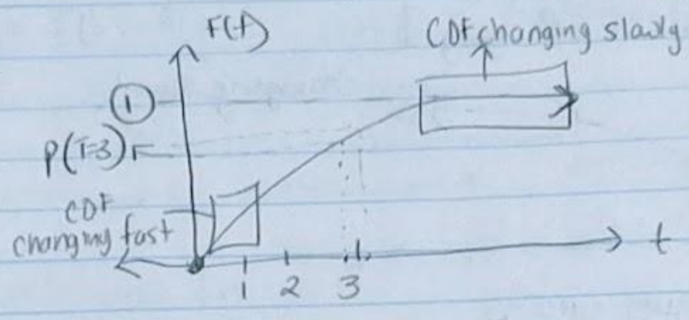
Lecture 17 11/07/2017

Limiting CDF

$F(t) = 1 - e^{-\lambda t}$



$\lim_{t \to -\infty} F(t) = 0$

$F'(t) = \lambda e^{-\lambda t} = \frac{\lambda}{\underbrace{e^{\lambda t}}_{t}} > 0 \Rightarrow F(t)$ is montonically increasing

$F(t)$ is a CDF

$* \ p(t) = 0 \ \forall t \ *$ any number

$Supp[T] = (0, \infty)$

the r.v doesn't matter

$|Supp(T)| = |\mathbb{R}| > |\mathbb{N}| \rightarrow T$ is not discrete

has to be continuous

10 25

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0 1

planck length $= 1.62 \times 10^{-35} m$

cant subdivide space smaller than this

$t = 5.3 \times 10^{-44} s$

$p(3) = 0$

$= P(T=3)$

$= P(T = 3.000000 \ldots s)$

infinite zeros, infinite precision

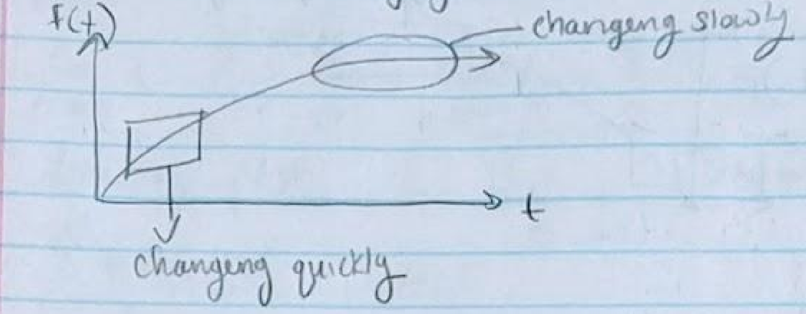
$P(T = 3.000) = P(T \in [2.995\bar{0}, 3.004\bar{9}]) = F(3.004\bar{9}) - F(2.995\bar{0}) > 0$

3 digits of precision ↑3 ↓3

$P(X \in [a,b]) = P(X \le b) - P(X \le a) = F(b) - F(a)$

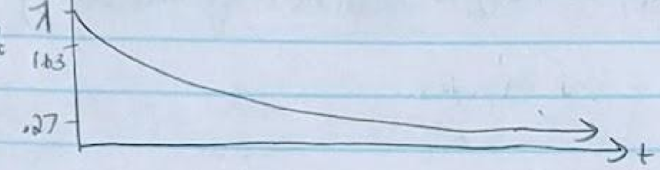
a b

How fast is CDF changing ?



$F'(t) = \lambda e^{-\lambda t}$

$f(t)$ → probability density function (PDF) $\neq$ PMF !
(Not a probability function)



"how much probability is in certain regions"

$\lambda = 2 = \quad F(t) = 1 - e^{-2t}$

$f(t) = 2e^{-2t}$

$f(0.1) = 1.63 \neq p(0.1) = 0$

value itself means nothing w/ no context

$f(1) = 0.27 \neq p(1) = 0$

$$\frac{p(T \approx 0.1)}{p(T \approx 1)} = \frac{1.63}{0.27} \approx 6$$

$$\lim_{\varepsilon \to 0} \frac{P(T \in [0.1, 0.1 + \varepsilon])}{P(T \in [1, 1+\varepsilon])} = \frac{F(0.1+\varepsilon) - F(0.1)}{F(1+\varepsilon) - F(1)} , \frac{\varepsilon}{\varepsilon}$$

$$= \frac{\lim_{\varepsilon \to 0} \dfrac{F(0.1+\varepsilon) - F(0.1)}{\varepsilon}}{\lim_{\varepsilon \to 0} \dfrac{F(1+\varepsilon) - F(1)}{\varepsilon}} = \frac{f(0.1)}{f(1)}$$

Analogous to $\sum_{x \in supp[X]} p(x) = 1$ for a discrete r.v X

PDF is legit if

$P(T \in (-\infty, \infty)) = P(T > 0) = 1$ (a) (b) $f(t) \geq 0$

$F(\infty) - F(-\infty) = \boxed{\int_{\mathbb{R}} f(t)\,dt = 1}$

$$P(T \in [a,b]) = \underbrace{F(b) - F(a)}_{\text{CDF}} = \int_a^b \underbrace{f(t)}_{\text{PDF}}\,dt$$ Fundamental Theory of Calculus

Definition = Continuous r.v X

(a) $|Supp(X)| = |\mathbb{R}|$

(b) $F(x)$ is valid (and no jumps)

(c) $p(x)$ is not a PMF

(d) $f(x) = F'(x)$ (i) $\int_{Supp[X]} f(x) = 1$

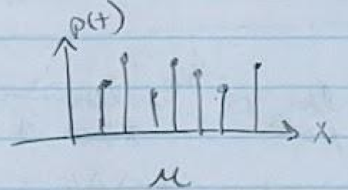
(ii) $f(x) \geq 0$

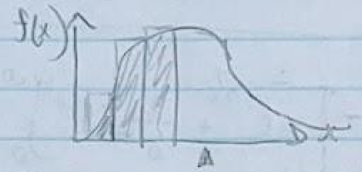
Definition $X \overset{d}{=} Y$ for continuous X, Y

if (i) $f(x) = f(y)$

(ii) $F(x) = F(y)$

$E(X)$

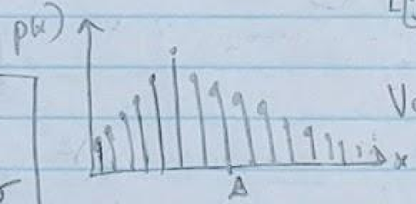




$$E(X) = \int_{Supp[X]} x f(x)\,dx$$

$$E[g(X)] = \int_{Supp[X]} g(x) f(x)\,dx$$

$$Var[X] = E[(x-\mu)^2] = \int_{Supp[X]} (x-\mu)^2 f(x)\,dx$$



$$\sum x p(x) = \sum x\,(F(b)_i - F(a)_i)$$

$E[aX+c] = a\mu + c$

$Var[aX+c] = a^2\sigma^2 \Rightarrow SE[aX+c] = |a|\sigma$

$E[\sum X_i] = \sum E[X_i] = n\mu$ ← if identicall distributed

$E(\bar{X}) = \mu$ ← if identically distributed

$Var[\sum X_i] = \sum Var[X_i] = n\sigma^2$ ← if all independent; if iid

$Var[\bar{X}] = \frac{\sigma^2}{n}$ ← if iid

Discrete → no PDF → no derivative

Continuous → no PMF

$$X \sim \text{Exp}(\lambda) := \lambda e^{-\lambda x}$$

exponential — PDF

set all possible values a r.v can realize to ← $\text{Supp}[X] = (0, \infty)$

$\lambda \in (0, \infty)$

$\lambda = np$
$p \in (0,1)$
$n \in \mathbb{N}$

$$E(X) = \int_0^\infty x\lambda^{-\lambda x}\,dx = \lambda\int_0^\infty x e^{-\lambda x}\,dx = \lambda\left[(x)\left(-\frac{1}{\lambda}e^{-\lambda x}\right) - \left(\frac{1}{\lambda^2}e^{-\lambda x}\right)\right]_0^\infty$$

$\int u\,dv = uv - \int v\,du$

let $u = x \Rightarrow du = dx$

let $dv = e^{-\lambda x}dx$

$\Rightarrow v = \int e^{-\lambda x}dx = -\frac{1}{\lambda}e^{-\lambda x} \Rightarrow \int v\,du = \int -\frac{1}{\lambda}e^{-\lambda x}dx = \frac{1}{\lambda^2}e^{-\lambda x}$

$$= -\left[xe^{-\lambda x} + \frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty = -\left(\lim_{x\to\infty} xe^{-\lambda x} + \lim_{x\to\infty}\frac{1}{\lambda}e^{-\lambda x}\right) - \left(0e^{\infty} + \frac{1}{\lambda}e^{-\lambda(0)}\right)$$

$$= -\left((0-0) - \left(0 + \frac{1}{\lambda}\right)\right) = \boxed{\frac{1}{\lambda}}$$

Memoryless Property

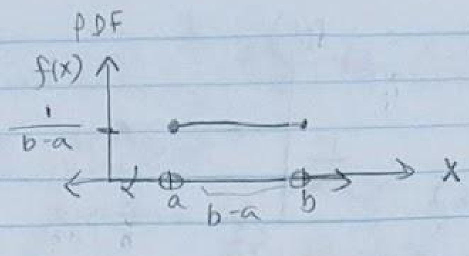
$$P(X>a) = P(X>a+b \mid X>b) = \frac{P(X>a+b \cap X>a)}{P(X>b)}$$

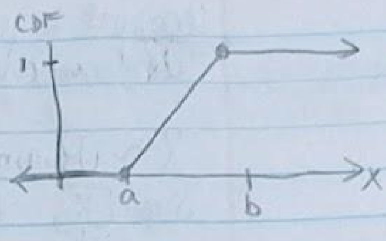
$$= \frac{P(X>a+b)}{P(X>b)} = \frac{\overset{e^{-\lambda a}e^{-\lambda b}}{\overset{\shortparallel}{e^{-\lambda(a+b)}}}}{e^{-\lambda b}} = e^{-\lambda a} = P(X>a)$$

$X \sim U(a,b)$

uniform

$: = \frac{1}{b-a}$





$Supp[X] = (a,b)$

Parameter Space

$a \in \mathbb{R}$
$b \in \mathbb{R}$ but $a < b$

$$\text{CDF} \rightarrow F(x) = \int f(x)\,dx + C = \int \frac{1}{b-a}\,dx + C = \frac{x}{b-a} + C + \frac{a}{b-a} = \boxed{\frac{x-a}{b-a} \Downarrow F(x)}$$

integrale
indefinite integral

$a \in \mathbb{R}$
$b \in \mathbb{R}$

$F(a) = 0$

$$\frac{a}{b-a} + C = 0 \Rightarrow \frac{a}{a-b} = C \Rightarrow \frac{a-b}{a} = \frac{1}{C} = 1 - \frac{b}{a} = \frac{1}{C}$$

$$\Rightarrow 1 - \frac{1}{C} = \frac{b}{a} \Rightarrow \frac{1}{C-1} = \frac{-b}{a} \Rightarrow \frac{1}{C} = 1 - \frac{b}{a} = \frac{a-b}{a} = C = \frac{a}{a-b} = \frac{-a}{b-a}$$