$$F(b) = 1 \Rightarrow \frac{b}{b-a} + c = 1$$

$$\Rightarrow \frac{b}{b-a} = 1 - c \Rightarrow \frac{b}{a-b} = c - 1$$

$$\Rightarrow c = \frac{b}{a-b} + 1 = \frac{b}{a-b} + \frac{a-b}{a-b} = \frac{a}{a-b} = c$$

here $\Rightarrow \frac{x}{b-a} + \frac{a}{a-b} = \frac{x}{b-a} + \frac{-a}{b-a} = \boxed{\frac{x-a}{b-a}}$

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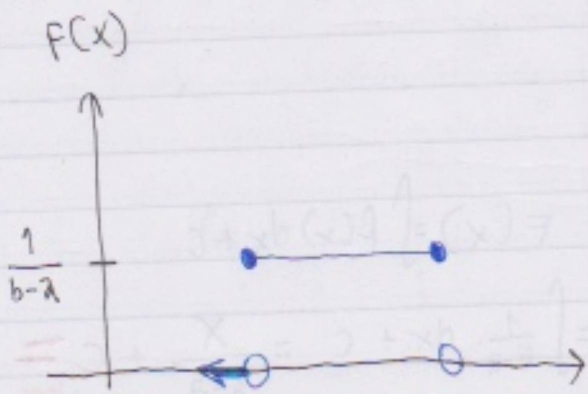
continuous R.V.

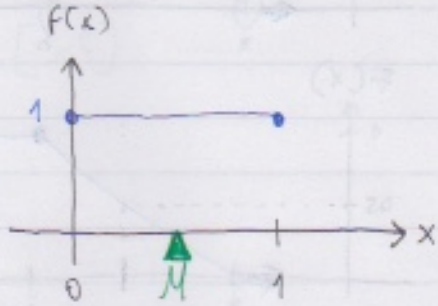
$\rightarrow$ Uniform

$X \sim U(a,b)$



$$1 = \int_{Supp(x)} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1$$

$X \sim U(0,1)$



$$\boxed{Supp[x] = (0,1)}$$

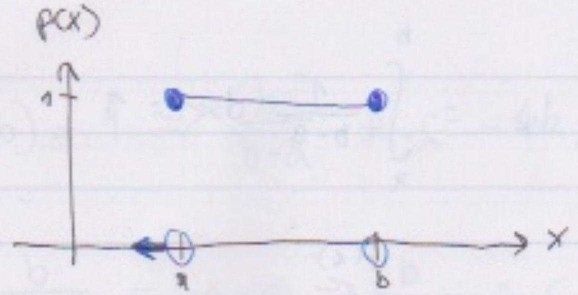
$$\boxed{E[x]} = \int_{Supp(x)} x\, f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

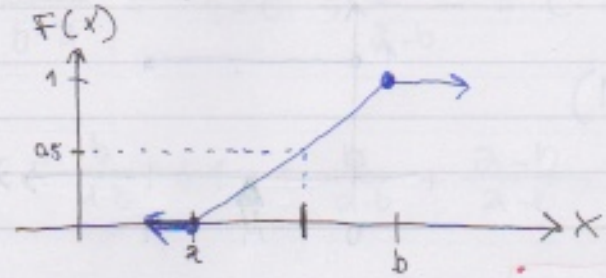
$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2\,(b-a)} = \boxed{\frac{a+b}{2}} = E[x]$$

$$\boxed{Med[x] = Quantile\left[x, \tfrac{1}{2}\right]} \neq \operatorname{argmin}\left\{F(x) \geq \tfrac{1}{2}\right\}$$

for discrete R.V.

For continuos it will be exact





to do inverse your function needs to be one-to-one

:. it has to be extrictly monotonic

inverse CDF

$$\text{Quantile}[x, p] = F^{-1}(p) \qquad p = F(x) = \frac{x-a}{b-a}$$

$$\Rightarrow x = p(b-a)+a$$

$$\text{Med}(x) = F^{-1}\left(\tfrac{1}{2}\right) = \frac{1}{2}(b-a)+a = \frac{1}{2}b - \frac{1}{2}a + \frac{2}{2}a$$

$$= \frac{1}{2}b + \frac{1}{2}a = \boxed{\frac{a+b}{2}} = \text{Med}[x]$$

$$\text{Var}[x] = E[(X-\mu)^2] = E[x^2] - \mu^2 = \left(\frac{a+b}{2}\right)^2$$

$$E[x^2] = \int_{\text{supp}(x)} x^2 f(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

pref time :

$$
\begin{array}{r|l}
 & b^2+ab+a^2 \\
b-a & b^3 - a^3 \\
 & -(b^3 - ab^2) \\
\hline
 & ab^2 - a^3 \\
 & -(ab^2 - a^2 b) \\
\hline
 & a^2 b - a^3 \\
 & -(a^2 b - a^3) \\
\hline
 & 0
\end{array}
$$

. . .

more

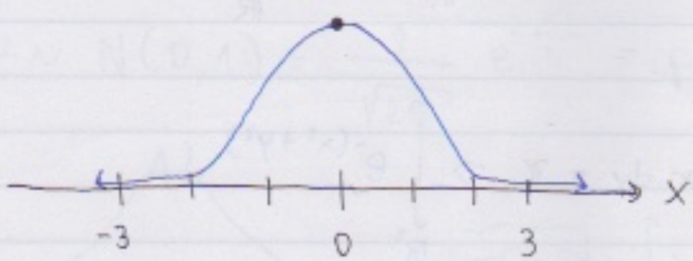
$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \boxed{\frac{(b-a)^2}{12}} \longleftarrow E[x^2]$$

$$\boxed{SE(x) = \frac{b-a}{\sqrt{12}}}$$

$$\boxed{Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$$

"normal", "gaussian", "bele"



1. $f(x) \geq 0$ ✓
2. $\int_{Supp(x)} f(x)\,dx = 1$

proof of (2) before we get back to prob.

WTS $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$

Gaussian Integral

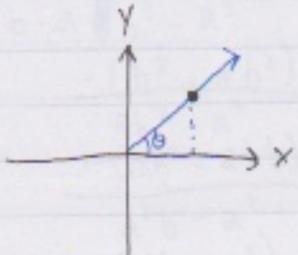
$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} = \int_{u_0}^{u_f} e^{-u^2} \sqrt{2}\, du = \sqrt{\cancel{2}\pi} = \sqrt{\pi}$$

$$\boxed{\text{let } u = \frac{1}{\sqrt{2}} x \Rightarrow \frac{x^2}{2} = u^2 \quad du = \frac{1}{\sqrt{2}} dx \Rightarrow dx = \sqrt{2}\, du}$$

$$\left(\int_{u_0}^{u_f} e^{-u^2} du\right)^2 = \pi \Rightarrow \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \pi$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2} e^{-y^2} dx\, dy = \pi \Rightarrow \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$$

Arc Integral



$x = r \cos\theta$

$y = r \sin\theta$

$dA \neq dr\, d\theta$

$r^2 = x^2 + y^2$

$$dx\,dy = dA = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi}\int_{u_0}^{u_f} e^{-u}\,\cancel{r}\,\frac{1}{2\cancel{r}}\,du\,d\theta$$

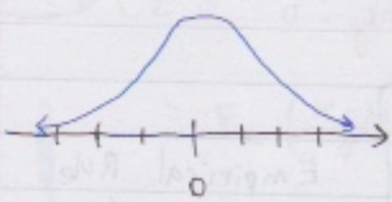
let $u = r^2 \Rightarrow du = 2r\,dr \Rightarrow dr = \frac{1}{2r}\,du$

$$= \frac{1}{2}\int_0^{2\pi}\int_{u_0}^{u_f} e^{-u}\,du\,d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \frac{1}{2}\left[\theta\right]_0^{2\pi}$$

$= \boxed{\pi}$

back to prob.

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x)$ valid



$$\boxed{E[Z]} = \int_{supp(x)} x\,f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{u_0}^{u_f} \cancel{x}\,e^{-u}\,\frac{1}{\cancel{x}}\,du$$

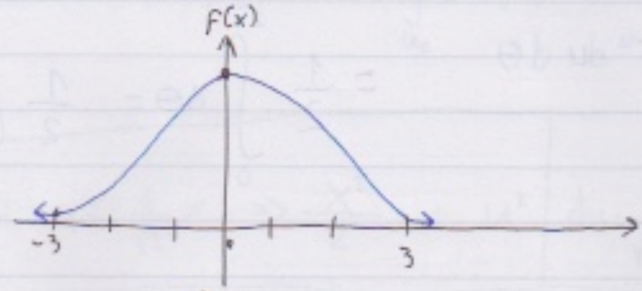
$$\boxed{\text{let } u = \frac{x^2}{2} \Rightarrow \frac{du}{dx} = x \Rightarrow dx = \frac{1}{x}\,du}$$

$$= -\frac{1}{\sqrt{2\pi}}(0-0) = \boxed{0}$$

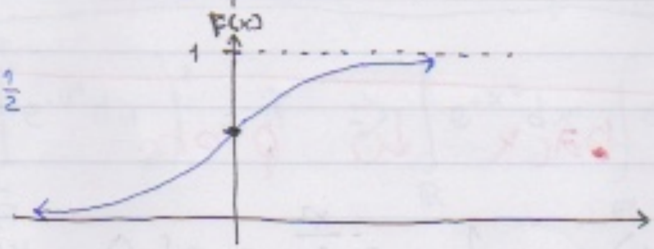
$$\boxed{Var[Z]} = E[Z^2] - \cancel{\mu^2}^{\,0} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \boxed{1}$$

$$F_Z(x) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx + C$$

not possible in closed form



$F(0) = \frac{1}{2}$



Memorize the following:

$P(Z \in [-1, 1]) \approx .68$

$P(Z \in [-2, 2]) \approx .95$

$P(Z \in [-3, 3]) \approx .997$

Empirical Rule

3σ Rule (3 sigma)

68-95-997 Rule

let's say

$$X \doteq \sigma Z + \mu$$

$$\boxed{E[X] = \sigma E[Z] + \mu} = \mu$$

$$\boxed{Var[X] = \sigma^2 \underbrace{Var[Z]}_{1}} = \sigma^2$$

$$\boxed{SE[X] = \sigma}$$

---

another proof:

$$F_X(x) = P(X \leq x) = P(\sigma Z + \mu \leq x)$$

$$= P\left(Z \leq \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$= F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$f_X(x) = F_X'(x) = \frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right]$$

$$= \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$= N(\mu, \sigma^2)$$

The normal distribution

Question

American male height is normally distr. with mean 70" (= 5'10") and std. error 4".

What is the probability a random American male is taller than 78" (= 6' 6")?

$$X \sim N(70", 4"^2)$$

$$P(X > 78) = P\left(\frac{X-70}{4} > \frac{78-70}{4}\right)$$

$$= P(Z > 2) = 2.5\%$$

↖ remember the 3σ Rule