previously…

1 coin experiment

- Probability

Ω – sample space

outcomes $\Omega = \{\omega_1, \omega_2, \omega_3, \dots\}$ all possible outcomes

events $A := \{\omega_1, \omega_2\}$, $B := \{\omega_1, \omega_3\}$

ω : lower case Omega

e.g.

coin flip $\Omega = \{H, T\}$ 2 possible outcomes: heads , tails

$|\Omega| = 2$ cardinality of Ω = number of possible outcomes

In a coin flip only Heads or Tails could happen and nothing else. Heads and Tails are mutually exclusive – (be careful outcomes are NOT sets)

$\{H\}, \{T\}$ events — H, T outcomes

$\{H\} \subseteq \Omega$ events are subsets — $H \in \Omega$ outcomes are members

Sets in which all elements are outcomes are called 'events'. An event is defined by:

$A \subseteq \Omega \rightarrow A \in 2^\Omega$

set of outcomes – **event**

event space i.e. all possible events

here, $2^\Omega = \{\varnothing, \{H\}, \{T\}, \{H,T\}\}$ all must boil down to these 4 things

$$P(A) = \frac{|A|}{|\Omega|}$$ if Ω is finite

working def for probability of A

proportion of A to Ω

for the time being

Def: $P: 2^\Omega \rightarrow [0,1]$ the probability of all the possible events is between 0 and 1 (inclusively)

$P(H) = \frac{|H|}{|\Omega|}$ this does NOT compute ILLEGAL !! | member | ???

$P(\{H\}) = \frac{|\{H\}|}{|\Omega|} = 1/2$ probability of an event (or $|\{H,T\}|$)

$P(A) = \frac{\text{\# of outcomes of interest}}{\text{\# of all possible outcomes}}$ — Event = A, B, etc… — Sample Space Ω

$$\forall \omega_i \quad P(\omega_i) = \frac{1}{|\Omega|}$$ probability of an outcome

def of 'equally likely'

$P(\{H,T\}) = 1$

$P(\{H\} \cup \{T\}) = 1$

$P(\{H\} \cap \{T\}) = 0$ absurd – coin edge?

for $\Omega = \{H, T\}$: $P(\{H\}) = 1/2$, $P(\{T\}) = 1/2$ both events are equally likely

$P(\Omega) = 1 \qquad P(\varnothing) = 0$

$P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$

$P(\varnothing) = \frac{|\varnothing|}{|\Omega|} = 0$ trivial events

Complement Rule – useful triviality

$$P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{|\Omega| - |A|}{|\Omega|} = 1 - \frac{|A|}{|\Omega|} = 1 - P(A)$$

$$P(A) = 1 - P(A^c)$$

$|2^\Omega| = 2^{|\Omega|}$ numb of possible events $\neq |\Omega|$

3

2 coin experiment

Duplicates : YES Order : YES

What does $|2^\Omega|$ mean ? – the number of ALL possible events and does NOT equal the number of possible outcomes $|2^\Omega| = 2^{|\Omega|}$

- Lets perform a new experiment in which we will be tossing 2 coins. Now our sample space Ω has 4 possible outcomes :

<HH>	<HT>
<TH>	<TT>

$2^2 = 4$ 2 variables: coins; 2 choices: Heads, Tails; 4 combinations HH, HT, TH, TT

we are flipping 2 coins into the air simultaneously

$\Omega = \{\langle HH\rangle, \langle HT\rangle, \langle TH\rangle, \langle TT\rangle\}$ $|\Omega| = 4$

1 out of 4 possible outcomes

- what is the probability that 2 coins will produce an event: "2 Heads" ?

$P(\{\langle H,H\rangle\})$ official format can be colloquially expressed as $P(H,H)$ unofficial format – "abuse of notation" but okay for convenience

$P(H,H) = \frac{|\{\langle H,H\rangle\}|}{|\Omega|} = 1/4$ event contains 1 outcome a tuple – order matters

- what is the probability that 2 coins will produce at least one H ?

lets call this event $A := \{\langle HH\rangle, \langle HT\rangle, \langle TH\rangle\}$

$P(A) = \frac{|A|}{|\Omega|} = 3/4$

self-note: the size of this event is easily determined by a simple observation of the Truth Table. Three out of four 2-coin-flips can produce an event with at least one H. However, what if the Truth Table was much larger ?

- what is the 'event space' – 2^Ω ? aka power-set of Ω

$2^\Omega = \{$
$\{\langle HH\rangle\}, \{\langle HT\rangle\}, \{\langle TH\rangle\}, \{\langle TT\rangle\},$
$\{\langle HH\rangle,\langle HT\rangle\}, \{\langle HH\rangle,\langle TH\rangle\}, \{\langle HH\rangle,\langle TT\rangle\}, \{\langle HT\rangle,\langle TH\rangle\},$
$\{\langle HT\rangle,\langle TT\rangle\}, \{\langle TH\rangle,\langle TT\rangle\}, \{\langle HH\rangle,\langle HT\rangle,\langle TH\rangle\},$
$\{\langle HH\rangle,\langle HT\rangle,\langle TT\rangle\}, \{\langle HH\rangle,\langle TH\rangle,\langle TT\rangle\}, \{\langle HT\rangle,\langle TH\rangle,\langle TT\rangle\},$
$\{\langle HH\rangle,\langle HT\rangle,\langle TH\rangle,\langle TT\rangle\}, \{\}$
$\}$

2 coin experiment has: - 4 outcomes - 2^4 = 16 events

$|\Omega| = 4 \neq |2^\Omega| = 16$ sample space ≠ event space

- what is the size of our 'event space 2^Ω' ? $|2^\Omega| = 2^{|\Omega|} = 2^4 = 16$

- what is the probability of an event in which 2 coins will produce at least one T ?

Lets call this event $B := \{\langle TT\rangle, \langle TH\rangle, \langle HT\rangle\}$

$P(B) = \frac{|B|}{|\Omega|} = 3/4$ thus the probability of events A and B is the same

- what is the probability of an event represented by the Union of A and B ?

$P(A \cup B) = P(A) + P(B) = 3/4 + 3/4 = 6/4 = 1.5$?? – FALSE

A & B are NOT mutually exclusive

another way of looking at this: $A \cup B = \Omega$

$P(A \cup B) = \frac{|\{\langle TT\rangle, \langle TH\rangle, \langle HT\rangle, \langle HH\rangle\}|}{|\Omega|} = 4/4 = 1$ you will always get at least one H or one T

- what is the probability of an event represented by the Intersection of A and B ?

$A \cap B := \{\langle TH\rangle, \langle HT\rangle\}$

$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = 2/4 = 1/2$

other ways of expressing this: P(A and B), P(A & B), P(A,B), P(AB)

4 coin experiment

Duplicates : YES Order : YES

- Lets take this further and add two more coins to our experiment. A single toss now consist of four coins. e.g. $\omega_i = \langle HTHT\rangle$

now our 'sample space' will have 16 possible outcomes $|\Omega| = 2^4 = 16$

$2^4 = 16$ 4 variables: coins; 2 choices: Heads, Tails; 16 possible combinations HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT

self-note: as the Truth Table is becoming larger the size of events is becoming harder to count.

- what is the probability that 4 coins will produce ⟨TTTT⟩ ?

$P(TTTT) = \frac{1}{16}$ unofficial format — which is the same for each of the remaining 15 outcomes $\forall \omega\ P(\omega) = \frac{1}{|\Omega|}$

- what is the probability that 4 coins will produce event: "2H and 2T" ?

HHTT, TTHH, HTHT, THTH, HTTH, THHT

$P(2H2T) = \frac{|2H2T|}{|\Omega|} = 6/16$

using concepts that we cover later $|2H2T| = \frac{_4P_4}{2!2!} = 6$

note that: $P(2H2T) > P(HTHT)$ ($6/16 > 1/16$)

4 coin experiment has: 16 outcomes (sample space), 2^{16} = 65,536 events (event space)

- what about the probability of having at least one H ? Lets call this event A

$P(A) = \frac{|A|}{|\Omega|} = 15/16$

also notice that there is only one possible outcome that does NOT contain H and which can be expressed as $A^c := \{\langle T,T,T,T\rangle\}$

Remember the 'Complement Rule' ?

$P(A) = 1 - P(A^c) = 1 - \frac{1}{16} = \frac{15}{16}$

4

10 coin experiment

Duplicates : YES Order : YES

- Imagine having total of ten coins in our experiment. A single toss will now consist of ten coins. e.g. $\omega_i = \langle HTHTHTHTHT\rangle$

now our 'sample space' will have 1024 possible outcomes $|\Omega| = |\Omega|^{10} = 2^{10} = 1024$

and our 'event space' will have 2^{1024} possible events $|2^\Omega| = 2^{|\Omega|} = 2^{1024}$

$2^{10} = 1024$ 10 variables: coins; 2 choices: Heads, Tails

CS: $2^x \approx 1000^{x/10}$

$2^{20} \approx$ 1 million, $2^{30} \approx$ 1 billion, $2^{40} \approx$ 1 trillion, $2^{50} \approx$ 1 quadrillion … $2^{1024} \approx$ some gargantuan number

- what is the probability that 10 coins will produce (5H5T) ?

difficult – NO obvious way to do this !

$P(5H5T) = \frac{|\{\langle HHHHHTTTTT\rangle, \dots\}|}{|\Omega|}$

number of outcomes in our desired event / number of all outcomes

$= \frac{\frac{10!}{5!5!}}{2^{10}} = 0.2460938$ * using concepts that we cover later

10 coins 10 flips; 5H – indistinct, 5T – indistinct, invariance

thus event (5H5T) has 252 outcomes

1000 coin experiment

- Imagine having total of thousand coins in our experiment. A single toss will now consist of thousand coins. e.g. $\omega_i = \langle HTHTHTHTHT\dots H\rangle$

now our 'sample space' will have 2^{1000} possible outcomes $|\Omega| = |\Omega|^{1000} = 2^{1000}$

and our 'event space' will have $2^{2^{1000}}$ possible events $|2^\Omega| = 2^{|\Omega|} = 2^{2^{1000}}$

- what is the probability that 10 coins will produce (600H400T) ?

$$P(600H400T) = \frac{\frac{1000!}{600!\,400!}}{2^{1000}} = \in [0,1]$$

calculator says – out of bounds; interval notation

in probability this is always the case

since $n! = \prod_{i=1}^{n} i \rightarrow \ln(n!) = \sum_{i=1}^{n} \ln(i)$

$n! \approx \sqrt{2\pi n}\,(\frac{n}{e})^n$ – Stirling's approximation

$\approx -23.79 \rightarrow p = e \approx 4.6 \cdot 10^{-11}$

probability of 600H400T rare!!

$\frac{1000!}{600!\,400!} = \frac{n!}{k!(n-k)!}$ such that $n \in \mathcal{N}$, $k \in \mathcal{N}$, $k \le n$

order does matter $_nP_k$ vs order does NOT matter $_nC_k := \binom{n}{k} := \frac{n!}{k!(n-k)!}$ invariance is built in

out of n total things select k — n – objects, entities, items; k – samples

$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$

$\binom{n}{n-1} = \frac{n!}{(n-1)!\,(n-(n-1))!} = n$

$\binom{n}{n} = \frac{n!}{n!\,0!} = 1$

$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$

$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \binom{n}{k}$

these combinatorial formulas appear everywhere

power-set size

Set A s.t. $|A| = n$, $n \in \mathcal{N}$ — we have a set-A such that the cardinality (size) of set-A is a natural number

$|2^A| = 2^{|A|} = 2^n$ the cardinality of the Power-Set of A is $2^{\text{natural\_numb}}$

$2^A := \{B: B \subseteq A\} = \{B: B\subseteq A \ \&\ |B|=0\} \cup \dots \cup \{B: B\subseteq A \ \&\ |B|=3\} \cup \{B: B\subseteq A \ \&\ |B|=4\} \cup \dots \{B: B\subseteq A \ \&\ |B|=n\} = \bigcup_{i=0}^{n} \{B: B\subseteq A \ \&\ |B|=i\}$

the Power-Set of A contains set-B such that set-B is a subset of set-A – order doesn't matter.

combinatorial identity

why is this true ? b/c all of the subsets (Bs) are : – collectively exhaustive (all possible Ω-subsets are included) – mutually exclusive (they got nothing in-common)

how many subsets of the Power-Set of A ? Lets add them up:

$$|2^A| = \sum_{i=0}^{n} |\{B: B\subseteq A \ \&\ |B|=i\}| = \sum_{i=0}^{n} \binom{n}{i} = 2^n$$

binomial theorem

$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$

$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3 = 1a^3b^0 + 3a^2b + 3ab^2 + 1a^0b^3$

$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$

$= \binom{4}{4}a^4b^0 + \binom{4}{3}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{1}a^1b^3 + \binom{4}{0}a^0b^4$

$\binom{n}{n} \ \binom{n}{n-1} = \binom{n}{1} \ \binom{n}{0}$

binomial theorem represented using our formulas

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

the most ubiquitous combinatorial theorem in mathematics !!!

$(1+x)^n = (1+x)(1+x)^{n-1}$

$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i 1^{n-i} = \sum_{i=0}^{n}\binom{n}{i}x^i = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

$\binom{n}{0} = 1$; $x^0 = 1$

$= 1 + \sum_{i=1}^{n-1}\binom{n}{i}x^i + x^n$

$(1+x)(1+x)^{n-1} = (1+x)\sum_{i=0}^{n-1}\binom{n-1}{i}x^i = \sum_{i=0}^{n-1}\binom{n-1}{i}x^i + \sum_{i=0}^{n-1}\binom{n-1}{i}x^{i+1}$ reindexing trick

$= \sum_{i=0}^{n-1}\binom{n-1}{i}x^i + \sum_{i=1}^{n}\binom{n-1}{i-1}x^i$

$= \binom{n-1}{0}x^0 + \sum_{i=1}^{n-1}\left[\binom{n-1}{i} + \binom{n-1}{i-1}\right]x^i + \binom{n-1}{n-1}x^n$

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$$ **Pascal Rule**

Pascal's Triangle

$\binom{0}{0}$ 1

$\binom{1}{0}$ 1 $\binom{1}{1}$ 1

$\binom{2}{0}$ 1 $\binom{2}{1}$ 2 $\binom{2}{2}$ 1

$\binom{3}{0}$ 1 $\binom{3}{1}$ 3 $\binom{3}{2}$ 3 $\binom{3}{3}$ 1

$\binom{n}{0} \ \binom{n}{1} = \binom{n}{n-1} \ \binom{n}{n}$

{JBSR} {JBSM} {JBSA} {JBRM} {JBRA} {JBMA} {JSRM} {JSRA} {JSMA} {JRMA} — 10

{BSRM} {BSRA} {BSMA} {BRMA} — 4

{SRMA} — 1

$\binom{4}{3}$ 4 $\binom{4}{4}$ 1 = 15; $\binom{5}{3}$ 10 $\binom{5}{4}$ 5; $\binom{6}{4}$ 15 $\binom{6}{5}$ $\binom{6}{6}$

6 choose 4

$\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{720}{24 \cdot 2!} = 15$

$\binom{5}{3} + \binom{5}{4} = 15$

$\binom{5}{3} + \binom{4}{3} + \binom{4}{4} = 15$

6 ppl – 4 seats

Bob, Jane, Richard, Susan, Max, and Alice

We have 6 people and only 4 seats.

$\Omega := \{\langle person1, person2, person3, person4\rangle, \langle 6,5,3,1\rangle, \langle 9,1,4,3\rangle, \dots\}$

- in how many ways can we seat 6 people in 4 seats – ORDER MATTERS ? (Duplicates : NO Order : YES)

6	5	4	3
seat-1	seat-2	seat-3	seat-4

$_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$ order matters – no replacement – all distinct

- in how many ways can we seat 6 people in 4 seats – ORDER MATTERS NOT ? (Duplicates : NO Order : NO)

$\frac{_6P_4}{4!} = \frac{\frac{6!}{2!}}{4!} = \frac{6!}{4!\,2!}$ order matters NOT – no replacement – all distinct

invariance

choose notation $_6C_4$ or $\binom{6}{4}$ "6 choose 4" = 15

{JBSR} {JBSM} {JBSA} {JBRM} {JBRA} {JBMA} {JSRM} {JSRA} {JSMA} {JRMA} {BSRM} {BSRA} {BSMA} {BRMA} {SRMA}

order matters NOT ? {JBSR} = {JSBR} = {RBSJ} = {BJSR}

- in how many ways can we seat 4 people s.t. Jane sits ?

ORDER Matters

$$\frac{4\,(_5P_3)}{_6P_4} = \frac{4\,(5\cdot4\cdot3)}{6\cdot5\cdot4\cdot3} = \frac{240}{360} = 2/3$$

1	5	4	3
Jane	seat-2	seat-3	seat-4
5	1	4	3
seat-1	Jane	seat-3	seat-4
5	4	1	3
seat-1	seat-2	Jane	seat-4
5	4	3	1
seat-1	seat-2	seat-3	Jane

ppl remaining — 4 (5·4·3) = 4 (₅P₃)

ORDER Matters NOT

$$\frac{\binom{5}{3}}{\binom{6}{4}} = \frac{5\cdot4\cdot3!\ /\ 2!\ 3!}{6\cdot5\cdot4!\ /\ 2\ 4!} = 10/15 = 2/3$$

Quest for the new sample space

3 ppl – 3 seats

Duplicates : NO Order : YES

- Lets go back to Female names: F={Jane, Mary, Susan}. We have 3 seats.

$\Omega := \{\langle JMS\rangle, \langle JSM\rangle, \langle SMJ\rangle, \dots\}$

sampling without replacement

- in how many ways can we seat Jane, Mary, Susan ?

$3^3 = 27$ 3 variables: seats; 3 choices: Jane, Mary, Susan; 27 possible combinations – replacement YES; 6 desired combinations – replacement NO

probability tree

seat-1 / seat-2 / seat-3: J → M → S, J → S → M; M → J → S, M → S → J; S → M → J, S → J → M ($P = 1/3$, $P = 1/2$, $P = 1$)

sampling without replacement | events we are interested in | < |Ω| — sampling with replacement – truth table

$|\Omega| < |F^3|$ why ? $|\Omega| = |F^3| = 3^3 = 27$ possible events

$\{\langle JMS\rangle, \langle JSM\rangle, \langle MJS\rangle, \langle MSJ\rangle, \langle SMJ\rangle, \langle SJM\rangle\}$ all equally likely

events we are interested in = 6

self-note: Warning ! This 'sample space' contains ONLY outcomes which are fitting to our needs. Thus can NOT be represented by a Truth Table, since Truth Table allows for duplicates. Here we are dealing with people which are unique. As a consequence the cardinality of our 'sample space without replacement' is greatly reduced.

seat-1 3 possibilities · seat-2 2 possibilities · seat-3 1 possibility: 3 · 2 · 1 = **3!**

$n! := \prod_{i=1}^{n} i$ note: $0! = 1$

$10! = 3.6m$; $20! = 2.7 \cdot 10^{32}$ diameter of observable universe in feet

note: if human cloning was allowed the entire 'sample space' would contain 27 possible outcomes. However, human cloning aside, we are only interested in the subset of 6. Jane, Mary, and Susan represent unique non-duplicable entities which we want to seat in ALL of the 3 available seats. 6 is the number of all possible seating arrangements. 6 can be represented by 3! = 1·2·3

10 ppl – 3 seats

Duplicates : NO Order : YES

- We have 10 people and only 3 seats.

replacement NO – thus a Truth Table ≠ | our sample space |

$\Omega := \{\langle person1, person2, person10\rangle, \langle 6,5,3\rangle, \langle 9,1,4\rangle, \dots\}$

| sample space | – ???

- in how many ways can we seat 10 people in 3 seats ?

seat-1 10 possibilities · seat-2 9 possibilities · seat-3 8 possibilities = 10·9·8

we can represent this as a fraction

$\frac{10!}{7!} = \frac{10\cdot9\cdot8\cdot7!}{7!} = 10\cdot9\cdot8$

$= \frac{10\cdot9\cdot8\cdot7!}{(10-3)!} = \frac{10!}{(10-3)!} = {_{10}P_3}$ (ppl, seats)

self-note: 19·18·17 = 5814 > 7!; 18·17·16 = 4896 < 7!; 10·9·8 = 720 = 6! coincidence ?; 9·8·7 = 504 ≠ 5!; 8·7·6 = 336 ≠ 5!; 7·6·5 = 210 ≠ 5!; 6·5·4 = 120 = 5!; 5·4·3 = 60 ≠ 4!; 4·3·2 = 24 = 4! YES

general formula for: **n-total objects k-sampled objects** which are sampled without replacement

$$_nP_k = \frac{n!}{(n-k)!}$$

permutation – the order matters

MNEMONIC permute n-things into k-spaces

$$_nP_n = \frac{n!}{(n-n)!} = n!$$

10 ppl – 10 seats

Duplicates : NO Order : YES

We have 10 people and 10 seats.

$\Omega := \{\langle person1, person2, \dots, person10\rangle, \langle 2,5,1,6,3,4,10,7,9,8\rangle, \dots\}$

- in how many ways can we seat 10 people in 10 seats ? lets use the formula:

$$_{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = 10! = 3.6 \text{ million}$$ | sample space |

3 couples – 6 seats

Duplicates : NO Order : YES

Couples: Bob – Jane, Richard – Susan, Max – Alice. We have 6 seats.

$\Omega := \{\langle Bob, Jane, Richard, Max, Alice, Susan\rangle, \langle Jane, Bob, Max, Richard, Susan, Alice\rangle, \langle Alice, Jane, Bob, Max, Susan, Richard\rangle, \dots\}$

- in how many ways can we seat Bob, Jane, Richard, Susan, Max, and Alice ? condition: **couples MUST sit together**.

Lets examine the probability of couples sitting together. Lets call this event 'A'

$P(A) = \frac{|A|}{|\Omega|} = \frac{?}{6!} = \frac{48}{6!} = 0.666\dots$ or $2/3$ — this is usually difficult to compute; 6-ppl , 6-seats $_nP_n = \frac{n!}{(n-n)!} = n!$

~~sampling with replacement~~ $6^6 = 46{,}656$ 6 variables: seats; 6 choices: Bob, Jane, Richard, Susan, Max, Alice; 46,656 possible combinations : Bob,Jane,Richard,Susan,Max,Alice; Bob,Jane,Richard,Susan,Alice,Max; Bob,Jane,Susan,Richard,Max,Alice; Bob,Jane,Susan,Richard,Alice,Max; … Bob,Bob,Bob,Bob,Bob,Bob – watch out !!

sampling without replacement: n – 6ppl , k – 6seats $_nP_k = \frac{n!}{(n-k)!} = \frac{6!}{(6-6)!} = 6!$

self-note: Warning ! Redefinition of Ω ! This 'sample space' contains ONLY outcomes which are fitting to our experiment.

approach I

Lets treat a couple as a single person and two seats as a single bench. Thus 3 ppl and 3 seats -> 3!

seat-1,2 3 couples · seat-3,4 2 couples · seat-5,6 1 couple = 3!

Within each bench couples can swap = 3! · 2

and there are 3 couples = 3! · 2³ = **48**

order matters Bob–Jane ≠ Jane–Bob

approach II

seat-1	seat-2	seat-3	seat-4	seat-5	seat-6	
Bob	Jane	Richard	Susan	Alice	Max	
6 ·	1 ·	4 ·	1 ·	2 ·	1	= **48**

- in how many ways can we seat Bob, Jane, Richard, Susan, Max, and Alice ? condition: **opposite genders MUST sit together**. (Duplicates : NO Order : YES)

Once we seat a Male we must follow up with a Female

seat-1	seat-2	seat-3	seat-4	seat-5	seat-6	event A_1
Male	Female	Male	Female	Male	Female	
3 ·	3 ·	2 ·	2 ·	1 ·	1	= 36

both cases are mutually exclusive (nothing in common) thus we can add them

Once we seat a Female we must follow up with a Male

seat-1	seat-2	seat-3	seat-4	seat-5	seat-6	event A_2
Female	Male	Female	Male	Female	Male	
3 ·	3 ·	2 ·	2 ·	1 ·	1	= 36

$$P(A_1) + P(A_2) = \frac{(|A_1| + |A_2|)}{|\Omega|} = \frac{36}{6!} + \frac{36}{6!} = \frac{72}{6!} = 0.1 \text{ or } 1/10$$

due to disjointedness of A_1 and A_2; 6-ppl , 6-seats $_nP_n = \frac{n!}{(n-n)!} = n!$

- what is the probability of Bob and Jane sitting together ? (Duplicates : NO Order : YES)

seat-1&2		seat-3	seat-4	seat-5	seat-6	
Bob	Jane	person	person	person	person	
1 ·	1 ·	4 ·	3 ·	2 ·	1	= 24 = 4!

Bob and Jane can swap 4! · 2

move the seats around 5 different ways 4! · 2 · 5

order matters

$$P(B,J) = \frac{|\{(B,J)\}|}{|\Omega|} = \frac{4!\cdot2\cdot5}{6!} = 0.333\dots \text{ or } 1/3$$

Orchids – Chrysanthemum

We have 5 flowers: 3 Orchids 'O' , 2 Chrysanthemum 'X'

- if all flowers are distinct, in how many ways can we arrange them ? (Duplicates : NO Order : YES)

$_5P_5 = \frac{5!}{(5-5)!} = 5!$ (flowers, position)

- Orchids are indistinguishable, Chrysanthemum are distinct – ways ? (invariance)

$O_1O_2O_3X_1X_2$, $O_1O_3O_2X_1X_2$, $O_2O_1O_3X_1X_2$, $O_2O_3O_1X_1X_2$, $O_3O_1O_2X_1X_2$, $O_3O_2O_1X_1X_2$ = 1

since 'Os' are now the same this entire group counts as 1. Sub-indices used only to denote the previous uniqueness

MNEMONIC divide by invariance

$\frac{5!}{3!} = 20$ ways ($_5P_5$ / $_3P_3$)

- Orchids are distinct, Chrysanthemum are indistinguishable – ways ? (invariance)

$\frac{5!}{2!} = 60$ ways

- Orchids and Chrysanthemum are indistinguishable – ways ? (invariance)

$\frac{5!}{3!2!} = 10$ ways

OOOXX, OOXOX, OXOOX, XOOOX, XOOXO, XOXOO, XXOOO, OXXOO, OOXXO, OXOXO

now the order can only be spotted by analyzing the relation between each element – sub-indices are gone

X – distinct: X_1OOX_2O (collapse / expand): $X_1O_1O_2X_2O_3$, $X_1O_1O_3X_2O_2$, $X_1O_2O_1X_2O_3$, $X_1O_2O_3X_2O_1$, $X_1O_3O_2X_2O_1$, $X_1O_3O_1X_2O_2$ — Os distinct = 3! rows

O – distinct: X_2OOX_1O: $X_2O_1O_2X_1O_3$, $X_2O_1O_3X_1O_2$, $X_2O_2O_1X_1O_3$, $X_2O_2O_3X_1O_1$, $X_2O_3O_2X_1O_1$, $X_2O_3O_1X_1O_2$ — Os distinct = 3! rows

Xs distinct = 2! rows

self-note: expanding is like multiplying by the permutation of unique elements e.g. $\frac{5!}{3!2!} \cdot 2! \rightarrow \cdot 3! = 5!$ (10 -> 60 -> 120)

100 balls we take out 7 -> $_{100}P_7$

1000,000 balls we take out 300,000 -> $_{1000,000}P_{300,000}$

number of ways to sample n-objects without replacement

seat-1	seat-2	seat-3	seat-4	…	seat-(n-1)	seat-n	
n ·	n-1 ·	n-2 ·	n-3 ·	⋯ ·	2 ·	1	= n!

number of ways to sample n-objects with replacement

seat-1	seat-2	seat-3	seat-4	…	seat-(n-1)	seat-n	
n ·	n ·	n ·	n ·	⋯ ·	n ·	n	= n^n

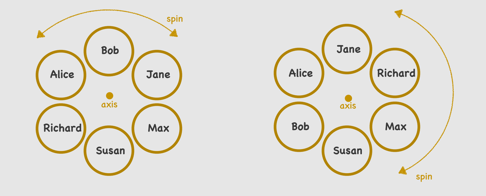
$n^n > n!$

n^n is $O(n!)$

6 ppl – 6 seats

Duplicates : NO Order : YES

Lets go back to Bob, Jane, Richard, Susan, Max, and Alice. Imagine arranging the seats in a ring. How many ways can we sit them. It is a ring so it does not matter where you start.



B J R S M A
A B J R S M
M A B J R S
S M A B J R
R S M A B J
J R S M A B

= 1 6 collapse into 1

spin

order by relation only

all these permutations are consistent 'same' , 'indistinguishable' , 'non-distinct' , 'invariant' , 'non-unique'

Principle of dividing out invariance

$$\frac{6!}{6}$$

size matter ?

We have 100 balls – select 3 without replacement. How many unique orders ? (Duplicates : NO Order : YES)

ball 100 | ball 99 | ball 98 $= {_nP_k} = {_{100}P_3}$

$\frac{_{100}P_3}{100^3} = .9702$

We have 100 balls – select 3 with replacement. How many unique orders? (Duplicates : YES Order : YES)

ball 100 | ball 100 | ball 100 $= n^k = 100^3$

≈

We have 10,000 balls – select 3 without replacement. How many unique orders ? (Duplicates : NO Order : YES)

ball 10,000 | ball 9,999 | ball 9,998 $= {_nP_k} = {_{10{,}000}P_3}$

$\frac{_{10000}P_3}{10000^3} = .9997$

We have 10,000 balls – select 3 with replacement. How many unique orders ? (Duplicates : YES Order : YES)

ball 10,000 | ball 10,000 | ball 10,000 $= n^k = 10{,}000^3$

Does the size-k matter ?

$$\lim_{n\to\infty} \frac{_nP_k}{n^k} = 1$$

$$\lim_{n\to\infty}\frac{\frac{n!}{(n-k)!}}{n^k} = \lim_{n\to\infty}\frac{\frac{n(n-1)(n-2)\cdots(n-(k-1))(n-k)(n-(k+1))}{(n-k)(n-(k+1))}}{n\cdot n\cdot n\cdot\cdots\cdot n}$$

$$= \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-(k-1))}{n\cdot n\cdot n\cdot\cdots\cdot n}$$ k terms / k terms

$$= \lim_{n\to\infty}\frac{n}{n}\cdot\lim_{n\to\infty}\frac{(n-1)}{n}\cdot\lim_{n\to\infty}\frac{(n-2)}{n}\cdots\lim_{n\to\infty}\frac{(n-(k-1))}{n}$$

1 1 1 1

$\lim_{n\to\infty}\frac{n}{n} - \lim_{n\to\infty}\frac{2}{n} = 1 - 0 = 1$ etc...

ans: Nope

self-note: **What is ORDER ?** – position of objects in reference to each-other.

e.g. alphabet ABCD ⋯Z (1 2 3 4 … 26)

A is at index 1 and Z is at index 26. All letters are sorted in an alphabetic order.

Truth Tables have duplicate values that are in order which allows us to see the difference between each row.

self-note: **order matters or not ?** – yes the order does matter. But imagine that the indices of each placeholder have been removed and the only way to see the order is by the references between each of the objects. – Who are my neighbors?

$6^6 = 46{,}656$ 6 variables: seats; 6 choices: J R S M A B; 46656 combinations JJJJJJ … BBBBBB (Duplicates : YES Order : YES)

$_6P_6 = \frac{6!}{(6-6)!} = 720$ order: placeholder-indices fixed

$\frac{_6P_6}{6} = 120$ order: relation to each other only

$\frac{_6C_6}{6} = 1$

LIMIT LAWS Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$