Lecture :- 16

~~E[X2]~~ $T_2 = X_1 + X_2$

$E[T_2] = E(X_1] + E[X_2]$

$Var[T_2] = ?$

$Var[X] = E[(X-\mu)^2]$

$E[X^2] = \sigma^2 + \mu^2$

$Var[aX+c] = a^2\sigma^2$

$SE[aX+C] = |a|.\sigma$

$$
\begin{aligned}
Var[X_1+X_2] &= E[((X_1+X_2)-(\mu_1+\mu_2))^2] \\
&= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2 + 2\mu_1\nu_2] \\
&= E[X_1^2] + E[X_2^2] + \mu_1^2 + \mu_2^2 + 2E[X_1 \cdot X_2] - 2\mu^2 - 2\mu_1 \cdot \mu_2 - 2\mu_2 \cdot \mu_1 - 2\mu_2^2 + 2\mu_1 \cdot \nu_2 \\
&= \sigma_1^2 + \sigma_2^2 + 2\, E([X_1 \cdot X_2] - \mu_1 \cdot \mu_2)
\end{aligned}
$$

only when independent covariance = $Cov[X_1, X_2]$

$$E[X_1, X_2] = \sum_{X_1}\sum_{X_2} X_1 X_2\, P(X_1, X_2) = \sum_{X_1}\cdot\sum_{X_2} X_1, X_2\, P(X_1)P(X_2)$$

$$= \underbrace{\sum_{X_1} x\, P(X_1)}_{\mu_1} \cdot \underbrace{\sum_{X_2} X_2 \cdot P(X_2)}_{\mu_2}$$

If X_1, X_2 are independent, $\Rightarrow P(X, X_2) = P(X_1).P(X_2)$

Then covariance = 0,

$$\boxed{\begin{aligned} &Cov[X_1 X_2] = E[X_1 X_2] - \mu_1\mu_2 = \mu_1\nu_2 - \mu_1\nu_2 = 0 \\ &Var(X_1+X_2) = \sigma_1^2 + \sigma_2^2 \text{ if } X_1 X_2 \text{ are independent} \end{aligned}}$$

If $X_1, \ldots X_n$ are independent,

$$Var[X_1 + \ldots + X_n] = \sum_{i=1}^{n} Var[X_i] \overset{iid}{=} n\sigma^2$$

$$Var[\bar{X}_n] = Var[\tfrac{1}{n} T_n] = \frac{1}{n^2} Var[T_n] = \frac{1}{n^2}\sum Var(X_i]$$

$$= \frac{1}{n^2}\, n\sigma^2 = \frac{\sigma^2}{n} \cdot \text{ if } \lim_{n\to\infty} \frac{\sigma^2}{n} = 0,$$

$$\Rightarrow S.E[\bar{X}_n] = \frac{\sigma}{\sqrt{n}} \quad \text{if } \lim_{n\to\infty}\frac{\sigma}{\sqrt{n}} = 0,$$

$E[\bar{X}] = E[\frac{1}{n} \cdot T_n] = \frac{1}{n} E[T_n] = \frac{1}{n} \cdot n\mu = \mu$

$X \sim Bin(n,p)$, $X = X_1 + \dots + X_n$ where $X_1 \dots X_n \overset{iid}{\sim} Bern(p)$

$= E(X) = \sum E[X_i] = np$

$E(X) = np$

$Var(X) = E[(X-\mu)^2] = \sum_{x=0}^{n} (x-np)^2 \binom{n}{x} p^x (1-p)^{n-x}$

$Var(X) = \sum Var[X_i] = n\sigma^2 = np(1-p) = \boxed{SE[\bar{X}] = \sqrt{np(1-p)}}$

$X \sim Geom(p) = (1-p)^{x-1} p$

$E(X) = \frac{1}{p} = \mu$

$Var(X) = E(X^2) - \mu^2 = E(X^2) - \frac{1}{p^2}$

$E[X^2] = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} \cdot p$

Let $d = x-1 \Rightarrow x = d+1$

$E(X^2) = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} p \Rightarrow \sum_{d=0}^{\infty} (d+1)^2 (1-p)^d p = \underbrace{\sum_{d=0}^{\infty} d^2 (1-p)^d p}_{(1-p)\, E(X^2)} +$

$2 \underbrace{\sum_{d=0}^{\infty} d(1-p)^d p}_{2(1-p)E(X) = \frac{2(1-p)}{p}} + \sum_{d=0}^{\infty} (1-p)^d \Rightarrow 1/p$

$E[X^2] = (1-p)E(X^2) + \frac{2(1-p)}{p} + 1 \cdot \frac{p}{p}$

$E(X^2) - (1-p)E(X^2) = \frac{2(1-p)+p}{p}$

$E(X^2)p = \frac{2(1-p)+p}{p}$; $\Rightarrow E(X^2) = \frac{2-2p+p}{p^2} = \boxed{\frac{2-p}{p^2}}$

$Var(X) = E(X^2) - \frac{1}{p^2} = \frac{2-p}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$

$X \sim Hyper(n, k, N)$

$Var(X) = \sum_{n \in supp(X)} (x - n\frac{k}{N})^2 \cdot \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$ (not in this class)

$X_1, X_2, \dots \overset{iid}{\sim} Bern(p)$

$X \sim Geom(p)$ "stopping time"

$P(X=17) = (1-p)^{16} p$

$P(X=7) = (1-p)^{6} p$

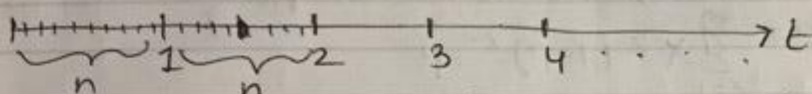
$$P(X=17 \mid X>10) = \frac{P(X=17 \cap X>10)}{P(X>10)} = \frac{P(X=17)}{P(X>10)} = \frac{(1-p)^{16}p}{(1-p)^{10}} = (1-p)^6 p$$

$$P(X = a+b \mid X>b) = \frac{P(X=a+b \cap X>b)}{P(X>b)} = \frac{P(X=a+b)}{P(X>b)}$$

$$= \frac{(1-p)^{a+b-1} - p}{(1-p)^{b}} = P(X=a)$$

"Memory less property" (∵ it's like first 10 didn't happen)

$X \sim Geom(p)$



$p(x) = (1-p)^{x-1} p$ | $p(t) = (1-p)^{t-1} p$ (in sec)

Run iid Bern(p)'s at every $\frac{1}{n}$ the period

$p(t) = (1-p)^{nt-1} p$, $t = 0.78$

FINAL MATERIAL ↓

$\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$$\lim_{n\to\infty} p(t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n} = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt-1} \left(\because \lim_{n\to\infty} \frac{\lambda}{n} = 0\right)$$

limiting PMF $p(t) = 0 \ \forall t$, $\sum_{t \in supp(t)} p(t) = 0$ ($p(t) \neq$ PMF b/c you are using continuous experiment) not discrete r.v.!

$$\lim_{n\to\infty} P(t) : \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \Big(\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}_{e^{-\lambda}}\Big)^t$$

$= 1 - e^{-\lambda t}$

$= 1 - \frac{1}{e^{\lambda t}}$ $\quad t \to \infty = 1$, $t \to -\infty = 0$

↑ legitimate CDF