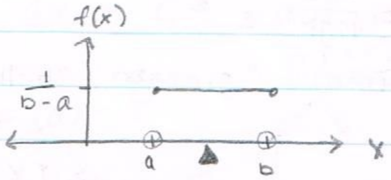
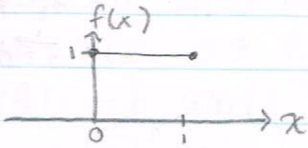
Math 241 Lecture 18 — Nov 20th



$E[x]$

$x \sim \text{Uniform}(a,b)$

$$1 = \int_{\text{supp}[x]} f(x)\,dx = \int_a^b \frac{1}{b-a}\cdot dx = 1$$



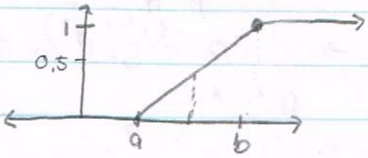
$x \sim U(0,1)$

$\text{supp}[x] = (0,1)$

$$E[X] = \int_{\text{supp}[x]} x\, f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

$\text{Median}[x] = \text{Quantile}[x, \tfrac{1}{2}] = \text{argmin}\{F(x) \geq \tfrac{1}{2}\}$

for discrete random variables



$\text{Quantile}[x,p] = F^{-1}(p)$ CDF inverse

$= \text{argmin}\{F(x) \geq p\}$

$$p = F(x) = \frac{x-a}{b-a} \Rightarrow x = p(b-a)+a$$

$$\text{Median}[x] = F^{-1}(\tfrac{1}{2})(b-a) + a = \tfrac{1}{2}b - \tfrac{1}{2}a + \tfrac{2}{2}a$$

$$= \tfrac{1}{2}b + \tfrac{1}{2}a = \frac{a+b}{2}$$

$$\text{Var}[x] = E[(X-M)^2] = E[x^2] - \underbrace{M^2}_{\left(\frac{a+b}{2}\right)^2}$$

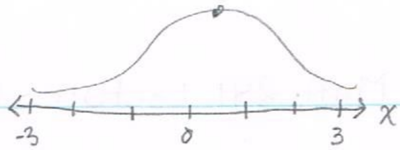
$$E[x^2] = \int_{\text{supp}[x]} x^2 f(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)} \quad \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4} = 4b^2+4ab+a^2-3a^2-6ab-3b^2 \,/12$$

$$= \frac{(b-a)^2}{12}$$

$$SE[x] = \frac{b-a}{\sqrt{12}}$$

$$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



"normal", "graussian", "belle"

① $f(x) \geq 0$ ✓

② $\int_{supp[X]} f(x)\,dx = 1$ $\quad$ $supp[X] = \mathbb{R}$

WTS $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 1$ $\quad$ let $u = \frac{1}{\sqrt{2}} x \Rightarrow \frac{x^2}{2} = u^2$

$du = \frac{1}{\sqrt{2}} dx \Rightarrow dx = \sqrt{2}\,du$

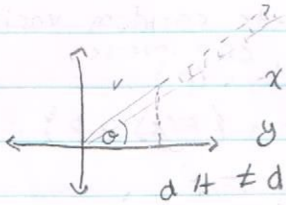
$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi} = \int_{u_0}^{u_f} e^{-u^2}\cancel{\sqrt{2}}\,du = \sqrt{\cancel{2}\pi} = \sqrt{\pi}$$

Area Integral

$$\left(\int_{u_0}^{u_f} e^{-u^2}\,dx\right)^2 = \pi \Rightarrow \int_{\mathbb{R}} e^{-x^2}\,dx \int_{\mathbb{R}} e^{-y^2}\,dy = \pi = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2} e^{-y^2}\,dx\,dy = \pi$$

Not responsible for

$$\Rightarrow \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$$



$x = r\cos\theta$

$y = r\sin\theta$

$dA \neq dr\,d\theta$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_{u_0}^{u_f} e^{-u} r \frac{1}{2r}\,du\,d\theta = \frac{1}{2}\iint e^{-u}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} [-e^{-u}]_{u_0}^{u_f}\,d\theta = \frac{1}{2}\int_0^{2\pi} \left[-e^{-r^2}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{2\pi} -(0-1)\,d\theta = \frac{1}{2}\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}[\theta]_0^{2\pi}$$

$$= \pi \checkmark$$

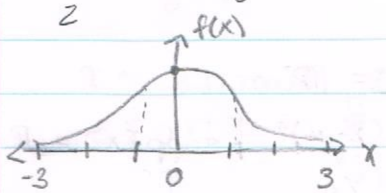
$$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x) \text{ (valid)}$$

$$E[Z] = \int x f(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{u_0}^{u_f} x e^{-u} \frac{1}{x}\,du$$

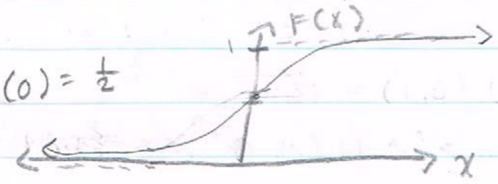
$$= \frac{1}{\sqrt{2\pi}} [-e^{-u}]_{u_0}^{u_f} = -\frac{1}{\sqrt{2\pi}} \left[e^{-x^2/2}\right]_{-\infty}^{\infty} = -\frac{1}{\sqrt{2\pi}}(0-0) = 0$$

$$Var[Z] = E[Z^2] - \cancel{\mu^2}^{\,0} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \ldots = 1$$

$$F_z(x) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + C$$ not possible in closed form



$F(0) = \frac{1}{2}$



CC ✗ $P(z \in [-1,1]) \approx .68$
$P(z \in [-2,2]) \approx .95$
$P(z \in [-3,3]) \approx .997$

Empirical Rule
3σ Rule
68-95-99.7 Rule

"standard normal"

$z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x)$ valid

$x = \sigma z + \mu$

$E[X] = \sigma E[z] + \mu = \mu$

$Var[X] = \sigma^2 \underbrace{Var[z]}_{1} = \sigma^2$ $\quad SE[X] = \sigma$

$F(x) = P(X \le x) = P(\sigma z + \mu \le x) = P\left(z \le \frac{x-\mu}{\sigma}\right)$
$= F_z\left(\frac{x-\mu}{\sigma}\right)$

$f_X(x) = F_X'(x) = \frac{d}{dx}\left[F_z\left(\frac{x-\mu}{\sigma}\right)\right] = \frac{1}{\sigma} f_z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{2\pi} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$

$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$

Assume male height is normally distr w/ mean 70" (= 5'10") and std error 4". What is the probability a random male is taller than 78" (= 6'6")?

$x \sim N(70", 4"^2)$

$P(X > 78) = P\left(\frac{X-70}{4}\right) > \left(\frac{78-70}{4}\right) = P(z > 2)$