Math 241 Lec11 10/17/17

$$X \sim \text{Hyper}(n, p, N) := \frac{\overset{\overset{K}{\shortparallel}}{\binom{pN}{x}} \; \overset{\overset{N-K}{\shortparallel}}{\binom{(1-p)N}{n-x}}}{\binom{N}{x}}$$

$p = \frac{K}{N}$ this is a reparameterization of the original $\text{Hyper}(n, K, N)$

$$\lim_{N \to \infty} \text{Hyper}(n, p, N) = \lim_{N \to \infty} \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{x}}$$

$$= \binom{n}{x} \lim_{N \to \infty} \frac{\frac{(pN)!}{(pN-x)!} \; \frac{((1-p)N)!}{((1-p)N-(n-x))!}}{\frac{N!}{(N-n)!}}$$

$\boxed{3}$

n terms, x terms, n-x terms

$$= \binom{n}{x} \lim_{N\to\infty} \frac{(pN)(pN-1)(pN-2)\cdots(pN-x+1)(1-p)N((1-p)N-1)((1-p)N-2)\cdots((1-p)N-n+x+1)}{\underbrace{(N)(N-1)(N-2)\cdots\cdots(N-n+1)}_{n \text{ terms}}}$$

Split up n terms with a line

$$\lim f(x)g(x) = \lim f(x) \lim g(x)$$

$$= \binom{n}{x} \underbrace{\underbrace{\lim_{N\to\infty} \frac{pN}{N}}_{p} \underbrace{\lim \frac{pN-1}{N-1}}_{p} \cdots \underbrace{\lim \frac{pN-x+1}{N-x+1}}_{p}}_{x} \cdot \underbrace{\underbrace{\lim \frac{(1-p)N}{N-x}}_{1-p} \cdot \underbrace{\lim \frac{(1-p)N-1}{N-x-1}}_{1-p} \cdots \underbrace{\lim \frac{(1-p)N-n+x+1}{N-n+1}}_{1-p}}_{n-x}$$

$$= \binom{n}{x} p^x (1-p)^{n-x} =: p(x)$$

$X \sim \text{Binomial}(n, p)$

Hypergeometric $(n, p, N) \longrightarrow$ Binomial$(n, p)$

Recall $\lim_{N\to\infty} \frac{{}_NP_K}{N^K} = 1$
sampling without replacement is the same as sampling with replacement if N large

$\text{Supp}[X] = \{0, 1, \ldots, n\}$

Parameter space

$n \in \mathbb{N}$

$p \in (0, 1)$

$\binom{n}{x} 0^x 1^{n-x}$ $\quad P(X=0) = \binom{n}{0} 0^0 1^n$

$\boxed{0^0 := 1}$ ? this is one of the motivations

$X \sim \text{Binomial}(n, 0) = \text{Deg}(0)$

$X \sim \text{Binomial}(n, 1) = \text{Deg}(n)$

$X \sim \text{Binomial}(1, p) = \underbrace{\binom{1}{x}}_{1} p^x (1-p)^{1-x} = p^x (1-p)^{1-x} = \text{Bern}(p)$

$\text{Supp}(X) = \{0, 1\}$ $\quad \binom{1}{0} = 1, \binom{1}{1} = 1$

of course it is!

~~Review~~ $X \sim \text{Binomial}(n,p) := \binom{n}{x} p^x (1-p)^{n-x}$ ~~param? Free variable?~~

~~what values could free variable take?~~

$X \sim \text{Hyper}(n,K,N) := \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$ $\quad X \sim \text{Hyper}(n,p,N) := \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$

$\lim_{N \to \infty} \text{Hyper}(n,p,N) = \text{Binom}(n,p)$ ~~Conceptually...~~

---

Property of PMF $\quad \sum_{x \in \text{Supp}(X)} p(x) = 1$

$$\sum_{x \in \{0,\ldots,n\}} \binom{n}{x} p^x (1-p)^{n-x} = 1 \quad \text{How?}$$

Recall: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ binomial thm.

let... $a = p$, $b = 1-p$, $i = x$ $\quad \underbrace{(p + (1-p))^n}_{1^n = 1} = \sum_{i=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

DONE ... this is why the binomial is named so...

---

$X_1$ and $X_2$ are ind $\quad X_1, X_2 \overset{ind}{\sim}$

$$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$$

$$P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2) \quad \forall x_1 \in \text{Supp}(X_1),$$

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)\,P(X_2 = x_2) \quad \forall x_2 \in \text{Supp}(X_2)$$

Joint mass function (JMF)

$X_1, X_2 \stackrel{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$

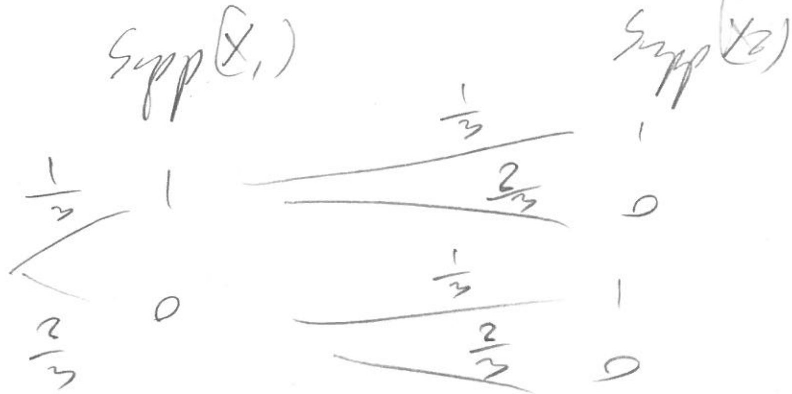
a function of two r.v.'s

let $T_2 := X_1 + X_2 = g(X_1, X_2)$ Conceptually... what is this?

a new r.v.

$\text{Supp}[T_2] = \{0, 1, 2\}$

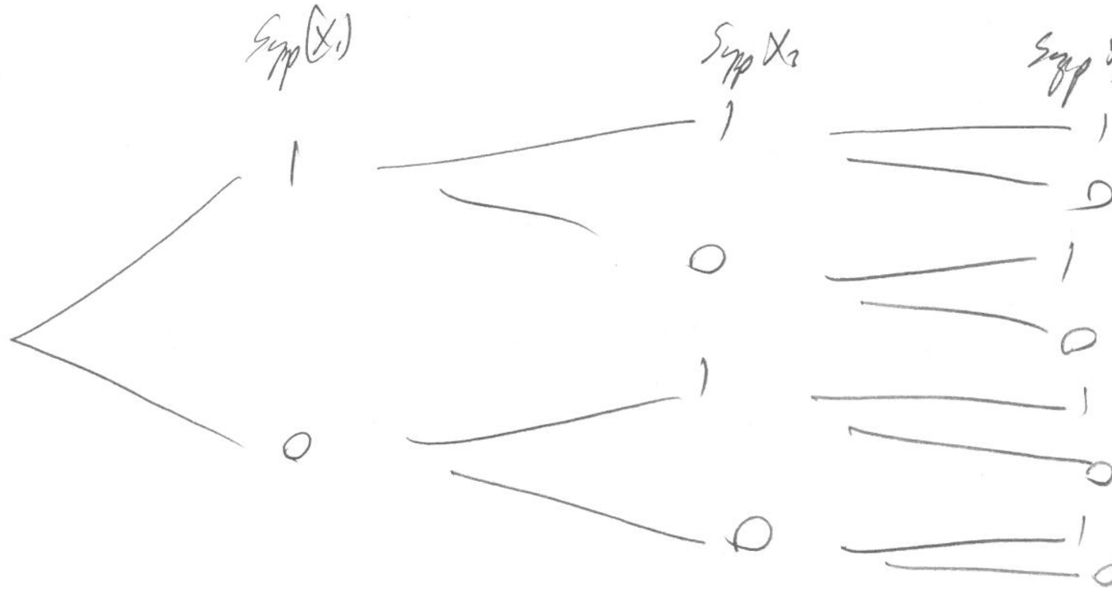
$T_2 \sim ?$ Let's use a tree to figure this out




| | $T_2$ |
|---|---|
| $P(X_1 = 1, X_2 = 1) = \frac{1}{9}$ | 2 |
| $P(X_1 = 1, X_2 = 0) = \frac{2}{9}$ | 1 |
| $P(X_1 = 0, X_2 = 1) = \frac{2}{9}$ | 1 |
| $P(X_1 = 0, X_2 = 0) = \frac{4}{9}$ | 0 |
| $1$ | |

$$\Rightarrow T_2 \sim \begin{cases} 0 & \text{wp } \frac{4}{9} \\ 1 & \text{wp } \frac{4}{9} \\ 2 & \text{wp } \frac{1}{9} \end{cases}$$

Let's do $T_3 := X_1 + X_2 + X_3$




| | $T$ |
|---|---|
| $\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0$ | 3 |
| $\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1$ | 2 |
| $\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1$ | 2 |
| $\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2$ | 1 |
| $\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1$ | 2 |
| $\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2$ | 1 |
| $\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2$ | 1 |
| $\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3$ | 0 |

$$T_3 \sim \begin{cases} 0 & \text{w.p.} \quad 1 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3 \\ 1 & \text{w.p.} \quad 3 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2 \\ 2 & \text{w.p.} \quad 3 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 \\ 3 & \text{w.p.} \quad 1 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 \end{cases}$$

1 - 3 - 3 - 1 pattern

| $0\ 0\ 0$ | $1\ 0\ 0$ | $1\ 1\ 0$ | $1\ 1\ 1$ |
|---|---|---|---|
| | $0\ 1\ 0$ | $1\ 0\ 1$ | |
| | $0\ 0\ 1$ | $0\ 1\ 1$ | |
| $\binom{3}{0}$ | $\binom{3}{1}$ | $\binom{3}{2}$ | $\binom{3}{3}$ |

$$\Rightarrow T_3 \sim \binom{3}{x}\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{3-x} = \text{Binom}\left(3, \frac{1}{3}\right)$$

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right) \qquad T = \sum_{i=1}^{n} X_i$$

$$T \sim \begin{cases} 0 & \text{w.p.} \quad \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n \\ 1 & \text{w.p.} \quad \binom{n}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1} \\ 2 & \text{w.p.} \quad \binom{n}{2} \\ \vdots & \qquad \vdots \\ n-2 & \\ n-1 & \\ n & \end{cases} = \text{Binom}\left(n, \frac{1}{3}\right)$$

$$X_1, \ldots X_n \overset{iid}{\sim} \text{Bern}(p) \qquad T = \sum_{i=1}^{n} X_i$$

$$T \sim \begin{cases} 0 & \text{w.p.} \quad \binom{n}{0} \\ 1 & \text{w.p.} \quad \binom{n}{1} \\ 2 & \text{w.p.} \quad \binom{n}{2} p^2 (1-p)^{n-2} \\ \vdots & \qquad \vdots \\ n-2 & \\ n-1 & \\ n & \end{cases}$$

do this first

$$\Rightarrow T \sim \text{Binom}(n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

Two ways to look at binomial:

$$\lim_{N\to\infty} \text{Hyper}(n,p,N)$$

— or —

$$X_1,\dots,X_n \stackrel{iid}{\sim} \text{Bern}(p)$$
$$X_1+\dots+X_n$$

for binomial..

$$F(x) := P(X \le x) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$$ best we can do!

no closed form

$$= I_{1-p}(n-k, 1+k)$$ regularized incomplete beta function

$$= (n-k)\binom{n}{k}\int_0^{1-p} t^{n-k-1}(1-t)^k \, dt$$

not tested

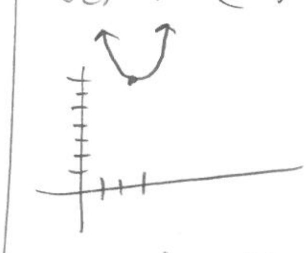
no closed form

$$X_1, X_2, \dots \stackrel{iid}{\sim} \text{Bern}(p)$$

possibly infinite series of binary experiments w/ same prob.
independent / one another

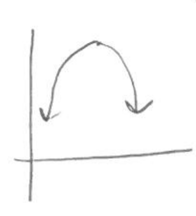
min, max
argmin, argmax

$f(x) = 7 + (x-3)^2$



max{f(x)} undefined
argmax{f(x)} undefined

$f(x) = 7 - (x-3)^2$



max{f(x)} = 7
argmax{f(x)} = 3