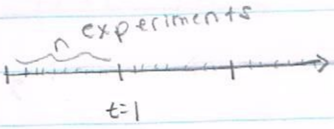
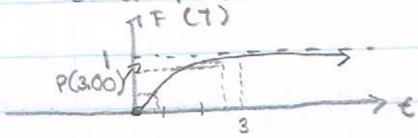
Math 241 Lecture 17 Nov 8^{th}



limiting PMF
$p(t) = 0 \; \forall t$
not a PMF

limiting CDF
$F(t) = 1 - e^{-\lambda t}$



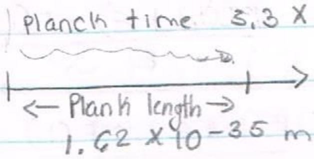
$\chi \sim \text{Geom}(p) = \left(1 - \frac{\lambda}{n}\right)^{nt} \frac{\lambda}{n}$

$\lambda = np$

$\text{supp}[T] = (0, \infty) \Rightarrow |\text{supp}[T]| = |\mathbb{R}| > |\mathbb{N}|$

$\Rightarrow$ T is not a discrete random variable

Is time continuous? We actually don't know



Prob. stopping $t = 3s$

$p(3) = P(T=3) = 0$

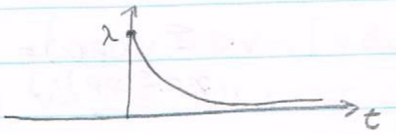
$p(T = 3.00000\ldots) = 0$

infinite precision

$P(T = 3.000$ rounded to the $) = P(T \in [2.995\bar{0}, 3.004\bar{9}])$

$= P(T < 3.004\bar{9}) - P(T \leq 2.995\bar{0}) = F(3.004\bar{9}) - F(2.995\bar{0}) > 0$

(difference of CDFs)

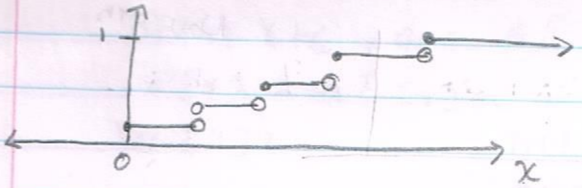


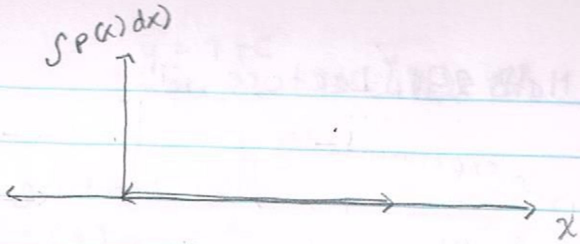
$F'(t) = \lambda e^{-\lambda t} = \frac{\lambda}{e^{\lambda t}}$

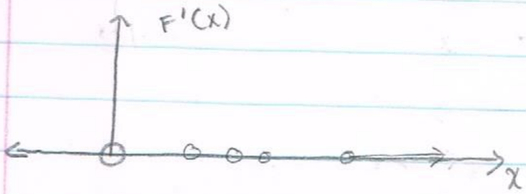
probability density function (PDF)

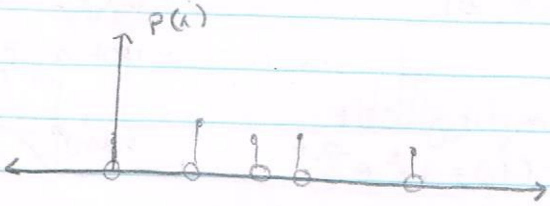
$P(T \in [a, b]) = F(b) - F(a) = \int_a^b f(t)\, dt$

Fundemental Thm of Calculus
Relates PDF and CDF









$\lambda = 2$

slopes

$f(t) = 2e^{-2t}$

$f(0.1) \approx 1.63 \neq p(0.1) = 0$

$f(1) \approx 0.27 \neq p(1) = 0$

$\frac{f(0.1)}{f(1)} \approx \frac{1.63}{0.27} \approx 6$

relative likelihood

$$\lim_{\varepsilon \to 0} \frac{P(T \in [0.1, 0.1+\varepsilon])}{P(T \in [1, 1+\varepsilon])}$$

$$\lim_{\varepsilon \to 0} \frac{F(0.1+\varepsilon) - F(0.1)}{F(1+\varepsilon) - F(1)}$$

$$\frac{\lim_{\varepsilon \to 0} \frac{F(0.1+\varepsilon) - F(0.1)}{\varepsilon}}{\lim_{\varepsilon \to 0} \frac{F(1+\varepsilon) - F(1)}{\varepsilon}} = \frac{f(0.1)}{f(1)}$$

$P(T \in (-\infty, \infty)) = P(T \in (0, \infty)) = 1$ probability it eventually stops

$\Rightarrow \int_0^\infty f(t)\,dt = 1$ $\boxed{\int_{supp[X]} f(x)\,dx = 1}$ $\sum_{x \in Supp[X]} p(x) = 1$

Definition Constant R.V. X

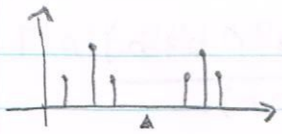
① $|Supp(X)| = |\mathbb{R}|$

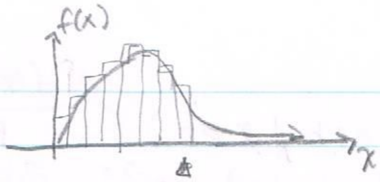
② $F(x)$ is a valid CDF w/wo jumps

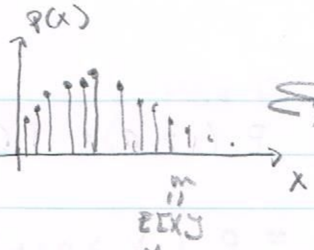
③ PDF DNE

④ PDF (i) $\int_{supp[X]} f(x)\,dx = 1$ (ii) $f(x) \geq 0 \ \forall x$

$X_1 \overset{d}{=} X_2 \Rightarrow f_{X_1}(x) = f_{X_2}(x)$ or $F_{X_1}(x) = F_{X_2}(x)$







$\Sigma x p(x)$

$$m \approx \Sigma x p(x) = \Sigma_x \big(\underbrace{F(x+f) - F(x)}_{p(x)}\big) = \int_{supp[X]} x f(x)\,dx$$

Assume $E[g(x)] = \int_{supp[X]} g(x) f(x)\,dx$

$Var[X] = E\left((x-m)^2\right) = \int (x-m)^2 f(x)\,dx$

$E[aX+c] = am + c$

$Var[aX+c] = a^2 \sigma^2$

$SE[aX+c] = |a|\,\sigma$

$E[\Sigma X_i) = \Sigma E(X_i) = nm$

$Var[\Sigma X_i] = \Sigma\, Var(X_i) = n\sigma^2$

independence of $X_1, \dots X_n$ iid

$X \sim Exp(\lambda) := \underbrace{\lambda e^{-\lambda t}}_{f(t)}$ $supp[X] = (0,\infty) = [0,\infty)$

exponential   PDF $\lambda \in (0,\infty)$

$F(x) = 1 - e^{-\lambda t}$

$1 - F(x) = e^{-\lambda t}$

$\lambda = np$ $n \in \mathbb{N}$ $p \in (0,1)$

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty \underbrace{x}_{u} \underbrace{e^{-\lambda x}\,dx}_{dv} = \lambda\left[(x)\left(-\tfrac{1}{\lambda}e^{-\lambda x}\right) - \tfrac{1}{\lambda^2} e\right.$$

$\int u\,dv = uv - \int v\,du$ $du = dx$

$v = \int e^{-\lambda x}dx = -\frac{1}{\lambda} e^{-\lambda x} = \frac{1}{\lambda^2} e^{-\lambda x}$

$$= \left[\frac{x}{e^{\lambda x}} + \frac{1}{\lambda e^{\lambda x}}\right]_0^\infty = -\left(\left(\lim_{x\to\infty} \frac{x}{e^{\lambda x}} + \lim_{x\to\infty}\frac{1}{\lambda e^{\lambda x}}\right) - \left(\frac{0}{e^{\lambda(0)}} + \frac{1}{\lambda e^{\lambda(0)}}\right)\right)$$

$$= \left((0+0) - \left(0 + \frac{1}{\lambda}\right)\right) = \boxed{\frac{1}{\lambda}}$$

p

$$P(x > a) = P(x > a+b \mid x > b) = \frac{P(x > a+b \cap a+b)}{P(x > b)}$$

$= e^{-\lambda a} \quad e^{-\lambda b}$

$$= \frac{P(x > a+b)}{P(x > b)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = P(x > a)$$

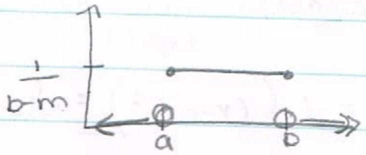
Exponential is

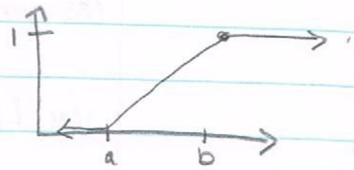
$x \sim U(a, b) = \frac{1}{b-a}$ $\quad$ $\text{supp}[x] = (a, b)$

unidom

$a \in \mathbb{R}$ $\quad$ but, $a < b$

$b \in \mathbb{R}$





$$\int f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}[x]_a^b = 1$$

$$F(x) = \int f(x)\,dx + c = \int \frac{1}{b-a}\,dx + c = \frac{x}{b-a} + C = \frac{x}{b-a} + \frac{a}{a-b} = \frac{x}{b-a} + \frac{-a}{b-a} = \boxed{\frac{x-a}{b-a}}$$

$F(b) = 1 \implies \frac{b}{b-a} + c = 1$

$\implies \frac{b}{b-a} = 1 - c \implies \frac{b}{a-b} = c - 1 \implies c = \frac{b}{a-b} + 1$

$= \frac{b}{a-b} + \frac{a-b}{a-b} = C$