Consider the pattern:

"Pascal's Δ"

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$3 = \binom{3}{1}$, $3 = \binom{3}{2}$, $1 = \binom{4}{0}$, $4 = \binom{4}{1}$, $6 = \binom{4}{2}$, $4 = \binom{4}{3}$, $1 = \binom{4}{4}$

$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$ Is this true?

Recurrence Relation $\forall n \in \mathbb{N}, k \in \{0, \dots, n-1\}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Identity (Formula/Rule/Theorem)

$$\frac{n!}{k!(n-k)!} \stackrel{?}{=} \left(\frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} + \frac{(n-1)!}{\underbrace{(n-k-1)!}_{(n-1-k)!}k!}\right)\frac{n}{n}$$

$$= \frac{n!}{n}\left(\frac{1}{(n-k)!(k-1)!}\cdot\frac{k}{k} + \frac{1}{(n-k-1)!k!}\cdot\frac{n-k}{n-k}\right)$$

$$= \frac{n!}{n}\left(\frac{k}{(n-k)!k!} + \frac{n-k}{(n-k)!k!}\right) = \binom{n}{k} \checkmark$$

- Let $S = \{\spadesuit, \clubsuit, \diamondsuit, \heartsuit\}$ called "suit" $|S| = 4$
- Let $R = \{2, 3, \dots, 10, J, Q, K, A\}$ called "rank" $|R| = 13$
- Let $D = S \times R$ called the "deck" of cards $\Rightarrow |D| = 52$
- Consider the "game" where you are given ("dealt") 5 cards (all equally likely) without replacement such that order doesn't matter. These 5 cards are called a "hand"

$P(\text{Royal Flush}) = \frac{|A|}{|\Omega|} = \frac{4}{\binom{52}{5}}$

10, J, Q, K, A all of same suit

$\binom{52}{5} = 2{,}598{,}960$

$P(4\text{ of a kind}) = \frac{\binom{13}{1}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}$

#1 of 4 of a kind

order card $\binom{48}{1}$

2 2 2 2
7 7 7 7
⋮
K K K K

$P(\text{Straight Flush}) = \frac{\binom{10}{1}\binom{4}{1} - 4}{\binom{52}{5}}$

beginning #; the suit; minus the royal flush

ALL SAME SUIT

A 2 3 4 5
2 3 4 5 6
⋮
9 10 J Q K
10 J Q K A

$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$

Rank of the 3 of a kind; RANK of the 2 of a kind; 3 suits; suits

$\underbrace{7\,7\,7}_{\text{3 of same rank}}\ \underbrace{Q\,Q}_{\text{2 of same rank}}$

$$P(\text{Flush}) = \frac{\binom{4}{1}\binom{13}{5} - \binom{Q}{1}\binom{4}{1} - 4}{\binom{52}{5}}$$

all same suit

but <u>not</u> straight

$$P(\text{Straight}) = \frac{\binom{10}{1}\binom{4}{1}^5 - \binom{Q}{1}\binom{4}{1}4}{\binom{52}{5}}$$

# of straights

$$P(3 \text{ of a kind}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

777 Q9

7♡, 7◇, 7♧   7◇, 7♧, 7♤

7♡, 7◇, 7♤   7♡, 7♧, 7♤

$$P(\text{Two-Pair}) = \frac{\binom{13}{2}\binom{4}{2}^2 \overbrace{\binom{11}{1}\binom{4}{1}}^{5^{th}\text{ card}}}{\binom{52}{5}}$$

or $\binom{44}{1}$

77 QQ3

<u>Note #1</u>

$${}_{13}P_2 = \binom{13}{1}\binom{12}{1} \neq \binom{13}{2}$$

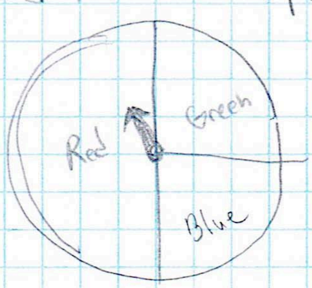
for the Full house we use permutations - order matters!
for the Two-Pair we use Combo

Revisit the "working" definition of probability

$P(A) := \frac{|A|}{|\Omega|}$ "the classic definition" in use through the 1800's

had a hidden assumption that all outcomes are equally likely

Consider the random experiment of spinning



$A = \{R\}$

$P(A) \neq \frac{|\{R\}|}{|\Omega|} = \frac{1}{3}$

$$\underbrace{\forall \omega \in \Omega \quad P(\{\omega\}) = \frac{1}{|\Omega|}}_{\text{equally likely outcomes}}$$

e.g. flipping coins, rolling dice
seating people, drawing cards

<u>We need a new definition of probability</u>

(I) Limiting Frequency Def

First, define $\mathbb{1}_{\omega \in A} := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

indicator function

$$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n} = \frac{\#\{\omega_i \in A\}}{n}$$

Von Mises, 1928 as n gets larger P(A) becomes more "stable"

<u>Problems with this new definition</u>

1. Requires experimentation, infinite elements (not possible)

   ⇒ We can only know an approximation ⇒ always wrong and could be very wrong

2. Not general

   P(OJ Guilty)   P(N. Korea nukes Guam)   P(Irma hits Miami