MAT 241 - DAY BEFORE THANKSGIVING - NOV 22nd - Lecture 19

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\quad$ $\text{Supp}[Z] = \mathbb{R}$

$X = \mu + \sigma Z \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $\text{supp}[X] = \mathbb{R}$

$\Downarrow$

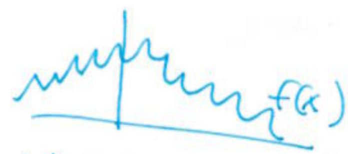
$Z = \frac{X-\mu}{\sigma}$

Z-score

$P(Z \in [-1,1]) = 68\%$

$P(Z \in [-2,2]) = 95\%$

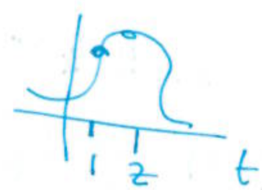
$P(Z \in [-3,3]) = 99.7\%$



$L = B[f]$

$L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ $\quad$ Bilateral Laplace Theorem $\Downarrow$

Thm $L(t)$ and $f(x)$ are 1-1 if $L(t)$ exists.



Define: the moment generating function (mgf)

$M_X(t) = L(-t) = \int_{\mathbb{R}} e^{tx} f(x)\,dx = E[e^{tX}]$

For discrete,

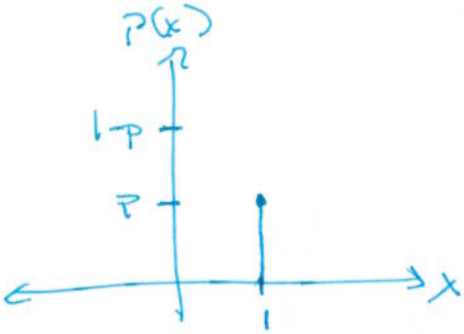
$M_X(t) = E[e^{tX}] = \sum_{x \in \text{supp}[X]} e^{tx} p(x)$

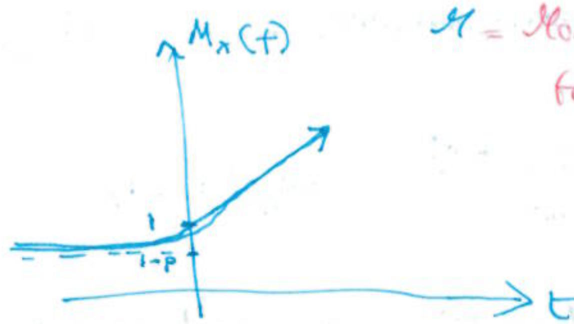
$\oplus$ if $M_X(t) = M_Y(t)$

$\Rightarrow X \overset{d}{=} Y$

$\Rightarrow f_X(x) = f_Y(x)$ for continuous. cont.

$\Rightarrow p_X(x) = p_Y(x)$ for discrete





M = Moment generating function.

Moments

$X \sim \text{Binomial}(n, p)$

$$E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$$

17th moment

Assume X is cont.

$$M_X(t) = E[e^{tX}]$$

$$M_X'(t) = \frac{d}{dt}\left[E[e^{tX}]\right] = \frac{d}{dt}\left[\int_{\mathbb{R}} e^{tx} f(x)\,dx\right] = \int_{\mathbb{R}} \frac{d}{dt}\left[e^{tx} f(x)\right] dx$$

$$= \int x e^{tx} f(x)\,dx = E[Xe^{tX}]$$

$$M_X'(0) = E[X]$$

$$M_X''(t) = E\left[\frac{d}{dx}(X e^{tX})\right] = E[X^2 e^{tX}]$$

$$M_X''(0) = E[X^2] \quad \dots$$

$$\boxed{M_X^{(k)}(0) = E[X^k] \quad k^{\text{th}} \text{ moment} \quad \text{II}}$$

$Y = aX + c$

$M_Y(t) = E[e^{tY}] = E[e^{t(aX+c)}] = E[e^{taX+tc}]$

$= E[e^{taX} e^{tc}]$

$= e^{tc} E[e^{taX}] =$ if $t' = ta$

$= e^{tc} E[e^{t'X}]$

$= e^{tc} M_X(t') = \boxed{e^{tc} M_X(at)}$ III

X_1, X_2 indedependent

$Y = X_1 + X_2$

$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}]$

$= E[e^{tX_1} e^{tX_2}]$

$= E[e^{tX_1}] E[e^{tX_2}]$

$= \boxed{M_{X_1}(t) M_{X_2}(t) \quad \text{IV}}$

if iid

$= (M_X(t))^2$

Binomial theorem

Recall

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

$X \sim$ Binomial(n, p) ~~Not needed Don't need to know~~

$M_X(t) = \sum_{x=0}^{n} \underbrace{e^{tx}}_{=(e^t)^x} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$

$= (1-p+pe^t)^n$

Prove

$X_1, \dots, X_n \overset{iid}{\sim} Bern(p)$

$T = X_1 + \dots + X_n \sim ?$ Binomial (n, p)

$M_T(t) = (M_X(t))^n = (1 - p + pe^t)^n \Rightarrow T \sim$ Binomial (n, p)

IV from Bern I

$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \lambda \frac{1}{t-\lambda} e^{(t-\lambda)x} \Big]_0^\infty$

$(t-\lambda) < 0$ only when $(t-\lambda) < 0$

$= \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x} \right]_0^\infty = \frac{-\lambda}{t-\lambda} \quad \forall\, t-\lambda < 0$

$$= \begin{cases} \frac{\lambda}{\lambda - t} & \text{IF } t < \lambda \\ \text{dne} & \text{otherwise.} \end{cases}$$

$X \sim Exp(\lambda)$

$Y = aX$ st $a \in (0, \infty)$

$Y \sim ?$

let $\lambda' = \frac{\lambda}{a}$

$M_Y(t) = M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t}$

III

$\Rightarrow Y \sim Exp(\lambda') = Exp\left(\frac{\lambda}{a}\right)$

II

$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2} - tx\right)} dx$$

Remember

ax^2+bx+c

$=(x-d)^2+e$

$(x-t)^2 = x^2 - 2tx + t^2$

$\Rightarrow x^2 - 2tx = (x-t)^2 - t^2$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((x-t)^2 - t^2)} dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}} dx = e^{\frac{t^2}{2}} \underbrace{\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx}_{\text{PDF for } V \sim N(t,1)}}_{1} = e^{\frac{t^2}{2}}$$

Want to show $E[X] = 0$

$$M_X'(0) = te^{t^2/2} \Big|_0 = 0$$

WTS $Var[X] = 1$ $\quad Var[X] = E[X^2] - \cancel{\mu^2} = E[X^2]$

$$M_X''(0) = t^2 e^{t^2/2} + e^{t^2/2} \Big|_0 = 1$$