Lecture 11

10/17/2017

$$X \sim Hyper(n, K, N) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Remember:

$$\binom{n}{k} = \frac{n!}{(n-x)!\,n!}$$

$$X \sim Hyper(n, p, N) = \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

s.t $p = \frac{K}{N}$

$$\lim_{N \to \infty} Hyper(n\,p\,N) = \lim_{N \to \infty} \frac{\frac{(pN)!}{x!(pN-x)!}\ \frac{((1-p))N)!}{(n-x)!(1-pN-(n-x))!}}{\frac{N!}{n!(N-n)!}} =$$

"limiting PMF"

↑ value of N

only thing changing is N

can pull out $\frac{1}{x!}\ \frac{1}{(n-x)!}\ \frac{1}{\frac{1}{n}} = \frac{n!}{(x!)(n-x)!} = \binom{n}{x}$

x terms ⇒ $(pN)(pN-1)\ldots(pN-x+1)$ $((1-p)N)((1-p)N-1)\ldots$ $(1-p)N-(n-x+1)$

↑ n-x terms total

$$= \binom{n}{x} \lim_{N \to \infty} \frac{\frac{(pN)!}{(pN-x)!}\ \frac{((1-p)N)!}{(1-p)N-(n-x))!}}{\frac{N!}{(N-n)!}}$$

$$\frac{10!}{(10-7)!} = \underbrace{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}_{7\ \text{terms}}$$

$$\underbrace{(N)(N-1 \ldots (N-n+1)}_{n\ \text{total terms}}$$

$\lim f(x)g(x) = \lim f(x) \cdot \lim g(x)$

free variable

x is a constant

$$= \binom{n}{x} \underbrace{\underbrace{\lim_{N \to \infty} \frac{pN}{N}}_{p}\ \underbrace{\lim_{N \to \infty} \frac{pN-1}{N-1}}_{p} \ldots \underbrace{\lim_{N \to \infty} \frac{pN-x+1}{N-x+1}}_{p}}_{p^x}\ \overbrace{\lim_{N \to \infty} \frac{(1-p)N}{N-x}}^{1-p}\ \overbrace{\lim_{N \to \infty} \frac{((1-p)N-1)}{N-x-1}}^{1-p} \ldots$$

$$\underbrace{\lim_{N \to \infty} \frac{(1-p)N-(n-x)+1}{N-n+1}}_{1-p}$$

$$\Rightarrow \binom{n}{x} p^x (1-p)^{n-x}$$

won't be affected

sampling w/ replacement

$X \sim Binomial(n, p) = \binom{n}{x} p^x (1-p)^{n-x}$

and asking "how many special balls did I see" $(1-p)^{n-x}$

Parameter Space

$n \in \{1, 2, 3 .. \infty\} = \mathbb{N}$

$p \in (0, 1)$

$Supp[X] = \{0, 1, ... n\}$

Want to Show

$$\sum_{x \in Supp(x)} p(x) = 1 \Rightarrow \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = ((p) + (1-p))^n$$

$$= 1^n = 1 \checkmark$$

Recall: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Independent Random Variables

X_1 and X_2 are independent r.v.'s if

(a) $P(X_1 = x_1 | X_2 = x_2) = P(X_1 = x_1)$ for all $x_1 \in Supp[X_1]$, for all $x_2 \in Supp[X_2]$

(b) $P(X_2 = x_2 | X_1 = x_1) = P(X_2 = x_2)$

(c) $P(X_1 = x_1 \underset{\text{AND}}{,} X_2 = x_2) = P(X_1 = x_1) P(X_2 = X_2)$ ⇒ multiplication Rule

Joint mass function

X_1 and X_2 are "iid" ⇒ independent and identically distributive (PMF are the same)

$X_1, X_2 \overset{iid}{\sim} Bern\left(\frac{1}{3}\right)$ "same process completely disconnected starting over again"

weighted coin

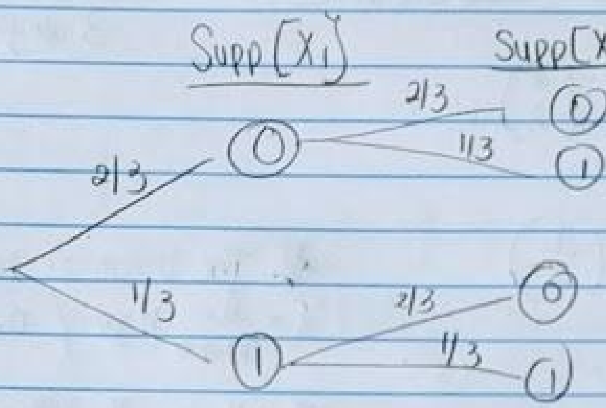
same Supp, same PMFs

$T_2 := X_1 + X_2$

$Supp[X_1] = \{0,1\}$

$Supp[X_2] = \{0,1\}$

$Supp[T_2] = \{\underbrace{0,1,2}_{PMF}\}$

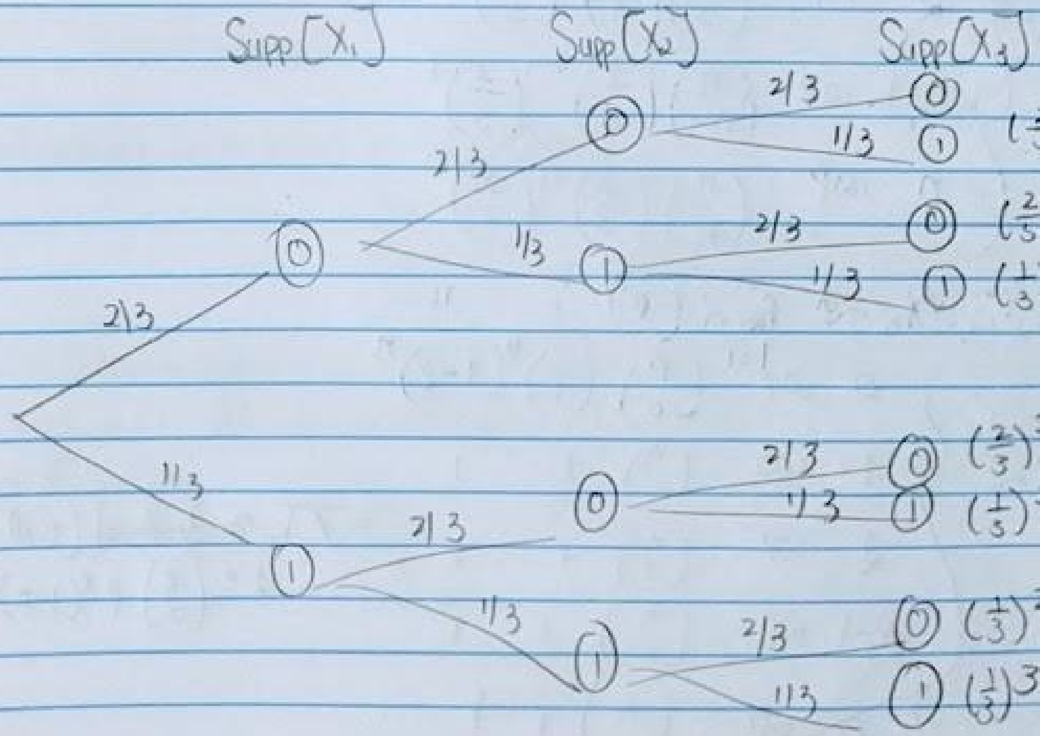


	T
$P(X_1=0, X_2=0) = 4/9$	0
$P(X_1=0, X_2=1) = 2/9$	1
$P(X_1=1, X_2=0) = 2/9$	1
$P(X_1=1, X_2=1) = 1/9$	2
1	

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(\frac{1}{3})$

$T_3 = X_1 + X_2 + X_3$

$Supp[T_3] = \{0,1,2,3\}$



	T
$(\frac{2}{3})^3(\frac{1}{3})^0$	0
$(\frac{2}{3})^2(\frac{1}{3})^1$	1
$(\frac{2}{3})^2(\frac{1}{3})^1$	1
$(\frac{1}{3})^2(\frac{2}{3})^1$	2
$(\frac{2}{3})^2(\frac{1}{3})^1$	1
$(\frac{1}{3})^2(\frac{2}{3})^1$	2
$(\frac{1}{3})^2(\frac{2}{3})^1$	2
$(\frac{1}{3})^3(\frac{2}{3})^0$	3

$$T_3 \sim \begin{cases} 0 \text{ wp } \left(\frac{2}{3}\right)^3 = \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 \\ 1 \text{ wp } \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = 3\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = \binom{3}{1} \\ 2 \text{ wp } \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 = \binom{3}{2} \\ 3 \text{ wp } \left(\frac{1}{3}\right)^3 = \binom{3}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 \end{cases}$$

3 ways to get 1/2

$$\binom{3}{1}\left\{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right. \qquad \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix} \binom{3}{2}$$

$X_1, X_2 \ldots X_n \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$

$T_n = \sum_{i=1}^{n} X_i$

$\text{Supp}[T_n] = \{0, 1, \ldots n\}$

★ The binomial conceptually is
$T = \lim_{N \to \infty} \text{Hyper}(n, p, N)$

$T = X_1 + \ldots X_n$ where $X_1, \ldots X_n \overset{iid}{\sim} \text{Bern}(p)$

$$T_n \sim \begin{cases} 0 \text{ wp } \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n \\ 1 \text{ wp } \binom{n}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1} \\ 2 \text{ wp } \binom{n}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{n-2} \\ n-1 \text{ wp } \binom{n}{n-1}\left(\frac{1}{3}\right)^{n-1}\left(\frac{2}{3}\right)^1 \\ n \text{ wp } \binom{n}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^0 \end{cases}$$

$X_1, X_2 \ldots X_n \overset{iid}{\sim} \text{Bern}(p)$

$$T_n \sim \begin{cases} 0 \text{ wp } \binom{n}{0}(p)^0(1-p)^n \\ 1 \text{ wp } \binom{n}{1} \quad " \quad " \\ 2 \text{ wp } \binom{n}{2} \quad " \quad " \\ n-1 \text{ wp } \binom{n}{n-1} \quad " \quad " \\ n \text{ wp } \binom{n}{n} \quad " \quad " \end{cases}$$

$\Rightarrow T_n \sim \text{Binomial}(n, p)$
$:= \binom{n}{x} p^x (1-p)^{n-x}$