11/28/2017

Lecture 20

$M_X(t) := E[e^{tX}]$
mgf

$M_X(0) = 1$

Properties

I $M_X(t) = M_Y(t) \iff X \overset{d}{=} Y$ as long as $M_X(t)$ exists

II $M_X^{(k)}(0) = E[X^k]$ i.e. k^{th} moment

III $Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_X(at)$

IV If X_1, X_2 are independent

$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) \overset{\text{if ident. distributed via I}}{=} (M_X(t))^2$

V Levine's Continuity Theorem

$X \sim Bern(p) \Rightarrow M_X(t) = 1 - p + pe^t$

$X \sim Binom(n, p) \Rightarrow M_X(t) = (1 - p + pe^t)^n$

$X \sim Exp(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$

standard normal $\leftarrow$ $Z \sim N(0,1) \Rightarrow M_Z(t) = e^{t^2/2}$

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2) \Rightarrow M_X(t) \overset{III}{=} e^{t\mu} M_Z(\sigma t)$$
$$= e^{t\mu} e^{\frac{\sigma^2 t^2}{2}}$$
$$= e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

$$X \sim Deg(c) = M_X(t) = E[e^{tX}] = \sum_{x \in Supp[X]} e^{tx} p(x) = e^{tc}$$

$X_1 \sim N(\mu_1, \sigma_1^2)$ ind. of $X_2 \sim N(\mu_2, \sigma_2^2)$

$X_1 + X_2 \sim ?$

$$M_{X_1+X_2}(t) \overset{IV}{=} M_{X_1}(t) M_{X_2}(t) = \left(e^{t\mu_1 + \frac{\sigma_1^2 t^2}{2}}\right)\left(e^{t\mu_2 + \frac{\sigma_2^2 t^2}{2}}\right)$$
$$= e^{t(\mu_1+\mu_2) + \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}} \Rightarrow X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2, \sigma_2^2)$$

Levine's Continuity Theorem

* equality in distribution, CDFs are equal *

IF $X_1, X_2, \dots$ sequence of r.v's

$\lim_{n\to\infty} M_{X_n}(t) = M_X(t) \Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ converge in distribution

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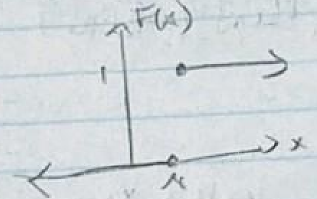
Law of Large Numbers

NOT ON FINAL

$\bar{X}_n \to \mu = E[X]$ ← long run average looks more like the average

Let $X_1, X_2, \dots \overset{iid}{\sim}$ with mean μ

Note: $\mu \sim \text{Deg}(\mu)$
μ ⇓ constant



WTS $\lim M_{\bar{X}_n}(t) = e^{t\mu}$

$$\lim_{n\to\infty} M_{\bar{X}_n}(t) = \lim_{n\to\infty} M_{\frac{X_1+\dots+X_n}{n}}(t) \overset{\text{III}}{=} \lim_{n\to\infty} M_{X_1+\dots+X_n}\left(\tfrac{t}{n}\right) \overset{\text{IV}}{=} \lim_{t\to\infty} \left(M_X\left(\tfrac{t}{n}\right)\right)^n$$

$$= \lim_{n\to\infty} e^{\ln\left(M_X\left(\frac{t}{n}\right)^n\right)} = \lim_{n\to\infty} e^{n\ln\left(M_X\left(\frac{t}{n}\right)\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{1/n}} = e^{\lim_{n\to\infty} \frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{1/n}}$$

let $u = \frac{1}{n}$ $= e^{\lim_{u\to 0} \frac{\ln(M_X(ut))}{u}}$ $\underset{=}{\text{L'Hopital ?}}$ $e^{\lim_{n\to\infty} \frac{t M_X'(ut)}{M_X(ut)}}$

$n\to\infty \Rightarrow u\to 0$

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x} \cdot \frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)} \quad \frac{d}{dx}[\ln(F(g(x)))] = \frac{g'(x)F'(g(x))}{F(g(x))}}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$e^{\frac{t(M_X'(0))}{\underbrace{M(0)}_{1}}} \quad M_X'(0) \to \mu \text{ by II}$$

$= e^{t\mu} \overset{\text{V}}{\Rightarrow} \bar{X} \to \mu$

let $X_1, X_2, \ldots \overset{iid}{\sim}$ w/ mean μ and w/ variance σ^2

Consider:

$$C_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} = \frac{\sqrt{n}\left(\frac{X_1 + \ldots X_n}{n}\right)}{\sigma} = \frac{\sqrt{n}\left(\frac{X_1 + \ldots X_n}{n} - \frac{\mu + \ldots \mu}{n}\right)}{\sigma}$$

Standardized average

$E[C_n] = 0, SE[C_n] = 1$

$$= \frac{(X_1 + \ldots X_n) - (\mu + \ldots \mu)}{\sigma\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}}\left(\frac{X_1 - \mu}{\sigma} + \ldots + \frac{X_i - \mu}{\sigma}\right)$$

let $V_i = \frac{X_i - \mu}{\sigma}$

$$= \frac{1}{\sqrt{n}}(V_1 + \ldots V_n)$$

Standardization of X

$E[V_i] = 0, SE[V_i] = 1$

$$\lim_{n\to\infty} M_{C_n}(t) = \lim_{n\to\infty} M_{\frac{1}{\sqrt{n}}(V_1 + \ldots V_n)}(t)$$

$$\overset{\text{III}}{=} \lim_{n\to\infty} M_{V_1 + \ldots V_n}\left(\frac{t}{\sqrt{n}}\right) \overset{\text{IV}}{=} \lim_{n\to\infty}\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)^n = \lim_{n\to\infty} e^{\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)^n\right)} =$$

$$= \lim_{n\to\infty} e^{n\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)}{1/n}}$$

$$= e^{\lim_{n\to\infty}\frac{\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)}{1/n}\frac{t^2}{t^2}} = e^{t^2\lim_{n\to\infty}\frac{\ln\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)}{\frac{t^2}{\mu}}}$$

let $\mu = \frac{1}{\sqrt{n}}$ $n\to\infty \Rightarrow \mu\to 0$

L'Hopitale   L'Hopitale

$$e^{t^2\lim_{\mu\to 0}\frac{\ln(M_V(\mu t))}{t^2\mu^2}} = e^{t^2/2\lim_{\mu\to 0}\frac{\frac{t(M_V'(\mu t))}{M_V(\mu t)}}{t^2\mu}} = \underbrace{0}$$

$$= e^{\frac{t^2}{2}\lim_{n\to 0}} \quad M_V(\mu t) + M_V'' t \quad = e^{t^2/2\frac{\overbrace{M_V(0)M_V''(0)}^{1} - M_V'(0)^2}{\underbrace{M_1(0)^2}_{1}}}$$

central Limit theorem $\Rightarrow$ $\boxed{e^{\frac{t^2}{2}} \overset{\text{①}}{\Rightarrow} \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}$

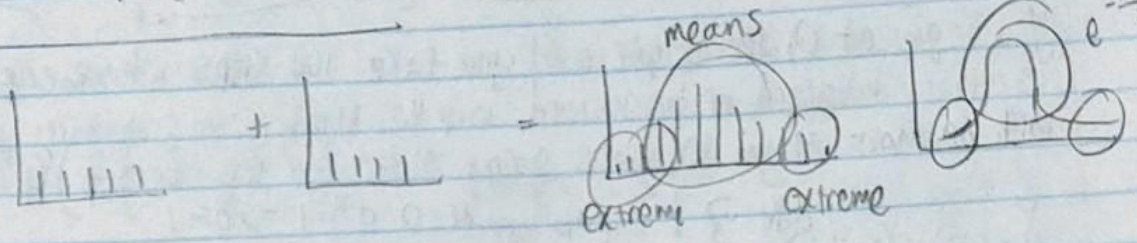
How to use Central Limit Theorem (CLT)

$n \to \infty$ is impractical, IF n is "large"...

① $\dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\approx} N(0,1)$

② $\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

③ $T \overset{d}{\approx} N\left(n\mu, (\sigma\sqrt{n})^2\right)$



$X_1 \cdots X_{30}$ iid Geom$(\frac{1}{2}) \Rightarrow \mu = \frac{1}{\frac{1}{2}} = 2, \sigma = \frac{\sqrt{1-p}}{p} = \frac{\sqrt{1-\frac{1}{2}}}{\frac{1}{2}} \approx 1.414$

$P(\bar{X} \geq 2.75)$ by CLT $\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

by CLT

$\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) = N\left(2, \left(\frac{1.414}{\sqrt{30}}\right)^2\right) = N(2, .258^2)$

$\rightarrow P\left(\frac{\bar{X}-2}{.250} \geq \frac{2.75-2}{.250}\right) = P(Z > 3) \approx 0.0015$