The Geometric Growth Model of Complex Networks

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Abstract

The model by Watts and Strogatz shows the possibility that a small-world network results from a few long-range edges. Barabási and Albert introduced the scale-free model driven by two mechanisms-the linear growth and the preferential attachment, which predicts the emergence of a power-law connectivity distribution. With these models, the range ($2 \sim 3$) of exponent is still indescribable property. Based on the properties of real networks, we introduce the new geometric growth model for many real networks. We characterize the dynamical evolution of the new model with geometrically growth rate and two parameters: addition and multiplication. Analysis of our model shows that it displays a power-law distribution in connectivity and has the range ($2 \sim 3$) of exponent in order to be the more efficient small-world networks.

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I. INTRODUCTION

The complex systems are ubiquitous in nature. Examples of the complex networks include the Internet [1], the World-Wide Web [2], collaborating movie actors [3], the scientific collaboration [4], the power grid [3], and the metabolic network [5]. Each component in complex systems interacts with others and develops a self-organized network structure in time. The simplest approach to the complex systems is to represent their components with vertices and the edges with the interactions between them.

The random graph theory of Erdös-Rényi was the first attempt to describe the topology of complex systems [6]. If all components connect blindly with others, the random graph theory will be valid. But the networks in nature consist of organized interactions. Watts and Strogatz [3] showed that the small-world networks can be found in the regime between regular and random networks. They defined two measures to characterize the network; the characteristic path length and the clustering coefficient. The characteristics of small-world networks is the short characteristic path length and high clustering coefficient of the networks. They proposed that small-world networks result from the introduction of a few long-range connections.

Those models, though stimulating, was not able to explain that the recent discovery that the probability of a vertex in the network being connected to k other vertices decays as a power law [7]. The power-law characteristics of the networks are independent of the characteristic of the system and the properties of components. To resolve the disagreement between distributions of the former models and those of systems in nature, two ingredients were introduced by Barabási and Albert [7]: the linear growth of networks and the preferential attachment. In their scale-free model, it is assumed that the size of network grows linearly, one by one, by adding one new node and probability that a new node will connect with others depends on the existing number of connections for vertices, mimicking the growth rule in real networks [8,9]. However, some networks in nature is known to display a geometric growth [10,11].

The goal of this paper is to introduce the new geometric growth model with the generic aspects of the real-world networks; the geometric growth of networks and the preferential attachment. Our model is based on the geometric growth rule with two intrinsic parameters; the addition parameter and the multiplication parameter. The simplicity of our model allows the exact calculation of the characteristic path length and the power-law exponent for the connective distribution. We find that the connectivity distribution from our model can vary from power-law to exponential-like with increasing random attachment and the exponent of the connectivity distribution varies from 1 to 3 for the geometric growth. This variation in the exponent of the connectivity distribution is not possible in the context of the simple linear growth model, requiring the fine-tuning of probability distributions in preferential attachment [7]. The often observed range ($2 \sim 3$) of exponents in real world connectivity distributions may be due to the efficiency in the communication between components. It is also exactly shown that the behavior of the power law and the short characteristic path length of the networks result from the self-organized network by multiplication not from a few long-range edges. Our model can be generalized to random networks with probability distributions associated with the addition and multiplication of connections. This generalized model can be analyzed using the master equation approach.

In section II, we find the evidence how the network in nature evolves and introduce the new simple model characterized by the geometric growth with two parameters for addition and multiplication. The analytic results on the characteristic path length and the exponent of power-law distribution of the complex network are obtained in section III. In section IV, the results in the previous section are verified by numerical simulations. The master equation approach for the probabilistic model with the geometric growth was shown in section V. Finally we end with a conclusion.

II. NEW SIMPLE GROWTH MODEL FOR REAL NETWORKS

A. The geometric growth in nature

Many networks in nature expand the population geometrically at least for early developing periods. The world population is a good example to explain how the number of social components grow. A social network is a collection of people. With a set of vertices denoting people, joined in pairs by edges denoting acquaintance. Nearly all mature individuals can produce offsprings, that is, new nodes. Then newly born individuals can connect to other individuals in the view of social acquaintance. The Population Division of the Department of Economic and Social Affairs at the United Nations Secretariat provided scientifically objective information on population [10]. "The World at Six Billion", a recent publication, provides in graphic form salient characteristic of past and current world population growth as in Fig. 1. Both population size and rate grow geometrically, which provides an evidence that social networks can grow geometrically at least at an early stage.

There is another of evidence to verify the growth pattern in the real networks. It is the world wide web. It is a common belief that the number of web sites grow exponentially. For examples, the Network Wizards company uses the Zone program to determine the approximate number of Internet hosts shown in Fig. 2. In fact, most networks expand geometrically at an early stage when there is no limitation on the growth resources and no inhibitory mechanisms against network growth.

When the complex networks grow in nature, the connections between components are not randomly created a priori. Like the model proposed by Barabási and Albert, the connectivity may be created by preferential attachment. To demonstrate the scale-free behavior of connectivity, there can be a variety of methods. We propose the complex networks with simple two ingredients in the growth rule; addition rule and multiplication rule. The multiplication rule is consistent with preferential attachment. It indicates that the node in network expands its connections by multiplying some factor with its previous number of connectivity. In contrast, the addition rule corresponds to equal attachment or random attachment.

B. New Geometric growth model

The previous models demonstrate the complex networks with the preferential attachment and linear growth of nodes. The preferential attachment was found in many real-world networks [7,14,13], but linear growth of nodes is adopted to simplify the simulation. Many real-world networks have the geometric growth rate on the number of vertices. Therefore, we introduce the new geometric growth model to describe the real-world networks. This model is characterized by the addition rule and multiplication rule and results in the geometric growth rate on the number of nodes.

1. Addition rule

The addition rule is that all components in networks get the same number of connections in time. When networks generate the new nodes, they will link with the existing vertices. In other words, the existing vertices will increase their connections as the number of the component in the network grows on the existing vertices. In case that the newly generated vertices randomly stretch their connections on the existing vertices, all existing vertices expand their connections in equal number. We classify this process as addition.

For example, we consider $(1+\alpha)$ vertices as the initial condition, where α is the addition parameter. For convenience, the vertex C has α connections to other vertices I which connects only to the vertex C. After one time step, every vertex increases its connections following the rule that total connections to each vertex are added by α connections.

For example, if the addition parameter, $\alpha = 3$, after each step, each vertex has three more vertices independently of where it is positioned and how many connections it has. The evolution of the network for a few time steps is shown in Fig. 4.

The above addition rule will generate the expanding network. To investigate the properties of the network, we want to know how fast the system, with the addition rule grows in time. For the first step, we calculate the number of newly generated nodes $I(t)$ at the time step t.

$$
\begin{aligned}
I(t=0) &= 1+\alpha \\
I(t=1) &= (1+\alpha)\alpha \\
&= \alpha+\alpha^2 \\
I(t=2) &= \{(1+\alpha)\alpha+1+\alpha\}\alpha \\
&= \alpha+2\alpha^2+\alpha^3 \\
\vdots\ &\ \vdots
\end{aligned}
$$

In general, after the time step T, we get

$$
\begin{aligned}
I(t=T) &= \alpha\sum_{p=0}^{T}\{\begin{pmatrix} T \\ p \end{pmatrix}\alpha^p\} \\
&= \begin{cases} \alpha(1+\alpha)^T & \text{when T} \geq 1 \\ 1+\alpha & \text{when T} = 0 \end{cases}.
\end{aligned}
$$

At each time step, the number of nodes that will be born next time, are geometrically increasing with the base, $(1+\alpha)$. The total number of nodes $N(T)$ at $t=T$ is given by

$$
\begin{aligned}
N(T) &= \sum_{j=0}^{j=T} I(t=j) \\
&= (1+\alpha)^{T+1}. \qquad (1)
\end{aligned}
$$

Note that $N(T)$ also shows the geometrical expansion in size. Equal number of nodes are added to every node independent of both its position and the number of its connections. The network which grows under the addition rule can be expressed as

$$
\frac{\partial k_i}{\partial t} = \alpha, \qquad (2)
$$

When k_i is the number of connections at site i. The form of growth is closely related to the random attachment in contrast to the preferential attachment in models of Barabási et al [7,14].

2. Multiplication rule

The scale-free model shows that the probability of their links is proportional to their own connections. We generalize this kind of growth which is controlled by a parameter, called as multiplication parameter, m that multiply the number of connected component at a fixed rate after each time step.

We consider an initial configuration with $(1+m)$ vertices For convenience, the vertex C is assumed to have m connections to other vertices I, where I is connected only to the vertex C. After one time step, each vertex increases its connections following the simple rule that total connections are m times the number of previous connections. Figure 5 shows an example with $m = 2$.

To find the growth rate of the network, we calculate the number of newly generated nodes $I(t = T)$ at the time step t. It is given by

$$\begin{aligned} I(t=T) &= 2m(m-1)\sum_{p=0}^{T-1}\begin{pmatrix} T-1 \\ p \end{pmatrix} m^p (m-1)^{T-1-p} \\ &= \left\{ \begin{array}{ll} 2m(m-1)(2m-1)^{T-1} & \text{when T} \geq 1 \\ 1+m & \text{when T} = 0 \end{array} \right), \end{aligned} \tag{3}$$

The total number of nodes $N(T)$ at $t = T$ is given by

$$\begin{aligned} N(T) &= \sum_{j=0}^{j=T} I(t=j) \\ &= 1 + m(2m-1)^T. \end{aligned} \tag{4}$$

From above, both the number of newly generated nodes and the size of nodes in the system, grow geometrically with the basis $(2m - 1)$. The network which grows under the multiplication rule can be expressed as

$$\frac{\partial k_i}{\partial t} = (m-1)k_i, \tag{5}$$

which is analogous to the preferential attachment. Note that the preferential attachment is described as [4]

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}. \tag{6}$$

We can prove that these two equations (Eq. 5 and 6) are equivalent. First, the relation between a continuous rate of change of k_i and the probability $\Pi(k_i)$ is written as

$$\frac{\partial k_i}{\partial t} = A\Pi(k_i), \tag{7}$$

where the denominator in Eq.6 is given by

$$\sum_j k_j = 2m(2m-1)^{t-1}. \tag{8}$$

Therefore, the system generates new nodes geometrically at each time t. If the number of connections from the newly generated node is the only source, the number of new connections toward the old vertices is the same as the number of the generated vertices. This can be expressed as

$$g(t) = \frac{\partial \sum_i k_i}{\partial t} = 2m(m-1)(2m-1)^{t-1}. \tag{9}$$

In our model, the growth rate $g(t)$ is the same as the summed connectivities from 0 to $t-1$ times $(m-1)$. Summing over all vertices in Eq. 7, we get

$$A = \frac{\partial \sum_i k_i}{\partial t} = 2m(m-1)(2m-1)^{t-1}. \tag{10}$$

Substituting Eq.6, Eq.8, and Eq.10 into Eq.7, we obtain

$$\frac{\partial k_i}{\partial t} = (m-1)k_i, \tag{11}$$

which proves the equivalence between the multiplication rule and the preferential attachment.

III. ANALYSIS OF THE GEOMETRIC GROWTH MODEL

A. Exponent in the power-law connectivity distribution

In our model, the size of vertices is geometrically increasing. However, the total number of vertices in the scale-free model proposed by Barabási and Albert [7,14] grows at a constant rate. The difference in growing rates affects power-law exponents in the connective distribution of the complex networks. If the distributions in connectivity are a power-law tail, these networks are evolved under the only preferential attachment which is identical with the multiplication rule [7,14]. Therefore, the networks only by the multiplication rule were considered in order to study on the exponent.

1. Growth under the multiplication rule

First, we assume the systems that generate new vertices at the constant rate. We will define these networks as the simple-growth network. The previous scale-free models belong to this class of networks [7,14].

- *Simple growth rule* :

  If the network follows the simple growth rule, with $g(t) \cong (G(t-1))'$ where $G(t-1) \equiv \sum_{t_j=0}^{t-1} g(t_j)$.
  For example, $g(t) = k$ satisfies the above condition. In this case, $G(t-1) = kt$.

- *Connections* :

  When the addition degree $\alpha = 0$, the probability $\Pi(k_i)$ that the vertex i receives more connections is given by

$$\Pi(k_i) = \frac{(m-1)k_i}{\sum_j (m-1)k_j}. \tag{12}$$

The rate of change for k_i is proportional to the probability $\Pi(k_i)$. So, the continuous form is expressed as

$$\frac{\partial k_i}{\partial t} = A\Pi(k_i) = A\frac{k_i}{\sum_j k_j}. \tag{13}$$

Summing both sides of Eq. 13 over all nodes, we get

$$g(t) = \frac{\partial \sum_i k_i}{\partial t} A, \tag{14}$$

where $g(t)$ is the number of newly generated connections at time t. We assume that new nodes have one connections. So, $g(t)$ is equal to the number of new nodes at the given time except $t = 0$. The number of total connections is given by the number of nodes minus one.

To find the number of connections that belong to the vertex i, we compute the denominator in Eq. 13 as

$$\begin{aligned}\sum_j k_i &= 2\sum_{t_j=0}^{t-1} g(t_j) \\ &= 2G(t-1),\end{aligned} \tag{15}$$

which leads to

$$\frac{\partial k_i}{\partial t} = \frac{g(t)}{2G(t-1)} k_i. \tag{16}$$

Therefore, it follows

$$k_i(t) = \left(\frac{G(t-1)}{G(t_i-1)}\right)^{\frac{1}{2}}, \tag{17}$$

where t_i is the time at which vertex i was added to the network. The probability that a vertex has a connectivity $k_i(t)$ larger than k is expressed as

$$P(k_i(t) > k) = P(G(t_i - 1) < G(L(k,t)). \tag{18}$$

where $L(k,t)$ is the upper limit of t_i and $G(L(k,t)) \equiv k^{-2}G(t-1)$. The probability density of t_i, $P_i(t_i)$ is given by

$$P_i(t_i) = \frac{g(t_i)}{\sum_{t_j=0}^{t} g(t_j)} = \frac{g(t_i)}{G(t)}, \tag{19}$$

so that we get

$$\begin{aligned} P(k_i(t) > k) &= \int\limits_{0}^{L(k,t)} \frac{g(t_i)}{G(t)} dt_i \\ &= \frac{G(L(k,t))}{G(t)} - \frac{G(0)}{G(t)} \\ &\propto k^{-2}. \end{aligned} \tag{20}$$

The probability for an arbitrary connectivity k, $P(k)$, is obtained as

$$\begin{aligned} P(k) &= \frac{\partial[1 - P(k_i(t) > k)]}{\partial k} \\ &\propto k^{-\gamma}. \end{aligned} \tag{21}$$

Note that the exponent for the connectivity distribution γ is always 3 independent of m if the network satisfies the simple-growth rule $g(t) \cong dG(t-1)/dt$.

In previous section, it was shown that some natural networks exhibit geometric growth rules. Now, we investigate the effect of the geometric growth rule with found on the behavior of the exponent of connectivity distribution.

- *Geometric growth rule* :

  At each time step, we add $2m(m-1)(2m-1)^{t-1}$ vertices, which leads to the geometric growth.

- *Connections* :

  Assume that each new vertex makes only one new connection to other vertices, following the multiplication rule,

$$\frac{\partial k_i}{\partial t} = (m-1)k_i. \tag{22}$$

Similarly as before, the probability for an arbitrary connectivity k, $P(k)$, can be obtained as

$$P(k) \propto k^{-\frac{1}{m-1}\log(2m-1)-1}. \tag{23}$$

So, the exponent γ is a function of the multiplication parameter m, so that $\gamma(m) = \frac{1}{m-1}\log(2m-1)+1$. Fig. 6 shows how the exponent varies as a function of the multiplication parameter m.

In the limit of $m \to 1$, we get $\gamma(1) = 3$. In the opposite case that m goes to infinity, we get $\gamma(\infty) = 1$. Numerical simulations in the next section verify the validity of these results. It explains why the exponents of real networks are bounded above with the maximum exponent of [16]. The geometric property and multiplication rule show the proper range of exponent of connectivity distribution.

In contrast to the case of the simple growth of the network, the geometric growth rule allows the exponent of the power-law connectivity distribution vary in the range of $1 \sim 3$. This variation in γ is consistent with the fact that nearly scale-free power-law networks in nature shows a range of the exponents, $\gamma = 2 \sim 3$ [16]. Though the higher exponent value of $\gamma \approx 4$ was reported in some networks [7], the connectivity distribution in this case has an exponential tail.

2. Growth under the addition rule

We know that network with only the multiplication parameter shows the scale-free distribution in connectivity mainly due to preferential attachment. But the non-preferential attachment, which is achieved by the addition parameter, can generate different types of connectivity distribution, which is explored in this section.

- *Growth rule under addition* :

  At each time step, we add $\alpha(1+\alpha)^t$ vertices.

- *Connections* :

  Based on the addition rule, every existing nodes increase the connections from the newly generated nodes by the same number. It is assumed that each new node is linked to only one existing node.

$$\frac{\partial k_i}{\partial t} = \alpha. \tag{24}$$

Therefore, we get

$$k_i(t) = \alpha(t - t_i). \tag{25}$$

Note that k_i increases linearly in time. Similarly as before, the probability for an arbitrary connectivity k, $P(k)$, can be obtained as

$$\begin{aligned} P(k) &= \frac{\partial[1 - P(k_i(t) > k)]}{\partial k} \\ &\propto (1+\alpha)^{-(\frac{k}{\alpha}+1)}. \end{aligned} \tag{26}$$

So, the connectivity distribution decays exponentially in k. While the multiplication rule leads to the power-law behaviors of connectivity distribution, the addition rule leads to the single-scale distribution with an exponentially decaying tail. But the exponential tail in log-scale is not easily distinguishable with the power-law tail for finite data size.

B. Characteristic path length

The characteristic path length is defined as the average value of minimal distance between any pair of vertices. Now focus on the multiplication rule which leads to the power-law connectivity distribution. The characteristic path length $l(t)$ at time step t is given by

$$l(t) = \frac{1}{N(t)(N(t)-1)} \sum_{i \neq j} d_{ij}(t), \tag{27}$$

where $N(t)$ is the total number of nodes at time t and $d_{ij}(t)$ is the shortest distance between vertex i and the vertex j. Note that $N(t)$ can be obtained from Eq. 1 and Eq. 4. The total sum of distances, $L(t)$, between two nodes is

$$L(t) = \sum_{i \neq j} d_{ij}(t), \tag{28}$$

where

$$
\begin{aligned}
L(t=0) &= N_C \cdot 1m + N_I \cdot [1 + 2(m-1)], && (29)\\
L(t=1) &= N_C \cdot [1m^2 + 2m(m-1)]\\
&\quad + N_I \cdot [1m + 2\{m(m-1) + (m-1)\} + 3(m-1)(m-1)]\\
&\quad + N_a \cdot [1 + 2\{m^2 - (m+1) + m\} + 3m(m-1)]\\
&\quad + N_b \cdot [1 + 2(m-1) + 3(m^2-1) + 4(m-1)^2], && (30)\\
L(t=2) &= N_C \cdot [1m^3 + 2\{(m^2-m)(m-1) + m(m-1) + m(m^2-m)\}\\
&\qquad +3m(m-1)(m-1)]\\
&\quad + N_I \cdot [1m^2 + 2\{m^3 - 1 + (m-1)(m-1)\}\\
&\qquad +3\{(m-1)^2 + (m^2-m)(m-1) + (m-1)(m^2-m)\} + 4(m-1)^3]\\
&\quad + N_a \cdot [1m + 2(m^3-1)\\
&\qquad +3\{(m(m-1)-1)(m-1) + m(m^2-m) + m(m-1)\} + 4m(m-1)^2]\\
&\quad + N_b \cdot [1m + 2(m^2-1) + 3\{(m^3-1) + (m-1)(m-2)\}\\
&\qquad +4\{(m^2-m)(m-1) + (m-1)^2 + (m-1)(m^2-m)\} + 5(m-1)^3]\\
&\quad + N_1 \cdot [1 + 2(m^3-1) + 3\{m^2(m-1) + m(m^2-m)\} + 4m(m-1)^2]\\
&\quad + N_2 \cdot [1 + 2(m-1) + 3(m^3-1)\\
&\qquad +4\{(m(m-1)-1)(m-1) + m(m^2-1)\} + 5m(m-1)^2]\\
&\quad + N_3 \cdot [1 + 2(m^2-1) + 3\{(m^3-1) + (m-1)(m-1)\}\\
&\qquad +4\{(m^2-1)(m-1) + (m-1)(m^2-m)\} + 5(m-1)^3]\\
&\quad + N_4 \cdot [1 + 2(m-1) + 3(m^2-1) + 4\{(m^3-1) + (m-1)(m-1-1)\}\\
&\qquad +5\{(m^2-1)(m-1) + (m-1)(m^2-1)\} + 6(m-1)^3], && (31)\\
&\vdots \quad \vdots
\end{aligned}
$$

where

$$N_C = 1, N_I = m,\ N_a = m^2 - m,\ N_b = m(m-1),\ N_1 = m^3 - m^2,$$

$$N_2 = (m^2 - m)(m-1),\ N_3 = m(m^2 - m),\ N_4 = m(m-1)(m-1), \tag{32}$$

and the subscripts for N are defined in Fig. 5. From above equations, $L(t)$ can be expanded in terms of m up to $m^{2(T+1)}$. Though it is difficult to obtain a general expression for L, we can compute two asymptotic limits.

1. $m \cong 1$

   Let $m = 1+\epsilon$, $\epsilon \ll 1$. The number of nodes $N(t=T)$ at time step T is given by

$$\begin{aligned} N(t=T) &= \sum_{j=0}^{j=T} I(t=j) \\ &= 1 + (1+2\epsilon)^T (1+\epsilon) \\ &\approx 2 + (2T+1)\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{33}$$

   It follows that

$$N(N-1) \approx 2 + 3(2T+1)\epsilon + \mathcal{O}(\epsilon^2) \tag{34}$$

   and from Eq. 33, we get

$$T = \frac{\log(N-1) - \log(1+\epsilon)}{\log(1+2\epsilon)}. \tag{35}$$

   Therefore, to the lowest order in ϵ, we get

$$\begin{aligned} L(t=0) &= (1+\epsilon) + (1+\epsilon+2\epsilon) + \mathcal{O}(\epsilon^2), \\ L(t=1) &= (1+2\epsilon+2\epsilon) + (1+2\epsilon+2(2\epsilon)) + 2(\epsilon) + \mathcal{O}(\epsilon^2), \\ L(t=2) &= (1+3\epsilon+2(2\epsilon)) + (1+3\epsilon+2(3\epsilon)) + 2(2\epsilon) + \mathcal{O}(\epsilon^2), \\ \vdots \quad & \quad \vdots \\ L(t=T) &= (1+(T+1)\epsilon+2(T\epsilon)) + (1+(T+1)\epsilon+2((T+1)\epsilon)) + 2(T\epsilon) + \mathcal{O}(\epsilon^2) \\ &= 2 + 4(2T+1)\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{36}$$

   From Eq.27, the characteristic length for a given N is

$$
\begin{aligned}
l &= \frac{2+4(2T+1)\epsilon}{2+3(2T+1)\epsilon} + \mathcal{O}(\epsilon^2), \\
&\cong \frac{-8}{7+6\log(N-1)} + \frac{4}{3}.
\end{aligned} \tag{37}
$$

The characteristic length has the upper limit of 4/3 even though the number of vertices, N, go to the infinity.

2. $m \gg 1$,

   The total number of nodes $N(t=T)$ at time step T is given by

$$
\begin{aligned}
N(t=T) &= \sum_{j=0}^{j=T} I(t=j), \\
&= 1 + m(2m-1)^T.
\end{aligned} \tag{38}
$$

   It follows that

$$
N(N-1) \approx 2^{2T} m^{2(T+1)} \tag{39}
$$

   and from Eq. 38, we get

$$
T = \frac{\log(N-1) - \log m}{\log(2m-1)}. \tag{40}
$$

   Since m is a large number, the highest order terms in m dominate, we obtain

$$
\begin{aligned}
L(t=0) &= 2m^2 + \text{lower order terms} \\
L(t=1) &= [(2+3)+(3+4)]m^4 + \text{lower order terms} \\
L(t=2) &= [(2+2\cdot 3+4)+2\cdot(3+2\cdot 4+5)+(4+2\cdot 5+6)]m^6 \\
&\quad + \text{lower order terms} \\
\vdots \quad & \vdots \\
L(t=T) &= \sum_{p=0}^{T} \begin{pmatrix} T \\ p \end{pmatrix} \sum_{k=0}^{T} \begin{pmatrix} T \\ k \end{pmatrix} (k+p+2) m^{2(T+1)},
\end{aligned} \tag{41}
$$

   The total sum of distances between all pairs for time T is given by

$$L(t=T) = \sum_{p=0}^{T} \begin{pmatrix} T \\ p \end{pmatrix} m^{2(T+1)} \left[(T+2(2+p))2^{T-1} \right]$$
$$= (T+2)2^{2T} m^{2(T+1)}. \tag{42}$$

Therefore, the characteristic path length $l(T)$ at time step T becomes

$$l \cong T+2$$
$$\cong \frac{\log(N-1)}{\log(2m-1)} + 2 - \frac{\log m}{\log(2m-1)}. \tag{43}$$

The first term on the right hand side of Eq.43 shows a logarithmic N-dependence. This shows that our model exhibits small-world behavior with logarithmic characteristic path length dependence on N as suggested by Watts and Strogatz [3].

Note that we focused on the large multiplication parameter limit because we are interested in the maximum value of the characteristic path length with fixed number of vertices. The maximum value of characteristic path length $l(T)$ is found to be logarithmically proportional to the number of vertices, so that it displays the small-world phenomena; the characteristic path length remains small even though there exit a huge number of vertices in the network. To our knowledge, it is the first analytic proof that the network that has a scale-free power-law distribution leads to the small-world network.

As m approaches 1, the characteristic path length of the network become smaller, so that the network becomes efficient in the communication among components. The range of exponents in real networks lie between 2 and 3, which corresponds to the efficient network with small multiplication parameters.

IV. NUMERICAL ANALYSIS OF THE GEOMETRIC GROWTH MODEL

In this section, we simulate the complex networks under the multiplication and geometric growth rules in section II. First, we vary the addition parameter with the multiplication parameter fixed. It is found that as the addition degree is increased, the tail part of distributions with exponential decay shrinks in size as shown in Fig. 7. These types of distributions were found in examples such as the movie-actor networks , the acquaintance network of Mormons and so on [15].

When the addition degree is increased further, the distributions are no longer scale-free and show exponential-like behavior. This indicates the addition degree controls the shape of distributions of connectivity from the scale-free to the exponential decay. The electric-power grid of Southern California and the network of world airport [15] displays this type of distribution similar to our results for larger addition degrees.

Next we will numerically investigate how the distribution changes as the multiplication parameter is changed with the addition parameter fixed. Fig. 8 shows that the increase in the multiplication degrees turns the distribution from the exponential decay to the scale-free power law. Note that large m leads to the exponent $\gamma \approx 1$.

In our new model, we can find the power-law tail of connectivity when the network evolves only by multiplication rule. In Fig. 9, the exponent of the power-law distribution changes from 1 to 3 as the multiplication parameter is decreased to 1.

Two asymptotic values of the exponent γ calculated in Section III and II agree with the results of numerical simulations. In the intermediate regions the numerically obtained exponents are somewhat smaller than those of analytic calculations. This small discrepancy between analytic and numerical result is a subject of more detailed investigations.

V. MASTER EQUATION WITH GEOMETRIC GROWTH

The master equation approach is good for this kind of problems [12]. We consider the network with the properties as mentioned in Section II. Geometric growth model has tree-like structure. But, we want to study on the more general networks with loops and the arbitrary geometric growth rate. This master equation approach gives the probabilistic model which represents more general networks in nature.

There is the existing network and the new generated nodes. At every time step, the number of new nodes in network is an arbitrary geometric function with time. Each new node is assumed to have the equal number (M) of connections. The existing node will receive links with the probability which depends on the two system parameters; multiplication and addition. The Fig. 3 shows this situation.

Let us derive the equation for the distribution $p(k,i,t)$ of connections k on the site i at the t time. The probability that a new vertex will be connected to vertex i depends on the two system parameters of that vertex, such that

$$\Pi(i) = \frac{mk_i + \alpha}{\sum_j mk_j + \alpha}. \tag{44}$$

The total sum of connections $\sum_j k_j$ will be $2M$ times the total number of nodes $N(t)$. So, $\sum_j mk_j + \alpha = (2mM+\alpha)N(t)$, where $N(t)$ is the total number of existing nodes in networks at t time. The probability for the site to receive exactly l new links among W new links is

$$\binom{W}{l}\Pi(i)^l(1-\Pi(i))^{W-l}. \tag{45}$$

Like this, the connectivity distribution of a site follows the master equation:

$$p(k,i,t+1) = \sum_{l=0}^{m}\binom{W}{l}[\frac{m(k_i-l)+\alpha}{(2mM+\alpha)N(t)}]^l[1-\frac{m(k_i-l)+\alpha}{(2mM+\alpha)N(t)}]^{W-l}p(k-l,i,t). \tag{46}$$

where W, the total number of new coming links are M times c^t.

$$P(k,t) = \frac{1}{N(t)}\sum_{x=1}^{x=\text{last node}} p(k,x,t), \tag{47}$$

Summing up Eq. 46 over all nodes,

$$\begin{aligned} N(t+1)P(k,t+1) = {} & [N(t) - \frac{Mc^t}{2mM+\alpha}(mk+\alpha)]P(k,t) \\ & + \frac{Mc^t}{2mM+\alpha}(m(k-1)+\alpha)P(k-1,t) + O(\frac{P}{N(t)}). \end{aligned} \tag{48}$$

This equation transformed into continuous form. And we use the condition for the stationary connectivity distribution [12].

$$\frac{(2mM+\alpha)}{M}\frac{N(t)\log d}{c^t}P(k)+(mk+\alpha)P(k)-(m(k-1)+\alpha)P(k-1)=(2m+\alpha)\delta(k),\tag{49}$$

where d is the basis of $N(t)$. Our model is geometrically growing with time. c^t is the newly generated node at t time and $N(t)$ is related to the c^t.

$$\begin{aligned} N(t) &= \sum_{T=0}^{T=t-1} c^T \\ &\cong \frac{c^t}{c-1} \end{aligned}\tag{50}$$

So, the basis of $N(t)$ is c. This geometric condition makes the equation more simple one because the $\frac{(2mM+\alpha)}{M}\frac{N(t)\log d}{c^t}$ is constant. Let's define the constant, $\beta \equiv \frac{(2mM+\alpha)}{M}\frac{\log c}{(c-1)}$.

$$\beta P(k)+(mk+\alpha)P(k)-(m(k-1)+\alpha)P(k-1)=\frac{(2mM+\alpha)}{M}\delta(k),\tag{51}$$

We may use Z-transform $\Phi(z)=\sum_{k=0}^{\infty}P(k)z^k$, and solve this equation. So, we obtain the probability that node will have k connections.

$$P(k)=\frac{2mM+\alpha}{mM}\frac{\Gamma(\frac{\alpha}{m}+\frac{\beta}{m})}{\Gamma(\frac{\alpha}{m})}\frac{\Gamma(\frac{\alpha}{m}+2+k)}{\Gamma(\frac{\alpha}{m}+\frac{\beta}{m}+3+k)}\tag{52}$$

When k is large value, $\Gamma(z)\sim z^z e^{-z}$. We are interested in the exponent in the regime of large connectivity.

$$P(k)\cong\frac{2mM+\alpha}{mM}\frac{\Gamma(\frac{\alpha}{m}+\frac{\beta}{m})}{\Gamma(\frac{\alpha}{m})}k^{-(1+\frac{\beta}{m})}.\tag{53}$$

Therefore, the exponent of connectivity distribution is $1+\frac{(2mM+\alpha)}{mM}\frac{\log c}{c-1}$. The most real complex networks show high preferential attachment and small random attachment [13]. And each new node in real networks has many links to the existing nodes. Therefore, $\alpha \ll 2mM$ is naturally satisfied. If $m\cong 1$, the asymptotic value of exponent is 3. Otherwise, it is 1. So, the exponent of connectivity distribution varies 1 to 3. This result of the networks which have the arbitrary geometric growth of nodes and loops in networks' structure agrees with the previous one in Section III.

VI. SUMMARY AND DISCUSSION

In this article we have introduced the new geometric growth model that can explain many properties of complex networks with generic aspects of the geometric growth in terms of two simple evolving parameters. The multiplication parameter can be interpreted as the degree of preferential attachment. The previous works used the only one kind of preferential attachment, that is, the case that the probability that a new node will link with a vertex i is proportional to the number of links that a vertex i has. But our model adopts the multiplication parameter as the tuner of preferential attachment. The addition parameter corresponds to the non-preferential attachment. The effect of this parameter is that each vertex expands its size of connections in the same number in time. It generates the exponential-like behavior of connectivity distributions whereas the scale-free distribution is obtained from the multiplication parameter.

We found that the exponent of connectivity distribution varies between 1 and 3. This characteristics was shown in both geometric growth model and master equation approaches. It is one of our objectives that the simple exponent-varying model is constructed with the generic aspects; geometric growth rate and preferential attachment. The network of the electrical power grid of the western United States shows a power law with an exponent $\gamma \approx 4$. Though this is obviously out of our expected range of the exponent, it is more exponential-like distribution than scale-free power law. When these kinds of networks are excluded, most scale-free networks in nature show the exponent between 2 and 3. This regime results from the small multiplication parameter. The reason why most real networks tend to have the small multiplication parameter is explainable in the view of the characteristic path length of networks. The distances between any pairs in networks with the small multiplication parameter are smaller than those in higher multiplication parameter. In some sense, shorter distances can make the system more efficient because the information travels along the fewer connections from one to the other vertex. Interestingly, the small multiplication parameter which results in the efficient network, may provide a way to form the real networks with the observed range of exponents in connectivity distributions.

Watts and Strogatz suggested that a small-world network results from both the preserved local neighborhood and a few long-range connections [3]. But our result shows that the small-world effect can be generated not by a few long-range connections but by the self-organized networks. The characteristic path length of the network that is self-organized by multiplication rule is logarithmically proportional to the number of vertices, which has the hierarchical structure found in many natural networks.

In this work, the characteristic path length was calculated using a simple toy model which is a tree-like network. It would be interesting to calculate the characteristic path length of the general networks with loop structures and show the persistence of the small-world behaviors.

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FIGURES

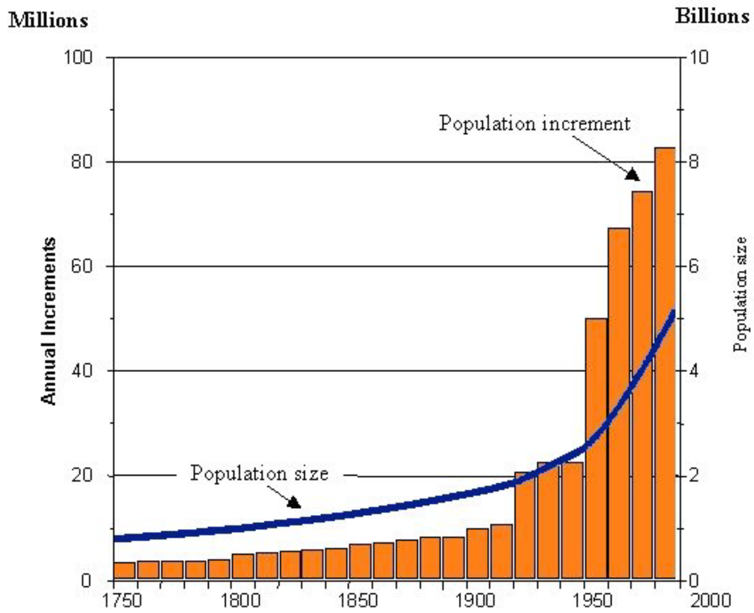

FIG. 1. Both population growth and the rate of population growth increase geometrically.

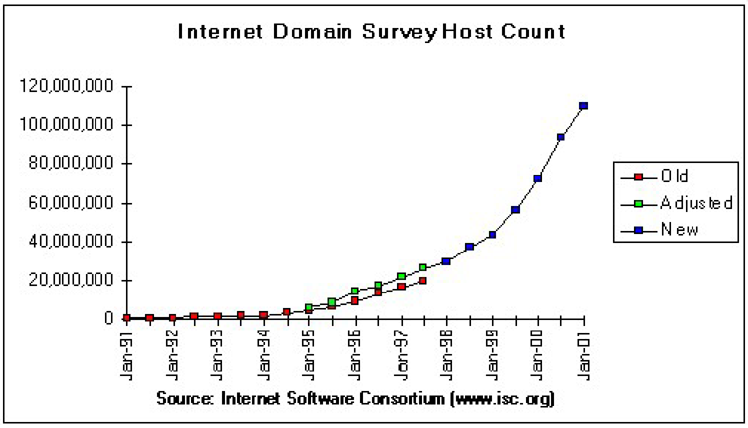

FIG. 2. The increase in Internet domain sites was shown. The number of Internet domain sites grows geometrically. Source: Internet Software Consortium (http://www.isc.org/)

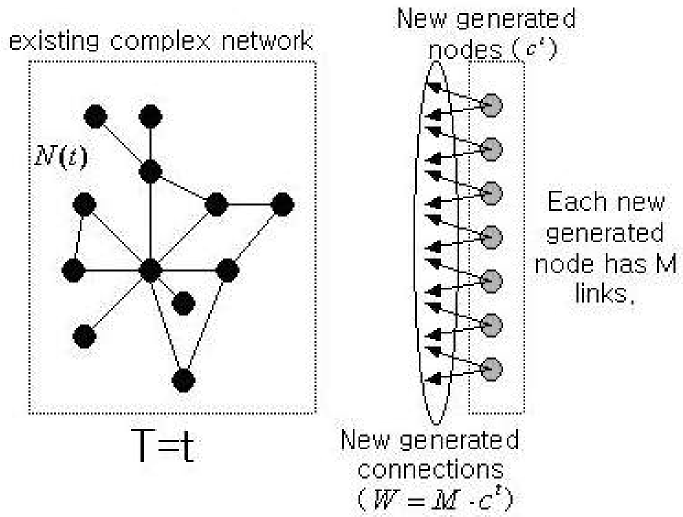

FIG. 3. The schematic picture of the geometric growth network : At time t, the existing network has $N(t)$ nodes and the new generated node is c^t. Therefore, the network grows geometrically. If each new node generates M links to the existing nodes, the total number of new connections is Mc^t.

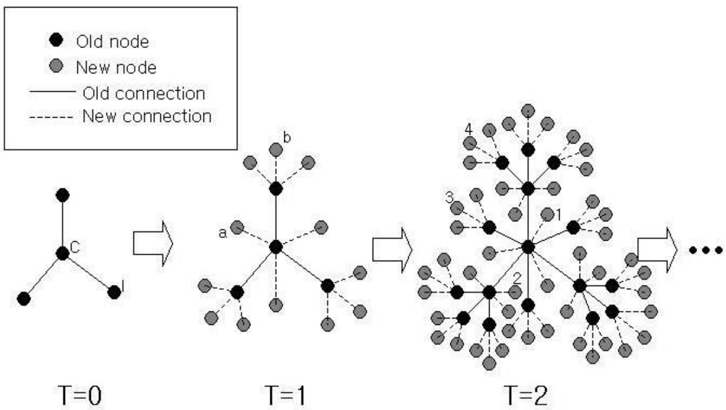

FIG. 4. The geometric growth model with addition rule only : Each node receives α additional connections. The size of networks grows geometrically. At T=1, vertex C have three more connections from vertices a and vertex I have three more from vertices b. At T=2, each vertex C, I, a, b has three more connections from vertices 1, 2, 3, 4.

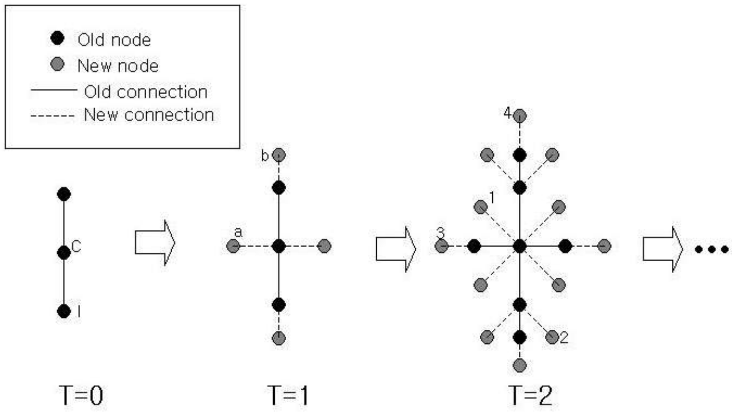

FIG. 5. The geometric growth model with the Multiplication rule only : Each node receives the $m-1$ times its own connections. Every node, therefore, has m times the number of its previous connections.

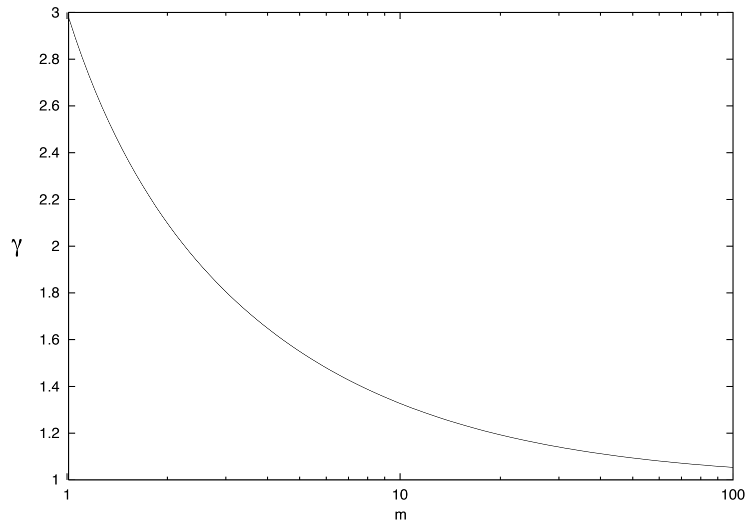

FIG. 6. The plot of the multiplication parameter m versus the exponent of the scale-free distribution γ.

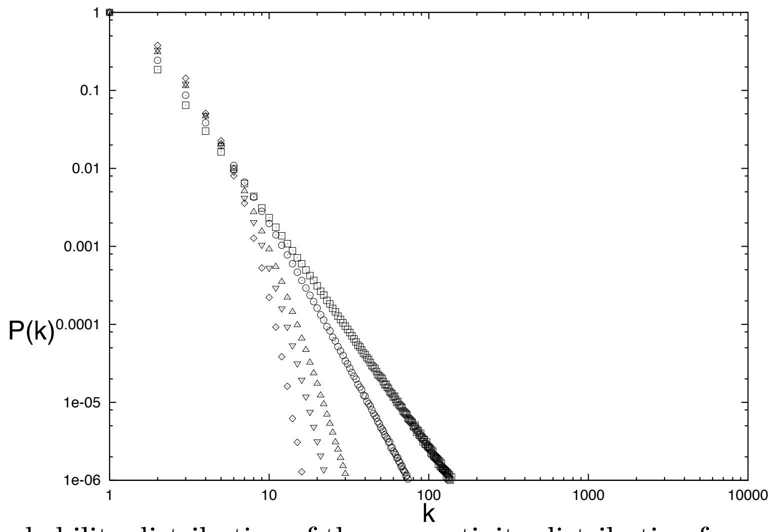

FIG. 7. The probability distribution of the connectivity distribution for our network model in log-log scale. The multiplication rule with $m = 1.001$ and no addition rule, $\alpha = 0$ with iteration of 7500 (square); $m = 1.001, \alpha = 0.001$ with iteration of 5000(circle); $m = 1.001, \alpha = 0.005$ with iteration of 5000(upward triangle); $m = 1.01, \alpha = 0.01$ with iteration of 2000 (downward triangle);$m = 1.01, \alpha = 0.1$ with iteration of 2000(diamond). The connectivity distributions become more exponential-like as the addition parameter increases.

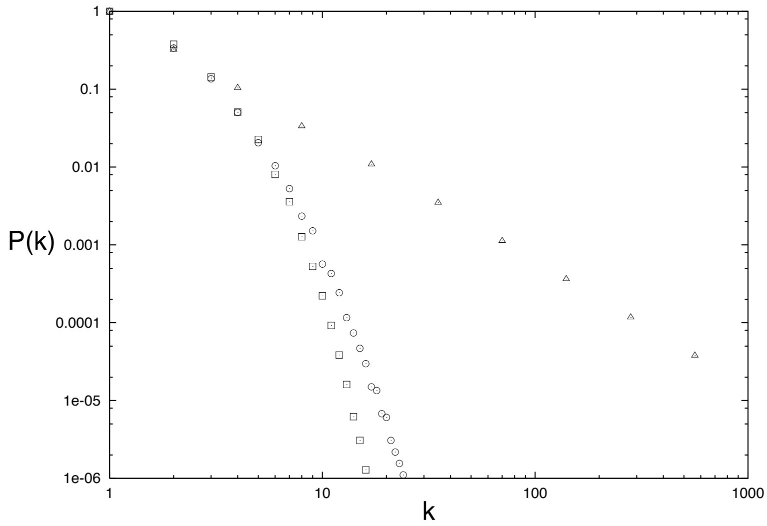

FIG. 8. The probability distribution of the connectivity for our network model in log-log scale. The addition parameter is fixed to $\alpha = 0.1$ and the multiplication parameter varies from $m = 1.001$ to $m = 2$: $m = 1.001, \alpha = 0.1$ with iteration of 2000(square); $m = 1.01, \alpha = 0.1$ with iteration of 2000(circle); $m = 2, \alpha = 0.1$ with iteration of 40(upward triangle). The behaviors of connectivity distributions change into the power-law distribution as the multiplication parameter increases.

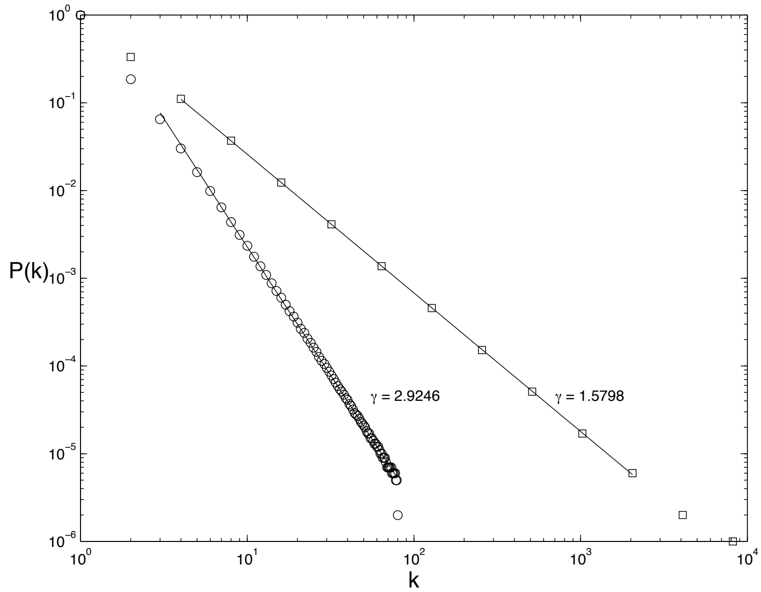

FIG. 9. The probability distribution of the connectivity for our network model in log-log scale. $m = 1.001$ with iteration of 10000 (circle); $m = 2$ with iteration of 50 (downward triangle). A least-squares curve-fitting method is used to find the straight line on power-law connectivity distributions. The exponent for the connectivity distributions varies from 1 to 3.