MAT 241 - 11/20/17 - B

$X \sim$ uniform(a,b) Supp$[X] = (a,b)$ $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a < b$

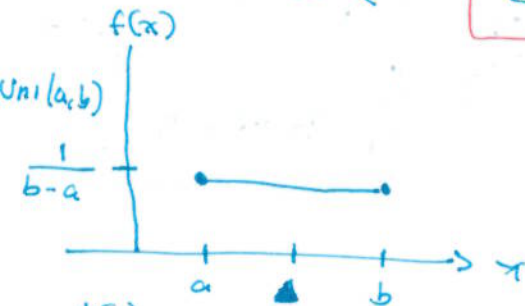
$1 = \int_{Supp[X]} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx$

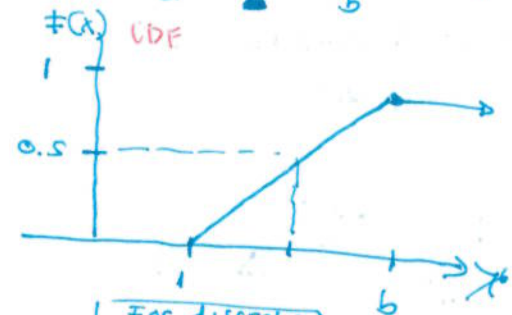
$E[X] = \int_{Supp[X]} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \boxed{\frac{a+b}{2}}$

$Med[X] = Quantile[X, \frac{1}{2}]$

~~argmin $\{F(x) \geq p\}$ for discrete r.v's~~

$X \sim Uni(a,b)$





"inverse CDF"

~~$Quantile[X, p] = F^{-1}(p)$ = argmin $\{F(x) \geq p\}$ for discrete r.v's~~

$Med[X] = F^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}(b-a) + a$

Quantile$[X, \frac{1}{2}]$

$= \frac{1}{2}b - \frac{1}{2}a + \frac{2}{2}a$

$= \frac{1}{2}b + \frac{1}{2}a$

$= \frac{a+b}{2}$

F^{-1}

$F(x) = \frac{x-a}{b-a} = p \Rightarrow x - a = p(b-a)$

$\Rightarrow x = p(b-a) + a = F^{-1}(p)$

For discrete
Quantile$[X,P] :=$ argmin $\{F(x) \geq p\}$

Inverse CDF
For continuous r.v
Quantile$[X,P] = F_X^{-1}(p)$ ← inverse function

Inverse function when one-to-one

$Var[X] = E[(X-\mu)^2]$

$= E[X^2] - \mu^2$

$E[X^2] = \int_{Supp[X]} x^2 f(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$

$= \frac{b^3-a^3}{3(b-a)}$

$\left(\frac{a+b}{2}\right)^2$

$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$

$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$

$$\begin{array}{r|l} & b^2+ab+a^2 \\ b-a & b^3 - a^3 \\ & -(b^3 - ab^2) \\ & ab^2 - a^3 \\ & -(ab^2 - a^2b) \\ & a^2b - a^3 \\ & -a^2b - a^3 \\ & 0 \end{array}$$

$\frac{b^2+ab+a^2}{3} = E[X^2]$

$\boxed{Var[X] = \frac{(b-a)^2}{12}}$

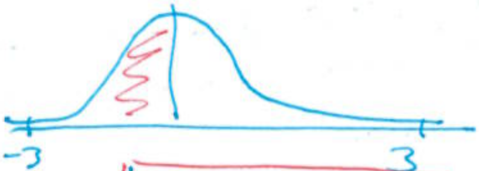
$\boxed{SE[X] = \frac{b-a}{\sqrt{12}}}$

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ *NEW Random variable*

"normal" "gaussin" "belle curve"

① $f(x) \geq 0$ ~~∀x in~~ Supp[X]

② $\int_{supp(X)} f(x)\,dx = 1$



$\boxed{Supp[X] = \mathbb{R}}$

Prove ② $\int_{supp(X)} f(x)\,dx = 1$ (Not necessary for class)

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 2\sqrt{\pi}$$

$$\boxed{\begin{aligned} &\text{let } u = \frac{1}{\sqrt{2}} x \Rightarrow \frac{x^2}{2} = u^2 \\ &du = \frac{1}{\sqrt{2}} dx \Rightarrow dx = \sqrt{2}\,du \end{aligned}}$$

Gaussian Integral

$$= \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{2\pi} = \pi$$

$$= \left(\int_{\mathbb{R}} e^{-u^2} du \right)^2 = \pi \Rightarrow \int_{\mathbb{R}} e^{-u^2} du \int_{\mathbb{R}} e^{-u^2} du = \pi$$

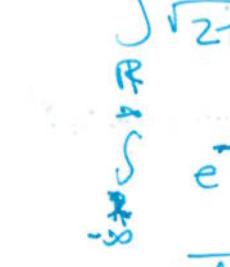
call x call y.

~~Let u=x~~

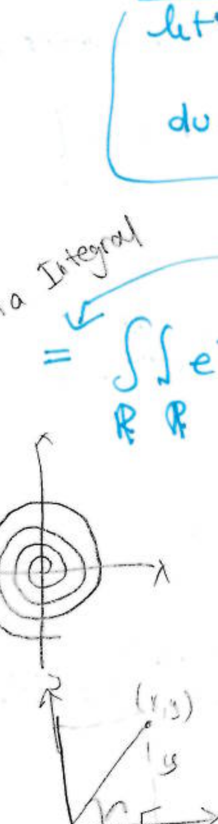
$$= \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \pi$$

Aria Integral

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2} e^{-y^2} dx\,dy = \pi \Rightarrow \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$$



$$dx\,dy = dA = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$



$x = r\cos\theta \quad y = r\sin\theta$

$dA \neq dr\,d\theta$

$= \int \rightarrow$ NEXT PAGE

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-r^2}\, r\, dr\, d\theta \qquad \text{let } u = r^2 \Rightarrow du = 2r\,dr \Rightarrow dr = \frac{1}{2r}\,du$$

$$= \int_0^{2\pi}\int_{u_0}^{u_f} e^{-u}\, r\, \frac{1}{2r}\, du\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_{u_0}^{u_f} e^{-u}\, du\, d\theta = \frac{1}{2}\int_0^{2\pi} \left[-e^{-u}\right]_{u_0}^{u_f} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[-e^{-r^2}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{2\pi} \underbrace{-(0-1)}_{1}\, d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \frac{1}{2}\left[\theta\right]_0^{2\pi} = \boxed{\pi}$$

Valid PDF →

$$\boxed{Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x) \quad (\text{valid})}$$

$$E[Z] = \int_{\mathrm{supp}(x)} x\, f(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

let $u = \frac{x^2}{2}$

$\frac{du}{dx} = x$

$dx = \frac{x}{du}$

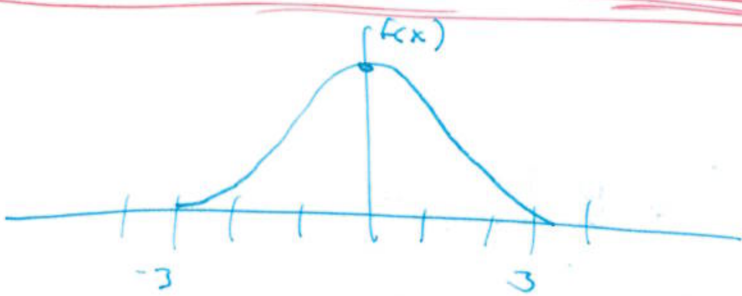
$$= \frac{1}{\sqrt{2\pi}}\int_{u_0}^{u_f} x e^{-u} \frac{1}{x}\, du = \frac{1}{\sqrt{2\pi}}\left[-e^{-u}\right]_{u_0}^{u_f} = \frac{1}{\sqrt{2\pi}}\left[e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}$$

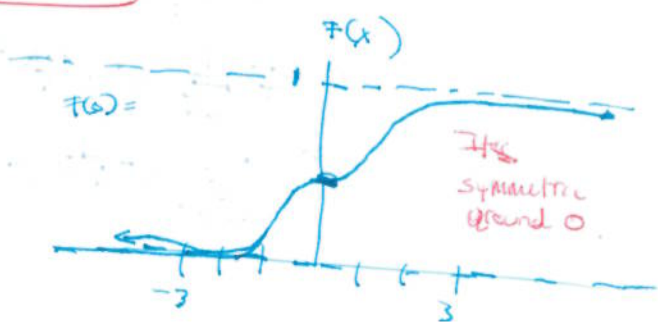
$$E[Z] = \frac{1}{\sqrt{2\pi}}[0-0] = 0$$

$$\mathrm{Var}[Z] = E[Z^2] - \overset{0}{\cancel{\mu^2}} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \ldots = 1$$

$$F(x) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx + C$$

~~Not possible~~ in closed form ← no formula for CDF





Integrals that you must know — Cheat sheet material

$$P(Z \in [-1,1]) \approx .68$$
$$P(Z \in (-2,2)) \approx .95$$
$$P(Z \in (-3,3)) \approx .997$$

Empirical Rule
3σ Rule
68, 95, 99.7 Rule

"Standard Normal"

$$X = \sigma Z + \mu \quad \boxed{\sigma > 0, \mu \in \mathbb{R}}$$

$$E[X] = \sigma \underbrace{E[Z]}_{0} + \mu = \mu$$

$$\text{var}[X] = \sigma^2 \underbrace{\text{var}[Z]}_{1} = \sigma^2$$

$$SE[X] = \sigma$$

Not responsible

$$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x)$$
$$= P\left(Z \le \frac{x-\mu}{\sigma}\right)$$
$$= F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$f_X(x) = F_X'(x) = \frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right]$$
$$= \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$$
$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}_{\text{the normal dist}} = \mathcal{N}(\mu, \sigma^2)$$

American male height is normally distributed with Mean = 70 inches. (5'10")
and Standard Error is 4 inches.

What is the prob. a random American male is taller than 78" (=6'6")?

$X \sim N(70", 4"^2)$

$P(X > 78) = P\left(\frac{X-70}{4} > \frac{78-70}{4}\right) = P(Z > 2) = 2.5\%$