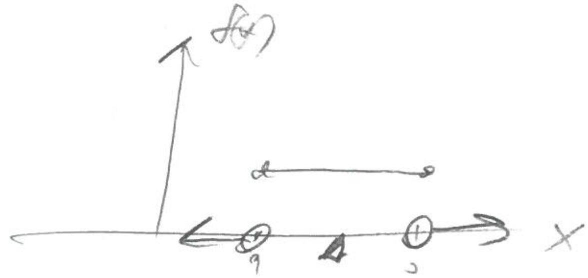
Lec 18 Math 241 11/9/17

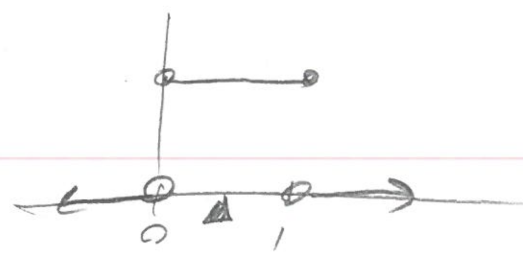
$X \sim U(a,b) = \frac{1}{b-a}$



$F(x) = \frac{x-a}{b-a}$

$X \sim U(0,1) = 1$

"Standard Uniform"



$F(x) = x$

$Supp(X) = (0,1)$

$$E(X) = \int_a^b x\, f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

std. uniform

$$E(X) = \frac{0+1}{2} = \frac{1}{2}$$

$Med(X) = ?$

if continuous since there is a value where $F(x) = p$

$$Quantile[X, p] = \arg\min\{F(x) \geq p\} = F^{-1}(p)$$

$$p = \frac{x-a}{b-a} = p(b-a) = x-a \Rightarrow x = p(b-a)+a = F^{-1}(p)$$

$$Med(X) = Quantile(X, .5) = F^{-1}(0.5) = 0.5(b-a)+a \Rightarrow \frac{1}{2}b - \frac{1}{2}a + a = \frac{1}{2}b + \frac{1}{2}a$$

$$= \boxed{\frac{a+b}{2}}$$ Same as μ

2

$$\sigma^2 = \mathrm{Var}(X) = \int_{\mathrm{Supp}(X)} (X-\mu)^2 f(x)\,dx = E(X^2) - \mu^2$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \frac{(a+b)^2}{4}$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$\begin{array}{r|l} & b^2 + ab + a^2 \\ \hline b-a & b^3 - a^3 \\ & -(b^3 - ab^2) \\ \hline & ab^2 - a^3 \\ & -(ab^2 - a^2b) \\ \hline & a^2b - a^3 \\ & -(a^2b - a^3) \\ \hline & 0 \end{array}$$

$$\frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4} = \frac{(4b^2 + 4ab + 4a^2) - (3a^2 + 6ab + 3b^2)}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \boxed{\frac{(b-a)^2}{12}}$$

$$SE(X) = \sqrt{\sigma^2} = \frac{b-a}{\sqrt{12}}$$

New r.v..

$$Z \sim \underset{\sim}{N}(0,1) := \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{f(x) \text{ PDF}}$$

normal r.v.,
gaussian r.v.
"bell curve"

$Supp(Z) := \{x : f(x) > 0\}$
Note... always > 0
$= \mathbb{R}$
"Anything could happen"

Is this a r.v.?

(a) $f(x) \geq 0$ ✓

(b) $\int_{Supp(x)} f(x)\,dx = 1 \Rightarrow \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$ proof...

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

let $u = \frac{1}{\sqrt{2}} x \Rightarrow \frac{x^2}{2} = u^2$

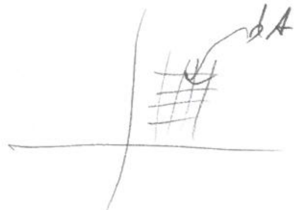
$du = \frac{1}{\sqrt{2}} dx \Rightarrow dx = \sqrt{2}\, du$

$$\Rightarrow \int_{\mathbb{R}} e^{-u^2} \sqrt{2}\, du = \sqrt{2\pi} \Rightarrow \overbrace{\int_{\mathbb{R}} e^{-u^2} du}^{\text{Gaussian Integral}} = \sqrt{\pi}$$

still $(-\infty, \infty)$

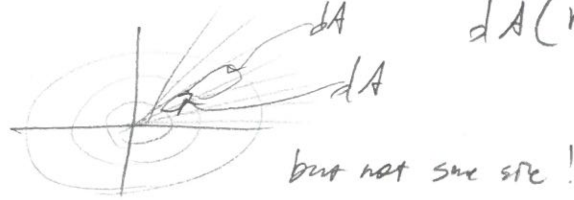
$$\left(\int_{\mathbb{R}} e^{-u^2} du\right)^2 = \pi \Rightarrow \int_{\mathbb{R}} e^{-u^2} du \int_{\mathbb{R}} e^{-u^2} du = \pi \Rightarrow \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \pi$$

$$\Rightarrow \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy = \pi}_{\text{height of as rect. prisms (parallelepipeds)}}$$



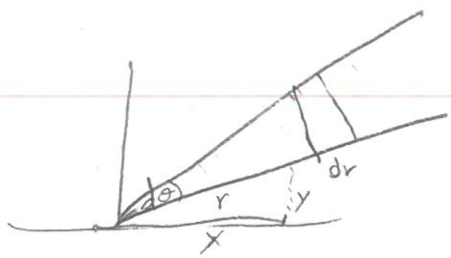
$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \pi$$

polar coord. switch



$dA(r)$

$x^2+y^2=r^2$



$x = r\cos\theta$

$y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$

$$\left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right) dr\,d\theta = \cos\theta\,(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r\cos^2\theta + r\sin^2\theta = r$$

$$\Rightarrow \underset{(r)}{\int_0^{\infty}}\underset{(\theta)}{\int_0^{2\pi}} e^{-r^2}\,r\,dr\,d\theta = \pi = \int_0^{\infty} e^{-r^2}\,r\,dr \int_0^{2\pi} d\theta = \pi = 2\pi\int_0^{\infty} e^{-r^2}\,r\,dr = \pi\,\frac{1}{2}$$

$u = r^2 \Rightarrow du = 2r\,dr \Rightarrow \frac{du}{2}\, r\,dr$

$$\int_0^\infty e^{-u}\frac{du}{\cancel{2}} = \cancel{\tfrac{1}{2}}\,1 \Rightarrow \int_0^\infty e^{-u}\,du = 1 = -\left[e^{-u}\right]_0^\infty = 1 \quad \overset{1}{e^{-0}} - \overset{0}{\lim_{x\to\infty} e^{-u}} = 1 \checkmark$$

$u = \frac{x^2}{2} \quad du = x\,dx$

$$E[Z] = \int_{Supp(X)} x f(x)\,dx = \int_{\mathbb{R}} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-u}\,du = \frac{1}{\sqrt{2\pi}}\left[-e^{-u}\right]_{x=-\infty}^{x=\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} = \frac{1}{\sqrt{2\pi}}\left(\lim_{x\to-\infty} e^{-\frac{x^2}{2}} - \lim_{x\to\infty} e^{-\frac{x^2}{2}}\right) = 0 \Rightarrow \boxed{\mu = 0}$$

$$Var(Z) = E(Z^2) - \cancel{(E(Z))^2}^{\,\to 0} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \underset{\text{Int. by parts}}{\ldots} = 1 \Rightarrow \boxed{\sigma^2 = \sigma = 1}$$

$$F(x) = \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx}_{\substack{\text{Antiderivative}\\ \text{impossible}\\ \text{(Risch algorithm)}}} + C$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, y\, e^{-\frac{y^2}{2}}\,dy \quad \text{(numerically)}$$

$$F(0) = \frac{1}{2}$$

$$P(Z \in (-1,1)) = F(1) - F(-1) \approx 0.68$$

$$P(Z \in [-2,2]) = F(2) - F(-2) \approx 0.95$$

$$P(Z \in (-3,3)) = F(3) - F(-3) \approx 0.997$$

"3σ rule", "empirical rule", "68-95-99.7 rule"

$Z \sim N(0,1)$ let $X = \sigma Z + \mu$ why? $\Rightarrow E(X) = \mu,\ SE(X) = \sigma$

$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$ unknown since $F_Z(x)$ unknown!

but... $f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} F_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{d}{du} \frac{1}{\sigma} F_Z(u) = \frac{1}{\sigma} f_Z(u)$

let $u = \frac{x-\mu}{\sigma}$ $\quad \frac{du}{dx} = \frac{1}{\sigma} \Rightarrow dx = \sigma\, du$ $\quad = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$

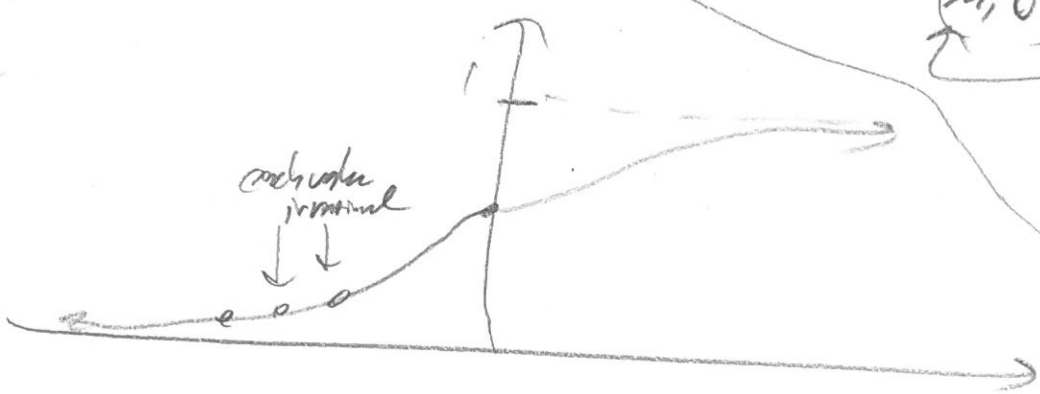
$\frac{d}{du} \frac{du}{dx}$ $\quad = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$

not covered on exam

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

general normal distr.

$X \sim N(\mu, \sigma^2)$ $\quad \sigma^2$: Var(X) why? $\quad \mu$: E(X)



Param Space

$\mu \in \mathbb{R}$

$\sigma^2 \in (0, \infty)$

$Supp(X) = \mathbb{R}$

$P(Z \le -1) = .16$

$P(Z \le -2) = .025$

$P(Z \le -3) = .0015$

But why so important?

let $L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\, dx$ Bilateral Laplace Transform

$$X \sim N(70'', 4''^2)$$

$$\Rightarrow Z = \frac{X-70''}{4''} \sim N(0,1)$$

$78'' =$ z-score $= 2$

$$P(X \geq 78'') = P\left(\frac{X-70''}{4''} \geq \frac{78''-70''}{4''}\right) = P(Z \geq 2) \approx 2.5\%$$

Lots of things are normally distr... why?... Patience...

let $L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\, dx$

L is called the "Bilateral Laplace Transform" of f

What does this look like?