LECTURE - 03 09/05/2017

* Working Def. of Prob. :-

$$P : 2^{\Omega} \rightarrow [0, 1]$$

"degree"

event space, the power of outcome space

where 1 is coping $P(\Omega) = 1$.

where 0 is impossibility $P(\phi) = 0$.

$A \subseteq \Omega$

$A \leftarrow 2^{\Omega}$

$$P(A) = \frac{|A|}{|\Omega|}$$

- What is the prob. of getting $\underbrace{\text{a sum} = 3}_{\text{info. abt } A}$ $\underbrace{\text{on two die rolls?}}_{\text{info. abt } \Omega}$

Step 1: translate from English $\rightarrow \Omega$

$$\Rightarrow \Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}$$

$$\therefore |\Omega| = 36$$

Step 2: count $|\Omega|$

Step 3: Translate from English $\rightarrow A$

$$A = \{\langle 1,2\rangle, \langle 2,1\rangle\}$$

Step 4: Compare $|A|$

Step 5: Divide

$$\therefore P(A) = \frac{|A|}{|\Omega|} = \frac{3}{36} = \boxed{\frac{1}{12}}$$

- What is prob. of getting $\underbrace{2 \text{ Heads}}_{A}$ on a $\underbrace{4 \text{ coin flips}}_{\Omega}$?

$$\therefore P(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{6}{16}}$$

$\therefore \Omega = \{H, T\}^4$

$|\Omega| = 2^4 = 16$

$\therefore A = \{\langle H, H, T, T\rangle, \langle T, T, H, H\rangle,$
$\langle T, H, T, H\rangle, \langle H, T, H, T\rangle,$
$\langle H, T, T, H\rangle, \langle T, H, H, T\rangle\}$

** $P(HHHH) \stackrel{?}{=} P(HHTT) \neq$
$P(2H, 2T)$

- Prob. of at least 1H on 4 tosses?

$$\therefore P(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{15}{16}}$$

$\therefore A = \{HTTT, HHHH, \ldots\}$

** Recall $|\Omega| = |A| + |A^c|$

$$\Rightarrow |A| = |\Omega| - |A^c|$$
$$= 16 - 1$$
$$= 15$$

$\therefore A^c = \{\text{no heads}\} = \{T, T, T, T\}$

* <u>Complement Rule</u> :-

$$\therefore P(A) = \frac{|A|}{|\Omega|}$$

$$= \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|}$$

$$= 1 - P(A^c)$$

- Flip 10 coins. What is prob. of 4H?

$\therefore P(A) = \frac{|A|}{|\Omega|}$ 4H, 6T

4! 6!

$\therefore |\Omega| = 2^{10} = 1024$

→ How many ways to order the 10 flips? 10!

$$\therefore P(A) = \frac{\frac{10!}{4!\,6!}}{2^{10}} \approx \boxed{0.205}$$

* Counting Method :-

$F = \{ \text{Jane}, \text{Mary}, \text{Susan} \}$

There are 3 chairs. How many ways to seat those 3 women?

Seat 1	Seat 2	Seat 3
Jane	Mary	Susan
	Susan	Mary
Mary	Jane	Susan
	Susan	Jane
Susan	Mary	Jane
	Jane	Mary

** This is the "TREE" illustration.

- Total no. of ways :

$$\underset{\text{seat \#1}}{\underline{3}}^{\text{1st seating}} \cdot \underset{\text{seat \#2}}{\underline{2}}^{\text{if Jane seats out of 3,}} \cdot \underset{\text{seat \#3}}{\underline{1}}^{\text{if both seats down}} = \boxed{6}$$

$\therefore \Omega = \{ \langle J, M, S \rangle, \langle J, S, M \rangle, \langle M, J, S \rangle, \langle M, S, J \rangle, \langle S, M, J \rangle, \langle S, J, M \rangle \}$

Note: $\Omega \subseteq F^3$ ** $|\Omega| \neq F^3$

$|F^3| = |3^3| = 27$ $\{ \langle J, J, J \rangle \}$

$\therefore |\Omega| \neq F^3$

sampling 3 times "without replacements" → sampling 3 times "with replacements"

* <u>Sampling with & without Replacement:-</u>

- Sample n ($n \in \mathbb{N}$) items without replacements. How many possible outcomes?

  no. of choices:

  $\underset{1^{st}\text{ sample}}{\underline{n}}$ $\underset{2^{nd}\text{ sample}}{\underline{n-1}}$ ... $\underset{n-1^{th}\text{ sample}}{\underline{2}}$ $\underset{n^{th}\text{ sample}}{\underline{1}}$

  $$= \prod_{i=1}^{n} i := n!$$
  $$= {}_nP_n = n!$$

- Sample n with replacements. How many outcomes?

  $\underset{1^{st}\text{ sample}}{\underline{n}}$ $\underset{2^{nd}\text{ sample}}{\underline{n}}$ ... $\underline{n}$ $\underline{n}$

  $$= n^n > n! \quad \text{for } n \geq 2$$

- 5 people, 3 chairs. How many seating arrangements?

  $\underset{1^{st}\text{ chair}}{\underline{5}}$ $\underset{2^{nd}\text{ chair}}{\underline{4}}$ $\underset{3^{rd}\text{ chair}}{\underline{3}}$ $= \boxed{\dfrac{5!}{2!}}$

- Sample n items k times without replacements. How many outcomes?

$$\underset{1^{st}\text{ sample}}{\underline{n}} \quad \underset{2^{nd}\text{ sample}}{\underline{n-1}} \quad \_\_ \quad \underset{k^{th}\text{ sample}}{\underline{n-k+1}}$$

$$= \frac{n!}{(n-k)!} \quad \Big\} \text{ Permutation } {}_nP_n := \frac{n!}{(n-k)!}$$

** Permutation is about orderings which are unique.

$$\frac{n!}{(n-n)!} = \frac{n!}{0!} \rightarrow \text{undefined}$$

Convention $\boxed{0! := 1}$

- 3 couples (6 people) : Bob - Jane, Richard - Susan, Charles - Mary

out ace- nt

$$\therefore P\left(\begin{array}{c}\text{every couple} \\ \text{sits together}\end{array}\right) = \frac{|A|}{|\Omega|} \; \Big\{ \begin{array}{l}\text{all orders} \\ \text{of six people}\end{array}$$

$$= \frac{6.4.2}{6!} = \boxed{\frac{1}{15}}$$

$$\underset{\#1}{\underline{6}} \cdot \underset{\#2}{\underline{1}} \cdot \underset{\#3}{\underline{4}} \cdot \underset{\#4}{\underline{1}} \cdot \underset{\#5}{\underline{2}} \cdot \underset{\#6}{\underline{1}} = 6.4.2$$

OR $\rightarrow \therefore P(A) = \frac{|A|}{|\Omega|} = \frac{3!\,2^3}{6!} = \boxed{\frac{1}{15}}$

Using Love Seat (LS) :-

with replacement: $\dfrac{3}{LS\#1}\quad\dfrac{2}{LS\#2}\quad\dfrac{1}{LS\#3}$

$$\therefore P\left(\begin{matrix}\text{alternating}\\\text{gender}\end{matrix}\right) = \frac{(3!)^2 \cdot 2}{6!} = \boxed{\frac{1}{10}}$$

Male A_{MF} : $\dfrac{3}{\#1}\cdot\dfrac{3}{\#2}\cdot\dfrac{2}{\#3}\cdot\dfrac{2}{\#4}\cdot\dfrac{1}{\#5}\cdot\dfrac{1}{\#6}$

Female A_{FM} : $\dfrac{3}{\#1}\cdot\dfrac{3}{\#2}\cdot\dfrac{2}{\#3}\cdot\dfrac{2}{\#4}\cdot\dfrac{1}{\#5}\cdot\dfrac{1}{\#6}$

* Addition Rule of Prob. :-

$$\therefore P(A) = P(A_{MF}) + P(A_{FM})$$

- $P\left(\begin{matrix}\text{Richard \& susan}\\\text{sitting together}\end{matrix}\right) = \dfrac{4!\cdot 5\cdot 2}{6!} = \boxed{\dfrac{1}{3}}$

$\dfrac{1}{LS}\cdot\dfrac{4}{\#1}\cdot\dfrac{3}{\#2}\cdot\dfrac{2}{\#3}\cdot\dfrac{1}{\#4}$

switch LS: $\dfrac{4}{\#1}\cdot\dfrac{1}{LS}\cdot\dfrac{3}{\#2}\cdot\dfrac{2}{\#3}\cdot\dfrac{1}{\#4}$

$\underline{LS}$ — — — — — 4!

— $\underline{LS}$ — — — —

— — $\underline{LS}$ — — — } 5

— — — $\underline{LS}$ — —

— — — — $\underline{LS}$

Using Love Seat (LS) :-

with replacement $\left\{ \dfrac{3}{LS\#1} \quad \dfrac{2}{LS\#2} \quad \dfrac{1}{LS\#3} \right.$

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Male A_{MF} : $\dfrac{3}{\#1} \cdot \dfrac{3}{\#2} \cdot \dfrac{2}{\#3} \cdot \dfrac{2}{\#4} \cdot \dfrac{1}{\#5} \cdot \dfrac{1}{\#6}$

Female A_{FM} : $\dfrac{3}{\#1} \cdot \dfrac{3}{\#2} \cdot \dfrac{2}{\#3} \cdot \dfrac{2}{\#4} \cdot \dfrac{1}{\#5} \cdot \dfrac{1}{\#6}$

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- $P\left(\begin{matrix}\text{Richard \& susan}\\\text{sitting together}\end{matrix}\right) = \dfrac{4! \cdot 5 \cdot 2}{6!} = \boxed{\dfrac{1}{3}}$

$\dfrac{1}{LS} \cdot \dfrac{4}{\#1} \cdot \dfrac{3}{\#2} \cdot \dfrac{2}{\#3} \cdot \dfrac{1}{\#4}$

switch LS: $\dfrac{4}{\#1} \cdot \dfrac{1}{LS} \cdot \dfrac{3}{\#2} \cdot \dfrac{2}{\#3} \cdot \dfrac{1}{\#4}$

LS LS — — — — 4! ⎫
— — LS — — — ⎬ 5
— — — LS — — ⎭
LS
2!

- 100 balls, sample without replacement
  $= {}^{100}P_3 \approx 0.3202$

- 100 balls, sample with replacement
  $= 100^3$

$$\lim_{n\to\infty} \frac{{}^{n}P_{k}}{n^{k}} = \lim \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}$$

$$= \lim \frac{n}{n} \cdot \lim \frac{n-1}{n} \cdots \lim \frac{n-k+1}{n} = \boxed{1}$$

If n is large, sampling with replacement $\approx$ sampling without replacement

—— x ——