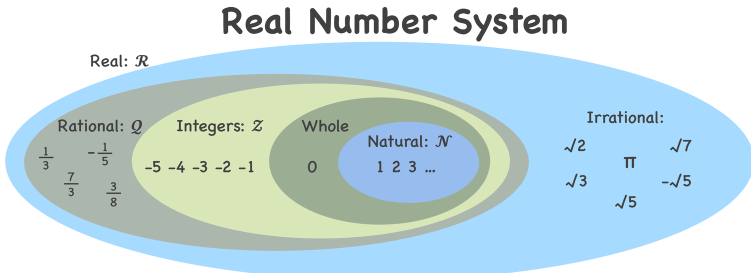
# 1 Basic Set Theory

**Set theory** is the branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used in the definitions of nearly all mathematical objects.
The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with the axiom of choice, are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

- wikipedia.org

## Real Number System




## important sets

- $\mathcal{N}$ : natural numbers := { 1,2,3,... } — some mathematicians also consider 0 as a natural number
- $\mathcal{Z}$ : integers := { ..., -2, -1, 0, 1, 2 ,... }
  $\mathcal{Z}^+$ : positive integers := { 1, 2, 3, ... }
- $\mathcal{R}$ : real numbers
  $\mathcal{R}^+$ : positive real numbers
- $\mathcal{Q}$ : rational numbers := { p/q | p,q ∈ $\mathcal{Z}$ , and q≠0 }
- $\mathcal{C}$ : complex numbers := { a+bi | a,b ∈ R }
  a,b - real numbers | variables
  i = √-1 - not a real number | constant

## Set Notation

set name — elements aka members — set braces {} denote the beginning / end of enumeration

F for female names F := { Jane , Mary , Susan , Dana }

assignment aka definition

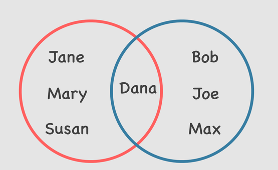
- order is irrelevant
- duplicates are irrelevant and count as 1

M for male names M := { Bob , Joe , Max , Dana }

## Venn Diagram

Venn Diagram Examples

| Jane , Mary , Susan |
|---|
| Dana |
| Bob , Joe , Max |




Sets can contain an infinite number of elements:

$\mathcal{N}$ := { 1 , 2 , 3 , 4 , ... }

ellipses ... are used when the general pattern of the elements is obvious and reads "and so forth"

$\mathcal{Z}$ := { ... , -2 , -1 , 0 , 1 , 2 , ... }

s/t: such that
b/c: because

- **Operations on Sets** — MNEMONIC: element ∈/∉ set

  set inclusion - Jane is a member of F

  Jane ∈ F

  set exclusion - Bob is NOT a member of F

  Bob ∉ F

- **Set Equality / Inequality**

  equality - LHS & RHS have the same elements

  { Jane , Mary , Dana , Susan } = F (order irrelevant)

  inequality - LHS & RHS do NOT have the same elements

  { Jane , Mary } ≠ F    M ≠ F

- **Subsets**

  subset - ALL elements of LHS are in RHS

  { Jane , Mary } ⊆ F

  proper subset - ALL elements of LHS are in RHS

  { Jane , Mary } ⊂ F

  ⊆ same as ⊂ or =

- **Predicates**

  ∈ ∉ = ≠ ⊂ ⊆ are predicate functions which return T/F

  ≠ ( M , F ) → True — are sets M and F not equal? True

  ≠ : set → { T , F } — if ≠ is a set, T & F are its members

  =( M , F ) → False — are sets M and F equal? False

## Union

set_Union ( param1:Set1 , param2:Set2 ) -> Set3

- **Set Union** - combines the elements of both sets

  set union — MNEMONIC: set ∪ set = set

  F ∪ M

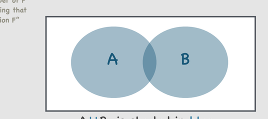
  { { Jane , Mary , Susan , Dana } { Bob , Joe , Max , Dana } }

  collapse or flatten (Ruby)

  { Jane , Mary , Susan , Bob , Joe , Max , Dana }

  x ∈ F or x ∈ M → x ∈ M∪F — if x is a member of F or x is a member of M, then x is a member of the Union of M and F

  Dana ∈ F ≡ Dana ∈ M∪F — saying that "Dana is a member of F" is logically equivalent to saying that "Dana is a member of M Union F"




A∪B is shaded in blue

∪ is "inclusive or" aka "non-exclusive or"

a member is in M **or** in F

### singleton set

singleton set : a set with one element

e.g. { a } , { ∅ }

ex

| | |
|---|---|
| { Jane } $\overset{?}{\subset}$ F (singleton set - set with only 1 member) | T - since there are elements in RHS that are not in LHS |
| Jane $\overset{?}{\subset}$ F | F - in order to be a proper subset, member Jane would have to be a set. ONLY a Set can be a Subset of another Set, this this notation is ILLEGAL |
| { Jane } $\overset{?}{\in}$ F | F - LHS is a set and RHS does NOT contain LHS among its members |
| { Jane } $\overset{?}{\subseteq}$ F | T - LHS is a subset of RHS because element Jane is contained by both LHS and RHS |
| Jane $\overset{?}{\in}$ F | T - Jane is a member of F |
| Jane $\overset{?}{\subseteq}$ F | F - Jane is NOT a set - notation illegal |

self-note:
remember: subsets are NOT members.
A:={ 1 , 2 , 3 , {4} , 5 , {6,7} }
subsets of A: {1} {1,2} {{4}} {3,{4}} etc...

ex

{ Jane } ∪ { Jane } $\overset{?}{=}$ { Jane }

T - since the duplicate elements count as one , and as a consequence the union of LHS && RHS flattens into a singleton set equivalent to either of the sides.

F ∪ M → Dana $\overset{?}{\in}$ F∪M    T - since ∪ is inclusive

Let's use a definition of $\mathcal{N}$ which does NOT include zero. We want to include zero among the elements of $\mathcal{N}$

$\mathcal{N}_0$ := $\mathcal{N}$∪{ 0 } = { 0 , 1 , 2 , 3 , ... }

## Intersection

set_Intersect ( param1:Set1 , param2:Set2 ) -> Set3

- **Set Intersection** - returns only the elements in-common

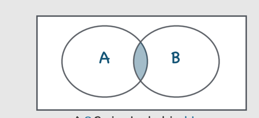
  set intersection — MNEMONIC: set ∩ set = set

  F ∩ M

  { { Jane , Mary , Susan , Dana } { Bob , Joe , Max , Dana } }

  { Dana }

  x ∈ F and x ∈ M → x ∈ M ∩ F — if x is a member of F and x is a member of M, then x is a member of the Intersection of M and F




A∩B is shaded in blue

a member is in M **and** in F

ex

F ∩ { Bob , Joe } = { }    "empty set" aka "null set"  ∅ := { }

T - since LHS && RHS have NO elements in common (they are mutually exclusive)

odds ∩ evens = ∅

## Difference

set_Diff ( param1:Set1 , param2:Set2 ) -> Set3

- **Set Difference / Subtraction** - LHS wipes away RHS

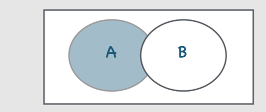
  set difference — MNEMONIC: set \ set = set 'wipe away'

  F \ M

  { { Jeane , Mary , Susan , Dana } { Bob , Joe , Max , Dana } }

  { Jeane , Mary , Susan } — All elements in F that are NOT in M

  x ∈ F and x ∉ M → x ∈ F\M — if x is a member of F and x is NOT a member of M, then x is a member of the Difference of F and M




A\B is shaded in blue

a member is in F **and NOT** in M

### mutual exclusivity

Def: A & B are mutually exclusive if
A ∩ B = ∅

"Mutually exclusive" is a statistical term describing two or more events that cannot occur simultaneously. For example, it is impossible to roll a five and a three on a single die at the same time.
In logic, two mutually exclusive propositions are propositions that logically cannot be true in the same sense at the same time. - wikipedia.org

ex

| | |
|---|---|
| ∅ ⊆ F | T - vacuously true |
| ∅ ∈ F | F - ∅ is a not a member but a set |
| ∅ ∉ F | T - look above |

### empty set

the Empty Set or Null Set is the set with no elements

symbolized ∅ or {}    ∅ ≠ {∅}

∅ is a subset of any set : ∅ ⊆ A

two sets are called disjoint if their intersection is the empty set

every non-empty set S has at least 2 subsets: ∅ and itself
1. b/c a∈∅ is always false ∅⊆S for every set S
2. b/c a∈S → a∈S is true S⊆S for every set S

- **Set Builder Notation** - use when it is impossible to list all elements of a set

  E := { 2n : n ∈ $\mathcal{Z}$ }

  "all elements 2n (even)" "such that" "n is an element of the integers"

  E := { ... , -4 , -2 , 0 , 2 , 4 , ... } - even numbers

- **Power Set** - set of subsets

  A := { 1 , 2 , 3 }    MNEMONIC: $2^{\#\text{ of elements}}$ = $2^3$ = 8 subsets

  $2^A$ := { B : B ⊆ A } = { ∅ , {1} , {2} , {3} , {1,2} , {1,3} , {2,3} , A }

  power set of A - ALL subsets of A

  CS binary representation:

  TTT TTF TFT TFF FTT FTF FFT FFF

  FFF TFF FFT TFT
  { ∅ , {1,2,3,} , {1} , {2} , {3} , {1,2} , {1,3} , {2,3} }
  TTT FTF TTF FTT

  $2^3$ = 8 — 3 variables: 1,2,3 — 2 choices: - present T - absent F — 8 possible combinations TTT TFT etc···

- **Set Cardinality** - set size (number of elements)

  overloaded |abs val| symbol aka 'pipe' character

  |A| = 3

  |F∪M| ≠ |F| + |M|
  7    4    4

  Heuristic Method of Counting :

  for any set S ... $|2^S| = 2^{|S|}$ — compare

heuristic: enabling a person to discover or learn something for themselves ; "a hands on" or interactive heuristic approach to learning
- an approach to problem solving, practical method, not guaranteed to be optimal but sufficient to get the job done···

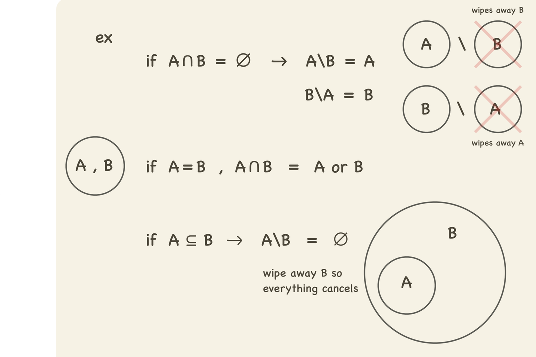
ex

if A∩B = ∅ → A\B = A    A \ B (wipes away B)
B\A = B    B \ A (wipes away A)

if A=B , A∩B = A or B

if A ⊆ B → A\B = ∅ — wipe away B so everything cancels

if A\B = ∅ , A ∩ B = A



self-note:
We have a Set Named 'A' . Set 'A' is defined as: A:={1,2,3} therefore our set A has 3 members: 1, 2, and 3. Since Set A is NOT an 'Empty Set' it can have multiple subsets. If we take ALL of the possible subsets of A we can create another set called the 'Power Set' and denoted as $2^A$. This Power Set contains as members all of the possible subsets of set A.

parent-set
A:={1,2,3}

all possible subsets
B:={1} C:={2} D:={3} E:={1,2} F:={1,3} G:={2,3}
A
{}
— every non-empty set has at least 2 subsets: - itself - empty set

power set
$2^A$ := { A, {}, B, C, D, E, F, G }

ex

$|2^A| = 2^{|A|} = 2^3 = 8$

$|2^{F∪M}| = 2^{|F∪M|} = 2^7 = 128$

|∅| = 0

|F∩M| = 1

|F\M| = 3

|F|-|M| = 0

# 1 cont...

### Universal Set U / domain

the Universal Set U is the set containing everything currently under consideration
- sometimes implicit
- sometimes explicitly stated
- contents depend on the context

## Special Set Ω

- **Special Set Ω**

  universe of discourse , sample space , CS - scope

  - all elements we are limited to
  - you define it

  Ω := F∪M = { names } — lets define Ω as the F Union M, the set of male and female names

  note:
  F ⊆ Ω , M ⊆ Ω , $2^F$ ⊆ Ω

  Dice Roll Ω := { 1,2,3,4,5,6 } — all possible outcomes that can happen
  Coin Flip Ω := { Heads , Tales }

- **Probability**

  working def for probability of A: $P(A) = \frac{|A|}{|\Omega|}$

  Ω := F∪M

  What is the probability a "random" name is a female ?

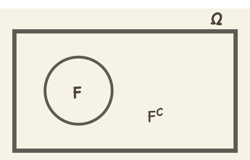
  $P(F) = \frac{|F|}{|\Omega|} = \frac{4}{7}$

ex

| | |
|---|---|
| F ∩ Ω = F | |
| F ∪ Ω = Ω | |
| ∅ ∩ Ω = ∅ | higher order of triviality |
| ∅ ∪ Ω = Ω | |
| $∅^C$ = Ω | everything (not empty set - nothing -) |
| F ∈ $2^Ω$ = T | |
| F \ Ω = ∅ | wipe away Ω |
| Ω \ F = $F^C$ | everything NOT in F |
| $(F^C)^C$ = F | double negation |
| A ∪ $A^C$ = Ω | |
| A ∩ $A^C$ = ∅ | mutually exclusive |
| A ⊆ $A^C$ - False | |

$|A| + |A^C| = |Ω|$ for |Ω| finite
cardinality of A + cardinality of everything else other than A = cardinality of everything

$|A| = |Ω| - |A^C|$ useful fact




{ A , $A^C$ } where $A^C$ := Ω\A

collectively exhaustive — mutually exclusive

{$A_1$,$A_2$,$A_3$,···,$A_n$} : n≤∞    {$A_1$,$A_2$,$A_3$,···,$A_n$} : n≤∞

$\bigcup_{i=1}^{n} A_i = Ω$    $A_i ∩ A_j = ∅$ for all i≠j

all possible Ω-subsets are included — they got nothing in-common

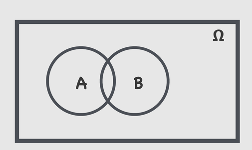
**collectively exhaustive:**

In probability theory and logic, a set of events is jointly or collectively exhaustive if at least one of the events must occur. For example, when rolling a six-sided die, the outcomes 1, 2, 3, 4, 5, and 6 are collectively exhaustive, because they encompass the entire range of possible outcomes. - wikipedia.org

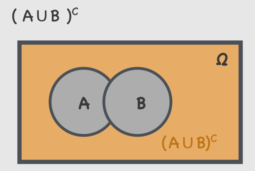
# 2

## De Morgan's laws

consider A , B . Immediately, you know A,B ⊆ Ω



consider $(A∪B)^C$

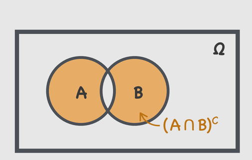


can I express this differently ?

algebra $(3·4)^7 = 3^7 · 4^7$

$(A∪B)^C \overset{?}{=} A^C ∪ B^C$ — False
$A^C ∪ B^C = (A∩B)^C$ — True
$(A∪B)^C ≠ (A∩B)^C$

$(A∪B)^C = A^C ∩ B^C$
de Morgan Laws
$(A∩B)^C = A^C ∪ B^C$

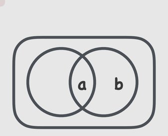


- **Ordered Pair - Tuples ()**

  < a , b > := { {a} , {a,b} }

  - order matters
  - duplicates matter

  < a,b > ≠ < b,a >
  < a,b > ≠ { a,b }



- **Cartesian Product**

  A x B := { <a,b> : a∈A , b∈B }

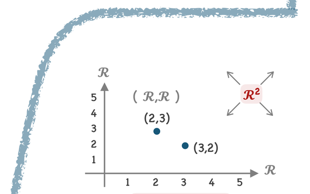
  A = { 1,2 }    B = { 3,4 }

  A x B = { <1,3> , <1,4> , <2,3> , <2,4> }

  | AxB | = |A| · |B| = 4    if finite

  $A^2$ = A x A = { <1,1> , <1,2> , <2,1> , <2,2> }
  $A^n$ = A x A x A x ... x $A^n$    n∈$\mathcal{N}$

  $|A^n| = |A|^n$    $|\bigtimes_{i=1}^{n} A_i| = \prod_{i=1}^{n} |A_i|$




$\mathcal{N}$ := { 1 , 2 , 3 , ··· }

$|\mathcal{N}| = \aleph_0$ — aleph - countable infinity — infinity that you can count to

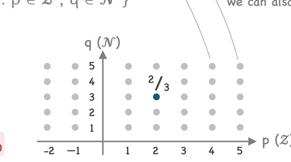
$\mathcal{Z}$ := {···,-2,-1,0,1,2,···}
$|\mathcal{Z}| = 2\aleph_0 + 1$ ?? seams reasonable but FALSE
$|\mathcal{Z}| = \aleph_0$

why ? Lets map {···,-2,-1,0,1,2,···}
{1 , 2 , 3 , 4 , 5 , ···} — back to $\mathcal{N}$

rational numbers

$\mathcal{Q}$ := { $\frac{p}{q}$ : p ∈ $\mathcal{Z}$ , q ∈ $\mathcal{N}$ } — we can also map them to $\mathcal{N}$

$|\mathcal{Q}|$ = ?




$|\mathcal{Q}| = \aleph_0$

are all numbers ∈ $\mathcal{Q}$ ? No! e.g. √2 , π ∉ $\mathcal{Q}$

real numbers

$\mathcal{R}$ := $\mathcal{Q}$ ∪ { all holes } (rationals, irrationals)

where is √2 , π , etc... ?
no holes - the continuum

[3,7] := { x: x≥3 , x≤7 } ⊂ $\mathcal{R}$
(10,17] := { x: x>10 , x≤17 } ⊂ $\mathcal{R}$

$|\mathcal{R}|$ = ?
assume $|\mathcal{R}| = \aleph_0$

consider x ∈ (0,1) , b(x) = x in base 2
0.0101101010···
0.1101001110···
0.0111100000···
0.0000101111···
— counting enumeration — we keep listing till infinity

lets flip the diagonal
0.1101101010···
0.1001001110···
0.0101100000···
0.0001101111···

(0,1) ⊂ $\mathcal{R}$ (interval notation (not a tuple))    | (0,1) | ≤ $|\mathcal{R}|$

0.1001··· — we get a real number that is not equal to any of the above numbers meaning the above list is incomplete and we can not enumerate all of the numbers

$|\mathcal{R}| ≠ \aleph_0$    $|\mathcal{R}| > \aleph_0$

$\mathcal{C}$ := $|\mathcal{R}|$ — uncountable infinity = the cardinality of the continuum

$\aleph_0 < \mathcal{C}$ — uncountable infinity is larger than countable infinity

# Probability

- **Probability**

  Ω - sample space

  outcomes: Ω = { $\omega_1$ , $\omega_2$ , $\omega_3$ , ... } (all possible outcomes)

  events: A := {$\omega_1$ , $\omega_2$}    B := {$\omega_1$ , $\omega_3$}

  e.g.

  coin flip Ω = { H,T } 2 possible outcomes: heads , tails
  |Ω| = 2 cardinality of Ω = number of possible outcomes

  In a coin flip only Heads or Tails could happen and nothing else. Heads and Tails are mutually exclusive - (be careful outcomes are NOT sets)

  { H } , { T } (events)    H , T (outcomes)

  { H } ⊆ Ω (events are subsets)    H ∈ Ω (outcomes are members)

  Sets in which all elements are outcomes are called 'events'. An event is defined by:

  A ⊆ Ω → A ∈ $2^Ω$

  set of outcomes - event    event space i.e. all possible events

  here, $2^Ω$ = { ∅ , { H } , { T } , { H,T } } — all must boil down to these 4 things

  $P(A) = \frac{|A|}{|\Omega|}$ if Ω is finite
  working def for probability of A

  $P(H) = \frac{|H|}{|\Omega|}$ — this does NOT compute ILLEGAL !! | member | ???

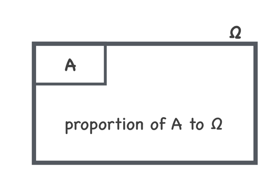
  $P(\{H\}) = \frac{|\{H\}|}{|\Omega|}$ = 1/2 probability of an event (or | {H,T} |)

  $∀\omega_i\ P(\omega_i) = \frac{1}{|\Omega|}$ probability of an outcome
  def of 'equally likely'

ω : lower case Omega

In probability theory, the sample space of an experiment or random trial is the set of all possible outcomes or results of that experiment. A sample space is usually denoted using set notation, and the possible outcomes are listed as elements in the set. It is common to refer to a sample space by the labels S, Ω, or U (for "universal set"). - wikipedia.org

In probability theory, an event is a set of outcomes of an experiment (a subset of the sample space) to which a probability is assigned. A single outcome may be an element of many different events, and different events in an experiment are usually not equally likely, since they may include very different groups of outcomes.
Typically, when the sample space is finite, any subset of the sample space is an event (i.e. all elements of the power set of the sample space are defined as events). However, this approach does not work well in cases where the sample space is uncountably infinite, most notably when the outcome is a real number. So, when defining a probability space it is possible, and often necessary, to exclude certain subsets of the sample space from being events. - wikipedia.org




for the time being

Def: P: $2^Ω$ → [ 0,1 ] — the probability of all the possible events is between 0 and 1 (inclusively)

$P(A) = \frac{\#\text{ of outcomes of interest}}{\#\text{ of all possible outcomes}}$ — Event = A, B, etc··· — Sample Space Ω

| | |
|---|---|
| Ω - sample space | is a 'special set' often known as the universe of discourse. It contains all possible outcomes as its members |
| $2^Ω$ - event space | is a 'power set of Ω'. It contains all possible subsets of Ω i.e. all possible events. |
| A , B - event | is a 'subset of Ω'. It contains outcomes of a given experiment aka trial. e.g. H-heads , T-tails → {H} , {H,T} , etc... |
| H , T - outcome | is a member of a set. Where the set is either Ω or and event. |

P( {H,T} ) = 1
P( {H} ∪ {T} ) = 1
P( {H} ∩ {T}) = 0    absurd - coin edge?

P( Ω ) = 1    P( ∅ ) = 0

$P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$
trivial events
$P(\varnothing) = \frac{|\varnothing|}{|\Omega|} = 0$

Complement Rule - useful triviality

$P(A^C) = \frac{|A^C|}{|\Omega|} = \frac{|\Omega| - |A|}{|\Omega|} = 1 - \frac{|A|}{|\Omega|} = 1 - P(A)$

$P(A) = 1 - P(A^C)$

$|2^Ω| = 2^{|Ω|}$ numb of possible events
≠ |Ω|

for Ω = { H,T }
P( {H} ) = $^1/_2$
P( {T} ) = $^1/_2$ — both events are equally likely

## Complement

set_Comp ( param1:Set1 , param2:Set2 ) -> Set3

- **Set Complement** - negation

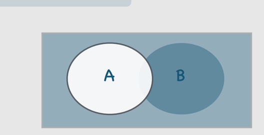
  set complement

  $F^C$ := Ω \ F

  { { Jane , Mary , Susan , Dana } { Bob , Joe , Max , Dana } }

  { Bob , Joe , Max } everything that is NOT in F

  x ∉ F → x ∈ $F^C$




$A^C$ is shaded in blue

everything **NOT** in F