MATH 241 LECTURE #3 9/5/17

Working Definition of Probability

$P: 2^{\Omega} \longrightarrow [0,1]$

event space, the power set of outcome space

"degree"

where 1 is covering

where 0 is impossibility

$A \subseteq \Omega$

$A \in 2^{\Omega}$

$P(\Omega) = 1$

$P(\emptyset) = 0$

$P(A) = \frac{|A|}{|\Omega|}$

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What is the probability of getting the $\underbrace{\text{sum of 3}}_{\text{info about } A}$ on the $\underbrace{\text{two die rolls}}_{\text{about } \Omega}$?

Step 1: translate from English $\rightarrow \Omega$. $\Omega = \{1,2,\dots,6\} \times \{1,2,\dots,6\}$ $|\Omega| = 36$

Step 2: Count $|\Omega|$

Step 3: Translate from English $\longrightarrow A$. $A = \{\langle 1,2\rangle, \langle 2,1\rangle\}$

Step 4: Compare $|A| = 2$

Step 5: Divide $P(A) = \frac{|A|}{|\Omega|} = \frac{2}{36}$

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What is the probability of getting 2 Heads on 4 coin flips?

$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{16}$

$\Omega = \{H,T\}^4$, $|\Omega| = |\{H,T\}|^4 = 2^4 = 16$

$A = \{\langle H,H,T,T\rangle, \langle T,T,H,H\rangle, \langle T,H,T,H\rangle$

$\langle H,T,H,T\rangle, \langle H,T,T,H\rangle, \langle T,H,H,T\rangle\}$

$P(HHHH) \stackrel{?}{=} P(HHHT) \neq P(2H2T)$

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Prob of at least 1 H on 4 tosses?

$P(A) = \frac{|A|}{|\Omega|} = \frac{?}{16}$

$A = \{HTTT, HHHH, \dots$

Recall $|\Omega| = |A| + |A^c| \Rightarrow |A| = |\Omega| - |A^c| = 16 - 1$

$A^c = \{\text{not at least } 1H\} = \{< 1H\} = \{TTTT\}$

$n \geq H$

Flip 10 coins. What is probability of 4 H?

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{2^{10}}$$

Unfortunately, we cannot answer this until we learn some more counting tools. We will return to this

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$F = \{ \text{Jane}, \text{Mary}, \text{Susan} \}$

There are 3 chairs. How many ways to seat those 3 women?

| Seat | 1 | 2 | 3 |
|---|---|---|---|

TREE

- Jane
  - Mary — Susan
  - Susan — Mary
- Mary
  - Jane — Susan
  - Susan — Jane
- Susan
  - Jane — Mary
  - Mary — Jane

total # of ways

$$\underset{\text{Seat 1}}{\underline{3}} \cdot \underset{\text{Seat 2}}{\underline{2}} \cdot \underset{\text{Seat 1}}{\underline{1}} = 6$$

$\Omega = \{ \langle J, M, S \rangle, \langle J, S, M \rangle, \langle M, J, S \rangle, \langle M, S, J \rangle, \langle S, M, J \rangle, \langle S, J, M \rangle \}$

$|\Omega| = 6$

Note $\Omega \subseteq F^3$ $\quad |F^3| = 27$

Sampling 3 times "without replacement"

Sampling 3 times "with replacement"

$|\Omega| \neq F^3$

Sample $n$ items without replacement ($n \in \mathbb{N}$). How many possible outcomes?

\# choices

$$\underset{\text{1st sample}}{\underline{n}} \quad \underset{\text{2nd sample}}{\underline{n-1}} \quad \cdots \quad \underset{(n-1)^{th}\text{ sample}}{\underline{2}} \quad \underset{n\text{th sample}}{\underline{1}} = \prod_{i=1}^{n} i$$

$n! :=$

Sample $n$ items with replacement. How many outcomes?

$$\underset{1^{st}\text{ sample}}{\underline{n}} \quad \underset{2^{nd}\text{ sample}}{\underline{n}} \cdots \underline{n} \quad \underline{n} = n^n > n! \quad \text{for } n \geq 2$$

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5 people. 3 chairs. How many seating arrangement?

$$\underset{1^{st}\text{ Chair}}{\underline{5}} \quad \underset{2^{nd}\text{ Chair}}{\underline{4}} \quad \underset{3^{rd}\text{ Chair}}{\underline{3}} = \frac{5!}{2!}$$

Sample $n$ items $k$ times without replacement. How many?

$$\underset{1^{st}\text{ Sample}}{\underline{n}} \quad \underset{2^{nd}\text{ sample}}{\underline{n-1}} \cdots \underset{k^{th}\text{ sample}}{\underline{n-k+1}} = \frac{n!}{(n-k)!}$$

$$\boxed{\begin{array}{l}\text{Permutations (a.k.a. Unique Orderings)} \\ {}_nP_k := \dfrac{n!}{(n-k)!}\end{array}}$$

Note: Convention dictates $\boxed{0! = 1}$

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1}$$

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3 couples (6 people): Bob - Jane, Richard - Susan, Charles - Mary

$$P(\text{every couple sits together}) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 4 \cdot 2}{6!} = \frac{\cancel{6} \cdot \cancel{4} \cdot \cancel{2}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2} \cdot 1} = \frac{1}{15}$$

$\Omega$ = "all seating arrangements of the 6 individuals"

$$A = \begin{array}{c|cccccc} & 6 & 1 & 4 & 1 & 2 & 1 \\ \text{Seat} & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

How many people seat sit in these chairs

<u>Alternative Method</u>

$$P(A) = \frac{|A|}{|\Omega|} = \frac{3!\,2^3}{6!} \Longleftarrow \underset{\text{Love Seat 1}}{\underline{3}} \cdot \underset{2}{\underline{2}} \cdot \underset{3}{\underline{1}}$$

switch possible

? ?

$$P(\text{alternating gender}) = \frac{(3!)^2 \cdot 2}{6!} = \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot 2 \cdot 1} = \frac{1}{10}$$

METHOD #1

$$\frac{3}{1} \quad \frac{3}{2} \quad \frac{2}{3} \quad \frac{2}{4} \quad \frac{1}{5} \quad \frac{1}{6} \qquad A_{MF}$$

$$P(A) = P(A_{MF}) + P(A_{FM})$$

$$3 \quad 3 \quad 2 \quad 2 \quad 1 \quad 1 \qquad A_{FM}$$

METHOD #2

$$\frac{6}{1} \quad \frac{3}{2} \quad \frac{2}{3} \quad \frac{2}{4} \quad \frac{1}{5} \quad \frac{1}{6} \Rightarrow P(A) = \frac{6 \cdot 3 \cdot 2 \cdot 2}{6!} = \frac{\cancel{6} \cdot \cancel{4} \cdot \cancel{3}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot 2 \cdot 1} = \frac{1}{10}$$

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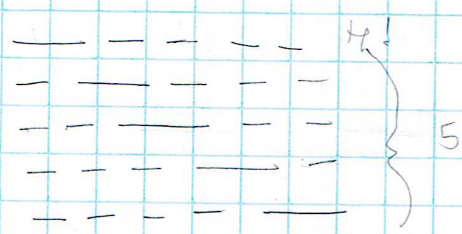
$$P\left(\begin{matrix}\text{Richard \& Susan} \\ \text{sit together}\end{matrix}\right) = \frac{\overbrace{4! \, 5}^{5!} \cdot 2}{6!} = \frac{2}{6} = \frac{1}{3}$$

$$\underset{\text{love seat}}{1} \quad \underset{\text{seat 1}}{4} \quad \underset{\text{seat 2}}{3} \quad \underset{\text{seat 3}}{2} \quad \underset{\text{seat 4}}{1}$$




$$\underset{\text{seat \# 1}}{4} \quad \underset{\text{love seat}}{1} \quad \underset{\text{seat \#2}}{3} \quad \underset{3}{2} \quad \underset{4}{1}$$

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$n = 100$ balls, sample $k = 3$ without replacement $\Rightarrow \dfrac{{}_{100}P_3}{100^3} \approx .9702$

" " with replacement

$$\lim_{n \to \infty} \frac{{}_nP_k}{n^k} = \lim \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}$$

$$= \underbrace{\lim \frac{n}{n}}_{1} \cdot \underbrace{\lim \frac{n-1}{n}}_{1} \cdots \lim \frac{n-k+1}{n} = 1$$

If $n$ is large, sampling with replacement $\approx$ without replacement