Math 2(?) 10/31/17 Lec 15

$\mu := E(X) := \sum_{x \in \mathrm{Supp}(X)} x\, p(x)$

$\sigma^2 := \mathrm{Var}(X) := \sum_{x \in \mathrm{Supp}(X)} (x-\mu)^2 p(x)$ ← spread $\mathrm{\$}^2$

Roulette: Bet \$1 on 7

$X_7 \sim \begin{cases} \$35 & \text{w.p. } \frac{1}{38} \\ -\$1 & \text{w.p. } \frac{37}{38} \end{cases}$

$\mu = -0.053$, $\sigma^2 = (35 - -0.053)^2 \frac{1}{38} + (-1 - -0.053)^2 \frac{37}{38} = 33.207\,\$^2$

" " " " w/ Bet on Black

$X_B \sim \begin{cases} \$1 & \text{w.p. } \frac{18}{38} \\ -\$1 & \text{w.p. } \frac{20}{38} \end{cases}$

$\mu = -0.053$, $\sigma^2 = (1 - -0.053)^2 \frac{18}{38} + (-1 - -0.053)^2 \frac{20}{38} = 0.997\,\$^2$

$\bar{X}_7 \to \mu$

$\bar{X}_B \to \mu$

which goes faster?
the one with lowest variance...
(we will prove why later)

Units! $\$^2$ ... has no meaning!

Easiest way to solve: $\sqrt{\$^2} = \$$ ← nice and interpretable

let $\sigma := SE(X) := \sqrt{Var(X)} = \sqrt{\sigma^2}$

"standard error" or "standard deviation"

$\sigma_7 = \$5.79$, $\sigma_B = \$1.00$

σ^2 interpretation

σ interpretation ... not so clear!

It is not an expression... it is just a practical strategy to report spread.

But... it will be useful later...

$T_2 := X_1 + X_2$ $\quad E(T) = \sum_{t \in Supp(T)} t\, p(t)$

$Supp(X_1)$	$Supp(X_2)$	T	$p(t)$
x_1		...	...
x_2			
x_3			
⋮	⋮	aggregate	

hard to get

Is there a better way?

X_1, X_2 indep $p(x_1, x_2) = p(x_1)\,p(x_2)$

~~$T = X_1 + X_2$ $\quad E(T) = \sum_{t \in Supp(T)} t\, p(t)$ req. complicated space structure~~

Note $T = g(X_1, X_2)$ $\cancel{= g(\vec{X})}$ s.t. $g: \mathbb{R}^2 \to \mathbb{R}$

$$E[g(X)] = \sum_{(x_1, x_2) \in Supp(X_1) \times Supp(X_2)} g(x_1, x_2) \overbrace{p(x_1, x_2)}^{\text{joint mass function (jmf)}}$$

$$= \sum_{x_1 \in Supp(X_1)} \sum_{x_2 \in Supp(X_2)} g(x_1, x_2)\, p(x_1, x_2)$$

$$E(T) = E(X_1 + X_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2)\, p(x_1, x_2) = \sum_{x_1} \sum_{x_2} x_1\, p(x_1, x_2) + \sum_{x_2} \sum_{x_1} x_2\, p(x_1, x_2)$$

$$= \sum_{x_1} x_1 \sum_{x_2} p(x_1, x_2) + \sum_{x_2} x_2 \sum_{x_1} p(x_1, x_2)$$

Note if X_1, X_2 ind. $\Rightarrow p(x_1, x_2) = p(x_1)\, p(x_2)$

$$= \overbrace{\sum_{x_1} x_1\, p(x_1)}^{E(X_1)} \underbrace{\sum_{x_2} p(x_2)}_{1} + \overbrace{\sum_{x_2} x_2\, p(x_2)}^{E(X_2)} \underbrace{\sum_{x_1} p(x_1)}_{1}$$

$$= E(X_1) + E(X_2)$$

If not... we need to figure out

$$\sum_{x_2} p(x_1, x_2) \quad \& \quad \sum_{x_1} p(x_1, x_2)$$

Consider X_1, X_2 s.t. $Supp(X_1) = \{1, 7, 19\}$, $Supp(X_2) = \{5, 23, 88\}$

$p(x_1, x_2)$

X_2 \ X_1	1	7	19	
5	$\frac{1}{15}$	$\frac{1}{3}$	$\frac{2}{15}$	$\frac{16}{30}$
23	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{1}{30}$	$\frac{5}{30}$
88	$\frac{1}{30}$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{9}{30}$
	$\frac{4}{30}$	$\frac{19}{30}$	$\frac{7}{30}$	1

$p(x_2)$ "marginal"

$p(x_1)$ "marginal"

$\sum \sum p(x_1, x_2) = ?$ 1 why?

Is X_1 ind X_2?

$\frac{4}{30} = P(X_1 = 1) \overset{?}{\neq} P(X_1 = 1 \mid X_2 = 23) = \frac{1}{30} \Rightarrow$ No...

What do we see here? $\sum_{x_1} p(x_1, x_2) = p(x_2)$, $\sum_{x_2} p(x_1, x_2) = p(x_1)$

Similar to $g(x) = \int f(x, y)\, dy$

Where'd y go?? "Integrated out"

$$E(T) = \sum_{x_1} x_1 \underbrace{\sum_{x_2} p(x_1, x_2)} + \sum_{x_2} x_2 \underbrace{\sum_{x_1} p(x_1, x_2)}$$

$$= \sum_{x_1} x_1 p(x_1) + \sum_{x_2} x_2 p(x_2) = E(X_1) + E(X_2)$$

for any r.v.'s $X_1, X_2, \ldots, X_n$,

$$E(T) = E(\Sigma X_i) = \Sigma E(X_i) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

for any r.v.'s $X_1, X_2, \ldots, X_n$ ident. distr. (not necessarily indep.)

$$E(T) = \Sigma E[X_i] = nE[X_i] = n\mu$$

$$\Rightarrow E(\bar{X}) = E\left(\frac{T}{n}\right) = \frac{1}{n}E(T) = \frac{1}{n}n\mu = \mu$$

~~Recall $X_1, X_2, \ldots, X_r \overset{iid}{\sim} \text{Geom}(p)$~~

~~$T = X_1 + X_2 + \ldots + X_r \sim \text{NegBin}(r, p)$ (by def.)~~

~~$E(T) = r\mu = r\frac{1}{p} = \boxed{\frac{r}{p}}$~~

$X \sim \text{Hyper}(n, K, N)$

Imagine $X_1, X_2, \ldots, X_n$ are the r.v.'s for a single draw without replacement

$$X = X_1 + X_2 + \ldots + X_n$$

$X_1 \sim \text{Bern}\left(\frac{K}{N}\right)$

$X_2 \sim \text{Bern}\left(\frac{K}{N}\right)$

$\vdots$

$X_n \sim \text{Bern}\left(\frac{K}{N}\right)$

$$\boxed{E(X) = n\mu = n\frac{K}{N}}$$

but each of them are not indep.

Back to variance ... Let's derive a useful rule

$$Var(X) := E\left[(X-\mu)^2\right]$$

$$= E\left[X^2 - 2\mu X + \mu^2\right]$$

$$= E(X^2) + E\left[-2\mu X\right] + E(\mu^2) \quad \text{(1st power)}$$

$$= E\left[X^2\right] - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$$

$$\boxed{Var(X) = E(X^2) - \mu^2} \Rightarrow E(X^2) = \sigma^2 + \mu^2$$

$E(X^2)$ second moment

$E[(X-\mu)]$ second central moment

$\dfrac{E[(X-\mu)^3]}{\sigma^3}$ third std moment   skewness

$E(X^k)$

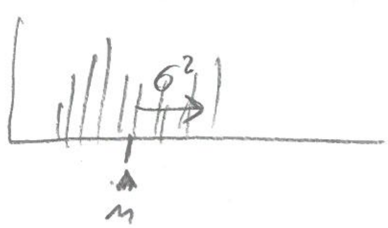
$E[(X-\mu)^k]$

$\dfrac{E[(X-\mu)^4]}{\sigma^4}$ kurtosis

Recall linear transformation $Y = aX + c$ s.t. $a \in \mathbb{R}$, $c \in \mathbb{R}$

$Y = X + c$, $Var(X) = \sigma^2$

What is $Var(Y)$



σ^2 shouldn't change..

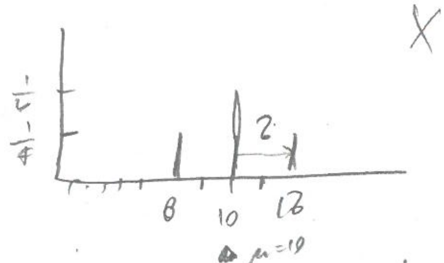
$$Var\left[X+c\right] = E\left[((X+c) - (\mu+c))^2\right] = E\left[(X-\mu)^2\right] = Var(X)$$

$E[X+c] = \mu + c$

$Y = aX \quad Var(X) = \sigma^2$

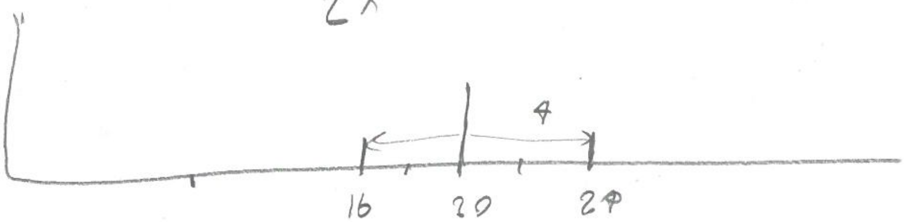
What is $Var(Y)$?

X



$$Var(X) = 2^2 \cdot \tfrac{1}{2} + 2^2 \cdot \tfrac{1}{2} = 4$$

$2X$



$\mu = 2\mu = 20$

$$Var(2X) = 4^2 \cdot \tfrac{1}{2} + 4^2 \cdot \tfrac{1}{2}$$

$$= 16 = 4\,Var(X) = 2^2\,Var(X)$$

Why? Variance is a prob weighted square error

$$Var(aX) = E\left[(aX - a\mu)^2\right] = E\left[(a(X-\mu))^2\right] = E\left[a^2(X-\mu)^2\right] = a^2 E\left[(X-\mu)^2\right]$$

$$= a^2\,Var(X)$$

$E(aX) = a\mu$

$$Var[aX + c] = a^2\sigma^2$$

$$\Rightarrow SD(aX+c) = \sqrt{Var(aX+c)} = |a|\sigma$$

X_1, X_2 two r.v.'s $Var(T)$?

$$Cov(X_1, X_2) := E[X_1X_2] - \mu_1\mu_2$$

$$Var[X_1 + X_2] = E\left[\left((X_1 + X_2) - (\mu_1 + \mu_2)\right)^2\right]$$

$$= E\left[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_2\mu_2 - 2X_1\mu_2 - 2X_2\mu_1 + 2\mu_1\mu_2\right]$$

$$= \underbrace{E(X_1^2)}_{\sigma_1^2 + \mu_1^2} + \underbrace{E(X_2^2)}_{\sigma_2^2 + \mu_2^2} + \cancel{\mu_1^2} + \cancel{\mu_2^2} + 2E[X_1X_2] - \cancel{2\mu_1^2} - \cancel{2\mu_2^2} - \cancel{2\mu_1\mu_2} - 2\mu_2\mu_1 + \cancel{2\mu_1\mu_2} = \sigma_1^2 + \sigma_2^2 + 2\left(E[X_1X_2] - \mu_1\mu_2\right)$$