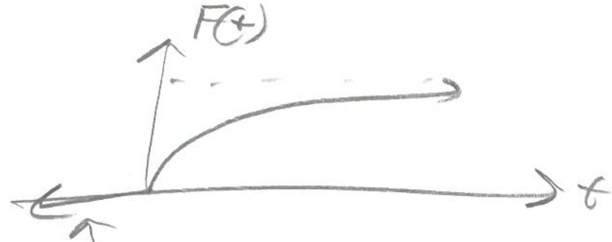
$F(t) \in [0,1]$?



why

comes from geometric

$F(t) \geq 0 \quad 1 - e^{-\lambda t} \geq 0$?

$1 \geq e^{-\lambda t}$?

$0 \geq -\lambda t$

$t \geq 0$? Yes

$F(t) \leq 1$?

$1 - e^{-\lambda t} \leq 1$

$-e^{-\lambda t} \leq 0$

$e^{-\lambda t} \geq 0$

$\frac{1}{e^{\lambda t}} \geq 0$ Yes

$\lim_{t \to -\infty} F(t) = 0$

$\lim_{t \to \infty} F(t) = \lim_{t \to \infty} 1 - e^{-\lambda t} = 1 - \lim_{t \to \infty} e^{-\lambda t} = 1 - \lim_{t \to \infty} \frac{1}{e^{\lambda t}} = 1$ ✓

Is $F(y) \geq F(x)$ if $y > x$?

Check derivative ... ensure always ≥ 0

$$\frac{d}{dt}\left[F(t)\right] = \frac{d}{dt}\left[1 - e^{-\lambda t}\right] = \lambda e^{-\lambda t} = \frac{\lambda}{e^{\lambda t}} \geq 0$$ ✓

Since there is a CDF $\Rightarrow$ T is a r.v. Discrete? No... no PMF. So what is it?

$Supp[T] = (0, \infty) \qquad |Supp(T)| = |\mathbb{R}| > |\mathbb{N}|$

size of the continuum (uncountable ∞)

$\Rightarrow$ T is a cont. r.v.

Why did this happen?

$h \to 0$ means no "space" in between #'s. Infinite division means no missing #'s.

Is continuous time real? Quantum Theory again.

Planck length 1.62×10^{-35} m, not possible to tell difference in location

$d_1 = d_2$! Crazy

$\rightsquigarrow$ speed of light

time it takes to cross Planck length: 5.3×10^{-44} s

Time may be discrete! We don't know! Until we understand quantum gravity...

$p(3) = 0$ why? ← infinite resolution

$p(3.0000000\ldots) = 0$

but $p(3.000) = P(T \in [2.995\bar{0}, 3.004\bar{9}])$ $\quad F(3.005) - F(2.995) > 0$

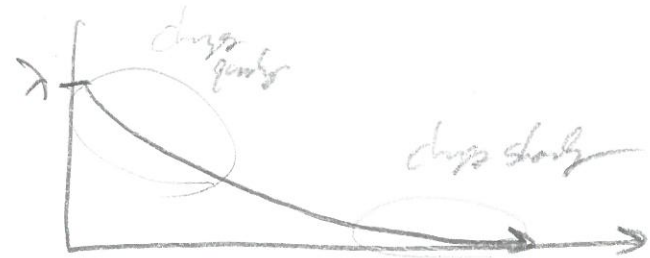
$= P(T \leq 3.004\bar{9}) - P(T \leq 2.995\bar{0})$

How "fast" does the CDF change?

$f(t) := F'(t) = \lambda e^{-\lambda t}$

probability density function (PDF)

$\neq$ prob mass function (PMF)



$$P(T \in [a,b]) = \int_a^b f(t)\, dt = F(b) - F(a)$$ Fund. Thm. Calculus...

PDF: answers the question: How dense is the probability in a certain region?

let $\lambda = 2$

$f(1) = 0.27 \neq p(1) = 0$

$p(1) = P(T = 1) = \int_1^1 f(t)\,dt = 0$

$f(0.1) = 1.63 \neq p(0.1) = 0$

But $f(0.1) > 1$! Possible? Yes ... PDF is not a prob.

PDF is completely abstract! It is good for

(a) Integration to find probs of regions

(b) Comparison $\quad \dfrac{f(0.1)}{f(1)} = \dfrac{1.63}{0.27} \approx 6 \Rightarrow$ realizations near 0.1 are 6x more likely than realizations near 1

why $\lim_{\varepsilon \to 0} \dfrac{P(T \in [0.1, 0.1+\varepsilon])}{P(T \in [1, 1+\varepsilon])} = \lim_{\varepsilon \to 0} \dfrac{F(0.1+\varepsilon) - F(0.1)}{F(1+\varepsilon) - F(1)}$

def of deriv. $F'(t) = f(t)$

$= \lim_{\varepsilon \to 0} \dfrac{\dfrac{F(0.1+\varepsilon) - F(0.1)}{\varepsilon}}{\dfrac{F(1+\varepsilon) - F(1)}{\varepsilon}} = \dfrac{f(0.1)}{f(1)}$

$P(T \in (-\infty, \infty)) = 1$ why? T has to realize something!

$= \int_{-\infty}^{\infty} f(t)\,dt$ by F.T.C. Analogous to $\sum_{x \in \text{Supp}(X)} p(x) = 1$ for the discrete case

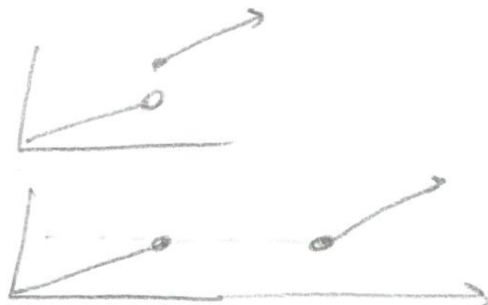
Def Cont r.v. X

(1) $|Supp(X)| = |\mathbb{R}|$

(2) $F(x)$ is a valid CDF with no "jumps". bends allowed

(3) PMF does not exist

(4) $f(x) \geq 0$ and $\int_{Supp(X)} f(x)\,dx = 1$



Def of Identical Distr for Cont. r.v.'s $X_1 \stackrel{d}{=} X_2$ if $f_1(x) = f_2(x)$

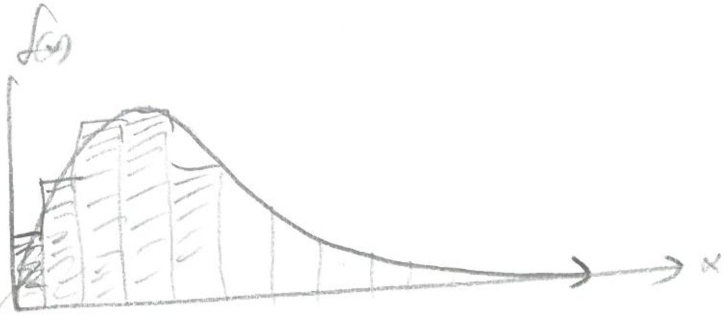
Def of iden. distr of both discrete & cont r.v.'s $X_1 \stackrel{d}{=} X_2$ iff $F_1(x) = F_2(x)$

CDF's always exist

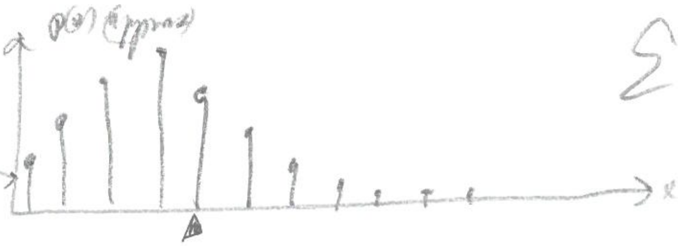
What is $E[X]$? If X is discrete

$$\sum_{x \in Supp(X)} x\, p(x)$$

Now..



Image



$$\sum x\, p(x)$$

Make the rectangles thinner & thinner ...

$$\sum_{Supp(X)} \underset{\text{midpoint}}{x}\; \underset{\text{area of rectangle}}{p(x)} \longrightarrow \boxed{\mu := E(X) = \int_{Supp(X)} x\, f(x)\, dx}$$

$$E[g(X)] = \int_{Supp(X)} g(x)\, f(x)\, dx$$

Not proven.... need Lebesgue integration to demonstrate

$$\Rightarrow Var(X) := E\left[(X-\mu)^2\right] = \int_{Supp(X)} (x-\mu)^2 f(x)\, dx$$

All rules from before apply now...

$$E[aX+c] = a\mu + c$$

$$Var(aX+c) = a^2\sigma^2 \Rightarrow SE(aX+c) = |a|\sigma$$

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = n\mu$$

If iden. distr

$$Var\left(\sum X_i\right) = \sum_{i=1}^{n} Var(X_i) = n\sigma^2$$

if indep

if iid

Why was this scrambled geometric?

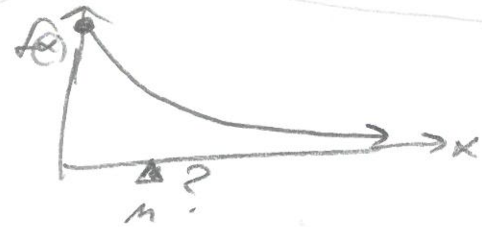
$P(X \le x) =$ $F(x) = 1 - e^{-\lambda x} \Rightarrow$ $P(X > x)$ $1 - F(x) = e^{-\lambda x}$

$$X \sim Exp(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x) \text{ not } p(x)}$$

$Supp(X) = (0, \infty)$ or $[0, \infty)$ but not $[0, \infty]$ why?

Param space $\lambda = np$ $n \to \infty$, $p \in (0,1)$

$\lambda \in (0, \infty)$



$$E(X) = \int_{Supp(X)} x f(x)\, dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty x e^{-\lambda x}\, dx$$

recall $\int u\, dv = uv - \int v\, du$

let $u = x$, $dv = e^{-\lambda x} dx$

$\Rightarrow du = dx \Rightarrow v = -\frac{1}{\lambda} e^{-\lambda x} \Rightarrow \int v\, du = \int -\frac{1}{\lambda} e^{-\lambda x} dx = \frac{1}{\lambda^2} e^{-\lambda x}$

$$\Rightarrow E(X) = \lambda \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty$$

$$= -\left(\left(\lim_{x \to \infty} x e^{-\lambda x} + \lim_{x \to \infty} \frac{1}{\lambda} e^{-\lambda x} \right) - \left(0 e^{-\lambda(0)} + \frac{1}{\lambda} e^{-\lambda(0)} \right) \right)$$

$X \sim Geom(p) \Rightarrow E(X) = \frac{1}{p}$ if $n = 1 \Rightarrow p = \lambda$

$$= -\left((0 + 0) - (0 + \tfrac{1}{\lambda}) \right) = \boxed{\frac{1}{\lambda}}$$

So makes sense!

	Single Success	Multiple Successes
Discrete	Geom	Neg Bin
Cont	Exponential	Erlang ε

Math 242/621 (type of gamma)

Geometric was called "memoryless" ... Is exponential also memoryless?

$$P(X > a+b \mid X > b) = \frac{P(X > a+b \;\&\; X > b)}{P(X > b)} = \frac{P(X > a+b)}{P(X > b)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = \frac{e^{-\lambda a}\cancel{e^{-\lambda b}}}{\cancel{e^{-\lambda b}}} = e^{-\lambda a}$$

$$= P(X > a) \Rightarrow \text{Yes!}$$

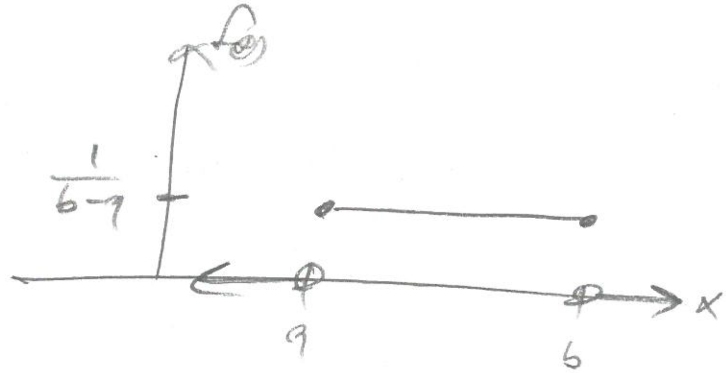
Recall $X \sim \text{Unif}(\{1, 7, 13\}) = \begin{cases} 1 & \text{wp } \frac{1}{3} \\ 7 & \text{wp } \frac{1}{3} \\ 13 & \text{wp } \frac{1}{3} \end{cases}$

discrete unif

Let $X \sim \text{Unif}(a, b) := f(x) = \frac{1}{b-a}$

is the cont. uniform

$\text{Supp}(X) := [a, b]$



Param Space

$a \in \mathbb{R}, b \in \mathbb{R}$ but $a < b$

Is this a PDF? Is $f(x) \geq 0$ always? Yes

Does $\int_{\text{Supp}(X)} f(x)\,dx = 1$? $\int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}(x)_a^b = \frac{b-a}{b-a} = 1$ ✓

Find CDF ... the antiderivative

$$F(x) = \int f(x)\,dx + C$$

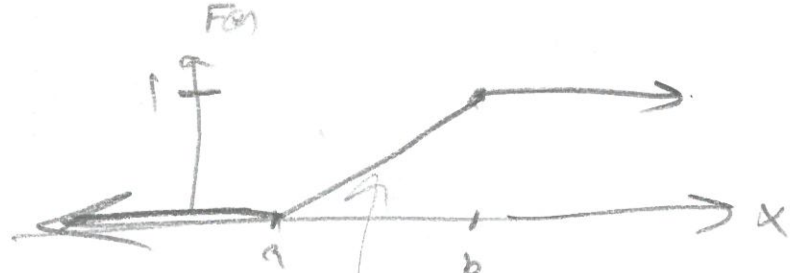
$$= \int \frac{1}{b-a}\,dx + c$$

$$= \frac{x}{b-a} + C \qquad \text{value of } C?$$

$F(a) = 0$ if $F(a) = 0 \Rightarrow \frac{a}{b-a} + C = 0 \Rightarrow C = \frac{-a}{b-a}$

$F(b) = 1$

$$\Rightarrow F(x) = \frac{x-a}{b-a}$$



slope? $\frac{1}{b-a}$ of course

if $a = 0$, $b = 1$

$\Rightarrow X \sim \text{Unif}(0,1)$ AKA the standard uniform

$f(x) = 1$

$F(x) = x$

$\text{Supp}(X) = [0,1]$

CS people... this is the rand function!

μ? $E(X) = \int_{\text{Supp}(X)} x\, f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \boxed{\frac{a+b}{2}}$$

makes sense

$\text{Med}(X) = \text{argmin}\{x : F(x) = 0.5\} \Rightarrow \frac{x-a}{b-a} = \frac{1}{2} \Rightarrow 2x - 2a = b - a \Rightarrow x = \frac{b+a}{2}$ sure!!