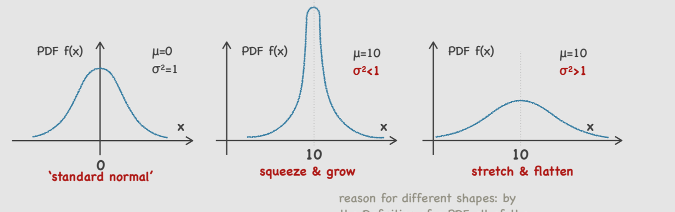
General Normal r.v. cont...

$X \sim \text{Normal}(\mu,\sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

by manipulating the parameters you can make any bell curve you want



reason for different shapes: by the Definition of a PDF all of the above areas must equal to 1

Supp[X]=$\mathcal{R}$
$\mu \in \mathcal{R}$
$\sigma^2 \in (0,\infty)$

$\sigma^2=0$?
Not possible
if $\sigma^2<0$, Var[X]≥0 so not possible

infinitesimally skinny bell curve is NOT a bell curve but a vertical line , thus under the plank-length x would have multiple values which excludes a possibility of a function

recall: $X=\mu+\sigma Z \rightarrow Z=\frac{X-\mu}{\sigma}$

I can take any normal r.v. and do this linear transformation and get standard normal

typical test question

e.g.
X is a rand. var model for male height. X is normally distributed with mean-μ=70" and SE-σ= 4". What is the probability a male is more than 78" tall ?

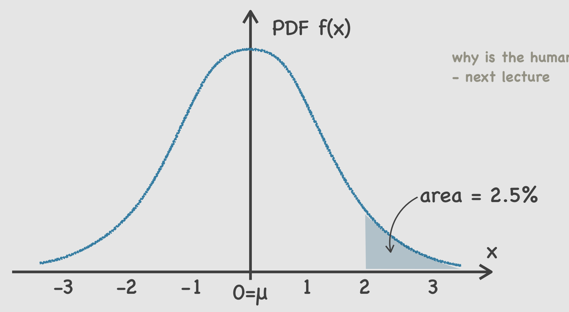
model: $X \sim \text{Normal}(70'', 4''^2)$ → $X=\mu+\sigma Z$: $X=70''+4''Z \rightarrow$ standardization $Z=\frac{X-70''}{4''} \rightarrow Z \sim \text{Normal}(0,1)$

4 is far more important than 16

probability statement
$P(X \geq 78'') = P\left(\frac{X-70''}{4''} \geq \frac{78''-70''}{4''}\right) = P(Z \geq 2) \approx 2.5\%$

what is the prob. of this 8" deviation from the 'standard' 70" ?

subtract the mean and divide by S.E. μ=70 σ=4



why is the human Height 'normally' distributed ? – next lecture

MGF – Moment Generating Function

$$M_X(t) := E[e^{tX}] = \begin{cases} \sum_{x \in Supp[X]} e^{tx}\, p(x) & \text{if discrete r.v.} \\ \int_{x \in Supp[X]=\mathcal{R}} e^{tx} f(x)\, dx & \text{if continuous r.v.} \end{cases}$$

g(X)

MGF – Moment Generating Function of r.v. X
which is an Expectation of g(X)

if two MGFs are equal, then X & Y are 'equal in distribution' (their PMFs (if existent) , CDFs and PDFs are equal)

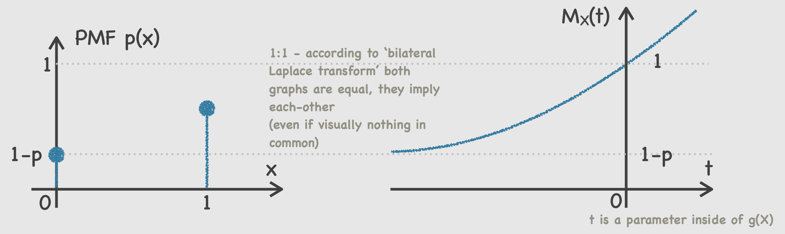
property **1**

if $M_X(t) = M_Y(t) \rightarrow X \overset{d}{=} Y$ identically distributed r.v.'s have the same MGF

$M_X(t)$ is 1:1 with PDF or PMF

MGF – Bernoulli

e.g.
$X \sim \text{Bernoulli}(p) := p^x(1-p)^{1-x}$



$M_X(t) = E[e^{tX}] = \sum_{x \in Supp[X]} e^{tx} p(x)$

$= e^{t\cdot 0} p(0) + e^{t \cdot 1} p(1)$ prob. of failure, prob. of success

$= e^0\ 1-p + e^t\ p$

$M_X(t) = 1-p+pe^t$

MGF – usefulness

Why is $M_X(t)$ useful?

consider $X \sim \text{Binomial}(n,p) := \binom{n}{x} p^x (1-p)^{n-x}$

find $\underbrace{E[X^{17}]}_{17^{th}\text{ moment}} = \sum_{x=0}^{n} \underbrace{x^{17}\binom{n}{x} p^x(1-p)^{n-x}}_{\text{not possible to figure out}}$

recall Taylor Series of f(x) near x=c

$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$

$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$ 3rd degree approximation

$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2$ 2nd degree approximation

let $f(x) = e^x$ near $x \approx 0 \rightarrow c=0$

$e^x \approx \frac{e^0}{0!}x^0 + \frac{e^0}{1!}x^1 + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{i=0}^{\infty}\frac{x^i}{i!}$

$e^{tX} = 1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \frac{t^5X^5}{5!} + \ldots$

$M_X(t) = E[e^{tX}] = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \frac{t^5X^5}{5!} + \ldots\right]$ E[X+Y] = E[X]+E[Y]

$= \underset{1}{E[1]} + \frac{t}{1!}E[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \frac{t^4}{4!}E[X^4] + \ldots$

$\frac{d}{dt}[M_X(t)] = \frac{d}{dt}E[e^{tX}] = E\left[\frac{d}{dt}[e^{tX}]\right]$

d/dt: anything that's a function of the r.v. X is a constant when differentiating with respect to t

$M'_X(t) = 0 + 1 \cdot E[X] + \frac{t}{1!}E[X^2] + \frac{t^2}{2!}E[X^3] + \frac{t^3}{3!}E[X^4] + \ldots$

let $t=0 \rightarrow M'_X(0) = E[X] = \mu$ '1st moment'

$M''_X(t) = 0 + 0\cdot E[X] + 1\cdot E[X^2] + \frac{t}{1!}E[X^3] + \frac{t^2}{2!}E[X^4] + \ldots$

let $t=0 \rightarrow M''_X(0) = E[X^2]$ '2nd moment'

$M'''_X(t) = 0 + 0 + 0\cdot E[X^2] + 1\cdot E[X^3] + \frac{t}{1!}E[X^4] + \ldots$

let $t=0 \rightarrow M'''_X(0) = E[X^3]$ '3rd moment'

so $M^{(17)}_X(0) = E[X^{17}]$
the 17th derivative of Binomial's MGF is equivalent to its 17th moment. While we do NOT have the 17th moment, we certainly can calculate the 17th derivative of the Binomial MGF

property **2**

$M^{(k)}_X(0) = E[X^k]$

this is how MGF gets its name ! it generates moments ... you can just crank out moments by taking derivatives

the kth derivative of MGF at t=0 gives you the 'kth moment'

linear transformation of r.v.
$Y = aX + c$

$M_Y(t) = M_{aX+c}(t) = E[e^{t(aX+c)}] = E[e^{taX}e^{tc}] = e^{tc}E[e^{taX}]$ (by $M_X(t) = E[e^{tX}]$; $E[e^{taX+tc}]$ & $x^{m+n}=x^mx^n$)

let $t'=ta$ $= e^{tc}E[e^{t'X}]$

$= e^{tc}M_X(t')$

$= e^{tc}M_X(ta)$

thus as soon as you say that X is linearly transformed you immediately know its an MGF

property **3**

$M_{aX+c}(t) = e^{tc}M_X(ta)$

$M_Y(t) = e^{tc}M_X(ta)$

if $X_1, X_2 \overset{iid}{\sim}$ $Y=X_1+X_2$

$M_Y(t) = M_{X_1}(t)M_{X_2}(t)$ property 4

$= (M_X(t))^2$ property 1

X_1, X_2 are independent r.v.'s

let $Y = X_1 + X_2$ ($E[e^{tX_1+tX_2}]$ & $x^{m+n}=x^mx^n$)

$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1}\cdot e^{tX_2}]$

$= E[e^{tX_1}]\cdot E[e^{tX_2}]$

$M_Y(t) = M_{X_1}(t)\cdot M_{X_2}(t) \leftarrow = M_{X_1}(t)\cdot M_{X_2}(t)$

property **4**

$M_{X_1+X_2}(t) = M_{X_1}(t)\cdot M_{X_2}(t)$

if iid : $M_Y(t) = (M_X(t))^2$

the MGF of a sum of independent r.v.'s is the product of MGF's of the constituent r.v.'s

previously you couldn't just add two PMFs , here you just multiply two MGFs very convenient since PMF and its MGF are the same 1:1

MGF – Binomial

$X \sim \text{Binomial}(n,p) = X\sim\text{Bernoulli}(p), X\sim\text{Bernoulli}(p), \ldots, X\sim\text{Bernoulli}(p)$ — n

We have a cup with 1 coin. We draw the coin out. Record whether H or T and put the coin back in the cup. We repeat the experiment n-times then see # of Hs.
OR
We have a cup with n coins. We draw them ALL out and record how many Hs.

if $X_1,\ldots,X_n \overset{iid}{\sim} \text{Bernoulli}(p)$ $T = X_1+X_2+\ldots+X_n$

$M_T(t) = M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t)\cdot M_{X_2}(t)\cdot\ldots\cdot M_{X_n}(t) = (M_X(t))^n$ property 4, property 1

$= (1-p+pe^t)^n$

and we already know the MGF of Bernoulli

$M_X(t) = (1-p+pe^t)^n$

MGF – Geometric

$X \sim \text{Geometric}(p) := (1-p)^{x-1}p$

$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}(1-p)^{x-1}p$ | pull cont. p

$= p\sum_{x=0}^{\infty} e^{tx}(1-p)^{x-1}$ | $\cdot\frac{1-p}{1-p}$

$= \frac{p}{1-p}\sum_{x=1}^{\infty}(e^t)^x(1-p)^x$ | update index

$= \frac{p}{1-p}\sum_{x=1}^{\infty}(e^t(1-p))^x$ | $a^mb^m=(ab)^m$

$= \frac{p}{1-p}\sum_{x=0}^{\infty}(e^t(1-p))^x - 1$ | update index

Geometric Series $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$, $a = e^t(1-p)$

$= \frac{p}{1-p}\left[\frac{1}{1-e^t(1-p)} - 1\right]$ | adjust denominator & −

$= \frac{p}{1-p}\,\frac{e^t(1-p)}{1-e^t(1-p)}$ | cancel (1-p)

$= \frac{pe^t}{1-e^t(1-p)}$

$M_X(t) = \frac{pe^t}{1-e^t(1-p)}\ \mathbb{1}\ t < \ln(1/_{1-p})$ true if (indicator function)

if $e^t(1-p) < 1$
$e^t < 1/_{1-p}$ | ln both sides
$t < \ln(1/_{1-p})$

MGF – Exponential

$X \sim \text{Exponential}(\lambda) := \lambda e^{-\lambda x}$

$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx$ | pull cont. λ

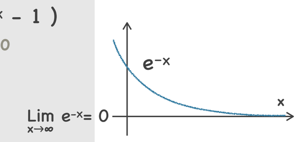
$= \lambda\int_0^\infty e^{tx}e^{-\lambda x}dx$ | $a^mb^m=(ab)^m$

$= \lambda\int_0^\infty(e^{t-\lambda})^x dx = \frac{\lambda}{t-\lambda}\left[(e^{t-\lambda})^x\right]_0^\infty \frac{\lambda}{t-\lambda}\left(\lim_{x\to\infty} e^{(t-\lambda)x} - 1\right)$

$= \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{\lambda-t}$

if $t-\lambda<0 \rightarrow t<\lambda$

$M_X(t) = \frac{\lambda}{\lambda-t}\ \mathbb{1}\ t<\lambda$



$Y = aX$

$M_Y(t) = M_{aX}(t) = M_X(at) = \frac{\lambda}{\lambda-at} = \frac{\lambda' a}{\lambda' a - at} = \frac{\lambda'}{\lambda'-t} \rightarrow Y\sim\text{Exponential}(\lambda')$ property 3

$Y \sim \text{Exponential}(\frac{\lambda}{a})$

or · let $\lambda' = \frac{\lambda}{a}$, then $\lambda = \lambda' a$; $\frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a}-t}$

using MGF

remember ?

using CDF

let $X\sim\text{Exponential}(\lambda)$, Y=2X   how is this distributed ?

a clever trick – look at the CDF of Y

$F_Y(x) = P(Y\leq x) = P(2X\leq x) = P(X\leq\frac{x}{2}) = F_X(\frac{x}{2})$ now we can use the CDF of X

$= 1 - e^{-\lambda\frac{x}{2}}$

$= 1 - e^{-\frac{\lambda}{2}x}$

$= 1 - e^{-\lambda' x}$ if $\lambda' = \frac{\lambda}{2}$

$= F_Y(x)$

Exponential CDF $F(x) = 1-e^{-\lambda x}$

$Y\sim\text{Exponential}(\frac{\lambda}{2})$ ↑ $Y\sim\text{Exponential}(\lambda')$ ← s.t.

MGF – Normal

$M_X(t) = E[e^{tX}] = \int_{\mathcal{R}} e^{tx}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}dx$ | pull cont. through

$= \frac{1}{\sqrt{(2\pi)}}\int_{\mathcal{R}} e^{tx}e^{-\frac{x^2}{2}}dx$ | $x^mx^n = x^{m+n}$

$= \frac{1}{\sqrt{(2\pi)}}\int_{\mathcal{R}} e^{-\frac{1}{2}x^2+tx}dx$

$-1/_2x^2 + tx = -1/_2(x^2-2tx) = -1/_2((x-t)^2-t^2) = -1/_2(x-t)^2 + 1/_2t^2$

complete the square
$(x-t)^2 = x^2-2tx+t^2$
$(x-t)^2 - t^2 = x^2-2tx$

$= \frac{1}{\sqrt{(2\pi)}}\int_{\mathcal{R}} e^{-\frac{1}{2}(x-t)^2}e^{\frac{1}{2}t^2}dx$

$= e^{\frac{1}{2}t^2}\underbrace{\int_{\mathcal{R}}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{1}{2}(x-t)^2}dx}_{=1}$ if $X\sim\text{Normal}(t,1) \rightarrow f(x)$

Standard Normal $M_Z(t) = e^{\frac{t^2}{2}}$

if $X\sim\text{Normal}(\mu,\sigma^2) \rightarrow X=\mu+\sigma Z$

property 3
$M_{aX+c}(t) = e^{tc}M_X(ta)$
$M_{\sigma Z+\mu}(t) = e^{t\mu}M_Z(t\sigma)$
$= e^{t\mu}e^{\frac{t^2\sigma^2}{2}}$

$\rightarrow M_X(t) = e^{t\mu}e^{\frac{\sigma^2t^2}{2}}$

$= e^{t\mu+\frac{\sigma^2t^2}{2}}$

General Normal $M_X(t) = e^{t\mu+\frac{\sigma^2t^2}{2}}$

if $X\sim\text{Deg}(c) \rightarrow M_X(t) = e^{tc}$

Lévy's Continuity Theorem

if $X_1,\ldots,X_n$ is a sequence of r.v.'s

$\lim_{n\to\infty} M_{X_n}(t) = M_Y(t) = e^{t\mu} \iff X_n \rightarrow Y$

if the Limit of LHS-MGF approaches RHS-MGF, then X_n and Y are the same

if n large, $M_{X_n}(t) \approx M_Y(t) \rightarrow X_n \overset{d}{\approx} Y$ approx. equally distributed; if the MGFs are equal then you have the same r.v.

if you have the same MGFs then you have the same r.v. (same PMF, PDF, CDF) b/c MGF and their corresponding PMF, CDF or PDF are the same by 'bilateral Laplace transform'

Law of Large Numbers LLN

if $X_1,\ldots,X_n \overset{iid}{\sim}$ something s.t. $E[X]<\infty$, $\bar{X}_n \rightarrow \mu$ (r.v. sequence converges to μ; Lec-12)   thus if $Y\sim\text{Deg}(\mu) \rightarrow \mu$, we are done (converges)

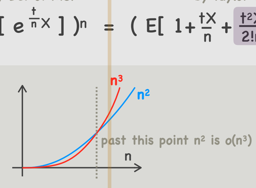
$\lim_{n\to\infty} M_{\bar{X}_n}(t) = M_Y(t) = e^{t\mu}$

$M_{\bar{X}_n}(t) = M_{\frac{X_1+\ldots+X_n}{n}}(t) = M_{\frac{T}{n}}(t) = M_T(\tfrac{t}{n}) = (M_X(\tfrac{t}{n}))^n = (E[e^{\frac{t}{n}X}])^n = \left(E\left[1+\frac{tX}{n}+\frac{t^2X^2}{2!n^2}+\frac{t^3X^3}{3!n^3}+\ldots\right]\right)^n$

(by Def of $\bar{X}_n$; T total r.v.; by property 3; by property 4; by Def of MGF; by Taylor Series)

We say $f(n) = o(g(n))$ if $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$ f(n) is 'little o' of g(n); 'little o' similar to CS 'Big O'

g(n) 'grows' faster then f(n)
e.g. $n^2 = o(n^3)$
$\lim_{n\to\infty}\frac{n^2}{n^3} = \lim_{n\to\infty}\frac{1}{n} = 0$ ✔



We can say $\frac{t^2X^2}{2!n^2}+\frac{t^3X^3}{3!n^3}+\ldots \geq o(\tfrac{1}{n})$ b/c $\lim_{n\to\infty}\frac{\frac{t^2X^2}{2!n^2}+\frac{t^3X^3}{3!n^3}+\ldots}{\frac{1}{n}} = \lim_{n\to\infty}\frac{t^2X^2}{2!n^1}+\frac{t^3X^3}{3!n^2}+\ldots = 0$

$= (E[1+\frac{tX}{n}+o(\tfrac{1}{n})])^n$ E[X+Y] = E[X] + E[Y]

$= (E[1] + E[\frac{tX}{n}] + E[o(\tfrac{1}{n})])^n$ $E[\frac{tX}{n}] = E[\frac{t}{n}X] = \frac{t}{n}E[X] = \frac{t}{n}\mu = \frac{t\mu}{n}$

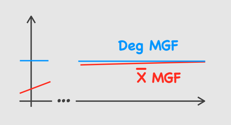
$= (1 + \frac{t\mu}{n} + E[o(\tfrac{1}{n})])^n$ E[1] = 1; this E can be ignored

$= (1+\frac{t\mu}{n}+o(\tfrac{1}{n}))^n$

recall $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$
↓
$\lim_{n\to\infty}(1+\frac{a}{n})^n = e^a$
↓
$\lim_{n\to\infty}(1+\frac{1}{n}+\frac{1}{n^2})^n = e$
↓
$\lim_{n\to\infty}(1+\frac{a}{n}+\frac{1}{n^2})^n = e^a$

fact
anything that grows faster than 1/n does not matter e.g $1/n^{1.0001}$
but if n=1billion , f(n)=3.08 ≠ 2.718
n=1trillion, f(n)=2.90 ≠ 2.718
slow !!

$\lim_{n\to\infty}M_{\bar{X}_n}(t) = \lim_{n\to\infty}(1+\frac{t\mu}{n}+o(\tfrac{1}{n}))^n = e^{t\mu} \longrightarrow \bar{X}_n \rightarrow \mu$



if $X\sim\text{Deg}(c) \rightarrow M_X(t) = e^{tc}$

if $X\sim\text{Deg}(\mu) \rightarrow M_X(t) = e^{t\mu}$

b/c the long run behavior of the MGF of $\bar{X}$ gets closer and closer to MGF of Deg(μ) , $\bar{X}$ converges to μ

Central Limit Theorem CLT Crown Jewel of Math 241

if $X_1,\ldots,X_n \overset{iid}{\sim}$ mean-μ and SE-σ

$C_n := \frac{\bar{X}_n-\mu}{\frac{\sigma}{\sqrt{n}}}$ C_n is the standardized $\bar{X}_n$ μ=0, σ=1

subtract the mean and divide by S.E.   S.E. of $\bar{X} = \frac{\sigma}{\sqrt{n}}$

how does C_n look like as n gets large ?   let's examine the long-run behavior of C_n MGF and see if it matches anything we know.

$= \frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} = \frac{\sqrt{n}(\frac{X_1+\ldots+X_n}{n}-\mu)}{\sigma} = \frac{\sqrt{n}(\frac{X_1+\ldots+X_n-\mu n}{n})}{\sigma} = \frac{\frac{X_1+\ldots+X_n-\mu n}{\sqrt{n}}}{\sigma} = \frac{X_1+\ldots+X_n-\mu n}{\sigma\sqrt{n}} = \frac{(X_1+\ldots+X_n)-(\mu+\ldots+\mu)}{\sigma\sqrt{n}}$ (by Def of $\bar{X}_n$; $\frac{n^{1/2}}{n^1} = n^{1/2-1} = n^{-1/2} = 1/\sqrt{n}$)

$= \frac{(X_1-\mu)+\ldots+(X_n-\mu)}{\sigma\sqrt{n}}$

$= \frac{1}{\sqrt{n}}\left[\frac{(X_1-\mu)}{\sigma}+\frac{(X_2-\mu)}{\sigma}+\ldots+\frac{(X_n-\mu)}{\sigma}\right]$

$= \frac{1}{\sqrt{n}}[Z_1+\ldots+Z_n]$

Var[X] = E[X²]−μ² ; if μ=0 , Var[X]=E[X²] (σ^2)
if σ=1 ,σ²=1 and E[X²]=1

we know that E[Z]=0, SE[Z]=1, E[Z²]=1

standardization: subtract the mean and divide by S.E.   let $Z_i = \frac{(X_i-\mu)}{\sigma}$

MGF of C_n

$M_{C_n}(t) = M_{\frac{1}{\sqrt{n}}(Z_1+\ldots+Z_n)}(t) = M_{(Z_1+\ldots+Z_n)}(\tfrac{t}{\sqrt{n}}) \underset{\text{b/c iid}}{=} (M_Z(\tfrac{t}{\sqrt{n}}))^n = (E[e^{\frac{t}{\sqrt{n}}Z}])^n = \left(E\left[1+\frac{tZ}{\sqrt{n}}+\frac{t^2Z^2}{2!n}+\frac{t^3Z^3}{3!n^{1.5}}+\frac{t^4Z^4}{4!n^2}+\ldots\right]\right)^n$

(by property 3; by property 4; by Def of MGF; by Taylor Series; $\sqrt{n}^2=n$, $\sqrt{n}^3=n^{1.5}$, $\sqrt{n}^4=n^2$; just as before $= o(\tfrac{1}{n})$)

$= (E[1+\frac{tZ}{\sqrt{n}}+\frac{t^2Z^2}{2!n}+o(\tfrac{1}{n})])^n$

$= (\underset{1}{E[1]} + \underset{0}{E[\frac{tZ}{\sqrt{n}}]} + E[\frac{t^2Z^2}{2!n}] + E[o(\tfrac{1}{n})])^n$ this E can be ignored

$= (1+0+\frac{t^2}{2!n}+o(\tfrac{1}{n}))^n$

remember μ standard normal = 0
$E[\frac{tZ}{\sqrt{n}}] = E[\frac{t}{\sqrt{n}}Z] = \frac{t}{\sqrt{n}}E[Z] = \frac{t}{\sqrt{n}}\mu = 0$

$E[\frac{t^2Z^2}{2!n}] = E[\frac{t^2}{2!n}Z^2] = \frac{t^2}{2!n}E[Z^2] = \frac{t^2}{2!n}\cdot 1$

What happens as n→∞

$\lim_{n\to\infty}M_{C_n}(t) = \lim_{n\to\infty}(1+\frac{t^2}{2!n}+o(\tfrac{1}{n}))^n = e^{\frac{t^2}{2}} = M_Z(t) \longrightarrow C_n \rightarrow Z\sim\text{Normal}(0,1)$

by the same 'e' transformation as we did in LLN

our C_n converges to the 'standard normal'

Central Limit Theorem – CLT

if $X_1,\ldots,X_n \overset{iid}{\sim}$ with mean-μ , SE-σ , and large n

$C_n := \frac{\bar{X}_n-\mu}{\frac{\sigma}{\sqrt{n}}} \rightarrow Z\sim\text{Normal}(0,1)$

MNEMONIC: if you want Z → subtract the mean and divide by S.E.

1. the standardized avg. of n r.v.'s looks like the 'standard normal' or the standardized average r.v. is approximately distributed as 'standard normal'

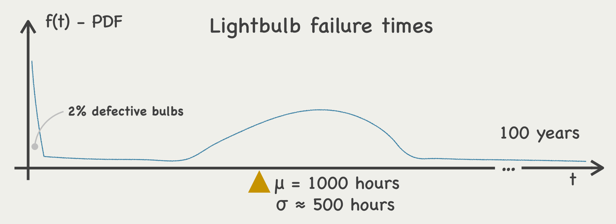
corollaries:

2. if μ large $C_n \overset{d}{\approx} Z\sim\text{Normal}(0,1)$ approx. equally distributed   if the mean is large, C_n and 'standard normal' are approx. equally distributed

3. $C_n := \frac{\bar{X}_n-\mu}{\frac{\sigma}{\sqrt{n}}} \approx Z \Rightarrow \bar{X}_n \approx \frac{\sigma}{\sqrt{n}}Z+\mu \Rightarrow \bar{X} \overset{d}{\approx} \text{Normal}(\mu, (\frac{\sigma}{\sqrt{n}})^2)$ (S.E. of $\bar{X}$)   the 'avg. r.v.' is approx. equally distributed as 'general normal' with $\sigma = \sigma/\sqrt{n}$ which is the S.E. of $\bar{X}$ 'average'

4. $n\bar{X} \approx \sigma\sqrt{n}Z+n\mu \Rightarrow T \approx \sigma\sqrt{n}Z+n\mu \Rightarrow T \overset{d}{\approx} \text{Normal}(n\mu, (\sigma\sqrt{n})^2)$ (total r.v.)   'how far away' from where you begin   the 'total r.v.' is approx. equally distributed as 'general normal' with μ=nμ and σ=σ√n

recall
$\bar{X} := \frac{X_1+\ldots+X_n}{n}$
$T := X_1+\ldots+X_n$



You buy 50 lightbulbs. What is the probability they last on **average** (keyword) more than 1300 hours?

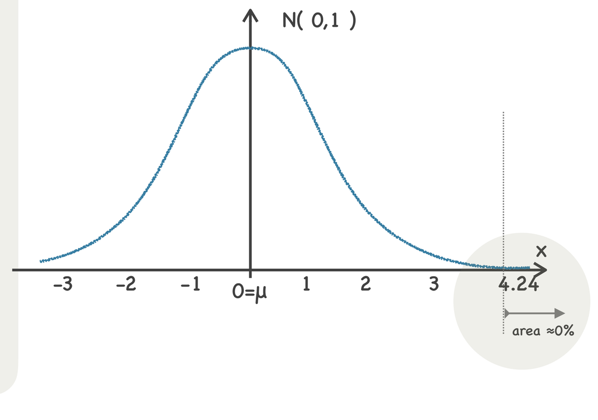
$X_1+\ldots+X_{50} \overset{iid}{\sim} f(\mu=1000, \sigma\approx 500)$

$\bar{X} \approx \text{Normal}(\mu, (\frac{\sigma}{\sqrt{n}})^2)$ S.E. $= \frac{\sigma}{\sqrt{n}} = \frac{500}{\sqrt{50}} = 70.7$

$\Rightarrow \bar{X} \approx \text{Normal}(1000, (70.7)^2)$

standardization: subtract the mean and divide by S.E. μ=1000 σ=70.7

$P(\bar{X}\geq 1300) = P\left(\frac{\bar{X}-1000}{70.7} \geq \frac{1300-1000}{70.7}\right) \approx P(Z\geq 4.24) \approx 0$



You are to flip a fair coin 30 times. You stop on the 1st Head. What is the probability that the **average** (keyword) stopping time is more than 2.75 ?

$X_1+X_2+\ldots+X_{30} \overset{iid}{\sim} \text{Geometric}(1/_2)$ $\mu = \frac{1}{p} = \frac{1}{1/_2} = 2$

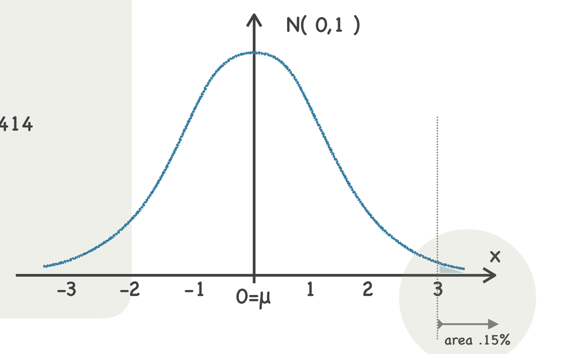
$\sigma^2 = \frac{1-p}{p^2} \Rightarrow \sigma = \sqrt{\frac{1-p}{p^2}} \approx 1.414$

$\bar{X} \approx \text{Normal}(\mu, (\frac{\sigma}{\sqrt{n}})^2)$ S.E. $= \frac{\sigma}{\sqrt{n}} = \frac{1.414}{\sqrt{30}} = .258$

$\Rightarrow \bar{X} \approx \text{Normal}(2, (.258)^2)$

standardization: subtract the mean and divide by S.E. μ=2 σ=.258

$P(\bar{X}\geq 2.75) = P\left(\frac{\bar{X}-2}{.258} \geq \frac{2.75-2}{.258}\right) \approx P(Z \geq \approx 3) = .0015$ by CLT



Take 100 random equally likely Left/Right steps. What is the probability you are more than 10 steps **away from** (keyword) where you began ?

$X_1+X_2+\ldots+X_{100} \overset{iid}{\sim} \text{Rademacher}\begin{cases}1 & \text{w.p. } 1/2\\ -1 & \text{w.p. } 1/2\end{cases}$

let $T = X_1+X_2+\ldots+X_{100} \underset{\text{by CLT}}{\approx} N(n\mu, (\sqrt{n}\,\sigma)^2) = N(0, 10^2)$ long-run asymptotic distribution; where we begin – mean (μ); standard deviation 'away from' μ (σ²)

$P(|T|>10) = P(T<-10) + P(T>10)$ (10 to the Left, 10 to the Right)

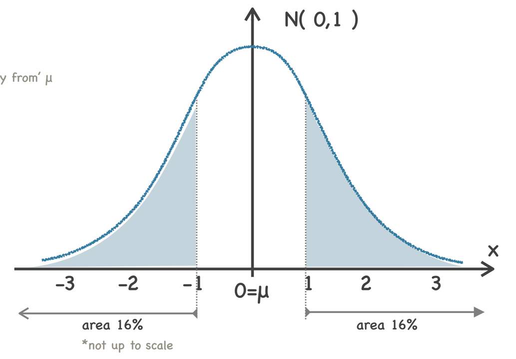
$= P\left(\frac{T-0}{10} < \frac{-10-0}{10}\right) + P\left(\frac{T-0}{10} > \frac{10-0}{10}\right)$

$= P(Z<-1) + P(Z>1)$ symmetric bell curve

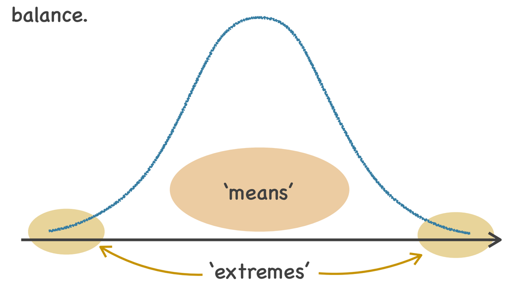
$= P(Z\notin[-1,1])$ remember the quantiles '68-95-997' rule ?

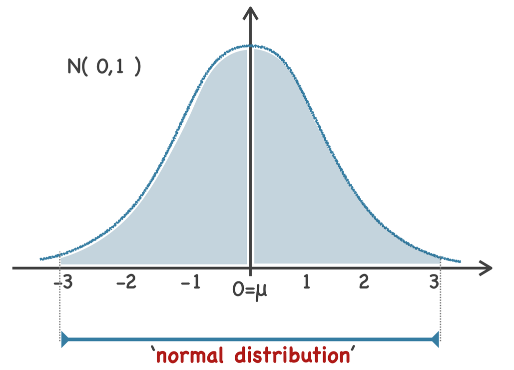
$= 100 - 68 = 32\%$

standardization: subtract the mean and divide by S.E. μ=0 σ=10



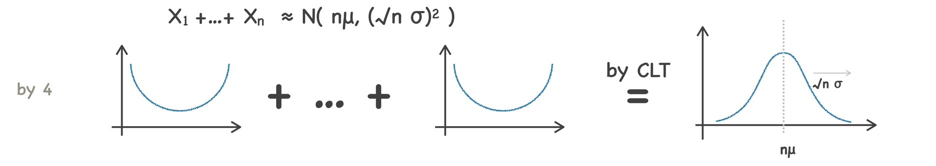
There is something very special about this shape. This function is very good at making the harmony between the 'means' and the 'extremes'. It creates a mathematically natural balance.



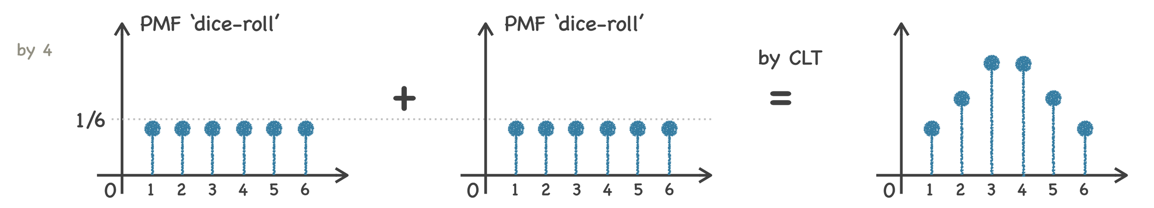


it defines what 'normalness' means, anything outside of this range is considered abnormal

$X_1+\ldots+X_n \approx N(n\mu, (\sqrt{n}\,\sigma)^2)$



anything creates a 'bell' 😱!



already starting to look like a bell