Math 241 11/2/17 Lec16

From last time...

$$Var(X) = E[(X-\mu)^2] = \sum_{x \in Supp(X)} (x-\mu)^2 p(x) \qquad Var(X) = \sigma^2 + \mu^2$$

$$Var[aX+c] = a^2\sigma^2$$

$$SE[aX+c] = |a|\sigma$$

$T_2 = X_1 + X_2$ Recall $E(T_2) = \mu_1 + \mu_2$

$$Var(T_2) = ? \quad E[((X_1+X_2) - (\mu_1+\mu_2))^2]$$

$$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2E(X_1X_2) - 2X_1\mu_1 - 2X_2\mu_2 - 2X_1\mu_2 - 2X_2\mu_1 + 2\mu_1\mu_2]$$

$$= E(X_1^2) + E(X^2) + 2E(X_1X_2) - 2\mu_1E(X_1) - 2\mu_2E(X_2) - 2\mu_2E(X_1) - 2\mu_1E(X_2) + 2\mu_1\mu_2$$

$$= \sigma_1^2 + \cancel{\mu_1^2} + \sigma_2^2 + \mu_2^2 \cancel{+\mu_1^2+\mu_2^2} + 2E(X_1X_2) - 2\mu_1^2 - 2\mu_2^2 - 2\mu_2\mu_1 \cancel{- 2\mu_2\mu_1 + 2\mu_1\mu_2}$$

$$\cancel{-\mu_1^2} \quad \cancel{-\mu_2^2}$$

$$= \sigma_1^2 + \sigma_2^2 + 2(\underbrace{E(X_1X_2) - \mu_1\mu_2}_{:= Cov[X_1,X_2]})$$

Covariance not covered in class

$$E[X_1X_2] = \sum_{x_1}\sum_{x_2} x_1x_2\, p(x_1,x_2) \qquad \text{If } X_1, X_2 \text{ indep} \Rightarrow p(x_1,x_2) = p(x_1)p(x_2)$$

$$= \sum_{x_1}\sum_{x_2} x_1x_2\, p(x_1)p(x_2) = \sum_{x_1} x_1 p(x_1) \sum_{x_2} x_2 p(x_2) = \mu_1\mu_2$$

$\Rightarrow Cov[X_1,X_2] = \mu_1\mu_2 - \mu_1\mu_2 = 0$ if X_1, X_2 are independent

$$\Rightarrow Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 \quad \text{if } X_1, X_2 \text{ indep.}$$

If $X_1, \dots, X_n$ indep.

if identically distr.

$$Var(T_n) = \sum_{i=1}^{n} \sigma_i^2 = n\sigma^2$$

if indep.

$$\Rightarrow Var[\overline{X}_n] = Var\left[\frac{1}{n} T_n\right] = \frac{1}{n^2} Var(T_n) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 \underset{\text{if iid}}{=} \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

$$\Rightarrow SE[\overline{X}_n] \underset{\text{if iid}}{=} \frac{\sigma}{\sqrt{n}}$$

$$T_n \sim Binom(n, p) \Rightarrow T_n = X_1 + \dots + X_n, \quad X_1, \dots, X_n \overset{iid}{\sim} Bern(p)$$

$$Var(T_n) \underset{\text{since iid}}{=} n\sigma^2 = np(1-p), \quad SE(T_n) = \sqrt{np(1-p)}$$

$$\underset{\text{or}}{=} \sum_{x=0}^{n} (x - np)^2 \binom{n}{x} p^x (1-p)^{n-x}$$

You choose!

σ^2 is the variance of a Geom(p)

$Y \sim \text{Geom}(p)$

$$Var(Y) = E(Y^2) - \left(\frac{1}{p}\right)^2$$

$$E(Y^2) = \sum_{y=1}^{\infty} y^2 (1-p)^{y-1} p$$

let $z = y-1 \Rightarrow y = z+1$

$y = 1 \ldots \infty$

$z = 0 \ldots \infty$

$$= \sum_{z=0}^{\infty} (z+1)^2 (1-p)^z p$$

$$= p\left(\sum_{z=0}^{\infty} (z^2 + 2z + 1)(1-p)^z\right)$$

$$= \underbrace{\sum_{z=0}^{\infty} z^2 (1-p)^z p} + \underbrace{2p \sum_{z=0}^{\infty} z(1-p)^z} + \overbrace{p \overbrace{\sum_{z=0}^{\infty} (1-p)^z}^{\frac{1}{p}}}^{1}$$

$$= \underbrace{(1-p) \sum_{z=1}^{\infty} z^2 (1-p)^{z-1} p}_{E(Y^2)} + 2(1-p) \underbrace{\sum_{z=1}^{\infty} z(1-p)^{z-1}}_{\substack{\text{Exp Geom} \\ \frac{1}{p}}}$$

$$\Rightarrow (1) E(Y^2) = (1-p) E[Y^2] + \frac{2(1-p)}{p} + 1$$

$1 - (1-p) = p$

$$\Rightarrow p\, E[Y^2] = \frac{2(1-p)}{p} + 1$$

$$\Rightarrow E(Y^2) = \frac{2(1-p)}{p^2} + \frac{1}{p}$$

$$\Rightarrow Var(Y) = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2-2p-1}{p^2} + \frac{p}{p^2}$$

$$\boxed{\sigma^2 = \frac{1-p}{p^2}}$$

$X \sim \text{Hyper}(n, K, N)$

$Var(X) = \ldots$ HARD... not covered!

L4

$X_1, X_2, \ldots \stackrel{iid}{\sim} \text{Bern}(p)$

$X \sim \text{Geom}(p)$ if X is stopping time

$$P(X=17 \mid X>10) = \frac{P(X=17 \ \& \ X>10)}{P(X>10)} = \frac{P(X=17)}{1-F(10)} = \frac{(1-p)^{16}p}{(1-p)^{10}}$$

$$= (1-p)^6 p$$

$$= P(X=7)$$

$$P(X=a+b \mid X>a) = \frac{P(X=a+b \ \& \ X>a)}{P(X>a)} = \frac{P(X=a+b)}{1-F(a)}$$

$$= \frac{(1-p)^{a+b-1}p}{(1-p)^a} = (1-p)^{b-1}p = P(X=b)$$

"Memorylessness" ... due to the iid Bernoullis. If you failed 1000 times, the 1001st begins a new geometric r.v. with parameter p.

Clinton Trump Nate Silver said $P(\text{Clinton Win}) = 0.75$

$X \sim \text{Bern}(0.75)$ Need a decision $\in \text{Supp}(X)$ choose $\text{Mode}[X]$!

$E(X) = 0.75$ Any meaning?

$P(\text{Clinton win} \mid \text{NS's models being correct})$

Models:

$$\underset{\text{electoral votes}}{Y} = f(\underbrace{X_1, \ldots, X_p}_{\text{covariates}} \mid \underbrace{\beta_1, \ldots, \beta_K}_{\text{params}}) + \mathcal{E}, \quad \mathcal{E} \sim h(\ldots)$$

MCMC Simulate all sim's many times...

$\beta_1 = g_1(X_1, \ldots X_p \mid \gamma_1, \ldots, \gamma_L)$

$\beta_2 = g_2(\cdot \ \ldots \mid \delta_1, \ldots \delta_J)$ etc...

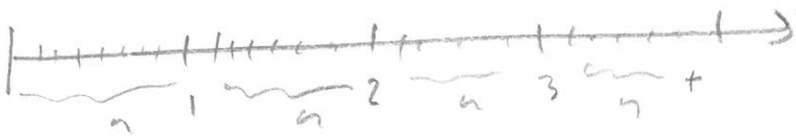
Bayes 341

More about the geometric r.v. ...

$T \sim \text{Geometric}(p) :: p(t) = (1-p)^{t-1}p$, $F(t) = 1-(1-p)^t$ $\quad E(T) = \frac{1}{p}\text{ exp} \cdot 1 \frac{\text{sec}}{\text{exp}}$

every second one iid Bernoulli...

Now... every second has n Bernoullis



$p(t) = (1-p)^{nt-1} p$, $F(t) = 1-(1-p)^{nt}$

$E[T] = \frac{1}{p}\text{ exp} \cdot \frac{1\text{ sec}}{n\text{ exp}} = \frac{1}{np}$

If n is large... immediately stop...

But also if $p \approx 0$ really small

let $\lambda = np$, n large, p small but the product of the two is constant

$\Rightarrow p = \frac{\lambda}{n}$ Using this substitution...

$\Rightarrow p(t) = \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n}$, $F(t) = 1-\left(1-\frac{\lambda}{n}\right)^{nt}$

Midterm 2 ↑

Final ↓

Now let $n \to \infty$ so that in every second there's infinite experiments; experiments occur continuously. What is the PMF?

$$\lim_{n\to\infty} p(t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n} = \underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt-1}}_{?} \underbrace{\lim_{n\to\infty} \frac{\lambda}{n}}_{0} = 0$$

No valid PMF!

$\sum_t p(t) = 0! \neq 1$

What about CDF?

$$\lim_{n\to\infty} F(t) = \lim 1-\left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt} = 1-\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\right)^{t}$$

$$\lim f(x)^a = \left(\lim f(x)\right)^a$$

famous limit!

Consider

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

n	$f(n)$
10	2.594
100	2.705
1000	2.717
10000	2.718
⋮	⋮

Converges... $\Rightarrow e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

or $e := \sum_{i=0}^{\infty} \frac{1}{i!}$ or $\int_1^e \frac{1}{x}\,dx = 1$

How about $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$ s.t. $a \in \mathbb{R}$ constant

let $\frac{1}{m} = \frac{a}{n} \Rightarrow n = ma$ if $n\to\infty \Rightarrow m \to \infty$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{ma} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^a = e^a$$

$$\Rightarrow \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \qquad \Rightarrow \lim F(t) = 1 - e^{-\lambda t}$$

Is this a valid CDF?