Lecture -11

$X \sim \text{Hyper}(n, K, N) = \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$ | $X \sim \text{hyper}(n, p, N) = \dfrac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$ s.t. $p := \frac{K}{N}$

$$\lim_{N\to\infty} \text{hyper}(n, p, N) = \lim_{N\to\infty} \frac{\dfrac{(pN)!\,((1-p)N)!}{x!\,(pN-x)!\,(n-x)!\,((1-p)N-(n-x))!}}{\dfrac{N!}{n!\,(N-n)!}}$$

"limiting PMF"
↳ only thing that's changing is N

$$= \binom{n}{x} \lim_{N\to\infty} \frac{\dfrac{(pN)!\,((1-p)N)!}{(pN-x)!\,((1-p)N-(n-x))!}}{\underbrace{\dfrac{N!}{(N-n)!}}_{N(N-1)\cdots(N-n+1)}}$$

$\lim f(x) \cdot g(x)$
$= \lim f(x) \cdot \lim g(x)$

$$= \binom{n}{x} \overbrace{\underbrace{\lim_{N\to\infty}\frac{pN}{N}}_{p}\ \underbrace{\lim_{N\to\infty}\frac{pN-1}{N-1}}_{p}\cdots\underbrace{\lim_{N\to\infty}\frac{pN-x+1}{N-x+1}}_{p}}^{p^x}\ \overbrace{\underbrace{\lim_{N\to\infty}\frac{(1-p)N}{N-x}}_{1-p}\ \underbrace{\lim_{N\to\infty}\frac{(1-p)N-1}{N-x-1}}_{1-p}\cdots\underbrace{\lim_{N\to\infty}\frac{(1-p)N-(n-x)+1}{N-n+1}}_{1-p}}^{(1-p)^{n-x}}$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

$X \sim \text{Binomial}(n, p) := \binom{n}{x} p^x (1-p)^{n-x}$
↳ Sampling n with replacement and asking "how many special balls did I guess?"

Parameter space
$n \in \mathbb{N}$
$p \in (0, 1)$

Support $[X] = \{0, 1, \dots n\}$
We want to show,

$$\sum_{x \in \text{Supp}(X)} P(x) = 1 \Rightarrow \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = ((p)+(1-p))^n = 1^n = 1$$

Recall: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Independent Random Variables

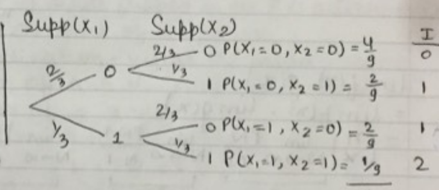
X_1 & X_2 are independent r.v.'s if

a) $P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$ for all $x_1 \in \text{Supp}[X_1]$, for all $x_2 = \text{Supp}(X_2)$
b) $P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$
c) $P(X_1 = x_1 \,;\, X_2 = x_2) = P(X_1 = x_1) \cdot P(X_2 = x_2)$ (∵ multiplication rule)
AND

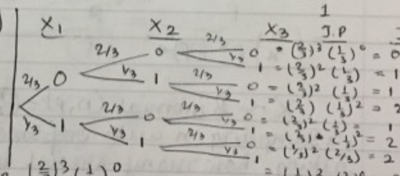
JOINT MASS FUNCTION

X_1, X_2 are "iid" (∵ iid = independent & identically distributed)
$X_1, X_2 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$ (PMF are the same)

Q $T_2 := X_1 + X_2$
$\text{Supp}[X_1] = \{0, 1\}$
$\text{Supp}[X_2] = \{0, 1\}$
$\text{Supp}[T_2] = \{0, 1, 2\}$



Q $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$
$T_3 := X_1 + X_2 + X_3$
$\text{Supp}[T_3] = \{0, 1, 2, 3\}$



$$\therefore T_3 \sim \begin{cases} 0 & \text{wp } \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 \\ 1 & \text{wp } \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \\ 2 & \text{wp } \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \\ 3 & \text{wp } \left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 \end{cases}$$

$X_1, X_2, \dots X_n \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$
$T_n = \sum_{i=1}^{n} X_i$
$\text{Supp}(T_n) = \{0, 1, \dots, n\}$

$$T_n \sim \begin{cases} 0 & \text{wp } \binom{n}{0}(p)^0(1-p)^n \\ 1 & \text{wp } \binom{n}{1}(p)^1(1-p)^{n-1} \\ 2 & \text{wp } \binom{n}{2}(p)^2(1-p)^{n-2} \\ n-1 & \text{wp } \binom{n}{n-1}(p)^{n-1}(1-p)^1 \\ n & \text{wp } \binom{n}{n}(p)^n(1-p)^0 \end{cases}$$

$T_n \sim \text{Bern}(n, p) := \binom{n}{x} p^x (1-p)^{n-x}$

The binomial conceptually is

$T = \lim_{N\to\infty} \text{Hyper}(n, p, N)$

$T = X_1 + \dots + X_n$ where $X_1, \dots X_n \overset{iid}{\sim} \text{Bern}(p)$