9/6

Working Definition of Probability

$P: 2^{\Omega} \longrightarrow [0, 1]$

domain:
event space
all $A \subseteq \Omega$

range
1: certainty $P(\Omega) = 1$
0: improbability $P(\emptyset) = 0$

$$P(A) = \frac{|A|}{|\Omega|}$$

$A \subseteq \Omega$
$A \in 2^{\Omega}$

What's the probability of getting a sum of 3 if you roll 2 dice?

Step 1: Translate from English to obtain Ω
$\Omega = \{1, 2, 3, 4, 5, 6\}^2$
Step 2: Find $|\Omega|$
$|\Omega| = |\{1, \ldots, 6\}|^2 = 6^2 = 36$
Step 3: Translate from English to obtain A
$A = \{(1, 2), (2, 1)\}$

Step 4: Compute $|A|$
$|A| = 2$
Step 5: Divide

$$P(A) = \frac{|A|}{|\Omega|}$$

What's the prob. of getting 2 Heads on 4 coins tossed

1: $\Omega = \{\text{Heads}, \text{Tails}\}^2$

2: $|\Omega| = 16$

3: $A = \{HHTT, HTHT, HTTH, TTHH, THTH, THHT\}$

4: $|A| = 6$

$P(HHHH) = P(HHTT) \neq P(2H)$

COMPLEMENT RULE

Recall: $\Omega = A \cup A^c$ and $\{A, A^c\}$ are mutually exclusive
$|\Omega| = |A| + |A^c|$
$\Rightarrow |A| = |\Omega| - |A^c|$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - P(A^c)$$

$A = \{\geq 1 \; H\}$ $\quad A^c = \{< H = \text{zero } H\}$
$A^c = \{(T, T, T, T)\}$ $\quad P(A) = 1 - \frac{1}{16} = \frac{15}{16}$

Flip 10 coins. What's the prob. of 4 H?

① $P(A) = \frac{|A|}{|\Omega|} = \frac{\quad}{1024}$

$|\Omega| = 2^{10} = 1024$

$A = \{HHHHTTTTTT, \ldots$

We need more tools

$F = \{Jane, Mary, Susan\}$

How many ways to sit these women in 3 chairs?

Let's ilustrate

Seat #1	Seat #2	Seat #3	Outcome
J	M	S	$\{(J,M,S)\}$
	S	M	$\{(J,S,M)\}$
M	J	S	$\{(M,J,S)\}$
	S	J	$\{(M,S,J)\}$
S	J	M	$\{(S,J,M)\}$
	M	J	$\{(S,M,J)\}$

$|\Omega| = 6$

$\underset{Seat\#1}{\underline{3}} \cdot \underset{Seat\#2}{\underline{2}} \cdot \underset{Seat\#3}{\underline{1}} = 6$

Note

$\Omega \neq F^3$

$6 = |\Omega| \neq |F^3| = 2\,?$

$\Omega \subset F^3$

$$\boxed{\begin{array}{c}(J,J,J) \in F^3 \\ (J,J,J) \notin \Omega\end{array}}$$

Ω represents the set of F without replacements

β represents the set of F with replacements

\# ways to sample n objects without replacement.

$$\underset{1^{st}\text{ sample}}{\underline{n}} \cdot \underset{2^{nd}}{\underline{n-1}} \cdot \dots \underset{n-1\text{ sample}}{\underline{2}} \cdot \underset{n^{th}}{\underline{1}} = \prod_{i=1}^{n} i$$

$$n!$$

\# ways to sample n object with replacement.

$$\underset{1st}{\underline{n}} \cdot \underset{2nd}{\underline{n}} \cdot \underset{3rd}{\underline{n}} \dots$$

$$n^n$$

# of ways to seat 10 people in 3 chairs?

$$\underset{\text{seat}\#1}{\underline{10}} \cdot \underset{\#2}{\underline{9}} \cdot \underset{\#3}{\underline{8}} = \frac{10!}{7!}$$

# of ways to sample k objects from a set of n objects without replacements

$$= \frac{n!}{(n-k)!}$$

$$\underset{\#1}{\underline{n}} \cdot \underset{\#2}{\underline{n-1}} \cdots \underset{\#k}{\underline{n-(k+1)}} = \frac{n!}{(n-k)!}$$

$$\boxed{{}_nP_k := \frac{n!}{(n-k)!}}$$

${}_n$Permute${}_k$

Permutations

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

$$\boxed{0! := 1}$$

What's the prob. that Jane and Susan sit together?

P(J and S sit together)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\{JSM, MJS, SJM, MSJ\}|}{3!} = \frac{4}{6}$$

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3 couples (6 people)

Bob - Jane
Richard - Susan
Charles - Mary

They are seated into 6 chairs

✓

P(the couples sit next to each other)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 4 \cdot 2}{6!} = \frac{1}{15}$$

$$\underset{1}{\underline{6}} \quad \underset{2}{\underline{1}} \quad \underset{3}{\underline{4}} \quad \underset{4}{\underline{1}} \quad \underset{5}{\underline{2}} \quad \underset{6}{\underline{1}}$$

Alternative way

Loveseat # $\quad \underset{\underset{2}{1}}{\underline{\text{3 couples}}} \quad \underset{\underset{2}{2}}{\underline{\text{2 couples}}} \quad \underset{\underset{2}{3}}{\underline{\text{1 couple}}}$

$$\frac{(\cancel{3} \cdot \cancel{2} \cdot \cancel{1})\, \cancel{2^3}}{\underset{3}{\cancel{6}} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = \frac{1}{15}$$

What's the probability of alternating gender

P(alternating gender)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 3 \cdot 2 \cdot 2}{6!} = \frac{1}{10}$$

seat# $\quad \underset{1}{\underline{6}} \quad \underset{2}{\underline{3}} \quad \underset{3}{\underline{2}} \quad \underset{4}{\underline{2}} \quad \underset{5}{\underline{1}} \quad \underset{6}{\underline{1}}$

Alternative Way | This will Fail Later?

$A = A_{1st\ Boy} \cup A_{1st\ Girl}$

mutually Exclusive

$|A| = |A_{1st\ Boy}| + |A_{1st\ Girl}|$ divide by $|\Omega|$

$P(A) = P(A_{1st\ B}) + P(A_{1st\ G})$ Addition rule

$\underset{\#1}{\underline{3}}\ \underline{3}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1}$

$\underset{\#1}{\underline{3}}\ \underline{3}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1}$

$P(A) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 3 \cdot 2 \cdot 2}{6!} = \frac{1}{10}$

P(R-S sit together)

$P(A) = \frac{|A|}{|\Omega|} = \frac{}{6!}$

$\underbrace{\underline{1} \quad \underline{1}}$ $\underline{4}$ $\underline{3}$ $\underline{2}$ $\underline{1}$

seat #1 #2

love sit

$\underline{\qquad\quad} \; \_ \; \_ \; \_ \; \_$

$\underline{4} \; \underline{\qquad\quad} \; \_ \; \_ \; \_$

$\_ \; \_ \; \underline{\qquad\quad} \; \_ \; \_$

$\_ \; \_ \; \_ \; \underline{\qquad\quad} \; \_$

$\_ \; \_ \; \_ \; \_ \; \underline{\qquad\quad}$

Plus they can flip

$P(A) = \frac{|A|}{|\Omega|} = \frac{4!\ 2\ 5}{6!} = \frac{1}{3}$

# of ways to sample 3 marbles from a bag of 100 without replacement

$${}_{100}P_3$$

Same with replacement

$$\frac{{}_{100}P_3}{100^3} \approx 0.97$$

If $n \to \infty$ what does the $\dfrac{\text{\# ways to sample without replacement}}{\text{with replacement}}$ look like?

$$\lim \frac{{}_nP_k}{n^k} = \lim \frac{(n)(n-1)\cdots(n-k+1)}{(n)(n)\cdots(n)}$$

$$\lim \frac{n}{n} \lim \frac{n-1}{n} \cdots \lim \frac{n-k+1}{n} = 1$$