Lec 19 Math 241 11/16/17

Previously, $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\quad Supp(Z) = \mathbb{R}$ "Standard normal"

$X = \mu + \sigma Z \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ "normal"

$\mu \in \mathbb{R}$, $\sigma^2 \in (0,\infty)$

$P(Z \in [-1,1]) = 0.68$, $P(Z \in [-2,2]) = 0.95$, $P(Z \in [-3,3]) = 0.997$

Mystery as to why this r.v. is so important... this is our mission

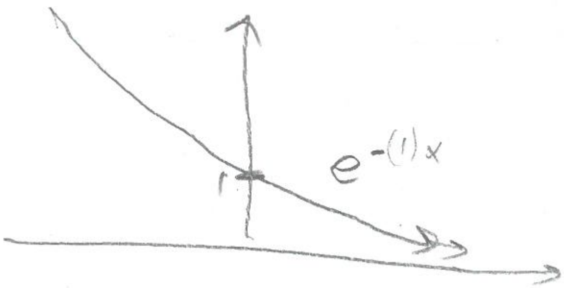
NEW TOPIC

Let $L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\, dx$

Bilateral Laplace Transform of f

What does this look like?

$L(1) = \int e^{-(1)x} f(x)\, dx$



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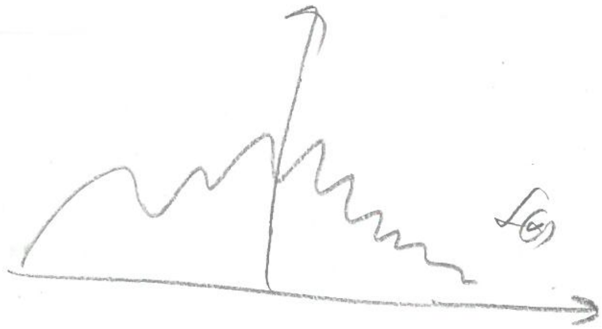

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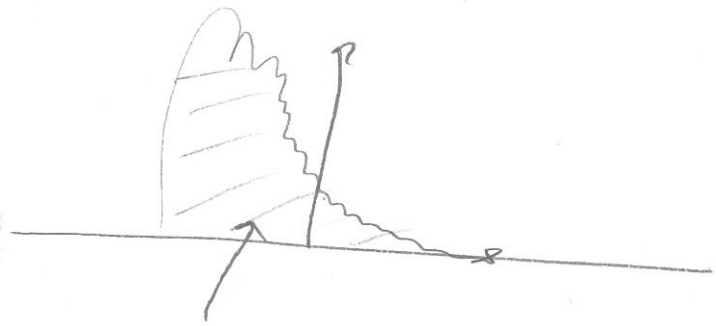
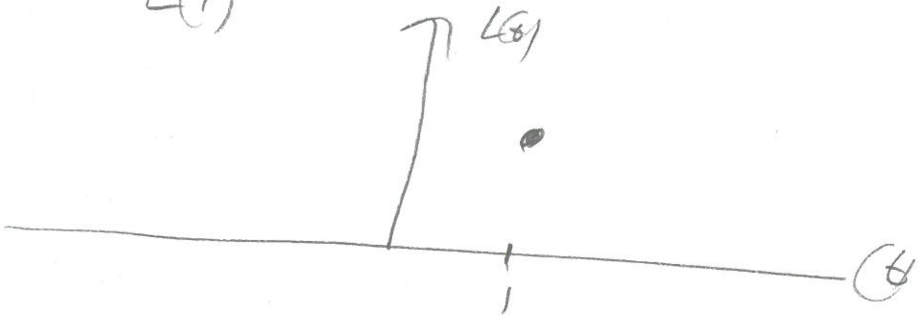


$L(1) =$



Do this for all values of t..

Thm: if $L(t)$ exists... $L(t)$ & $f(x)$ are 1:1.

$L(t) \Rightarrow f(x)$, $f(x) \Rightarrow L(t)$

Note: if $f(x)$ is PDF then

since $E g(x) = \int g(x) f(x)\, dx$

$$\int_{\mathbb{R}} e^{-tx} f(x)\, dx = E\left[e^{tX}\right]$$

define $\underbrace{M_X(t)}$ $:= E[e^{tX}] = \int_{\text{supp}(X)} e^{tx} f(x)\,dx$ cont.

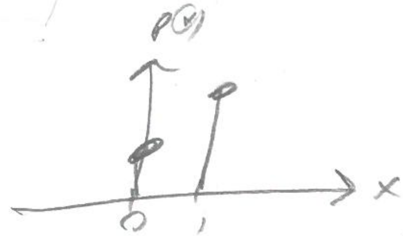
$= \sum_{x \in \text{supp}(X)} e^{tx} p(x)$ discrete.

moment generating function (MGF) of r.v. X

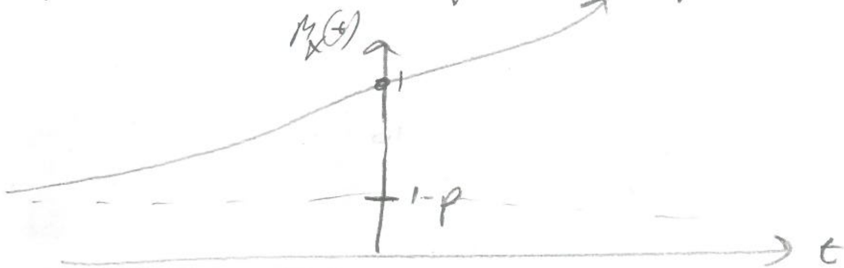
$\boxed{\text{I} \quad M_X(t) \text{ is 1:1 with } f(x) \text{ or } p(x)}$

identically distr. r.v.'s have same mgf!

e.g. $X \sim \text{Bern}(p) := p^x (1-p)^{1-x}$



$M_X(t) = E[e^{tX}] = e^{t(0)} p(0) + e^{t(1)} p(1) = 1-p + pe^t$



are the same!!

Why imply each other..

genotype vs phenotype

$\Rightarrow \boxed{M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y}$

Why is $M_X(t)$ useful..

Consider $X \sim \text{Binomial}(4, p)$ $\quad E[X^{17}] = \underbrace{\sum_{x=0}^{4} x^{17} \binom{4}{x} p^x (1-p)^{4-x}}_{\text{not possible to figure out...}}$

Recall $f(x)$ when $x \simeq c$ can be approximated by...

$f(x) \quad f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$ $\quad$ 3rd degree approx.

$= \sum_{i=0}^{\infty} \frac{f^{i}(c)}{i!}(x-c)^i$

$$M_X(t) = E[e^{tX}]$$

$$M_X'(t) = \frac{d}{dt} E(e^{tX})$$

$$= E\left(\frac{d}{dt} e^{tX}\right)$$

exchanging diff., int requires more advanced math. Assume it is okay here.

$$= E(Xe^{tX})$$

$$M_X'(0) = E(Xe^{0X}) = E(X)$$

$$M_X''(t) = E[X^2 e^{tX}]$$

$$M_X''(0) = E(X^2 e^{0X}) = E(X^2)$$

$$\vdots$$

II $$\boxed{M_X^{(k)}(0) = E[X^k]}$$

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Other cool properties

$Y = aX + c$

III

$M_Y(t) = M_{aX+c}(t) = E[e^{t(aX+c)}] = E[e^{atX} e^{tc}] = e^{tc} E[e^{t(aX)}] = e^{tc} M_X(at)$

$Y = X_1 + X_2$ and X_1, X_2 indep

IV

$M_Y(t) = M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}] = E[e^{tX_1} e^{tX_2}] = E[e^{tX_1}] E[e^{tX_2}] = M_{X_1}(t) M_{X_2}(t)$

the mgf of a sum of indep. r.v.'s is the product of the mgfs of the constituent r.v.'s

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$X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p)$

$T = X_1 + \dots + X_n \sim \text{Binom}(n, p)$ what is mgf of binomial?

$M_T(t) = M_{X_1+\dots+X_n}(t) = M_{X_1}(t) \cdot \ldots \cdot M_{X_n}(t) = \overset{\text{why? I}}{(M_X(t))^n} = \boxed{(1-p+pe^t)^n}$

$X \sim \text{Bern}(p)$

$M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1} p = p \sum e^{tx}(1-p)^{x-1} = \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t)^x (1-p)^x = \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t(1-p))^x$

$= \frac{p}{1-p} \left(\sum_{x=0}^{\infty} (e^t(1-p))^x - 1 \right)$

if $e^t(1-p) < 1 \Rightarrow$

$\Rightarrow e^t < \frac{1}{1-p}$

$\Rightarrow t < \ln\left(\frac{1}{1-p}\right)$

$= \frac{p}{1-p} \left(\frac{1}{1-e^t(1-p)} - 1 \right) = \frac{p}{\cancel{1-p}} \frac{e^t \cancel{(1-p)}}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)}$ II $t < \ln\left(\frac{1}{1-p}\right)$

TOO HARD

$X \sim Exp(\lambda)$

$$m_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty = \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{\lambda - t}$$

if $t - \lambda < 0 \Rightarrow t < \lambda$

$Y = a X$

$$m_Y(t) = m_{aX}(t) = m_X(at) = \frac{\lambda}{\lambda - at} = \frac{\lambda' a}{\lambda' a - ta} = \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim Exp\left(\overset{\lambda'}{\overset{\shortparallel}{\frac{\lambda}{a}}}\right)$$

$\lambda' = \frac{\lambda}{a}$

$X \sim Deg(c) \Rightarrow m_X(t) = E[e^{tX}] = e^{tc}$

LLN

$Z \sim N(0,1)$

$$m_Z(t) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2 + tx} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{1}{2}t^2} dx$$

$$-\frac{1}{2}x^2 + tx = -\frac{1}{2}(x^2 - 2tx) = -\frac{1}{2}\left((x-t)^2 - t^2\right) = -\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2$$

complete the square

$$e^{\frac{t^2}{2}} \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx}_{1} = e^{\frac{t^2}{2}}$$

$X \sim N(t,1) \Rightarrow f(x) =$

$X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma Z$

$$\Rightarrow m_X(t) = e^{t\mu} e^{\frac{(\sigma t)^2}{2}} = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

Prove:

$$SE(Z) = 1 \Rightarrow SE(Z) = \sqrt{Var(Z)} = \sqrt{E(Z^2) - (m_Z'(0))^2} = \sqrt{E(Z^2)} = \sqrt{m_Z''(0)} = 1$$

$m_Z'(t) = t e^{t^2/2}$ $\quad m_Z'(0) = 0$

$m_Z''(t) = e^{t^2/2} + t^2 e^{t^2/2}$, $\quad m_Z''(0) = 1$