A discrete random variable (rv) X has a prob. mass function (PMF) $p(x) := P(X = x)$ and cumulative distri function (CDR) $F(x) = (X \leq x)$. The r.v X has "support"

$$\text{Supp}[X] := \{x : p(x) > 0, x \in \mathbb{R}\}$$

Since X is discrete, $|\text{supp}(X)| \leq |\mathbb{N}|$

Support and PMF are related in:

$$\sum_{x \in \text{supp}(X)} P(x) = 1.$$

The most fundamental discrete r.v is the Bernoulli:

$$X \sim \text{Bern}(p) := \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

What is p? P is a parameter. Parameters have parameter spaces, e.g. $p \in (0,1)$, so $p \neq 0$ and $p \neq 1$.

"degenerate" If $X \sim \text{Deg}(c) = \{c \text{ w.p. } 1.$

a.k.a. $\text{Deg}(c) = \mathbb{1}_{x=c}$, where $\mathbb{1}_{x=c}$ is an <u>indicator function</u>

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c. \end{cases}$$

<u>Def.</u> The r.v. X_1, X_2 are <u>independent</u> if

$$\underbrace{p(x_1, x_2)}_{\text{joint mass function.}} = p_{x_1}(x_1)\, p_{x_2}(x_2)$$

$\forall x_1, x_2$ in their supports.

Indicated as $X_1, X_2 \overset{\text{ind.}}{\sim}$

Def. $X_1 \overset{d}{=} X_2$. The r.v.'s X_1, X_2 are <u>equal in distribution</u> if
$$p_{X_1}(x) = p_{X_2}(x)$$

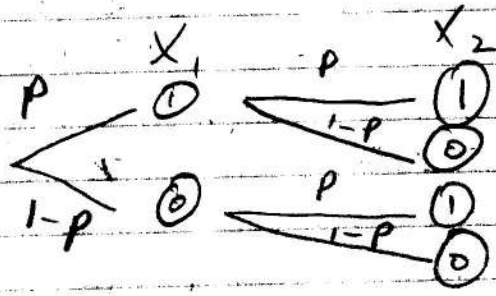
Def. $X_1, X_2 \overset{iid}{\sim}$. The r.v.'s X_1, X_2 are <u>independent and identically distributed</u> if $X_1, X_2 \overset{ind}{\sim}$ and $X_1 \overset{d}{=} X_2$

Let $T_2 = X_1 + X_2$ where $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$

$\text{Supp}[T_2] = \{0, 1, 2\} = \text{Supp}[X_1] + \text{Supp}[X_2]$

$A + B = \{a + b \mid a \in A, b \in B\}$.

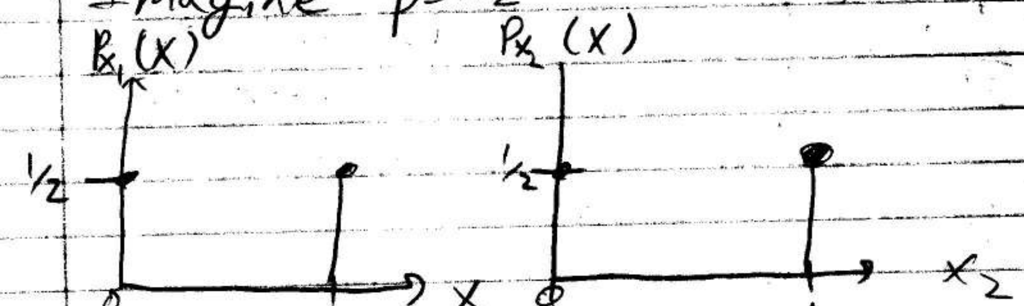
Probability Tree:

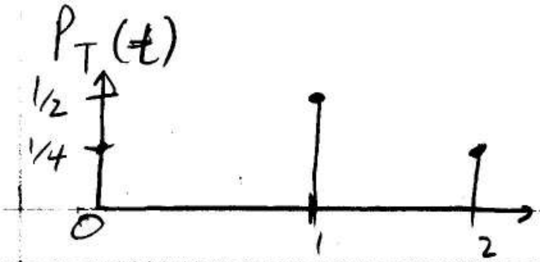


$p_{X_1, X_2}(x_1, x_2)$	T_2
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0

$$\Rightarrow p_T(2) = p^2$$
$$p_T(0) = (1-p)^2$$
$$p_T(1) = 2p(1-p)$$

Imagine $p = \frac{1}{2}$





$$P_{T_2}(t) := P(T_2 = t) = \sum_{x \in \text{Supp}[X]} P_{X_1}(x)\, P_{X_2}(t-x)$$

$$T_2 = X_1 + X_2 \quad = \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x}\right)\left(p^{t-x}(1-p)^{1-t+x}\right)$$

$$= p^t \sum_{x \in \{0,1\}} (1-p)^{2-t} = p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} 1$$

$$= 2p^t (1-p)^{2-t}$$ But This is WRONG!

$$P_{T_2}(2) = 2p^2 \neq p^2 \;!$$

Again: $$p(t) = P(T_2 = t) = \sum_{x \in \text{Supp}(X)} P_{X_1}(x)\, P_{X_2}(t-x)$$

$$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x}\, \mathbb{1}_{x \in \{0,1\}}\; p^{t-x} (1-p)^{1-t+x}\, \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{x \in \{0,1\}}\, \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \Big(\underbrace{\mathbb{1}_{0 \in \{0,1\}}}_{1}\, \mathbb{1}_{t-0 \in \{0,1\}} + \underbrace{\mathbb{1}_{1 \in \{0,1\}}}_{1}\, \mathbb{1}_{t-1 \in \{0,1\}} \Big)$$

$$\Big(\underbrace{\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t-1 \in \{0,1\}}}_{\binom{2}{t}} \Big)$$

$p(0) = (1-p)^2$
$p(2) = p^2$
$p(1) = 2p(1-p)$

$$\Rightarrow \binom{2}{t} p^t (1-p)^{2-t}$$

$X \sim Bern(p) = Bern(1,p) = \binom{1}{x} p^{x}(1-p)^{1-x}.$

Now $\binom{n}{k}$ only valid with $k \le n$. Otherwise, 0.

Now back to $P_{T_2}(t)$;

$$P(T_2 = t) = \sum_{x \in Supp[X]} P_{X_1}(x)\, P_{X_2}(t-x)$$

$$= \sum_{x \in \{0,1\}} \binom{1}{x} p^{x}(1-p)^{1-x} \binom{1}{t-x} p^{t-x}(1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in [0,1]} \binom{1}{x}\binom{1}{t-x} \qquad \binom{1}{0}\binom{1}{t} + \binom{1}{1}\binom{1}{t-1}$$

$$= \binom{2}{t} p^t (1-p)^{2-t} \quad \text{by } \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \square$$

$$p(t) = P(T_2 = t) = P_{X_1}(x) * P_{X_2}(x) := \sum_{x \in Supp[X]} P_{X_1}(x)\, P_{X_2}(t-x)$$

convolution of two independent PMF's.

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(p)$

Now, $T_3 = X_1 + X_2 + X_3 = X_3 + T_2 = P_{X_3}(x) * P_{T_2}(x)$

$$= \sum_{x \in Supp[X]} P_{X_3}(x)\, P_{T_2}(t-x) =$$

$$= \sum_{x \in \{0,1\}} \binom{1}{x} p^{x}(1-p)^{1-x} \binom{2}{t-x} p^{t-x}(1-p)^{2-t+x}$$

$$= p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{1}{x}\binom{2}{t-x} = p^t (1-p)^{3-t}\left(\binom{2}{t} + \binom{2}{t-1}\right)$$

$$= \binom{3}{t} p^t (1-p)^{3-t}.$$