Lecture 19

11/22/17

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

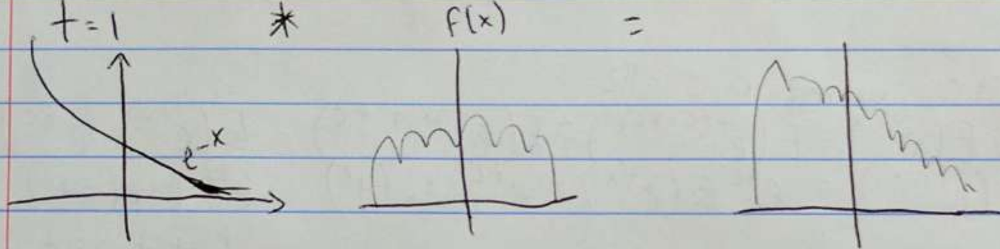
$X = \mu + \sigma Z \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$\mathrm{Supp}(Z) = \mathrm{Supp}(X) = \mathbb{R}$

$\to Z = \frac{X-\mu}{\sigma}$

Z scores

$L(t) = B(f) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ - Bilateral Laplace Transformation



<u>Thm</u> $L(t)$ and $f(x)$ are 1:1 (if $L(t)$ exists).

$M_X(t) := E(e^{tX}) = \int_{\mathbb{R}} e^{tx} f(x)\,dx$ if X is continuous

$= \sum_{x \in \mathrm{supp}(x)} e^{tx} p(x)$ if X is discrete

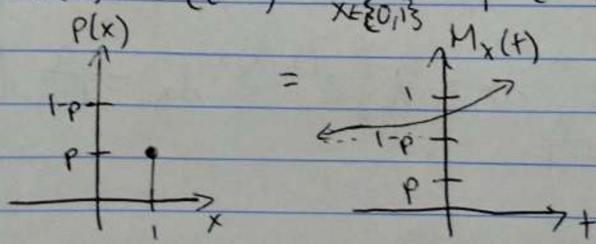
(F) $M_X(t) = M_Y(t) \Leftrightarrow X \overset{d}{=} Y$

$\Leftarrow$ uses properties we know

$\Rightarrow$ theorem: equation of Bilateral Laplace Transformation

$X \sim \mathrm{Bern}(p)$

$M_X(t) = E(e^{tX}) = \sum_{x \in \{0,1\}} e^{tx} p^x (1-p)^{1-x} = 1-p+pe^t$



Sometimes yes
Sometimes no

II $M_X(t) = E(e^{tX})$

$$M'_X(t) = \frac{d}{dt}\left(E(e^{tx})\right) = \frac{d}{dt}\left(\int_{\mathbb{R}} e^{tx} f(x)\,dx\right) \overset{?}{=} \int \frac{d}{dt}\left(e^{tx} f(x)\,dx\right) dx$$

$$= \int_{\mathbb{R}} x e^{tx} f(x)\,dx = E(Xe^{tX})$$

$M'_X(0) = E(X)$ - First moment

$M''_X(t) = E(X^2 e^{tX})$

$M''_X(0) = E(X^2)$ - Second moment

$\vdots$

$M_X^{(k)}(0) = E(X^k)$ - k^{th} moment

III $Y = aX + c$, where $a, c \in \mathbb{R}$

$$M_Y(t) = E(e^{tY}) = E(e^{t(aX+c)}) = E(e^{taX + tc}) = E(e^{taX} \cdot e^{tc})$$

$$= e^{tc} E(e^{taX}) = e^{tc} E(e^{t'X}) = e^{tc} M_X(t') = e^{tc} M_X(at)$$

Let $t' = at$

IV $Y = X_1 + X_2$ such that X_1, X_2 are independent

$$M_Y(t) = E(e^{tY}) = E(e^{t(X_1+X_2)}) = E(e^{tX_1} e^{tX_2}) = E(e^{tX_1}) E(e^{tX_2})$$

$$= M_{X_1}(t) M_{X_2}(t) \underset{\text{if iid}}{=} (M_X(t))^2$$

$X \sim \text{Binomial}(n, p)$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} \overbrace{e^{tx}}^{=(e^t)^x} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= (1 - p + pe^t)$$

$X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p)$

$T = X_1 + \dots + X_n \sim \text{Binomial}$

$$M_T(t) \underset{IV}{=} (M_X(t))^n = (1 - p + pe^t)^n \underset{I}{\Rightarrow} T \sim \text{Binomial}(n, p)$$

$X \sim \text{Exp}(\lambda)$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda} \left(e^{(t-\lambda)x}\right)\Big|_0^\infty$$

If $\lambda > t \Rightarrow \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{t-\lambda}$

$X \sim \text{Exp}(\lambda)$

$Y = aX$, $a > 0$ (scaling)

$$M_Y(t) \underset{III}{=} M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{1/a}{1/a} = \frac{\lambda/a}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim \text{Exp}(\lambda') \underset{I}{=} \text{Exp}\left(\frac{\lambda}{a}\right)$$

$Z \sim N(0,1)$

$$M_Z(t) = E(e^{tZ}) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + tx} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx$$

$$ax^2 + bx + c = (x+d)^2 + e \qquad (x-t)^2 = x^2 - 2tx + t^2$$

$$ax^2 + bx = (x+d)^2 + e \qquad x^2 - 2tx = (x-t)^2 - t^2$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((x-t)^2 - t^2)} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}} dx = e^{\frac{t^2}{2}} \underbrace{\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx}_{\text{PDF of } N(t,1)}}_{1} = e^{\frac{t^2}{2}}$$

Verify $E(Z) = 0$

$$M_Z'(0) = t e^{\frac{t^2}{2}} \Big|_0 = 0$$

Verify $SE(Z) = 1 \Rightarrow Var(Z) = 1$

$E(Z^2) = 1$

$$M_Z''(0) = e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}} \Big|_0 = 1$$