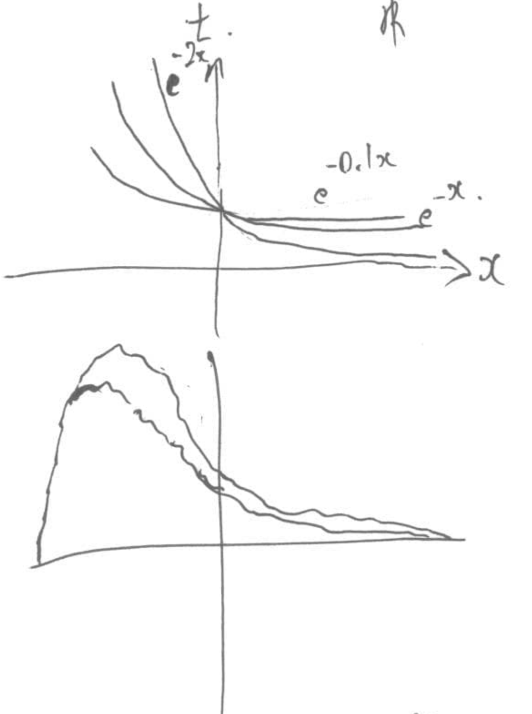
#19 $Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

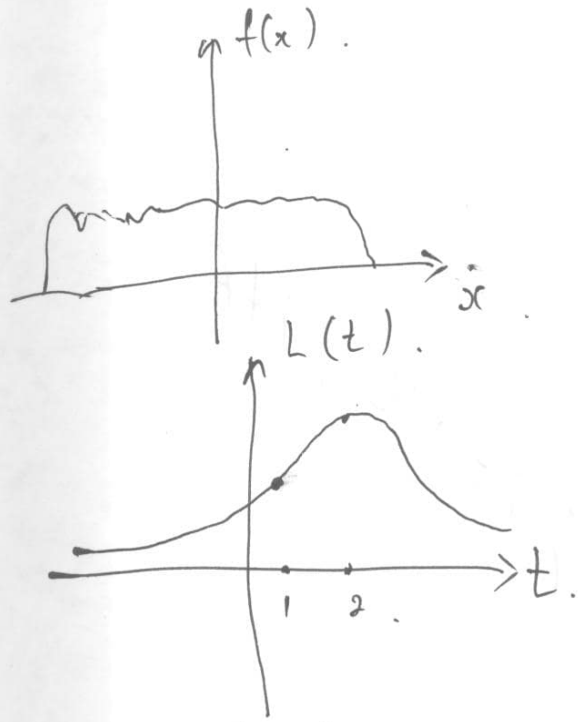
$$X = \mu + \sigma Z \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}.$$

$\mathrm{Supp}(Z) = \mathrm{Supp}(X) = \mathbb{R}$

$$Z = \frac{X-\mu}{\sigma}$$

$$L(t) = B[f] = \int_{\mathbb{R}} e^{-tx} f(x)\,dx.$$





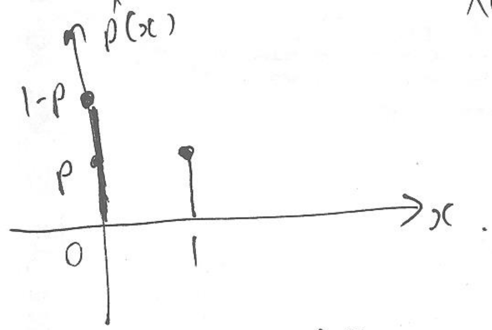
Moment Generation Function (MGF). of R.V X:

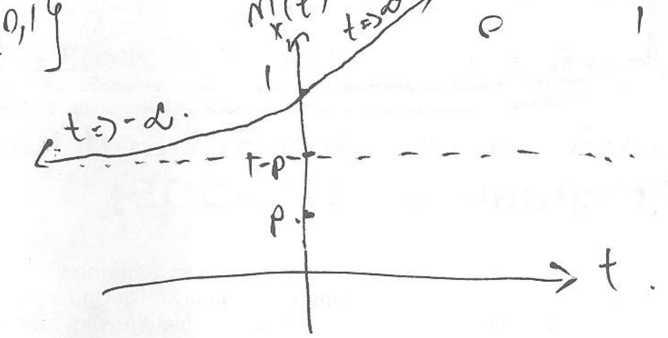
$$M_X(t) := E\left[e^{tX}\right] \begin{cases} \int_{\mathbb{R}} e^{tx} f(x)\,dx = L(t) & \text{if } X \text{ continuous. (PDF)} \\ \sum_{x \in \mathrm{Supp}[X]} e^{tx} p(x) & \text{if } X \text{ is discrete. (PMF)} \end{cases}$$

(I). $M_X(t) = M_Y(t) \iff X \overset{d}{=} Y$.

$X \sim \text{Bern}(p)$.

$$M_X(t) = E[e^{tX}] = \sum_{x \in \{0,1\}} e^{tx} p^x (1-p)^{1-x} = \underbrace{1-p}_{0} + \underbrace{pe^t}_{1}.$$





$$M_X(t) = E[e^{tX}]$$

$$M_X'(t) = \frac{d}{dt}\left(E[e^{tX}]\right) = \frac{d}{dt}\left[\int_{\mathbb{R}} e^{tx} f(x)\,dx\right]$$

Yes & No Sometime. $\overset{?}{=}$ $\int_{\mathbb{R}} \frac{d}{dt}\left[e^{tx} f(x)\right] dx := \int_{\mathbb{R}} x e^{tx} f(x)\,dx = E\left[Xe^{tX}\right]$

if $M_X'(0) = E[X] \Leftarrow$ "first Moment".

$M_X''(t) = E[X^2 e^{tX}]$, $M_X''(0) = E[X^2] \Leftarrow$ "Second moment"

(II) $M_X^{k}(0) = E[X^k] \Leftarrow$ "k^{th} Moment"

Linear transformation of X.

$Y = aX + c$ where a and c are constant.

$$M_Y(t) = E[e^{tY}] = E[e^{t(aX+c)}] = E[e^{taX+tc}] = E[e^{taX} e^{tc}]$$

$$= e^{tc} E[e^{taX}]$$

let $t' = at$.

(III) $\therefore M_Y(t) = e^{tc} E[e^{t'X}] = e^{tc} M_X[t'] = e^{tc} M_X(at)$.

$Y = X_1 + X_2$. s.t X_1, X_2 are independent.

two dices.

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1+tX_2}]$$

X_1, X_2 independent $\Rightarrow = E[e^{tX_1}] \cdot E[e^{tX_2}] = M_{X_1}(t)\, M_{X_2}(t)$.

if iid $\Rightarrow (M_X(t))^2$. IV

$X \sim Binom(n, p)$ ← Discrete

$$M_X(t) = E[e^{tX}] = \sum_{X=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}.$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}. \Rightarrow (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$= (pe^t + 1-p)^n.$$

$X_1, \dots, X_n \overset{iid}{\sim} Bern(p)$

$T = X_1 + \dots + X_n$.

$$M_T(t) \underset{IV}{=} (M_X(t))^n = (1-p+pe^t)^n$$

$\Rightarrow T \sim Binom(n, p)$.

I

$X \sim Exp(\lambda)$ ← continuous.

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx.$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x}\, dx = \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty.$$

$$= \frac{-\lambda}{t-\lambda} \quad \text{if} \quad t<\lambda$$

$$= DNS \quad \text{if} \quad t>\lambda$$

let $\lambda' = \frac{\lambda}{a}$,

$Y = aX$ where $a>0$.

$$M_Y(t) \underset{(\text{III})}{=} M_X(at) = \frac{\lambda}{\lambda - at}\,\frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a}-t} = \frac{\lambda'}{\lambda'-t}.$$

$\Downarrow$ (I).

$$Y \sim Exp(\lambda') = Exp\left(\frac{\lambda}{a}\right)$$

$Z \sim N(0,1)$.

$$M_Z(t) = E\left[e^{tZ}\right] = \int_{\mathbb{R}} e^{tx}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx.$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}+tx}\,dx. = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x^2-2tx)}\,dx.$$

$x^2-2tx = (x-t)^2 - t^2$.

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-t)^2 - \frac{t^2}{2}}\,dx = e^{\frac{t^2}{2}} \underbrace{\int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-t)^2}}_{N(t,1)}\,dx}_{1}.$$

$$= e^{\frac{t^2}{2}}. \quad \#$$

Verify $E[Z] = 0$. , $M_Z'(0) = t e^{t^2/2}\Big|_0 = 0.$ ✓.

Verify $SE[Z] = 1 \Rightarrow Var[Z] = 1$.

$$E[Z^2] - \cancel{\mu^2}\,0 = 1.$$

$$M_Z''[0] = e^{t^2/2} + t^2 e^{t^2/2}\Big|_0 = 1.$$