Math 241 Lecture 19

Nov. 22nd

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\quad$ $supp[Z] = \mathbb{R}$

$X = \mu + \sigma Z \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ $\quad$ $supp[X] = \mathbb{R}$

$Z = \frac{X-\mu}{\sigma}$

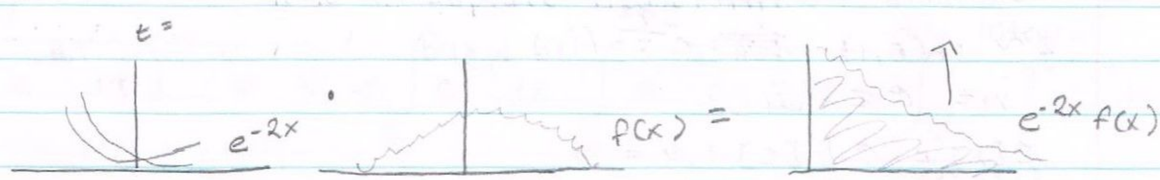
$P(Z \in (-1,1)) \approx 68\%$

$P(Z \in (-2,2)) \approx 95\%$

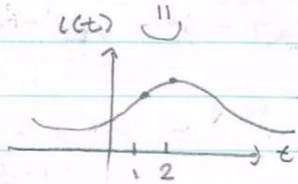
$P(Z \in (-3,3)) \approx 99.7\%$

$L = B[f]$

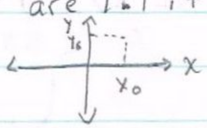
$$L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$$



Thm: Let (t) and f(x) are 1:1 if L(t) exists



f is 1:1



Define: the moment generating function (mgf)

$$M_X(t) = L(-t) = \int_{\mathbb{R}} e^{tx} f(x)\,dx = E[e^{tX}]$$

(for cont)

For discrete, $M_X(t) = E[e^{tX}] = \sum_{x \in supp[X]} e^{tx} p(x)$

① If $M_X(t) = M_Y(t)$

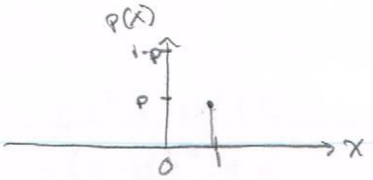
$\Rightarrow X \overset{d}{=} Y$

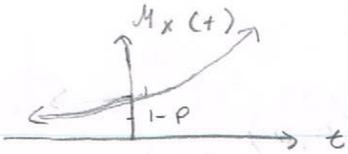
$\Rightarrow f_X(x) = f_Y(x)$ for cont

$\Rightarrow P_X(x) = P_Y(x)$ for discrete

$X \sim Bern(p) = p^x (1-p)^{1-x}$

$M_X(t) = E[e^{tX}] = \sum_{x \in [0,1]} e^{tx} p^x (1-p)^{1-x} = 1 - p + e^t p$





$X \sim \text{Binomial}(n, p)$

$E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$

$\uparrow$

17^{th} moment

$M_X(t) = E[e^{tX}]$ assume X cont.

$M_X'(t) = \frac{d}{dt}\left[E[e^{tX}]\right] = \frac{d}{dt}\left[\int_{\mathbb{R}} e^{tx} f(x)\,dx\right] =$

$= \int_{\mathbb{R}} \frac{d}{dt}\left[e^{tx} f(x)\right] dx = \int_{\mathbb{R}} x e^{tx} f(x)\,dx = E[X e^{tX}]$

$M_X'(0) = E[X]$

$M_X''(t) = E\left[\frac{d}{dt}(X e^{tX})\right] = E[X^2 e^{tX}]$

$M_X''(0) = E[X^2]$

III $M_X^{(k)}(0) = E[X^k]$ k^{th} moment

$Y = aX + c$

$M_Y(t) = E[e^{tY}]$

$= E[e^{t(aX+c)}]$

$= E[e^{taX + tc}]$

$= e^{tc} E[e^{taX}]$ if $t' = ta$

$= e^{tc} \quad E[e^{t'X}]$

$= e^{tc} M_X(t') = \boxed{e^{tc} M_X(at)}$ I

X_1, X_2 independent

$Y = X_1 + X_2$

$M_Y(t) = E[e^{tY}] = E[e^{tX_1} e^{tX_2}] = E[e^{tX_1}]\, E[e^{tX_2}]$

$= M_{X_1}(t) M_{X_2}(t)$ II

$= M_Y(t)$

$X \sim$ Binomial (n, p)

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p x)(1-p)^{2A-x}$$

$$(a+b)^n = \sum_{i=0}^{m} \binom{m}{i} a^i$$

$X_1, \dots X_n \overset{iid}{\sim}$ Bernoulli(p)

$T = X_1 + \dots X_n \sim ?$

$$M_T(t) = M_X(t)^n = (1-p+pe^t) \Rightarrow T \sim \text{binom}$$

$E \sim Exp(\lambda) := \lambda e^{-\lambda x}$

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int e^{(t-\lambda)x} dx$$

$$= \lambda \left[\frac{1}{t-\lambda} e^{(t-\lambda)x}\right]_0^\infty = \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x}\right]_0^\infty$$

$$= \begin{cases} \frac{\lambda}{\lambda - t} & \text{if } t < \lambda \\ \text{dne} & \text{o/t (otherwise)} \end{cases}$$

$X \sim Exp(\lambda)$

$Y = aX$ such that $a \in (0, \infty)$

$Y = ?$

$$M_Y(t) \underset{\text{III}}{=} M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{1/a}{1/a} = \frac{\lambda/a}{\lambda/a - t} = \frac{\lambda'}{\lambda' - t}$$

$$\Rightarrow Y \sim Exp(\lambda') = Exp\left(\frac{\lambda}{a}\right)$$

I

$$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2} + tx} dx$$

$ax^2 + bx + c$
$= (x-d)^2 + e$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-1/2} (x^2 - 2tx)\, dx =$$

$$(x-t)^2 = x^2 - 2tx + t^2 \Rightarrow x^2 - 2tx = (x-t)^2 - t^2$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((x-t)^2 - t^2)}\, dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}}\, dx$$

$$= e^{\frac{t^2}{2}} \int_{\mathbb{R}} \underbrace{\underbrace{\frac{1}{\sqrt{2\pi}} e^{-1/2(x-t)^2}}_{\text{PDF for } Y \sim \mathcal{N}(t,1)}}_{=1} dx = e^{\frac{t^2}{2}}$$

WTS $E[X] = 0$ $\quad M_X'(0) = t e^{t^2/2} \Big|_0 = 0$ ✓

WTS $Var[X] = 1$ $\quad Var[X] = E[X^2] - \mu^2 = E[X^2]$

$$M_X''(0) = t^2 e^{t^2/2} + e^{t^2/2} \Big|_0 = 1$$