Recall

11/27 - Lecture 20

$M_X(t) = E[e^{tX}]$

mgf

(I) $M_X(t) = M_Y(t) \Leftrightarrow X \stackrel{d}{=} Y$

(II) $E[X^k] = M_X^{(k)}(0)$

(III) $Y = aX + c \Leftrightarrow M_Y(t) = e^{tc} M_X(at)$

(IV) if X, Y independent $\Rightarrow M_{X+Y}(t) = M_X(t) M_Y(t)$

$X \sim Bern(p) \Rightarrow M_X(t) = 1 - p + pe^t$

$X \sim Binomial(n, p) \Rightarrow M_X(t) = (1 - p + pe^t)^n$

$X \sim Exp(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$

$Z \sim N(0,1) \Rightarrow M_Z(t) = e^{t^2/2}$

$X \sim N(\mu, \sigma^2)$

$X = \mu + \sigma Z \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{t\mu} M_Z(\sigma t) = e^{t\mu} e^{(\sigma t)^2/2} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$X_1 \sim N(\mu_1, \sigma_1^2)$ independence of $X_2 \sim N(\mu_2, \sigma_2^2)$

$Y = X_1 + X_2 \sim ?$

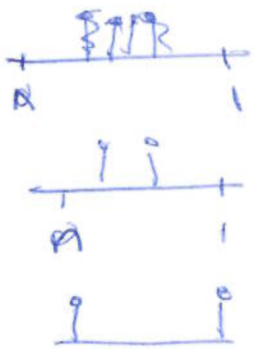
$$\begin{aligned} M_Y(t) &= M_{X_1}(t) M_{X_2}(t) \\ &= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} \\ &= e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2) t^2}{2}} \end{aligned}$$

$\Rightarrow Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

V) Levy's Continuing Theorem

let x_1, x_2 be a ~~se~~sequence of rv's

if $\lim_{n\to\infty} M_{x_n}(t) = M_X(t) \Rightarrow \underbrace{\lim_{n\to\infty} F_{x_n}(x) = F_x(x)}_{\text{convergence in distribution.}}$



n = 1

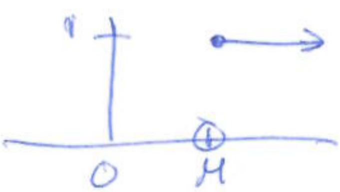
n = 2

n = ∞

$\overline{X} \to \mu = E(x)$ ← As long as iid, $\bar{x}$ will equal μ.

Law of large #'s

$\mu \sim Deg(\mu)$



Assume

$x_1, x_2, \ldots \overset{iid}{\sim}$ with mean μ.

$\lim M_{\overline{X}_n}(t) = e^{t\mu}$ ← mgf for the $Deg(\mu)$

$\Rightarrow \lim M_{\frac{x_1 + \cdots + x_n}{n}}(t) \overset{III}{=} \lim M_{x_1 + \cdots + x_n}\left(\frac{t}{n}\right)$

$= \lim M_{x_1}\left(\frac{t}{n}\right) \overset{IV}{\cdot} M_{x_2}\left(\frac{t}{n}\right) \cdots M_{x_n}\left(\frac{t}{n}\right) =$

$= \lim_{n\to\infty} \left(M_X\left(\frac{t}{n}\right)\right)^n = \lim_{n\to\infty} e^{\ln\left(\left(M_X\left(\frac{t}{n}\right)\right)^n\right)}$

$= \lim_{n\to\infty} e^{n \ln\left(M_X\left(\frac{t}{n}\right)\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}} = e^{\lim_{n\to\infty} \frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}}$

let $u = \frac{1}{n}$

$\Rightarrow n \to \infty \Rightarrow u \Rightarrow 0$

$$= e^{\lim_{u \to 0} \frac{\ln(M_X(ut))}{n}} =$$

L'Hospital Rule

$$= e^{\lim_{u \to \infty} \frac{t M_X'(ut)}{M_X(ut)}} = e^{\frac{t(M_X'(0))}{M_X(0)}} = e^{t\mu}$$

Note

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

V) $\lim_{n \to \infty} M_{X_n}(t) = M_X(t) \Rightarrow X_n \to X$

CDF's are limiting i.e. $\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \; \forall x$

Assume

$X_1, X_2, \ldots \overset{iid}{}$ with mean μ and variance σ^2

Consider

standardized Average

$$\boxed{C_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \quad \Rightarrow E[C_n] = 0 \quad \Rightarrow SE[C_n] = 1}$$

$$C_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} = \frac{\sqrt{n}\left(\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right)}{\sigma}$$

$$= \frac{\sqrt{n}\left(\frac{X_1 + \ldots + X_n}{n} - \frac{\mu + \ldots + \mu}{n}\right)}{\sigma}$$

$$= \frac{(X_1 - \mu) + \ldots + (X_n - \mu)}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}}\left(\frac{X_1 - \mu}{\sigma} + \frac{X_2 - \mu}{\sigma} + \ldots + \frac{X_n - \mu}{\sigma}\right)$$

let $Z_i = \frac{X_i - \mu}{\sigma}$ $= \frac{1}{\sqrt{n}}(Z_1 + \ldots + Z_n)$

$E[Z_i] = 0$

$SE[Z_i] = \sigma$

$$M_{Z_n}(t) = M_{\frac{Z_1+\cdots+Z_n}{\sqrt{n}}}(t) \overset{\mathrm{III}}{=} M_{Z_1+\cdots+Z_n}\left(\frac{t}{\sqrt{n}}\right)$$

$$\overset{\mathrm{IV}}{=} \left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)^n = e^{\ln\left(\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)^n\right)} = e^{n\ln\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)}$$

$$= \lim_{n\to\infty} M_{Z_n}(t) = e^{\lim_{n\to\infty} n \ln\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)}$$

$$= e^{\lim_{n\to\infty} \frac{\ln\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{t^2}{t^2}} = e^{t^2 \lim_{n\to\infty} \frac{\ln\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)}{\frac{t^2}{n}}}$$

let $u = \frac{1}{\sqrt{n}}$ $\quad n\to\infty \Rightarrow u\to 0$

$$= e^{t^2 \lim_{u\to 0} \frac{\ln\left(M_Z(ut)\right)}{t^2 u^2}} = e^{\frac{t^2}{2}\lim_{u\to 0} \frac{\frac{t M_Z'(ut)}{M_Z(ut)}}{t^2 u}} = e^{\frac{t^2}{2}\lim_{u\to 0}}$$

L Hospital Rule $\qquad$ L Hospital Rule

Note Look up Quotient Rule

$$\boxed{\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) + g'(x)f(x)}{g(x)^2}}$$

$$= e^{\frac{t^2}{2}\lim_{u\to 0} \frac{\frac{t M_Z''(ut) + t M_Z'(ut)^2}{M_Z(ut)^2}}{t}} = e^{\frac{t^2}{2}}$$

$$= e^{\frac{t^2}{2}\lim_{u\to 0} \frac{\frac{M_Z(ut) \cdot M_Z''(ut) - M_Z'(ut)^2}{M_Z(ut)^2}}{t}} = e^{\frac{t^2}{2}\frac{M_Z(0)M_Z''(0) - M_Z'(0)^2}{M_Z(0)^2}}$$

$1 = Var[Z] = E[Z^2] - E[Z]^2$

$\overset{\mathrm{V}}{\Rightarrow} M_{Z_n} \to e^{\frac{t^2}{2}}$ $\boxed{Z_n = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \Rightarrow N(0,1) \quad \text{Central limit theorem (CLT)}}$

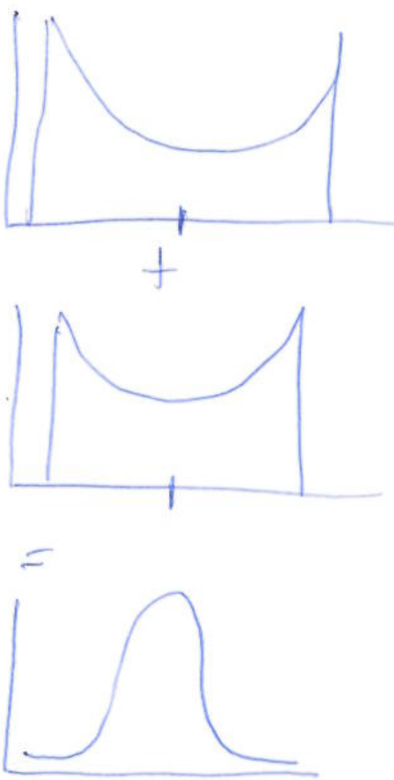
How to use CLT to solve problems

first note that $n \to \infty$ is impossible
so it never truly converges

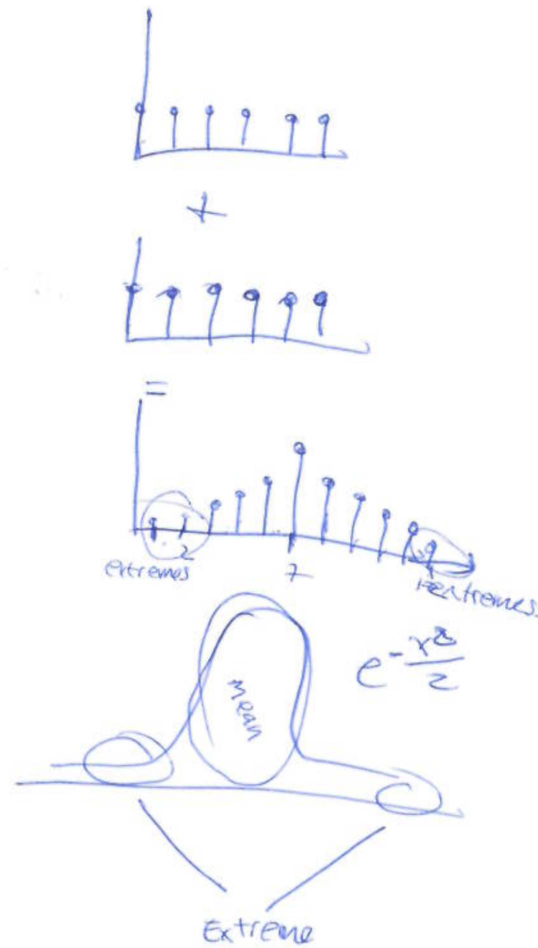
① If n is "large enough" then $\frac{\bar{x}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\approx} N(0,1)$

Ⅱ $\bar{X} \overset{d}{\approx} N(\mu, (\frac{\sigma}{\sqrt{n}})^2)$

Ⅲ $T \overset{d}{\approx} N(n\mu, (\sigma\sqrt{n})^2)$



Bathtub



$X_1 \ldots X_{30} \overset{iid}{\sim} Geom(\frac{1}{2}) = \mu = \frac{1}{\frac{1}{2}} = 2$, $\sigma = \frac{\sqrt{1-p}}{p} = \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}} \approx 1.414$

What is the probability the avg wait time is more than 2.75?

$P(\bar{X} \geq 2.75) = P\left(\frac{\bar{X} - 2}{.258} \geq \frac{2.75 - 2}{.258}\right) = P(Z \geq 3) \approx .0015$

$n = 30 \Rightarrow$ "large" $=$ CLT

$\bar{X} \overset{d}{\approx} N(\mu, (\frac{\sigma}{\sqrt{n}})^2) = N(2, (1.414)^2) = N(2, 0.258^2)$

Take 100 steps with probability forward ~~and~~ backward being $\frac{1}{2}$

$$x \sim \begin{cases} +1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$$

what is the prob. you are more than 10 steps away from ~~at~~ ~~where you~~ starting position after 100 steps

$$T = X_1 + \dots + X_{100} \underset{\text{III}}{\overset{d}{\approx}} \left(N(n\mu, \sigma\sqrt{n}\right) = N(0, 1\sqrt{100}) = N(0, 10^2)$$

$$P(|T| \geq 10)$$