MAT 241 - LECTURE 16 - Nov 6, 2017

$$\boxed{\begin{aligned} Var[X] &= E[(X-\mu)^2] \quad \text{Mean or Expectation} \\ E[X^2] &= \sigma^2 + \mu^2 \\ Var[aX+c] &= a^2\sigma^2 \\ SE[aX+c] &= |a|\sigma \end{aligned}}$$

$$T_2 = X_1 + X_2$$

Recall

$$E[T_2] = E[X_1] + E[X_2]$$

is a constant

$$\boxed{E[\mu_1^2] = \mu_1^2}$$

ex $E[17] = 17$

$(X_1 + X_2 - \mu_1 - \mu_2)(X_1 + X_2 - \mu_1 - \mu_2)$

$$Var[T_2] = E[((X_1+X_2) - (\mu_1+\mu_2))^2]$$

$$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2 + 2\mu_1\mu_2]$$

$$= E[X_1^2] + E[X_2^2] + \mu_1^2 + \mu_2^2 + 2E[X_1X_2] - 2\mu_1^2 - 2\mu_1\mu_2 - 2\mu_2\mu_1 - 2\mu_2^2 + 2\mu_1\mu_2$$

$$= \sigma_1^2 + \sigma_2^2 + 2(\underbrace{E[X_1X_2] - \mu_1\mu_2}_{\text{Covariance } Cov[X_1,X_2]}) = \cancel{\sigma_1^2 + \sigma_2^2 + 2Cov[X_1,X_2]}$$

we can't get anything better than this, so we assume X_1, X_2 are independent to simplify more.

Assume X_1, X_2 are independent

$$E[X_1X_2] = \sum_{x_1}\sum_{x_2} x_1 x_2\, p(x_1)p(x_2)$$

$$= \underbrace{\sum_{x_1} x_1\, p(x_1)}_{\mu_1}\underbrace{\sum_{x_2} x_2\, p(x_2)}_{\mu_2} = \mu_1\mu_2 \Rightarrow Cov[X_1, X_2]$$

continue after assumption

$$= \sigma_1^2 + \sigma_2^2 + 2(E[X_1X_2] - \mu_1\mu_2) = \sigma_1^2 + \sigma_2^2 + 2(\mu_1\mu_2 - \mu_1\mu_2)$$

if X_1, X_2 are independent

$$\boxed{Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2}$$

if X_1, X_2 are independent

No covariance calculation question on Exam

Can you add variances if they are not independent? *NO*

If $X_1, \dots, X_n$ are independent

$$\mathrm{Var}[X_1 + \dots + X_n] \underset{\text{if independent}}{=} \sum_{i=1}^{n} \mathrm{Var}[X_i] \underset{\text{if iid}}{=} n\sigma^2$$

If $X_1, \dots, X_n$ are independent and identically distributive

$$\sum_{i=1}^{n} \mathrm{Var}[X_i] = \mathrm{Var}[X_1 + \dots + X_n] \underset{\text{if iid}}{=} n\sigma^2$$

$\bar{X}_n = X_1 + \dots + X_n$

$$\mathrm{var}[\bar{X}_n] = \mathrm{Var}[\tfrac{1}{n} T_n] = \frac{1}{n^2}\mathrm{Var}[T_n] \overset{\text{if independent}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}[X_i] \overset{\text{if iid}}{=} \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

if $\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$ are independent

$$\mathrm{Var}[\bar{X}_n] = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{var}[X_i]$$

if $\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$ are iid

$$\mathrm{var}[\bar{X}_n] = \frac{\sigma^2}{n}$$

Note $\mathrm{Var}[\bar{X}_n]$ goes to 0 as $n \to \infty$

means $\bar{X} \to \mu$

Law of large #'s under iid.

$$E[\bar{X}_n] = E[\tfrac{1}{n}(X_1 + \dots + X_n)] = \tfrac{1}{n} E[\Sigma X_i] = \frac{1}{n} n\mu = \mu. \Rightarrow SE(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow SE(\bar{X}_n) = \frac{\sigma}{\sqrt{n}} \underset{n\to\infty}{\to} 0$$

$X \sim \mathrm{Binom}(n, p)$ $\qquad X = X_1 + \dots + X_n$ s.t. $X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Bern}(p)$

$\boxed{E[X] = np}$

$$\mathrm{Var}[X] = \sum_{x=0}^{n} (x - np)^2 \binom{n}{x} p^x (1-p)^{n-x} \quad \text{(Hard)}$$

"the binomial is the sum of Bernoulli trials"

$$\mathrm{Var}[X] = \Sigma\, \mathrm{Var}[X_i] = n\sigma^2 = np(1-p) \Rightarrow SE[X] = \sqrt{np(1-p)}$$

$$\boxed{\begin{aligned} \mathrm{Var}[X] &= np(1-p) \\ SE[X] &= \sqrt{np(1-p)} \end{aligned}}$$

$\leftarrow$ for $X \sim \mathrm{Bin}(n, p)$

$X \sim \text{Geometric} = \underbrace{(1-p)^{x-1} p}_{\text{PMF}}$

$E[X] = \frac{1}{p} = \mu$

$Var[X] = E[X^2] - \mu^2 = E[X^2] - \frac{1}{p^2}$

$\mu^2 = \frac{1}{p^2}$

$\mu = \frac{1}{p}$ of Geometric

$E[X^2] = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} p$

* have to start at 1, need to get a success, if you do 0 trials, it won't be possible to have a success. *

Let $y = x-1 \Rightarrow x = y+1$

$E[X^2] = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} p = \sum_{y=0}^{\infty} (y+1)^2 (1-p)^y p$

$= \sum_{y=0}^{\infty} y^2 (1-p)^y + p + 2\sum_{y=0}^{\infty} y(1-p)^y p + p\sum_{y=0}^{\infty} (1-p)^y$

$= (1-p)\underbrace{\sum_{y=1}^{\infty} y^2 (1-p)^{y-1} p}_{E[X^2]} + 2(1-p)\underbrace{\sum_{y=1}^{\infty} y(1-p)^{y-1} p}_{\mu = \frac{1}{p}} + \underbrace{p\sum_{y=0}^{\infty} (1-p)^y}_{p \cdot \frac{1}{p} = 1}$

$\sum_{x=0}^{\infty} a^x = \frac{1}{1-a}$ if $a \in (0,1)$

$E[X^2] = (1-p)E[X^2] + \frac{2(1-p)}{p} + \frac{p}{p}$

$E[X^2] - (1-p)E[X^2] = \frac{2(1-p)+p}{p}$

$E[X^2]p = \frac{2(1-p)+p}{p}$

$E[X^2] = \frac{2(1-p)+p}{p^2} = \frac{2-2p+p}{p^2} = \frac{2-p}{p^2}$

Variance of Geometric

$Var[X] = \frac{1-p}{p^2}$

$Var[X] = E[X^2] - \frac{1}{p^2}$

$= \frac{2-p}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$

Variance of Hyper

$X \sim \text{Hyper}(n, K, N)$

$$\text{var}[X] = \sum_{k \in \text{supp}[X]} \left(x - n\frac{K}{N}\right)^2 = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

very hard, not ~~cover~~ covered in class.

$X \sim \text{Geometric}(p)$ = "stopping time"

Geometric is memoryless, it doesn't remember where it's been

$p(X=7) = (1-p)^6 p$

$p(X=17) = (1-p)^{17} p$

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

$p(X = 17 \mid X > 10) =$

probability that $X = 17$ given $X > 10$

restarting geometric, we are able to do this because they are iid Bern(p)

$$\star\ p(X=17 \mid X>10) = p(X=7) = (1-p)^6 p\ \star$$

$\star$ Want to show that $P(X=a) = P(X = a+b \mid X > b)$ $\star$

Example first

~~$P(X=a+b \mid X>b) =$~~

$$P(X=17 \mid X>10) = \frac{P(X=17 \cap X>10)}{P(X>10)} = \frac{P(X=17)}{P(X>10)} = \frac{(1-p)^{16} p}{(1-p)^{10}} = (1-p)^6 p$$

$P(X>10) = 1 - F(X \le 10) = 1 - (1 - (1-p)^{10}) = (1-p)^{10}$

General proof

$$p(X = a+b \mid X > b) = \frac{P(X = a+b \cap X > b)}{P(X>b)} = \frac{P(X=a+b)}{P(X>b)} = \frac{(1-p)^{a+b-1}}{(1-p)^b}$$

$$= (1-p)^{a-1} = \boxed{P(X=a)}$$

Memoryless property of the Geometric

$X \sim Geo(p)$ represents stopping time. $\boxed{n = 10}$

n n n n n

$\to$ time.

1 2 3 4

$F(x) = 1 - (1-p)^t \Rightarrow$ CDF

$P(t) = (1-p)^{t-1} p$ within each time period, we run n Bern(p) $\overset{iid}{\sim}$ experiments.

$p(t) = (1-p)^{nt-1} p$ "squeezing more experiments in each time period."
"Run iid Bern(p) at every $\frac{1}{n}$ time period."

Jam infinite experiments in every time period. $\boxed{\text{END OF Midterm 2}}$

let $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ $\leftarrow$ "means we are making n really big"

$$\lim_{n\to\infty} p(t) = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nt-1} \left(\frac{\lambda}{n}\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nt-1} \underbrace{\lim_{n\to\infty} \frac{\lambda}{n}}_{0} = 0$$

limiting PMF $p(t) = 0$

$\boxed{X \sim Geo(p) \quad \text{CDF} \quad F(x) = 1 - (1-p)^{\lambda}}$

$$\sum_{t \in supp[t]} p(t) = 0$$

$p(t)$ is not a PMF
~~A~~ problem not a DISCRETE ~~PMF~~

limiting CDF

$$\lim_{n\to\infty} F(t) = \lim_{n\to\infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{nt} = 1 - \underbrace{\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nt}}_{e^{-\lambda}} = 1$$

$$= 1 - e^{-\lambda t}$$

$$= \boxed{1 - \frac{1}{e^{\lambda t}} = F(t)} \Leftarrow \text{valid CDF}$$

Recall

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

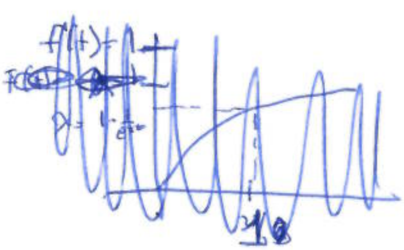
$$e^x = \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right)^x$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nx}$$

Let $c = nx \Rightarrow n = \frac{c}{x}$

$$\lim_{c\to\infty} \left(1 + \frac{x}{c}\right)^c = e^x$$

Is this CDF valid



$$\lim_{t\to\infty} F(t) = 1$$

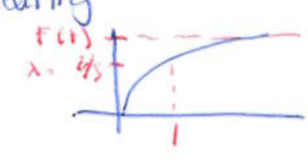
$$\lim_{t\to\infty} F(t) = F(0) = 0$$

because supports begins at 0

$F'(t) = \lambda e^{-\lambda t} \geq 0$ "monotonically increasing

$\lambda = np$ $(+)$ $(+)$ $(+)$

$\boxed{\text{No support}}$



-