Lecture 3.

Definition of probability

$p: 2^{\Omega} \to [0,1]$

even space "degree"

the power set of value 1 is $p(\Omega)=1$

outcomespace value 0 is impossibility $p(\phi)=0$

$A \subseteq \Omega \Rightarrow A \in 2^{\Omega}$

$\boxed{Ex}$ What is the probability of getting a sum $=3$ on two die roll?

Step 1: Translate from English $\to \Omega$, $\Omega = \{1,2 \cdots 6\} \times \{1,2 \cdots 6\}$, $\Omega = 36$

Step 2: Count $|\Omega|$

Step 3: Translate from English $\to A$, $A = \{\langle 1,2 \rangle, \langle 2,1 \rangle\}$

Step 4: Compute $|A|$, $|A| = 2$

Step 5: Divide $P(A) = \frac{|A|}{|\Omega|} = \frac{2}{36}$

Ex What is prob of getting 2 heads on 4 coins flip?

$p(A) = \frac{|A|}{|\Omega|} = \frac{6}{16}$ $\Omega = \{H,T\}^4$, $|\Omega| = |\{H,T\}|^4 = 2^4 = 16$

$A = \{\langle H,H,T,T\rangle, \langle T,T,H,H\rangle, \langle T,H,T,H\rangle$
$\langle H,T,H,T\rangle, \langle H,T,T,H\rangle, \langle T,H,H,T\rangle\}$

Q: $p(H,H,H,H) \stackrel{?}{=} P(H,H,T,T)$

Yes because LhS is one outcome, RhS is one outcome. All outcome are equal propabitities. Noticed that above probability are not the same as $P(2H, 2T)$

Ex. Prob of at least 1 H on 4 tosses?

$P(A) = \frac{|A|}{|\Omega|} = \frac{15}{16}$ Recall $|\Omega| = |A| + |A^c|$

$|A| = |\Omega| - |A^c| = 16 - 1$

$|A^c| = \{\text{Not at least 1H} \sim \geq 1H\} = \{T,T,T,T\}$

Therefore Complement Rule. can help us

$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - P(A^c)$

Ex $F = \{\text{Jane, Mary, Susan}\}$. There are 3 chairs. How many ways to seat those 3 seats?

Total of ways: $\underset{\text{Seat \#1}}{\underline{3}} \cdot \underset{\text{Seat \#2}}{\underline{2}} \cdot \underset{\text{seat \#3}}{\underline{1}} = 3 \times 2 \times 1 = 6$ ways $= |\Omega|$

Noted that $\Omega \subseteq F^3$. $|F^3| = 27$ and $|\Omega| \neq F^3$

$|\Omega|$: without replacement

F^3: replacement/ with repeatation

Ex. {Jane, Jane, Jane}

Ex Sample n items without replacement. How many possible outcome?

$\underset{\text{1st Sample}}{\underline{n}} \cdot \underset{\text{2nd Sample}}{\underline{n-1}} \cdots \underset{\text{n-1th Sample}}{\underline{2}} \cdot \underset{\text{nth Sample}}{\underline{1}} = n! = \prod_{i=1}^{n} i$

Ex Sample n items with replacement. How many possible outcome?

$\underset{\text{1st sample}}{\underline{n}} \cdot \underset{\text{2nd Sample}}{\underline{n}} \cdots \underset{\text{n-1th Sample}}{\underline{n}} \cdot \underset{\text{nth Sample}}{\underline{n}} = n^n > n!$

Ex. 5 people 3 chairs. How many ways? (without Replacement)

$\frac{5}{1st} \cdot \frac{4}{2^{nd}} \cdot \frac{3}{3^{rd}} =$ 60 ways.

Ex. Sample n items k times without replacement. ?

$\frac{n}{1st} \cdot \frac{n-1}{2^{nd}} \frac{n-2}{3^{rd}} \cdots \frac{n-k+1}{k^{th}\text{ sample}} = \frac{n!}{(n-k)!}$

Permutation (unique ordering)

$_nP_k = \frac{n!}{(n-k!)}$ (without replacement)

Ex. 3 couples, Bob-Jane, Richard-Susan, Charles-Mary

$P(\text{Every couple sit together}) = \frac{6}{\#1} \cdot \frac{1}{\#2} \frac{4}{\#3} \frac{1}{\#4} \frac{2}{\#5} \frac{1}{\#6}$

Alterative: $\frac{3}{\#1\,\&\,\#2} \cdot \frac{2}{\#3\,\&\,\#4} \frac{1}{\#5\,\&\,\#6} = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1$

2 2 2

Ex. 6 people alternative gender sit together.

$P(A) = \frac{6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1}{\#1\ \#2\ \#3\ \#4\ \#5\ \#6}$

Alternative: Male First $\frac{3}{\#1} \cdot \frac{3}{\#2} \cdot \frac{2}{\#3} \frac{2}{\#4} \frac{1}{\#5} \frac{1}{\#6}$ + Female First $\frac{3}{\#} \frac{3}{} \frac{2}{} \frac{2}{} \frac{1}{} \frac{1}{}$

$P(A) = P(A_{\text{male first}}) + P(A_{\text{female first}})$ ⇒ Addition between two probabilites Works because the two possibilities are disjoint.

Ex. P(Richard & Susan) sit together

$\frac{R-S}{\#1\#2} \frac{4}{\#3} \frac{3}{\#4} \frac{2}{\#5} \frac{1}{\#6} \times 5 \times 2$

Ex. $n = 100$ balls. Sample $k=3$ without replacement $\Rightarrow {}_{100}P_3$

with replacement $\Rightarrow 100^3$ ≈ 0.9706

$\lim_{n \to \infty} \frac{_nP_k}{n^k}$

If n is large, sample without replacement