Math 241 Fall 2017
Final Examination

Solutions

Professor Adam Kapelner

December 18, 19 and 20, 2017

Full Name ____________________________________ Section (A, B or C) _____

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Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an *unauthorized* cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

________________________________ ________________

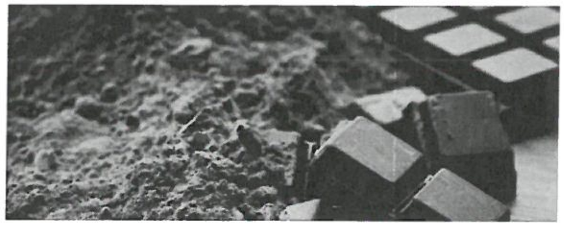
signature date

Instructions

This exam is 120 minutes and closed-book. You are allowed three 8.5" $\times$ 11" pages (front and back) of a "cheat sheet." You may use a graphing calculator of your choice but *no cell phones*. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in choose, permutation, factorial, summation or any other notation which could be resolved to a number with a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1

You are the marketing executive for a food company that is developing a new granola bar. You are so busy worrying about sales and distribution that you cannot work too hard on flavor, hence you pick the most loved flavor: chocolate.



You want to indeed verify this will sell so you do a survey as follows: each person in the survey tastes the bar and tells you whether or not they would buy it from you. Call the true proportion of people who would buy it p.

(a) [4 pt / 4 pts] We survey 200 unique people at random from a very expensive supermarket in Manhattan. Under what circumstance(s) would this be an *unbiased* survey?

If their target market is people who shop at upscale supermarkets in Manhattan, this survey would be unbiased.

(b) [4 pt / 8 pts] We survey 200 unique people at random from a very expensive supermarket in Manhattan. Under what circumstance(s) would this be a *biased* survey?

'' '' '' '' '' '' not '' '' '' '' '' '' '' '' biased

(c) [3 pt / 11 pts] We survey 200 unique people at random from a very expensive supermarket in Manhattan and 173 of them would buy the bar. Provide your best guess of p.

$\hat{p} = .865$

(i) [4 pt / 42 pts] What step(s) could you have taken to reduce the chance of this error?

Survey more people (raise n) or increase α.

(j) [3 pt / 45 pts] Regardless of any of the questions above, assume that truly 90% of people would buy the bar. Create a r.v. model that reflects how many people need to answer the survey before finding one who likes the bar.

$X \sim \text{Geom}(0.9)$

(k) [6 pt / 51 pts] Regardless of any of the questions above, assume that truly 90% of people would buy the bar. If you survey 15 people, what is the probability more than 13 like the bar? Compute explicitly. Round to two significant digits.

$X \sim \text{Binom}(15, 0.9)$

$$P(X > 13) = P(X = 14) + P(X = 15)$$
$$= \binom{15}{14} 0.9^{14} 0.1^1 + \binom{15}{15} 0.9^{15} 0.1^0$$
$$= 15 \cdot 0.9^{14} \cdot 0.1 + 0.9^{15} = \boxed{.55}$$

(l) [10 pt / 61 pts] Regardless of any of the questions above, assume that truly 90% of people would buy the bar if they go to the store. If you make \$2 profit on each sold bar, create a r.v. that reflects the approximate total profit after 100 people enter the store. Compute parameters explicitly and provide the PMF (or PDF).

$$X_1, \ldots, X_{100} \overset{iid}{\sim} \begin{cases} \$2 & \text{wp } 0.9 \\ \$0 & \text{wp } 0.1 \end{cases} \Rightarrow \mu = \$2 \cdot 0.9 + \$0 \cdot 0.1 = \$1.80$$
$$\sigma^2 = (\$2 - \$1.80)^2 \cdot 0.9 + (\$0 - \$1.80)^2 \cdot 0.1 = .36\,\$^2$$
$$\Rightarrow \sigma = \$0.60$$

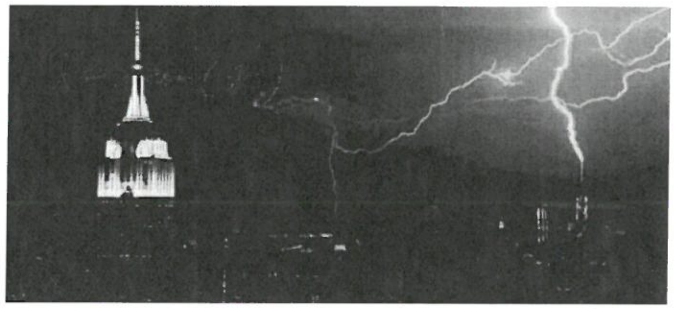
$$\underset{\text{by CLT}}{T \approx} N(n\mu, (\sigma\sqrt{n})^2) = N(100 \cdot \$1.80, (\$0.60 \cdot \sqrt{100})^2) = N(\$180, \$6^2)$$
$$= \frac{1}{\sqrt{2\pi 6^2}} e^{-\frac{1}{2 \cdot 6^2}(x - 180)^2}$$

(m) [2 pt / 63 pts] What theorem did you use in the previous question?

Central Limit Thm.

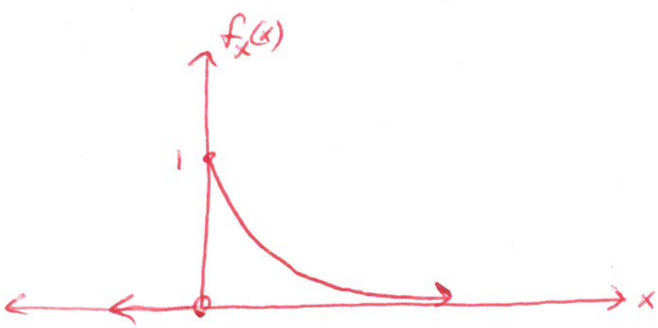
Problem 2

You are in your office building in Manhattan mesmerized by the lightning storm outside

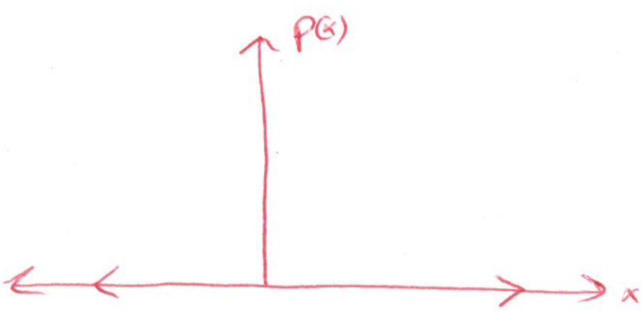


and you are waiting for the next beautiful bolt of lightning. By your calculations, lightning is approximately distributed exponentially with rate parameter $\lambda = 1$ min.

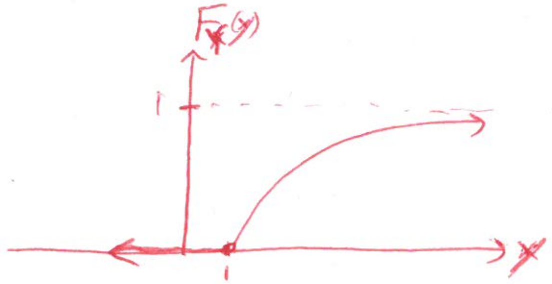
(a) [4 pt / 67 pts] Let $X \sim \text{Exp}(1)$. Draw the PDF of X below. Notate the axes and indicate important points on the plot.



(b) [4 pt / 71 pts] Let $X \sim \text{Exp}(1)$. Draw the PMF of X below. Notate the axes and indicate important points on the plot.

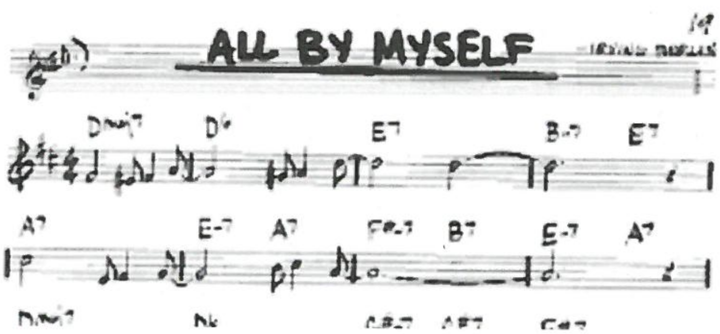


(c) [6 pt / 77 pts] Let Y denote the r.v. of total waiting time until the next lightning bolt given that 1 minute has already elapsed. Draw the CDF of Y below. Notate the axes and indicate important points on the plot.



Problem 3

You are composing jazz music for the piano (example below).



The typical "American songbook" style music is four-bar in the form AABA where each bar is four measures. There is typically one chord per measure. Thus there are 8 unique measures in a single song and hence 8 chords.

To pick a chord, you first pick a key and then a quality. There are 12 keys to choose from: $C, C\natural, D, D\natural, E, F, F\natural, G, G\natural, A, A\natural, B$ and 8 chord qualities: major, minor, augmented, half diminished, full diminished, dominant 7th, minor 7th and major 7th.

Pretend all chords are picked at random for the purpose of this question and that all 8 chords are unique.

(a) [3 pt / 80 pts] How many unique chords are there? You need one key and one quality. Compute explicitly.

$\binom{12}{1}\binom{8}{1} = 96$

(b) [3 pt / 83 pts] How many ways are there to choose 8 unique chords where order doesn't matter? No need to compute explicitly.

$$\binom{96}{8}$$

(c) [3 pt / 86 pts] How many ways are there to choose 8 unique chords where order *does* matter? No need to compute explicitly.

$${}_{96}P_{8}$$

(d) [5 pt / 91 pts] If 8 unique chords are picked at random, what is the probability they are all minor 7th? No need to compute explicitly.

$$\frac{\binom{12}{8}}{\binom{96}{8}} = \frac{{}_{12}P_{8}}{{}_{96}P_{8}}$$

Problem 4

Below are some theoretical questions.

(a) [4 pt / 95 pts] Let $X \sim \text{Poisson}(\lambda) = \lambda^x e^{-\lambda}/x!$. The mgf of X is $M_X(t) = e^{\lambda(e^t-1)}$ and $\text{Supp}[X] = \mathbb{N}_0$ i.e. all natural numbers including zero. Provide an expression for $\mathbb{E}[X]$ using the definition of expectation for discrete r.v.'s but do not solve.

$$E(X) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

(b) [5 pt / 100 pts] Let $X \sim \text{Poisson}(\lambda) = \lambda^x e^{-\lambda}/x!$. The mgf of X is $M_X(t) = e^{\lambda(e^t - 1)}$ and $\text{Supp}[X] = \mathbb{N}_0$ i.e. all natural numbers including zero. Find $\mathbb{E}[X]$ as function of the parameter(s) of X using any method you wish.

$M_X(t) = e^{\lambda e^t} e^{-\lambda}$

$M'_X(t) = \lambda e^t e^{\lambda e^t} e^{-\lambda}$

$M_X'(0) = \lambda e^0 e^{\lambda e^0} e^{-\lambda} = \lambda \cancel{e^{\lambda}} \cancel{e^{-\lambda}} = \boxed{\lambda = E(X)}$

(c) [3 pt / 103 pts] [Extra credit] Let $X \sim \text{Poisson}(\lambda)$ Show that $Y = aX$ where $a > 1$ is *not* Poisson distributed.

(d) [5 pt / 108 pts] [Extra credit] Let $X_n \sim \text{Poisson}\left(\frac{1}{n}\right)$. What r.v. does X_n become as $n \to \infty$? Prove it below.