MATH 241 Fall 2017 Homework #1

Professor Adam Kapelner

Due 5PM outside my office KY604, Wednesday, Sept 13, 2017

(this document last updated Monday 4$^{\text{th}}$ September, 2017 at 10:41pm)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out". Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, read the section about sample spaces in Chapter 2 and relevant parts of Chapter 1 in Ross. Chapter references are from the 7th edition.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 15 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks not on this printout. Keep this first page printed for your records. Write your name and section below (A or B).

NAME: ______________________________ SECTION (A or B): ______

Set Theory Problems below are related to set theory. The sets we talk about in class are composed of outcomes in a universe that are events. Some of the problems below will be about abstract sets that are divorced from the sets used in probability.

Problem 1

These are questions on abstract set theory. Assume capital letters are arbitrary sets and Ω is the universe for all the following questions. Answer as succinctly as possible.

(a) [easy] Answer the following as best as possible.
$A \cup A =$
$A \cap A =$
$A \cap \varnothing =$
$A \cup \Omega =$
$A \cap \Omega =$
$A \cup A^C =$
$A \cap A^C =$
$(A^C)^C =$
$\varnothing^C =$
$\Omega^C =$
$A \backslash A =$
$\Omega \backslash A =$
$A \backslash \Omega =$
$A \backslash \varnothing =$

(b) [easy] Are the following true (T) or false (F) for arbitrary sets $A,\ B,\ C$? The last one is extra credit and requires an explanation for bonus points.
$A \subseteq \Omega$
$A \subset \Omega$
$\varnothing \subseteq A$ and $A \subseteq \Omega$
$A \subseteq A \cup B$
$A \subseteq A \cap B$
$A \in A$

(c) [harder] Are the following true (T) or false (F) for the arbitrary set A?
$A \subseteq A$
$A \subset A$
$\varnothing \subseteq A$
$\varnothing \subset A$
$\varnothing \subseteq \varnothing$
$\varnothing \subset \varnothing$

(d) [harder] Are the following true (T) or false (F)? The symbol "$\Rightarrow$" denotes logical implication *i.e.* if the conditions on the l.h.s are met, the statement on the r.h.s is always true. Commas should be interpreted to mean "and."
$A \subseteq B \Rightarrow A \cap B = A$
$A \subseteq B,\ B \subseteq C \Rightarrow A \subseteq C$

$A \subseteq B, \; B \subseteq C \Rightarrow A \subset C$
$A \subseteq B, \; A \subseteq C \Rightarrow A \subset B \cap C$
$A \subset A \cup B$

(e) [harder] Express $A \cap B$ only in terms of set subtraction (by using the symbol "\").

(f) [easy] If $\{A, B, C\}$ are collectively exhaustive, simplify $A \cup B \cup C$ as best as you can.

(g) [harder] If $\{A, B, C\}$ are collectively exhaustive, simplify $A \cap B \cap C$ as best as you can.

(h) [harder] If $\{A, B, C\}$ are mutually exclusive, simplify $A \cup B \cup C$ as best as you can.

(i) [easy] If $\{A, B, C\}$ are mutually exclusive, simplify $A \cap B \cap C$ as best as you can.

(j) [difficult] In class we played fast and loose with the definitions of set operators. Define $A \cup B$ using set builder notation. You can use the words "or" or "and" or "and / or".

(k) [difficult] Define $A \backslash B$ using set builder notation. You can use the words "or" or "and" or "and / or".

(l) [E.C.] Prove DeMorgan's laws from the ground up.

Problem 2

Consider the sample space Ω where you flip a fair coin and roll a fair die.

(a) [easy] Draw this outcome space in a Venn Diagram. Use a rectangle for Ω.

(b) [easy] What is $|\Omega|$?

(c) [harder] How many unique probability questions could you ask about this random experiment?

(d) [easy] Are singleton sets (sets of size 1) of the outcomes in Ω mutually exclusive? Explain why.

(e) [harder] Using our "working definition" of probability, what is the probability of heads (H) or an even number (E)?

Problem 3

A "full deck of cards" has 52 cards where each card has two characteristics: (1) one of four suits $\spadesuit$, $\heartsuit$, $\clubsuit$ and $\diamondsuit$ and (2) one of 13 ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K and each card is unique. The game Euchre (see `http://en.wikipedia.org/wiki/Euchre` for more information), 24 playing cards are used consisting of only aces, kings, queens, jacks, tens, and nines.

(a) [harder] Construct Ω_E, the event space of a Euchre deck by using set notation and operations on Ω, the event space of a full deck of cards. Use the "..." notation used in class to specify your sets explicitly and use rank and suit such as $4\clubsuit$ to denote the ω's $\in \Omega$. Hint: use the Cartesian product (denoted by $\times$) on two sets.

(b) [difficult] Let B be the set of black cards, F the set of face cards and $\spadesuit$ the set of spades. List all the elements in this set:

$$\left((B \cap F)^C \cup \spadesuit\right)^C \setminus (\{10\spadesuit, 10\diamondsuit, 10\heartsuit\} \cap \Omega)$$

(c) [difficult] Do this problem after completing the last questions since it has to do with counting. Given 5 Euchre cards, how many ways is there to order them?

(d) [difficult] You are dealt five Euchre cards out of the 24 total hands. How many ways is there to order all hands?

Problem 4

We will review the notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ as well as their subsets.

(a) [easy] Draw a number line for x and shade in the area that represents the set $[1, 3] \cup (4, 9]$. If the set includes a number on the endpoint, draw a solid circle "$\bullet$" and if does not include the number, draw an open circle "$\circ$."

(b) [easy] Draw a number line for $Z \subset \mathbb{R}$ where $Z := \{x \in \mathbb{R} : |x| > 2\}$. This Z notation we'll be using in a couple months when we get to the normal distribution. Note: Z is not $\mathbb{Z}$; Z is just an arbitrary letter to denote this set and it could have been any other letter.

(c) [easy] Draw on the number line the set $[0, 1] \cap \left(0, \frac{1}{2}\right) \cap \left[0, \frac{1}{4}\right]$.

(d) [harder] Find the set $A := \bigcup_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$. Hint: draw out the first few expressions and evaluate them to see the pattern.

(e) [difficult] Find the set $B := \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$. Hint: draw out the first few expressions and evaluate them to see the pattern

(f) [easy] Find the set $\mathbb{Z} \backslash \mathbb{N}$.

(g) [harder] Describe the set $\mathbb{R} \backslash \mathbb{Q}$ as best as you can in English and give an example of an element of this set.

(h) [E.C.] Prove $|\mathbb{R}^2| = |\mathbb{R}|$.

Counting Problems below are related to counting. We will review the methods learned in class and expand our horizons.

Problem 5

In this problem, we imagine rolling different sized-dice. Assume the outcomes (each face of each die) are equally likely for that die.

Let R be a standard 6-sided die, let S be an 10-sided die, let T be a 12-sided die, and let U be a 18-sided die. What is the sample size of Ω (*i.e.* $|\Omega|$) for the experiment where we...

(a) [easy] roll R 3 times?

(b) [harder] roll R then S then T then U?

(c) [harder] roll R 34 times, then roll S 45 times, then roll T 12 times, then roll U 76 times.

(d) [harder] Roll R and then roll S only if R rolled greater than or equal to 4. Construct the universe of discourse in this situation by enumerating each outcome of Ω below.

(e) [harder] Would each $\omega \in \Omega$ here be equally likely? If yes, explain why. If no, provide a counterexample.

Problem 6

Examine the following words and tell me how many *permutations* there are of the letters. We do not care about keeping track of the individual common letters. For example, in the word *dad*, there are two $d's$ and we want to treat the permutation d_1d_2a the *same* as d_2d_1a.

(a) [easy] town

(b) [easy] tsktsk (yes, this is a real word!)

(c) [harder] mississippi

(d) [difficult] supercalifragilisticexpialidocious

Problem 7

Below is a standard chessboard. Rows one and eight have the following pieces: two rooks, two knights, two bishops, a king and a queen. Rows two and seven have 8 pawns. Rows one and two have all black pieces and rows seven and eight have all white pieces.

(a) [easy] How many ways are there to place the black queen on a white square?

(b) [harder] How many ways are there to set up the pieces in the back ranks of both white and black *i.e.* arrange the two rooks, two knights, two bishops, king and queen on the first row of 8 squares. Note that this game is called "Fischer Random Chess" after the famous grandmaster Bobby Fischer who proposed the idea to make standard chess more exciting.

(c) [difficult] The game progresses and white takes two black pawns and black takes two white pawns. How many ways are there to arrange the pieces on the board? We don't care about pieces of a type being unique (*i.e.* all white pawns are the same, all black rooks are the same, etc).

(d) [difficult] Are all arrangements "equally likely" during an actual chess game? Explain why or why not.