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Consider the Pattern

Pascal's Triangle

$$1 = \binom{0}{0}$$
$$\binom{1}{0} = 1 \qquad 1 = \binom{1}{1}$$
$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad 1 = \binom{2}{2}$$
$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$
$$\begin{matrix} 1 & 4 & 6 & 4 & 1 \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \end{matrix}$$

$(a+b)^3$

$(a+b)^4$

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

CONJECTURE

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Pascal's Identity, Rule, Formula, thm...

Prove $\forall n \in \mathbb{N}, k \in \{1, \ldots, n-1\}$

$n-k-1 = (n-1)-(k)$

$$\frac{n!}{(n-k)!\,k!} \overset{?}{=} \left(\frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!}\right)\left(\frac{n}{n}\right)$$

$$\shortparallel$$

$$\frac{n!}{n}\left(\frac{1}{(n-k)!\,(k-1)!}\left(\frac{k}{k}\right) + \frac{1}{(n-k-1)!\,k!}\left(\frac{n-k}{n-k}\right)\right) = \frac{n!}{n}\left(\overbrace{\frac{k}{(n-k)!\,k!} + \frac{n-k}{(n-k)!\,k!}}^{n}\right) = \frac{n!}{(n-k)!\,k!}$$

Set $\underset{\text{suit}}{S} = \{\underset{\text{spade}}{\spadesuit}, \underset{\text{club}}{\clubsuit}, \underset{\text{diamond}}{\diamondsuit}, \underset{\text{heart}}{\heartsuit}\}$

$R = \{2, 3, \ldots, 10, \underset{\text{Jack}}{J}, \underset{\text{Queen}}{Q}, \underset{\text{King}}{K}, \underset{\text{Ace}}{A}\}$

$\underset{\text{deck}}{D} = R \times S$

consider the "game" where you are dealt (sample without replacement) 5 cards so that order does not matter

(5 cards = Hand)

$$P(\text{Royal Flush}) = \frac{|R.F.|}{|\Omega|} = \frac{4}{\binom{52}{5}}$$

$$|\Omega| = \binom{52}{5} = 2{,}598{,}960$$

10, J, Q, K, A

All same suit

$$P(\text{straight Flush}) = \frac{\binom{10}{1}\binom{4}{1} - 4}{\binom{52}{5}}$$

← subtract our royal Flush

same suit

10 total

A, 2, 3, 4, 5

2, 3, 4, 5

3, 4, 5, 6, 7

$$P(4\text{ of a kind}) = \frac{\binom{13}{1}\overbrace{\binom{12}{1}\binom{4}{1}}^{\binom{48}{1}}}{\binom{52}{5}}$$

7777Q

$$P(\text{Full House}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

$\binom{4}{2} = 6$

$\underbrace{QQQ}_{3\text{ of a kind}}$ $\underbrace{77}_{2\text{ of a kind}}$

$$P(\text{Flush}) = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} - \binom{9}{1}\binom{4}{1} - 4$$

$$P(\text{Straight}) = \frac{\binom{10}{1}\binom{4}{1}^5 - \binom{9}{1}\binom{4}{1} - 4}{\binom{52}{5}}$$

$\binom{48}{1}($

$$P(3 \text{ of a kind}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

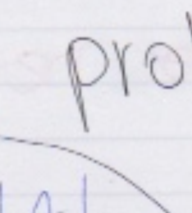
$$P(2 \text{ pair}) = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

Recall the "working" definition of probability

$$P(A) = \frac{|A|}{|\Omega|}$$

consider the following random experiment.

spin a spiner, return ω



$$P(\{R\}) \neq \frac{|\{R\}|}{|\Omega|} = \frac{1}{3}$$

→ Valid with an implicit assumption ....

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$$\forall \omega \in \Omega \quad P(\{\omega\}) = \frac{1}{|\Omega|}$$

equally likely outcome $\boxed{\text{assumption}}$

When is this valid?
toss coins, roll die, deal cards...

$\Omega = \{$Struck by lightening, Not struck$\}$

$P(\{\text{struck}\}) = \frac{1}{2}$

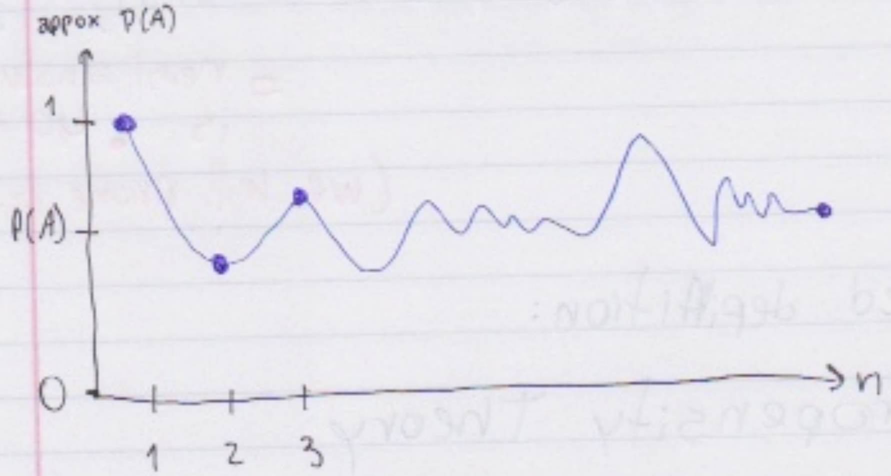
We need a new definition

(I) Limiting Frequency Definition

First, $\mathbb{1}_{\omega \in A} := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

Indicador function

$$P(A) :\approx \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n} = \lim_{n\to\infty} \frac{\#\{A \text{ happens}\}}{n}$$



10000 coin toss

$$\frac{5067 \text{ Heads}}{10000} = 5.067$$

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Problems

(1) Need ability to run experiments

(2) You can never run ∞ experiments
=> you are limited to an approximation (never exact) which can be error-prone

P(Irma hits Miami)
P(OJ Simpson is guilty)

In 1654, Chevalier de Mere wrote a letter to Pascal & Fernet claiming:

$$P(\{\geq 1 \text{ double-6 is 24 die rolls}\}) < \frac{1}{2}$$

= real answer is .4914 (we will prove later)

Accepted defiAition:

II Propensity Theory

Karl Popper, 1957

There's an inherent property inside the random experiments device which indices the long run prequency.