Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$$3 = \binom{3}{1}, \quad 3 = \binom{3}{2}$$

$$1 = \binom{4}{0}, \quad 4 = \binom{4}{1}, \quad 6 = \binom{4}{2}, \quad 4 = \binom{4}{3}, \quad 1 = \binom{4}{4}$$

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

Recurrence Relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Is this true for $\forall n \in \mathbb{N}, k \in \{0 \cdots n\}$

$$\frac{n!}{k!(n-k)!} \overset{?}{=} \frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!} \qquad \text{multiply by } \frac{n}{n}$$

$$= \frac{n!}{n}\left(\frac{1}{(n-k)!(k-1)!} \cdot \frac{k}{k} + \frac{1}{(n-k-1)!\,k!} \cdot \frac{n-k}{n-k} \right)$$

$$= \frac{n!}{\cancel{n!}}\left(\frac{\cancel{k}}{(n-k)!\,k!} + \frac{\cancel{n-k}}{(n-k)!\,k!} \right)$$

$$= \binom{n}{k}$$

Pascal's Identity

$\boxed{Ex}$ Let $S = \{♠, ♣, ♢, ♡\}$ call suit $\quad |S| = 4$

Let $R = \{2, 3, 4 \cdots 10, J, Q, K, A\}$ call rank. $\quad |R| = 13$

Let $D = S \times R$ called the deck of cards $\quad |D| = 52$

Consider the game where you are give (deal) 5 cards without replacement s.t. order doesn't matter. These 5 cards of called a "hand".

$$P(\text{Royal Flash}) = \frac{|A|}{|\Omega|} = \frac{4}{\binom{52}{5}} \leftarrow \text{ways to pull out 5 cards}$$

10, J, Q, K, A
with same suit

P(4 of a kind) = $\dfrac{\binom{13}{1}\binom{48}{1}}{\binom{52}{5}}$ ← $\binom{13}{1}$ means 4 same rank at once

4 same rank

$\binom{48}{1}$ means remain one card can be anything else.

which make more sense to write $\binom{12}{1}\binom{4}{1}$

such that $\binom{4}{1}$ is any ~~rank~~ suit and $\binom{12}{1}$ is any rank

P(Full House) = $\dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$

3 same rank & 2 same rank

-4 means reduce Royal flash. $\binom{10}{1}-4=\binom{9}{1}$

P(Straight Flash) = $\dfrac{\binom{9}{1}\binom{4}{1}\cancel{-4}}{\binom{52}{5}}$ ← $\binom{10}{1}-4$ mean first number for A to 10 (J-K will be illegal, because we don't have ~~number~~ next to them to form Straight Flash)

5 order number with same suit

(A can be see the smallest)

$\binom{4}{1}$ means the different suit.

P(flash) = $\dfrac{\binom{4}{1}\binom{13}{5}-\binom{9}{1}\binom{4}{1}-4}{\binom{52}{5}}$ any suit (same) → $\binom{4}{1}$ any 5 card ~~fom~~ 13 $\binom{13}{5}$

without straight flash & Royal flash.

all same suit

But not straight

P(Straight) = $\dfrac{\binom{10}{1}\binom{4}{1}^5-\binom{9}{1}\binom{4}{1}-4}{\binom{52}{5}}$ $\binom{10}{1}$ A-5 through 10-A

$\binom{4}{1}^5$ any suit for each 5 cards

P(3 of kind) = $\dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$ any one from 13 card with 3 same $\binom{13}{1}\binom{4}{3}$

Ex. 777Q9

any one from 12 remain card with any ~~suit~~ ×2

$\binom{12}{2}\binom{4}{1}^2$

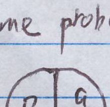
P(2 pairs) = $\dfrac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$ $\binom{13}{2}$ means two different rank from 13 rank.

$\binom{4}{2}$ means 4 suit choose any 2 suit for certain rank.

Notice that $\binom{13}{1}\binom{12}{1}$ in Full House $\neq$ $\binom{13}{2}$ in 2 pairs

order matter order doesn't matter.

Revisit the "working" definition that $\dfrac{|A|}{|\Omega|}$, which assume every outcome has the same probability. But in a experiment of spin a wheel



that $\dfrac{|\{R\}|}{|\Omega|} \neq \dfrac{1}{3}$, thus we need a new definition of probability