Lecture 16

11/8/17

$$Var(x) = E((X-\mu)^2) = \sum_{x \in supp(x)} (x-\mu)^2 p(x)$$

$$Var(aX+c) = a^2\sigma^2 \Rightarrow SE(aX+c) = |a|\sigma$$

$$E(X_1+X_2) = E(X_1)+E(X_2)$$

$$Var(X_1+X_2) = E(((X_1+X_2)-(\mu_1+\mu_2))^2)$$
$$= E(X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_2$$
$$- 2X_2\mu_2 + 2\mu_1\mu_2)$$
$$= E(X_1^2) + E(X_2^2) + \mu_1^2 + \mu_2^2 + 2E(X_1X_2) - 2\mu_1E(X_1) - 2\mu_2E(X_1)$$
$$- 2\mu_1E(X_2) - 2\mu_2E(X_2) + 2\mu_1\mu_2$$
$$= \sigma_1^2 + \sigma_2^2 + 2\underbrace{(E(X_1X_2) - \mu_1\mu_2)}_{= \text{Covariance}(X_1,X_2)}$$

Assume X_1, X_2 independent

$$P(X_1,X_2) = P(X_1)P(X_2)$$
$$E(X_1X_2) = \sum_{x \in supp(X_1)} \sum_{x \in supp(X_2)} X_1X_2\, p(X_1,X_2) = \sum_{X_1}\sum_{X_2} p(X_1)p(X_2)$$

$$= \sum_{X_1} X_1 p(X_1) \sum_{X_2} X_2 p(X_2)$$

$$Var(\Sigma X_i) = \Sigma Var(X_i) = n\sigma^2$$
$$Var(\bar{x}) = Var(\tfrac{1}{n}\Sigma X_i) = \tfrac{1}{n^2} Var(\Sigma X_i) = \tfrac{1}{n^2}\Sigma Var(X_i) = \tfrac{\sigma^2}{n}$$
$$SE(\bar{x}) = \tfrac{\sigma}{\sqrt{n}} \Rightarrow \bar{x} \to \mu \text{ (Law of Large Numbers)}$$

$X \sim Binomial(n,p)$

$$Var(x) = \sum_{x=0}^{n} (x-np)^2 \binom{n}{x} p^x (1-p)^{n-x} = \ldots = np(1-p)$$

$$Var(x) = Var(\Sigma X_i) = n\sigma^2 = np(1-p)$$
$$SE(X) = \sqrt{np(1-p)}$$

$X \sim Geometric(p) = (1-p)^{x-1} p$

$Supp(x) = \mathbb{N}$

$Var(x) = E(X^2) - \mu^2$

$E(X^2) = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} p = \sum_{y=0}^{\infty} (y+1)^2 (1-p)^y p = \sum_{y=0}^{\infty} y^2 (1-p)^y p$

$+ \sum_{y=0}^{\infty} 2y(1-p)^y + p \sum_{y=0}^{\infty} (1-p)^y$

$= (1-p) \sum_{y=1}^{\infty} y^2 (1-p)^{y-1} p + 2(1-p) \sum_{y=1}^{\infty} y(1-p)^{x-1} p + 1$

$\Rightarrow E(X^2) = (1-p) E(X^2) + \frac{2(1-p)}{p} + 1$

$\Rightarrow E(X^2) - (1-p) E(X^2) = \frac{2-p}{p}$

$E(X^2)(1-(1-p)) = \frac{2-p}{p}$

$E(X^2) = \frac{2-p}{p^2}$

$Var(x) = E(X^2) - \frac{1}{p^2} = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

$P(X=7) = (1-p)^6 p$

$P(X=17) = (1-p)^{16} p$

$P(X=17 \mid X>10) = P(X=7) = \frac{P(X=17 \cap X>10)}{P(X>10)} = \frac{P(X=17)}{P(X>10)} = \frac{(1-p)^{16} p}{1-F(10)}$

$P(X=a) = P(X=a+b \mid X>b) = \frac{P(X=a+b)}{P(X>b)} = \frac{(1-p)^{a+b+1}}{(1-p)^b} = (1-p)^{a-1} p = P(X=a)$

Memoryless property

$T \sim Geometric(p) = (1-p)^{t-1} p$

Every sec we do $n \overset{iid}{\sim} Bern(p)$'s

If $n=10$, $Supp(t) = \{0.1, 0.2, \ldots\}$

For arbitrary n, $p(t) = (1-p)^{nt-1} p$, $Supp(t) = \{\frac{1}{n}, \frac{2}{n}, \ldots\}$

Let $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$P(t) = (1 - \frac{\lambda}{n})^{nt-1} \cdot \frac{\lambda}{n}$

$$\lim_{n\to\infty} P(t) = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt-1}\cdot\lim_{n\to\infty}\frac{\lambda}{n} = 0 \;\forall t$$

Limiting CDF

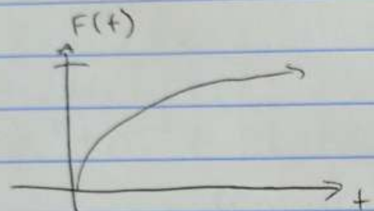
$$\lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nt} = 1-\left(\lim\left(1-\frac{\lambda}{n}\right)^{nt}\right) = 1-\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\right)^{t} = 1-e^{-\lambda t}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n} = \left(\lim_{c\to\infty}\left(1+\frac{1}{c}\right)^{c}\right)^{a} = e^{a}$$

Let $c = \frac{n}{a} \Rightarrow n = ac$

$n \to \infty \Rightarrow c \to \infty$



$F(0) = 0$

$\lim_{t\to\infty} F(t) = 1$

$F'(t) = \lambda e^{-\lambda t} = \frac{\lambda}{e^{\lambda t}} > 0$

$\Rightarrow F(t)$ is monotonically increasing