Math 241 Fall 2017
Midterm Examination One

Solutions

Professor Adam Kapelner

October 3/4, 2017

Full Name ____ Section (A, B or C) _____

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Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using a cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

______________________________________ __________________
signature date

Instructions

This exam is seventy five minutes and closed-book. You are allowed one page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in choose, permutation, exponent, factorial or any other notation which could be resolved to a number with a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1 Below are some multiple choice theoretical exercises. Circle the correct answer.

(a) [3 pt / 3 pts] If non-empty events A_1, A_2, A_3 are disjoint, then the statement "$\mathbb{P}\left(\cup_{i=1}^{3} A_i\right) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3)$" is

(i) always true

(ii) sometimes true

(iii) never true

(b) [3 pt / 6 pts] If non-empty events A_1, A_2, A_3 are disjoint, then the statement "$\mathbb{P}\left(\cap_{i=1}^{3} A_i\right) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3)$" is

(i) always true

(ii) sometimes true

(iii) never true

(c) [3 pt / 9 pts] If non-empty events A_1, A_2, A_3 are disjoint, then the statement "$\mathbb{P}\left(\cap_{i=1}^{3} A_i\right) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$" is

(i) always true

(ii) sometimes true

(iii) never true

(d) [3 pt / 12 pts] If non-empty events A_1, A_2, A_3 are disjoint, then the statement "A_1, A_2, A_3 collectively exhaust the experimental outcome space Ω" is

(i) always true

(ii) sometimes true

(iii) never true

(e) [3 pt / 15 pts] If non-empty events A_1, A_2, A_3 are disjoint, then the statement "$\mathbb{P}(A_1 \mid A_2) = \mathbb{P}(A_1)$" is

(i) always true

(ii) sometimes true

(iii) never true

(f) [3 pt / 18 pts] If non-empty events A_1, A_2, A_3 are *independent*, then the statement "$\mathbb{P}\left(\cap_{i=1}^{3} A_i\right) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$" is

(i) always true

(ii) sometimes true

(iii) never true

(g) [3 pt / 21 pts] If non-empty events A_1, A_2, A_3 are independent, then the statement "$\mathbb{P}(A_1 \mid A_2) = \mathbb{P}(A_1)$" is

(i) always true

(ii) sometimes true

(iii) never true

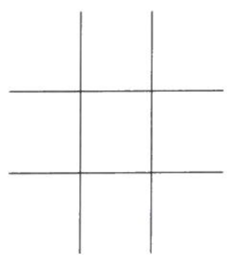
(h) [3 pt / 24 pts] If non-empty events A_1, A_2, A_3 are independent, then the statement "$\mathbb{P}(A_1 \mid A_2) = \mathbb{P}(A_1 \mid A_3)$" is

(i) always true

(ii) sometimes true

(iii) never true

Problem 2 The game of Tic-Tac-Toe is played with two players. The first player has the mark "X" and the second player has the mark "O". They play on the following 3×3 board with nine *distinct* spaces for a player to place their mark:



Each player takes turns by writing their mark in one of the nine spaces until one player gets "3 in a row" which means three of their mark in a row, a column, or diagonally across. So an X is placed, then a O is placed, then an X is placed, then an O is placed, etc until one player wins or until all spaces are filled with no winner.

(a) [3 pt / 27 pts] How many ways are there for the first player to place their mark?

9

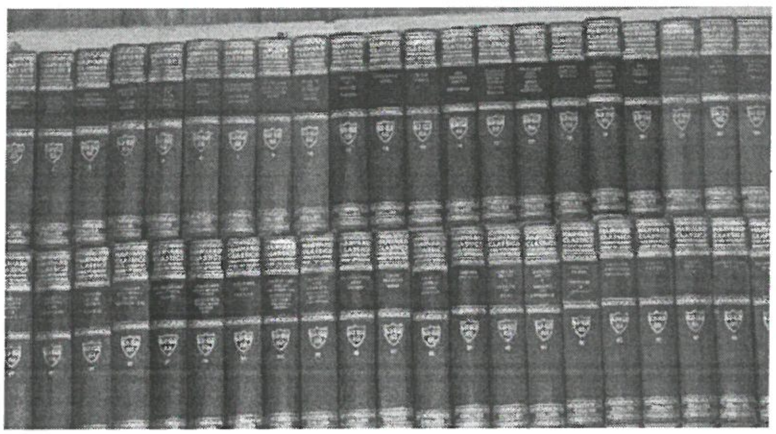
(b) [3 pt / 30 pts] How many ways are there for the first two turns to be played? In other words, how many ways are there to place one X and one O?

$_9P_2 = 9 \cdot 8$

(c) [4 pt / 34 pts] Imagine the marks are created by a computer randomly and the game doesn't stop upon a win. Assume all 9 marks are written (still X then O then X then O, etc). How many possible boards are there with all 9 spaces occupied?

$$\frac{9!}{5!\,4!} = \binom{9}{5} = \binom{9}{4}$$

Problem 3 Professor Bob's library of Harvard classics is pictured below:



He has 16 red books, 9 blue books, 7 green blooks and 10 purple books. $n = 42$

(a) [4 pt / 38 pts] If he randomly takes 4 books off the shelf *with* replacement, what is the probability he gets all red books?

$$\left(\frac{16}{42}\right)^4$$

(b) [4 pt / 42 pts] If he randomly takes 4 books off the shelf *without* replacement, what is the probability he gets all red books?

$$\frac{\binom{16}{4}}{\binom{42}{4}}$$

(c) [4 pt / 46 pts] If he randomly takes 4 books off the shelf without replacement, what is the probability he gets two blue books and two purple books?

$$\frac{\binom{9}{2}\binom{10}{2}}{\binom{42}{4}}$$

(d) [5 pt / 51 pts] If he randomly takes 4 books off the shelf without replacement, what is the probability he gets two blue books then two purple books *in that order*?

$$\frac{{}_9P_2 \; {}_{10}P_2}{{}_{42}P_4}$$

(e) [4 pt / 55 pts] If he randomly takes 4 books off the shelf without replacement, what is the probability he gets books of all different colors?

$$\frac{\binom{16}{1}\binom{9}{1}\binom{7}{1}\binom{10}{1}}{\binom{42}{4}}$$

(f) [4 pt / 59 pts] If he randomly takes 4 books off the shelf without replacement, what is the probability he gets two books of one color and two books of a different color (e.g. two green books and two purple books)? This problem is hard, save it for last.

$$\frac{\binom{16}{2}\binom{9}{2} + \binom{16}{2}\binom{7}{2} + \binom{16}{2}\binom{10}{2} + \binom{9}{2}\binom{7}{2} + \binom{9}{2}\binom{10}{2} + \binom{7}{2}\binom{10}{2}}{\binom{42}{4}}$$

(g) [3 pt / 62 pts] He assigns each specific book to a specific student in his class to read. If he hands them out randomly, what is the probability at least one student will get their assigned book?

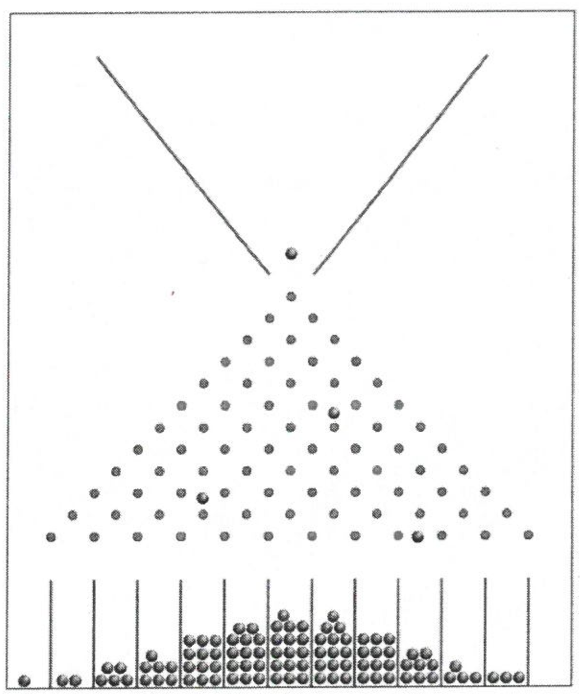
$1 - \frac{1}{e} \approx \frac{2}{3}$

Problem 4 This problem is about the philosophical theory of probability.

(a) [3 pt / 65 pts] Experts at a think tank estimate war with North Korea at 50-50. What definition of probability was employed here?

Subjective

(b) [3 pt / 68 pts] Pictured below is the "bean machine". A marble gets dropped in from the top and it makes its way down to a bin on the bottom.



Only 0.8% of the marbles wind up in the leftmost bin. What definition of probability was most likely employed to arrive at this number?

long-run frequency

(c) [3 pt / 71 pts] If a marble was dropped down, would Laplace believe its final destination bin to be truly deterministic? Yes/no.

Yes

$P(C) = 0.00\overset{5}{\cancel{3}}7 \Rightarrow P(C^c) = .9943$

$P(A) = \frac{8}{323} = 0.0248 \Rightarrow P(A^c) = .9752$

$P(C|A) = 0.15$

Problem 5 The incidence of cirrhosis in America is approximately 0.57% but among alcoholics, it's 15%. There are about 8 million alcoholics in America, a country with a total population of 323 million. Let the event C denote cirrhosis and the event A denote alcoholism. → 0.57%

(a) [5 pt / 76 pts] In a random sample of 12 Americans, what is the probability at least one of them is alcoholic? Compute and round to 3 significant digits.

$P(\geq 1 \text{ A in } 12) = 1 - P(0 \text{ A in } 12) = 1 - P(A^c)^{12} = 1 - .9752^{12} = .260$

(b) [3 pt / 79 pts] [Extra Credit] In a random sample of 86 Americans, what is the probability 59 of them are alcoholic? Compute and round to 3 significant digits.

(c) [4 pt / 83 pts] Find $\mathbb{P}(AC)$. Compute and round to 3 significant digits.

$P(AC) = P(C|A)\,P(A) = 0.15 \cdot 0.0248 = .00372$

(d) [4 pt / 87 pts] We do not know if Joe is an alcoholic or not. What is the best guess of the probability he develops cirrhosis?

$P(C) = .0057$

(e) [4 pt / 91 pts] Prove or disprove: A, C are independent events.

$P(C|A) \stackrel{?}{=} P(C)$

$.15 \neq$ ~~.0037~~ $.0057$

$\Rightarrow$ not independent

(f) [4 pt / 95 pts] Find the probability of not having cirrhosis and not being an alcoholic. Compute and round to 3 significant digits.

$P(C^c A^c) = 1 - P(AC) = 1 - .00372 = .996$

(g) [8 pt / 103 pts] Find the risk ratio of alcoholism in cirrhosis. Compute and round to 3 significant digits. Provide a one-sentence interpretation in English.

$$\frac{P(C|A)}{P(C|A^c)} = \frac{.15}{.002} = 75$$

$P(C, A^c) = P(C) - P(C, A) = .0057 - .00372 = .00198$

$$P(C|A^c) = \frac{P(C, A^c)}{P(A^c)} = \frac{.00198}{.9752} = .002$$

You are 75 times more likely to get cirrhosis if you are an alcoholic vs. if you are not an alcoholic.