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Continuous r.v.

a Geometric experiment is equivalent to performing many iid Bernoulli experiments with the 'stopping time' of the 1st successful Bernoulli. e.g. you keep flipping that proverbial coin until it lands Heads. It can happen on the 1st flip or on the n^{th}.

$T \sim Geometric(p) := (1-p)^{t-1} p$ $F(t) = 1-(1-p)^t$ $T_n = X_1 + X_2 + ... + X_n$

- Now imagine we are conducting these iid Bernoulli experiments $X_1, X_2, ..., X_n$ each every second.
- Next imagine we are able to perform n-number of iid Bernoulli experiments $X_1, X_2, ..., X_n$ in a single second. Meaning, our Geometric experiment might be over in under one second.

$E[T] = \frac{1}{p}$ experiments $\cdot \frac{1 sec}{n \text{ experiments}} = \frac{1}{np}$ sec

doing so many iid Bernoullis alters our PMF $p(t) = (1-p)^{nt-1} p$ and CDF $F(t) = 1-(1-p)^{nt}$

- Imagine n-large but p-small

Let $\lambda = n\,p \rightarrow p := \frac{\lambda}{n}$ reparametrization

$p(t) = (1-\frac{\lambda}{n})^{nt-1}\frac{\lambda}{n}$ $F(t) = 1-(1-\frac{\lambda}{n})^{nt}$

PMF

- Lets make n-humongous $n \rightarrow \infty$, BUT λ remains a reasonable constant e.g. 1,...,10, which means that p vanishes at the same rate as $n \rightarrow \infty$. If the experiments occur continuously, what happens to the PMF (remember PMF is a discrete function)?

$\lim_{n\to\infty} p(t) = \lim_{n\to\infty}(1-\frac{\lambda}{n})^{nt-1}\frac{\lambda}{n} = 1 \cdot 0 = 0$ NO valid PMF!

$\sum p(t) = 0 \neq 1$

CDF

- So at the moment as $n \rightarrow \infty$ there in no valid PMF, meaning T is not a valid discrete r.v. – how about CDF?

$\lim_{n\to\infty} F(t) = \lim_{n\to\infty} 1-(1-\frac{\lambda}{n})^{nt} = 1 - \lim (1-\frac{\lambda}{n})^{nt}$

$e := \lim (1+\frac{1}{n})^n$

let $\frac{1}{m} = \frac{a}{n} \rightarrow n = ma$

a = some reasonable const. 1...10 so if $n\to\infty$, $m\to\infty$

$\lim_{m\to\infty} [(1+\frac{1}{m})^m]^a = e^a$, $-\lambda = a$, $= e^{-\lambda}$

$= 1 - [e^{-\lambda}]^t$

$= 1 - e^{-\lambda t}$

CDF $F(t) = 1 - e^{-\lambda t}$ ← $\lim_{n\to\infty} = 1 - e^{-\lambda t}$ is this a valid CDF ?

proving a valid CDF

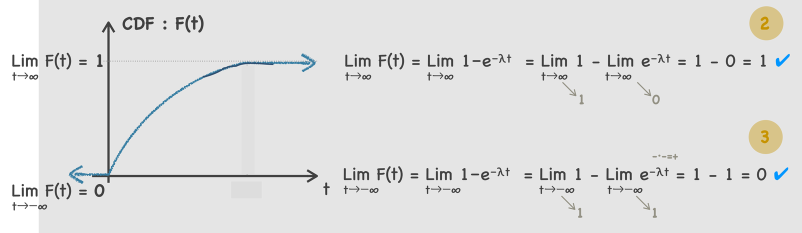
4 properties every CDF must satisfy

1. $F(t) \in [0,1]$ by Def probabilities must be between 0 and 1
2. $\lim_{t\to\infty} F(t) = 1$
3. $\lim_{t\to-\infty} F(t) = 0$
4. $a \le b \rightarrow F(a) \le F(b)$; $a < b \rightarrow F(a) < F(b)$ False — monotonically increasing, slope is ≥ 0

$1-e^{-\lambda t} \ge 0$, $1 \ge e^{-\lambda t}$ | ln both sides, $0 \ge -\lambda t$, $-\lambda t \le 0$, $t \in (0,\infty)$ — we begin at time 0, thus $0 \notin$; geometric never ends ∞ ✔

$1-e^{-\lambda t} \le 1$, $-e^{-\lambda t} \le 0$, $e^{-\lambda t} \ge 0$, $\frac{1}{e^{\lambda t}} \ge 0$ ✔

ok, so we satisfy the [0,1] (1)



(2) $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 1-e^{-\lambda t} = \lim 1 - \lim e^{-\lambda t} = 1 - 0 = 1$ ✔

(3) $\lim_{t\to-\infty} F(t) = \lim_{t\to-\infty} 1-e^{-\lambda t} = \lim 1 - \lim e^{-\lambda t} = 1 - 1 = 0$ ✔

remember calc-1 ? – slope is the derivative

$\frac{d}{dt}[F(t)] = \frac{d}{dt}[1-e^{-\lambda t}] = 0 - -\lambda e^{-\lambda t} = \frac{\lambda}{e^{\lambda t}} \ge 0$ ✔ (4) — Chain Rule

slope = 0; positive slope; negative slope

PDF – Probability Density Function

$f(t) := \frac{d}{dt}[F(t)] = \frac{d}{dt}[1-e^{-\lambda t}] = 0 - -\lambda e^{-\lambda t} = \lambda e^{-\lambda t}$

PDF: answers the question: how dense is the probability in a certain region? how quickly is the CDF collecting probabilities?

exponential decay function

let $\lambda=2$: $f(0.1) = 2e^{-2\cdot 0.1} \approx 1.63$; $f(1) = 2e^{-2\cdot 1} \approx 0.27$; $p(1) = 0$; $p(0.1) = 0$

$p(1) = P(T=1) = \int_1^1 f(t)\,dt = 0$

$f(0.1) > 1$! Possible – Yes... PDF is NOT a probability

PDF is a completely abstract metric that is good for 2 things:
1. Integrating to find the probabilities of a region via the Fund Theorem of Calc
2. Compare two points relative likelihoods

$\frac{f(0.1)}{f(1)} = \frac{1.63}{0.27} \approx 6$ meaning the realizations near 0.1 are 6 times more likely then realizations near 1 (e.g. your geometric stopping time is to occur 6 times more likely at 0.1 than 1)

why?

$\lim_{\varepsilon\to0}\frac{P(T\in[0.1, 0.1+\varepsilon])}{P(T\in[1, 1+\varepsilon])} = \lim_{\varepsilon\to0}\frac{F(0.1+\varepsilon)-F(0.1)}{F(1+\varepsilon)-F(1)}$ $|\cdot \frac{1/\varepsilon}{1/\varepsilon}$ — ε(epsilon) tiny bit

$= \lim_{\varepsilon\to0}\frac{\frac{F(0.1+\varepsilon)-F(0.1)}{\varepsilon}}{\frac{F(1+\varepsilon)-F(1)}{\varepsilon}} = \frac{\lim \frac{F(0.1+\varepsilon)-F(0.1)}{\varepsilon}}{\lim \frac{F(1+\varepsilon)-F(1)}{\varepsilon}}$ — $F'(0.1) = f(0.1)$, $F'(1) = f(1)$

$= \frac{f(0.1)}{f(1)}$

PDF is a derivative of CDF $f(t) := \frac{d}{dt}[F(t)]$, $F'(t) = f(t)$

$g'(x) = \lim_{h\to0}\frac{g(x+t)-g(x)}{h}$ remember calc 1?

what is the prob. of success occurring somewhere between −∞ and +∞

$P(T\in[-\infty,\infty]) = 1$ why? $= \int_{-\infty}^{\infty} f(t)\,dt = 1$ T has to realize somehow !

$P(T\in Supp[T]) = 1 = \int_{t\in Supp[t]} f(t)\,dt = 1$

PDF propensity a la $\sum_{x\in Supp[X]} p(x) = 1$ discrete r.v.

what is the prob. of success occurring at the exactly 3rd try

$P(T=3) = p(3) = 0$ just proved — Def: PMF p(x) P(X=x)

$P(T=4.9) = p(4.9) = 0$

$P(T=2.3876) = p(2.3876) = 0$

$P(T=300000) = p(300000) = 0$

??? how can a probability of everything be 0 ? paradox ?

discrete vs continuous

$P(T=3) = P(T=3.00000000...) = 0$ 0s never stop – this contains infinite information

but if we round to the nearest millisecond...

$P(T=3.000) = P(T\in[2.99950..., 3.000499...])$

$= P(T\le3.00049...) - P(T\le2.99950...)$ — Def: CDF F(x) P(X≤x)

$F(3.0005) - F(2.9995) > 0$ we got some positive value out of this

$P(T\in[a,b]) = \int_a^b f(t)\,dt = F(b) - F(a)$ Fundamental Theorem of Calculus

where did this f(t) come from? $f(t) := \frac{d}{dt}[F(t)]$ what does this mysterious derivative of CDF represent?

Does Continuity exist?

Is Time Discrete or Continuous? It is a big open question in Physics. We do not know. It goes back to whether quantum gravity is real or not (Quantum Theory).

if quantum gravity is real:

plank length $1.62\cdot10^{-35}$m in comparison electrons diameter = $2.82\cdot10^{-15}$m

at plank length both positions d_1 and d_2 are at the same exact place, we can not distinguish the difference $d_1 = d_2$ crazy ! analogous to saying 2=5

if something was here then it would generate mass equal to a black hole capable to swallowing the entire known universe

light — photon (under plank length we can NOT tell the exact position of the photon)

time quantity where you can not tell exactly where the photon is → plank time $5.3\cdot10^{-44}$s

light speed $3\cdot10^8$ m/s

if gravity is quantized, then time is discrete , which means there is no such thing as Continuous r.v. 🤔

Support of Cont. r.v.

Thus squeezing infinitely many Bernoullis into a single second broke our PMF, which means that our Geometric experiment is NOT a discrete r.v. . However, it does have a CDF. If you have a CDF , you have a r.v. . It is not like any random variable we have seen before, it is a new kind of a r.v. . T is a **Continuous r.v.**

$Supp[T] = (0,\infty) = [0,\infty)$ it is uncountably close to 0

$|Supp[T]| = |\mathcal{R}| > |\mathcal{N}|$ size of a discrete r.v.

$\mathfrak{c} > \aleph_0$ — uncountable infinity, countable infinity

Properties of Continuous r.v.

1. $|Supp[X]| = |\mathcal{R}|$ continuum
2. Has valid CDF F(x) no jumps
3. PMF $p(x):=P(X=x)$ does not exist
4. PDF exists $f(x)\ge0$ and $\int_{Supp[X]} f(x)\,dx = 1$

Def. of Identical Distribution for Cont. r.v.s

X_1, X_2 Cont. r.v. and $X_1 \stackrel{d}{=} X_2$ if $f_1(x) = f_2(x)$ → PDFs are the same; if $F_1(x) = F_2(x)$ → CDFs are the same; no PMFs

PDF – probability densities

lets discretize this cont. r.v. by taking 'Midpoint Rule' approximations 'backwards'

rectangle areas are probabilities

we got an approximation of a PDF in form of a PMF approximated continuous r.v. is a discrete r.v.

increasing the resolution gives us a better approximation, but no matter how good the resolution is, it is still an approximation

remember? Riemann sums

allowing the separation between each value to approach 0 will allow us to calculate the exact area which is the probability

area = 1

$E[X] = \sum x\,p(x)$ → $E[X] = \int x\,f(x)\,dx$

Recall... Also the following is true: $E[g(x)] = \sum g(x)\,p(x)$; $E[g(x)] = \int g(x)\,f(x)\,dx$

$Var[X] = E[(X-\mu)^2]$ → $Var[X] = \int (x-\mu)^2 f(x)\,dx$

same as before

$E[aX+c] = aE[X]+c$

$Var[aX+c] = a^2\sigma^2$

$SE[aX+c] = |a|\,SE[X] = |a|\,\sigma$

$E[\Sigma X_i] = \Sigma E[X_i] = n\,E[X]$ if identically distributed

$Var[\Sigma X_i] = \Sigma Var[X_i] = n\,Var[X]$ — if independent; if iid

$Var[Z] = E[Z^2] - \mu^2$

Exponential r.v.

thus we took a Geometric and drove an infinite amount of iid Bernoullis into each second. It turns out that by doing that we arrived at a very famous Cont. r.v. called **Exponential r.v.**

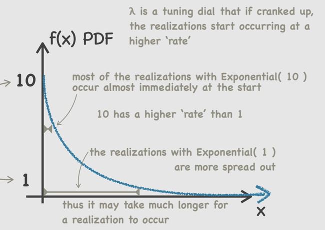
$\lambda = \mathcal{N}$; $\lambda = n\cdot p$ i.e. ∞#of_trials · prob.

$X \sim Exponential(\lambda) := \lambda e^{-\lambda x}$ — PDF f(x) NOT PMF p(x)

$Supp[X] = (0,\infty)$

Parameter Space $\lambda \in (0,\infty)$ 'rate'

$E[X] = \frac{1}{\lambda}$ $Var[X] = \frac{1}{\lambda^2}$ $SE[X] = \frac{1}{\lambda}$



λ↑ more experiments , higher probability; λ↓ less experiments , lesser probability

λ is a tuning dial that if cranked up, the realizations start occurring at a higher 'rate'

Geometric is called 'memoryless' ... Is Exponential also 'memoryless' ?

Exponential has a **memorylessness** property

$P(X>a+b \mid X>b) = P(X>a)$

$P(X>a) = e^{-\lambda a}$ ✔

$\frac{P(X>a+b \,\&\, X>b)}{P(X>b)} = \frac{P(X>a+b)}{P(X>b)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = \frac{e^{-\lambda a}e^{-\lambda b}}{e^{-\lambda b}} = e^{-\lambda a}$

$P(X\le x) = F(x) = 1-e^{-\lambda x}$ (CDF) ← use CDF to measure probability

$P(X>x) = 1-F(x) = e^{-\lambda x}$

$P(X>x) = 1 - P(X\le x)$ by Complement Rule $P(A^c) = 1 - P(A)$

$E[X] = \int_0^\infty x\,f(x)\,dx = \int_0^\infty x\,\lambda e^{-\lambda x}\,dx$ | pull the const. through — $Supp[X] = (0,\infty)$

$= \lambda\int_0^\infty x\,e^{-\lambda x}\,dx$

integration by parts $\int u\,dv = uv - \int v\,du$

let $u=x$, $dv=e^{-\lambda x}dx$ → $du=dx$ → $v=\frac{1}{-\lambda}e^{-\lambda x}$

→ $\int v\,du = \int\frac{1}{-\lambda}e^{-\lambda x}dx = \frac{1}{\lambda^2}e^{-\lambda x}$

$= \lambda\left[-\frac{1}{\lambda}x\,e^{-\lambda x} - \frac{1}{\lambda^2}e^{-\lambda x}\right]_0^\infty$

$= -\left(\left(\lim_{x\to\infty} xe^{-\lambda x} + \lim_{x\to\infty}\frac{1}{\lambda}e^{-\lambda x}\right) - \left(0\cdot e^{-\lambda\cdot0} + \frac{1}{\lambda}e^{-\lambda\cdot0}\right)\right)$

$= -((0+0) - (0+\frac{1}{\lambda}))$

$E[X] = \frac{1}{\lambda}$

also remember our Geometric journey? $X\sim Geometric(p)$

$E[X] = \frac{1}{p}$ if n=1 → p=λ; $= \frac{1}{np}$; $= \frac{1}{\lambda}$ so makes sense !

alternative way: $\int u\,v'\,dx = u\,v - \int u'\,v\,dx$; $u = x$, $u' = 1$, $v' = e^{-\lambda x}$, $v = \frac{1}{-\lambda}e^{-\lambda x}$

$= \lambda\left[x\frac{1}{-\lambda}e^{-\lambda x} - \int_0^\infty 1\,\frac{1}{-\lambda}e^{-\lambda x}dx\right]$

$= \lambda\left[x\frac{1}{-\lambda}e^{-\lambda x} - \frac{1}{-\lambda}\frac{1}{-\lambda}e^{-\lambda x}\right]_0^\infty$

	single stop	multi stop
discrete	Geometric	NegativeBinomial
continuous	Exponential	Erlang

Var[X] is derived also using the integration by parts – hw

Standard Normal r.v.

$Z\sim N(0,1) := \frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}$ — PDF f(x) = CDF' (CDF is a mystery)

'Standard Normal r.v.' aka 'Bell Curve' aka 'Gaussian r.v.'

is this a PDF ? → continuous r.v.
1. $f(x) \ge 0$ (slope of CDF) yes b/c $e^{-\frac{x^2}{2}} = +$ $x\in\mathcal{R}$
2. $\int_{x\in Supp[X]} f(x)\,dx = 1$ ← proof

$SE[Z] = \sigma = 1$ $Var[Z] = \sigma^2 = 1$ $E[Z] = \mu = 0$ $Supp[Z] = \mathcal{R}$

- find σ^2

$Var[X] = E[X^2] - \mu^2 = \int_{x\in Supp[X]}(x-\mu)^2 f(x)\,dx$ — $\mu=0$

$= \int_{\mathcal{R}} x^2\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}dx$ — $Supp[X]=\mathcal{R}$

= integration by parts ...

$= 1 = Var[X] = \sigma^2$

$SE[X] = \sqrt{\sigma^2} = \sqrt{1} = 1$

- find μ

$E[X] = \int_{x=-\infty}^{x=\infty} x\,f(x)\,dx = \int_{x=-\infty}^{x=\infty} x\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}dx$ | pull the const. through

$= \frac{1}{\sqrt{(2\pi)}}\int_{x=-\infty}^{x=\infty} x\,e^{-\frac{x^2}{2}}dx$

let $u=\frac{x^2}{2}$, du=x dx; $\int e^{-u}du = -e^{-u} = -e^{-\frac{x^2}{2}}$

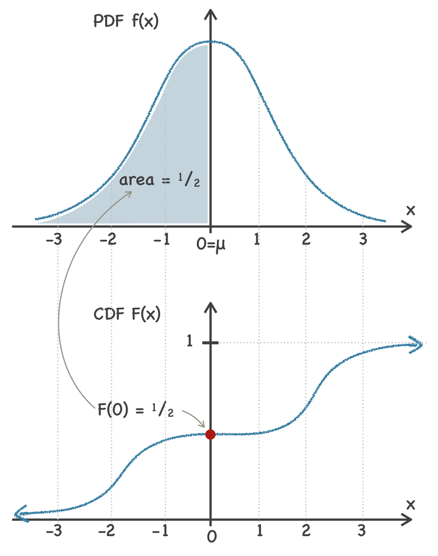
$= \frac{1}{\sqrt{(2\pi)}}\left[-e^{-\frac{x^2}{2}}\right]_{x=-\infty}^{x=\infty} = \frac{1}{\sqrt{(2\pi)}}\left[\lim_{x\to-\infty}e^{-\frac{x^2}{2}} - \lim_{x\to\infty}e^{-\frac{x^2}{2}}\right] = 0 = \mu$

- find CDF

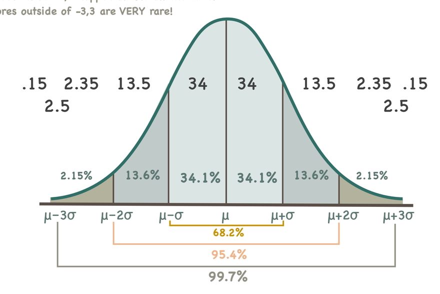
$F(x) = \int f(x)\,dx + C = \int\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}dx + C$

Risch Algorithm gives you the rules what f(x) has to look like in order to be anti-differentiable

it has been proven that it is impossible to find this antiderivative ! – NO CLOSED FORM; got to use a computer to do this



Z-scores inside -1,1 happen often!
Z-scores inside -2,2 happen very often!
Z-scores inside -3,3 happen almost all the time!
Z-scores outside of -3,3 are VERY rare!



'68-95-997 rule'

$\int_{-1}^{1}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}dx$ $\int_{-2}^{2}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}dx$ $\int_{-3}^{3}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}dx$

	PMF	PDF	CDF	E[X]	Var[X]	| Supp[X] |	Quantile[X]
Discrete r.v.	p(x)=[0,1]; $\sum_{x\in Supp[X]} p(x)=1$	no PDF	Yes (not always in closed form)	$\sum_{x\in Supp[X]} x\,p(x)$	$\sum_{x\in Supp[X]}(x-\mu)^2 p(x)$	≤ |𝒩|	min {x: F(x) ≥ p}
Continuous r.v.	no PMF	$f(x)\ge0$; $\int_{x\in Supp[X]} f(x)\,dx = 1$ 'integrates to unity'; $f(x) = d/dx$ [CDF] (b/c CDF is monotonically increasing)	Yes	$\int_{x\in Supp[X]} x\,f(x)\,dx$	$\int_{x\in Supp[X]}(x-\mu)^2 f(x)\,dx$	= |ℛ| > |𝒩|	x s.t. F(x) = p

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68-95-997 rule

$P(Z[a,b]) = F(b) - F(a)$

$P(Z\in[-1,1]) = F(1) - F(-1) \approx .68$
$P(Z\in[-2,2]) = F(2) - F(-2) \approx .95$
$P(Z\in[-3,3]) = F(3) - F(-3) \approx .997$

'68-95-997 rule' aka '3 sigma rule' aka 'empirical rule'

the amount of probability between -3SE and 3SE is .997 or 99.7%

let $X\sim Exponential(\lambda)$, Y=2X how is this distributed ?

a clever trick – look at the CDF of Y

$F_Y(x) = P(Y\le x) = P(2X\le x) = P(X\le\frac{x}{2}) = F_X(\frac{x}{2})$ now we can use the CDF of Exponential X

$= 1 - e^{-\lambda\frac{x}{2}} = 1 - e^{-\frac{\lambda}{2}x} = 1 - e^{-\lambda' x} = F_Y(x)$ — Exponential CDF $F(x) = 1-e^{-\lambda x}$

$Y\sim Exponential(\frac{\lambda}{2})$ — if $\lambda' = \frac{\lambda}{2}$ — $Y\sim Exponential(\lambda')$ s.t.

let $X\sim Uniform(a,b)$, Y=cX+d how is this distributed ?

self-note: why the heights are different? b/c by the Def of a valid PDF the areas of the rectangles formed under both intervals must be equal to 1

multiplication stretches, addition shifts

$F_Y(x) = P(Y\le x) = P(cX\le x-d) = P(X\le\frac{x-d}{c}) = F_X(\frac{x-d}{c})$

$= \frac{(\frac{x-d}{c})-a}{b-a}\cdot\frac{c}{c} = \frac{x-d-ac}{cb-ca} = \frac{x-(ca+d)}{cb-ca} = \frac{x-(ca+d)}{bc+d-ca-d} = \frac{x-(ac+d)}{(bc+d)-(ac+d)} = \frac{x-a'}{b'-a'} = F_Y(x)$

let $a' = ac+d$, $b' = bc+d$ — Uniform CDF $F(x) = \frac{x-a}{b-a}$

$Y\sim Uniform(ac+d, bc+d)$ — $Y\sim Uniform(a',b')$

General Normal r.v.

$Z\sim N(0,1)$ Let $X=\mu+\sigma Z$ how is this distributed ?

we know this $\frac{1}{\sqrt{(2\pi)}}e^{-\frac{x^2}{2}}$; we want this μ=10, σ=1/2

self-note: what did we do here? – we de-standardized the Standard Normal. We took the Standard Normal's σ and multiplied it by half and then we took the Standard Normal's μ and added 10 to it. The resulting graph is also a Normal distribution but of a different kind.

$F_X(x) = P(X\le x) = P(\mu+\sigma Z\le x) = P(\sigma Z\le x-\mu) = P(Z\le\frac{x-\mu}{\sigma}) = F_Z(\frac{x-\mu}{\sigma})$

unknown since $F_Z(x)$ ('bell curve' CDF) is unknown

but...

$f_X(x) = d/dx[F_X(x)] = d/dx[F_Z(\frac{x-\mu}{\sigma})]$

let $u=\frac{x-\mu}{\sigma}$, $\frac{du}{dx}=\frac{1}{\sigma}$, dx=σdu

$= \frac{d}{du}\cdot\frac{du}{dx}[F_Z(u)] = \frac{d}{du}\cdot\frac{1}{\sigma}[F_Z(u)] = \frac{1}{\sigma}f_Z(u) = \frac{1}{\sigma}\frac{1}{\sqrt{(2\pi)}}e^{-\frac{u^2}{2}} = \frac{1}{\sqrt{(2\pi\sigma^2)}}e^{-\frac{u^2}{2}} = \frac{1}{\sqrt{(2\pi\sigma^2)}}e^{-\frac{(\frac{x-\mu}{\sigma})^2}{2}}$

de-standardization of the Standard Normal

$X\sim N(\mu,\sigma^2) = f_X(x) = \frac{1}{\sqrt{(2\pi\sigma^2)}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ 'General Normal Distribution' PDF

Param Space: $\mu\in\mathcal{R}$ we can center it anywhere; $\sigma^2\in(0,\infty)$ squeeze / stretch/flatten

μ=0 , σ=1 special case 'Standard Normal Distribution'

$Supp[X] = \mathcal{R}$

we can easily find E[X] and SE[X] by plugging in for X=μ+σZ

$E[X] = E[\mu+\sigma Z] = \mu$

$SE[X] = SE[\mu+\sigma Z] = \sigma$

gives a very flexible 'bell curve' (squeeze/stretchable σ and centered dynamically) μ

Uniform Cont. r.v.

discrete Uniform r.v. $X\sim Uniform(\{1,7,28\})$

continuous Uniform r.v. $X\sim Uniform(a,b) := \frac{1}{b-a}$ — PDF f(x) = CDF' F'(x)

$Supp[X] = [a,b]$ any value between a & b

Parameter Space $a\in\mathcal{R}$, $b\in\mathcal{R}$, a<b

$Var[X] = \frac{(b-a)^2}{12}$ $E[X] = \frac{b+a}{2}$ CDF $P(X\le x) = F(x) = \frac{x-a}{b-a}$

Look at the relationship between the PDF & the CDF. (function and its derivative) On the CDF graph between a & b the slope is rising at the constant rate which is unequivocally depicted on the graph of the PDF.

- find μ

$E[X] = \int_a^b x\,f(x)\,dx = \int_a^b x\frac{1}{b-a}dx$ | pull the const. through — Supp[X] = [a,b]

$= \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$

$\frac{1}{b-a}(\frac{b^2}{2}-\frac{a^2}{2}) = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} =$

- find CDF

$F(x) = \int f(x)\,dx + C$ — BUT 'C' can NOT just be anything b/c in our scenario the CDF has strict limits for what it can be

$= \int\frac{1}{b-a}dx + C = \frac{1}{b-a}\int 1\,dx + C = \frac{1}{b-a}x + C = \frac{x}{b-a} + C$

Find C. Lets use what we know: F(a) = 0, F(b) = 1; $\frac{a}{b-a} + C = 0$, $C = -\frac{a}{b-a}$

$= \frac{x}{b-a} + -\frac{a}{b-a} = \frac{x-a}{b-a}$

- find σ²

expected long-run average squared error

$\sigma^2 = Var[X] = \int_{Supp[X]}(x-\mu)^2 f(x)\,dx = E[X^2] - \mu^2$

$= \int_a^b x^2\frac{1}{b-a}dx - (\frac{b+a}{2})^2 = \frac{1}{b-a}\int_a^b x^2dx - (\frac{b+a}{2})^2 = \frac{1}{b-a}\frac{x^3}{3}\Big|_a^b - (\frac{b+a}{2})^2 = \frac{x^3}{3(b-a)}\Big|_a^b - (\frac{b+a}{2})^2$

$= \frac{b^3}{3(b-a)} - \frac{a^3}{3(b-a)} - (\frac{b+a}{2})^2 = \frac{b^3-a^3}{3(b-a)} - \frac{(b+a)^2}{4} = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$

$= \frac{(4b^2+4ab+4a^2)-(3b^2+6ab+3a^2)}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$

$(b^3-a^3) \div (b-a) = b^2+ab+a^2$

→ $SE[X] = \sqrt{\sigma^2} = \sqrt{\frac{(b-a)^2}{12}} = \frac{b-a}{\sqrt{12}}$

$SE[X] = \frac{b-a}{\sqrt{12}}$

$X\sim Uniform(0,1) = 1$ (PDF) — if a=0, b=1

Standard Uniform

pick a number between 0 and 1 at random – it is the algorithmic foundation for all randomness inside computers , every programing language has a way of pulling a realization out of this.

Java – Math.random() returns a number from zero to one.