Lecture 5 09/12/2017

Consider the pattern:

"Pascals $\triangle$"

$$\begin{array}{lccccccccc} & & & & & 1 & & & & \\ (a+b)^2 & & & & 1 & 2 & 1 & & & \\ & & & 1 & 3 & & 3 & 1 & & \\ (a+b)^4 & & 1 & 4 & & 6 & & 4 & 1 & \end{array}$$

$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$

$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$

Recurrence Relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Is this true $\forall n \in \mathbb{N}_0, k \in \{0 \dots n-1\}$

$$\frac{n!}{k!(n-k)!} \stackrel{?}{=} \left(\frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!} \right) \frac{n}{n}$$

$$= \frac{n!}{n} \left(\frac{1}{(n-k)!\,(k-1)!} \cdot \frac{k}{k} + \frac{1}{(n-k-1)!\,k!} \cdot \frac{n-k}{n-k} \right)$$

$$= \frac{n!}{\cancel{n}} \left(\frac{k}{(n-k)!\,k!} + \frac{n-k}{(n-k)!\,k!} \right)$$

$$= \binom{n}{k}$$

Pascals
Identity
Formula
Rule
Theorem

let $S = \{♠, ♣, ◇, ♡\}$ called suit $|S| = 4$

let $R = \{2, 3, \dots 10, J, Q, K, A\}$ called rank $|R| = 13$

let $D = S \times R$ called the "deck of cards $|D| = 52$

all equally likely

consider the "game" when you are given (dealt) 5 cards w/o replacement such that order doesn't matter. These 5 cards are called a "hand"

(1) $P(\text{Royal Flush}) = \frac{|A|}{|\Omega|} = \frac{4}{\binom{52}{5}}$

~~has to be~~
10, J, Q, K, A
all of the same suit

$\binom{52}{5}$ = 2,598,960

---

(3) $P(4 \text{ of a Kind}) = \frac{\binom{13}{1}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}$

7777K

$\binom{13}{1}$: # of 4 of a kind

$\binom{12}{1}$: rank

$\binom{4}{1}$: suit

other card $\binom{48}{1}$

---

(2) $P(\text{straight flush}) = \frac{\binom{10}{1}\binom{4}{1} - 4}{\binom{52}{5}}$

$\binom{10}{1}$: beginning number

$\binom{4}{1}$: suit

~~all same suit~~

A 2 3 4 5
2 3 4 5 6
3 4 5 6 7
4 5 6 7 8
"
9 10 J Q K
10 J Q K A

---

(4) $P(\text{Full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$

77700

3 same rank, 2 of same rank

$\binom{13}{1}$: rank of 3 of a kind

$\binom{4}{3}$: 3 suits

$\binom{12}{1}$: rank of 2 of a kind

$\binom{4}{2}$: suits of 2 of a kind

(5) $P(\text{Flush}) = \dfrac{\binom{4}{1}\binom{13}{5} - \binom{9}{1}\binom{4}{1} - 4}{\binom{52}{5}}$

all same suit
but not straight

$P(\text{straight}) = \dfrac{\binom{10}{1}\binom{4}{1}^5 - \binom{9}{1}\binom{4}{1} - 4}{\binom{52}{5}}$

3 4 5 6 7

rank

---

order doesn't matter

$P(3 \text{ of a Kind})$ $\dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$

7 7 7 Q 9

5th card

$P(\text{Two Pair})$ $\dfrac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$

7 7 Q Q 3

$_{13}P_2 = \binom{13}{1}\binom{12}{1} \neq \binom{13}{2} \cdot 2$

order matters

order doesn't matter

7 7 Q Q ? = Q Q 7 7 ?

7 7 Q Q ≠ Q Q 7 7

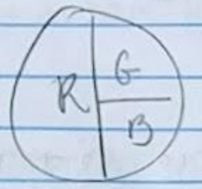
Revisit the working definition of Probability

$P(A) = \frac{|A|}{|\Omega|}$ "the classic def" in use through 1800s

→ had a hidden assumption

Consider the random experiment of spinning

For all $\forall \omega \in \Omega$ $P(\{\omega\}) = \frac{1}{|\Omega|}$



$A = \{R\}$

$P(A) = \frac{|\{R\}|}{|\Omega|} = \frac{1}{3}$ X

equally likely outcome

e.g. flip coins
roll die
seating people
drawing cards

Need a new definition of probability

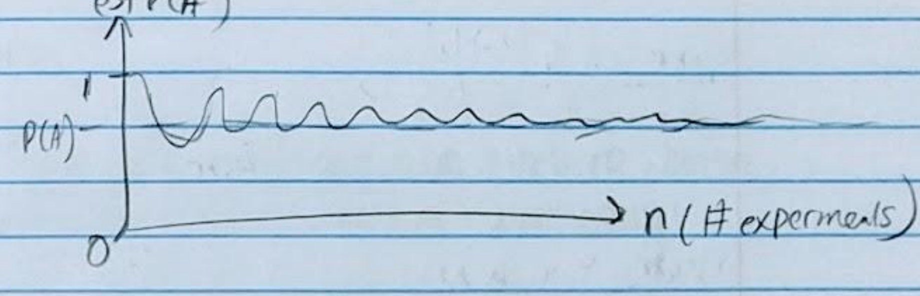
(I) Limiting Frequency Def.

First define $\mathbb{1}_{\omega \in A} := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

indicator function

$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n} = \frac{\#\{\omega_i \in A\}}{n}$

Von Mises, 1928
as n gets larger
P(A) becomes more
"stable



1654 → Chevalier de Mere who wrote a letter to Pascal and Fermat and said I think $P(\{\geq 1 \text{ double-6 in 24 rolls of the die}\}) \approx \frac{1}{2}$

True prob = .4914

Problems

1) requires experimentation, infinite experiments
$\underbrace{\text{infinite experiments}}_{\text{not possible}}$

=> we can only learn an approximation => always wrong and could be very wrong

2) not general
P(OJ Simpson guilty) =
P(Irma hits Miami)
P(North Korea Nukes Guam)