09/05/2017

Lecture 3

Working Definition of Probability

$P : 2^{\Omega} \to [0,1]$

event space — the powerset of the outcome space

"degree" range
where 1 is certainty $P(\Omega)=1$
where 0 is impossibility $P(\emptyset)=0$

$A \subseteq \Omega$ $\quad P(A) = \frac{|A|}{|\Omega|}$
$A \in 2^{\Omega}$

What is the probability of getting a $\underbrace{\text{sum of 3}}_{\text{info about } A}$ on $\underbrace{\text{two die rolls}}_{\Omega}$?

Step 1: Translate from English $\to \Omega$, $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

Step 2: Count $|\Omega| = |\Omega_0||\Omega_0| = 6 \cdot 6 = 36$

Step 3: Translate from English $\to A$ $\quad A = \{\langle 1,2\rangle, \langle 2,1\rangle\}$

Step 4: Compute $|A|$

Step 5: Divide $\quad P(A) = \frac{|A|}{|\Omega|} \quad \frac{|\{\langle 2,1\rangle, \langle 1,2\rangle\}|}{36} = \frac{2}{36}$

What is the probability of getting 2 heads on 4 coin flips?

$\Omega = \{H,T\}^4 \quad |\Omega| = |\{H,T\}|^4 = 2^4 = 16$

$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{16}$

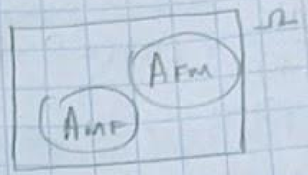
$A = \{\langle H,H,T,T\rangle$
$\langle T,T,H,H\rangle$
$\langle T,H,T,H\rangle$
$\langle H,T,H,T\rangle$
$\langle H,T,T,H\rangle$
$\langle T,H,H,T\rangle\}$

$P(H,H,H,H) \stackrel{?}{=} P(H,H,H,T) \neq P(2H,2T)$

all outcomes are equal

$P(\text{alternating gender}) = \frac{(3!)^2}{6!} \cdot 2 \qquad \frac{\cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 2 \cdot \cancel{1}}{\cancel{6} \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = \frac{2}{20} = \frac{1}{10}$

$P(A) = P(A_{MF}) + P(A_{FM}) \qquad A = A_{MF} \cup A_{FM}$



METHOD #1

MALE $\frac{3}{1}\ \frac{3}{2}\ \frac{2}{3}\ \frac{2}{4}\ \frac{1}{5}\ \frac{1}{6}$ A_{MF}

FEMALE $\frac{3}{1}\ \frac{3}{2}\ \frac{2}{3}\ \frac{2}{4}\ \frac{1}{5}\ \frac{1}{6}$ A_{FM}

METHOD #2 $\frac{6}{1}\ \frac{3}{2}\ \frac{2}{3}\ \frac{2}{4}\ \frac{1}{5}\ \frac{1}{6} \Rightarrow P(A) = \frac{6 \cdot 3 \cdot 2 \cdot 2}{6!} = \frac{\cancel{6} \cdot \cancel{4} \cdot \cancel{3}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot 2 \cdot 1} = \frac{1}{10}$

$P(\text{Richard + Susan sit together}) = \frac{4! \cdot \overset{5!}{5 \cdot 2}}{6!} = \frac{2}{6} = \frac{1}{3}$

$\frac{1}{\text{Loveseat}}\ \ \frac{4}{\text{seat 1}}\ \ \frac{3}{\text{seat 2}}\ \ \frac{2}{\text{seat 3}}\ \ \frac{1}{\text{seat 4}}$

$\frac{4}{\text{seat 1}}\ \ \frac{1}{\text{LS}}\ \ \frac{3}{\text{seat 2}}\ \ \frac{2}{\text{seat 3}}\ \ \frac{1}{\text{seat 4}}$

$\left.\begin{array}{l} \text{\_\_\_ \_ \_ \_ \_} \\ \text{\_ \_\_\_ \_ \_ \_} \\ \text{\_ \_ \_\_\_ \_ \_} \\ \text{\_ \_ \_ \_\_\_ \_} \\ \text{\_ \_ \_ \_ \_\_\_} \end{array}\right\} 5$, $\underbrace{\qquad\qquad}_{4!}$

$n = 100$ balls, sample $K = 3$ without replacement $\Rightarrow {}_{100}P_3 \approx .9702$

with replacement $\Rightarrow 100^3$

$$\lim_{n \to \infty} \frac{{}_nP_K}{n^K} = \lim \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-K+1}{n}$$

$$= \underbrace{\lim \frac{n}{n}}_{1} \cdot \underbrace{\lim \frac{n-1}{n}}_{1} \cdot \underbrace{\lim \frac{n-K+1}{n}}_{1} = 1$$

IF n is large, sampling w/ replacement $\approx$ sampling w/o replacement

Sample n items w/o replacement ($n \in \mathbb{N}$). How many possible outcomes?

\# choices

$\underset{\text{1st sample}}{\underline{n}} \quad \underset{\text{2nd sample}}{\underline{n-1}} \quad \cdots \quad \underset{\text{n-1 sample}}{\underline{2}} \quad \underset{\text{nth sample}}{\underline{1}} = \boxed{n! = \prod_{i=1}^{n} i}$

Sample n items w/ replacement. How many outcomes?

$\underset{\text{1st sample}}{\underline{n}} \quad \underset{\text{2nd sample}}{\underline{n}} \quad \cdots \quad \underline{n} \quad \underline{n} = n^n > n!$ for $n \geq 2$

5 people, 3 chairs, how many seating arrangements?

$\underset{\text{1st chair}}{\underline{5}} \quad \underset{\text{2nd chair}}{\underline{4}} \quad \underset{\text{3rd chair}}{\underline{3}} = \frac{5!}{2!}$

Sample n items K times w/o replacement. How many?

$\underset{\text{1st sample}}{\underline{n}} \quad \underset{\text{2nd sample}}{\underline{n-1}} \quad \cdots \quad \underset{\text{Kth sample}}{\underline{n-K+1}} = \frac{n!}{(n-K)!}$

Permutations ⇒ unique ordering

$_nP_K = \frac{n!}{(n-K)!}$

NOTE: convention dictates $0! = 1$

$_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1}$

3 couples (6 people): Bob-Jane, Richard-Susan, Charles-Mary

$P(\text{that every couple sits together}) = \frac{|A|}{|\Omega|} = \frac{6 \cdot 4 \cdot 2}{6!} = \frac{6 \cdot 4 \cdot 2}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{15}$

$|\Omega|$ ⇒ all orders of 6 people

$\underset{\text{Seat 1}}{\underline{6}} \quad \underset{2}{\underline{1}} \quad \underset{3}{\underline{4}} \quad \underset{4}{\underline{1}} \quad \underset{5}{\underline{2}} \quad \underset{6}{\underline{1}}$

Alternative Method

$P(A) = \frac{|A|}{|\Omega|} = \frac{3! \, 2^3}{6!}$

$\underset{\text{Loveseat 1}}{\underline{3}} \quad \underset{2}{\underline{2}} \quad \underset{3}{\underline{1}}$

2 2 2

Switch possible

Probability of at least 1 H on 4 tosses?

$$P(A) = \frac{|A|}{|\Omega|} = \frac{15}{16}$$

$A = \{HTTT, HHHH, \dots\} = 15$

Recall $|\Omega| = |A| + |A^c| \Rightarrow |A| = |\Omega| - |A^c| = 16 - 1 = 15$

$A^c = \{\text{not at least } 1H\} = \{<1H\} = \{T,T,T,T\}$

$\Omega \geq 1H$

$|A^c| = 1 \Rightarrow P(A^c) = \frac{1}{16}$

Complement Rule

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - P(A^c)$$

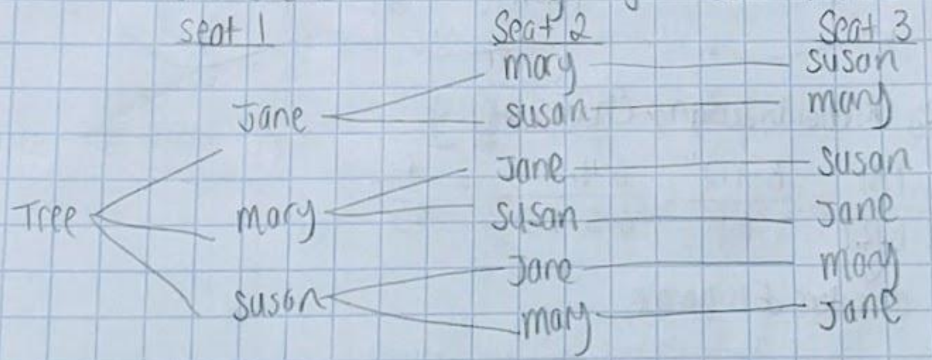
Flip 10 coins

$$P(A) = \frac{|A|}{|\Omega|} = \frac{}{2^{10}} = \frac{}{1024}$$

HUGE, will come back to this

$F = \{\text{Jane, Mary, Susan}\}$

There are 3 chairs. How many ways to seat those 3 women



Total # of ways

$$\underset{\text{Seat \#1}}{3} \cdot \underset{\text{Seat \#2}}{2} \cdot \underset{\text{Seat \#1}}{1} = 6$$

$\Omega = \{\langle J,M,S\rangle, \langle J,S,M\rangle, \langle M,J,S\rangle, \langle M,S,J\rangle, \langle S,M,J\rangle, \langle S,J,M\rangle\}$ $|\Omega| = 6$

NOTE = $\Omega \subseteq F^3$ $|F^3| = 27$

NOTE $\Omega \neq F^3$

no repetition (Ω) — sampling 3 times "without replacement" "don't put back in jar"

repetition (F^3) — Sampling 3 times w/ replacement "putting back in jar"