Lec 29 Math 241 11/23/17

Recap: $g(x)$ special rep...

$m_X(t) := E[e^{tX}]$ def of mgf

Properties

I) $m_X(t) = m_Y(t) \Rightarrow X \overset{d}{=} Y$

Note: $E[X^0] = 1 = m_X(0)$ (the k=0 case)

II) $E[X^k] = m_X^{(k)}(0)$

III) if $Y = aX + c \Rightarrow m_Y(t) = e^{tc}\, m_X(at)$

IV) if X, Y indep   if X, Y iid

$m_{X+Y}(t) = m_X(t)\, m_Y(t) = (m_X(t))^2$

~~V) Levy's Continuity thm~~

~~if~~ $\cancel{\lim_{n\to\infty} m_{X_n}(t) = m_Y(t) \not\Leftrightarrow X \to Y}$ ~~the r.v. X converges to the r.v. Y~~

~~⇒ if n large~~ $\cancel{X_n \overset{d}{\approx} Y}$ ~~X, Y are approx. equally distr.~~

$\cancel{\Rightarrow f_{X_n} \approx f_Y}$ ~~PDFs approx~~

if $X \sim Bern(p) \Rightarrow m_X(t) = 1 - p + pe^t$ (from def)

$X \sim Binom(n,p) \Rightarrow m_X(t) = (1 - p + pe^t)^n$ (rule IV)

$\cancel{X \sim Geom(p) \Rightarrow m_X(t) = \frac{pe^t}{1 - e^t(1-p)} \text{ if } t < \ln\left(\frac{1}{1-p}\right)}$ ~~(from def) (HW)~~

$X \sim Exp(\lambda) \Rightarrow m_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$ (from def)

$Z \sim N(0,1) \Rightarrow m_Z(t) = e^{\frac{t^2}{2}}$, $X = \mu + \sigma Z$, $m_X(t) = \dots$

Prove this: $X \sim N(\mu, \sigma^2) \Rightarrow m_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Prove this: $X \sim Deg(c) \Rightarrow m_X(t) = e^{ct}$

$X_1 \sim N(\mu_1, \sigma_1^2)$, and $X_2 \sim N(\mu_2, \sigma_2^2)$

(V) Levy's Continuity Thm

$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

let $X_1, X_2, \dots$ be a sequence of r.v.'s

if $\lim_{n\to\infty} M_{X_n}(t) = M_Y(t) \Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_Y(x) \ \forall x$ i.e. $\lim_{n\to\infty} X_n \overset{d}{=} Y$

if the mgf looks more and more like the mgf of another r.v., the limit is that other r.v.

Assume $X_1, X_2, \dots, X_n$ iid some distr with finite μ.

Recall $\overline{X}_n \to \mu$ "the law of large #'s (LLN)"

Note: $\mu \sim Deg(\mu)$

So if $\lim M_{\overline{X}_n}(t) = M_\mu(t) = e^{t\mu}$, then this is proven

$x = e^{\ln(x)}$

$$M_{\overline{X}_n}(t) = M_{\frac{X_1 + \dots + X_n}{n}}(t) = M_{X_1 + \dots + X_n}\left(\tfrac{t}{n}\right) = \left(M_X\left(\tfrac{t}{n}\right)\right)^n = e^{n \ln\left(M_X\left(\frac{t}{n}\right)\right)} = e^{\frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}}$$

$$\lim M_{\overline{X}_n}(t) = e^{\lim_{n\to\infty} \frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}} = e^{\lim_{d\to 0} \frac{\ln(M_X(td))}{d}} = e^{\lim_{d\to 0} \frac{t M_X'(td)}{M_X(td)}} = e^{t\mu} \checkmark$$

let $d = \frac{1}{n}$ $\quad n \to \infty \Rightarrow d \to 0$

L'Hopital's Rule

Recall

$M_X(0) = E[e^{t(0)}] = E[1] = 1$

$M_X'(0) = E[X] = \mu$

~~$\Rightarrow \lim_{n\to\infty} M_{\bar X}(t) = e^{t\mu} = M_\mu(t) \Rightarrow \bar X \to \mu$ proof of ...~~

Still... why is $Z \sim N(0,1)$ so special?

$X_1, \dots, X_n \overset{iid}{\sim}$ Anything with mean μ and s.e. σ

Consider $C_n := \dfrac{\bar X_n - \mu}{\frac{\sigma}{\sqrt n}}$ C_n is the standardized $\bar X_n$

$$\left.\begin{aligned} E(C_n) &= 0 \\ SE(C_n) &= 1 \end{aligned}\right\} \text{ why? Recall the problem...}$$

Prestep:

$$\frac{\bar X_n - \mu}{\frac{\sigma}{\sqrt n}} = \frac{\sqrt n\,(\bar X - \mu)}{\sigma}$$

$$= \frac{\sqrt n \left(\frac{X_1 + \dots + X_n}{n}\right) - \mu}{\sigma} \qquad \mu = \frac{n\mu}{n}$$

$$= \frac{\sqrt n \; \frac{X_1 + \dots + X_n}{n} - \frac{\mu + \dots + \mu}{n}}{\sigma}$$

$$= \frac{(X_1 - \mu) + \dots + (X_n - \mu)}{\sigma \sqrt n}$$

$$= \frac{1}{\sqrt n}\left(\frac{X_1 - \mu}{\sigma} + \dots + \frac{X_n - \mu}{\sigma}\right) \qquad \text{let } Z_i := \frac{X_i - \mu}{\sigma}$$

$$= \frac{1}{\sqrt n}\left(Z_1 + \dots + Z_n\right)$$

Note: $E(Z_i) = 0$, $SE(Z_i) = 1$ why?

$$M_{C_n}(t) = M_{\frac{1}{\sqrt n}(Z_1 + \dots + Z_n)}(t) = M_{Z_1 + \dots + Z_n}\left(\frac{t}{\sqrt n}\right) = \left(M_Z\left(\frac{t}{\sqrt n}\right)\right)^n = e^{n \ln\left(M_Z\left(\frac{t}{\sqrt n}\right)\right)} = e^{\frac{\ln\left(M_Z\left(\frac{t}{\sqrt n}\right)\right)}{\frac{1}{n}}} = e^{t^2 \frac{\ln\left(M_Z\left(\frac{t}{\sqrt n}\right)\right)}{t^2/n}}$$

let $u = \frac{1}{\sqrt{n}}$

$$M_{C_n} = e^{t^2 \frac{\ln(M_Z(ut))}{u^2 t^2}}$$

$$\lim_{n\to\infty} M_{C_n} = \lim_{u\to 0} e^{t^2 \frac{\ln(M_Z(ut))}{u^2t^2}} = e^{t^2 \lim_{u\to 0} \frac{\ln(M_Z(tu))}{u^2 t^2}}$$

L'Hopital's

L'Hopital's

$$= \lim_{u\to 0} \frac{\cancel{t} M_Z'(tu)/M_Z(tu)}{2u t^{\cancel{2}}} = \frac{1}{2}\lim_{u\to 0} \frac{\frac{d}{du}\left[M_Z'(tu)/M_Z(tu)\right]}{t}$$

$$\frac{\cancel{t} M_Z'(u) M_Z'(u) + \cancel{t} M_Z(u) M_Z''(0)}{\frac{M_Z(u)^2}{\cancel{t}}}$$

$\mu^2 = 0$ $\quad 1 \quad \sigma^2 = 1$

$$= \frac{1}{2} \frac{M_Z'(0)^2 + M_Z(0) M_Z''(0)}{\underset{1}{M_Z(0)^2}} = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} M_{C_n}(t) = e^{\frac{t^2}{2}} \Rightarrow C_n \to N(0,1)$$

Central Limit Thm !! (CLT)

$X_1, X_2, \ldots, \overset{iid}{\sim} f(\mu, \sigma^2)$ ← Common situation

Common situation

$$\Rightarrow C_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \to N(0,1)$$ ← Super mysterious!

This is the mystery ... solved!

How to use the CLT.

Obviously $n \to \infty$ is impractical

if n is large, CLT approximately works. There are three formulas you should know. If $X_1, X_2, \ldots X_n \overset{iid}{\sim} f(\mu, \sigma^2)$

① $\dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\approx} N(0,1)$

② $\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$ *

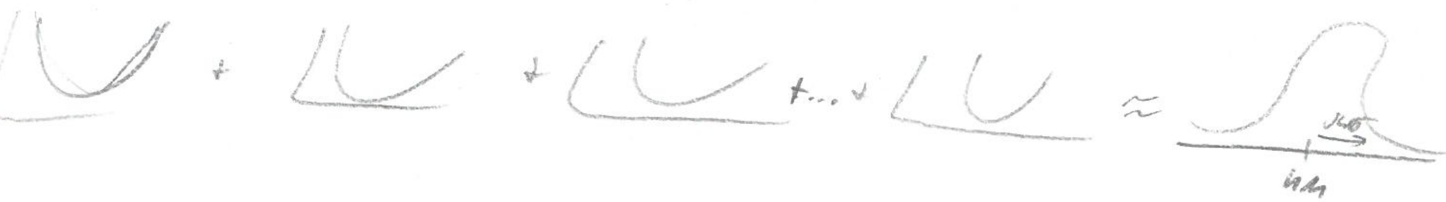
③ $T \overset{d}{\approx} N(n\mu, (\sigma\sqrt{n})^2)$ *

most useful

$X \sim \text{Bathtub}(\mu)$

Already it begins!

$X_1 + \dots + X_n \approx N(n\mu, (\sqrt{n}\,\sigma)^2)$



??

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \approx e^{-x^2}$$ makes tails small!

natural balance between means & errors

When is a certain value considered "normal"?

Why is it called the normal distr??

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example $X_1, \dots, X_{30} \overset{iid}{\sim} \text{Geom}(\frac{1}{2})$

What's the prob the avg wait time is more than 2.75?

standardize!!

$$P(\overline{X} \geq 2.75) \approx P\left(\frac{\overline{X} - 0.5}{.258} \geq \frac{2.75 - 0.5}{.258}\right) = P(Z \geq 3) = .0015$$

Recall $\overline{X} \approx N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \Rightarrow \overline{X} \approx N(0.5, .258^2)$

$\mu = p = \frac{1}{2}, \ \sigma = \sqrt{\frac{q}{p^2}} = \sqrt{\frac{1/2}{1/4}} = \sqrt{2} \approx 1.414 \Rightarrow \frac{\sigma}{\sqrt{n}} \approx \frac{1.414}{\sqrt{30}} \approx .258$