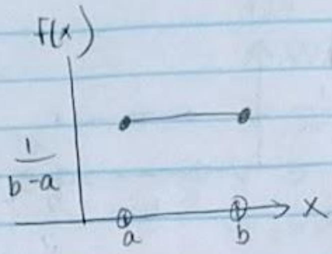
Lecture 18 11/09/2017

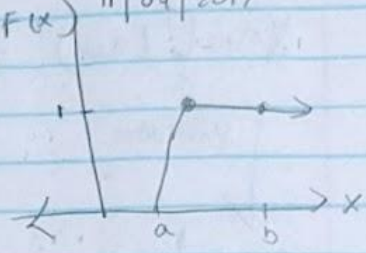
"Uniform r.v"

$X \sim U(a,b)$

$Supp[X] = (a,b)$

$a \in \mathbb{R},\ b \in \mathbb{R},\ a<b$

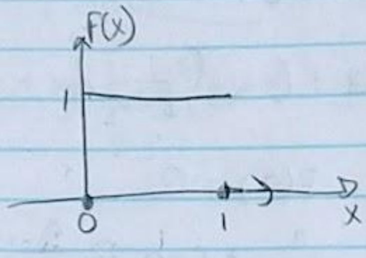


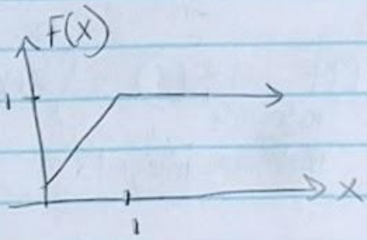


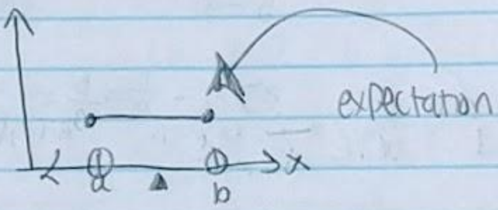
$$\int_{Supp[X]} f(x)\,dx = 1 \qquad \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}(x)\Big|_a^b = 1$$

$X \sim U(0,1) = 1$

CDF $F(x) = x$







$$E(x) = \int_a^b x\frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2} ✓$$

$$Med[X] = Quantile\left[X, \tfrac{1}{2}\right] = \tfrac{1}{2}(b-a)+a = \frac{b}{2}-\frac{a}{2}+\frac{2a}{2} = \boxed{\frac{b+a}{2}}$$

For X discrete r.v ... $Quantile[X,p] := \underset{x}{\operatorname{argmin}}\{F(x) \geq p\}$

For X continuous r.v $Quantile[X,p] = F_X^{-1}(p)$ ← inverse function

For $X \sim U(a,b)$

$$F(x) = \frac{x-a}{b-a} = p \Rightarrow x-a = p(b-a) \Rightarrow x = p(b-a)+a = F^{-1}(p)$$

$$\sigma^2 = E[X^2] - \mu^2 \quad , \quad E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{3}\frac{b^3-a^3}{b-a} = \frac{1}{3}\frac{(b-a)(b^2+ab+a^2)}{(b-a)}$$

$$\mu^2 = \left(\frac{a+b}{2}\right)^2 = \frac{a^2+2ab+b^2}{4}$$

$$\begin{array}{r|l} & b^2+ab+a^3 \\ \hline b-a & b^3-a^3 \\ & -(b^3-ab^2) \\ \hline & ab^2-a^3 \\ & -(ab^2-a^2b) \\ \hline & a^2b-a^3 \\ & -(ba^2-a^3) \\ \hline & 0 \end{array}$$

$$\sigma^2 = \frac{b^2+ab+a^2}{3}\cdot\frac{4}{4} - \frac{a^2+2ab+b^2}{4}\cdot\frac{3}{3}$$

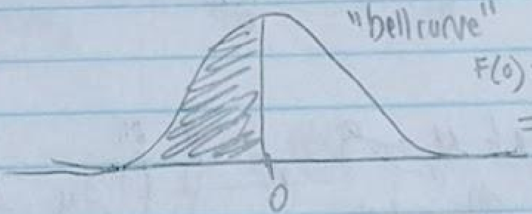
$$= \frac{4b^2+4ab+4a^2}{12} - \frac{3a^2+6ab+3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$$

$$\sigma = \frac{b-a}{\sqrt{12}}$$

$$Z \sim N(0,1) := \underbrace{\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}}_{f(x)}$$ always positive ≥ 0 checking if PDF ✓

Standard "normal"
"Gaussian"
"bell curve"



"bell curve"

$F(0) = P(Z<0) = \int_{-\infty}^{0} f(x)\,dx = \frac{1}{2}$

$Supp[Z] = \mathbb{R}$

$$\int_{\mathbb{R}} f(x)\,dx = 1 \qquad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,dx = 1 \quad = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}\,dx = \sqrt{2\pi} =$$

let $u = \frac{1}{\sqrt{2}}x = u^2 = \frac{x^2}{2}$

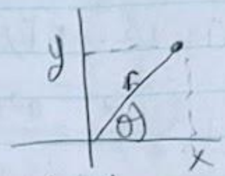
$du = \frac{1}{\sqrt{2}}dx$

$dx = \sqrt{2}\,du$

$$\int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}\,du = \sqrt{2\pi} = \sqrt{\pi}$$

Proof that $\frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \geq 0$ ✓

$$\left(\int_{-\infty}^{\infty} e^{-u^2} du\right)^2 = \pi$$



$x = r\cos\theta$
$y = r\sin\theta$

$$\int_{\mathbb{R}} e^{-u^2} du \int_{\mathbb{R}} e^{-u^2} du = \pi \Rightarrow \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2} e^{-y^2} dx\,dy$$

Area integral $\displaystyle\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$

$$dx\,dy = dA = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \partial r\, d\theta = r\,dr\,d\theta$$

$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial x}{\partial \theta} = r\cos\theta$

$$= \left(\frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right)\partial r\, d\theta$$

$$= \cos(\theta) r \cos(\theta) - -r\sin(\theta)\sin(\theta)$$

$$= r\cos^2(\theta) + r\sin^2(\theta) = r$$

let $u = r^2$

$\frac{du}{dr} = 2r$

$du = 2r\,dr$

$dr = \frac{1}{2r} du$

$$\int_0^{\infty}\int_0^{2\pi} e^{-u} \times \frac{1}{2x} du\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} e^{-u} du\, dr = \frac{1}{2}\int_0^{2\pi} \left[-e^{-u}\right]_0^{\infty} d\theta$$

$$\frac{1}{2}\int_0^{2\pi} d\theta = \frac{1}{2}\theta\Big]_0^{2\pi} = \pi$$

$$E[Z] = \int_{\mathbb{R}} x \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int x e^{-u} \frac{1}{x} du = \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} = \frac{1}{\sqrt{2\pi}}(0-0) = 0$$

let $u = \frac{x^2}{2}$

$\frac{du}{dx} = x$

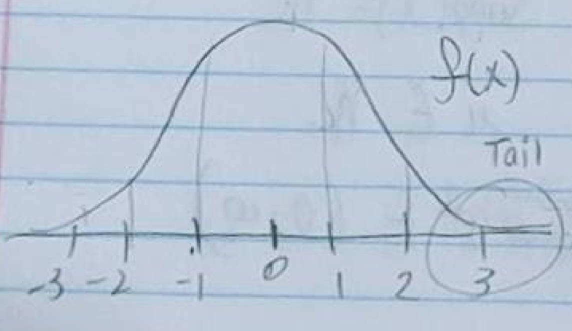
$dx = \frac{1}{x} du$

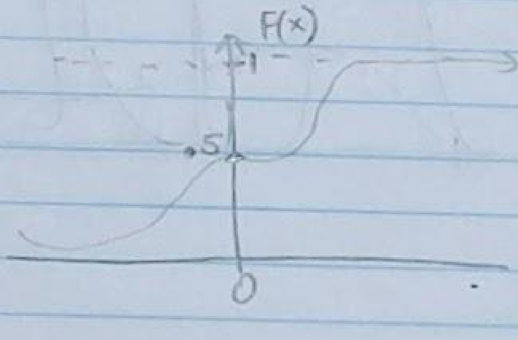
$$\sigma^2 = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x}{2}} dx = \ldots = 1 \Rightarrow \sigma = 1$$

$$F(x) = \int \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$ not possible in closed form

$$F(0) = P(Z < 0)$$
$$= \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{2}$$

$$\left.\begin{aligned} &P(Z \in [-1,1]) \approx 68\% \\ &P(Z \in [-2,2]) \approx 95\% \\ &P(Z \in [-3,3]) \approx 99.7\% \end{aligned}\right\}$$ "3σ Rule, empirical rule, "68-95-99.7%" rule





$Z \sim N(0,1)$

$X = \mu + \sigma Z$

$F_X(x) := P(X \leq x) = P(\mu + \sigma Z \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$

$$f_X(x) = F_X'(x) = \frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right]$$

$$= \frac{1}{\sigma} F_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$$

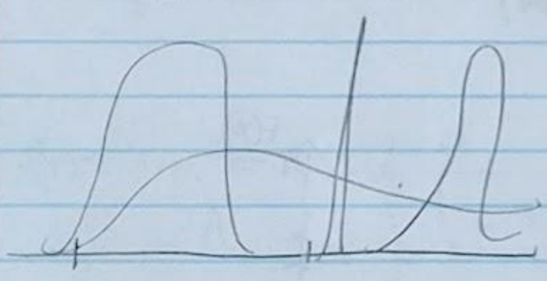
$$X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

"normal"

$E(X) = E[\mu + \sigma Z] = \mu + \sigma E[Z] = \mu$

$Var[X] = Var[\mu + \sigma Z] = \sigma^2 Var[Z] = \sigma^2$

$SE[X] = \sigma$



$Supp[X] = \mathbb{R}$

$\mu \in \mathbb{R}$

parameter space $\sigma^2 \in (0, \infty)$

$X \sim N(70", 4"^2)$ Whats the probability a random person is taller than 6'6"?

$P(X > 78) = P\left(\frac{X-70}{4} > \frac{78-70}{4}\right) = P(Z > Z) = 2.5\%$