11/27

Re call

$M_x(t) := E(e^{tx})$

Mgf

Important Rules

I $M_x(t) = M_Y(t) \Leftrightarrow X \overset{d}{=} Y$

II $E[X^k] = M_x^{(k)}(0)$

III $Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_x(at)$

IV if X, Y independent $\Rightarrow M_{X+Y}(t) = M_x(t) M_Y(t)$

New V $\lim_{n\to\infty} M_{X_n}(t) = M_X(t) \Rightarrow X_n \to X$ CDF's are limiting i.e. $\lim_{n\to\infty} F(x) = F_X(x)\ \forall x$

$X \sim Bern(p) \Rightarrow M_x = 1 - p + pe^t$

$X \sim Binom(n, p) \Rightarrow M_x(t) = (1 - p + pe^t)^n$

$X \sim Exp(\lambda) \Rightarrow M_x(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$

$X \sim N(0,1) \Rightarrow M_Z(t) = e^{t^2/2}$

$X \sim N(\mu, \sigma^2)$

Rule 3 $X = \mu + \sigma Z \sim N(\mu, \sigma^2) \Rightarrow M_x(t) = e^{t\mu} M_Z(\sigma t)$

$= e^{t\mu} e^{(\sigma t)^2/2} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$X \sim Deg(c),\ M_x(t) = E[e^{tX}] = e^{tc}$

$X_1 \sim N(\mu_1, \sigma_1^2)$

ind of

$X_2 \sim N(\mu_2, \sigma_2^2)$

$Y = X_1 + X_2 \sim ?$ $\quad M_Y(t) = M_{X_1}(t) M_{X_2}(t)$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2) t^2}{2}}$$

$$\Rightarrow Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

One mere Rule:

V) Levy's Continuity Theorem:

Let $X_1, X_2, \ldots$ be a sequence of r.v.'s

if $\lim_{n \to \infty} M_{X_n}(t) = M_X(t) \Rightarrow \underbrace{\lim_{n \to \infty} F_{X_n}(x) = F_X(x)}_{\text{conversion in distribution}}$

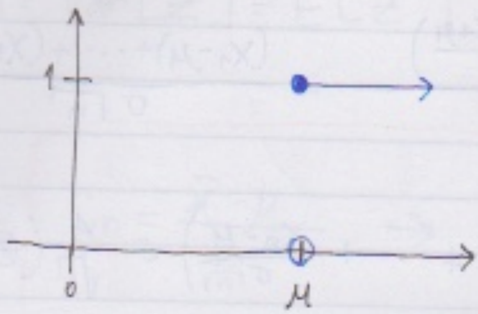
$\bar{X} \to \mu = E[X]$

Law of Large #'s

Assume $X_1, X_2, \ldots$ iid with mean μ.

$\lim M_{\bar{X}_n}(t) = e^{t\mu}$ ← mgf for the Deg(μ)

$M \sim \text{Deg}(\mu)$



$$\Rightarrow \lim M_{\frac{X_1+\cdots+X_n}{n}}(t) \overset{\text{Rule III}}{=} \lim M_{X_1+\cdots+X_n}\left(\tfrac{t}{n}\right) \overset{\text{Rule IV}}{=} \ldots$$

$$= \lim M_{X_1}\left(\tfrac{t}{n}\right) \cdot M_{X_2}\left(\tfrac{t}{n}\right) \cdot \ldots \cdot M_{X_n}\left(\tfrac{t}{n}\right) = \lim_{n\to\infty} \left(M_X\left(\tfrac{t}{n}\right)\right)^n$$

$$= \lim_{n\to\infty} e^{\ln\left(\left(M_X\left(\frac{t}{n}\right)\right)^n\right)} = \lim_{n\to\infty} e^{n \ln\left(M_X\left(\frac{t}{n}\right)\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}}$$

$$= e^{\lim_{n\to\infty} \frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}} = e^{\lim_{n\to 0} \frac{\ln(M_X(nt))}{n}} \overset{\text{L'Hospital Rule}}{=} e^{\lim_{n\to 0} \frac{t M_X'(nt)}{M_X(nt)}}$$

$$= e^{\frac{t M_X'(0)}{M_X(0)}} = \boxed{e^{t\mu}}$$

$$= e^{t^2/2} \lim_{n\to 0} \frac{M_z(ut)\,\cancel{t}\,M_z''(ut) - M_z'(ut)^2\,\cancel{t}}{\frac{M_z(ut)^2}{\cancel{t}}}$$

$$= e^{t^2/2}\, \frac{\cancel{M_z(0)}^{1}\,\cancel{M_z''(0)}^{1} - \cancel{M_z'(0)}^{0}{}^2}{\cancel{M_z(0)^2}_{1}} = \boxed{e^{\frac{t^2}{2}}}$$

or 2?

$$1 = Var[Z] = E[Z^n] - \cancel{E[Z]}^{0}$$

Rule
V

$$\Rightarrow \boxed{C_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \to N(0,1)}$$

Central Limit Theorem (CLT)

How to use CLT to solve problems

Note $n \to \infty$ is impossible
C_n now truly converges

Important

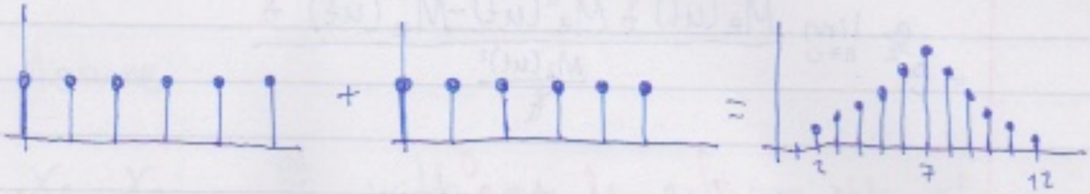
If n is "large enough"

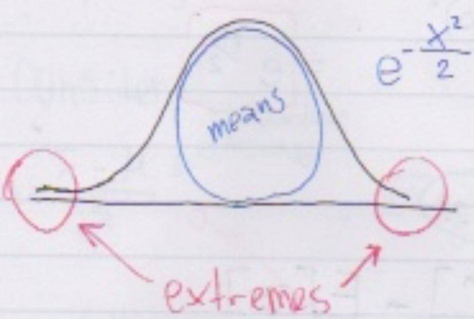
① $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\approx} N(0,1)$

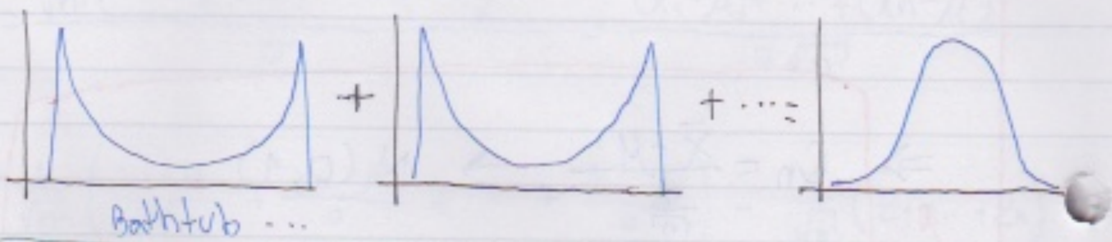
* ② $\bar{X} \overset{d}{\approx} N(\mu, (\frac{\sigma}{\sqrt{n}})^2)$

* ③ $T \overset{d}{\approx} N(n\mu, (\sigma\sqrt{n})^2)$

* Most used to solve problems







example

$X_1, \cdots, X_{30} \overset{iid}{\sim} \text{Geom}\left(\frac{1}{2}\right) \Rightarrow \mu = \frac{1}{\frac{1}{2}} = 2, \ \sigma = \frac{\sqrt{1-p}}{p}$

$= \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}} \approx \boxed{1.414}$

What's the probability the avg wait time is more than 2.75?

$P(\bar{X} \geq 2.75) = P\left(\frac{\bar{X}-2}{.258} \geq \frac{2.75-2}{.258}\right) = P(Z \geq 3) \approx \boxed{.0015}$

$n = 30 \Rightarrow$ "large" $\Rightarrow$ CLT

$\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) = N\left(2, \left(\frac{1.414}{\sqrt{30}}\right)^2\right) = N(2, 0.258^2)$

example:

Take 100 steps with probability forward & backward being $\frac{1}{2}$

$$X \sim \begin{cases} +1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$$

What's the probability you are more than 10 steps away from starting position after 100 steps

$$T = X_1 + \cdots + X_{100} \overset{d}{\underset{3}{\approx}} N(nm, (\sigma\sqrt{n})^2)$$

$$= N(0, 10^2) = P(|T| \geq 10)$$