MATH 24A Lec 2 8/30/17

Set builder notation

$E := \{2n : n \in \mathbb{Z}\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$

all elements $2n$ s.t. n is an integer

$A := \{1,2,3\}$

Lec 1
Lec 2

$2^A := \{B : B \subseteq A\} = \dots$

powerset of A ~~set~~ set of all subsets

$\{3,2\} \in 2^A$?

size of set / cardinality

abs. value sign

$|A| = 3$ just count elements! $f : \text{set} \to \mathbb{N}_0$

$|F \cup M| \overset{?}{=} |F| + |M|$ No... $7 \neq 4 + 4 = 8$

$|F \cap M| \overset{?}{=} |F| - |M|$ No... $1 \neq 4 - 4 = 0$

$|F \setminus M| = |F| - |M|$ No ... $3 \neq 4 - 4 = 0$

$|2^A| = 8$ why? $\left\{ \underset{1}{F}\ \underset{2}{F}\ \underset{3}{F},\ \underset{1}{T}\ \underset{2}{F}\ \underset{3}{F},\ \dots,\ \underset{1}{T}\ \underset{2}{T}\ \underset{3}{T} \right\}$

{true or false}
{true or false}
{true or false}

heuristic / method of counting
rule of thumb that generally is useful

$2 \cdot 2 \cdot 2 = 8 = 2^3$

$\Rightarrow |2^S| = 2^{|S|}$

rigorous proof requires graduate knowledge of set theory

A. 2

$$|2^A| = 8 \quad \{ \underset{\phi}{\underline{F}_1\,\underline{F}_2\,\underline{F}_3},\ \underset{\{1\}}{\underline{T}_1\,\underline{F}_2\,\underline{F}_3},\ \underset{\{2\}}{\underline{F}_1\,\underline{T}_2\,\underline{F}_3},\ \ldots,\ \underset{A}{\underline{T}_1\,\underline{T}_2\,\underline{T}_3} \}$$

$|2^F| = ?$ 16

What are all possible configurations?

$$|\{T,F\}| \cdot |\{T,F\}| \cdot |\{T,F\}| = 2 \cdot 2 \cdot 2 = 8$$
1 2 3

heuristic
method of counting

heuristic: an approach to problem solving, practical method, not guaranteed to be optimal but sufficient to get the job done...

for any set S...

$$|2^S| = 2^{|S|}$$

rigorous
proof requires ZF set axioms

Special set: Ω "universe", "sample space", "space of discourse",
CS: "scope". All elements we are limited to. You define it.

$$\Omega := F \cup M = \{ \ \_\_\_\_ \ \}$$

Note: $F \subseteq \Omega$, $M \subseteq \Omega$, $2^F \overset{?}{\subseteq} \Omega$ (HW)

~~Coin Flip $\Omega := \{H, T\}$, Die Roll: $\Omega := \{1,2,3,4,5,6\}$~~

~~What is the probability a "random" name is a female?~~

~~definition~~ ~~define~~

~~$P(F) = \frac{|F|}{|\Omega|} = \frac{4}{7}$~~

~~always assume scope!~~

~~name~~
~~working def: $P(A) = \frac{|A|}{|\Omega|}$~~

$F \cap \Omega =$

$F \cup \Omega =$

$\phi \cup \Omega =$

$\phi \cap \Omega =$

$F \setminus \Omega =$

$\Omega \setminus F = \{$Bob, Joe, Mia$\}$ i.e. everybody not in F!

$A^c := \Omega \setminus A$

Venn Diagram

$(A^c)^c = A$

$A \cup A^c = \Omega$ $\quad C = \{A, A^c\}$. C's elements are "collectively exhaustive" i.e. their union is Ω.

$A \cap A^c = \phi$ mutually exclusive

$A \overset{?}{\subseteq} A^c$ No...

For finite sets $|A| + |A^c| = |\Omega| \Rightarrow |A| = |\Omega| - |A^c|$

we will see this is important for the complement rule of prob.

$\{A_1, A_2, \ldots\}$ are collectively exhaustive $\Rightarrow \bigcup_{i=1}^{\infty} A_i = \Omega$

$\{A_1, A_2, \ldots\}$ are mutually exclusive $\Rightarrow A_i \cap A_j = \phi \quad \forall i \neq j$

Not covered on exams!

$\mathbb{N} = \{1, 2, \dots\}$

$|\mathbb{N}| = \infty$ ... but it is a special type of ∞ called $\aleph_0$. Countable infinity

$|\mathbb{Z}| \stackrel{?}{=} 2\aleph_0 + 1$

Def $|A| = |B|$ really means I can find a 1:1 function between A & B.

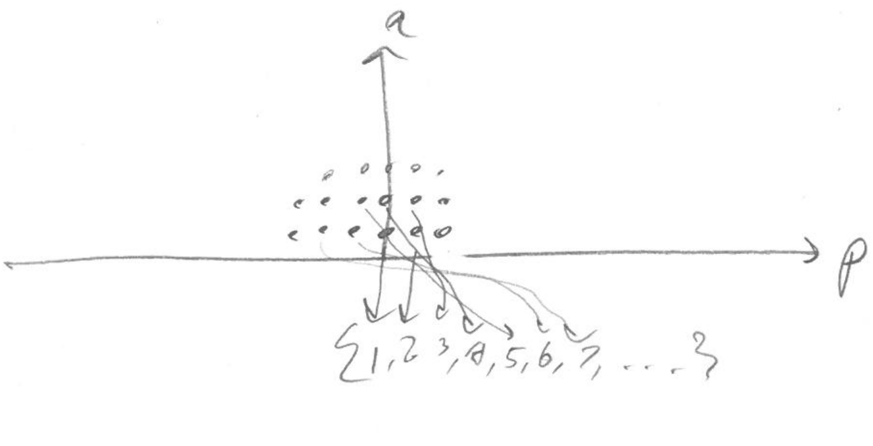
$\{1, 2, 3, 4, 5, \dots\}$

$\{\dots, -2, -1, 0, 1, 2, \dots\}$

$\Rightarrow |\mathbb{Z}| = |\mathbb{N}| = \aleph_0$

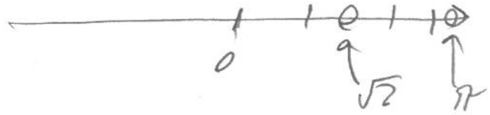
$\mathbb{Q} := \{\frac{p}{q} \text{ s.t. } p \in \mathbb{Z}, q \in \mathbb{N}\}$ i.e. the rationals

$|\mathbb{Q}| > \aleph_0$ all fractions! It must be!! No..



$\Rightarrow |\mathbb{Q}| = |\mathbb{N}| = \aleph_0$

But not all #'s $\in \mathbb{Q}$ e.g. $\sqrt{2} \notin \mathbb{Q}$, $\pi \notin \mathbb{Q}$, etc..



Holes!

a algebraic #, these are countable → root of polynomial

a transcendental #, infinite non-repeating decimal

$\mathbb{R} := \mathbb{Q} \cup \{\sqrt{2}, \pi\} \cup$ all others

Notation

$[a,b] := \{x : x \geq a \ \& \ x \leq b\}$

$(a,b) := \{x : x > a \ \& \ x < b\}$

Does $|\mathbb{R}| = \aleph_0$?

Let $A = [0,1]$ is $|A| = \aleph_0$? If not, then $|\mathbb{R}| \neq \aleph_0$

If $|A| = \aleph_0 \Rightarrow$ all $x \in A$ have a base 2 decimal expansion such as:

$\Rightarrow A = \{x_1, x_2, x_3, \dots\}$

x_1 $\quad 0.011010\dots$

x_2 $\quad 0.001010\dots$

x_3 $\quad 0.001111\dots$

x_4 $\quad 0.111010\dots$

$\vdots$

What if I "flip" the diagonal to produce $x^* = 0.1101\dots$

$x^* \notin A$ since $x^* \neq x_1$, $x^* \neq x_2, \dots$

$\Rightarrow |A| \neq \aleph_0 \Rightarrow |A| = c$ is uncountable infinity $|\mathbb{R}| = c > \aleph_0$

Ordered Pairs

$$\langle a,b\rangle := \{\{a\},\{a,b\}\}$$

↑
a, b in this order

Does this do its job??

$\langle a,b\rangle \neq \langle b,a\rangle = \{\{b\},\{a,b\}\}$ since $\{b\} \notin \{\{a\},\{a,b\}\}$

$\langle a,a\rangle \neq \{a\}$

Yes...

Cartesian Product

$$A \times B := \{\langle a,b\rangle : a \in A,\ b \in B\}$$

e.g.

$A = \{1,2\}$, $B = \{3,4\}$

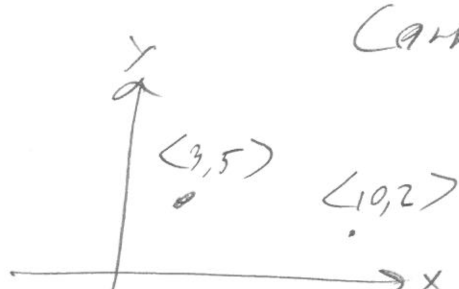
$A \times B = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 3,4\rangle\}$

$|A \times B| = 4 = |A|\,|B| = 2 \cdot 2$

$|A_1 \times A_2 \times \ldots \times A_n| = \prod_{i=1}^{n} |A_i|$

$A^2 := A \times A$

Cartesian plane



$|A^2| = |A||A| = |A|^2$

$\mathbb{R} \times \mathbb{R}$

Now for probability.

Ω is called the sample space or experimental space or outcome space. It's elements are called outcomes of the experiment.

$\omega_1 \in \Omega$, $\omega_2 \in \Omega$, etc are outcomes

e.g. coin toss "experiment"

$$\Omega = \{ \underset{\omega_1}{\overset{\shortparallel}{H}}, \underset{\omega_2}{\overset{\shortparallel}{T}} \}$$ where each outcome represents a possible outcome of the experimental trial.

Are H, T mutually exclusive? Be careful! They're not sets!

$\{H\}, \{T\}$ are mutually exclusive

Are they collectively exhaustive? Yes $\{H\} \cup \{T\} = \Omega$

Sets of outcomes are called events. What are all events of Ω?

$2^{\Omega} = \{\phi, \{H\}, \{T\}, \{H,T\}\}$

$\nwarrow$ event space

Recall $P(A) = \frac{|A|}{|\Omega|}$ working definition...

$\Rightarrow P : 2^{\Omega} \to [0,1]$ It's a set function!

$P(H)$ undefined!

$P(\{H\}) = \frac{|\{H\}|}{|\{H,T\}|} = \frac{1}{2}$

$\{H,T\}$ prob of either heads or tails

$P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$

$P(\phi) = \frac{|\phi|}{|\Omega|} = 0$

Ω, ϕ are the "trivial" events

Die Roll

$\Omega = \{1,2,3,4,5,6\}$

$|2^{\Omega}| = \cancel{32}\ 64$ Lots of prob questions to ask.

$P(\{1,3,5\})$ odd # $= \frac{3}{6}$

$P(\{1,2,5,6\}) = \frac{4}{6}$

$\vdots$ etc

$|2^{\Omega}| > |\Omega|$

size of event space is greater than size of outcome space!

If it were the same, you can never ask $P(\{1,3,5\})$ e.g.!

Let's do two coin flips

$\Omega' = \Omega^2 = \{\langle H,H\rangle, \langle H,T\rangle, \langle T,T\rangle, \langle T,H\rangle\}$

Ω'

HH	HT
TH	TT

$P(HH) = P(\{\langle H,H\rangle\}) = \frac{|\{\langle H,H\rangle\}|}{|\Omega'|} = \frac{1}{4}$

abuse of notation .. but who cares...

$2^{\Omega'} = \{\emptyset, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, TH\}, \dots, \Omega\}$ a lot!

$|2^{\Omega'}| = 2^{|\Omega'|} = 2^4 = 16$ Let $B :=$ at least one tail

$$P(B) = \frac{|\{\langle H,T\rangle, \langle T,H\rangle, \langle T,T\rangle\}|}{|\Omega'|} = \frac{3}{4}$$

Choosing this set and counting it is where all the action is!

$D :=$ one Head

$$P(D) = \frac{|\{\langle H,T\rangle, \langle T,H\rangle\}|}{|\Omega'|} = \frac{2}{4}$$