Math 241 Lec 5 9/13/17

Consider the following pattern

Pascal's Triangle

$1 = \binom{0}{0}$

$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$ ← Outsides are 1's, insides are sums of the #'s above

$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

$1 = \binom{3}{0} \quad 3 = \binom{3}{1} \quad 3 = \binom{3}{2} \quad 1 = \binom{3}{3}$

$1 \quad 4 \quad 6 \quad 4 \quad 1$ ← We've seen this pattern before when we were proving the binomial thm

$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$

$\vdots$

so $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$. This motivates the following recurrence relation

Pascal's
- identity
- rule
- thm
- formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Is this universally true for all $n \in \mathbb{N}_0$, $k \in \{0, \dots, n\}$?

$$\frac{n!}{k!(n-k)!} \stackrel{?}{=} \left(\frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!} + \overset{\text{switch}}{\frac{(n-1)!}{(n-k-1)!\,(k)!}} \right) \cdot \frac{n}{n}$$

$$= \frac{n!}{n}\left(\frac{1}{(n-k)!\,(k-1)!} \cdot \frac{k}{k} + \frac{1}{(n-k-1)!\,k!} \cdot \frac{n-k}{n-k} \right)$$

$$= \frac{n!}{n}\left(\frac{k}{(n-k)!\,k!} + \frac{n-k}{(n-k)!\,k!} \right) = \frac{n!}{(n-k)!\,k!}$$

Consider the set of playing cards, D.

Suits: $S = \{ ♠, ♣, ♡, ♢ \}$

rank $R = \{2, 3, \dots, 10, J, Q, K, A\}$

elements are called "cards"

$|S| = 4$, $|R| = 13$, $D = S \times R$, $|D| = 52$

Consider the situation where you are dealt 5 cards randomly where each card is equally likely ("a hand") order doesn't matter. How many hands are there?

$$\binom{52}{5} = 2,598,960$$

What is the prob. you are dealt a "royal flush" 10 J Q K A same suit.

$$P(\text{Royal Flush}) = \frac{|\text{Royal Flush}|}{|\Omega|} = \frac{4}{\binom{52}{5}}$$

$$P(\text{Str flush}) = \frac{\overset{\text{suit}}{\binom{4}{1}}\overset{\text{starting rank}}{\binom{9}{1}}}{\binom{52}{5}}$$

A2345
23456
34567
.
.
9 10 J Q K

$\Big\}$ 9

~~10 J Q K A~~
royal flush.

$$P(\text{4 of a kind}) = \frac{\overset{\text{rank}}{\binom{13}{1}}\overset{\text{any other card}}{\binom{48}{1}}}{\binom{52}{5}} = \binom{12}{1}\binom{4}{1}$$

7777J

choose rank in 3-of-a-kind

choose 3 of the 4 suits of that rank

choose a remaining rank

choose 2 of the 4 suits of other rank

$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

777JJ

3-of-a-kind 2-of-a-kind

pick suit

$$P(\text{flush}) = \frac{\binom{4}{1}\binom{13}{5} - \binom{4}{1}\binom{9}{1}}{\binom{52}{5}} \quad \Leftarrow \text{\# of str. flushes (from above)} \quad \binom{4}{1} \Leftarrow \text{\# royal flushes}$$

5 ranks w/out replacement

all same suit but NOT a str flush or royal flu

$$P(\text{Straight}) = \frac{\binom{9}{1}\binom{4}{1}^5}{\binom{52}{5}}$$

in order but not a str. flush

pick a rank ↓ ↓ pick 3 suits of 4 → why not $\binom{48}{2}$? Too many! You may get a pair!

$$P(\text{3 of a kind}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

$$P(\text{2 of a kind}) = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} \quad \text{"} \binom{44}{1} \quad \neq \quad \binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

$$\text{Since } \underbrace{\binom{13}{2}}_{\text{order does not matter}} \neq \underbrace{\binom{13}{1}\binom{12}{1} = 13 \cdot 12 = 13\,P_2}_{\text{order matters}}$$

AA 55 Q = 55 AA Q

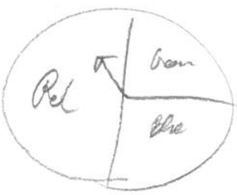
Contrast to
$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \neq \frac{\binom{13}{2}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

Since AAA77 ≠ AA777 saying they're equal ⇒ off by factor of 2.

Return to "working" def of prob. Indeed this is called the "classic def" and it was in use through the 1800's

$$P(A) = \frac{|A|}{|\Omega|}$$

Consider the following random experiment



$P(\{Red\}) \overset{?}{=} \frac{|\{Red\}|}{|\Omega|} = \frac{1}{3}$ No....

$\Rightarrow \Omega = \{R, G, B\}$

What hidden assumption did we employ this whole time?

$\forall \omega \in \Omega, P(\{\omega\}) = \frac{1}{|\Omega|}$ i.e. each outcome equally likely

e.g. flipping coins, rolling dice, selecting people at random, drawing cards randomly

But the spinner above... not so.

Imagine winning the lotto $\Omega = \{yes, no\}$ $P(yes) \overset{?}{=} \frac{1}{2}$ no..

Most non-toy problems do not have this property.

Even $\Omega = \{H, T\} \Rightarrow P(H) = P(T) = \frac{1}{2}$ Are you sure? Prob not each coin is weighted differently. At some decimal $P(H) = 0.500001$

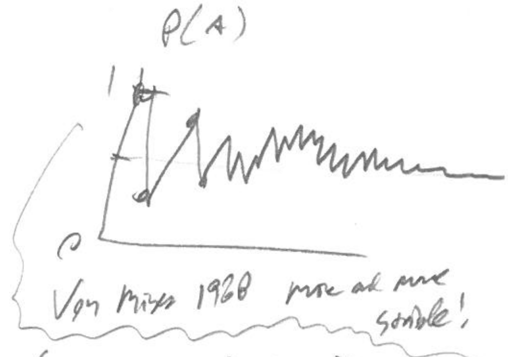
"Equally likely" is frequently an approximation to reality.

How do we define prob?

(I) Limiting Frequency Def.

Define $\mathbb{1}_{\omega \in A} := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

$$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n} = \frac{\#\{\omega \in A\}}{n}$$



John Kerrich was a statistician. Caught by the Nazis thrown in jail. In jail, he flipped a coin 10,000 times and got 5067 heads (published in 1946). $\Rightarrow 5067 \approx .5$

Problems

① $n \neq \infty$ thus we can only approx. prob's by using large n and actually running experiments. This is bad:

(i) we never know "the" prob's values.

(ii) our experiment can fail! we can have incorrect prob. values.

② Some random experiments are one-off.

P(OJ simpson guilty)

P(Irma hits Miami)

P(NK nukes Guam)

impossible to assess acc'd to this def.

Interesting story 1654 a guy named Chevalier de Mere wrote in a letter to Pascal & Fermat that he believed

$P(\{\geq 1$ double 6 in 24 rolls of two dice$\}) < \frac{1}{2}$.

Turns out $= .4914$ we will see this next class. He used the l.r.f. def!