Math 241 lec 3 9/5/17

Working Definition of Prob:

$$P : 2^{\Omega} \to [0,1]$$

domain: event space is powerset of outcome space

range: a number between 0 and 1 where 1 is certainty and 0 is impossibility

$$P(A) := \frac{|A|}{|\Omega|} \quad \text{where} \quad A \in 2^{\Omega} \Rightarrow A \subseteq \Omega$$

What is the prob of $\underbrace{\text{getting a sum of 3}}_{\text{this defines } A}$ when $\underbrace{\text{rolling two die}}_{\text{this tells you } \Omega}$?

$\{1,2,\ldots,6\}$

Step 1: translate from English into Ω. $\Omega = \Omega_D \times \Omega_D$

Step 2: count $|\Omega| = |\Omega_D||\Omega_D| = 6 \cdot 6 = 36$

Step 3: translate from English into A. $A = \{\langle 2,1\rangle, \langle 1,2\rangle\}$

Step 4: count $|A|$

Step 5: divide

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\{\langle 2,1\rangle, \langle 1,2\rangle\}|}{36} = \frac{2}{36} ✓$$

Step 4 is the hardest usually...

What is the prob of getting 2 H on 4 coin tosses?

① $\Omega = \{H,T\}^4$

② $|\Omega| = |\{H,T\}^4| = |\{H,T\}|^4 = 2^4 = 16$

③ $A = \{\langle H,H,T,T\rangle, \langle H,T,H,T\rangle, \ldots\}$

④ $|A| = 6$

$$P(A) = \frac{6}{16}$$

looks the same but are not!

$$P(HHHH) = P(HTHT) \overset{?}{=} P(2H)$$

$$\frac{1}{16} = \frac{1}{16} \neq \frac{6}{16}$$

$$P(\text{at least one } H) = \frac{|\{(H,H,H,H), (HHHT), \ldots\}|}{16}$$

How big? HARD. Is there another way?

Recall $|\Omega| = |A| + |A^c| \Rightarrow |A| = |\Omega| - |A^c|$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = \frac{|\Omega|}{|\Omega|} - \frac{|A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - P(A^c)$$

AKA the "complement rule"

If $|A^c|$ is easier to count, it immediately results in $P(A)$.

≥ 1

$A = \{\text{at least one } H\}$

$A^c = \{\text{zero heads}\} = \{(T,T,T,T)\}$

$\Rightarrow |A^c| = 1 \Rightarrow P(A^c) = \frac{1}{16} \Rightarrow P(A) = \frac{15}{16}$ ✓

Flip 10 coins

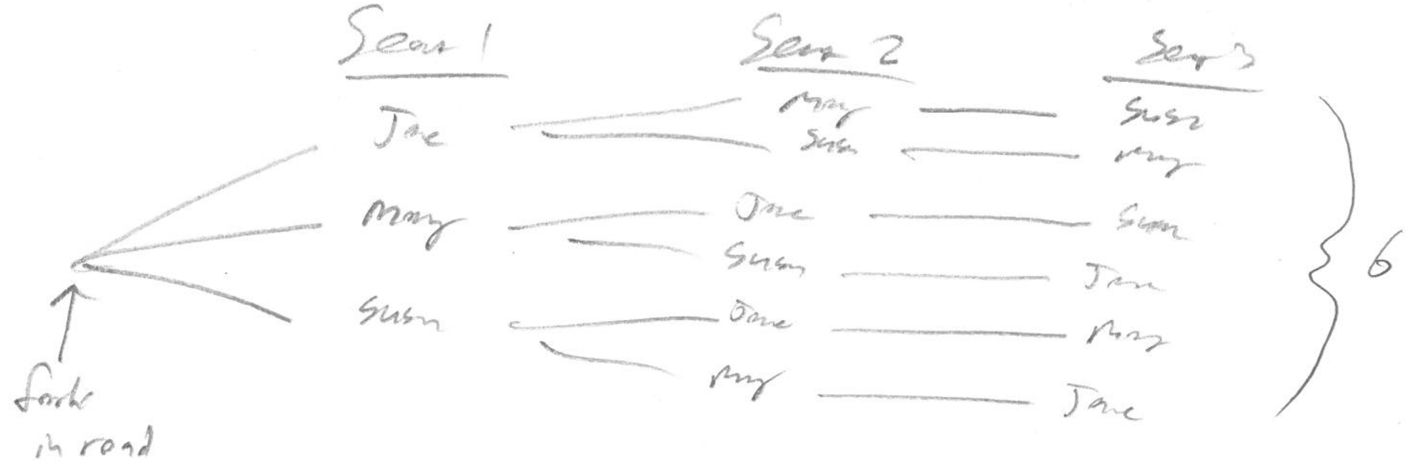
$|\Omega| = 2^{10} = 1024 \qquad |2^\Omega| = 2^{2^{10}} = \text{HUGE}$

What is prob of $\underbrace{\text{getting } 4H}_{A}$?

$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{1024}$ ← very hard... we need a better way to count...

$F = \{\text{Jane}, \text{Mary}, \text{Susan}\}$. We want to seat them in 3 chairs

How many ways to do this? Let's draw a "tree" illustration.



Each path above is a seating

$$\Omega = \{\langle J, M, S\rangle, \langle J, S, M\rangle, \langle M, J, S\rangle, \langle M, S, J\rangle, \langle S, J, M\rangle, \langle S, M, J\rangle\}$$

$|\Omega| = 6$

Without drawing the tree, you can multiply the # of options at every fork in the road:

$$\text{Tot arrangements} = \underset{\text{seat 1}}{3} \cdot \underset{\text{seat 2}}{2} \cdot \underset{\text{seat 3}}{1} = 6$$

Note $\Omega \neq F^3$, $\Omega \subset F^3$. $\langle J, J, J\rangle \notin \Omega$. Why?

$6 = |\Omega| \neq |F^3| = 27$ You can only sit J once!

Ω represents the set of F "sampled without replacement"

F^3 " " " " "sampled with replacement"

Sampling without replacement ex: seating people $\{T, M, S\}$ repeats not allowed

" " with " " " ex: flipping coin $\{H, T\}$ repeats allowed

Take a ball (ω) out of a basket (Ω). Do you get to put it back for the next selection? Yes — with replacement. No — without r.

\# ways to sample n objects without replacement? read "factorial"

$$n \cdot (n-1) \cdot (n-2) \cdots (2)(1) = n! := \prod_{i=1}^{n} i$$

" " " " " " with replacement

$$(n)(n)(n) \cdots (n) = n^n \quad \Rightarrow n! < n^n \text{ for } n > 1$$

\# ways to sit 5 people? $5! = 120$

10 people? $10! = 3.6$ m

20 people? 2.7×10^{32} = diam universe in fm

\# ways to seat 10 people in 3 chairs?

$$\underbrace{10}_{\text{seat 1}} \cdot \underbrace{9}_{\text{seat 2}} \cdot \underbrace{8}_{\text{seat 3}} = \frac{10!}{7!}$$

\# ways to sample k objects without replacement from a set of n objects

famous formula

$$\frac{n}{1} \frac{(n-1)}{2} \cdots \frac{(n-k+1)}{k} = \frac{n!}{(n-k)!} \qquad \underbrace{{}_nP_k}_{\text{"permutations"}} := \frac{n!}{(n-k)!}$$

$${}_nP_n = \frac{n!}{0!} = n!$$

this is $n!$ Thus, in order to make notation consistent, $0! := 1$

even though it makes no sense!

back to probability ...

$$P(\underbrace{J, S \text{ sit next to each other}}_{A}) = \frac{|A|}{|\Omega|} = \frac{\left|\left\{ \begin{array}{l} \langle S,J,M\rangle, \langle M,S,J\rangle \\ \langle J,S,M\rangle, \langle M,J,S\rangle \end{array} \right\}\right|}{3!} = \boxed{\frac{2}{3}}$$

sampling w/out replacement

New problem: (6 people) 3 couples, Bob-Jane, Richard-Susan, Charles-Mary

$$P(\underbrace{\text{every couple sits adjacent}}_{A}) = \frac{|A|}{|\Omega|} = \frac{?}{6!} = \frac{\cancel{6} \cdot \cancel{4} \cdot \cancel{2}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2}} = \boxed{\frac{1}{15}}$$

We need to use combinatorics to figure out $|A|$.

# options	6	1	4	1	2	1
Seat #	1	2	3	4	5	6

$$= \frac{\cancel{3 \cdot 2 \cdot 2 \cdot 2 \cdot 2}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2}} = \frac{1}{15}$$

or

3	2	1
Bench 1	Bench 2	Bench 3
2	2	2

$= 3!\, 2^3$

$$P(\underbrace{\text{alternating gender}}_{A}) = \frac{|A|}{|\Omega|} = \frac{?}{6!} = \frac{2(3!)}{6!} = \frac{\cancel{2}\cdot\cancel{3\cdot2}\cdot\cancel{1\cdot2}}{\cancel{6}\cdot5\cdot\cancel{4}\cdot\cancel{3\cdot2}} = \boxed{\frac{1}{10}}$$

B	G	B	G	B	G
3	3	2	2	1	1

G	B	G	B	G	B
3	3	2	2	1	1

$$A = \{\text{alternating gender}\} = \underbrace{\{\text{set of all BGBGBG}\}}_{A_{BG}} \cup \underbrace{\{\text{set of all GBGBGB}\}}_{A_{GB}}$$

$$P(A) = P(A_{BG}) + P(A_{GB})$$

Note: $A_{BG} \cap A_{GB} = \emptyset$ mutually exclusive

$\Rightarrow$ Probs of disjoint events can be added ... we will return to this later...

$$P(\underbrace{\text{Richard–Susan sit together}}_{A}) = \frac{|A|}{|\Omega|} = \frac{?}{6!} = \frac{\cancel{5\cdot4\cdot3\cdot2}\cdot4}{6\cdot5\cdot\cancel{4\cdot3\cdot2}} = \boxed{\frac{2}{3}}$$

R S 4 3 2 1

4 R S 3 2 1

...

_ _ _ _ R S

$5\cdot 4!\cdot 2$

100 balls, 3 samples without replacement. How many? $_{100}P_3 = \frac{100!}{97!} = 100 \cdot 99 \cdot 98$

" " " " " " " with " " " " " $100^3 = 100 \cdot 100 \cdot 100$

$$r = \frac{_{100}P_3}{100^3} = .9702 \approx 1$$

If n is large sampling without replacement $\approx$ " " " " " " with " " " . Proof:

$$\lim_{n\to\infty} r = \lim_{n\to\infty} \frac{_nP_k}{n^k} = \lim \overbrace{\frac{(n)}{n}\frac{(n-1)}{n}\cdots\frac{(n-k+1)}{n}}^{k \text{ terms}}$$

(k terms)

$$= \lim \frac{n}{n} \lim \underbrace{\frac{n-1}{n}}_{\underbrace{\frac{n}{n}}_{1} - \underbrace{\frac{1}{n}}_{0}} \cdots \lim \underbrace{\frac{n-k+1}{n}}_{\underbrace{\frac{n}{n}}_{1} - \underbrace{\frac{k}{n}}_{0} + \underbrace{\frac{1}{n}}_{0}} = 1$$

Now we sit all 6 people in a circle. How many ways to do so assuming you don't care which seat is first?

rotational invariance

In some sense...

$\Rightarrow$ ⟨B, J, R, S, C, M⟩
"=" ⟨J, R, S, C, M, B⟩
= ⟨R, S, C, M, B, J⟩
= ⟨S, C, M, B, J, R⟩
= ⟨C, M, B, J, R, S⟩
= ⟨M, B, J, R, S, C⟩

considered "same", "indistinguishable", "non-unique", "non-distinct", "indistinct"

$$\Rightarrow \frac{6!}{6}$$

the original answer for 6 chairs

? Invariance factor

Heuristic: divide out invariance factors.