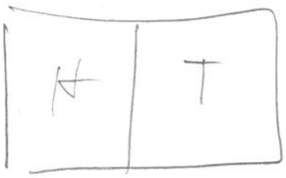
# Lecture 9 9/20/17 Math 271

Midterm 1 ↑

Midterm 2 ↓

$n = 3$ flips

$\Omega = \{H, T\}$




$\omega_1 = H$

$\omega_2 = T$

$\omega_3 = H$

What is the average of the 3 flips? How "randomly spread" are the 3 flips?

You can't perform computations easily on arbitrary sets

What if I created a function e.g.

$$\mathbb{1}_{\omega = H} = \begin{cases} 1 & \text{if } \omega = H \\ 0 & \text{o/t} \end{cases}$$

$n = 3 \quad 1, 1, 0 \Rightarrow \text{Avg } \bar{x} = \frac{1+1+0}{3} = \frac{2}{3}$

What did we do?


experiment

X: outcome → values you care ~~about~~ in your model

difficult to do computations

easy to do computations (functions with a unit like \$1, in, etc)

Generally, $X : \Omega \to \mathbb{R}$

X, is a function called a random variable ("r.v."), $X(\omega)$

$X(H) = 1$

$X(T) = 0$

What is $P(X = 1)$?

technically illegal since $P : 2^{\Omega} \to [0, 1]$

since $X(H)=1$

$$P(\{\omega : X(\omega) = 1\}) = P(\{H\}) = \frac{|\{H\}|}{|\Omega|} = \frac{1}{2} \Rightarrow \begin{array}{l} P(X=1) = \frac{1}{2} \\ P(X=0) = \frac{1}{2} \end{array}$$

we will use the abuse of notation

<u>Support</u>: r.v.'s have a "<u>range</u>" of <u>possible</u> values — the # of values is limited

$\subseteq \mathbb{R}$ $\quad x : P(X=x) > 0$

big X (function)

$$Supp[X] := \{x : P(X=x) > 0\} \subseteq \mathbb{R}$$

little x: an arbitrary value in the range of the function

why not $\geq 0$? Impossible realizations are not interesting

Def: a discrete r.v. is one s.t. $|Supp(X)| \leq |\mathbb{N}|$

Milestone 3 $\quad$ c+ble $\infty$

Assume $|\Omega|$ c+ble. $\Rightarrow \Omega = \{\omega_1, \omega_2, \ldots\}$ s.t. $P(\{\omega_i\}) > 0$

$$P(X=17) = 0$$

$$\sum_{x \in Supp(X)} P(X=x) = 1 \quad \text{Intuitively... why?}$$

SKIP?

$$\bigcup_{x \in Supp[X]} \{\omega : X(\omega) = x\} \stackrel{?}{=} \Omega$$

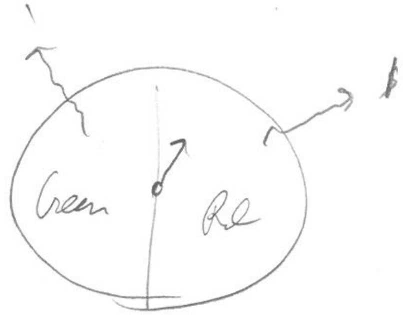
If not, $\exists\, \omega$ s.t. $X(\omega) \notin Supp(X) \Rightarrow P(\{\omega\}) = 0 \Rightarrow$ Coll contr.

$$\{\omega : X(\omega) = x_1\} \cap \{\omega : X(\omega) = x_2\} \stackrel{?}{=} \emptyset$$

Yes b/c $X(\omega_i) = x_1$ and $x_2$ which violates the def. of function must excl.

$$\Rightarrow P(\Omega) = P(\{\omega : X(\omega) = x_1\}) + P(\{\omega : X(\omega) = x_2\}) + \ldots$$

$$1 = \sum_{x \in Supp[X]} P(X=x) \quad \checkmark$$




$$X = \begin{cases} 1 & \text{if } \omega = \text{Red} \\ 0 & \text{if } \omega = \text{Green} \end{cases}$$

$P(X=1) = \frac{1}{2}$

$P(X=0) = \frac{1}{2}$

is $X$ the "same" as previously. Technically, NO since

$X: \{H,T\} \to \{1,0\}$ , $X: \{R,G\} \to \{1,0\}$

But when looking at the values and their prob's <u>ONLY</u>, it doesn't matter

"with probability"

$$\Rightarrow \quad X \sim \begin{cases} 1 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$$

"distributed as"

There are many $\Omega$'s that can produce this r.v.

$\Rightarrow$ we don't care about $\Omega$ anymore. We know it's there, we know there's some underlying experiment and sample space, but we don't need to know what it is

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This r.v. is very special. It is the first of the "brand name" r.v.'s we'll discuss

definition

$$X \sim \text{Bernoulli}\left(\frac{1}{2}\right) := \begin{cases} 1 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$$

$\text{Supp}[X] = \{0,1\}$

More generally,

$X \sim \text{Bernoulli}(p) := \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$ ← Why?

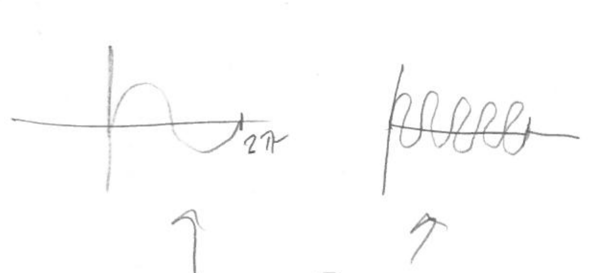
X is distributed Bernoulli w/ "parameter" $p$.

A parameter is a choice which defines a model.

For example

$f(x) = \sin(x)$ is a special case of $f(x) = \sin(ax)$ s.t. $a$ is a constant

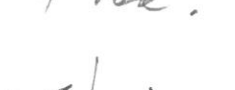
$a$ defines the period/ frequency of the wave


$2\pi$

Same? No, but same family

Valid values of $a$ here? $a \in \mathbb{R}$?

What if $a = 0$?



Is this a sine curve?

Here, we would argue "no" but some say "yes".

This is the trivial case which is uninteresting.

$a \in \mathbb{R} \setminus \{0\}$

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$X \sim \text{Bernoulli}(p)$ Def: Parameter space: Knob you can turn on an oven e.g.

What are the possible values of $p$? $p$ is a prob.

$p \overset{?}{\in} [0,1]$ What if $p = 1$? $X = \{1 \text{ w.p. } 1$ 1, 1, 1, 1, 1, 1

What if $p = 0$? $X = \{0 \text{ w.p. } 0$ 0, 0, 0, 0, 0, 0, 0 if $X \sim Deg(c) := \{ c$ w.p. $1$ $\quad Supp[X] = \{c\}$

the "degenerate" r.v.

X is a r.v. by definition but it is trivial and unnecessary. Why? it just spits out the same value each time.

Just like $f(x) = \sin(0x) = 0$ is unnecessary and $a = 0$ was not included in set of values defining the sine curve family, so too $p = 0$, $p = 1$ are not included in the parameter space

$\Rightarrow p \in (0,1)$

More notation:

defined anywhere

$p(x) := P(X = x) \qquad p : \mathbb{R} \to [0,1]$

1-dim discrete r.v.

prob. mass function (PMF)

if $x \in Supp(X)$

$p(x) > 0$

if $x \notin Supp(X)$

$p(x) = 0$

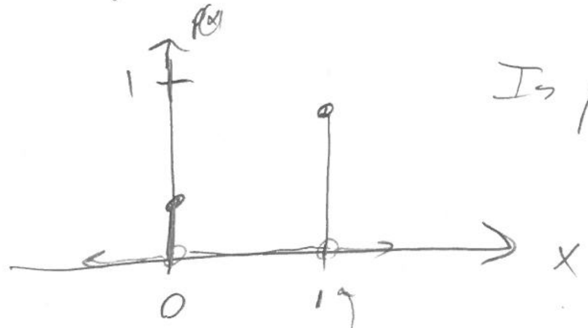
$\sum_{x \in Supp(X)} p(x) = 1$ (Same proof as before)

"Something has to happen"

"prob of anything happening is 1"

It is useful to plot PMF's.

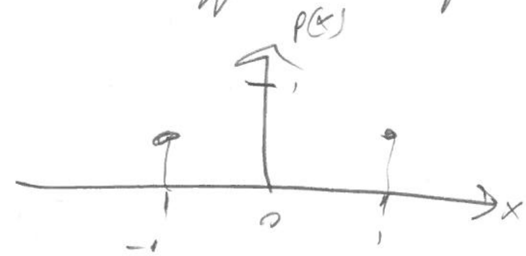
$X \sim \text{Bern}\left(\frac{3}{4}\right)$




Is $p(x)$ a cont. function?
# of discontinuities?
$\text{Supp}[X]$

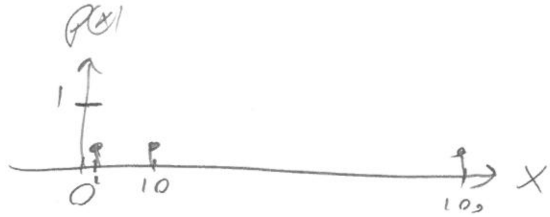
Usually no since it's implicitly understood that values not in the support have prob. 0.

$X \sim \text{Rademacher} := \begin{cases} -1 & \text{wp } \frac{1}{2} \\ 1 & \text{wp } \frac{1}{2} \end{cases}$




Random Walk is one dim.

$X \sim \text{Unif}(\{1, 10, 100\}) = \begin{cases} 1 & \text{wp } \frac{1}{3} \\ 10 & \text{wp } \frac{1}{3} \\ 100 & \text{wp } \frac{1}{3} \end{cases}$




Discrete Uniform; Generally $X \sim \text{Unif}(A)$ $\text{Supp}(X) = A$

Parameter space: $A \subset \mathbb{R}$ s.t. $|A| \in \mathbb{N} \setminus \{1\}$

if $|A| = 1 \Rightarrow$ Deg.!

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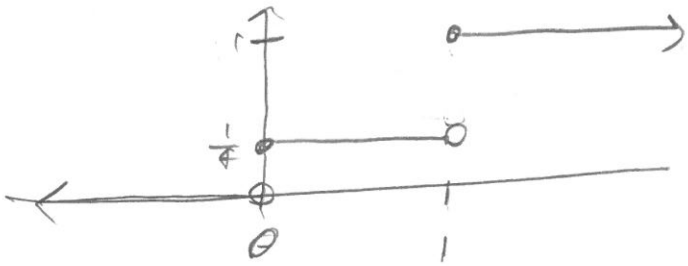
More notation

$F(x) := P(X \leq x)$ Cumulative Distribution Function (CDF)

$X \sim \text{Bernoulli}\left(\frac{3}{4}\right)$



$F(-37) = 0$

$F(1001) = 1$

Properties (derivable from def.)

① $\lim_{x \to \infty} F(x) = 1$

② $\lim_{x \to -\infty} F(x) = 0$

③ $x \le y \Rightarrow F(x) \le F(y)$ "monotonicity"

monotonically increasing $\Rightarrow$ ④ $F(x) \in [0,1]$

$x < y \not\Rightarrow$ necessarily $F(x) < F(y)$

"not strictly" monotonically increasing

$F(x)$ continuous if $X$ is a discrete r.v.?

$\text{Supp}(X) = \{x_1, x_2, \ldots\}$ ordered smallest to largest

$F(x_2) = p(x_1) + p(x_2)$

$\forall \varepsilon > 0 \quad F(x_2 - \varepsilon) = p(x_1) \Rightarrow$ discontinuous

# of discontinuities? $\text{Supp}(X)$

$X \sim \text{Bernoulli}(p) := \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

piecewise function
(ugly, difficult)

$p(x) = p^x(1-p)^{1-x}$ nice!

$X_1 \sim \text{Bernoulli}(p)$ $\quad p_1(x) = p^x(1-p)^{1-x}$ (equal)

$X_2 \sim \text{Bernoulli}(p)$ $\quad p_2(x) = p^x(1-p)^{1-x}$ $=$

**Def:** $X_1 \overset{d}{=} X_2$ if $p_1(x) = p_2(x)$ or $F_1(x) = F_2(x)$

$X_1$ and $X_2$ are "equal in distribution"

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10 cards 4 R, 6 B

Draw cards without replacement

$P(2R \text{ out of } 3 \text{ cards}) = \dfrac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$

$P(x\, R \text{ out of } 3 \text{ cards}) = \dfrac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$

$P(x\, R \text{ out of } n \text{ cards}) = \dfrac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$

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10 cards K R, 10-K B

$P(x\, R \text{ out of } n \text{ cards}) = \dfrac{\binom{K}{x}\binom{10-K}{n-x}}{\binom{10}{n}}$

N cards K R, N-K Blue

$P(\text{"}) = \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$