Math 281 Lec 4

ways to sample k objects out of a set of n objects without replacement

$$_nP_k = \frac{n!}{(n-k)!}$$

Recall B-J, R-S, C-M sitting in 6 seats. How many ways to seat them in a circle s.t. you don't care which seat is the first seat

$\langle B, J, R, S, C, M \rangle$ is equivalent to
$\langle J, R, S, C, M, B \rangle$ . . . . .

. . . .

$\langle M\ B\ J\ R\ S\ C \rangle$

These are part of the 6! but all 6 are equivalent.

$$\Rightarrow \frac{6!}{6}$$

Divide out the invariance factor

Imagine a basket of 5 flowers: 3 orchids: O_1, O_2, O_3,
2 chrysanthemums: X_1, X_2

How many ways to set up 5 flower pots?

$\Rightarrow$ Sample 5 objects without replacement $\Rightarrow 5! = 120$

This assumes each orchid is "distinct", "distinguishable", "unique" and each chrysanthemum is unique too.

What if the orchids are not unique?

In the list of 120, we find "collapsable subsets":

120

$O_1\ O_2\ O_3\ \ X_1\ X_2$
$O_1\ O_3\ O_2\ \ X_1\ X_2$
$O_2\ O_1\ O_3\ \ X_1\ X_2$
$O_2\ O_3\ O_1\ \ X_1\ X_2$
$O_3\ O_1\ O_2\ \ X_1\ X_2$
$O_3\ O_2\ O_1\ \ X_1\ X_2$

set of 6 collapse into 1!

$\Rightarrow$ 1 Only count as one!

$O_1\ O_2\ O_3\ \ X_2\ X_1$
$X_2\ X_1$
$X_2\ X_1$
$X_2\ X_1$
$X_2\ X_1$
$X_2\ X_1$

How many are there 6. Why?
# of ways to sample 3 orchids without replacement $\Rightarrow 3! = 6$

total # of ways

$$= \frac{5!}{3!}$$

duplicate eliminate factor

Now if chrysanthemums are indistinguishable...

$O_1 O_2 O_3 X_1 X_2$

$O_1 O_2 O_3 X_2 X_1$

this set of $2 = 2!$ # ways to order collapses into 1

$\Rightarrow$ total # of ways

$= \dfrac{5!}{2!}$

If both are indistinguishable $\Rightarrow \dfrac{5!}{3!\,2!}$

$OOOXX$ — $2!$ — $OOOX_1X_2$, $OOOX_2X_1$

$OOOX_1X_2$:
$O_1O_2O_3\,X_1X_2$
$O_1O_3O_2\,X_1X_2$
$O_2O_1O_3\,X_1X_2$
$O_2O_3O_1\,X_1X_2$
$O_3O_1O_2\,X_1X_2$
$O_3O_2O_1\,X_1X_2$
$\} 3!$

$OOOX_2X_1$ $\} 3!$

$\} 3!\,2!$

Both indistinct — X's are unique — All unique

Remember...

$P(4H \text{ in } 10 \text{ coin flips}) = \dfrac{|A|}{|\Omega|} = \dfrac{\frac{10!}{4!\,6!}}{2^{10}} \approx .205$

4H's, 6T's

s.t. H's are indistinct & T's are indistinct $\Rightarrow \dfrac{10!}{4!\,6!}$

ways to order 10 items

don't care about order of H's

don't care about order of T's

Let's go back to $\{J, B, S, R, M, C\}$ our favorite 6 people

How many ways to seat them in 6 chairs? $6! = {}_6P_6 = 720$

" " " " " 3 chairs? $\dfrac{6!}{(6-3)!} = {}_6P_3 = 120$

How many - - - - - - 3 chairs
s.t. their order doesn't matter in the chairs?

e.g. $\{B, S, C\}$ were chosen, we are now saying that

$$\langle B,S,C\rangle = \langle B,C,S\rangle = \langle S,B,C\rangle = \langle S,C,B\rangle = \langle C,B,S\rangle = \langle C,S,B\rangle$$

How many ways? 3 people, 3 chairs!

So for every $6P_3$ permutation, there are $3P_3$ that are considered identical

$$\frac{6P_3}{3P_3} = 20.$$ Can we list them?

JBS	BSR	SRM	RMC
JBR	BSM	SRC	
JBM	BSC	SMC	
JBC	BRM		
JSR	BRC		
JSM	BMC		
JSC			
JRM			
JRC			
JMC			
10 +	6 +	3 +	1 = 20

How about 6 people, 4 chairs? $\dfrac{6P_4}{4P_4} = \dfrac{\cancel{6}\cdot 5\cdot \cancel{4}\cdot 3}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = 15$

JBSR	BSRM	SRMC
JBSM	BSRC	
JBSC	BSMC	
JBRM	BRMC	
JBRC		
JBML		
JSRM		
JSRC		
JSMC		
JRMC		
10	+ 4	+ 1 = 15

This is the number of ways to "choose" 4 objects out of a set of 6.

i.e. the # of ways to sample k objects from a set of n without replacement s.t. order doesn't matter!

$$\binom{n}{k} := \frac{{}_nP_k}{{}_kP_k} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!\,k!}$$

$${}_nC_k =$$

A function of two params: $n \in \mathbb{N}_0$, $k \in \{0,1,\dots,n\}$

As of now, any illegal parameters returns 0 by definition.

Identities

① $\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n$

② $\binom{n}{n-1} = \quad = n$

③ $\binom{n}{n} = \quad = 1$

④ $\binom{n}{0} = \quad = 1$

⑤ $\binom{n}{k} = \binom{n}{n-k}$

each choice of k is a choice of $n-k$. Thus they must be equal.

Imagine 6 people B-J, R-S, C-M. 4 are seat down randomly. What is prob I get there?

$\{J, ?, ??\}$ from $\{B, R, S, C, M\}$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{5}{3}}{\binom{6}{4}} = \frac{\frac{5\cdot\cancel{4}\cdot\cancel{3}}{\cancel{3}\cdot\cancel{2}\cdot 1}}{\frac{\cancel{6}\cdot 5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}} = \frac{2}{3}$$

$$\frac{\frac{4({}_5P_3)}{4!}}{\frac{{}_6P_4}{4!}} = \frac{\frac{{}_5P_3}{3!}}{\binom{6}{4}} = \frac{\binom{5}{3}}{\binom{6}{4}}$$

order doesn't matter

$$= \frac{\underline{J}\,\underline{5}\,\underline{4}\,\underline{3} + \underline{5}\,\underline{J}\,\underline{4}\,\underline{3} + \underline{5}\,\underline{4}\,\underline{J}\,\underline{3} + \underline{5}\,\underline{4}\,\underline{3}\,\underline{J}}{{}_6P_4} = \frac{4({}_5P_3)}{{}_6P_4} = \frac{4\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}} = \frac{2}{3}$$

order does matter

A cool combinatorial identity

Recall

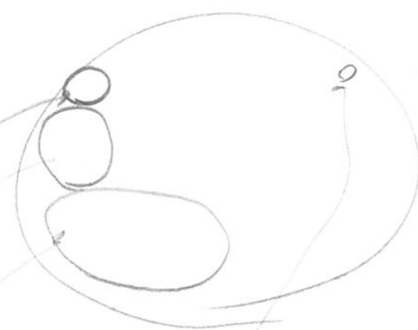
$2^A = \{B : B \subseteq A\}$, let $|A| = n$ total # elems

$|2^A| = 2^{|A|} = 2^n$

Cool proof

Note

2^A



$$\begin{aligned} 2^A = \{B : B \subseteq A\} = &\ \{B : B \subseteq A \;\&\; |B| = 0\} \cup \\ &\ \{B : B \subseteq A \;\&\; |B| = 1\} \cup \\ &\ \{B : B \subseteq A \;\&\; |B| = 2\} \cup \\ &\ \quad \vdots \\ &\ \cup \\ &\ \{B : B \subseteq A \;\&\; |B| = n\} \end{aligned}$$

these are mutually excl. & coll. exh.

$$\Rightarrow |2^A| = |\{B : B \subseteq A \;\&\; |B| = 0\}| + |\{B : B \subseteq A \;\&\; |B| = 1\}| + \dots + |\{B : B \subseteq A \;\&\; |B| = n\}|$$

$$= \sum_{i=0}^{n} |\{B : B \subseteq A \;\&\; |B| = i\}|$$

How many ways?
$|A| = n$ we choose i elems s.t. order doesn't matter

$$= \boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

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The most famous thm. using combinatorics:

$$(a+b)^2 = (a+b)(a+b) = \underbrace{a^2 + ab + ba + b^2}_{4 \text{ terms } = (2)(2)} \overset{\text{simplified}}{=} a^2 + 2ab + b^2$$

$$(a+b)^3 = (a+b)(a+b)(a+b) = \underbrace{a^3 + \ldots\ldots}_{8 \text{ terms } = (2)(2)(2)} \overset{\text{simplified}}{=} a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 \quad (a+b)(a+b)(a+b)(a+b) = \binom{4}{4}a^4b^0 + \binom{4}{3}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{1}a^1b^3 + \binom{4}{0}a^0b^4$$

$$\binom{4}{4} = \binom{4}{0}, \quad \binom{4}{3} = \binom{4}{1}, \quad \binom{4}{2} = \binom{4}{2}, \quad \binom{4}{1} = \binom{4}{3}, \quad \binom{4}{0} = \binom{4}{4}$$

$$\sum_{i=0}^{4} \binom{4}{i} = 2^4 \quad \text{\# of terms correct!}$$

$$(a+b)^n = \binom{n}{n}a^nb^0 + \binom{n}{n-1}a^{n-1}b^1 + \ldots + \binom{n}{1}a^1b^{n-1} + \binom{n}{0}a^0b^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Binomial Thm!

Consider

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} (1)^i x^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$= \binom{n}{0}x^0 + \underbrace{\binom{n}{1}x^1 + \ldots + \binom{n}{n-1}x^{n-1}} + \binom{n}{n}x^n$$

$$= 1 + \sum_{i=1}^{n-1} \binom{n}{i} x^i + x^n$$

$$= (1+x)(1+x)^{n-1} = (1+x)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}x^i\right) = \sum_{i=0}^{n-1}\binom{n-1}{i}x^i + \sum_{i=0}^{n-1}\binom{n-1}{i}x^{i+1}$$

$$= 1 + \sum_{i=1}^{n-1}\binom{n-1}{i}x^i + \sum_{i'=1}^{n}\binom{n-1}{i'-1}x^{i'}$$

let $i' = i+1 \Rightarrow i = i'-1$

$\Rightarrow i = n-1 \Rightarrow i' = n$

$\Rightarrow i = 0 \Rightarrow$

$$= 1 + \sum_{i=1}^{n-1}\binom{n-1}{i}x^i + \sum_{i'=1}^{n-1}\binom{n-1}{i'-1}x^{i'} + x^n$$