Lecture 10 Math 241 Oct 16^{th}

10 cards, 4R 6B

$$P\begin{pmatrix}\text{drawing} \\ \text{2R 3 cards} \\ \text{w/o replacement}\end{pmatrix} = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} \qquad P(xR \text{ in } 3) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$$

$$P(xR \text{ in } n \text{ cards}) = \frac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$$

10 cards, kR $\qquad$ N card, kR

$$P(xR \text{ in } n) = \frac{\binom{k}{x}\binom{10-k}{n-x}}{\binom{10}{n}} \qquad P(xR \text{ in } n) = \frac{\binom{k}{x}\binom{n-k}{n-x}}{\binom{N}{n}}$$

$$x \sim \text{hypegeometric}(n,k,N) := \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad p(x) := P(X = x)$$

the 1 param

Support $[x]$ $\qquad x \sim \text{Bein}(p) := \underbrace{p^x(1-p)^{1-x}}_{p(x)}$

$\text{supp}[x] = \{0,1\}$

$p \in (0,1)$

100 students, 53 female, pick 8 random (N, k, R)

what is the probability that 6 of them are female?

$P(6f \text{ in } 8)$

$$x \sim (\overset{n}{8}, \overset{k}{53}, \overset{N}{100}) = \frac{\binom{53}{x}\binom{100-53}{8-x}}{\binom{100}{8}} \qquad P(6 = x) = \frac{\binom{53}{6}\binom{47}{8-2x}}{\binom{100}{8}}$$

$\left\{ \begin{array}{l} X = \text{random variable} \\ x = \text{free variable} \end{array} \right.$

what if $N = 0$? It would be degenerate.

$\Rightarrow n = 0$

$\Rightarrow k = 0$

what if $N = 1$?

$\Rightarrow k \in \{0,1\}$

$\Rightarrow n = 1$

what if $N = 2$

$\Rightarrow k \in \{0,1,2\}$

$\Rightarrow n =$

$x \sim \text{hype}(1,1,2) = \dfrac{\binom{1}{x}\binom{1}{1-x}}{\binom{2}{1}}$ bag with 2 balls, taking 1 out, asking P(special)

$P(1) = \dfrac{\binom{1}{1}\binom{1}{0}}{\binom{2}{1}} = \dfrac{1}{2}$ $\quad P(0) = \dfrac{\binom{1}{0}\binom{1}{1}}{\binom{2}{1}} = \dfrac{1}{2}$

$x \sim \text{hyper}(1, k, N) = \dfrac{\binom{k}{x}\binom{n-k}{1-x}}{\binom{N}{1}}$ $\quad P(1) = \dfrac{\binom{k}{1}\binom{N-k}{0}}{\binom{N}{1}} = \dfrac{k}{N}$

$\text{supp}[x] = \{0, 1\}$

$= \text{Bern}\left(\frac{k}{N}\right)$ $\quad P(0) = \dfrac{\binom{k}{0}\binom{N-k}{1}}{\binom{N}{1}} = \dfrac{N-k}{N} = 1 - \dfrac{k}{N}$

Parm Space Hyper

$N \in \{2, 3, \dots\}$

$k \in \{1, 2, \dots, N-1\}$

$n \in \{1, 2, \dots, N-1\}$

(a) $x \sim \text{Hyper}(\overset{n}{2}, \overset{k}{4}, \overset{N}{10})$, $\text{Supp}[x] = \{0, 1, 2\}$

(10 balls, 4 special, (2) chosen, are special?)

(b) $x \sim \text{Hyper}(5, 4, 10)$, $\text{supp}[x] = \{0, 1, 2, 3, 4\}$

10 balls, 4 special, pick 5, P(of the 5 being special)?

(c) $x \sim \text{Hyper}(8, 4, 10)$, $\text{Supp}[x] = \{2, 3, 4\}$ $\quad$ $\begin{array}{r}10\\-8\\\hline 2\end{array}$ $\begin{array}{r}10\\-4\\\hline 6\end{array}$

$N-k = 6$

(d) $x \sim \text{Hyper}(5, 7, 10)$, $\text{supp}[x] = \{2, 3, 4, 5\}$ $\quad$ $\begin{array}{r}10\\-5\\\hline 5\end{array}$

(a) $n < k,\ n < N-k$ $\quad \text{supp}[x] = \{0, \dots, n\}$

(b) $n \geq k,\ n < N-k$ $\quad \text{supp}[x] = \{0, \dots, k\}$

(c) $n \geq k,\ n \geq N-k$ $\quad \text{supp}[x] = \{n-(N-k), \dots, k\}$

(d) $n < k,\ n \geq N-k$ $\quad \text{supp}[x] = \{n-(N-k), \dots, n\}$

	$n < k$	$n \geq k$
$n < N-k$	$0 \dots n$	$0 \dots k$
$n \geq N-k$	$n-(N-k) \dots n$	$n-(N-k) \dots k$

$\text{supp}[x] = \{\max\{0, n-(N-k)\}, \dots, \min\{n, k\}\}$

$\displaystyle\sum_{x \in \text{supp}[x]} p(x) = 1$ $\qquad x \sim \text{Hyper}(n, k, N) = \dfrac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$

let $p = \frac{k}{N} \Rightarrow k = pN$

bag of N marbles and a certain proportion of them are special

$$X \sim \text{hyper}(n, p, N) := \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

$N \in \{2, \dots\}$

$n \in \{1, \dots N-1\}$

$p \in \{\frac{1}{N}, \frac{2}{N}, \dots \frac{N-1}{N}\}$

reparametarization

Consider $p = 0.5, n = 6, N = 100$

$$p(3) = \frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = .3223$$

$N = 1000$

$$p(3) = \frac{\binom{500}{3}\binom{500}{3}}{\binom{1000}{6}} = .3134$$

$N = 10{,}000$

$$p(3) = \frac{\binom{5000}{3}\binom{5000}{3}}{\binom{10000}{6}} = .3126$$

what is the limiting random variable?

$$\lim_{N \to \infty} \text{Hyper}(n, p, N) = \lim_{N \to \infty} \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}} = \frac{\frac{(pN)!}{x!\,(pN-x)!}\,\frac{((1-p)N)!}{(n-x)!\,((1-p)N-(n-x))!}}{\frac{N!}{n!\,(N-n)!}}$$

$$= \frac{1}{x!}\,\frac{1}{(n-x)!}\,\frac{1}{n!}\,\lim_{n \to \infty} \frac{\frac{(pN)!}{(pN-x)!}\,\frac{((1-p)N)!}{(1-p)N-(n-x)!}}{\frac{N!}{(N-n)!}}$$