Working Def of probability

$P: 2^{\Omega} \to (0,1)$

domain: event space — all $A \subseteq \Omega$

range

1: certainty $P(\Omega)=1$

0: impossibility $P(\emptyset)=0$

$$P(A) = \frac{|A|}{|\Omega|} \quad A \subseteq \Omega, \quad A \in 2^{\Omega}$$

→ What is the prob. of getting a sum of 3 if you roll 2 die?

Step 1: translate from english to obtain Ω — $\Omega = \{1,2,..6\}^2$

Step 2: compute $|\Omega|$ — $|\Omega| = |\{1,...6\}|^2 = 6^2 = 36$

Step 3: translate from english to obtain A

"getting a sum of 3" $A = \{\langle 1,2\rangle, \langle 2,1\rangle\}$

Step 4: compute $|A|$ — $|A|=2$

Step 5: divide $\frac{|A|}{|\Omega|} = \frac{1}{18}$

→ What is the probability of getting 2 Heads on 4 coin tosses?

$\Omega = \{H,T\}^4 = \langle H,T,T,H\rangle \in \Omega$

$|\Omega| = 16$

$A = \{\langle H,H,T,T\rangle, \langle T,H,T,H\rangle, \langle T,T,H,H\rangle, \langle T,H,H,T\rangle, \langle H,T,H,T\rangle, \langle H,T,T,H\rangle\}$

$|A| = 6 \quad \frac{|A|}{|\Omega|} = \frac{3}{8}$

$P(HHHH) = P(HHTT) \neq P(2H)$

P(at least one H)

$A = \{\langle H,H,H,H \rangle, HTTH, THTT, HHTT \ldots\}$

$|\Omega| = |A| + |A^c|$

$\Omega = A \cup A^c$ and $\{A, A^c\}$ are mutually exclusive.

$= |A| = |\Omega| - |A^c|$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|}$$

$$= 1 - P(A^c)$$

$A = \{ \geq 1 \ H\}$

$A^c = \{ < H \cdot \text{zero } H\}$

$= \{TTTT\}$

$$P(A) = 1 - \frac{1}{16} = \frac{15}{16}$$

→ Flip 10 coins. What is prob of 4 H?

$$P(A) = \frac{|A|}{|\Omega|} = \frac{}{1024}$$

$|\Omega| = 2^{10} = 1024$

$A = \{HHHHTTTTTT, \ldots\ldots\}$

$F = \{Jane, Mary, Susan\}$

How many ways to sit them in 3 chairs?

Seat #1	Seat #2	Seat #3	Outcome
Jane	Mary	Susan	{ JMS,
	Susan	Mary	JSM,
Mary	Jane	Susan	MJS,
	Susan	Jane	MSJ,
Susan	Jane	Mary	SJM,
	Mary	Jane	SMJ }

$3 \times 2 \times 1 = 6$

Note: $\Omega \neq F^3$

$6 = |\Omega| \neq |F^3| = 27$

$\Omega \subset F^3$

$\langle J, J, J \rangle \in F^3$

$\langle J, J, J \rangle \notin \Omega$

Ω represents the set of F sampled "without replacement" 3 times.

F^3 " " " " " " " " "with replacement" 3 times.

\# of ways to sample n objects without replacement

$\underset{1^{st}\text{ sample}}{\underline{n}}$ $\underset{2^{nd}\text{ sample}}{\underline{n-1}}$ $\underset{n-1\text{ sample}}{\underline{2}}$ $\underset{n^{th}\text{ sample}}{\underline{1}}$

$$= \prod_{i=1}^{n} i = \boxed{n!}$$

\# of ways to sample n objects w/ replacement

$\underset{1^{st}\text{ sample}}{\underline{n}}$ $\underset{2^{nd}\text{ sample}}{\underline{n}}$ $\underset{n-1\text{ sample}}{\underline{n}}$ $\underset{n^{th}\text{ sample}}{\underline{\quad}} = \boxed{n^n}$

# of ways to seat 10 people in 3 chairs?

$$\underset{\text{seat 1}}{\underline{10}} \cdot \underset{\text{seat 2}}{\underline{9}} \cdot \underset{\text{seat 3}}{\underline{8}} = \frac{10!}{7!}$$

# of ways to sample k objects from a set of n objects w/o replacement.

${}_nP_k$ permutations $= \boxed{\frac{n!}{(n-k)!}}$ $\underset{\text{seat 1}}{\underline{n}} \; \underset{\text{seat 2}}{\underline{n-1}} \cdots \underset{\text{seat k}}{\underline{n-k+1}}$

every possible way of ordering/sorting

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0} = n!$$

→ P(J and S sit together)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\{J,S,M\},\{MJS\},\{SJM\},\{MSJ\}|}{3!}$$

$$= \frac{4}{6} = \frac{2}{3}$$

→ 3 couples

Bob - Jane
Richard - Susan
Charles - Mary

Seated in 6 chairs.

P(the couples sit next to each other)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 4 \cdot 2}{6!} = \frac{1}{15}$$

$$\underset{1}{\underline{6}} \cdot \underset{2}{\underline{1}} \cdot \underset{3}{\underline{4}} \cdot \underset{4}{\underline{1}} \cdot \underset{5}{\underline{2}} \cdot \underset{6}{\underline{1}}$$

seat #

alternative method:

$$\text{loveseat \#}\quad \underset{1}{\underline{3(2)}} \cdot \underset{2}{\underline{2(2)}} \cdot \underset{3}{\underline{1(2)}}$$

$$= \frac{(3\cdot 2\cdot 1)2^3}{6\cdot 5\cdot 4\cdot 3\cdot 2} = \frac{1}{15}$$

→ P(alternating gender)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6\cdot 3\cdot 2\cdot 2}{6!} = \frac{\cancel{6}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{6}\cdot 5\cdot\cancel{4}\cdot\cancel{3}\cdot 2} = \frac{1}{10}$$

$$\text{seat \#}\quad \underset{\substack{1\\ M}}{\underline{6}}\quad \underset{\substack{2\\ F}}{\underline{3}}\quad \underset{3}{\underline{2}}\quad \underset{4}{\underline{2}}\quad \underset{5}{\underline{1}}\quad \underset{6}{\underline{1}}$$

$$A = A_{1st\ is\ B} \cup A_{1st\ is\ G}$$

mutually exclusive

$$\frac{|A|}{|\Omega|} = \frac{|A_{1st\ B}|}{|\Omega|} + \frac{|A_{1st\ G}|}{|\Omega|}$$ divide by $|\Omega|$

$$P(A) = P(A_{1st\ B}) + P(A_{1st\ G})$$

$P(A_{1st}\ B)$

$$\text{seat \#}\quad \underset{1}{\underline{3}}\quad \underset{2}{\underline{3}}\quad \underset{3}{\underline{2}}\quad \underset{4}{\underline{2}}\quad \underset{5}{\underline{1}}\quad \underset{6}{\underline{1}}$$

$$= \frac{(3\cdot 3\cdot 2\cdot 2)2}{6!} = \frac{1}{10}$$

→ P(R-S sitting together)

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\quad}{6!}$$

$$\text{seat \#}\quad \underset{\text{loveseat}}{\underline{1(2)}} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$$

$\underset{\text{seat 1}}{\underline{4}}$ $\underset{\text{love seat}}{\underline{1}}$ $\underline{3}$ $\underline{2}$ $\underline{1}$

* there are 5 possible positions for the loveseat

$$= \frac{4!\,2\cdot 5}{6!} = \frac{5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 2}{6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = \frac{1}{3}$$

→ # of ways to sample 3 marbles from a bag of 100 w/o replacement $= {}_{100}P_3$

" " w/ replacement $= 100^3$

$$\frac{{}_{100}P_3}{100^3} \approx .97$$

if $n \to \infty$ what does the $\frac{\text{\# ways sample w/o rep}}{\text{\# ways w/ replacement}}$

look like?

$$\lim_{n\to\infty} \frac{{}_nP_k}{n^k} = \lim_{n\to\infty} \frac{(n)(n-1)\cdots(n-k+1)}{(n)(n)\cdots(n)}$$

$$= \lim_{n\to\infty}\frac{n}{n}\cdot\lim_{n\to\infty}\frac{n-1}{n}\cdots\lim_{n\to\infty}\frac{n-k+1}{n} = 1$$