09/07/2017

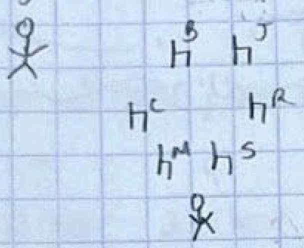
Lecture 4

# of ways to sample K objects out of a set of n objects w/o replacement

"Permutations"

$$_nP_K = \frac{n!}{(n-K)!} = (n)(n-1)\cdots(n-K+1)$$

Recall B-J, R-S, C-M sitting in 6 seats, How many ways to seat them in a circle and you don't care which chair is the first seat




How many ways to seat?

6! moving one notch is equivalent =

$$\left.\begin{array}{l} \langle B,J,R,S,M,C\rangle = \\ \langle C,B,J,R,S,M\rangle = \\ \langle M\ C\ B\ J\ R\ S\rangle = \\ \langle S\ M\ C\ B\ J\ R\rangle = \\ \langle R\ S\ M\ C\ B\ J\rangle = \\ \langle J\ R\ S\ M\ C\ B\rangle \end{array}\right\} \text{collapsable subset}$$

vs. unique when in one line

$6! = 720$

Every 6 collapses to one

$\frac{6!}{6}$ going to have 120

principle of dividing out the invariable factor

Imagine a basket of 5 flowers : 3 orchids $\{O_1, O_2, O_3\}$ and 2 chrysanthemum $\{X_1, X_2\}$

① How many ways to place them in 5 flower pots? $5! = 120$

This assures each orchid is "distinct", "distinguisable", "unique" and each chrysanthemum is unique too

How many ways to place them in 5 flower pots if each orchid is "not unique" "indistinguishable", "indistinct"

$$\left\{\begin{array}{l} O_1, O_2, O_3, X_1, X_2 \\ O_1, O_3, O_2, X_1, X_2 \\ O_2, O_1, O_3, X_1, X_2 \\ O_2, O_3, O_1, X_1, X_2 \\ O_3, O_1, O_2, X_1, X_2 \\ O_3, O_2, O_1, X_1, X_2 \end{array}\right\} OOOX_1X_2$$

120 $O_1, O_2, O_3, X_1, X_2$

$O_1, O_2, O_3, X_2, X_1$

______ $X_2, X_1$

______ $X_2, X_1$

______ $X_2, X_1$

______ $X_2, X_1$

______ $X_2, X_1$

3! # of ways you can arrange 3 orchids

$3! = 6$

set of 6 collapse into 1!

=> 1 only counts as one

Total # of ways $= \frac{5!}{3!}$

How many ways can we place if each orchid is indistinguishable and each chrysanthemum is indistinct

OOOXX $\frac{5!}{3!\,2!}$

Remember

$P(4 \text{ H in } 10 \text{ coin tosses}) = \frac{|A|}{|\Omega|} = \frac{\frac{10!}{4!\,6!}}{2^{10}} = \frac{\quad}{1029} \approx .205$

↓

4 Hs, 6 Ts

How many ways to order 10 flips? 10!

every head + tail is the same

B, J, R, S, C, M

How many ways to order them in 6 chairs? 6!

How many ways to order them in 3 chairs? ${}_6P_3$

Such that the order of these 3 doesn't matter

{B, J, R}

↑

collapse $\left\{\begin{array}{l} \langle B,R,J\rangle = \\ \langle R,B,J\rangle = \\ \langle R,J,B\rangle = \\ \langle J,B,R\rangle = \\ \langle J,R,B\rangle = \\ \langle B,J,R\rangle \end{array}\right\}$

$\frac{{}_6P_3}{{}_3P_3} = \frac{{}_6P_3}{3!}$

| | |
|---|---|
| JBS | BSR |
| JBR | BSM |
| JBM | BSC |
| JBC | BRM |
| JSR | BRC |
| JSM | BMC |
| JSC | SRM |
| JRM | SRC |
| JRC | SMC |
| JMC | RMC |

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How many ways to seat them in 4 chairs so order doesn't matter?

$$\frac{_6P_4}{4!} = \frac{6\cdot 5\cdot \cancel{4}\cdot \cancel{3}}{\cancel{4}\cdot \cancel{3}\cdot 2} = \frac{30}{2} = 15$$

| | | |
|---|---|---|
| JBSR | JSRM | BSRM |
| JBSM | JSRC | BSRC |
| JBSC | JSMC | BSMC |
| JBRM | JRMC | BRMC |
| JBRC | | SRMC |
| JBMC | | |

How many ways to sample K items out of a set of n w/o replacement such that order doesn't matter

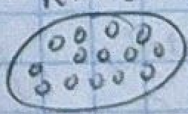
$$\binom{n}{k} := \frac{_nP_k}{_kP_k} = \frac{_nP_k}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!\,k!}$$

$$_nC_k :=$$

COMBINATIONS "CHOOSE" How many ways to choose

Combinatorial Identities

n balls



① $\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n$

④ $\binom{n}{0} = \frac{n!}{n!\,0!} = 1$

② $\binom{n}{n-1}$ $\frac{n!}{1!\,(n-1)!} = n$

⑤ $\binom{n}{n} = 1 = 1$

③ $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{\underbrace{(n-(n-k))!}_{K}\,(n-k)!}$

You seat 4 people randomly what is the probability Jane is seated?

B, ~~J~~, R, S, M, C

$P(A) = \frac{|A|}{|\Omega|}$ = order doesn't matter $\frac{\binom{5}{3}}{\binom{6}{4}} := \frac{\frac{5!}{3!\,2!}}{\frac{6!}{4!\,2!}} = \frac{2}{3}$

= order matters $\frac{4({}_5P_3)}{{}_6P_4} = \frac{4(5\cdot4\cdot3)}{6(5\cdot4\cdot3)} = \frac{2}{3}$

$\{J, \_, \_, \_\}$

| | J | 5 | 4 | 3 |
|---|---|---|---|---|
| Chair # | 1 | 2 | 3 | 4 |
| | _ | J | _ | _ |
| | _ | _ | J | _ |
| | _ | _ | _ | J |

${}_5P_3$

$\left(\frac{\frac{4({}_5P_3)}{{}_6P_4}}{4!}\right)$ how to choose from one another

$2^A := \{B : B \subseteq A\}$
$= \{B : B \subseteq A \text{ and } |B| = 0\} \cup$
$\{B : B \subseteq A \text{ and } |B| = 1\} \cup$
$\{B : B \subseteq A \text{ and } |B| = 2\} \cup$
$\vdots$
$\{B : B \subseteq A \text{ and } |B| = n\}$

mutually exclusive and collectively exhaustive

$2^A = \{ \underbrace{\{\}}_{\text{size } 0}\ \underbrace{\{1\}\{2\}\{3\}}_{\text{size } 1}\ \underbrace{\{1,2\}\{2,3\}\{1,3\}}_{\text{size } 2}\ \underbrace{\{1,2,3\}}_{\text{size } 3} \}$

mutually exclusive
no element in one set that is in another set $\emptyset \cap A = \emptyset$

$|A| = n$

$|2^A| = |\{B : B \subseteq A \text{ and } |B| = 0\}| + \dots + |\{B : B \subseteq A \text{ and } |B| = n\}|$

$$\sum_{i=0}^{n} |\{B : B \subseteq A \text{ and } |B| = i\}| = \boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

$(a+b)^2 = (a+b)(a+b) = \overbrace{a^2 + ab + ba + b^2}^{\text{4 items}} = a^2 + 2ab + b^2$

$(a+b)^3 = (a+b)(a+b)(a+b) = \underbrace{\qquad}_{\text{8 items}} = a^3 + 3a^2b + 3ab^2 + b^3$

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) \quad 16 \text{ terms} = a^4 b^0$$

$$2^4 = \sum_{i=0}^{4} \binom{4}{i} = \binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0}$$

$$(a+b)^n = \underbrace{\qquad}_{2^n \text{ terms}} = \sum_{i=0}^{n} \binom{n}{i} = \binom{n}{n} a^n b^0 + \binom{n}{n-1} a^{n-1} b^1 + \cdots$$

$$+ \binom{n}{2} a^2 b^{n-2} + \binom{n}{1} a^1 b^{n-1} + \binom{n}{0} a^0 b^n$$

$$= \boxed{\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}}$$

Binomial Theorem