HOW TO DO MULTIPLE REGRESSION BY HAND

THANK YOU

MAPLE Lab

Memory And Psycholinguistics in Learning & Education

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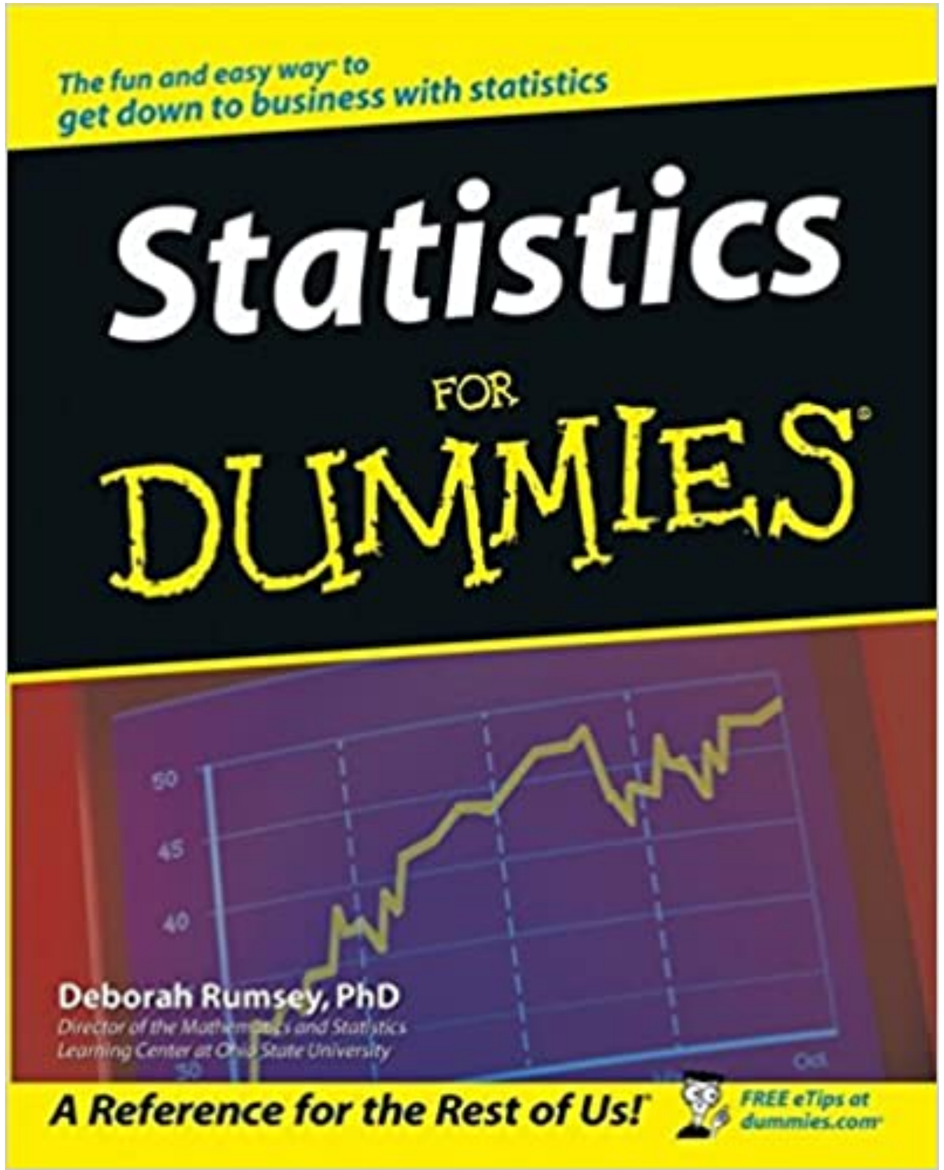
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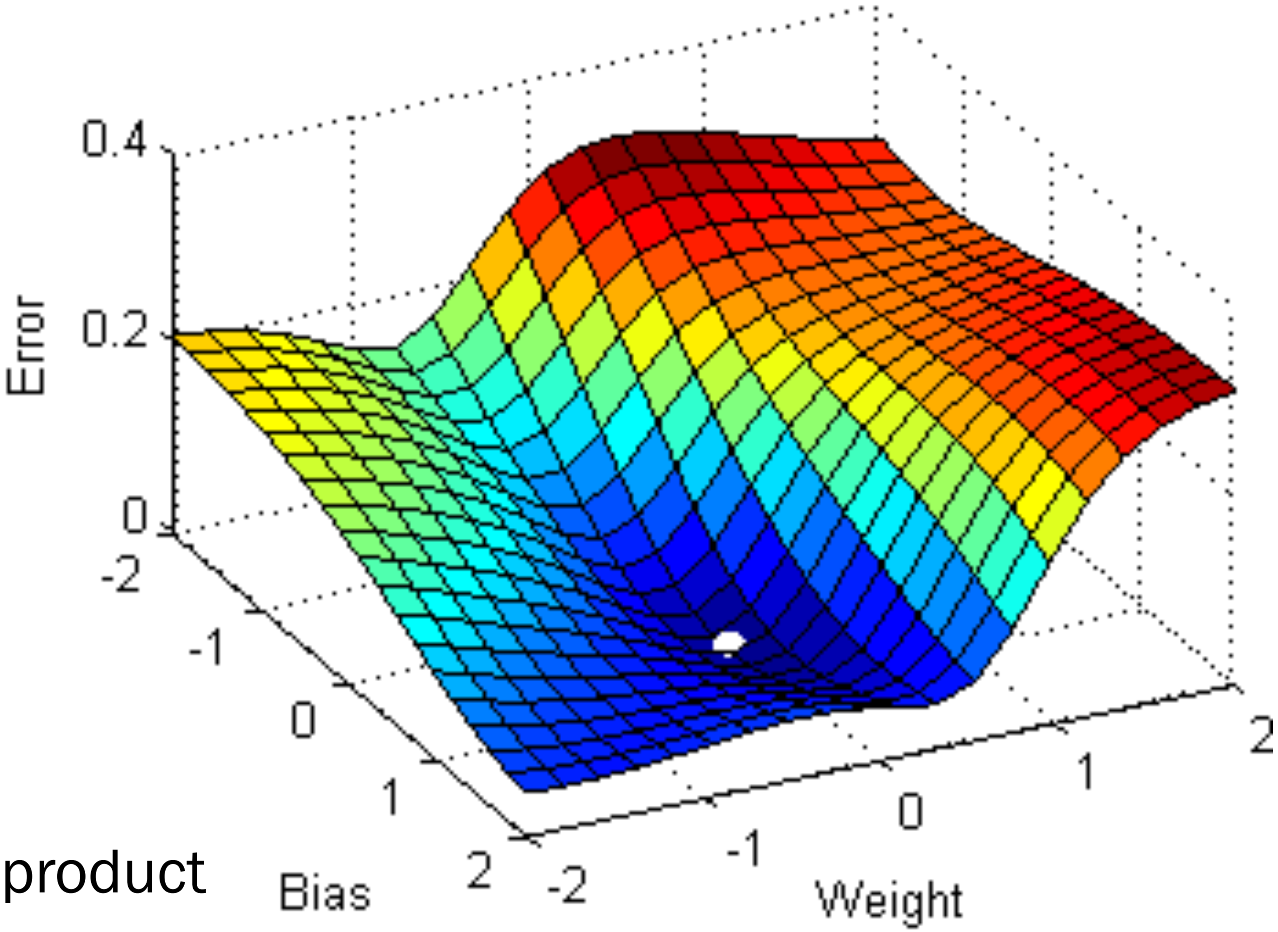
WHY DO THIS?

Partial eta squared

Derivative



Dot product

$\sum$

WHY DO THIS?

What we will cover

- Math: Calculus, Linear Algebra, Basic Algebra
- Beta coefficients and error
- Plugging in numbers to find real values

What we will not cover

- All the alternative ways of presenting the equations
- Standardizing data and why you really should
- Assumptions & Overfitting
- Logistic Regression & Integrals

The Regression Equation

(for 2 independent variables)

$$\hat{y} = \bar{y} - \frac{\sum x_2^2\left(\sum x_1 y - \frac{\sum x_1 \sum y}{N}\right) - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)\left(\sum x_2 y - \frac{\sum x_2 \sum y}{N}\right)}{\sum x_1^2 \sum x_2^2 - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)}\,\overline{x_1} -$$

$$\frac{\sum x_1^2 \left(\sum x_2 y - \frac{\sum x_1 \sum y}{N}\right) - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)\left(\sum x_1 y - \frac{\sum x_1 \sum y}{N}\right)}{\sum x_1^2 \sum x_2^2 - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)}\,\overline{x_2} +$$

$$\frac{\sum x_2^2\left(\sum x_1 y - \frac{\sum x_1 \sum y}{N}\right) - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)\left(\sum x_2 y - \frac{\sum x_2 \sum y}{N}\right)}{\sum x_1^2 \sum x_2^2 - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)}\,\mathrm{x}_1 +$$

$$\frac{\sum x_1^2 \left(\sum x_2 y - \frac{\sum x_1 \sum y}{N}\right) - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)\left(\sum x_1 y - \frac{\sum x_1 \sum y}{N}\right)}{\sum x_1^2 \sum x_2^2 - \left(\sum x_1 x_2 - \frac{\sum x_1 \sum x_2}{N}\right)}\,\mathrm{x}_2 + \epsilon$$

The Regression Equation
(for infinite independent variables)

Irreducible error

$$\hat{y} = (X^{T}X)^{-1}X^{T}\,\vec{y} + \boxed{\epsilon}$$

Minimizing Error

Hours of Sleep Per Night	Age
14	49
12	53
7	28
13	30
6	34
14	71
14	43
8	35
13	63
8	23
4	38
?	28

Minimizing Error

Hours of Sleep Per Night	Age
14	¼*49
12	¼*53
7	¼*28
13	¼*30
6	¼*34
14	¼*71
14	¼*43
8	¼*35
13	¼*63
8	¼*23
4	¼*38
?	¼*28

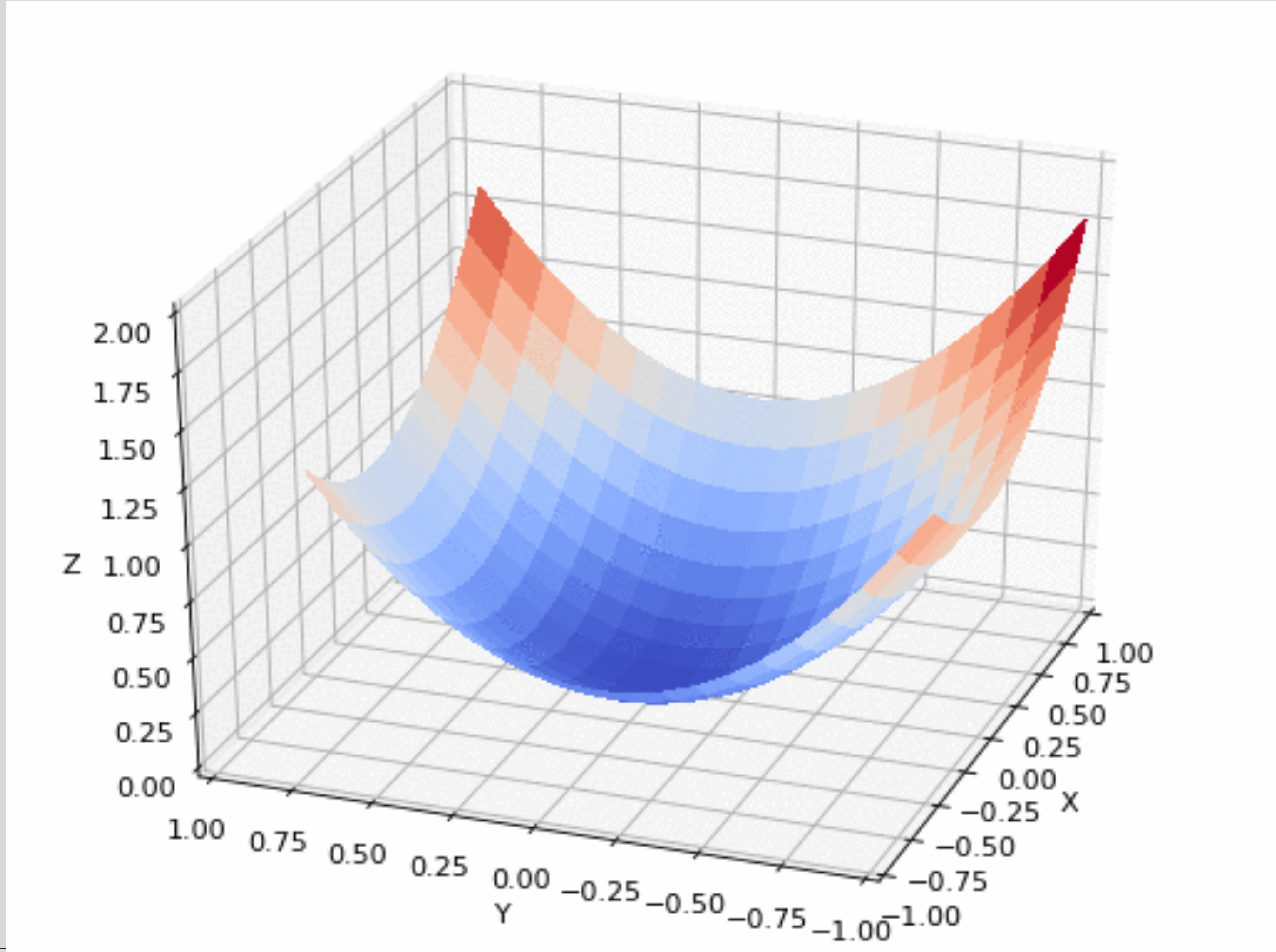
Minimizing Error

Hours of Sleep Per Night	Age	Predicted Sleep	Error	Error²
14	¼*49	12.25	-1.75	3.0625
12	¼*53	13.25	1.25	1.5625
7	¼*28	7	0	0
13	¼*30	7.5	-5.5	30.25
6	¼*34	8.5	2.5	6.25
14	¼*71	17.75	3.75	14.0625
14	¼*43	10.75	-3.25	10.5625
8	¼*35	8.75	0.75	0.5625
13	¼*63	15.75	2.75	7.5625
8	¼*23	5.75	-2.25	5.0625
4	¼*38	9.5	5.5	30.25
?	¼*28	7		

$$\sqrt{\frac{109.1875}{11}}$$

3.1506

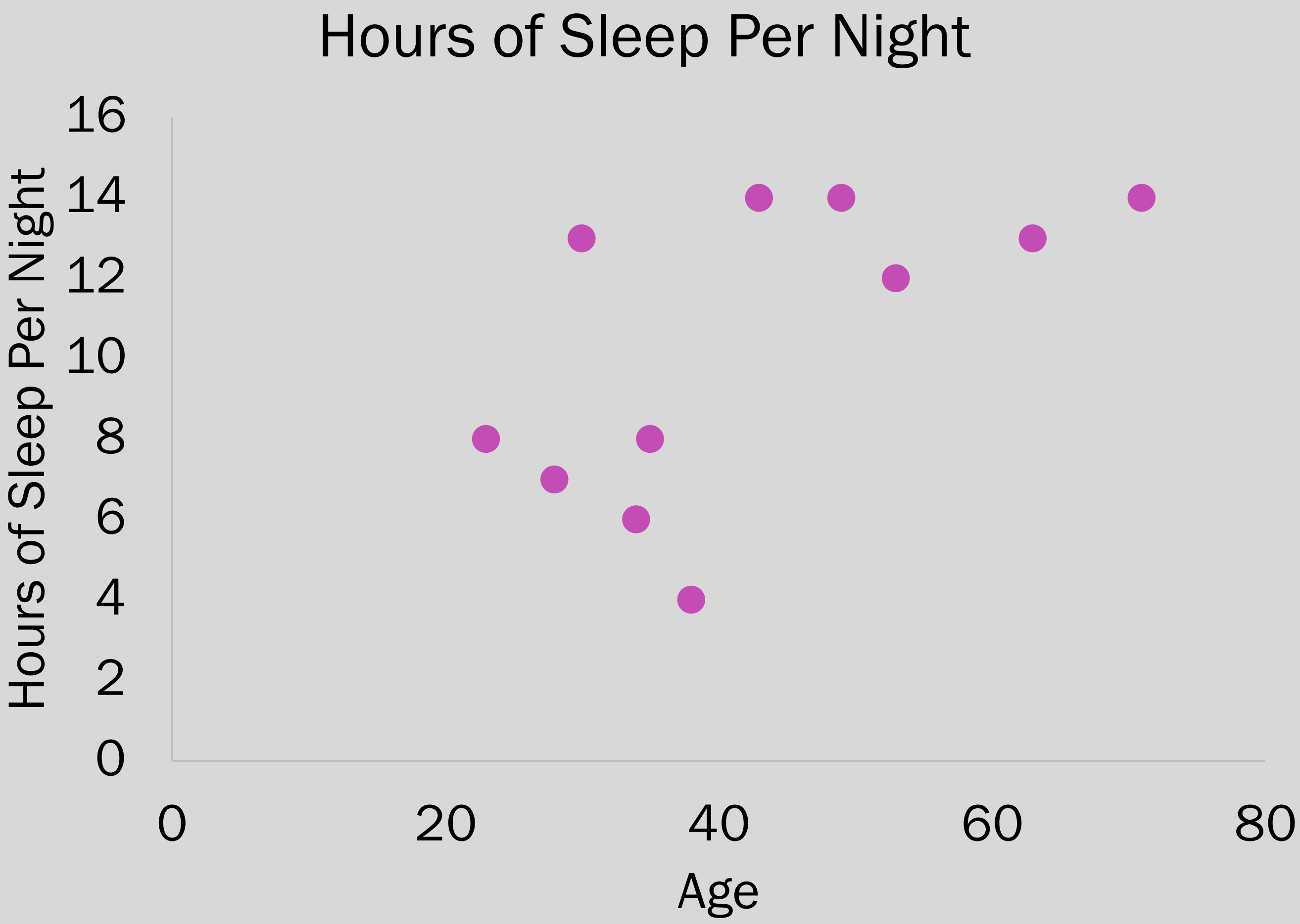
Minimizing Error



Minimizing Error

$$y = a + bx$$

Hours of Sleep Per Night	Age
14	¼*49
12	¼*53
7	¼*28
13	¼*30
6	¼*34
14	¼*71
14	¼*43
8	¼*35
13	¼*63
8	¼*23
4	¼*38
10.27	42.45

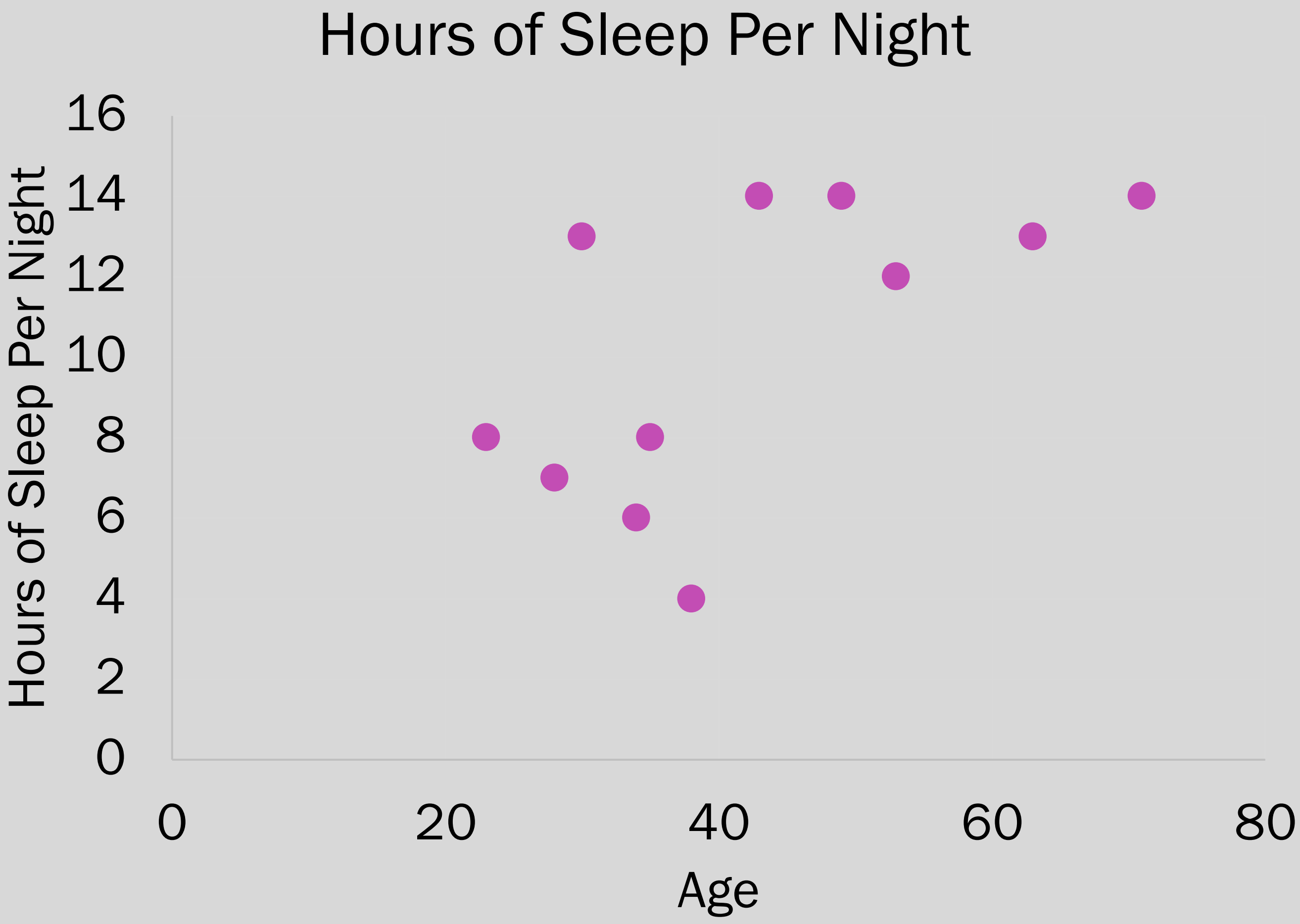


Minimizing Error

$$\bar{y} = a + b\bar{x} \qquad \bar{y} - b\bar{x} = a$$

$$10.27 = -0.34 + \frac{1}{4} * x$$

Hours of Sleep Per Night	Age
14	¼*49
12	¼*53
7	¼*28
13	¼*30
6	¼*34
14	¼*71
14	¼*43
8	¼*35
13	¼*63
8	¼*23
4	¼*38
10.27	42.45

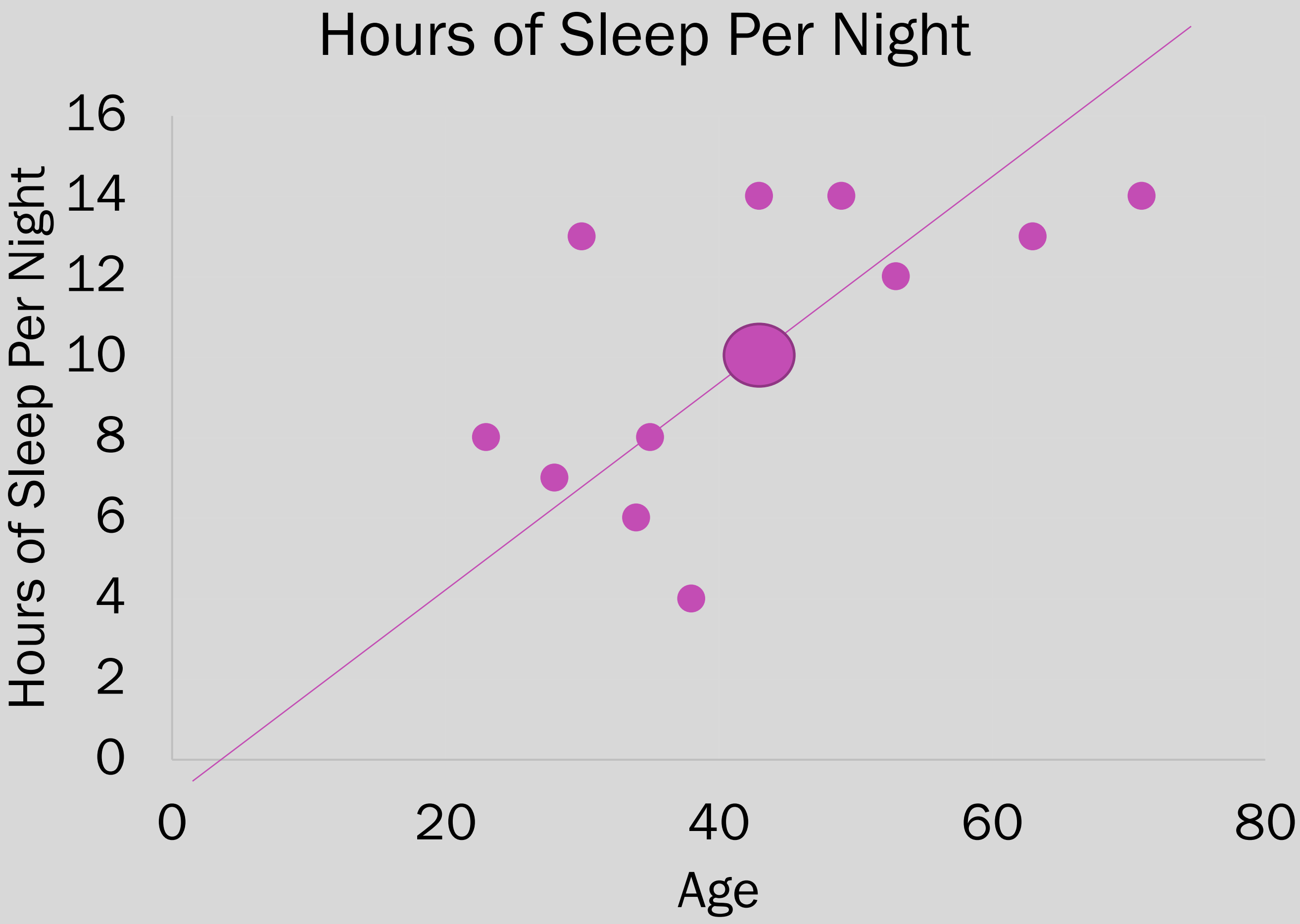


Minimizing Error

$$\bar{y} = a + b\bar{x} \qquad \bar{y} - b\bar{x} = a$$

$$10.27 = -0.34 + \frac{1}{4} * 42.45$$

Hours of Sleep Per Night	Age
14	¼*49
12	¼*53
7	¼*28
13	¼*30
6	¼*34
14	¼*71
14	¼*43
8	¼*35
13	¼*63
8	¼*23
4	¼*38
10.27	42.45

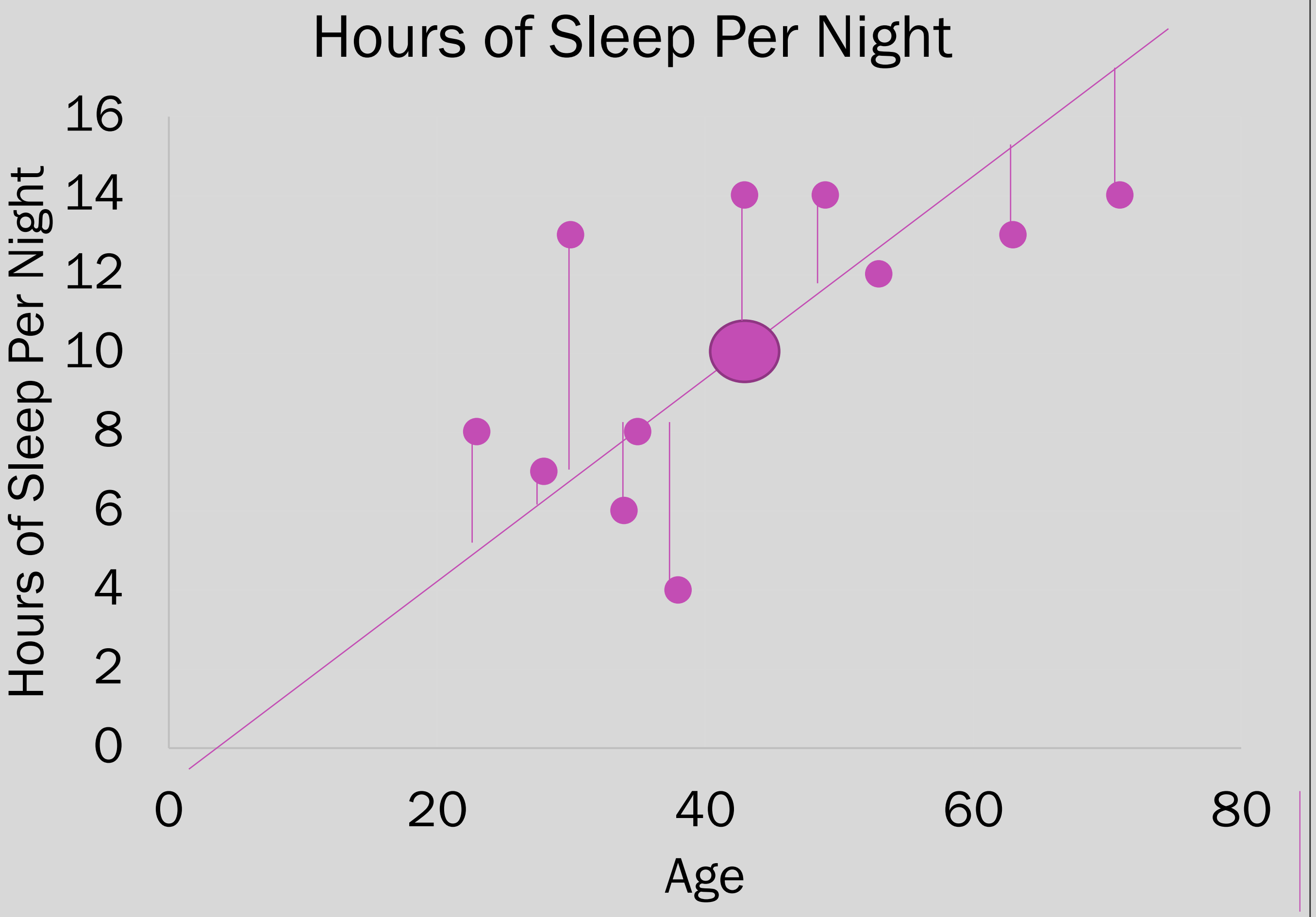


Minimizing Error

$$\bar{y} = a + b\bar{x} \qquad \bar{y} - b\bar{x} = a$$

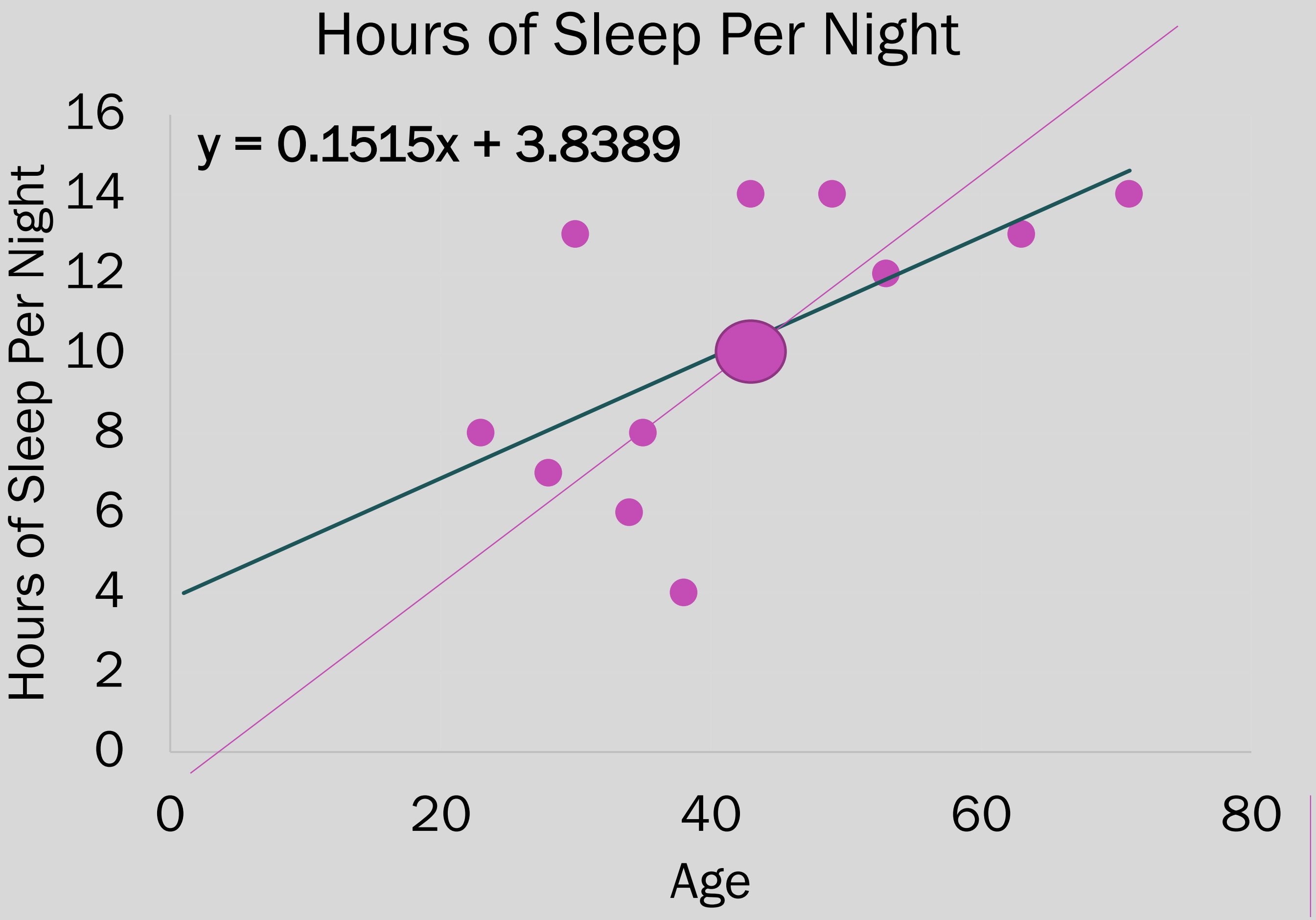
$$10.27 = -0.34 + \frac{1}{4} * 42.45$$

Hours of Sleep Per Night	Age
14	¼*49
12	¼*53
7	¼*28
13	¼*30
6	¼*34
14	¼*71
14	¼*43
8	¼*35
13	¼*63
8	¼*23
4	¼*38
10.27	42.45



Minimizing Error

Hours of Sleep Per Night	Age
14	¼*49
12	¼*53
7	¼*28
13	¼*30
6	¼*34
14	¼*71
14	¼*43
8	¼*35
13	¼*63
8	¼*23
4	¼*38
10.27	42.45



Understanding the Equation

$$\hat{y} = a + bx$$

$$(y - \hat{y})^2 = e$$

$$y = \hat{y} + \epsilon$$

$$\sqrt{\frac{\sum_{i=1}^{N}(y-\hat{y})^2}{N-1}} = \mathrm{e}$$

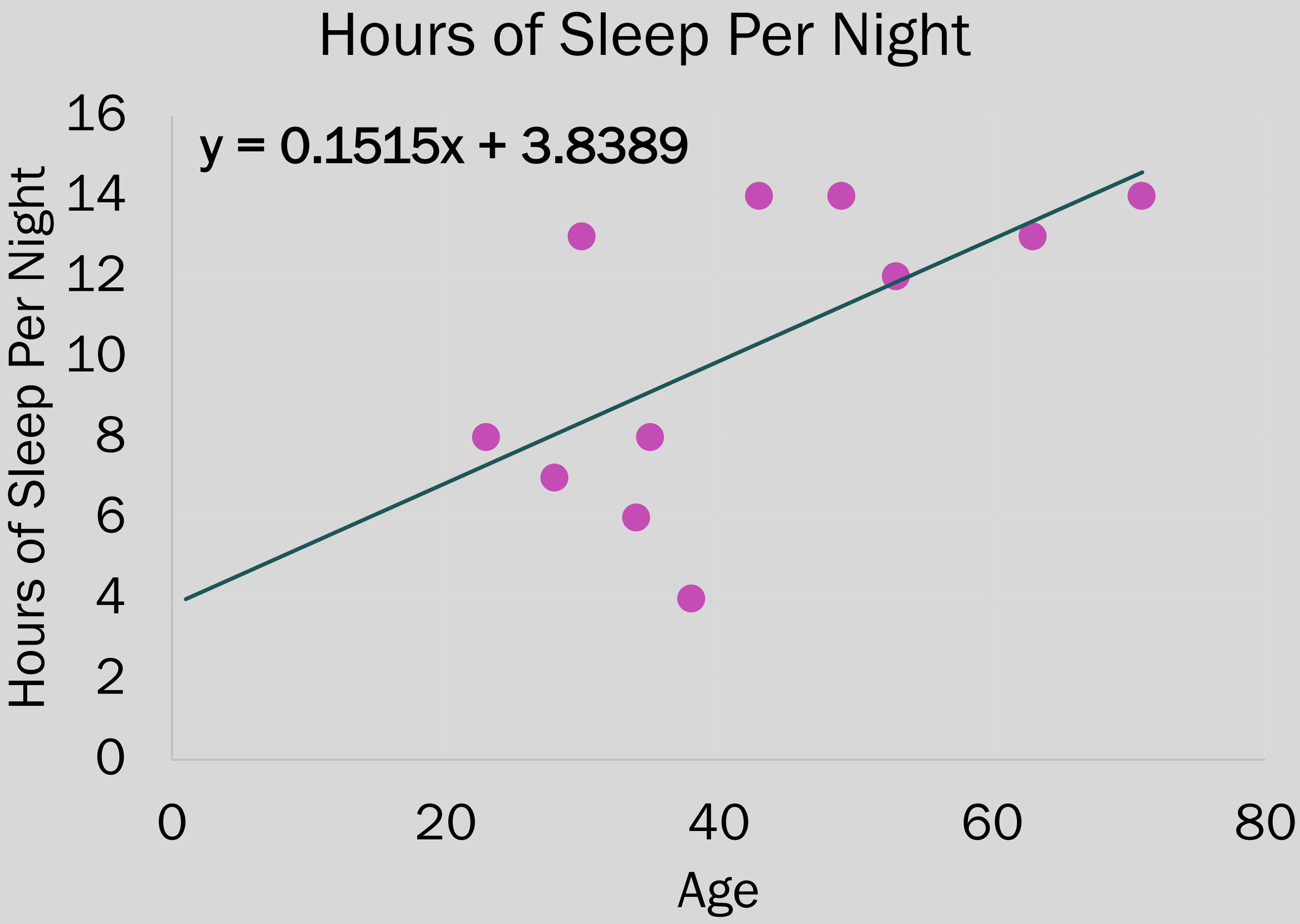
$$y - \hat{y} = e$$

Minimizing Error

$$e = \sqrt{\frac{\sum_{i=1}^{N}(y-\hat{y})^2}{N-1}} = 2.79$$

$$\hat{y} = 3.84 + .1515x$$

Hours of Sleep Per Night	Age
14	3.84+.15*49
12	3.84+.15*53
7	3.84+.15*28
13	3.84+.15*30
6	3.84+.15*34
14	3.84+.15*71
14	3.84+.15*43
8	3.84+.15*35
13	3.84+.15*63
8	3.84+.15*23
4	3.84+.15*38
8.04	3.84+.15*28



Hypothesis Testing

$$H_0: \beta_k = 0$$

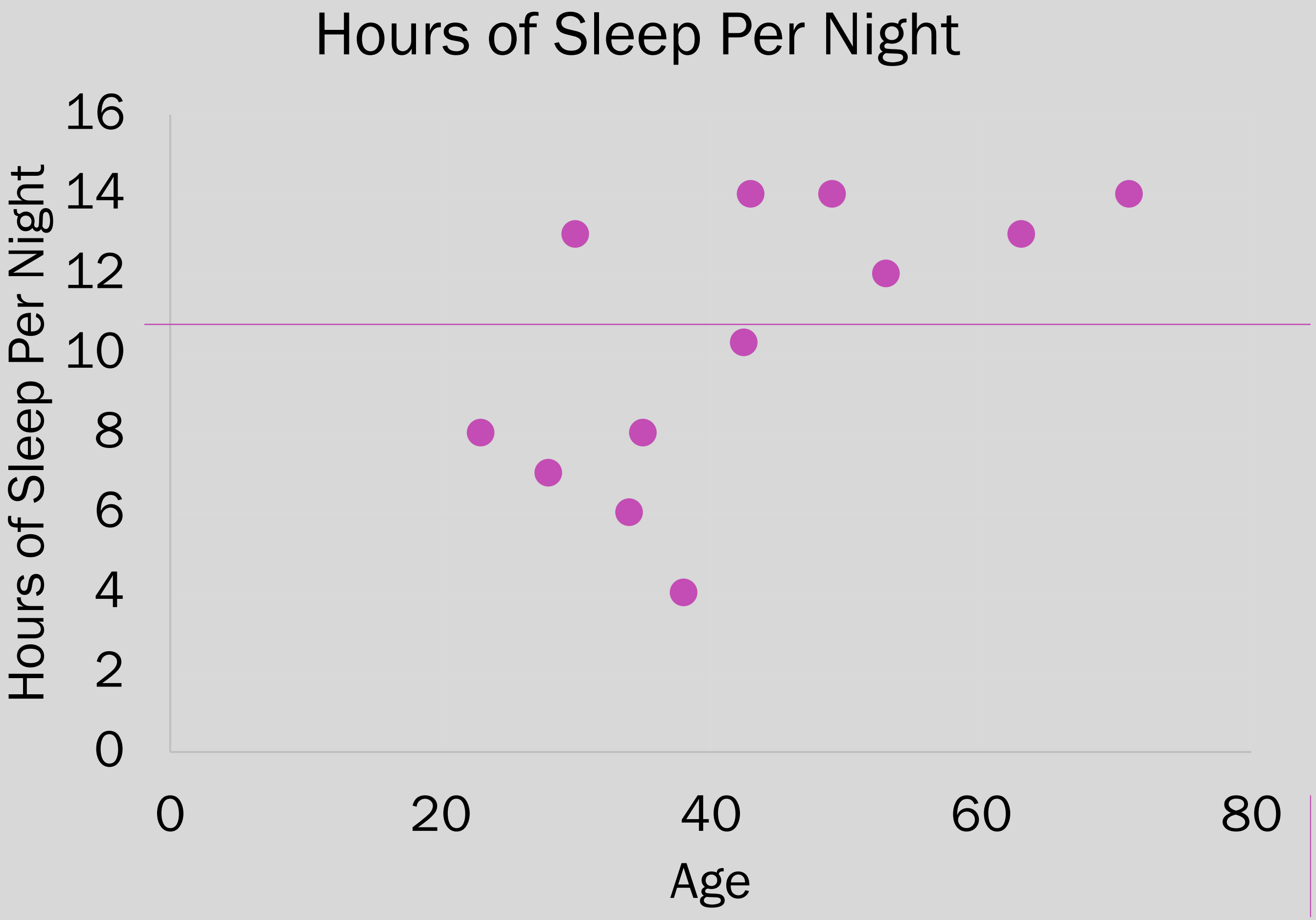
$$\bar{y} = a + 0\bar{x}$$

$$\bar{y} = a$$

$$H_A: \beta_k \neq 0$$

$$\hat{y} = 3.84 + .1515x$$

$$\widehat{\beta_k} = .1515 \pm error$$



How do we find Beta?

$$y = a + bx + \epsilon$$

How do we find Beta?

$$y = \beta_0 + \beta_k X_{ik} + \epsilon$$

How do we find Beta?

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 \cdots \beta_k X_k$$

How do we find Beta?

$$y = \beta_0 + \beta_k x_k$$
$$\beta_0 = \bar{y} - \beta_k \bar{x}_k$$

How do we find Beta?

$$slope = \frac{y_1 - y_2}{x_1 - x_2} \longrightarrow \beta_1 = r\frac{s_y}{s_x}$$

r = Pearson's Correlation
S = Standard Deviation

How do we find Beta?

Pearson's Correlation

Standard Deviation of y

$$\beta_1 = r\frac{s_y}{s_x} = \frac{Cov}{S_x S_y} \quad \frac{S_y}{S_x}$$

Standard Deviation of x

How do we find Beta?

Covariance

$$\beta_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{N} \quad \frac{1}{\sqrt{\frac{\sum(x_i - \bar{x})^2}{N}}\sqrt{\cancel{\frac{\sum(y_i - \bar{y})^2}{N}}}} \quad \frac{\sqrt{\cancel{\frac{\sum(y_i - \bar{y})^2}{N}}}}{\sqrt{\frac{\sum(x_i - \bar{x})^2}{N}}}$$

How do we find Beta?

Covariance

$$\beta_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{N} \quad \frac{1}{\sqrt{\frac{\sum (x_i - \bar{x})^2}{N}} \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}}$$

How do we find Beta?

Covariance

$$\beta_1 = \frac{\frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\cancel{N}}}{\frac{\sum(x_i - \bar{x})^2}{\cancel{N}}}$$

How do we find Beta?

$$\beta_1 = r\frac{s_y}{s_x} = \frac{cov(x,y)}{var(x)} = \frac{SC(x,y)}{SS(x)} = \frac{\sum_{i=1}^{N}(xi-\bar{x})(yi-\bar{y})}{\sum_{i=1}^{N}(xi-\bar{x})^2}$$

Understanding the Equation

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$\sum_{i=1}^{N} (y_i - \hat{y}_i)^2 = e$$

Understanding the Equation

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$\sum_{i=1}^{N} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2}))^2 = e$$

Understanding the Equation

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$\sum_{i=1}^{N} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} \_ \hat{\beta}_2 x_{i2})^2 = e$$

Understanding the Equation

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$arg \frac{min}{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2} \sum_{i=1}^{N} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} \_ \hat{\beta}_2 x_{i2_{)}2} = e$$

Understanding the Equation

$\hat{\beta}_0 =$ $\hat{\beta}_1 =$ $\hat{\beta}_2 =$

$$arg \frac{min}{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2} \sum_{i=1}^{N} (y_i - \hat{\beta}_0 1 - \hat{\beta}_1 x_{i1} \_ \hat{\beta}_2 x_{i2_{)}2} = 0$$

Solving with Calculus

$$\hat{\beta}_0 = \frac{\partial}{\partial \hat{\beta}_0} \sum_{i=1}^{N} (y_i - \hat{\beta}_0 1 - \hat{\beta}_1 x_{i1} \_ \hat{\beta}_2 x_{i2_{)}2} = 0$$

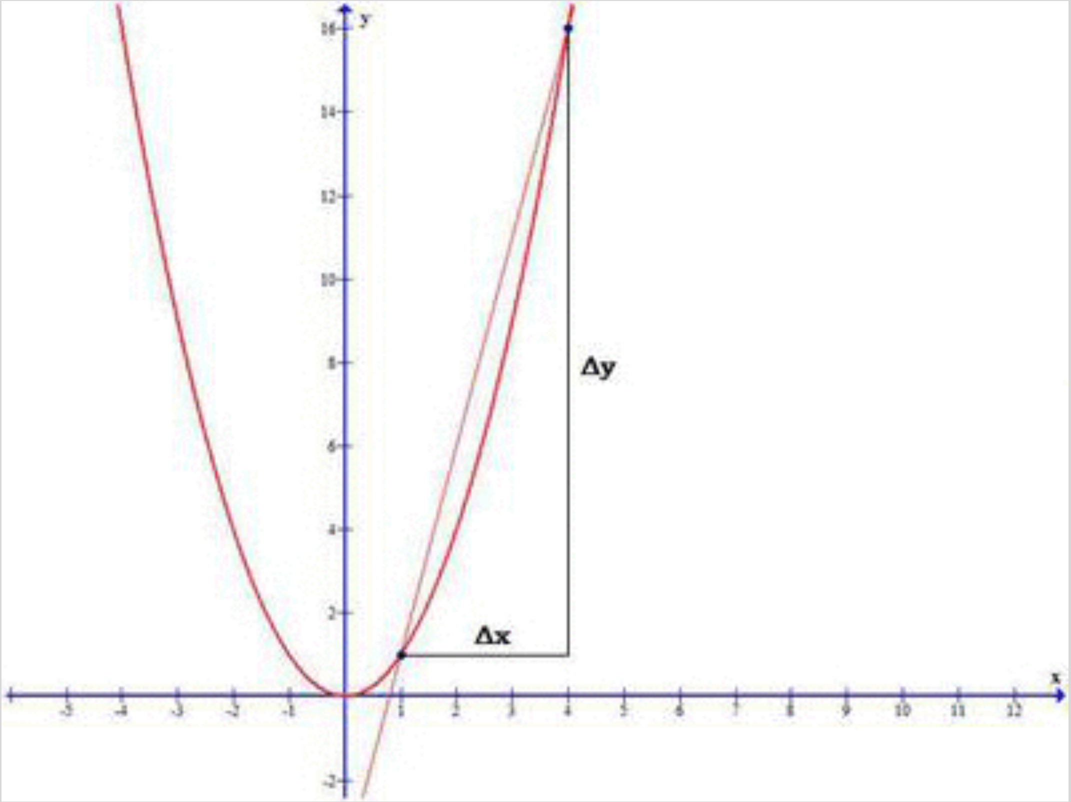
Solving with Calculus

What is a derivative?

Slope of the tangent

or

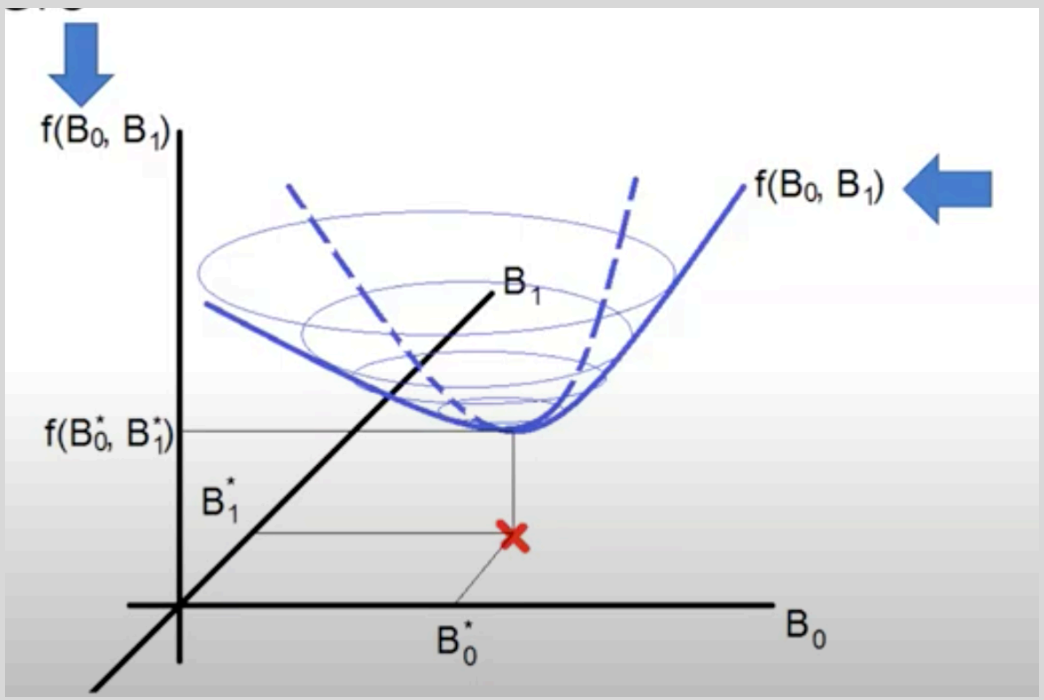
Instantaneous rate of change



Solving with Calculus

What is a derivative?

Tells us whether or not to increase or decrease the beta value



Solving with Calculus

$$\frac{\partial}{\partial \hat{\beta}_0} \sum_{i=1}^{N} (y_i - \hat{\beta}_0 1 - \hat{\beta}_1 x_{i1} \_ \hat{\beta}_2 x_{i2_{)}2} = 0$$

$$2 \sum_{i=1}^{N} (y_i - \hat{\beta}_0 1 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2_{)}}{}^{(2-1)} = 0$$

start with the outer part (chain rule), use the power rule

$$\cancel{-1 * 2} \sum_{i=1}^{N} (y_i - 1\hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2_{)}} = 0$$

Do the same with the constant on the coefficient (inner part).

Solving with Calculus

$$2\sum_{i=1}^{N}(y_i \_ \hat{\beta}_0 1 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2_)}{}^{(2-1)} = 0$$

start with the outer part (chain rule), use the power rule

$$\cancel{-1 *} 2\sum_{i=1}^{N}(y_i - 1\hat{\beta}_0 - \hat{\beta}_1 x_{i1} \_ \hat{\beta}_2 x_{i2_)} = 0$$

Do the same with the constant on the coefficient (inner part).

$$\sum_{i=1}^{N}(y_i) - \sum_{i=1}^{N}(\hat{\beta}_0) - \sum_{i=1}^{N}(\hat{\beta}_1 x_{i1_)} - \sum_{i=1}^{N}(\hat{\beta}_2 x_{i2_)} = 0$$

Solving with Calculus

$$\sum_{i=1}^{N}(y_i) - \sum_{i=1}^{N}(\hat{\beta}_0) - \sum_{i=1}^{N}(\hat{\beta}_1 x_{i1_{)}} - \sum_{i=1}^{N}(\hat{\beta}_2 x_{i2_{)}} = 0$$

$$\sum_{i=1}^{N}(y_i) - \sum_{i=1}^{N}(\hat{\beta}_1 x_{i1_{)}} - \sum_{i=1}^{N}(\hat{\beta}_2 x_{i2_{)}} = \sum_{i=1}^{N}(\hat{\beta}_0)$$

$$\sum_{i=1}^{N}(y_i) - \sum_{i=1}^{N}(\hat{\beta}_1 x_{i1_{)}} - \sum_{i=1}^{N}(\hat{\beta}_2 x_{i2_{)}} = n\hat{\beta}_0$$

Solving with Calculus

$$\sum_{i=1}^{N}(y_i) - \sum_{i=1}^{N}(\hat{\beta}_1 x_{i1_{)}} - \sum_{i=1}^{N}(\hat{\beta}_2 x_{i2_{)}} = n\hat{\beta}_0$$

$$\frac{\sum_{i=1}^{N}(y_i) - \sum_{i=1}^{N}(\hat{\beta}_1 x_{i1_{)}} - \sum_{i=1}^{N}(\hat{\beta}_2 x_{i2_{)}}}{n} = \frac{n\hat{\beta}_0}{n}$$

$$\bar{y} - \hat{\beta}_1\bar{x}_1 - \hat{\beta}_2\bar{x}_2 = \hat{\beta}_0$$

Solving with Calculus

$$\frac{\partial}{\partial \hat{\beta}_1} \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1} - \hat{\beta}_2 X_{i2})^2 = 0$$

$$\cancel{-2} X_{i1} \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1} - \hat{\beta}_2 X_{i2}) = 0$$

$$X_{i1} \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1} - \hat{\beta}_2 X_{i2}) = 0$$

Solving with Calculus

$$X_{i1}\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1} \_ \hat{\beta}_2 X_{i2)} = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - [\bar{Y} - \hat{\beta}_1\bar{X}_1 - \hat{\beta}_2\bar{X}_2] - \hat{\beta}_1 X_{i1} \_ \hat{\beta}_2 X_{i2)} = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y} - \hat{\beta}_1(X_{i1} \_ \bar{X}_1) - \hat{\beta}_2(X_{i2} \_ \bar{X}_2) = 0$$

Solving with Calculus

$$X_{i1} \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1} \_ \hat{\beta}_2 X_{i2_{)}} = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - [\bar{Y} - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 \bar{X}_2] - \hat{\beta}_1 X_{i1} \_ \hat{\beta}_2 X_{i2_{)}} = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y} - \hat{\beta}_1 (X_{i1} \_ \bar{X}_1) - \hat{\beta}_2 (X_{i2} \_ \bar{X}_2) = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y}) - \sum_{i=1}^{N} X_{i1} \hat{\beta}_1 (X_{i1} \_ \bar{X}_1) - \sum_{i=1}^{N} X_{i1} \hat{\beta}_2 (X_{i2} \_ \bar{X}_2) = 0$$

Solving with Calculus

$$X_{i1}\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{i1} \_ \hat{\beta}_2 X_{i2}) = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - [\bar{Y} - \hat{\beta}_1\bar{X}_1 - \hat{\beta}_2\bar{X}_2] - \hat{\beta}_1 X_{i1} \_ \hat{\beta}_2 X_{i2}) = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y} - \hat{\beta}_1(X_{i1} \_ \bar{X}_1) - \hat{\beta}_2(X_{i2} \_ \bar{X}_2) = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y}) - \sum_{i=1}^{N} X_{i1}\hat{\beta}_1(X_{i1} \_ \bar{X}_1) - \sum_{i=1}^{N} X_{i1}\hat{\beta}_2(X_{i2} \_ \bar{X}_2) = 0$$

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y}) - \sum_{i=1}^{N} X_{i1}\hat{\beta}_2(X_{i2} \_ \bar{X}_2) = \hat{\beta}_1\sum_{i=1}^{N} X_{i1}(X_{i1} \_ \bar{X}_1)$$

Solving with Calculus

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y}) - \sum_{i=1}^{N} X_{i1}\hat{\beta}_2(X_{i2} \_ \bar{X}_2) = \hat{\beta}_1 \sum_{i=1}^{N} X_{i1}(X_{i1} \_ \bar{X}_1)$$

$$\beta_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2}$$

$$\frac{\sum_{i=1}^{N} (Y_i - \bar{Y})X_{i1} \_ \hat{\beta}_2 \sum_{i=1}^{N}(X_{i2} - \bar{X}_{i2})X_{i1}}{\sum_{i=1}^{N}(X_{i1} \_ \bar{X}_1)^2} = \hat{\beta}_1$$

Solving with Calculus

$$\sum_{i=1}^{N} (Y_i - \bar{Y}) X_{i1}$$

$$(Y_i - \bar{Y})(X_{i1} \_ \bar{X})$$

$$(Y_i X_{i1} \_ Y_i \bar{X} - \bar{Y} X_{i1} \_ \bar{Y} \bar{X})$$

2+1+3 = 6/3 = 2
1+4+4 = 9/3 = 3
6+6+6
2*3+1*3+3*3 = 18

$$(Y_i X_{i1}) - (\bar{Y} X_{i1})$$

Solving with Calculus

$$\sum_{i=1}^{N} X_{i1}(Y_i - \bar{Y}) - \sum_{i=1}^{N} X_{i1}\hat{\beta}_2(X_{i2} \_ \bar{X}_2) = \hat{\beta}_1 \sum_{i=1}^{N} X_{i1}(X_{i1} \_ \bar{X}_1)$$

$$\beta_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2}$$

$$\frac{\sum_{i=1}^{N} (Y_i - \bar{Y})(X_{i1} \_ \bar{X}_1) - \cancel{\hat{\beta}_2 \sum_{i=1}^{N}(X_{i2} \quad \bar{X}_{i2})(Xi_1 \_ \bar{X}_{i1})}}{\sum_{i=1}^{N}(X_{i1} \_ \bar{X}_1)^2} = \hat{\beta}_1$$

How do we find Beta?

$$b_1 = \frac{(\sum x_2^2)(\sum x_1 y) - (\sum x_1 x_2)(\sum x_2 y)}{(\sum x_1^2)(\sum x_2^2) - (\sum x_1 x_2)^2}$$

and

$$b_2 = \frac{(\sum x_1^2)(\sum x_2 y) - (\sum x_1 x_2)(\sum x_1 y)}{(\sum x_1^2)(\sum x_2^2) - (\sum x_1 x_2)^2}$$

$$\sum x_1 y = \sum X_1 Y - \frac{(\sum X_1)(\sum Y)}{N}$$

$$\sum x_2 y = \sum X_2 Y - \frac{(\sum X_2)(\sum Y)}{N}$$

$$\sum x_1 x_2 = \sum X_1 X_2 - \frac{(\sum X_1)(\sum X_2)}{N}$$

Solving with Matrix Math

$$y = \beta_0 1 + \beta_1 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4}$$

$$
\begin{array}{l}
y_1 = \beta_0 1 + \beta_1 x_{12} + \beta_3 x_{13} + \beta_4 x_{14} \\
y_2 = \beta_0 1 + \beta_1 x_{22} + \beta_3 x_{23} + \beta_4 x_{24} \\
\;\vdots \qquad\;\; \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
y_n = \beta_0 1 + \beta_1 x_{n2} + \beta_3 x_{n3} + \beta_4 x_{n4}
\end{array}
$$

Solving with Matrix Math

$y = \beta_0 1 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3}$

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} 1 x_{11} x_{12} x_{13} \\ 1 x_{21} x_{22} x_{23} \\ 1 \ . \ \ . \ \ . \\ 1 \ . \ \ . \ \ . \\ 1 \ . \ \ . \ \ . \\ 1 x_{n1} x_{n2} x_{n3} \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ . \\ . \\ . \\ \epsilon_n \end{bmatrix}$$

Solving with Matrix Math

Hours of Sleep Per Night	Age	Hours of Exercise a Week	Interaction Between Age & Exercise
14	49	11	4911
12	53	1	531
7	28	17	2817
13	30	13	3013
6	34	3	343
14	71	20	7120
14	43	19	4319
8	35	14	3514
13	63	0	630
8	23	5	235
4	38	8	388
8.08	28	4	284

Solving with Matrix Math

6 x 2 Matrix

1	49
1	53
1	28
1	30
1	34
1	71

·

2 x 1 Matrix

8
3

6 x 1 Matrix

1*8 + 49*3
1*8 + 53*3
1*8 + 28*3
1*8 + 30*3
1*8 + 34*3
$B_0X_0 + B_1X_1$

Solving with Matrix Math

N x 1 Matrix | N x 4 Matrix | 4 x 1 Matrix

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_{11} & x_{12} & x_{13} \\ 1 & x_{21} & x_{22} & x_{23} \\ 1 & . & . & . \\ 1 & . & . & . \\ 1 & . & . & . \\ 1 & x_{n1} & x_{n2} & x_{n3} \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{X\vec{\beta}} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ . \\ . \\ . \\ \epsilon_n \end{bmatrix}$$

Solving with Matrix Math

$$\vec{y} = \mathrm{X}\vec{\beta} + \vec{\epsilon}$$

$$\vec{\epsilon} = \vec{y} - \mathrm{X}\vec{\beta}$$

Solving with Matrix Math

$$\sum_{i=1}^{N} \hat{\epsilon}_i^{\,2} = \vec{\epsilon}^{T}\vec{\epsilon}$$

N x 1 Matrix

1 x N Matrix

$$\begin{bmatrix}\epsilon_1 & \epsilon_2 & . & . & . & \epsilon_n\end{bmatrix}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ .\\ .\\ .\\ \epsilon_n\end{bmatrix} = \epsilon_1{}^2 + \epsilon_2{}^2 + . . . + \epsilon_n{}^2$$

Solving with Matrix Math

$$\vec{\epsilon} \;=\; \vec{y} - \mathrm{X}\vec{\beta}$$

$$\vec{\epsilon}^{T}\vec{\epsilon} = (\vec{y} - \mathrm{X}\vec{\beta})^{\mathrm{T}}\,(\vec{y} - \mathrm{X}\,\vec{\beta})$$

Solving with Matrix Math

$\vec{\epsilon} = \vec{y}$ -X$\vec{\beta}$

$\vec{\epsilon}^T\vec{\epsilon} = (\vec{y}$ -X$\vec{\beta}$)^T ($\vec{y}$ -X$\vec{\beta}$)

$\vec{\beta}$ = (k+1) x 1 -> Transposed -> 1 x (k+1)

X = N x (k+1) -> Transposed -> (k+1) x N

$\vec{y}$ = N x 1 -> Transposed -> 1 x N

(1 x (k+1))((k+1) x N)(N x 1)

(1 x N)(N x (k+1))((k+1) x 1)

(1 x N)(N x 1)

(1 x (k+1))((k+1) x 1)

(1x1)

(1x1)

$$\vec{y}^T\vec{y} - \vec{\beta}^T X^T \vec{y} - \vec{y}^T X \vec{\beta} + \vec{\beta}^T X^T X \vec{\beta} = 0$$

Solving with Matrix Math

$$\vec{\epsilon} = \vec{y} - X\vec{\beta}$$

$$\vec{\epsilon}^T \vec{\epsilon} = (\vec{y} - X\vec{\beta})^T (\vec{y} - X\vec{\beta})$$

$$0 = \vec{y}^T \vec{y} - \vec{\beta}^T X^T \vec{y} - \vec{y}^T X \vec{\beta} + \vec{\beta}^T X^T X \vec{\beta}$$

$$\frac{\partial}{\partial \vec{\beta}} \vec{y}^T \vec{y} - 2\vec{\beta}^T X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta} = 0$$

Solving with Matrix Math

$$\vec{\epsilon} = \vec{y} - X\vec{\beta}$$

$$\vec{\epsilon}^T\vec{\epsilon} = (\vec{y} - X\vec{\beta})^T (\vec{y} - X\vec{\beta})$$

$$0 = \vec{y}^T\vec{y} - \vec{\beta}^T X^T \vec{y} - \vec{y}^T X \vec{\beta} + \vec{\beta}^T X^T X \vec{\beta}$$

$$0 = \frac{\partial}{\partial \vec{\beta}} \vec{y}^T\vec{y} - 2\vec{\beta}^T X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta}$$

$$-1 * 2\vec{\beta}^{T1-0} X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta} = 0$$

Solving with Matrix Math

$\vec{\epsilon} = \vec{y} - X\vec{\beta}$

$\vec{\epsilon}^T\vec{\epsilon} = (\vec{y} - X\vec{\beta})^T (\vec{y} - X\vec{\beta})$

$0 = \vec{y}^T\vec{y} - \vec{\beta}^T X^T \vec{y} - \vec{y}^T X \vec{\beta} + \vec{\beta}^T X^T X \vec{\beta}$

$0 = \vec{y}^T\vec{y} - 2\vec{\beta}^T X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta}$

$-2\vec{\beta}^{T1-0} X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta} = 0$

$$-2X^T \vec{y} + 2X^T X \vec{\beta} = 0$$

Solving with Matrix Math

$\vec{\epsilon} = \vec{y}$ -X$\vec{\beta}$

$\vec{\epsilon}^T\vec{\epsilon} = (\vec{y}$ -X$\vec{\beta}$)$^{\mathrm{T}}$ ($\vec{y}$ -X$\vec{\beta}$)

$0 = \vec{y}^T\vec{y} - \vec{\beta}^{\mathrm{T}}\mathrm{X}^{\mathrm{T}}\vec{y} - \vec{y}^{\mathrm{T}}\mathrm{X}\vec{\beta} + \vec{\beta}^{\mathrm{T}}\mathrm{X}^{\mathrm{T}}\mathrm{X}\vec{\beta}$

$0 = \vec{y}^T\vec{y} - 2\vec{\beta}^{\mathrm{T}}\mathrm{X}^{\mathrm{T}}\vec{y} + \vec{\beta}^{\mathrm{T}}\mathrm{X}^{\mathrm{T}}\mathrm{X}$

$-2\vec{\beta}^{\mathrm{T}1\text{-}0}\mathrm{X}^{\mathrm{T}}\vec{y} + \vec{\beta}^{\mathrm{T}}\mathrm{X}^{\mathrm{T}}\mathrm{X}\vec{\beta} = 0$

$-2\mathrm{X}^{\mathrm{T}}\vec{y} + 2\mathrm{X}^{\mathrm{T}}\mathrm{X}\vec{\beta} = 0$

$$\mathrm{X}^{\mathrm{T}}\mathrm{X}\vec{\beta} = \mathrm{X}^{\mathrm{T}}\vec{y}$$

Solving with Matrix Math

$$\vec{\epsilon} = \vec{y} - X\vec{\beta}$$

$$\vec{\epsilon}^T\vec{\epsilon} = (\vec{y} - X\vec{\beta})^T(\vec{y} - X\vec{\beta})$$

$$0 = \vec{y}^T\vec{y} - \vec{\beta}^T X^T \vec{y} - \vec{y}^T X \vec{\beta} + \vec{\beta}^T X^T X \vec{\beta}$$

$$0 = \vec{y}^T\vec{y} - 2\vec{\beta}^T X^T \vec{y} + \vec{\beta}^T X^T X$$

$$-2\vec{\beta}^{T1-0} X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta} = 0$$

$$-2X^T \vec{y} + 2X^T X \vec{\beta} = 0$$

$$X^T X \vec{\beta} = X^T \vec{y}$$

$$(X^T X)^{-1} X^T X \vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

Solving with Matrix Math

$$\vec{\epsilon} = \vec{y} - X\vec{\beta}$$

$$\vec{\epsilon}^T\vec{\epsilon} = (\vec{y} - X\vec{\beta})^T(\vec{y} - X\vec{\beta})$$

$$0 = \vec{y}^T\vec{y} - \vec{\beta}^T X^T \vec{y} - \vec{y}^T X \vec{\beta} + \vec{\beta}^T X^T X \vec{\beta}$$

$$0 = \vec{y}^T\vec{y} - 2\vec{\beta}^T X^T \vec{y} + \vec{\beta}^T X^T X$$

$$-2\vec{\beta}^{T\,1-0} X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta} = 0$$

$$-2X^T \vec{y} + 2X^T X \vec{\beta} = 0$$

$$X^T X \vec{\beta} = X^T \vec{y}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad (X^T X)^{-1} X^T X \vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

Solving with Matrix Math

$$\vec{\epsilon} = \vec{y} - X\vec{\beta}$$

$$\vec{\epsilon}^T\vec{\epsilon} = (\vec{y} - X\vec{\beta})^T(\vec{y} - X\vec{\beta})$$

$$0 = \vec{y}^T\vec{y} - \vec{\beta}^T X^T \vec{y} - \vec{y}^T X \vec{\beta} + \vec{\beta}^T X^T X \vec{\beta}$$

$$0 = \vec{y}^T\vec{y} - 2\vec{\beta}^T X^T \vec{y} + \vec{\beta}^T X^T X$$

$$-2\vec{\beta}^{T1-0} X^T \vec{y} + \vec{\beta}^T X^T X \vec{\beta} = 0$$

$$-2X^T \vec{y} + 2X^T X \vec{\beta} = 0$$

$$X^T X \vec{\beta} = X^T \vec{y}$$

$$(X^T X)^{-1} X^T X \vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

$$\vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

Solving with Matrix Math

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \qquad AA^{-1} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\vec{\beta} = (X^T X)^{-1} X^T \, \vec{y}$$

Carrying out the Math

$$\vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

Hours of Sleep Per Night	Age	Hours of Exercise a Week	Undergrad, Grad, None	Undergrad vs Grad
14	49	11	0.33	-0.5
12	53	1	0.33	-0.5
7	28	17	0.33	-0.5
13	30	13	0.33	-0.5
6	34	3	0.33	0.5
14	71	20	0.33	0.5
14	43	19	0.33	0.5
8	35	14	0.33	0.5
13	63	0	-0.666666667	0
8	23	5	-0.666666667	0
4	38	8	-0.666666667	0
8.08	28	4	-0.666666667	0

Carrying out the Math

$$\vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

Intercept	Age	Hours of Exercise a Week	Undergrad, Grad, None	Undergrad vs Grad
1	49	11	0.33	-0.5
1	53	1	0.33	-0.5
1	28	17	0.33	-0.5
1	30	13	0.33	-0.5
1	34	3	0.33	0.5
1	71	20	0.33	0.5
1	43	19	0.33	0.5
1	35	14	0.33	0.5
1	63	0	-0.666666667	0
1	23	5	-0.666666667	0
1	38	8	-0.666666667	0
1	28	4	-0.666666667	0

Carrying out the Math

$$\vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

B0	1	1	1	1	1	1	1	1	1	1	1	1
B1	49	53	28	30	34	71	43	35	63	23	38	28
B2	11	1	17	13	3	20	19	14	0	5	8	4
B3	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	-0.66	-0.66	-0.66	-0.66
B4	-0.5	-0.5	-0.5	-0.5	0.5	0.5	0.5	0.5	0	0	0	0

Carrying out the Math

$$\vec{\beta} = (X^T X)^{-1} X^T \vec{y}$$

B0	1.142303548	-0.018526505	-0.030756717	0.336370451	0.21417592
B1	-0.018526505	0.000429388	8.49671E-05	-0.002773002	-0.0027664
B2	-0.030756717	8.49671E-05	0.002843668	-0.023163562	-0.0104414
B3	0.336370451	-0.002773002	-0.023163562	0.57382688	0.09701723
B4	0.214175915	-0.002766363	-0.010441401	0.097017226	0.55245149

Carrying out the Math

$$\vec{\beta} = (X^{T}X)^{-1}X^{T}\,\vec{y}$$

B0	-0.098783573	0.134677578	0.105732735	0.191706593	0.639343659	-0.569001228	-0.019502363	0.282493265	-0.24911325	0.338163374	-0.032004356	0.276287565
B1	0.003906972	0.00477485	-0.004600364	-0.004081458	-0.005979942	0.011351839	-0.00075598	-0.004615916	0.010373579	-0.006377088	0.000318627	-0.004315117
B2	0.002186539	-0.025910277	0.017464239	0.0062595	-0.032278717	0.019207431	0.013984682	-0.000913397	-0.009961412	0.000858244	0.010663757	-0.001560589
B3	0.088461205	0.30900482	0.007712865	0.09482111	0.412381952	-0.083999662	0.016807944	0.154809767	-0.220879899	-0.225777649	-0.336863358	-0.216479094
B4	-0.280117972	-0.18676942	-0.284672742	-0.248439867	0.397360177	0.117500919	0.205400497	0.279738407	-0.024783135	0.033664402	-0.039155252	0.030273985

B0	-0.098783573	0.134677578	0.105732735	0.191706593	0.639343659	-0.569001228	-0.019502363	0.282493265	-0.24911325	0.338163374	-0.032004356	0.276287565
B1	0.003906972	0.00477485	-0.004600364	-0.004081458	-0.005979942	0.011351839	-0.00075598	-0.004615916	0.010373579	-0.006377088	0.000318627	-0.004315117
B2	0.002186539	-0.025910277	0.017464239	0.0062595	-0.032278717	0.019207431	0.013984682	-0.000913397	-0.009961412	0.000858244	0.010663757	-0.001560589
B3	0.088461205	0.30900482	0.007712865	0.09482111	0.412381952	-0.083999662	0.016807944	0.154809767	-0.220879899	-0.225777649	-0.336863358	-0.216479094
B4	-0.280117972	-0.18676942	-0.284672742	-0.248439867	0.397360177	0.117500919	0.205400497	0.279738407	-0.024783135	0.033664402	-0.039155252	0.030273985

$$(X^TX)^{-1}X^T \vec{y}$$

Hours of Sleep Per Night
14
12
7
13
6
14
14
8
13
8
4
8.08

Carrying out the Math

$$\vec{\beta} = (X^TX)^{-1}X^T \vec{y}$$

B0	2.893654423
B1	0.15251752
B2	0.094434562
B3 (In School vs Not in School)	1.231000594
B4 (Under vs Grad)	-2.207496706

Carrying out the Math

$$\hat{y} = 2.89 + .15x_{i1} + .09x_{i2} + 1.23x_{i3} - 2.21x_{i4}$$

Carrying out the Math

Predicted Hours of Sleep
12.91987496
12.58559942
10.28361442
10.21091121
7.669138959
14.91767475
10.55274963
8.860436663
11.6815911
6.053063126
8.624129608
6.721216162

$$(y - \hat{y})^2 = e$$

Carrying out the Math

Predicted Hours of Sleep
1.166670105
0.342926675
10.78212364
7.779016289
2.786024864
0.84212694
11.8835351
0.740351251
1.738202017
3.790563193
21.38257463
1.846293518

Residuals Squared
8.008192282
6.228016443
0.037486543
0.01461952
5.860568181
23.30644346
0.214137222
1.511826
2.533162242
16.29685933
2.148776005
11.34870455

$$\sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y})^2}{N - (k+1)}}$$

Carrying out the Math

Standard Error of the Regression

$$\sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y})^2}{N - (k+1)}} = 3.049$$

Carrying out the Math

Standard Error of the Regression

$$\sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y})^2}{N-(k+1)}}^{2} = \sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y})^2}{N-(k+1)} * Diag} =$$

B0	1.142303548	-0.018526505	-0.030756717	0.336370451	0.21417592
B1	-0.018526505	0.000429388	8.49671E-05	-0.002773002	-0.0027664
B2	-0.030756717	8.49671E-05	0.002843668	-0.023163562	-0.0104414
B3	0.336370451	-0.002773002	-0.023163562	0.57382688	0.09701723
B4	0.214175915	-0.002766363	-0.010441401	0.097017226	0.55245149

Carrying out the Math

	Slope	$\sqrt{SE\ for\ Reg * Diag}$
Age	.15	.06
Exercise	.09	.16
In School vs Not in School	1.23	2.31
Under vs Grad	-2.21	2.27

Carrying out the Math

Usually 1.96

	Slope	SE	CI ($B_k \pm 2.365$*SE)
Age	.15	.06	[.003, .30]
Exercise	.09	.16	[-0.29, 0.48]
In School vs Not in School	1.23	2.31	[-4.23, 6.69]
Under vs Grad	-2.21	2.27	[-7.57, 3.15]

THANK YOU

MAPLE Lab

Memory And Psycholinguistics in Learning & Education

Scott Fraundorf

Doug Getty

Ciara Willett

Griffin Koch

Joseph Yurko and Machine Learning INFSCI 2595

Elizabeth Votruba-Drzal

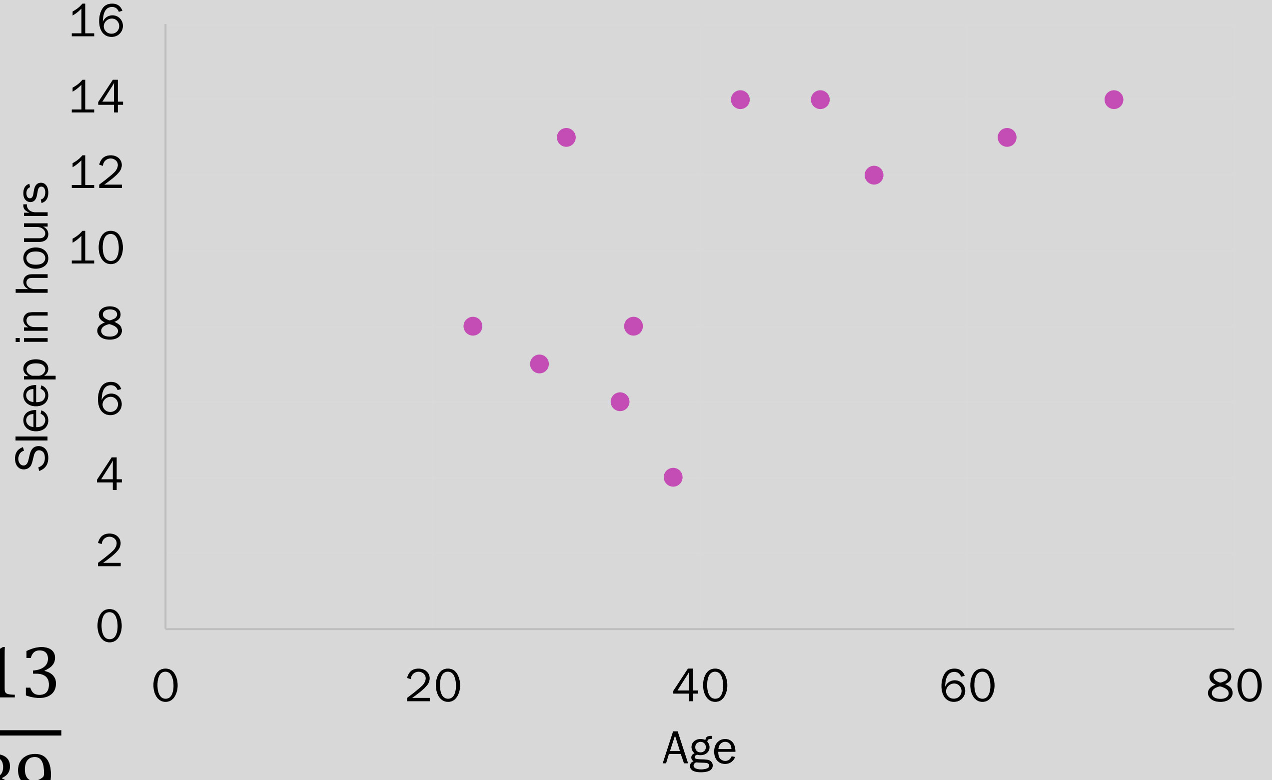
Starting simple

a = intercept
b = slope

$$\boxed{y = a + bx} + e$$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

Hours of Sleep Per Night	Age
14	49
12	53
7	28
13	30
6	34
14	71
14	43
8	35
13	63
8	23
4	38
?	28



$$b = \frac{11 * 5{,}143 - 467 * 113}{11 * 22{,}107 - 218{,}089}$$

How do we find Beta?

$$\beta_1 = r\frac{s_x}{s_y} = \frac{n\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{n\sum_{i=1}^{N} {x_i}^2 - \left(\sum_{i=1}^{N} x_{\,i}\right)^2}$$

$$\frac{\sum_{i=1}^{N} x_i y_i - \bar{x}\bar{y}}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \frac{\sum_{i=1}^{N} x_i y_i - \bar{y}n\bar{x}}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \frac{\sum_{i=1}^{N} x_i y_i - \bar{y}\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\frac{\sum_{i=1}^{N} x_i(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \frac{\sum_{i=1}^{N}(y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

How do we find Beta?

$$\beta_1 = r\frac{s_x}{s_y} = \frac{n\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{n\sum_{i=1}^{N} x_i^{\,2} - \left(\sum_{i=1}^{N} x_{\,i}\right)^2}$$

$$\frac{n\sum_{i=1}^{N} x_i y_i - n\bar{x}n\bar{y}}{n\sum_{i=1}^{N} x_i^{\,2} - (n\bar{x})^2}$$

How do we find Beta?

$$\sum_{i=1}^{N}(xi - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \bar{y} - \cancel{\sum_{i=1}^{N} \bar{x} y_i} + \cancel{\sum_{i=1}^{N} \bar{x}\bar{y}}$$

$$\sum_{i=1}^{N}(y_i - \bar{y})x_i$$

How do we find Beta?

$$\beta_1 = r\frac{s_x}{s_y} \quad = \quad \frac{n\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{n\sum_{i=1}^{N} {x_i}^2 - \left(\sum_{i=1}^{N} x_{\ i}\right)^2}$$

$$\frac{\cancel{n}\sum_{i=1}^{N} x_i y_i - n^{\cancel{2}}\bar{x}\bar{y}}{\cancel{n}\sum_{i=1}^{N} {x_i}^2 - n^{\cancel{2}}\bar{x}^2}$$

$$\beta_1 = \frac{cov(x,y)}{var(x)} = \frac{SC(x,y)}{SS(x)} = \frac{\sum_{i=1}^{N}(xi-\bar{x})(yi-\bar{y})}{\sum_{i=1}^{N}(xi-\bar{x})^2}$$

How do we find Beta?

$$\beta_1 = r\frac{s_x}{s_y} = \frac{n\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{n\sum_{i=1}^{N} {x_i}^2 - \left(\sum_{i=1}^{N} x_{\ i}\right)^2}$$

$$\frac{\sum_{i=1}^{N} x_i y_i - \bar{x}\bar{y}}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \frac{\sum_{i=1}^{N} x_i y_i - \bar{y}n\bar{x}}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \frac{\sum_{i=1}^{N} x_i y_i - \bar{y}\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\frac{\sum_{i=1}^{N} x_i(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \frac{\sum_{i=1}^{N}(y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

How do we find Beta?

$$\beta_1 = r\frac{s_x}{s_y} = \frac{n\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{n\sum_{i=1}^{N} {x_i}^2 - \left(\sum_{i=1}^{N} x_i\right)^2}$$

$$\frac{\cancel{n}\sum_{i=1}^{N} x_i y_i - n^{\cancel{2}}\bar{x}\bar{y}}{\cancel{n}\sum_{i=1}^{N} {x_i}^2 - n^{\cancel{2}}\bar{x}^2}$$

$$\beta_1 = \frac{cov(x,y)}{var(x)} = \frac{SC(x,y)}{SS(x)} = \frac{\sum_{i=1}^{N}(xi-\bar{x})(yi-\bar{y})}{\sum_{i=1}^{N}(xi-\bar{x})^2}$$

How do we find Beta?

$$\beta_1 = \frac{cov(x,y)}{var(x)} = \frac{SC(x,y)}{SS(x)} = \frac{\sum_{i=1}^{N}(xi - \bar{x})(yi - \bar{y})}{\sum_{i=1}^{N}(xi - \bar{x})^2}$$

$$\frac{\sum_{i=1}^{N}(xi - \bar{x})(yi - \bar{y})}{\sum_{i=1}^{N}(xi - \bar{x})^2} = \frac{\sum_{i=1}^{N}(xi - \bar{x})(yi - \bar{y})}{\cancel{N - 1}} \times \frac{\cancel{N - 1}}{\sum_{i=1}^{N}(xi - \bar{x})^2}$$