Lecture 20

11/27/17

$M_X(t) = E(e^{tX})$

m.g.f.

$M_X(0) = 1$

Properties

(I) $M_X(t) = M_Y(t) \iff X \stackrel{d}{=} Y$

(II) $M_X^{(k)}(0) = E(X^k)$

(III) $Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_X(at)$

(IV) X_1, X_2 independent $T = X_1 + X_2 \Rightarrow M_T(t) = M_{X_1}(t) M_{X_2}(t)$

$X \sim \text{Bern}(p) \Rightarrow M_X(t) = 1 - p + pe^t$

$X \sim \text{Binomial}(n,p) \Rightarrow M_X(t) = (1 - p + pe^t)^n$

$X \sim \text{Exponential}(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$

$Z \sim N(0,1) \Rightarrow M_Z(t) = e^{\frac{t^2}{2}}$

$X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

$\Rightarrow M_X(t) = e^{t\mu} M_Z(\sigma t)$

$= e^{t\mu} e^{\sigma^2 t^2/2}$

$= e^{t\mu + \sigma^2 t^2/2}$

$X \sim \text{Deg}(c)$, $M_X(t) = E(e^{tX}) = e^{tc}$

$X_1 \sim N(\mu_1, \sigma_1^2)$

independent of

$X_2 \sim N(\mu_2, \sigma_2^2)$

$X_1 + X_2 \sim$ (IV)

$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \left(e^{t\mu_1 + \frac{\sigma_1^2 t^2}{2}}\right)\left(e^{t\mu_2 + \frac{\sigma_2^2 t^2}{2}}\right)$

$= e^{t(\mu_1 + \mu_2) + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}} \Rightarrow X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

(V) Levy's Continuity Theorem

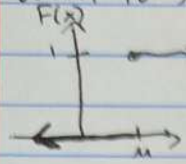
$X_1, X_2, \ldots$ sequence of r.v.s

$\lim_{n\to\infty} M_{X_n}(t) = M_Y(t) \Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_Y(x)$ or $X_n \xrightarrow{d} Y$

$\bar{X} \to \mu = E(X)$ – Law of Large Numbers

μ is a constant, ie. $Deg(\mu)$

Want to show $\lim_{n\to\infty} M_{\bar{X}}(t) = e^{t\mu} \Rightarrow \bar{X} \xrightarrow{d} \mu$



$\longrightarrow \lim_{n\to\infty} F_{\bar{X}_n}(x) = F_\mu(x)$

Assume $X_1, X_2, \ldots \overset{iid}{\sim}$ with mean μ

$\lim_{n\to\infty} M_{\bar{X}_n}(t) = \lim_{n\to\infty} M_{\frac{X_1+\ldots+X_n}{n}}(t) = \lim_{n\to\infty} M_{X_1+\ldots+X_n}\left(\frac{t}{n}\right) = \lim_{n\to\infty}\left(M_X\left(\frac{t}{n}\right)\right)^n$

$= \lim_{n\to\infty} e^{\ln((M_X(\frac{t}{n}))^n)} = \lim_{n\to\infty} e^{n\ln(M_X(\frac{t}{n}))} = e^{\lim_{n\to\infty} n\ln(M_X(\frac{t}{n}))} = e^{\lim_{n\to\infty}\frac{\ln(M_X(\frac{t}{n}))}{1/n}}$

Let $\mu = \frac{1}{n}$, $n\to\infty \Rightarrow \mu \to 0$ $= e^{\lim_{\mu\to 0}\frac{\ln(M_X(\mu t))}{\mu}} = e^{\lim_{\mu\to 0}\frac{tM_X'(\mu t)}{M_X(\mu t)}} = e^{tM_X'(0)/M_X(0)} = e^{t\mu} \Rightarrow \bar{X} \to \mu$

Assume $X_1, X_2, \ldots \overset{iid}{\sim}$ with mean μ, variance σ^2

Consider $C_n := \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$

$E(C_n) = 0$

$SE(C_n) = 1$

$\lim_{n\to\infty} M_{C_n}(t) = \lim_{n\to\infty} M_{\frac{V_1+\ldots+V_n}{\sqrt{n}}}(t) = \lim_{n\to\infty} M_{V_1+\ldots+V_n}\left(\frac{t}{\sqrt{n}}\right) = \lim_{n\to\infty}\left(M_V\left(\frac{t}{\sqrt{n}}\right)\right)^n$

$= \lim_{n\to\infty} e^{\ln((M_V(\frac{t}{\sqrt{n}}))^n)} = \lim_{n\to\infty} e^{n\ln(M_V(\frac{t}{\sqrt{n}}))} = e^{\lim_{n\to\infty} n\ln(M_V(\frac{t}{\sqrt{n}}))} = e^{\lim_{n\to\infty}\frac{\ln(M_V(\frac{t}{\sqrt{n}}))}{1/n}\cdot\frac{t^2}{t^2}}$

Let $u = \frac{1}{\sqrt{n}}$, $n\to\infty \Rightarrow u \to 0$ $= e^{t^2\lim_{n\to\infty}\frac{\ln(M_V(\frac{t}{\sqrt{n}}))}{t^2/n}} = e^{t^2\lim_{u\to 0}\frac{\ln(M_V(ut))}{u^2t^2}} = e^{t^2\lim_{u\to 0}\frac{\frac{tM_V'(ut)}{M_V(ut)}}{2ut^2}} = e^{\frac{t^2}{2}\lim_{u\to 0}\frac{M_V(ut)\,t(M_V''(ut)) - t(M_V'(ut))^2}{M_V(ut)^2}}$

$= e^{\frac{t^2}{2}\frac{M_V(0)M_V''(0) - M_V'(0)^2}{M_V(0)^2}} = e^{t^2/2}$

$\Rightarrow C_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$ – Central Limit Theorem

<u>How to use CLT</u>

n is never infinite, but if n is large, then

CLT1 ① $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \overset{d}{\approx} N(0,1)$

CLT2 ② $\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

CLT3 ③ $T \overset{d}{\approx} N\left(n\mu, (\sigma\sqrt{n})^2\right)$

$X_1, X_2, \ldots, X_{30} \overset{iid}{\sim} (Geom(\tfrac{1}{2}))$ – $\mu = \frac{1}{1/2} = 2$

What is $P(\bar{x} \geq 2.75)$? $\quad \sigma = \frac{\sqrt{1-p}}{p} = \frac{\sqrt{1-\frac{1}{2}}}{1/2} = \frac{\sqrt{\frac{1}{2}}}{1/2} = \frac{1}{\sqrt{1/2}} = \sqrt{2} \approx 1.414$

$\bar{x} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

$P\left(\frac{\bar{x}-2}{.258} \geq \frac{2.75-2}{.258}\right) = P(Z \geq 3) \approx .0015$