Custom R.V.'s

Roulette in America
"
Bet on black - make \$1

(Pay out is 1:1)

$X \sim \begin{cases} \$1 & wp \ 18/38 \\ \$1 & wp \ 18/38 \end{cases}$

→ ultimately you lose if you keep playing

$\mu = E[X] = \sum_{x \in Supp[X]} x \, p(x) = \$1 \cdot \frac{18}{38} + \$1 \cdot \frac{38}{38} = \$0.053$

$\bar{X} \longrightarrow \mu$ (n goes big)

$\lim_{n \to \infty} T_n = \lim_{n \to \infty} X_1 + \cdots + X_n = -\infty$

Bet on #7 Payout is 35:1

$X \sim \begin{cases} \$35 & wp \ 1/38 \\ -\$1 & wp \ 37/38 \end{cases}$

$\mu = 35 \cdot \frac{1}{38} + -1 \cdot \frac{37}{38} = \$0.053$

Bet on "the first dozen" (#1 - 12) Payout 2:1

$X \sim \begin{cases} \$2 & wp \ 12/38 \\ -\$1 & wp \ 26/38 \end{cases}$

$\mu = 2 \cdot 13/38 + -1 \cdot 26/38 = -\$0.053$

<u>Def</u>: "Fair Game" $\mu = 0$

If traffic, it takes 12 mins
If no traffic, it takes 7 mins

$P(\text{traffic}) = 0.3$ Construct a R.V. for time W

$W \sim \begin{cases} 12 & wp \ 0.3 \\ 7 & wp \ 0.7 \end{cases}$

$E(W) = 12 \times 0.3 + 7 \times 0.7 = 8.5$ min

In general: $E[X] \notin Supp[X]$

Uber charges \$0.40/min. How much does Uber charge for time in this ride?

r.v. B. $B = \$0.40/\text{min} \ W = g(W)$

$B \sim \begin{cases} \$4.80 & w.p. \ 0.3 \\ \$4.80 & wp \ 0.7 \end{cases}$

$E[B] = 4.80 * 0.3 + 2.80 * 0.7 = \$3.16$

$= 0.40 * 12 * 0.3 + 0.4 * 7 * 0.7$

$= 0.4\,(12 * 0.3 + 7 * 0.7)$

$= 0.4\,E[W]$

Uber charge \$3 as a base fee. Behold T as base fee plus cherge for time.

$T = B + \$3 = g(B)$

$$T \sim \begin{cases} \$7.80 \text{ wp } 0.3 \\ \$5.80 \text{ wp } 0.7 \end{cases}$$

$$\begin{aligned} E(T) &= 7.8 * 0.3 + 5.80 * 0.7 = \$6.12 \\ &= (4.80 + 3) * 0.3 + (2.80 + 3) * 0.7 \\ &= 4.8 * 0.3 + 3 * 0.3 + 2.80 * 0.7 + 3 * 0.7 \\ &= \underbrace{4.8 * 0.3 + 2.8 * 0.7}_{E(B)} + 3 \cdot 1 \end{aligned}$$

$E(B) + 3$

$Y = aX + c \;,\; a, c \in \mathbb{R}$

$$E[Y] = E[\underbrace{aX + c}_{g(x)}] = \sum_{x \in Supp[X]} (ax + c)\,p(x) + \sum axp(x) + cp(x)$$

$$= \sum axp(x) + \sum cp(x) = a \underbrace{\sum_{x \in Supp[X]} xp(x)}_{E(X)} + c \underbrace{\sum_{x \in Supp[X]} p(x)}_{1}$$

$$= a\,E[X] + c$$

$X \sim Bin(6, \frac{1}{2})$

$Y = X^2$

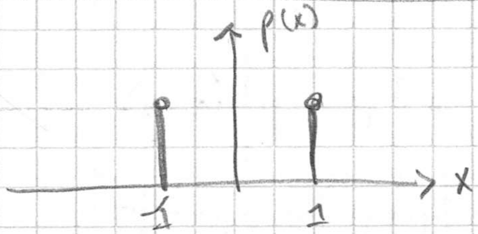
Conjecture

$E[X^2] \overset{?}{=} E(X)^2$

In general

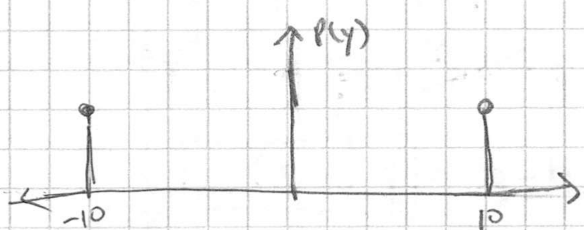
$E[g(X)] \neq g(E[X])$

$X \sim \begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$



$Y = 10X$

$Y \sim \begin{cases} 10 & \text{wp } \frac{1}{2} \\ -10 & \text{wp } \frac{1}{2} \end{cases}$



$X \overset{d}{=} Y$

$E[X] = E[Y] \not\Rightarrow p_X(x) = p_Y(y)$

$E[X] = 0, E[Y] = 10 \; \underbrace{E[X]}_{0} = 0$

Error Functions

$e(x, \mu) = |x - \mu|$

"L1 error"

"can't minimize this, analytically"

$e(x, \mu) = (x - \mu)^2$ "L2 error"

$L = (X - \mu)^2 = g(X)$

variance (always positive)

$Var[X] := E[L] = E[(X-\mu)^2] = E[(X - E[X])^2]$

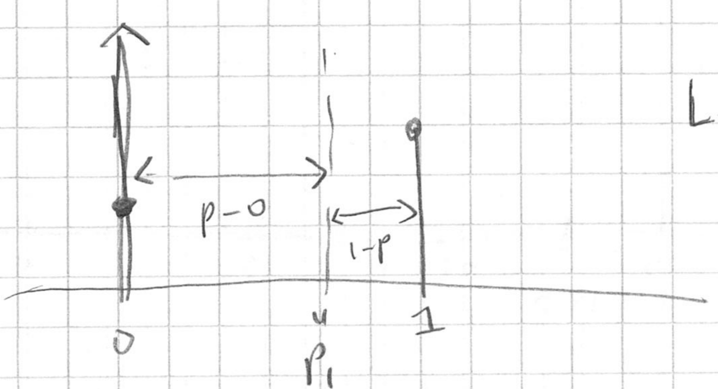
$Var[X] = E(L) = E[(X-\mu)^2] = (1-0)^2 * 0.5 + (-1-0)^2 * 0.5$

$= 1*0.5 + 1*0.5 = 1$

$Var[Y] = (10-0)^2 * 0.5 + (-10-0)^2 * 0.5 = 100 * 0.5 + 100 * 0.5 = \boxed{100}$

$X \sim Bern(p) = \begin{cases} 1 & wp\ p \\ 0 & wp\ 1-p \end{cases}$

$E[X] = 1 * p + 0(1-p) = p$



$L \sim \begin{cases} (1-p)^2 = 1-2p+p^2 & wp\ p \\ (0-p)^2 = p^2 & wp\ 1-p \end{cases}$

$$Var[X] = E[L] = (1-p)^2 p + p^2(1-p)$$

$$= (1-2p+p^2)p + p^2 - p^3$$

$$= p \cancel{-2p^2} \cancel{+p^3} + p^2 \cancel{-p^3}$$

$$= p - p^2 = \boxed{p(1-p)}$$