10/30

Custom R.V.

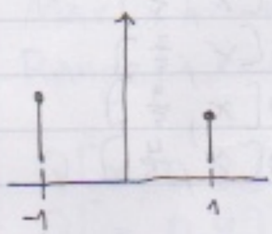
Roulette in America

\$1 Bet on Black Payout is 1:1

$X \sim$ \$1 w.p. $\frac{18}{38}$, $-\$1$ w.p. $\frac{20}{38}$

$$\boxed{E[X] = \sum_{x \in supp[X]} x \, p(x)}$$

general formula



$\bar{x} \to m$
Law of Large Numbers

$= (\$1)\left(\frac{18}{38}\right) + (-\$1)\left(\frac{20}{38}\right) = -\$0.053$

If you play many times the average of you playing will converge to this number.

The average converges to the expectation

$\mu \notin supp[X]$
generally speaking

"Don't roll a die and get 3.5"

$$\lim_{n\to\infty} T_n = X_1 + \dots + X_n = -\infty$$

$n\bar{x}$, $\bar{x} \to -\$0.053$, $n \to \infty$

Bet on a "lucky 7" Payout 35:1

$$X \sim \begin{cases} \$35 & \text{w.p. } \frac{1}{38} \\ -\$1 & \text{w.p. } \frac{37}{38} \end{cases}$$

$$E[X] = 35 \cdot \frac{1}{38} + (-1)\frac{37}{38} \approx -\$0.053$$

Bet on first dozen (1-12)

Payout is 2:1

$$X \sim \begin{cases} \$2 & \text{w.p. } \frac{12}{38} \\ -\$1 & \text{w.p. } \frac{26}{38} \end{cases}$$

$$E[X] = (2)\left(\tfrac{12}{38}\right) + (-1)\left(\tfrac{26}{38}\right) = \$0.053$$

"Roulette in Europe" (all numbers go to 37 instead of 38)

European Roulette is more fair

-\$0.027 -lose less on average compared to America's -\$0.053

Fair game $E[X]=0$

$P(\text{Traffic}) = 0.3$

If traffic, Uber takes 12 mins.

If no traffic, 7 mins.

Model time in car

Model of time spent in car

$$W \sim \begin{cases} 12 \text{ mins} & \text{w.p. } 0.3 \\ 7 \text{ mins} & \text{w.p. } 0.7 \end{cases} \qquad E[W] = 12 \cdot 0.3 + 7 \cdot 0.7 = 8.5 \text{ mins}$$

Support must be 12 and 7, Expectation not part of it.

Uber charges \$0.40/min. Model B, the price paid for the time is taxi. [same probability]

$$B \sim \begin{cases} \$0.40 \cdot 12 & \text{w.p. } 0.3 \\ \$0.40 \cdot 7 & \text{w.p. } 0.7 \end{cases} \qquad \$4.80 \cdot 0.3 + \$3.80 \cdot 0.7 = \$3.12$$

unneccessary work $\boxed{= \$0.40\, E[W]}$

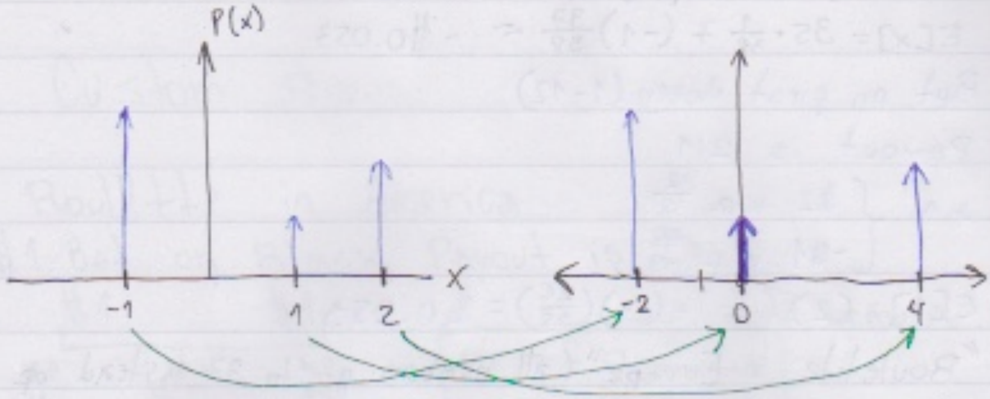
Linear Transformation

$$Y = aX + c \quad , B,$$

$Y = aX,\ a \in \mathbb{R}$ $\quad E[Y] = E[aX] = E[g(x)] = \sum_{x \in supp(x)} a x p(x) = a \sum x p(x) = aE[x]$

$$E[g(x)] = \sum_{x \in supp(x)} g(x)\, p(x)$$

Proof of Uber problem unnecessary work



$$\boxed{\text{Base Fare is } \$3}$$

Model T, the total price

$$T \sim \begin{cases} 3 + 4.80 = 7.80 & \text{w.p. } 0.3 \\ 3 + 2.80 = 5.80 & \text{w.p. } 0.7 \end{cases}$$

$$E(T) = 7.80 \cdot 0.3 + 5.80 + 0.7 = 6.12$$

$$= (3 + 4.80)\, 0.3 + (3 + 2.80) \cdot 0.7$$
$$= 3 \cdot 0.3 + 4.80 \cdot 0.3 + 3 \cdot 0.7 + 2.80 \cdot 0.7$$
$$= 3\,(0.3 + 0.7) + 4.80 \cdot 0.3 + 2.80 \cdot 0.7$$
$$= 3 + E(B)$$

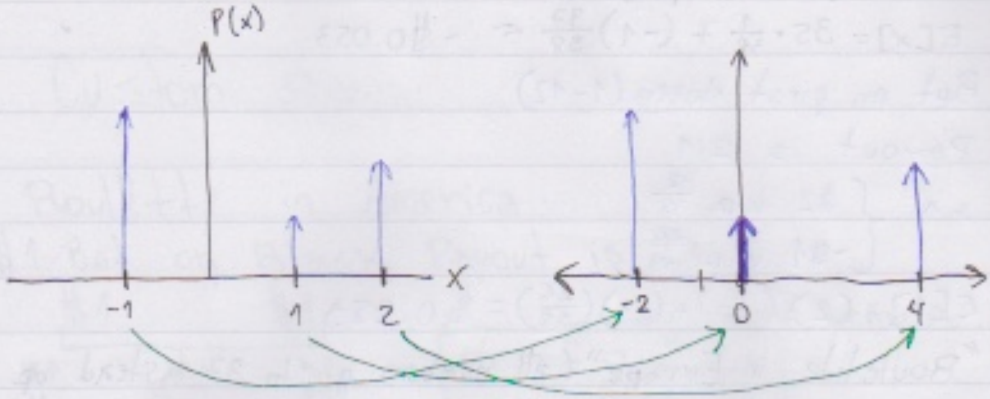
Linear Transformation

$Y = aX + c$, B,

$Y = aX,\ a \in \mathbb{R}$ $E[Y] = E[aX] = E[g(x)] = \sum_{x \in supp(x)} ax\,p(x) = a\sum x\,p(x) = aE[x]$

$E[g(x)] = \sum_{x \in supp(x)} g(x)\,p(x)$

Proof of Uber problem unnecessary work



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$= 3\,(0.3 + 0.7) + 4.80 \cdot 0.3 + 2.80 \cdot 0.7$

$= 3 + E(B)$

$Y = X + c, \quad c \in \mathbb{R}$

$E(Y) = E[X + c]$

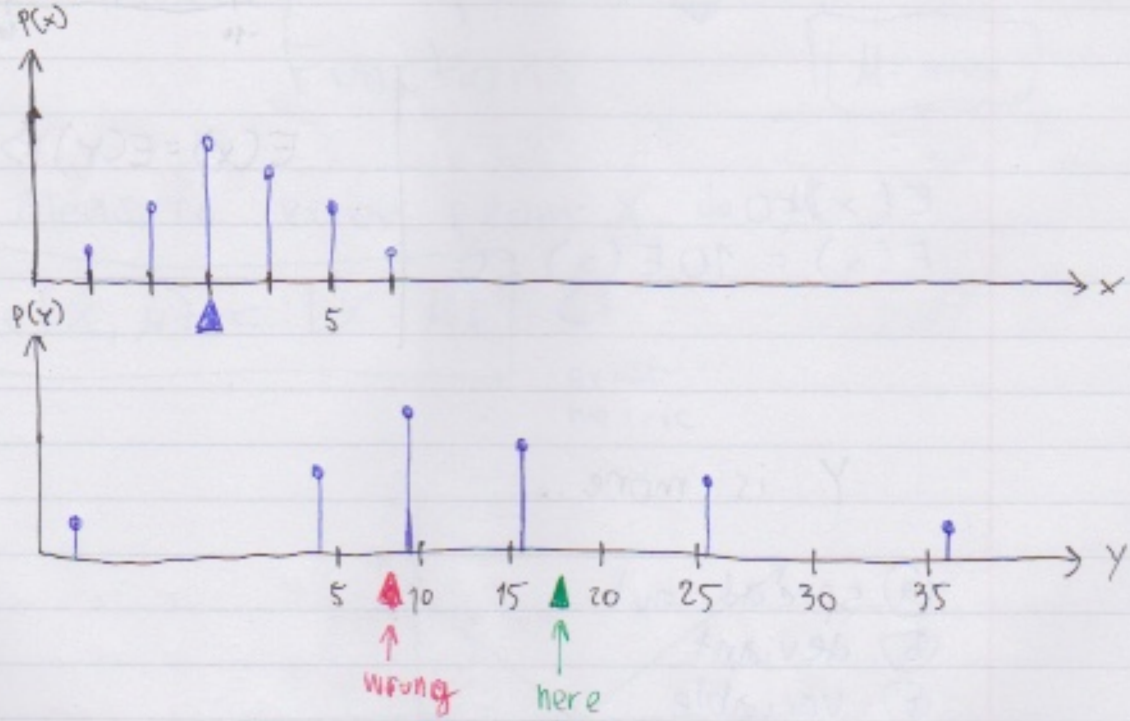
$$= \sum_{x \in Supp(X)} (x+c)\,p(x) = \underbrace{\sum x\,p(x)}_{E(X)} + \sum c\,p(x)$$

$$= E(X) + c \sum p(x)$$

$$= E(X) + c$$

$X \sim Bin(6, \tfrac{1}{2}) \Rightarrow E(X)$. Let $Y = X^2$

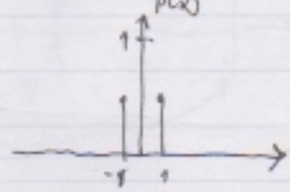
Does $E(X^2) \stackrel{?}{=} (E(X))^2$



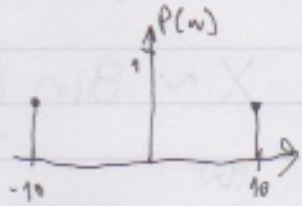
$$E(Y) = \sum_{x=0}^{6} x^2 \binom{n}{x} p^x (1-p)^{n-x} = \boxed{17.5}$$ look at previous picture

$\Rightarrow E(g(x)) \neq g(E(x))$
generally

$X \sim$ Raten $\begin{cases} 1 \text{ wp } \frac{1}{2} \\ -1 \text{ wp } \frac{1}{2} \end{cases}$



$Y = 10X \begin{cases} 10 & \text{wp } \frac{1}{2} \\ -10 & \text{wp } \frac{1}{2} \end{cases}$

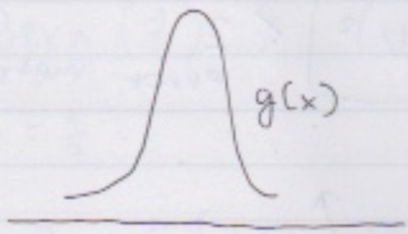


$E(x) = 0$
$E(x) = 10E(x) = 0$

$E(X) = E(Y) \not\Rightarrow X \overset{d}{=} Y$

Y is more ...

(a) spread out
(b) deviant
(c) variable
...



$\int_{\mathbb{R}} g(x)\,dx = 3$

$\int_{\mathbb{R}} h(x) = 3$

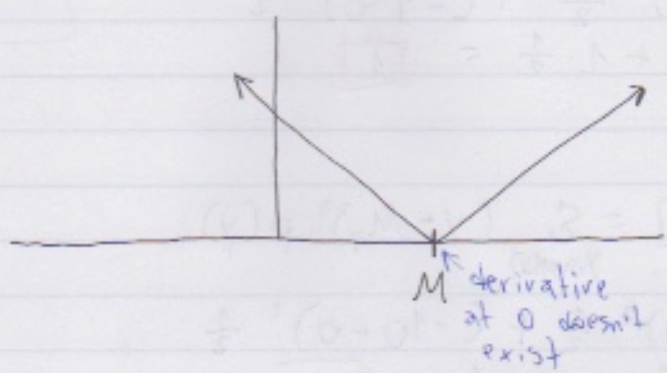
Theory of Error/Distance Functions

$\boxed{\mu = \text{mean}}$

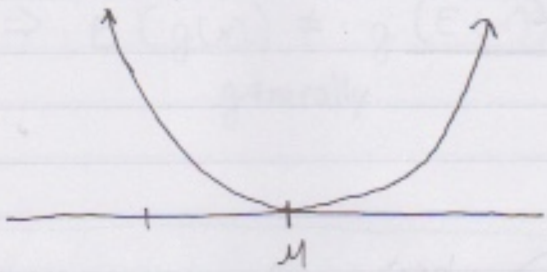
Measure error from X to μ

$\boxed{e(x,\mu) = |x-\mu|}$ L1

error metric



$\boxed{e(x,\mu) = (x-\mu)^2} < 2$ error metric



Let $L = (x-\mu)^2$

"squared error loss"

$$Var[X] := E[L] = E[(X-\mu)^2] = E[(X-E(X))^2]$$

$$Var[X] = \sum_{x \in Supp(X)} (x-\mu)^2 p(x)$$

$$= (1-0)^2 \tfrac{1}{2} + (-1-0)^2 \tfrac{1}{2}$$
$$= 1 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{2} = \boxed{1}$$

$$Var[Y] = \sum_{y \in Supp(Y)} (y-\mu_y)^2 p(y)$$

$$= (10-0)^2 \tfrac{1}{2} + (-10-0)^2 \tfrac{1}{2}$$
$$= 100 \cdot \tfrac{1}{2} + 100 \tfrac{1}{2} = \boxed{100}$$

average square dist.
from expectation

$X \sim \text{Bern}\left(\frac{1}{3}\right)$

$E(x) = \frac{1}{3}$

$\text{Var}[x] = E\left[(X-\mu)^2\right]$

$= \sum_{x \in \text{supp}(x)} (x-\mu)^2 \, p(x) = \left(1-\frac{1}{3}\right)^2 \frac{1}{3} + \left(0-\frac{1}{3}\right)^2 \frac{2}{3}$

$= \left(\frac{2}{3}\right)^2 \frac{1}{3} + \left(\frac{1}{3}\right)^2 \frac{2}{3}$

$= \frac{4}{9} + \frac{2}{6} = \boxed{\frac{2}{3}}$

$\sum (x-\mu)^2 \, p(x)$

$= (1-p)^2 p + (0-p)^2 (1-p)$
$= (1-2p+p^2)\, p + p^2(1-p)$
$= p - 2p^2 + \cancel{p^3} + p^2 - \cancel{p^3} = p - p^2 = \boxed{p(1-p)}$

$\boxed{\mu, \sigma^2}$