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Working Definition of Probability

$$P: 2^{\Omega} \rightarrow [0, 1]$$

↑
domain:
event space
all $A \subseteq \Omega$

↑
range
1: certainty $P(\Omega) = 1$
0: improbability $P(\emptyset) = 0$

$$P(A) = \frac{|A|}{|\Omega|}$$

$$A \subseteq \Omega$$

$$A \in 2^{\Omega}$$

What's the probability of getting a sum of 3 if you roll 2 dice?

Step 1: Translate from English to obtain Ω

$$\Omega = \{1, 2, 3, 4, 5, 6\}^2$$

Step 2: Find $|\Omega|$

$$|\Omega| = |\{1, \dots, 6\}|^2 = 6^2 = 36$$

Step 3: Translate from English to obtain A

$$A = \{(1, 2), (2, 1)\}$$

Step 4: Compute $|A|$

$$|A| = 2$$

Step 5: Divide

$$P(A) = \frac{|A|}{|\Omega|}$$

What's the prob. of getting 2 Heads on 4 coins tossed

1: $\Omega = \{\text{Heads, Tails}\}^2$

2: $|\Omega| = 16$

3: $A = \{\text{HHTT, HTHT, HTTH, TTHH, THTH, THHT}\}$

4: $|A| = 6$

$P(\text{HHHH}) = P(\text{HHTT}) \neq P(2H)$

Recall: $\Omega = A \cup A^c$ and $\{A, A^c\}$ are mutually exclusive

$$|\Omega| = |A| + |A^c|$$

$$\Rightarrow |A| = |\Omega| - |A^c|$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - P(A^c)$$

$$A = \{\geq 1 \text{ H}\}$$

$$A^c = \{< 1 \text{ H} = \text{zero H}\}$$

$$A^c = \{\text{T, T, T, T}\}$$

$$P(A) = 1 - \frac{1}{16} = \frac{15}{16}$$

COMPLEMENT
RULE

Flip 10 coins. What's the prob.
of 4 H?

$$(1) P(A) = \frac{|A|}{|S|} = \frac{1}{1024}$$

$$|S| = 2^{10} = 1024$$

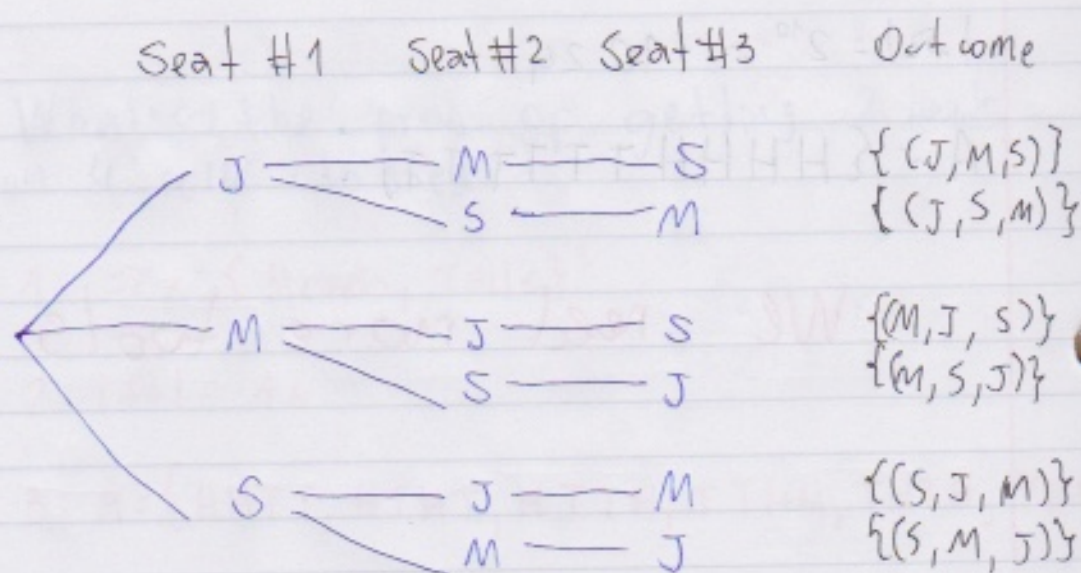
$$A = \{HHHH TTTT, \dots\}$$

We need more tools.

$F = \{ \text{Jane, Mary, Susan} \}$

How many ways to sit those women in 3 chairs?

Let's illustrate



$$|\Omega| = 6$$

$$\frac{3}{\text{Seat \#1}} \cdot \frac{2}{\text{Seat \#2}} \cdot \frac{1}{\text{Seat \#3}} = 6$$

Note

$$\Omega \neq F^3$$

$$6 = |\Omega| \neq |F^3| = 27$$

$$\Omega \subset F^3$$

$$(J, J, J) \in F^3$$

$$(J, J, J) \notin \Omega$$

S_2 represents the set of F without replacement

S_3 represents the set of F with replacement

ways to sample n objects without replacement.

$$\frac{n}{1^{\text{st}} \text{ sample}} \cdot \frac{n-1}{2^{\text{nd}}} \cdot \dots \cdot \frac{2}{(n-1)^{\text{th}} \text{ sample}} \cdot \frac{1}{n^{\text{th}}} = \prod_{i=1}^n i = n!$$

ways to sample n object with replacement.

$$\frac{n}{1^{\text{st}}} \cdot \frac{n}{2^{\text{nd}}} \cdot \frac{n}{3^{\text{rd}}} \cdot \dots = n^n$$

of ways to seat 10 people in 3 chairs?

$$\frac{10}{\text{seat \#1}} \cdot \frac{9}{\text{\#2}} \cdot \frac{8}{\text{\#3}} = \frac{10!}{7!}$$

of ways to sample K objects from a set of n objects without replacements

$$= \frac{n!}{(n-k)!}$$

$$\frac{n}{\text{\#1}} \cdot \frac{n-1}{\text{\#2}} \cdots \frac{n-(k+1)}{\text{\#K}} = \frac{n!}{(n-k)!}$$

$${}_n P_k := \frac{n!}{(n-k)!}$$

${}_n$ Permute k

Permutations

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

$$0! := 1$$

What's the prob. that Jane and Susan sit together?

$P(J \text{ and } S \text{ sit together})$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\{JSM, MJS, SJM, MSJ\}|}{3!} = \frac{4}{6}$$

3 couples (6 people)

Bob - Jane

Richard - Susan

Charles - Mary

They are seated into 6 chairs

$P(\text{the couples sit next to each other})$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 4 \cdot 2}{6!} = \frac{1}{15}$$

$$\begin{array}{cccccc} \frac{6}{1} & \frac{1}{2} & \frac{4}{3} & \frac{1}{4} & \frac{2}{5} & \frac{1}{6} \end{array}$$

Alternative way

Loveseat #	$\frac{3 \text{ couples}}{1 \curvearrowright 2}$	$\frac{2 \text{ couples}}{2 \curvearrowright 2}$	$\frac{1 \text{ couple}}{3 \curvearrowright 2}$
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$$\frac{(\cancel{6} \cdot \cancel{2} \cdot \cancel{4}) \cancel{2}^3}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = \frac{1}{15}$$

What's the probability of alternating gender

$P(\text{alternating gender})$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 3 \cdot 2 \cdot 2}{6!} = \frac{1}{10}$$

$$\begin{array}{cccccc} \text{seat \#} & \frac{6}{1} & \frac{3}{2} & \frac{2}{3} & \frac{2}{4} & \frac{1}{5} & \frac{1}{6} \end{array}$$

Alternative Way

This Will
Fail Later?

$$A = A_{1st \text{ Boy}} \cup A_{1st \text{ Girl}}$$

mutually Exclusive

$$|A| = |A_{1st \text{ Boy}}| + |A_{1st \text{ Girl}}| \text{ divide by } |S|$$

$$P(A) = P(A_{1st \text{ B}}) + P(A_{1st \text{ G}}) \text{ Addition rule}$$

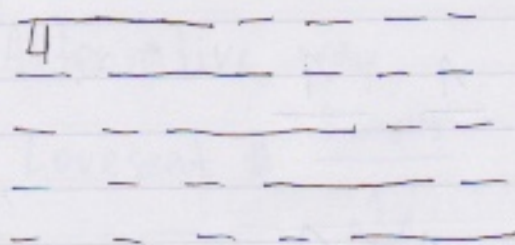
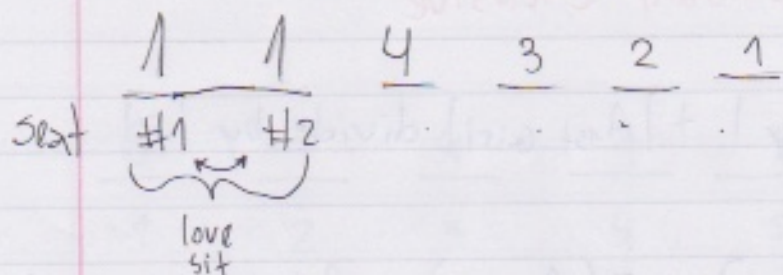
$$\frac{3}{\#1} \quad \frac{3}{\dots} \quad \frac{2}{\dots} \quad \frac{2}{\dots} \quad \frac{1}{\dots} \quad \frac{1}{\dots}$$

$$\frac{3}{\#1} \quad \frac{3}{\dots} \quad \frac{2}{\dots} \quad \frac{2}{\dots} \quad \frac{1}{\dots} \quad \frac{1}{\dots}$$

$$P(A) = \frac{|A|}{|S|} = \frac{6 \cdot 3 \cdot 2 \cdot 2}{6!} = \frac{1}{10}$$

$P(R-S \text{ sit together})$

$$P(A) = \frac{|A|}{|S|} = \frac{\quad}{6!}$$



plus
they
can flip

$$P(A) = \frac{|A|}{|S|} = \frac{4! \cdot 2 \cdot 5}{6!} = \left(\frac{1}{3} \right)$$

What's the probability of alternating gender?

$P(\text{alternating gender})$

$$P(A) = \frac{|A|}{|S|} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{6!} = \frac{1}{60}$$

0 3 2 2 1 1

seat 4 3 2 1 5 6

of ways to sample 3 marbles
from a bag of 100 without replacement

$${}_{100}P_3$$

Same with replacement

$$\frac{{}_{100}P_3}{100^3} \approx 0.97$$

If $n \rightarrow \infty$ what does the # ways
to sample without replacement look like?
with replacement

$$\lim_{n \rightarrow \infty} \frac{{}_n P_k}{n^k} = \lim_{n \rightarrow \infty} \frac{(n)(n-1) \cdots (n-k+1)}{(n)(n) \cdots (n)}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n} \quad \lim_{n \rightarrow \infty} \frac{n-1}{n} \cdots \lim_{n \rightarrow \infty} \frac{n-k+1}{n} = 1$$