$$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \sup(Z) = \mathbb{R}$$

$$X \sim \mu + \sigma Z \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \qquad \cancel{\sup Z}\ \sup(X) = \mathbb{R}$$

$$Z = \frac{X-\mu}{\sigma}$$

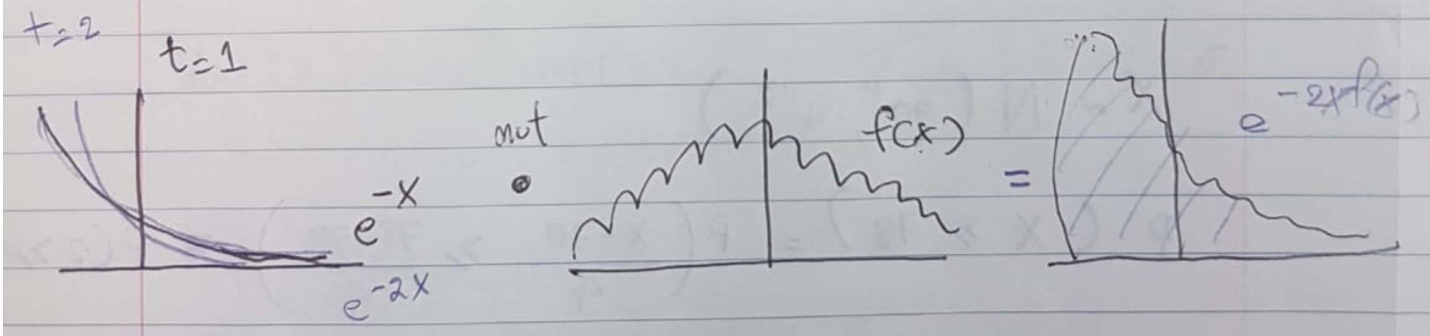
$P(Z \in [-1,1]) \simeq 68\%$

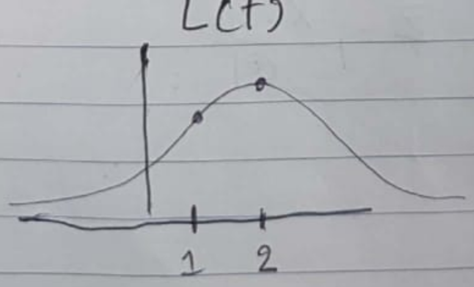
$P(Z \in [-2,2]) \simeq 95\%$

$P(Z \in [-3,3]) \simeq 99.7\%$

$$L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\, dx$$

Bilateral Laplace tranform





Thm : $L(t)$ and $f(x)$ are 1 to 1 if $L(t)$ exist

Define : the moment generating function (mgf)

For Continue. $M_X(t) = L(\#t) = \int_{R} e^{tx} f(x)\,dx = E[e^{tX}]$

For Discrete $M_X(t) = E[e^{tX}] = \sum_{x \in \sup p(x)} e^{tx} \, p(x)$

properties

(I) If $M_X(t) = M_Y(t)$

$\Rightarrow X \overset{d}{=} Y$

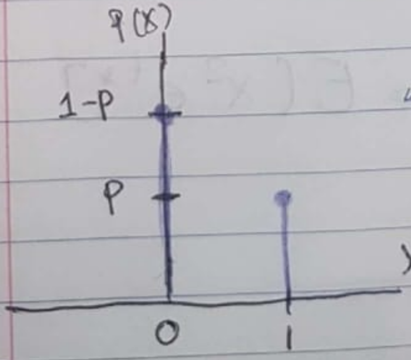
$\Rightarrow$ Continue $f_X(x) = f_Y(x)$

$\Rightarrow$ Discrete $P_X(x) = P_Y(x)$

Discrete

$X \sim Bern(p) = p^x (1-p)^{1-x}$

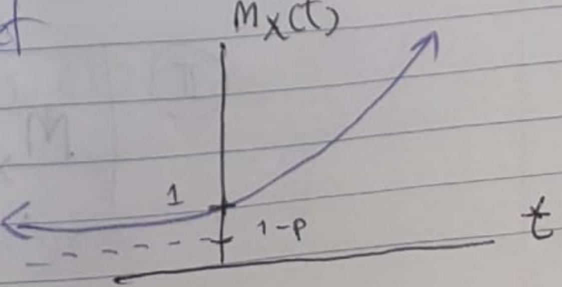
$M_X(t) = E[e^{tX}] = \sum_{x \in (0,1)} e^{tx} p^x (1-p)^{1-x} = \underset{x=0}{1-p} + \underset{x=1}{e^t p}$



put this in $L = B[f]$

$L(t) := \int_{R} e^{-tx} f(x)\,dx$

get



$X \sim \text{Binominal}(n, p)$

$$E[X^{17}] = \sum_{x=0}^{17} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$$

$\uparrow$

17 moment

$$M_X(t) = E[e^{tx}]$$

$$M'_X(t) = \frac{d}{dt}[E[e^{tx}]] = \frac{d}{dt}\left[\int_R e^{tx} f(x)\, dx\right]$$

$$\int_{\mathbb{R}} \frac{d}{dt}[e^{tx} f(x)]\, dx = \int_R x e^{tx} f(x)\, dx$$

$$= E[Xe^{tx}]$$

$$M'_X(0) = E(X]$$

$$M''_X(t) = E\left[\frac{d}{dt}(Xe^{tx}]\right] = E(X^2 e^{tx}]$$

$$M''_X(0) = E(X^2] \ldots$$

Ⅱ $\boxed{M_X^{(k)}(0) = E(X^k)}$ k^{th} moment hi

$Y = aX + c$

$M_y(t) = E[e^{ty}]$ *

$= E[e^{t(aX+c)}]$

$= E[e^{tax} * e^{tc}]$

$= e^{tc} E[e^{tax}]$ if $t' = ta$

$= e^{tc} E[e^{t'x}]$

$= e^{tc} M_x(t')$

$\boxed{\text{III} \quad = e^{tc} M_x(at)}$

X_1, X_2 independent

$Y = X_1 + X_2$

$M_y(t) = E[e^{ty}] = E[e^{t(x_1+x_2)}]$

$= E[e^{tx_1} \cdot e^{tx_2}]$

$= E[e^{tx_1}] E[e^{tx_2}]$

$\boxed{\text{IV} \quad = M_{x_1}(t) \; M_{x_2}(t)}$

if iid

$= (M_x(t))^2$

~~证 证明~~

$X \sim \text{Binom}(n, p)$

$$M_X(t) = \sum_{x=0}^{n} \underbrace{e^{tx}}_{=(e^t)^x} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p^x)(1-p)^{n-x}$$

Look like binomial Thm

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

$$= (1-p+pe^t)^n$$

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(p)$

prove

$T = X_1 + \ldots + X_n \sim \text{Binom}(n, p)$

$M_T(t) = (M_X(t))^n = (1-p+pe^t)^n \Rightarrow T \sim \text{Binom}(n,$

$X \sim Exp(\lambda)$: $\lambda e^{-\lambda x}$

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$= \lambda \left[\frac{1}{t-\lambda} \right] e^{(t-\lambda)x}$$

$$= \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x} \right] \Big|_0^\infty$$

$$= \frac{-\lambda}{t-\lambda}$$

$$= \frac{\lambda}{\lambda - t} \quad \text{if } t < \lambda$$

otherwise doesn't exist.

$X \sim Exp(\cancel{\lambda})$

$Y = aX \quad st. \quad a \in (0, \infty)$

$Y = ?$

$$M_Y(t) = M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{\cancel{a}} - t}$$

III

let $\cancel{\lambda}$ $\lambda' = \frac{\lambda}{a} \Rightarrow \frac{\lambda'}{\lambda' - t}$ I

$Y \sim Exp(\lambda') = Exp\left(\frac{\lambda}{a}\right)$

$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$M_X(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + tx} dx = \cancel{\int \frac{1}{\sqrt{2\pi}} e^{-(tx -}}$$

$(x-t)^2 = x^2 - 2tx + t^2$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx$$

$x - t$

$(x-t)^2 - t^2 = x^2 - 2tx$

$$= \int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left((x-t)^2 - t^2\right)} dx$$

$$= \int_R \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x-t)^2} \cdot e^{\frac{t^2}{2}} \, dx$$

$$= e^{\frac{t^2}{2}} \underbrace{\int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} \, dx}_{\text{PDF for } V \sim N(t,1) = 1} = e^{\frac{t^2}{2}}$$

$E[X] = 0$

$M'_X(0) = t e^{\frac{t^2}{2}} \Big|_0 = 0$ ✓

$Var[X] = E[X^2] - \cancel{\mu}^0 = E[X^2]$

$M''_X(0) = t^2 e^{\frac{t^2}{2}} + e^{\frac{t^2}{2}} \Big|_0 = 1$

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