain

$$\alpha_{3:3}=3\int_{-\infty}^{\infty}F(x)\frac{1}{\pi}e^{-x^2}\,dx=\frac{3}{\sqrt{\pi}}I_1(1)=\frac{1.5}{\sqrt{\pi}}=0.8462843753$$

from (3.9.8). Because of the symmetry of $f(x)$, we also have $\alpha_{2:3}=0$ and $\alpha_{1:3}=-\alpha_{3:3}=-0.8462843753$.

By differentiating the expression of $I_2(a)$ from (3.9.2) with respect to a, and by applying Fubini's theorem, we obtain

$$I_2'(a) = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} F(ax)x\, e^{-1/2x^2(2+a^2)}\, dx$$

$$= -\frac{\sqrt{2}}{\pi(a^2+2)} \int_{-\infty}^{\infty} F(ax) \frac{d}{dx}\{e^{-1/2x^2(2+a^2)}\},$$

which, upon integration by parts, yields

$$I_2'(a) = \frac{\sqrt{2}}{\pi(a^2+2)} \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2x^2(a^2+1)2}\, dx$$

$$= \frac{a}{\pi(a^2+2)(a^2+1)^{1/2}}. \tag{3.9.9}$$

By solving (3.9.9), we derive

$$I_2(a) = \frac{1}{\pi} \tan^{-1}(a^2+1)^{1/2}. \tag{3.9.10}$$

From (3.2.1) let us now consider the expression of $\alpha_{4:4}$ and integrate by parts as before. Since $f'(x) = -xf(x)$, we obtain

$$\alpha_{4:4} = 6 \int_{-\infty}^{\infty} \{F(x)\}^2 \frac{1}{\pi} e^{-x^2}\, dx = \frac{6}{\sqrt{\pi}} I_2(1)$$

$$= \frac{6}{\pi\sqrt{\pi}} \tan^{-1}\sqrt{2} = 1.0293753730$$

from (3.9.10). From Relation 3.3.1 we get

$$\alpha_{3:4} = 4\alpha_{3:3} - 3\alpha_{4:4} = \frac{6}{\sqrt{\pi}} - \frac{18}{\pi\sqrt{\pi}} \tan^{-1}\sqrt{2} = 0.2970113823,$$

and because of the symmetry of $f(x)$ we have $\alpha_{2:4} = -\alpha_{3:4} = -0.2970113823$ and $\alpha_{1:4} = -\alpha_{4:4} = -1.0293753730$.

By using the relation in (3.9.7), we get by setting $m = 1$

$$I_3(a) = \frac{3}{2} I_2(a) - \frac{3}{4} I_1(a) + \frac{1}{8} I_0(a)$$

$$= \frac{3}{2\pi} \tan^{-1}(a^2+1)^{1/2} - \frac{1}{4}. \tag{3.9.11}$$

Let us now consider the expression of $\alpha_{5:5}$ from (3.2.1) and integrate by parts by using $f'(x)=-xf(x)$. We then get

$$\alpha_{5:5}=10\int_{-\infty}^{\infty}\{F(x)\}^3\frac{1}{\pi}e^{-x^2}\,dx=\frac{10}{\sqrt{\pi}}I_3(1)=1.1629644736,$$

and from Relation 3.3.1 we get

$$\alpha_{4:5}=5\alpha_{4:4}-4\alpha_{5:5}=0.4950189705.$$

Also, because of the symmetry of $f(x)$ we have $\alpha_{3:5}=0$, $\alpha_{2:5}=-\alpha_{4:5}=-0.4950189705$ and $\alpha_{1:5}=-\alpha_{5:5}=-1.1629644736$. However, because $I_4(a)$ cannot be expressed in terms of elementary functions, the above method fails for $\alpha_{6:6}$. Moreover, Ruben (1954), who has shown that the moments of order statistics can be expressed as linear functions of the contents of certain hyperspherical simplices, has noted that for dimension greater than three these contents cannot be expressed in terms of elementary functions; this (as pointed out above) explains why the method fails for $\alpha_{6:6}$.

By using a similar but somewhat cumbersome approach, the product moments of order statistics may also be determined in terms of elementary functions for small sample sizes. First of all, we know that

$$\alpha_{1,2:2}=\alpha_{1:1}^2=0.$$

Next, let us consider the expression of $\alpha_{1,2:3}$ from (3.2.3) given by

$$\alpha_{1,2:3}=6\int_{-\infty}^{\infty}\int_{-\infty}^{y}xy\{1-F(y)\}f(x)f(y)\,dx\,dy. \tag{3.9.12}$$

By making use of the property that $f'(x)=-xf(x)$, we have

$$\int_{-\infty}^{y}xf(x)\,dx=-f(y), \tag{3.9.13}$$

which, when used in (3.9.12), gives

$$\alpha_{1,2:3}=-6\int_{-\infty}^{\infty}y\{1-F(y)\}f^2(y)\,dy. \tag{3.9.14}$$

By writing $yf(y)$ as $-f'(y)$ and integrating by parts, we get from (3.9.14)

$$\begin{aligned}\alpha_{1,2:3}&=6\int_{-\infty}^{\infty}f^3(y)\,dy+6\int_{-\infty}^{\infty}y\{1-F(y)\}f^2(y)\,dy\\&=6\int_{-\infty}^{\infty}f^3(y)\,dy-\alpha_{1,2:3},\end{aligned}$$

which immediately yields

$$\alpha_{1,2:3} = 3\int_{-\infty}^{\infty} f^3(y)\, dy = \frac{\sqrt{3}}{2\pi} I_0(a) = \frac{\sqrt{3}}{2\pi} = 0.2756644477$$

from (3.9.3). Because of the symmetry of $f(x)$ we have $\alpha_{2,3:3} = \alpha_{1,2:3} = 0.2756644477$. Also, by using Relation 3.3.4 we get

$$\alpha_{1,3:3} = 3\alpha_{1,2:2} - \alpha_{1,2:3} - \alpha_{2,3:3} = -\frac{\sqrt{3}}{\pi} = -0.5513288954.$$

Next, let us consider the expression of $\alpha_{1,2:4}$ from (3.2.3) given by

$$\alpha_{1,2:4} = 12\int_{-\infty}^{\infty}\int_{-\infty}^{y} xy\{1-F(y)\}^2 f(x)f(y)\, dx\, dy. \tag{3.9.15}$$

By applying (3.9.13) to the above equation we get

$$\alpha_{1,2:4} = -12\int_{-\infty}^{\infty} y\{1-F(y)\}^2 f^2(y)\, dy. \tag{3.9.16}$$

By writing $yf(y)$ as $-f'(y)$ and integrating by parts, we get from (3.9.16)

$$\begin{aligned}\alpha_{1,2:4} &= 24\int_{-\infty}^{\infty} \{1-F(y)\} f^3(y)\, dy + 12\int_{-\infty}^{\infty} y\{1-F(y)\}^2 f^2(y)\, dy\\ &= 24\int_{-\infty}^{\infty} \{1-F(y)\} f^3(y)\, dy - \alpha_{1,2:4},\end{aligned}$$

which immediately yields

$$\begin{aligned}\alpha_{1,2:4} &= 12\int_{-\infty}^{\infty} f^3(y)\, dy - 12\int_{-\infty}^{\infty} F(y) f^3(y)\, dy\\ &= \frac{2\sqrt{3}}{\pi}\left\{I_0(a) - I_1\left(\sqrt{\frac{2}{3}}\right)\right\}\\ &= \frac{\sqrt{3}}{\pi} = 0.5513288954.\end{aligned}$$

Similarly, let us now start with the expression of $\alpha_{2,3:4}$ from (3.2.3) given by

$$\alpha_{2,3:4} = 24\int_{-\infty}^{\infty}\int_{-\infty}^{y} xyF(x)\{1-F(y)\} f(x)f(y)\, dx\, dy. \tag{3.9.17}$$

By making use of the property that $f'(x) = -xf(x)$, we have

$$\int_{-\infty}^{y} xF(x)f(x)\, dx = -F(y)f(y) + \frac{1}{2\sqrt{\pi}} F(\sqrt{2}y), \tag{3.9.18}$$

which, when used in (3.9.17), gives

$$\alpha_{2,3:4} = -24K_1 + \frac{24}{2\sqrt{\pi}} K_2, \tag{3.9.19}$$

where

$$K_1 = \int_{-\infty}^{\infty} y\{1 - F(y)\}F(y)f^2(y)\, dy \tag{3.9.20}$$

and

$$K_2 = \int_{-\infty}^{\infty} y\{1 - F(y)\}F(\sqrt{2}y)f(y)\, dy. \tag{3.9.21}$$

Since the integrand in the expression for K_1 in (3.9.20) is an odd function in y, we immediately have $K_1 = 0$. Next, by writing $yf(y)$ as $-f'(y)$ in the expression for K_2 in (3.9.21), integrating by parts, and then simplifying the resulting expression, we get

$$K_2 = \frac{1}{\sqrt{3\pi}} I_0(a) - \frac{1}{2\sqrt{\pi}} I_1(\sqrt{2}) - \frac{1}{\sqrt{3\pi}} I_1\left(\sqrt{\frac{2}{3}}\right) = \frac{2-\sqrt{3}}{4\sqrt{3\pi}}.$$

Upon substituting for K_1 and K_2 in (3.9.19), we get

$$\alpha_{2,3:4} = \frac{\sqrt{3}(2-\sqrt{3})}{\pi} = 0.1477281323.$$

By using Relation 3.3.4 we get

$$\begin{aligned}\alpha_{1,3:4} &= 4\alpha_{1,2:3} - 2\alpha_{1,2:4} - \alpha_{2,3:4} \\ &= -\frac{\sqrt{3}(2-\sqrt{3})}{\pi} = -0.1477281323,\end{aligned}$$

and by symmetry of $f(x)$ we also have $\alpha_{3,4:4} = \alpha_{1,2:4} = 0.5513288954$ and $\alpha_{2,4:4} = \alpha_{1,3:4} = -0.1477281323$. Finally, $\alpha_{1,4:4}$ may be derived by using Relation 3.3.4 as

$$\begin{aligned}\alpha_{1,4:4} &= 2\alpha_{1,3:3} - \frac{1}{2}(\alpha_{1,3:4} + \alpha_{2,4:4}) \\ &= -\frac{3}{\pi} = -0.9549296586.\end{aligned}$$

A general approach, given by Godwin (1949a), is to express the product moments in terms of integrals of the form

$$J_n = \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{-Q(x_1,\ldots,x_n)}\, dx_1\, dx_2 \cdots dx_n,$$

where $Q(x_1, x_2, \ldots, x_n)$ is a quadratic form in the x_i's. For $n=1,2,3$, J_n can be written down explicitly in terms of elementary functions as follows:

$n=1$
$$Q(x_1)=a_{11}x_1^2$$
$$J_1=\sqrt{\pi}/2a_{11}$$

$n=2$
$$Q(x_1,x_2)=a_{11}x_1^2+a_{22}x_2^2+2a_{12}x_1x_2$$
$$J_2=\frac{1}{\sqrt{\Delta_2}}\left\{\frac{\pi}{2}-\tan^{-1}\left(\frac{a_{12}}{\sqrt{\Delta_2}}\right)\right\},$$
$$\text{where } \Delta_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}^2$$

$n=3$
$$Q(x_1,x_2,x_3)=a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2$$
$$+2a_{13}x_1x_3+2a_{23}x_2x_3$$
$$J_3=\frac{\sqrt{\pi}}{4\sqrt{\Delta_3}}\left\{\frac{\pi}{2}+\tan^{-1}\left(\frac{a_{12}a_{13}-a_{11}a_{23}}{\sqrt{a_{11}\Delta_3}}\right)\right.$$
$$\left.+\tan^{-1}\left(\frac{a_{12}a_{23}-a_{13}a_{22}}{\sqrt{a_{22}\Delta_3}}\right)+\tan^{-1}\left(\frac{a_{13}a_{23}-a_{12}a_{33}}{\sqrt{a_{33}\Delta_3}}\right)\right\},$$
$$\text{where } \Delta_3=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{vmatrix}$$
$$=a_{11}a_{22}a_{33}-a_{11}a_{23}^2-a_{22}a_{13}^2-a_{33}a_{12}^2+2a_{12}a_{13}a_{23}.$$

The values of $\alpha_{i:n}$ for all i and for $n=2(1)100(25)250(50)400$ have been tabulated to five decimals by Harter (1961a), and also by Harter (1970) for some more choices of n. The mean and variance of the ith quasi-range have been tabulated by Harter (1959) for n up to 100. For the sample range, Tippett (1925) has computed the expected value for $n \leq 1000$, whereas Harter (1960) has tabulated the mean, variance, and the coefficients of skewness and kurtosis for $n \leq 100$. By using the tables of means and product moments of order statistics for sample sizes up to 20 prepared by Teichroew (1956), Sarhan and Greenberg (1956) have tabulated the variances and covariances to ten decimals for $n \leq 20$. These tables have been extended to $n \leq 50$ by Tietjen *et al.* (1977). The values of the mean and standard deviation prepared by Yamauti (1972) for sample sizes up to 50 are contained in the tables of Tietjen *et al.* (1977). For the largest order statistic $X_{n:n}$, Ruben (1954) has tabulated the first 10 moments for $n \leq 50$ and Borenius (1966), the first two moments for $n \leq 120$.

In addition to the above results, the moments of normal order statistics also satisfy some interesting recurrence relations and identities. We present a few of them in the following theorems and also illustrate some of their applications.

Theorem 3.9.1. *If $g(x)$ is any differentiable function such that differentiation of $g(x)$ with respect to x and $E\{g(X)\}$ with respect to an absolutely continuous distribution are interchangeable, then for $1 \le i \le n$*

$$E\{g'(X_{i:n})\} = -\sum_{j=1}^{n} E\{g(X_{i:n}) f'(X_{j:n})/f(X_{j:n})\}, \tag{3.9.22}$$

where $f(x)$ is the pdf of the population.

Proof. By adopting a method of proof originally given by Seal (1956), Govindarajulu (1963) has proved this theorem as follows. For all real t we have for $1 \le i \le n$

$$\begin{aligned} E\{g(X_{i:n}+t)\} &= n! \underset{-\infty<x_{1:n}<\cdots<x_{n:n}<\infty}{\int\int\cdots\int} g(x_{i:n}+t) \prod_{j=1}^{n} f(x_{j:n})\, dx_{j:n} \\ &= n! \underset{-\infty<x_{1:n}<\cdots<x_{n:n}<\infty}{\int\int\cdots\int} g(x_{i:n}) \prod_{j=1}^{n} f(x_{j:n}-t)\, dx_{j:n}. \end{aligned} \tag{3.9.23}$$

By differentiating both sides of (3.9.23) with respect to t and setting $t = 0$, we derive

$$E\{g'(X_{i:n})\} = n! \underset{-\infty<x_{1:n}<\cdots<x_{n:n}<\infty}{\int\int\cdots\int} g(x_{i:n}) \left\{-\sum_{j=1}^{n} \frac{f'(x_{j:n})}{f(x_{j:n})}\right\} \prod_{k=1}^{n} f(x_{k:n})\, dx_{k:n},$$

which yields the relation in (3.9.22).

Corollary 3.9.1. *For $1 \le i \le n$, we have*

$$\sum_{j=1}^{n} E\{X_{i:n} f'(X_{j:n})/f(X_{j:n})\} = -1. \tag{3.9.24}$$

Proof. This follows directly from Theorem 3.9.1 if we take $g(x) = x$.

Corollary 3.9.2. *For the standard normal population with pdf $f(x)$ as in (3.9.1), we have for $1 \leq i \leq n$*

$$\sum_{j=1}^{n} \alpha_{i,j:n} = 1 \tag{3.9.25}$$

and

$$\sum_{j=1}^{n} \beta_{i,j:n} = 1. \tag{3.9.26}$$

Proof. (3.9.25) is derived from Corollary 3.9.1 simply by noting that $f'(x) = -xf(x)$ for the standard normal distribution. The identity in (3.9.26) then follows from (3.9.25) by using the fact that $\sum_{j=1}^{n} \alpha_{j:n} = n\alpha_{1:1} = 0$.

The two identities given in Corollary 3.9.2 may also be proved by using the independence of $\bar{X}$ and $X_{i:n} - \bar{X}$; see McKay (1935) and Daly (1946).

Theorem 3.9.2. *For the standard normal population, we have for $1 \leq i \leq n$*

$$\alpha_{i:n}^{(2)} = 1 + n \binom{n-1}{i-1} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{1}{i+j} \alpha_{1,2:i+j}. \tag{3.9.27}$$

Proof. From (3.2.1), we have for $1 \leq i \leq n$

$$\alpha_{i:n}^{(2)} = \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} x^2 \{F(x)\}^{i-1} \{1 - F(x)\}^{n-i} f(x)\, dx. \tag{3.9.28}$$

By writing $xf(x)$ as $-f'(x)$ in (3.9.28) and integrating by parts, we obtain

$$\begin{aligned} \alpha_{i:n}^{(2)} = 1 &+ \frac{n!}{(i-1)!(n-i)!} (i-1) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \int_{-\infty}^{\infty} x \{F(x)\}^{i+j-2} f^2(x)\, dx \\ &+ \frac{n!}{(i-1)!(n-i)!} (n-i) \sum_{j=0}^{n-i-1} (-1)^{j+1} \binom{n-i-1}{j} \\ &\times \int_{-\infty}^{\infty} x \{F(x)\}^{i+j-1} f^2(x)\, dx. \end{aligned} \tag{3.9.29}$$

Since

$$\int_x^{\infty} y f(y)\, dy = -\int_x^{\infty} f'(y)\, dy = f(x),$$

we have

$$\int_{-\infty}^{\infty} x\{F(x)\}^k f^2(x)\,dx = \int_{-\infty}^{\infty} x\{F(x)\}^k f(x)\left\{\int_x^{\infty} yf(y)\,dy\right\}dx$$

$$= \iint\limits_{-\infty<x<y<\infty} xy\{F(x)\}^k f(x)f(y)\,dy\,dx$$

$$= \alpha_{k+1,k+2:k+2}/\{(k+1)(k+2)\}$$

$$= \alpha_{1,2:k+2}/\{(k+1)(k+2)\}, \tag{3.9.30}$$

because of the symmetry of the standard normal distribution. By using the formula in (3.9.30) for the two integrals on the RHS of (3.9.29) and simplifying the resulting expression, we derive the relation in (3.9.27).

In particular, by setting $i=n$ in (3.9.27) we obtain the relation

$$\alpha_{n:n}^{(2)} = \alpha_{1:n}^{(2)} = 1 + \alpha_{1,2:n}. \tag{3.9.31}$$

The results given in Theorems 3.9.1 and 3.9.2 have been used for checking the computations of $\alpha_{i,j:n}$ and $\beta_{i,j:n}$. In addition, Davis and Stephens (1977, 1978) have applied (3.9.26) and (3.9.31) to improve the David–Johnson approximation of the variance–covariance matrix of normal order statistics. A detailed discussion of the David–Johnson approximation is presented in the next section.

By noting that the condition $f'(x) = -xf(x)$ is satisfied by both the standard normal and the half normal (or chi) distributions, and then proceeding exactly on the same lines as in the proof of Theorem 3.9.2, Joshi and Balakrishnan (1981) have established the following theorem.

Theorem 3.9.3. *For the standard normal and the half normal distributions, we have*

$$\sum_{j=i}^{n} \alpha_{i,j:n} = 1 + \sum_{j=i}^{n} \alpha_{i-1,j:n}, \qquad 1 \le i \le n, \tag{3.9.32}$$

and

$$\sum_{j=i+1}^{n} \alpha_{i,j:n} = \sum_{j=i+1}^{n} \alpha_{j:n}^{(2)} - (n-i), \qquad 1 \le i \le n-1, \tag{3.9.33}$$

with $\alpha_{0,j:n} = 0$ *for* $j \ge 1$.

For the case $i = n-1$, *the relation in* (3.9.33) *simply reduces to* (3.9.31).

Theorem 3.9.4. *For the standard normal and the half normal distributions, we have for* $1 \le i \le n$

$$\sum_{j=1}^{n} \alpha_{i,j:n} = 1 + n\alpha_{1:1}\alpha_{i-1:n-1} \tag{3.9.34}$$

and

$$\sum_{j=1}^{n} \beta_{i,j:n} = 1 - (n-i+1)\alpha_{1:1}(\alpha_{i:n} - \alpha_{i-1:n}), \tag{3.9.35}$$

with $\alpha_{0:j} = 0$ *for* $j \ge 1$.

Proof. For the case $i=1$, (3.9.34) is the same as (3.9.32). For $2 \le i \le n$, it is known that for any arbitrary distribution (Joshi and Balakrishnan, 1982)

$$\sum_{j=i}^{n} \alpha_{i-1,j:n} + \sum_{j=1}^{i-1} \alpha_{j,i:n} = n\alpha_{1:1}\alpha_{i-1:n-1}. \tag{3.9.36}$$

Upon adding Eqs. (3.9.32) and (3.9.36), we derive the relation in (3.9.34). The relation in (3.9.35) readily follows from (3.9.34).

Joshi and Balakrishnan (1981) have used the results of Theorems 3.9.3 and 3.9.4 to find a convenient expression for the variance of the selection differential or reach statistic, defined as $\Delta_k = \bar{X}_k - \bar{X}_n$, where $\bar{X}_n$ is the sample mean and $\bar{X}_k$ is the average of the k largest order statistics. Some properties of $\bar{X}_k$ have been investigated by several authors including Schaeffer *et al.* (1970) and Burrows (1972, 1975). They have observed that $\nu_k = k\,\mathrm{Var}(\bar{X}_k)$ remains almost constant for the selected fraction k/n. While Schaeffer *et al.* (1970) tabulated ν_k for $n \le 20$ and all choices of k by making use of Sarhan and Greenberg's (1956) tables, Burrows (1972, 1975) has provided approximations to $E(\bar{X}_k)$ and ν_k for large values of n. By following Joshi and Balakrishnan (1981), we now have

$$\begin{aligned} k^2 E(\bar{X}_k^2) &= \sum_{i=n-k+1}^{n} \sum_{j=n-k+1}^{n} \alpha_{i,j:n} \\ &= \sum_{i=n-k+1}^{n} \alpha_{i:n}^{(2)} + 2 \sum_{i=n-k+1}^{n-1} \sum_{j=i+1}^{n} \alpha_{i,j:n} \\ &= \sum_{i=n-k+1}^{n} \alpha_{i:n}^{(2)} + 2 \sum_{i=n-k+1}^{n-1} \left\{ \sum_{j=i+1}^{n} \alpha_{j:n}^{(2)} - (n-i) \right\} \end{aligned} \tag{3.9.37}$$

upon using (3.9.33). By rearranging the terms in (3.9.37) and then simplifying, we derive

$$k^2 E(\bar{X}_k^2) = \sum_{i=n-k+1}^{n} (2i - 2n + 2k - 1)\alpha_{i:n}^{(2)} - k(k-1). \tag{3.9.38}$$

As a result, the mean and the variance of $\bar{X}_k$ can be calculated from the tables of $\alpha_{i:n}$ and $\alpha_{i:n}^{(2)}$ alone; Joshi and Balakrishnan (1981) have tabulated them for n up to 50 and have pointed out that Burrows' (1975) approximation for ν_k is not satisfactory for small values of k even when $n=50$, and that the approximation improves with increasing values of k.

For the purpose of testing the presence of outliers in normal samples, Murphy (1951) has proposed the internally studentized version of the selection differential defined by

$$D_k = k(\bar{X}_k - \bar{X}_n)/s = k\Delta_k/s, \tag{3.9.39}$$

where $s^2 = \sum_{i=1}^{n} (X_{i:n} - \bar{X}_n)^2/(n-1)$ is the sample variance. Statistics related to D_k for small values of k have been studied in great detail by several authors. Interested readers may refer to Barnett and Lewis (1978) and Hawkins (1979) for more information on this topic.

Let us now consider the case when $X_1, X_2, \ldots, X_n$ is a random sample from a normal $N(\mu, \sigma^2)$ population, and $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ are the order statistics obtained from this sample. First of all, it is quite well known that $(\bar{X}, s)$ form a complete sufficient statistic for (μ, σ). Next, we note that the statistic D_k defined in (3.9.39) is invariant with respect to both location and scale, and hence its distribution does not involve (μ, σ). Then, by invoking Basu's (1955) theorem, we immediately find the variables D_k and s to be statistically independent. Consequently, we get from (3.9.39) that

$$\begin{aligned} E(D_k^\ell) &= k^\ell E(\Delta_k^\ell)/E(s^\ell) \\ &= \left\{ \frac{k^\ell \left(\frac{n-1}{2}\right)^{\ell/2} \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1+\ell}{2}\right)} \right\} E(\Delta_k^\ell). \end{aligned} \tag{3.9.40}$$

The mean and the variance of the internally studentized selection differential, D_k, can now be computed from (3.9.40) by using the results that

$$E(\Delta_k) = E(\bar{X}_k)$$

and

$$E(\Delta_k^2) = \mathrm{Var}(\bar{X}_k) - \frac{1}{n} + \{E(\bar{X}_k)\}^2.$$

These quantities have been tabulated for the case $k=1$ by Borenius (1966) for n up to 120.

3.10. Results for the Half Logistic Distribution

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics from a half logistic population with pdf

$$f(x) = 2e^{-x}/(1+e^{-x})^2, \qquad 0 \leq x < \infty, \tag{3.10.1}$$

and cdf

$$F(x) = (1-e^{-x})/(1+e^{-x}), \qquad 0 \leq x < \infty. \tag{3.10.2}$$

The density function $f(x)$ in (3.10.1), obtained by folding the logistic density function in (3.6.1) at $x = 0$, is a monotonic decreasing function of x in the interval $[0, \infty)$ and has an increasing hazard rate. It, therefore, serves as a good failure-time model in life-testing problems, as is shown later in Chapters 4 and 6.

From (3.10.1) and (3.10.2), we immediately observe the relations

$$f(x) = F(x)\{1-F(x)\} + \frac{1}{2}\{1-F(x)\}^2, \tag{3.10.3}$$

$$f(x) = \{1-F(x)\} - \frac{1}{2}\{1-F(x)\}^2 \tag{3.10.4}$$

and

$$f(x) = \frac{1}{2}\{1-F^2(x)\}. \tag{3.10.5}$$

By making use of the relations in (3.10.3)-(3.10.5), Balakrishnan (1985) has derived several recurrence relations satisfied by the single and the product moments of order statistics. These relations, when applied in a simple and systematic recursive way, will enable one to compute the means, variances, and covariances of all order statistics for all sample sizes simply by starting with the values of $\alpha_{1:1} = E(X) = \ln 4$ and $\alpha_{1:1}^{(2)} = E(X^2) = \pi^2/3$.

Theorem 3.10.1. *For the half logistic population with pdf as in* (3.10.1), *we have*

$$\alpha_{1:n+1}^{(k+1)} = 2\left\{\alpha_{1:n}^{(k+1)} - \frac{k+1}{n}\,\alpha_{1:n}^{(k)}\right\}, \qquad n \geq 1,\, k \geq 0, \tag{3.10.6}$$

with $\alpha_{1:n}^{(0)} \equiv 1$.

Proof. For $n \geq 1$, we have from (2.4.4) that

$$\alpha_{1:n}^{(k)} = n \int_0^\infty x^k \{1-F(x)\}^{n-1} f(x)\, dx.$$

Upon using the relation in (3.10.4) and splitting the above integral accordingly into two, we get

$$\alpha_{1:n}^{(k)} = n\left[\int_0^\infty x^k\{1-F(x)\}^n\,dx - \frac{1}{2}\int_0^\infty x^k\{1-F(x)\}^{n+1}\,dx\right]. \quad (3.10.7)$$

Integrating the RHS of (3.10.7) by parts, treating x^k for integration and the rest of the integrand for differentiation, we obtain for $n \geq 1$ and $k \geq 0$

$$\alpha_{1:n}^{(k)} = \frac{n}{k+1}\left[n\int_0^\infty x^{k+1}\{1-F(x)\}^{n-1}f(x)\,dx - \frac{n+1}{2}\int_0^\infty x^{k+1}\{1-F(x)\}^n f(x)\,dx\right]. \quad (3.10.8)$$

Equation (3.10.8), when simplified, yields the relation in (3.10.6).

Theorem 3.10.2. *For the half logistic population with pdf as in* (3.10.1), *we have*

$$\alpha_{i+1:n+1}^{(k+1)} = \frac{1}{i}\left[\frac{(n+1)(k+1)}{n-i+1}\,\alpha_{i:n}^{(k)} + \frac{n+1}{2}\,\alpha_{i-1:n}^{(k+1)} - \frac{n-2i+1}{2}\,\alpha_{i:n+1}^{(k+1)}\right],$$

$$1 \leq i \leq n,\ k \geq 0, \quad (3.10.9)$$

with $\alpha_{0:t}^{(k)} \equiv 0$ *for* $t \geq 1$ *and* $k \geq 0$, *and* $\alpha_{i:t}^{(0)} \equiv 1$ *for* $1 \leq i \leq t$.

Proof. From (2.4.4), we have for $1 \leq i \leq n$ and $k \geq 0$

$$\alpha_{i:n}^{(k)} = \frac{n!}{(i-1)!(n-i)!}\int_0^\infty x^k\{F(x)\}^{i-1}\{1-F(x)\}^{n-i}f(x)\,dx.$$

Upon using the relation in (3.10.3) and splitting the integral accordingly into two, we get

$$\alpha_{i:n}^{(k)} = \frac{n!}{(i-1)!(n-i)!}\left[\int_0^\infty x^k\{F(x)\}^i\{1-F(x)\}^{n-i+1}\,dx + \int_0^\infty x^k\{F(x)\}^{i-1}\{1-F(x)\}^{n-i+2}\,dx\right]. \quad (3.10.10)$$

Upon integrating the first integral on the RHS of (3.10.10) by parts, treating x^k for integration and the rest of the integrand for differentiation, we obtain

$$\int_0^\infty x^k\{F(x)\}^i\{1-F(x)\}^{n-i+1}\,dx$$

$$=\frac{1}{k+1}\left[(n-i+1)\int_0^\infty x^{k+1}\{F(x)\}^i\{1-F(x)\}^{n-i}f(x)\,dx\right.$$

$$\left.-i\int_0^\infty x^{k+1}\{F(x)\}^{i-1}\{1-F(x)\}^{n-i+1}f(x)\,dx\right]$$

$$=\frac{i!(n-i+1)!}{(n+1)!(k+1)}\{\alpha_{i+1:n+1}^{(k+1)}-\alpha_{i:n+1}^{(k+1)}\}. \tag{3.10.11}$$

By simply changing i to $i-1$ in the above equation, we obtain an expression for the second integral on the RHS of (3.10.10):

$$\int_0^\infty x^k\{F(x)\}^{i-1}\{1-F(x)\}^{n-i+2}\,dx=\frac{(i-1)!(n-i+2)!}{(n+1)!(k+1)}\{\alpha_{i:n+1}^{(k+1)}-\alpha_{i-1:n+1}^{(k+1)}\}. \tag{3.10.12}$$

By substituting the expressions of the two integrals in (3.10.11) and (3.10.12) in Eq. (3.10.10) and simplifying the resulting equation, we get

$$\alpha_{i:n}^{(k)}=\frac{(n-i+1)}{(n+1)(k+1)}[i\{\alpha_{i+1:n+1}^{(k+1)}-\alpha_{i:n+1}^{(k+1)}\}+(n-i+2)\{\alpha_{i:n+1}^{(k+1)}-\alpha_{i-1:n+1}^{(k+1)}\}]. \tag{3.10.13}$$

From Relation 3.3.1, we have

$$(n-i+2)\{\alpha_{i:n+1}^{(k+1)}-\alpha_{i-1:n+1}^{(k+1)}\}=(n+1)\{\alpha_{i:n+1}^{(k+1)}-\alpha_{i-1:n}^{(k+1)}\},$$

which, when used in Eq. (3.10.13) and simplified, yields the relation in (3.10.9).

Starting with the first k raw moments of X, Theorems 3.10.1 and 3.10.2 will allow one to evaluate the first k raw moments of all order statistics for all sample sizes in a very simple recursive manner. Thus, for example, starting with $\alpha_{1:1}=E(X)=\ln 4$ and $\alpha_{1:1}^{(2)}=E(X^2)=\pi^2/3$, one can employ Theorems 3.10.1 and 3.10.2 to complete the means and variances of all order statistics for all sample sizes.

Theorem 3.10.3. *For the half logistic distribution with pdf as in* (3.10.1), *we have*

$$\alpha_{i,i+1:n+1}=\alpha_{i:n+1}^{(2)}+\frac{2(n+1)}{n-i+1}\left\{\alpha_{i,i+1:n}-\alpha_{i:n}^{(2)}-\frac{1}{n-i}\,\alpha_{i:n}\right\},\qquad 1\le i\le n-1. \tag{3.10.14}$$

Proof. From (2.3.5), we may write for $1 \le i \le n-1$ that

$$\alpha_{i:n} = E(X_{i:n}X^0_{i+1:n})$$
$$= \frac{n!}{(i-1)!(n-i-1)!} \int_0^\infty \int_x^\infty x\{F(x)\}^{i-1}\{1-F(y)\}^{n-i-1} f(x) f(y)\, dy\, dx$$
$$= \frac{n!}{(i-1)!(n-i-1)!} \int_0^\infty x\{F(x)\}^{i-1} f(x) I(x)\, dx, \tag{3.10.15}$$

where

$$I(x) = \int_x^\infty \{1-F(y)\}^{n-i-1} f(y)\, dy. \tag{3.10.16}$$

Upon using the relation in (3.10.4), we may write

$$I(x) = \int_x^\infty \{1-F(y)\}^{n-i}\, dy - \frac{1}{2}\int_x^\infty \{1-F(y)\}^{n-i+1}\, dy,$$

which, when integrated by parts, yields

$$I(x) = (n-i)\int_x^\infty y\{1-F(y)\}^{n-i-1} f(y)\, dy - x\{1-F(x)\}^{n-i}$$
$$+ \frac{1}{2} x\{1-F(x)\}^{n-i+1} - \frac{n-i+1}{2}\int_x^\infty y\{1-F(y)\}^{n-i} f(y)\, dy.$$

The recurrence relation in (3.10.14) follows immediately when one substitutes the above expression of $I(x)$ in (3.10.15) and simplifies the resulting equation.

Theorem 3.10.4. *For the half logistic distribution with pdf as in* (3.10.1), *we have*

$$\alpha_{2,3:n+1} = \alpha^{(2)}_{3:n+1} + (n+1)\left\{\alpha_{2:n} - \frac{n}{2}\alpha^{(2)}_{1:n-1}\right\}, \qquad n \ge 2, \tag{3.10.17}$$

and

$$\alpha_{i+1,i+2:n+1} = \alpha^{(2)}_{i+2:n+1} + \frac{(n+1)}{i(i+1)}[2\alpha_{i+1:n} + n\{\alpha_{i-1,i:n-1} - \alpha^{(2)}_{i:n-1}\}],$$
$$2 \le i \le n-1. \tag{3.10.18}$$

Proof. From (2.3.5), let us write for $1 \le i \le n-1$

$$\alpha_{i+1:n} = E(X^0_{i:n} X_{i+1:n})$$

$$= \frac{n!}{(i-1)!(n-i-1)!} \int_0^\infty \int_0^y y\{F(x)\}^{i-1}\{1-F(y)\}^{n-i-1} f(x) f(y)\, dx\, dy$$

$$= \frac{n!}{(i-1)!(n-i-1)!} \int_0^\infty y\{1-F(y)\}^{n-i-1} f(y) J(y)\, dy, \tag{3.10.19}$$

where

$$J(y) = \int_0^y \{F(x)\}^{i-1} f(x)\, dx. \tag{3.10.20}$$

Upon using the relation in (3.10.5), we may write

$$J(y) = \frac{1}{2}\left[\int_0^y \{F(x)\}^{i-1}\, dx - \int_0^y \{F(x)\}^{i+1}\, dx\right],$$

which, when integrated by parts, yields for $i=1$,

$$J(y) = \frac{1}{2}\left[y - y\{F(y)\}^2 + 2\int_0^y xF(x) f(x)\, dx\right],$$

and for $2 \le i \le n-1$,

$$J(y) = \frac{1}{2}\left[y\{F(y)\}^{i-1} - (i-1)\int_0^y x\{F(x)\}^{i-2} f(x)\, dx \right.$$
$$\left. - y\{F(y)\}^{i+1} + (i+1)\int_0^y x\{F(x)\}^{i} f(x)\, dx\right].$$

The recurrence relations in (3.10.17) and (3.10.18) follow immediately upon substitution of the above expressions of $J(y)$ in (3.10.19) and simplification of the resulting equations.

By setting $i=n-1$ in (3.10.18), in particular, we obtain the relation

$$\alpha_{n,n+1:n+1} = \alpha^{(2)}_{n+1:n+1} + \frac{(n+1)}{(n-1)n}\left[2\alpha_{n:n} + n\{\alpha_{n-2,n-1:n-1} - \alpha^{(2)}_{n-1:n-1}\}\right],$$
$$n \ge 3. \tag{3.10.21}$$

It should be mentioned here that Theorems 3.10.3 and 3.10.4 will allow one to evaluate the product moments of order statistics for all sample sizes in a simple recursive manner. To see this, we first note from (3.3.8) that

$$\alpha_{1,2:2} = \alpha^2_{1:1} = (\ln 4)^2. \tag{3.10.22}$$

The relations in (3.10.14), (3.10.17), and (3.10.18), when used along with (3.10.22), will enable one to calculate the product moments $\alpha_{i,i+1:n}$ ($1 \le i \le n-1$). This is sufficient for the evaluation of all the product moments $\alpha_{i,j:n}$, as the remaining product moments, viz., $\alpha_{i,j:n}$ for $1 \le i < j \le n$ and $j-i \ge 2$, can all be systematically computed by using Relation 3.3.4.

Proceeding similarly, Balakrishnan (1985) has established some more recurrence relations satisfied by the product moments $\alpha_{i,j:n}$ ($1 \le i < j \le n$, $j-i \ge 2$), which are presented in the following two theorems.

Theorem 3.10.5. *For the half logistic distribution with pdf as in* (3.10.1), *we have*

$$\alpha_{i,j:n+1} = \alpha_{i,j-1:n+1} + \frac{2(n+1)}{n-j+2}\left\{\alpha_{i,j:n} - \alpha_{i,j-1:n} - \frac{1}{n-j+1}\,\alpha_{i:n}\right\},$$

$$1 \le i \le n-2,\ j-i \ge 2. \qquad (3.10.23)$$

Theorem 3.10.6. *For the half logistic distribution with pdf as in* (3.10.1), *we have*

$$\alpha_{2,j+1:n+1} = \alpha_{3,j+1:n+1} + (n+1)\left\{\alpha_{j:n} - \frac{n}{2}\,\alpha_{1,j-1:n-1}\right\}, \qquad 3 \le j \le n, \qquad (3.10.24)$$

and

$$\alpha_{i+1,j+1:n+1} = \alpha_{i+2,j+1:n+1} + \frac{(n+1)}{i(i+1)}[2\alpha_{j:n} - n\{\alpha_{i,j-1:n-1} - \alpha_{i-1,j-1:n-1}\}],$$

$$2 \le i \le n-2,\ j-i \ge 2. \qquad (3.10.25)$$

By making use of all these recurrence relations, Balakrishnan (1985) has tabulated the means, variances, and covariances of order statistics from the half logistic distribution with pdf as in (3.10.1) for sample sizes up to 15. Balakrishnan (1985) has also tabulated several percentage points of the smallest and the largest order statistics for sample sizes up to 15. He has further shown that the distribution of order statistics is unimodal and has also tabulated the modes of all order statistics for sample sizes 15 and less. Wong (1988) has recently extended the tables of means, variances, and covariances for sample sizes up to 20.

3.11. David and Johnson's Approximation

As explained already in Sections 2.4 and 3.4, the probability integral transformation $u = F(x)$ transforms the order statistic $X_{i:n}$ from a population with pdf $f(x)$ and cdf $F(x)$ into the uniform order statistic $U_{i:n}$ for $i = 1, 2, \ldots, n$. Hence, by inverting the above transformation we get for $1 \le i \le n$

$$X_{i:n} = F^{-1}(U_{i:n}) = G(U_{i:n}), \tag{3.11.1}$$

which, when expanded in a Taylor series around the point $E(U_{i:n}) = i/(n+1) = p_i$, gives

$$X_{i:n} = G_i + G_i'(U_{i:n} - p_i) + \frac{1}{2} G_i''(U_{i:n} - p_i)^2$$
$$+ \frac{1}{6} G_i'''(U_{i:n} - p_i)^3 + \frac{1}{24} G_i^{\text{iv}}(U_{i:n} - p_i)^4 + \cdots; \tag{3.11.2}$$

here, G_i denotes $G(p_i)$, G_i' denotes $d/du\, G(u)|_{u=p_i}$, and similarly G_i'', G_i''', $G_i^{\text{iv}}, \ldots,$ denote successive derivatives of $G(u)$ evaluated at $u = p_i$. Then, by taking expectation on both sides of (3.11.2) and by using the expressions of the central moments of uniform order statistics derived from (3.4.7) (written, however, in inverse powers of $n+2$ by David and Johnson (1954) for simplicity and computational ease), we obtain

$$\alpha_{i:n} \simeq G_i + \frac{p_i q_i}{2(n+2)} G_i'' + \frac{p_i q_i}{(n+2)^2}\left[\frac{1}{3}(q_i - p_i) G_i''' + \frac{1}{8} p_i q_i G_i^{\text{iv}}\right]$$
$$+ \frac{p_i q_i}{(n+2)^3}\left[-\frac{1}{3}(q_i - p_i) G_i''' + \frac{1}{4}\{(q_i - p_i)^2 - p_i q_i\} G_i^{\text{iv}}\right.$$
$$\left. + \frac{1}{6} p_i q_i (q_i - p_i) G_i^{\text{v}} + \frac{1}{48} p_i^2 q_i^2 G_i^{\text{vi}}\right], \tag{3.11.3}$$

where $q_i = 1 - p_i = (n - i + 1)/(n+1)$. Similarly, by taking expectation on the series expansion for $X_{i:n}^2$ obtained from (3.11.2), and then subtracting from it the expression of $\alpha_{i:n}^2$ obtained from (3.11.3), we get an approximate formula for the variance of $X_{i:n}$:

$$\beta_{i,i:n} \simeq \frac{p_i q_i}{n+2}(G_i')^2 + \frac{p_i q_i}{(n+2)^2}\left[2(q_i - p_i) G_i' G_i'' + p_i q_i \left\{G_i' G_i''' + \frac{1}{2}(G_i'')^2\right\}\right]$$
$$+ \frac{p_i q_i}{(n+2)^3}\left[-2(q_i - p_i) G_i' G_i'' + \{(q_i - p_i)^2 - p_i q_i\}\right.$$
$$\times \left\{2 G_i' G_i''' + \frac{3}{2}(G_i'')^2\right\} + p_i q_i (q_i - p_i)\left\{\frac{5}{3} G_i' G_i^{\text{iv}} + 3 G_i'' G_i'''\right\}$$
$$\left. + \frac{1}{4} p_i^2 q_i^2 \left\{G_i' G_i^{\text{v}} + 2 G_i'' G_i^{\text{iv}} + \frac{5}{3}(G_i''')^2\right\}\right]. \tag{3.11.4}$$

Next, by taking expectation on the series expansion for $X_{i:n}X_{j:n}$ obtained from (3.11.2), and then subtracting from it the expression of $\alpha_{i:n}\alpha_{j:n}$ obtained from (3.11.3), we derive an approximate formula for the covariance of $X_{i:n}$ and $X_{j:n}$ $(1 \le i < j \le n)$:

$$\begin{aligned}
\beta_{i,j:n} \simeq {} & \frac{p_i q_j}{n+2} G_i' G_j' + \frac{p_i q_j}{(n+2)^2}\Big[(q_i - p_i) G_i'' G_j' + (q_j - p_j) G_i' G_j'' \\
& + \frac{1}{2} p_i q_i G_i''' G_j' + \frac{1}{2} p_j q_j G_i' G_j''' + \frac{1}{2} p_i q_j G_i'' G_j''\Big] \\
& + \frac{p_i q_j}{(n+2)^3}\Big[-(q_i - p_i) G_i'' G_j' - (q_j - p_j) G_i' G_j'' + \{(q_i - p_i)^2 - p_i q_i\} G_i''' G_j' \\
& + \{(q_j - p_j)^2 - p_j q_j\} G_i' G_j''' + \Big\{\frac{3}{2}(q_i - p_i)(q_j - p_j) + \frac{1}{2} p_j q_i \\
& - 2 p_i q_j\Big\} G_i'' G_j'' + \frac{5}{6} p_i q_i (q_i - p_i) G_i^{\text{iv}} G_j' + \frac{5}{6} p_j q_j (q_j - p_j) G_i' G_j^{\text{iv}} \\
& + \Big\{p_i q_j (q_i - p_i) + \frac{1}{2} p_i q_i (q_j - p_j)\Big\} G_i''' G_j'' + \Big\{p_i q_j (q_j - p_j) \\
& + \frac{1}{2} p_j q_j (q_i - p_i)\Big\} G_i'' G_j''' + \frac{1}{8} p_i^2 q_i^2 G_i^{\text{v}} G_j' + \frac{1}{8} p_j^2 q_j^2 G_i' G_j^{\text{v}} \\
& + \frac{1}{4} p_i^2 q_i q_j G_i^{\text{iv}} G_j'' + \frac{1}{4} p_i p_j q_j^2 G_i'' G_j^{\text{iv}} \\
& + \frac{1}{12}(2 p_i^2 q_j^2 + 3 p_i p_j q_i q_j) G_i''' G_j'''\Big]. \qquad (3.11.5)
\end{aligned}$$

Similarly, series approximations for the first four cumulants and cross-cumulants of order statistics have been developed by David and Johnson (1954); also see K. Pearson and M. V. Pearson (1931). Clark and Williams (1958) have developed very similar series approximations by making use of the exact expressions of the central moments of uniform order statistics where the kth central moment is of order $\{(n+2)(n+3)\cdots(n+k)\}^{-1}$ instead of in inverse powers of $n+2$. A different kind of series approximation based on the logistic distribution rather than on the uniform distribution has been given by Plackett (1958). Saw (1960) has employed the Darboux form for the remainder in a Taylor series expansion in order to obtain bounds for the remainder term when the expansion for the first single moment $\alpha_{i:n}$ $(1 \le i \le n)$ in (3.11.3) is terminated after an even number of terms in the series. Details of these works may be had from Arnold and Balakrishnan (1989).

As pointed out by David and Johnson (1954), the evaluation of the derivatives of G_i is rather easy in most cases. First of all, we realize that

$$G_i' = \frac{d}{du} G(u)|_{u=p_i} = \frac{1}{f(F^{-1}(u))}\bigg|_{u=p_i} = \frac{1}{f(G_i)}, \tag{3.11.6}$$

which is just the reciprocal of the pdf of the population evaluated at G_i. The above expression of G_i' in (3.11.6) allows us to write down the higher-order derivatives of G_i without great difficulty in most cases.

For example, for the standard normal distribution, by making use of the property $f'(x) = -xf(x)$ we get

$$\begin{aligned}
G_i' &= 1/f(G_i),\\
G_i'' &= G_i/\{f(G_i)\}^2,\\
G_i''' &= \{1+2G_i^2\}/\{f(G_i)\}^3,\\
G_i^{\text{iv}} &= G_i(7+6G_i^2)/\{f(G_i)\}^4,\\
G_i^{\text{v}} &= \{7+46G_i^2+24G_i^4\}/\{f(G_i)\}^5,\\
G_i^{\text{vi}} &= G_i\{127+326G_i^2+120G_i^4\}/\{f(G_i)\}^6,
\end{aligned}$$

etc.

Similarly, for the logistic population with pdf and cdf as

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{and} \quad F(x) = \frac{1}{1+e^{-x}},$$

respectively, we obtain

$$\begin{aligned}
G_i &= F^{-1}(p_i) = \ln(p_i) - \ln(q_i),\\
G_i' &= \frac{1}{p_i} + \frac{1}{q_i},\\
G_i'' &= -\frac{1}{p_i^2} + \frac{1}{q_i^2},\\
G_i''' &= 2\left\{\frac{1}{p_i^3} + \frac{1}{q_i^3}\right\},\\
G_i^{\text{iv}} &= 6\left\{-\frac{1}{p_i^4} + \frac{1}{q_i^4}\right\},\\
G_i^{\text{v}} &= 24\left\{\frac{1}{p_i^5} + \frac{1}{q_i^5}\right\},\\
G_i^{\text{vi}} &= 120\left\{-\frac{1}{p_i^6} + \frac{1}{q_i^6}\right\},
\end{aligned}$$

etc.

As illustrated by David and Johnson (1954), and also by several other authors, this simple approximation procedure works well in most cases. However, this procedure may not provide satisfactory results for the extreme order statistics. In this case, the convergence of this approximation to the exact value may be very slow, and even nonexistent in some cases.