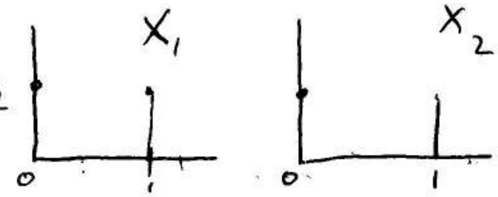
PMF's:




8/31/17

$X_1, X_2 \overset{iid}{\sim} Bern(\tfrac{1}{2})$, $T_2 = X_1 + X_2$

Tally of $T_2$:


This is an example of a convolution. Think of passing the PMF graph of $X_1$ through $X_2$ and see where the lines match.

$$X_1 + X_2 \sim P_{X_1}(x) * P_{X_2}(x) := \sum_{x \in Supp[x]} P_{X_1}(x) P_{X_2}(t-x)$$

For Bern(p): $\displaystyle\sum_{x \in \{0,1\}} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} = \sum_{x \in \{0,1\}} p^t (1-p)^{2-t}$

$$= p^t (1-p)^{2-t} \underbrace{\sum 1}_{2} = \cancel{2p^t(1-p)^{2-t}}$$ This was wrong.

$$p(2) = p^0 (1-p)^{1-0} \underbrace{p^{2-0} (1-p)^{1-2}}_{\text{Turned off using indicator function.}} + p^1 (1-p)^{1-1} p^{2-1} (1-p)^{1-2+1}$$

$X_1, X_2 \overset{iid}{\sim} Bin(n,p)$,

$$Y = X_1 + X_2 \sim P_{X_1}(x) * P_{X_2}(x) = \sum_{x \in Supp[x_2]} P_{X_1}(x) P_{X_2}(y-x)$$

$$= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \underbrace{\mathbb{1}_{x \in \{0,\dots,n\}}}_{\text{not needed.}} \binom{n}{y-x} p^{y-x} (1-p)^{n-y+x} \underbrace{\mathbb{1}_{y-x \in \{0,\dots,n\}}}_{\text{not needed}}$$

$$= \sum_{x \in \{0,\dots,n\}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{y-x} p^{y-x} (1-p)^{n-y+x}$$

$$= p^y (1-p)^{2n-y} \sum_{x \in \{0,\dots,n\}} \binom{n}{x}\binom{n}{y-x}$$

$$= p^y (1-p)^{2n-y} \binom{2n}{y}$$ by Vandermonde's Identity.

$$= Binom(2n, p)$$

Consider $B_1, B_2, \ldots \overset{iid}{\sim} Bern(p)$
Let $X := \min_t \{B_t = 1\} - 1$.

This is called a geometric r.v.

So $X \sim Geom(p)$. $Supp[X] = \{0, 1, \ldots\} = \mathbb{N}_0$

$P(X=0) = p$

$P(X=1) = (1-p)p$

$P(X=2) = (1-p)^2 p$

$P(X=x) = (1-p)^x p$.

Parameter Space: $0 < p < 1$.

Now, for the convolution of $Geom(p)$

$T_2 = X_1 + X_2 \sim p(t) = P_{X_1}(x) * P_{X_2}(x)$

$= \sum_{x \in Supp[X_1]} P_{X_1}(x) P_{X_2}(t-x) = \sum_{x \in \mathbb{N}_0} (1-p)^x p (1-p)^{t-x} p \mathbb{1}_{t-x \in \mathbb{N}_0}$

$t-x \in \mathbb{N}_0$ equivalent to $t \geq x$.

$= (1-p)^t p^2 \sum_{x \in \mathbb{N}_0} \mathbb{1}_{x \leq t} = (1-p)^t p^2 \sum_{x=0}^{t} 1$

$= (1-p)^t p^2 \cdot (t+1)$. Now $Supp[T_2] = \{0, 1, \ldots\}$.

Let $T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim p(t) = P_{X_3}(x) * P_{T_2}(x)$

$= \sum_{x \in Supp[X_3]} P_{X_3}(x) P_{T_2}(t-x) =$

$= \sum_{x \in \mathbb{N}_0} (1-p)^x p \, (t-x+1)(1-p)^{t-x} p^2 \mathbb{1}_{t-x \in Supp[T_2] = \mathbb{N}_0}$

$= p^3 (1-p)^t \sum_{x \in \mathbb{N}_0} (t-x+1) \mathbb{1}_{x \leq t}$

$$= (1-p)^t p^3 \left( (t+1) \sum_{x \in \mathbb{N}_0} \mathbb{1}_{x \le t} - \sum_{x \in \mathbb{N}_0} x \mathbb{1}_{x \le t} \right.$$

$$= (1-p)^t (p^3) \left( (t+1) \underbrace{\sum_{x=0}^{t} 1}_{t+1} - \underbrace{\sum_{x=0}^{t} x}_{\frac{t(t+1)}{2}} \right)$$

$$= (1-p)^t p^3 \left( \frac{t^2+3t+2}{2} \right)$$

$T_3$ = # failures until 3 successes.

$P(T_3 = t) = \binom{t+2}{2} (1-p)^t p^3$.

Note that $\binom{t+2}{2} = \frac{(t+2)!}{2!\,t!} = \frac{(2+t)(1+t)}{2} = \frac{t^2+3t+2}{2}$

Name: $T_2 \sim \text{Neg Bin}(2, p)$
$T_3 \sim \text{Neg Bin}(3, p)$

In HW, prove by induction that
$X \sim \text{NegBin}(k, p) = \binom{x+k-1}{k-1} (1-p)^x p^k$

Now, $\text{Supp}[X] = \{0, \dots, n\}$

$X \sim \text{Bin}(n, p)$. What if $n$ is really big?
What if $p$ is really small?
But they are related via $\lambda := np$
$\Rightarrow p = \frac{\lambda}{n}$

What is the pmf if $n \to \infty$

$$\lim_{n \to \infty} p(x) = \lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \lim_{n \to \infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n \to \infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n \to \infty} \frac{n!}{(n-x)!\,n^x} \underbrace{\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{-x}}_{1}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \quad . \text{ Now, } X \sim \text{Poisson}(\lambda) := \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\text{Supp}[X] = \{0, 1, \ldots\} = \mathbb{N}_0$$

Parameter Space: $\lambda \in (0, \infty)$

Convolution of Poisson:

$$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$$

$$T = X_1 + X_2 \sim P_{X_1}(x) * P_{X_2}(x)$$

$$= \sum_{x \in \text{Supp}[X]} P_{X_1}(x)\, P_{X_2}(t-x) = \sum_{x \in \mathbb{N}_0} \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{x \le t}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in \mathbb{N}_0} \frac{1}{x!\,(t-x)!} \cdot \mathbb{1}_{x \le t} \cdot \frac{t!}{t!}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \mathbb{N}_0} \binom{t}{x} \mathbb{1}_{x \le t}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x=0}^{t} \binom{t}{x} = \frac{\lambda^t e^{-2\lambda}}{t!} \cdot 2^t = \frac{(2\lambda)^t e^{-2\lambda}}{t!}$$

$$= \text{Poisson}(2\lambda)$$