Lecture 10 10/10/17

Review...

$X \sim \text{Bern}(p) := p^x(1-p)^{1-x}$

Verify $\sum_{x \in Supp(X)} p(x) = 1$

$= \sum_{x \in \{0,1\}} p^x(1-p)^{1-x} = p^0(1-p)^{1-0} + p^1(1-p)^{1-1}$

$= 1-p+p = \boxed{1}$ ✓

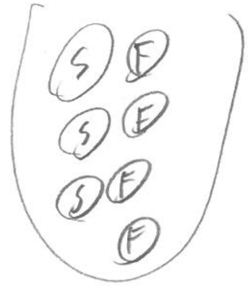
A new situation for a new r.v.

10 cards
4 R
6 B

$P(2R \text{ in } 3 \text{ cards}) = \dfrac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$

$P(xR \text{ in } 3 \text{ cards}) = \dfrac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$

$P(xR \text{ in } n \text{ cards}) = \dfrac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$



$N=7$
$K=3$
n ... up to you

K R, 10-K B

$P(xR \text{ in } n \text{ cards}) = \dfrac{\binom{K}{x}\binom{10-K}{n-x}}{\binom{10}{n}}$

n: sample size
K: #successes
N: population size

N cards N-K B

$P(xR \text{ in } n \text{ cards}) = \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$

3 kinds you can learn!

$X \sim \text{Hyper}(n, K, N) := p(x)$ → three parameters! Bern had just 1!

e.g. 100 students, 53 females

pick 8 at random. What's the prob of 6 being female?

Model $X \sim \text{Hyper}(8, 53, 100)$

Prob Query: $P(X=6) = \dfrac{\binom{53}{6}\binom{47}{2}}{\binom{100}{8}}$ → $6+2=8$ why?

use calculator... & Stirling's approx if needed

What can these knobs be?

$N=0$? Absurd $n=0$? Absurd as well $X \sim \text{Deg}(0)$

$N=1$? $\Rightarrow K=0$ or 1

$\Rightarrow n=1 \Rightarrow X \sim \text{Deg}(0)$ if $K=0$ or $X \sim \text{Deg}(1)$ if $K=1$

$N=2$ if $n=2$ $X \sim \text{Deg}(K)$ you get everything!

if $n=1$

$K=0 \Rightarrow X \sim \text{Deg}(0)$

$K=2 \Rightarrow X \sim \text{Deg}(1)$

$K=1$

$$X \sim \text{Hyper}(1,1,2) = \frac{\binom{1}{x}\binom{1}{1-x}}{\binom{2}{1}} \Rightarrow \begin{array}{l} P(X=0)=\frac{1}{2} \\ P(X=1)=\frac{1}{2} \end{array} = \text{Bern}\left(\tfrac{1}{2}\right) \quad \text{Why?}$$

$N=3$

$K=1,2$

$n=1,2$ Parameter space

$$\Rightarrow \boxed{\begin{array}{l} N \in \mathbb{N} \setminus \{1\} \\ K \in \{1,2,\ldots,N-1\} \\ n \in \{1,2,\ldots,N-1\} \end{array}}$$

It must be this...

$$X \sim \text{Hyper}(1,K,N) = \text{Bern}\Big(\overset{\overset{p}{\shortparallel}}{\tfrac{K}{N}}\Big) \qquad \text{Supp}(X)=\{0,1\}$$

Proof

$$= \frac{\binom{K}{x}\binom{N-K}{1-x}}{\binom{N}{1}} =$$

$$P(X=0) = \frac{\binom{K}{0}\binom{N-K}{1}}{N} = \frac{N-K}{N} = 1-\frac{K}{N} \quad \text{"}1-p\text{"}$$

$$P(X=1) = \frac{\binom{K}{1}\binom{N-K}{1-1}}{N} = \frac{K}{N} \quad \text{"}p\text{"}$$

general model

$X \sim \text{Hyper}(n,K,N)$ $\text{Supp}(X)$?

(a) $X \sim \text{Hyper}(2, 4, 10) \Rightarrow \text{Supp}[X] = \{0, 1, 2\}$

(b) $X \sim \text{Hyper}(5, 4, 10) \Rightarrow \text{Supp}(X) = \{0, 1, 2, 3, 4\}$ why $5 \notin \text{Supp}$?

(c) $X \sim \text{Hyper}(8, 4, 10) \Rightarrow \text{Supp}(X) = \{2, 3, 4\}$ why $1 \notin \text{Supp}$?

(d) $X \sim \text{Hyper}(5, 7, 10) \Rightarrow \text{Supp}(X) = \{2, 3, 4, 5\}$ why $6 \notin \text{Supp}$?

4 cases of $X \sim \text{Hyper}(n, K, N)$

(a) $n < K, \; n < N-K$ choose less than # successes & $n <$ # failures

$\text{Supp}(X) = \{0, 1, \ldots, n\}$

(b) $n \geq K, \; n < N-K$

$\text{Supp}(X) = \{0, 1, \ldots, K\}$

(c) $n \geq K, \; n \geq N-K$

$\text{Supp}(X) = \{n-(N-K), \ldots, K\}$

(d) $n < K, \; n \geq N-K$

$\text{Supp}(X) = \{n-(N-K), \ldots, n\}$

	$n < K$	$n \geq K$
$n < N-K$	$\{0, \ldots, n\}$	$\{0, \ldots, K\}$
$n \geq N-K$	$\{n-(N-K), \ldots, n\}$	$\{n-(N-K), \ldots, K\}$

$$\text{Supp}[X] = \{\max(0, n-(N-K)), \ldots, \min(n, K)\}$$

$$\sum_{x \in \text{Supp}[X]} p(x) = 1 \;?$$ Hard... HW...

"Equivalent parameterization" °F or °C? or K?

let $p = \frac{K}{N} \Rightarrow K = pN$

$N = 9$, $p = 0.7$ not legal! why $K = 6.3$ makes no sense

$$X \sim \text{Hyper}(n, p, N) := \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

param space

$N \in \mathbb{N} \setminus \{1\}$

$n \in \{1, \ldots, N-1\}$

$p \in \left\{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$

Consider $p = 0.5$, $n = 6$, $N = 100$

$$P(X=3) = \frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = .3223$$

Now $N = 1000$

$$P(X=3) = \frac{\binom{500}{3}\binom{500}{3}}{\binom{1000}{6}} = .3134$$

$N = 10{,}000$

$$P(X=3) = \frac{\binom{5000}{3}\binom{5000}{3}}{\binom{10000}{6}} = .3126$$

Converge?

$$\lim_{N\to\infty} P(X=3) = \;?$$

Generally, what is the limiting r.v.?

$X \sim \text{Hypergeom}(n, p, N)$ and $N \to \infty$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} = \frac{\frac{1}{x!(n-x)!}}{\frac{1}{n!}}$$

$$\lim_{N\to\infty} p(x) = \lim_{N\to\infty} \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}} = \lim_{N\to\infty} \frac{\frac{(pN)!}{x!(pN-x)!}\cdot\frac{((1-p)N)!}{(n-x)!((1-p)N-n+x)!}}{\frac{N!}{(N-n)!\,n!}} = \frac{\frac{1}{x!(n-x)!}}{\frac{1}{n!}} \lim_{N\to\infty} \frac{\frac{(pN)!}{(pN-x)!}\cdot\frac{((1-p)N)!}{((1-p)N-n+x)!}}{\frac{N!}{(N-n)!}}$$

factor out constants

$$\lim_{x\to\infty} a f(x) = a \lim_{x\to\infty} f(x)$$