Lec 12 Math 241 10/19/17

1

Two ways to look at binomial:

$$\lim_{N \to \infty} \text{Hyper}(n, p, N)$$

— or —

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(p)$

$X_1 + \ldots + X_n$

for binomial..

$$F(x) := P(X \le x) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$$

best we can do!

no closed form

$$= I_{1-p}(n-k, 1+k)$$

regularized incomplete beta function

$$= (n-k) \binom{n}{k} \int_0^{1-p} t^{n-k-1} (1-t)^k \, dt$$

not tested

no closed form

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

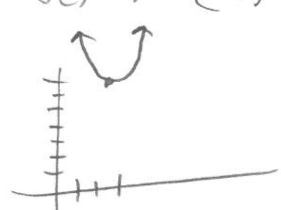
possibly infinite series of binary experiments w/ same prob.

independent / one another

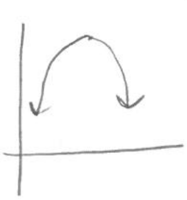
min, max

argmin, argmax

$f(x) = 7 + (x-3)^2$



$f(x) = 7 - (x-3)^2$



$\max\{f(x)\}$ undefined

$\arg\max\{f(x)\}$ undefined

$\min\{f(x)\} = 7$

$\arg\min\{f(x)\} = 3$

$\{3, 4, 7, 11, 12, 13, 18, \ldots\}$

let $T = \min\{t : X_t = 1\}$

T is the first time a success occurs

AKA the "stopping time"

$P(T=1) = p$ $\quad \frac{X = 1}{t=1}$ $\quad P(X=1)$

Why multiplication?

$P(T=2) = (1-p)p$ $\quad X = \underline{0}\ \underline{1}$, $t = 1\ 2$ $\quad P(X_1 = 0, X_2 = 1) = P(X_1 = 0)P(X_2 = 1)$

Independence of the Bernoulli experiments

$P(T=3) = (1-p)^2 p$ $\quad X\ \underline{0}\ \underline{0}\ \underline{1}$, $t\ 1\ 2\ 3$

$P(T=t) = (1-p)^{t-1} p$

$$\underbrace{\frac{0}{1}\ \frac{0}{2} \ldots \frac{0}{} \ \frac{0}{t-2}\ \frac{0}{t-1}}_{\substack{t-1 \\ \text{failures}}}\ \underbrace{\frac{1}{t}}_{\substack{1 \\ \text{success}}}$$

$X \sim \text{Geometric}(p) := \underbrace{(1-p)^{x-1} p}_{p(x)\ \text{PMF}}$

switch back to std notation

Param Space $p \in (0,1)$ why?

it's built from Bernoullis

$\text{Supp}(X) = \{1, 2, \ldots\} = \mathbb{N}$

$0 \notin$ Why?

$p = 1$ If... $X \sim \text{Deg}(1)$

$$\sum_{x \in Supp(X)} p(x) = 1 \; ?$$

$$\sum_{i=1}^{\infty} (1-p)^{i-1} p \stackrel{?}{=} 1 \Rightarrow \sum_{i=1}^{\infty} (1-p)^{i-1} \stackrel{?}{=} \frac{1}{p} \Rightarrow \sum_{i=0}^{\infty} (1-p)^{i} \stackrel{?}{=} \frac{1}{p}$$

let $q := 1-p$ since $p \in (0,1) \Rightarrow q \in (0,1)$

$$S := \sum_{i=0}^{\infty} q^i \stackrel{?}{=} \frac{1}{1-q}$$

$= q^0 + q^1 + q^2 + q^3 + \ldots$ "geometric series"

$= 1 + q + q^2 + q^3 + \ldots$

$= 1 + q(1 + q + q^2 + \ldots$

$= 1 + q(S) \Rightarrow S - qS = 1 \Rightarrow S(1-q) = 1 \Rightarrow S = \frac{1}{1-q}$ ✓

This is how it gets its name

$$F(x) := P(X \le x) = \sum_{i=1}^{x} (1-p)^{i-1} p$$ HARD METHOD

— or —

EASIER METHOD

$$F(x) = 1 - P(X > x)$$

$$P(X>x) = P\Big(\underbrace{\underset{1}{0}\ \underset{2}{0}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}\ \cdots\ \underset{x}{0}}_{\text{these are all 0's}}\ \overbrace{\underset{x+1}{\_}\ 0\ \_\ 0\ \_\ 1\ \_\ 0\ \_\ 0\ \_\ \cdots}^{\text{the success is somewhere here}}\Big)$$

$$P(X > x) = (1-p)^{x}$$

Seems quicker way

$$P(X>x) = P(X=x+1) + P(X=x+2) + P(X=x+3) + \ldots$$

$$= \sum_{i=x+1}^{\infty} (1-p)^{i-1} p$$

$$= \sum_{i=1}^{\infty} (1-p)^{i+x-1} p$$

$$= (1-p)^{x} \underbrace{\sum_{i=1}^{\infty} (1-p)^{i-1} p}_{1}$$

$X \sim \text{Geom}(p) := (1-p)^{x-1} p$ Valid PMF $\text{Supp}(X) = \mathbb{N}$, param space $p \in (0,1)$

you can think of this as the stopping time of $X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

e.g. prob of getting royal flush $= \frac{4}{\binom{52}{5}} = 1.53/\text{million}$

$= .00000153$ "p"

Scenario...

Play poker until you get a Royal Flush. On which # hand do you get it for the first time?

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(.00000153)$ stopping time

$X \sim \text{Geom}(.00000153)$

What is the prob. I get it on the millionth hand?

$P(X=1000000) = (.999985)^{999999} \cdot .00000153$

What is the prob I get it on the millionth time or sooner?

$F(x) = P(X \leq x) = 1-(1-p)^x$ powerful...

$P(X \leq 1000000) = 1 - .9999985^{1000000} = .777 \approx 80\%$

Usually I cover the Negative Binomial r.v. ... but we are ~~so~~ far behind now.

Philosophical Jump

$X \sim \text{Bern}(p)$ model

but $x = 0$ or $x = 1$

realization of the r.v.

realization means to "make real"

$P(X = x)$ prob. the r.v. model realized such a way

datum: realization of a r.v. $\in \text{Supp}(X)$? YES..

data: " " " " s of " r.v's

iid data " " " " " " iid r.v.'s

In class demos

$X_1, \ldots, X_8 \overset{iid}{\sim}$ Hypergeometric $(3, 3, 8)$ = Hypergeom $(3, 0.375, 8)$

$X_1, \ldots, X_8 \overset{iid}{\sim} \text{Bin}\left(8, \frac{1}{2}\right)$

$X_1, \ldots, X_8 \overset{iid}{\sim} \text{Geom}\left(\frac{1}{2}\right)$

$X_1, \ldots, X_? \overset{iid}{\sim}$ Rademacher $= \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$

$T_n = X_1 + \dots + X_n = \sum X_i$ "total r.v"

$$\bar X_n = \frac{X_1 + \dots + X_n}{n} = \frac{T_n}{n}$$ "avg r.v."

$$\bar x_n = \frac{\sum x_i}{n}$$

a realization from $\bar X_n$

Let's do many $X_1, \dots, \overset{iid}{\sim}$ Binomial

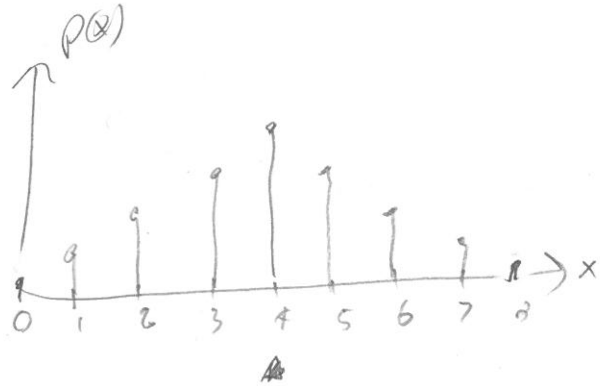
and calculate

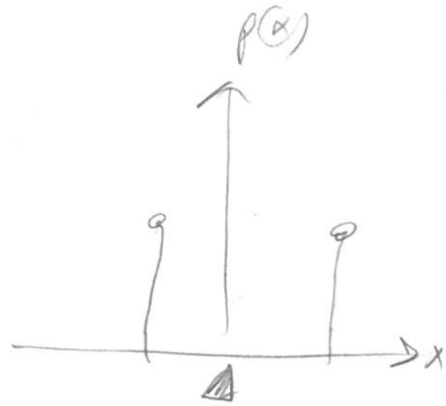
$\bar X_n$

do you think $\lim_{n \to \infty} \bar x_n = 4$?

do you think the limit for the rademacher $\bar x = 0$?

It appears





Balance pt

It appears $\bar x \xrightarrow{\text{on the page}}$ balance pt.