$$\boxed{11/22}$$

Review...

$$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad Supp(Z) = \mathbb{R}$$

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \qquad Supp(X) = \mathbb{R}$$

$$\Downarrow$$

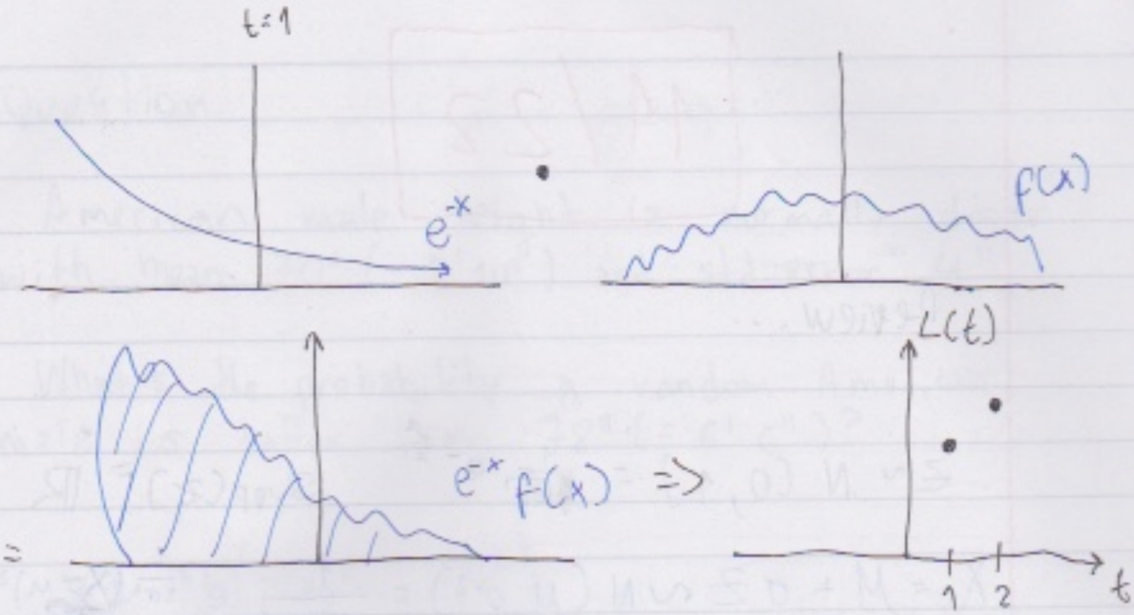
$$Z = \frac{X-\mu}{\sigma}$$

z-score

$$P(Z \in [-1,1]) \approx 68\%$$
$$P(Z \in [-2,2]) \approx 95\%$$
$$P(Z \in [-3,3]) \approx 99.7\%$$

$$L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\, dx$$

Bilateral Laplace Transform



Theorem: $L(t)$ and $f(x)$ are 1:1 if $L(t)$ exists.

1:1 (one to one). For every x there's just one y.

For every function f there's a function of L
It is a one to one Transform

Define: The moment generating function (mgf)

for continuous $\Rightarrow$ $$M_X(t) := L(-t) = \int_{\mathbb{R}} e^{tx} f(x)\, dx = E[e^{tX}]$$

for discrete $\Rightarrow$ $$M_X(t) := E[e^{tX}] = \sum_{x \in \mathrm{Supp}(X)} e^{tx} p(x)$$

. $M \rightarrow$ stands for Moment generating function

(I) If $M_x(t) = M_y(t)$

$\Rightarrow X \stackrel{d}{=} Y$

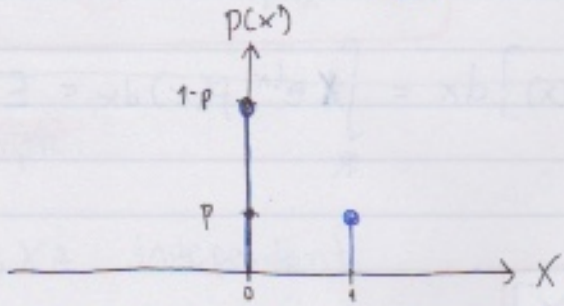
$\Rightarrow f_x(x) = f_y(x)$ for continuous

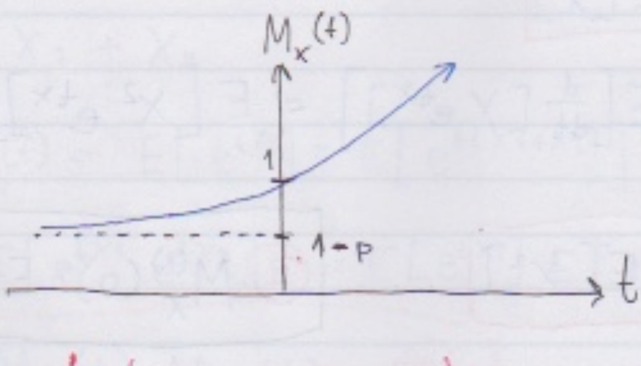
$\Rightarrow p_x(x) = p_y(x)$ for discrete

example:

$X \sim Bern(p) := p^x (1-p)^{1-x}$

$M_x(t) = E[e^{tX}] = \sum_{x \in \{0,1\}} e^{tx} p^x (1-p)^{1-x} = 1 - p + e^t p$





these two are the same in some way

t is a convention
It doesn't stand for time

let's talk about moments!

$X \sim \text{Binomial}(n, p)$

$$E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$$

17^{th} moment

$$M_X(t) = E[e^{tX}]$$

$$M'_X(t) = \frac{d}{dt}\left[E[e^{tx}]\right] = \frac{d}{dt}\left[\int_{\mathbb{R}} e^{tx} f(x)\,dx\right]$$

$$= \int_{\mathbb{R}} \frac{d}{dt}\left[e^{tx} f(x)\right] dx = \int_{\mathbb{R}} x\, e^{tx} f(x)\,dx = E[xe^{tX}]$$

$$\boxed{M'_X(p) = E[X]}$$

$$M''_X(t) = E\left[\frac{d}{dt}[Xe^{tX}]\right] = E\left[X^2 e^{tx}\right]$$

$$\boxed{M''_X(p) = E[X^2]} \;\ldots\; \boxed{\text{II} \quad M_X^{(k)}(0) = E[X^k] \;\; k^{th} \text{ moment}}$$

again... II $M_X^{(k)}(0) = E[X^k]$ k^{th} moment

$Y = aX + c$

$M_Y(t) = E[e^{tY}] = E[e^{t(aX+c)}] = E[e^{taX} e^{tc}]$

$= e^{tc} E[e^{taX}]$ if $t' = ta$

$= e^{tc} E[e^{t'X}]$

$= e^{tc} M_X(t') = \boxed{e^{tc} M_X(at)}$ III

again... III $e^{tc} M_X(at)$

~~Example~~:

X_1, X_2 independent

$Y = X_1 + X_2$

$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}]$

$= E[e^{tX_1} e^{tX_2}] = E[e^{tX_1}] E[e^{tX_2}]$

$= M_{X_1}(t) M_{X_2}(t)$ IV

if iid $\Rightarrow$ $(M_X(t))^2$

again.. IV $M_{X_1}(t)\, M_{X_2}(t)$

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example 6:

$X \sim \text{Binomial}(n, p)$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} \qquad (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$= (1-p+pe^t)^n$$

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example

$X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p)$

prove:

$T = X_1 + \dots + X_n \sim \text{Binom}(n, p)$

$$M_T(t) = \underbrace{(M_X(t))^n}_{\text{IV}} = \underbrace{(1-p+pe^t)^n}_{\text{from Bernoulli}} \underbrace{\Rightarrow}_{\text{I}} T \sim \text{Binomial}(n, p)$$

example:

$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

$$M_x(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty e^{(t-\lambda)x}\, dx$$

$$= \lambda \left[\frac{1}{t-\lambda} e^{(t-\lambda)x} \right]_0^\infty = \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x} \right]_0^\infty$$

$$= -\frac{\lambda}{t-\lambda} \mathbb{1}_{t-\lambda<0} \quad \text{or} \quad = \begin{cases} \frac{\lambda}{\lambda - t} & \text{if } t < \lambda \\ \text{d.n.e.} & \text{otherwise (o/t)} \end{cases}$$

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example:

$X \sim Exp(\lambda)$

$Y = aX$ s.t. $a \in (0, \infty)$

$\boxed{\text{let } \lambda' = \frac{\lambda}{a}}$

$Y \sim ?$

$$M_Y(t) \underset{\text{III}}{=} M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t}$$

I $\Rightarrow Y \sim Exp(\lambda') = Exp\left(\frac{\lambda}{a}\right)$

$$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + tx} dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}((x-t)^2 - t^2)} dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}} dx = e^{\frac{t^2}{2}} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx}_{\substack{\text{PDF for } V \sim N(t,1) \\ 1}} = e^{\frac{t^2}{2}}$$

solution

want To show

WTS $E[X] = 0$

$$M_X'(0) = t e^{t^2/2} \Big|_0 = 0 \quad \checkmark$$

WTS $Var[X] = 0$ $\quad Var[X] = E[X^2] - \mu_2 = E[X^2]$

$$M_X''(0) = t^2 e^{t^2/2} + e^{t^2/2} \Big|_0 = 1$$