10/16

Review

10 cards, 4 R 6 B

P(drawing 2R in 3 cards without replacement)

$$= \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$$

$$P(x R \text{ in } 3) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$$

$$P(x R \text{ in } n) = \frac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$$

10 cards, K R

$$P(x R \text{ in } n) = \frac{\binom{K}{x}\binom{10-K}{n-x}}{\binom{10}{n}}$$

N cards, K R

$$P(x R \text{ in } n) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Drawing without replacement

Now... resuming...

$$X \sim \text{Hypergeometric}(n, K, N) := \underbrace{\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}}_{p(x) := P(X=x)}$$

$$X \sim \text{Bern}(p) := \underbrace{p^x (1-p)^{1-x}}_{p(x)}$$

$\text{Supp}(x) = \{0, 1\}$ $\boxed{p \in (0, 1)}$

parameter space

degenerated 100 53 8 random 6

$$X \sim \text{Hypergeometric}(n, K, N) := \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

100 students
53 female
8 picked
p(6 female)

$$X \sim \text{Hypergeometric}(\overset{n}{8}, \overset{K}{53}, \overset{N}{100}) =$$

$$\frac{\binom{53}{x}\binom{47}{8-x}}{\binom{100}{8}}$$ x = free variable

$$p(X=6) = p(6) = \frac{\binom{53}{6}\binom{47}{2}}{\binom{100}{8}}$$

$$p(x) := P(X=x)$$

Parameter space

degenerated If $N=0$? $\to K=0$, $\to n=0$

degenerated If $N=1$? $\to K \in \{0,1\}$, $\to n=1$

If $N=2$? $K \in \{0,1,2\}$, $n \in \{0,1,2\}$

$$X \sim \text{Hyper}(1,1,2) = \frac{\binom{1}{x}\binom{1}{1-x}}{\binom{2}{1}} = \text{Bern}\left(\tfrac{1}{2}\right)$$

$$p(1) = \frac{\binom{1}{1}\binom{1}{0}}{\binom{2}{1}} = \frac{1}{2}$$

$$p(0) = \frac{\binom{1}{0}\binom{1}{1}}{\binom{2}{1}} = \frac{1}{2}$$

$$X \sim \text{Hyper}(1, K, N) = \frac{\binom{K}{x}\binom{N-K}{1-x}}{\binom{N}{1}}$$

how many are special?

$$\text{Supp}(X) = \{0, 1\}$$

$$p(1) = \frac{\binom{K}{1}\binom{N-K}{0}}{\binom{N}{1}} = \frac{K}{N}$$

$$p(0) = \frac{\binom{K}{0}\binom{N-K}{1}}{\binom{N}{1}} = \frac{N-K}{N} = 1 - \frac{K}{N}$$

Parameter Space Hyper

$N \in \{2, 3, \dots\}$
$K \in \{1, 2, \dots, N-1\}$
$n \in \{1, 2, \dots, N-1\}$

Examples

a) $x \sim \text{Hyper}(2, 4, 10)$, $\boxed{\text{Supp}(x) = \{0, 1, 2\}}$

b) $x \sim \text{Hyper}(5, 4, 10)$, $\boxed{\text{Supp}(x) = \{0, 1, 2, 3, 4\}}$

c) $x \sim \text{Hyper}(8, 4, 10)$, $\boxed{\text{Supp}(x) = \{2, 3, 4\}}$

d) $x \sim \text{Hyper}(5, 7, 10)$, $\boxed{\text{Supp}(x) = \{2, 3, 4, 5\}}$

Important

(a) $n < K$, $n < N-K \rightarrow \text{supp}(x) = \{0, \dots, n\}$

(b) $n \geq K$, $n < N-K \rightarrow \text{supp}(x) = \{0, \dots, K\}$

(c) $n \geq K$, $n \geq N-K \rightarrow \text{supp}(x) = \{n-(N-K), \dots, K\}$

(d) $n < K$, $n \geq N-K \rightarrow \text{supp}(x) = \{n-(N-K), \dots, n\}$

	$n<K$	$n \geq K$
$n<N-K$	$0 \cdots n$	$0 \cdots K$
$n \geq N-K$	$n-(N-K) \cdots n$	$n-(N-K) \cdots K$

$$\text{Supp}(x) = \left\{ \max\{0, n-(N-K)\} \dots, \min\{n, K\} \right\}$$

vandard identity

$X \sim \text{Hypper}(n, K, N)$

$$= \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

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let $p = \frac{K}{N} \Rightarrow K = pN$

$$X \sim \text{Hypper}(n, p, N) := \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

reparameterization

$N \in \{2, \ldots\}$
$n \in \{1, \ldots, N-1\}$
$p \in \{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$

Consider $p = 0.5$, $n=6$, $N=100$

$$p(3) = \frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = .3223$$

$N = 1000$

$$p(3) = \frac{\binom{500}{3}\binom{500}{3}}{\binom{1000}{6}} = .3134$$

$N = 10{,}000$

$$p(3) = \frac{\binom{5000}{3}\binom{5000}{3}}{\binom{10000}{6}} = .3126$$

What is the limiting r.v.

$$\lim_{N\to\infty} \text{Hypper}(n, p, N) = \lim_{N\to\infty} \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

$$= \lim_{N\to\infty} \frac{\frac{(pN)!}{x!\,(pN-x)!}\;\frac{((1-p)N)!}{(n-x)!\,((1-p)N-(n-x))!}}{\frac{N!}{n!\,(N-n)!}} = \underbrace{\frac{1}{x!}\;\frac{1}{(n-x)!}\;\frac{1}{\frac{1}{n!}}}_{\binom{n}{x}}$$

$$\lim_{N\to\infty} \frac{\dfrac{(pN)!}{(pN-x)!} \; \dfrac{((1-p)N)!}{((1-p)N-(n-x))!}}{\dfrac{N!}{(N-n)!}}$$