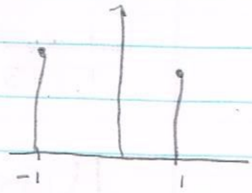
Math 241 Lecture 14

Oct. 30th



Custom r.v.'s

expectation

- Roulette in America

$E[x] = \sum_{x \in supp[x]} x \, p(x)$

$1 Bet on black payout is 1:1

$x \sim \begin{cases} \$1 & 18/38 \\ -\$1 & 20/38 \end{cases}$

$= (\$1)\left(\frac{18}{38}\right) + (-\$1)\left(\frac{20}{38}\right) = -\$0.053$

negative b/c odds are against you

$\bar{x} \Rightarrow \mu$ converge to the average (-0.05)

law of large #'s $\mu \notin supp(x)$ generally speaking

$\lim_{n \to \infty} T_n = \underbrace{x_1 + \dots x_n}_{n \bar{x}} = -\infty$

$n \to \infty$, $\bar{x} \to -0.053$

The casinos care about this number a lot b/c there are thousands of people each night so they see this number a lot.

- Bet on a lucky 7 Payons 35:1

$x \sim \begin{cases} \$35 & wp \ 1/38 \\ -\$1 & wp \ 37/38 \end{cases}$

$E[x] = 35 \cdot \frac{1}{38} + (-1)\frac{37}{38} = -\$0.053$

- Bet on first dozen (1-12) payout 2:1

$x \sim \begin{cases} \$2 & wp \ 12/38 \\ -\$1 & wp \ 26/38 \end{cases}$

$E[x] = 2 \cdot \frac{12}{38} + (-1)\frac{26}{38} = -\$0.053$

- Roulette in Europe

$x \sim \begin{cases} \$1 & 18/37 \\ -\$1 & 19/37 \end{cases}$

$E[x] = \$1\left(\frac{18}{37}\right) + (-\$1)\left(\frac{19}{37}\right) = -\$0.027$

European roulette is "more fair" then American roulette
Average of slighty less odds against you.

Fair Game

$E[x] = 0$

- $P(\text{traffic}) = 0.3$

If traffic, Uber takes 12mins. If not trafffic 7mins.

Model time in car w.

$$w \sim \begin{cases} 12\text{min} & \text{wp } 0.3 \\ 7\text{min} & \text{wp } 0.7 \end{cases}$$

$E[w] = 12 \cdot 0.3 + 7 \cdot 0.7 = 8.5$ min

In a large amount of trips, the average time spent is 8.5mins (long run average).

- Uber charges \$0.40/min Model B the price paid for time in taxis.

$$B \sim \begin{cases} \$0.40 \cdot 12 = \$4.80 & \text{wp } 0.3 \\ \$0.40 \cdot 7 = \$2.80 & \text{wp } 0.7 \end{cases}$$

$E[B] = \$4.80 \cdot 0.3 + \$2.80 \cdot 0.7 = \$3.12$

Another way to derive that number:

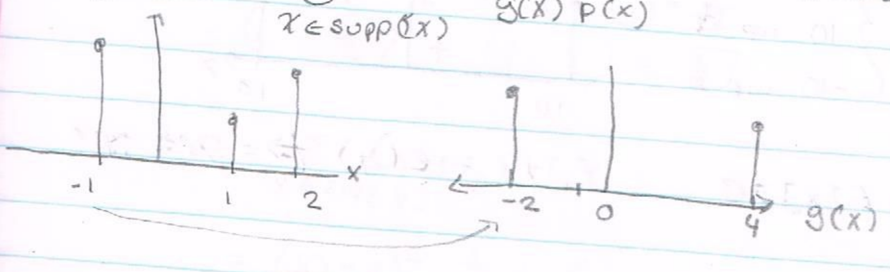
$= \$0.40 \cdot 12 \cdot 0.3 + \$0.40 \cdot 7 \cdot 0.7$

$= \$0.40\,(12 \cdot 0.3 + 7 \cdot 0.7)$

$= \$0.40 \quad E(n)$

$Y = aX,\ a \in \mathbb{R}$ $\quad E[Y] = E[aX] = E[g(X)] = \sum_{x \in supp[Y]} ax\,p(x) = a\sum x\,p(x) = a\,E(X)$

$E[g(X)] = \sum_{x \in supp(X)} g(x)\,p(x)$



- Base fare is \$3

Model T, the total price.

$$T \sim \begin{cases} 3 + 4.80 = 7.80 & \text{wp } 0.3 \\ 3 + 2.80 = 5.80 & \text{wp } 0.7 \end{cases}$$

$E(T) = 7.80 \cdot .3 + 5.80 \cdot .7 = 6.12$

$= (3 + 4.8)(0.3) + (3 + 2.80)(.7)$

$= 3(0.3 + 0.7) + 4.80 \cdot 0.3 + 2.80 \cdot .7$

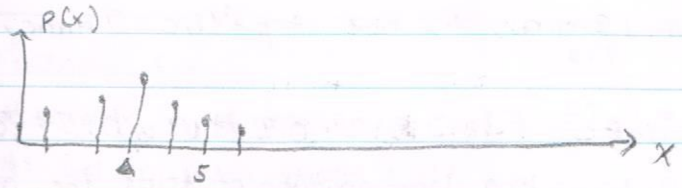
$= 3 + E[B]$

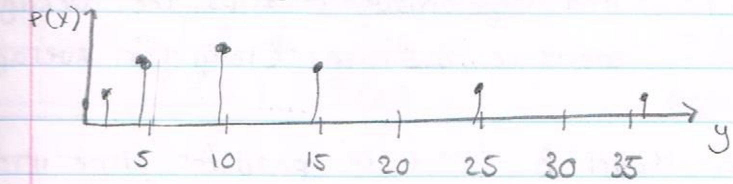
$Y = X + c,\quad c \in \mathbb{R}$

$E[Y] = E[X + c] = \sum (x+c)\,p(x) = \underbrace{\sum x\,p(x)}_{E[X]} + \sum c\,p(x)$

$$\boxed{\begin{array}{l} Y = aX + c, \quad a, c \in \mathbb{R} \\ E[Y] = a\,E[X] + c \end{array}}$$ Linear transformations

$X \sim \text{Bin}\left(6, \frac{1}{2}\right) \rightarrow E[X] = 3$ Let $Y = X^2$ Does $E(X^2) \stackrel{?}{=} (E[X])^2$
$\neq$

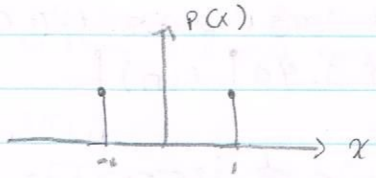




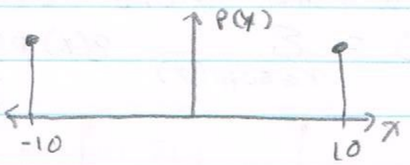
$$E[Y] = \sum_{x=0}^{6} x^2 \binom{n}{x} p^x (1-p)^{n-x} = 17.5$$

$$\Rightarrow \quad E[g(x)] \neq g(E(x))$$
generally

$X \sim$ Rademacher $\begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$

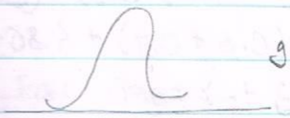


$y = 10x \sim \begin{cases} 10 & \text{wp } \frac{1}{2} \\ -10 & \text{wp } \frac{1}{2} \end{cases}$

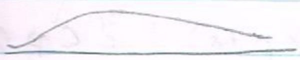


$E[X] = 0$
$E[Y] = 10\,E[X] = 0$ $E[X] = E(Y) \not\Rightarrow X \stackrel{d}{=} Y$

Y is more...
(a) spread out
(b) deviant
(c) variable



g(x) $\int_{\mathbb{R}} g(x)\, d = 3$



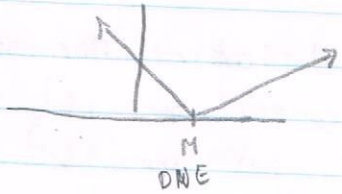
h(x) $\int_{\mathbb{R}} h(x) = 3$

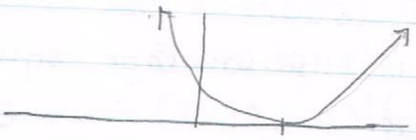
Theory of Error / Distance Function

measure error from x to μ

$\epsilon(x, m) = |x - m|$ L_1 error metric

(can't be negative. ex: can never be -20 feet away)





$\epsilon(x, m) = (x-m)^2$ L_2 error metric

Let $L = (x-m)^2$

"squared error loss"

$Var(x) := E[L] = E[(x-m)^2] = E[(x - E(x))^2]$

$$Var[x] = \sum_{x \in supp(x)} (x-m)^2 \, p(x)$$

$$= (1-0)^2 \tfrac{1}{2} + (-1-0)^2 \tfrac{1}{2}$$

$$= (1 \cdot \tfrac{1}{2}) + (1 \cdot \tfrac{1}{2}) = \boxed{1}$$

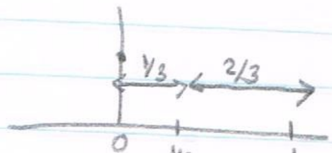
$$Var[y] = \sum_{y \in supp(y)} (y - m_y)^2 \, p(x)$$

$$= (10-0)^2 \tfrac{1}{2} + (-10-0)^2 \tfrac{1}{2}$$

$$= 100 \cdot \tfrac{1}{2} + 100 \cdot \tfrac{1}{2} = \boxed{100}$$

$x \sim Bern(\tfrac{1}{3})$

$E[x] = \tfrac{1}{3}$



$Var(x) = E((x-m)^2)$

$$= \sum_{x \in supp(x)} (x-m)^2 \, p(x) = (1 - \tfrac{1}{3})^2 \tfrac{1}{3} + (0 - \tfrac{1}{3})^2 \tfrac{2}{3}$$

$$= (\tfrac{2}{3})^2 \tfrac{1}{3} + (\tfrac{1}{3})^2 \tfrac{2}{3}$$

$$= \tfrac{4}{9} + \tfrac{2}{6} = \boxed{\tfrac{2}{3}}$$

$$= (1-p)^2 \, p$$

$\Sigma (x-m)^2 p(x)$

$= (1-p)^2 \; p + (0-p)^2 (1-p)$

$= (1-2p+p^2)p + p^2(1-p)$

$= p - 2p^2 \cancel{+p^3} + p^2 \cancel{-p^3} = p - p^2 = \boxed{p(1-p)}$

variance is the average squared distance from the expectation

μ, σ^2