#20. $M_X(t) = E[e^{tX}]$

$M_X(0) = E[1]$

Properties.

① $M_X(t) = M_Y(t) \Leftrightarrow X \overset{d}{=} Y$.

② $M_X^{(k)}(0) = E[X^k]$

③ $Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_X(at)$.

④ X_1, X_2 indep, $T = X_1 + X_2 \Rightarrow M_T(t) = M_{X_1}(t) M_{X_2}(t)$.

V $X \sim Bern(p) \Rightarrow M_X(t) \Rightarrow 1 - p + pe^t$.

$X \sim Binom(n, p) \Rightarrow M_X(t) = (1 - p + pe^t)^n$

$X \sim Expo(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$

$Z \sim N(0,1) \Rightarrow M_Z(t) = e^{t^2/2}$.

General Normal Derived $\quad X = \mu + \sigma Z$

$$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) \overset{③}{=} e^{t\mu} M_Z(\sigma t)$$
$$= e^{t\mu} e^{\sigma^2 t^2/2}$$
$$= e^{\sigma^2 t^2/2 + t\mu}.$$

$X \sim Deg(c) \quad \approx$ always c, not random.

$M_X(t) = E[e^{tX}] = e^{ct}$.

$X_1 \sim N(\mu_1, \sigma_1^2)$ ind of.

$X_2 \sim N(\mu_2, \sigma_2^2)$.

$$X_1 + X_2 \sim X_{X_1+X_2}(t) = M_{X_1}(t)\, M_{X_2}(t).$$

$$= \left(e^{t\mu_1 + \sigma_1^2 t^2/2}\right)\left(e^{t\mu_2 + \sigma_2^2 t^2/2}\right) = e^{t(\mu_1+\mu_2) + \frac{t^2}{2}(\sigma_1^2+\sigma_2^2)}$$

$$\Rightarrow X_1 + X_2 \sim N\left((\mu_1+\mu_2), (\sigma_1^2+\sigma_2^2)\right)$$

V $X_1, \dots, X_n$ sequence of R.V's.

$$\lim_{n\to\infty} M_{X_n}(t) = M_Y(t) \Rightarrow \lim_{n\to\infty} F_{X_n}(X) = F_Y(X).$$

or $X_n \xrightarrow{d} Y$. identically distributed.

converges to

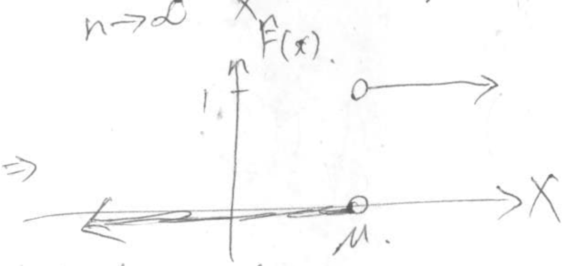
$\bar{X} \to \mu = E[X]$ Law of Large Numbers.

"μ" is a constant i.e. Deg (μ)

CWTS $\lim_{n\to\infty} M_{\bar{X}}(t) \overset{\text{Proof:}}{=} e^{t\mu} \overset{\text{V}}{\Rightarrow} \bar{X} \xrightarrow{d} \mu.$

CDF $\lim_{n\to\infty} F_{\bar{X}}(X) = F_\mu(X).$

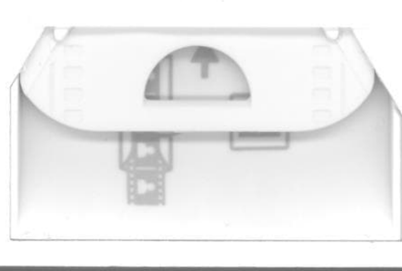
CDF $\Rightarrow$



Assume $X_1, X_2, \dots$ iid with mean μ.

$$\lim_{n\to\infty} M_{\bar{X}_n}(t) = \lim_{n\to\infty} M_{\frac{X_1+X_2+\dots+X_n}{n}}(t)$$

$$\overset{\text{III}}{=} \lim_{n\to\infty} M_{X_1+\dots+X_n}\left(\tfrac{t}{n}\right) \overset{\text{IV}}{=} \lim_{n\to\infty}\left(M_X\left(\tfrac{t}{n}\right)\right)^n$$



$$= \lim_{n\to\infty} e^{\ln\left(M_X\left(\frac{t}{n}\right)\right)^n} = \lim_{n\to\infty} e^{n\ln\left(M_X\left(\frac{t}{n}\right)\right)}$$

$$= e^{\lim_{n\to\infty} n\ln\left(M_X\left(\frac{t}{n}\right)\right)^n}$$

$$= e^{\lim_{n\to\infty} \frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)^n}{\frac{1}{n}}}$$

let $u = \frac{1}{n}$. $n \to \infty \Rightarrow u = 0$.

$$\therefore = e^{\lim_{u\to 0} \frac{\ln(M_X(tu))}{u}}$$

$$\frac{d}{dx} \Rightarrow = e^{\lim_{u\to 0} \frac{tM_X'(tu)}{M_X(tu)}} = e^{\frac{tM_X'(0)}{\cancel{M_X(0)} \Rightarrow 1}}$$

$$\stackrel{(II)}{=} e^{t\mu} \stackrel{(V)}{\Rightarrow} \bar{X} \to \mu.$$

WTS. $Z \sim N(0,1)$ is important.

Assume $X_1, X_2, \ldots$ iid with μ, σ^2.

Consider $C_n := \boxed{\dfrac{\bar{X}_n - \mu}{\underbrace{\frac{\sigma}{\sqrt{n}}}_{\text{Standard Error}}}}$

Standardise average.

$$E[C_n] = 0, \quad SE[C_n] = 1.$$

$$C_n = \frac{(\bar{X}_n - \mu)\sqrt{n}}{\sigma} = \frac{\sqrt{n}\left(\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right)}{\sigma}$$

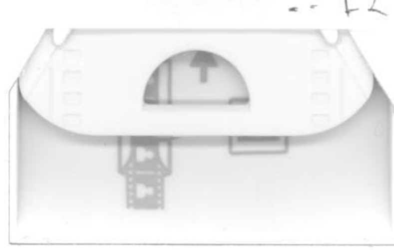
$$= \frac{\sqrt{n}\left(\frac{X_1 + \ldots + X_n}{n} - \frac{\mu + \ldots + \mu}{n}\right)}{\sigma}$$

$$= \frac{(X_1 + \ldots + X_n) - (\mu + \ldots + \mu)}{\sqrt{n}\ \sigma}$$

$$= \frac{1}{n}(V_1 + V_2 + \ldots + V_n)$$

let $V_n = \frac{X_n - \mu}{\sigma}$

$\therefore E[V_n] = 0$, $SE[V_n] = 1$.



$$\lim_{n\to\infty} M_{C_n}(t) = \lim_{n\to\infty} M_{\frac{V_1+\dots+V_n}{\sqrt{n}}}(t)$$

$$\overset{\text{III}}{=} \lim_{n\to\infty} M_{V_1+\dots+V_n}\left(\frac{t}{\sqrt{n}}\right)$$

$$\overset{\text{IV}}{=} \lim_{n\to\infty} \left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)^n = \lim_{n\to\infty} e^{\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)^n} = \lim_{n\to\infty} e^{n\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)}$$

$$= e^{\lim_{n\to\infty} n\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)} = e^{t^2 \lim_{n\to\infty} \frac{\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)}{\frac{t^2}{n}}}$$

let $u = \frac{1}{\sqrt{n}}$, $n\to\infty \Rightarrow u = 0$.

$$\therefore = e^{t^2 \lim_{u\to 0} \frac{\ln(M_V(ut))}{t^2u^2}}$$

$$\frac{d}{du} \Rightarrow = e^{t^2 \lim_{u\to 0} \frac{\frac{\cancel{t M_V'(ut)}\to 0}{\cancel{M_V(ut)}\to 1}}{2ut^2}}$$

$$\frac{d}{du} \Rightarrow = e^{\frac{t^2}{2}\lim_{u\to 0} \frac{-t M_V'(ut) M_V'(ut) + M_V(ut)\, t\, M_V''(ut)}{(M_V(ut))^2\, t}}$$

$$= e^{\frac{t^2}{2}\lim_{u\to 0} \frac{-t(\cancel{M_V'(ut)})^2 + \cancel{M_V(ut)}\cancel{M_V''(ut)}}{t\cancel{(M_V(ut))^2}}} = e^{\frac{t^2}{2}}$$

$$C_n := \frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1).$$

Central Limit Theorem (CLT).

How to use CLT.

n is never ∞, but if n is "large", then.

① $\dfrac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\approx} N(0,1)$.

② $\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

③ $T \overset{d}{\approx} N\left(n\mu, (\sigma\sqrt{n})^2\right)$

Problem: $X_1, X_2, \ldots, X_{30} \overset{iid}{\sim} Geo\left(\frac{1}{2}\right)$. what is $P(\bar{X} \geq 2.75)$

$$\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

$$= N\left(2, \left(\frac{\sqrt{2}}{\sqrt{30}}\right)^2\right) = N\left(2, (0.25)^2\right)$$

$$\approx P\left(\frac{\bar{X}-2}{0.25} \geq \frac{2.75-2}{0.25}\right) = P(Z \geq 3) \approx .0015$$

If $X_1, \ldots, X_n$ iid n is large.

$$\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

$$T \overset{d}{\approx} N\left(\mu, (\sigma\sqrt{n})^2\right).$$