LECTURE - 5 09/12/2017

* Consider the Pattern :-

"Pascal's Δ"

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad \underset{\binom{3}{1}}{\overset{3}{\shortparallel}} \quad \underset{\binom{3}{2}}{\overset{3}{\shortparallel}} \quad 1$$
$$\underset{\binom{4}{0}}{\overset{1}{\shortparallel}} \quad \underset{\binom{4}{1}}{\overset{4}{\shortparallel}} \quad \underset{\binom{4}{2}}{\overset{6}{\shortparallel}} \quad \underset{\binom{4}{3}}{\overset{4}{\shortparallel}} \quad \underset{\binom{4}{4}}{\overset{1}{\shortparallel}}$$

Recurrance Relation :

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$\therefore$ This is true $\forall n \in \mathbb{N}$, $k \in \{0, \ldots, n-1\}$

- $$\frac{n!}{k!(n-k)!} \stackrel{?}{=} \frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} + \frac{(k-1)!}{(n-1-k) = (n-k-1)!\,k!}$$

Multiplying RHS with $\frac{n}{n}$:

$$= \frac{n!}{n}\left(\frac{1}{(n-k)!\,(k-1)!}\cdot\frac{k}{k} + \frac{1}{(n-k-1)!\,k!}\cdot\frac{n-k}{n-k}\right)$$

$$= \frac{n!}{n}\left(\frac{k}{(n-k)!\,k!} + \frac{n-k}{(n-k)!\,k!}\right)$$

$$= \binom{n}{k}$$ → Pascal's / Identity / Rule / Formula Theorem.

- Let $S = \{ \spadesuit, \clubsuit, \diamondsuit, \heartsuit \}$ called "suit"
  Let $R = \{ 2, 3, \ldots, 10, J, Q, K, A \}$ called "rank"
  Let $D = S \times R$ called "deck" of cards.

  $|S| = 4$, $|R| = 13$, $|D| = 52$

Consider the "game" when you are given ("delt") 5 cards (all equally likely) with / without replacement such that order doesnot matter. These 5 cards are called "hand".

- $P(\text{Royal Flush}) = \dfrac{|A|}{|\Omega|}$

Ex: 10, J, Q, K, A
all of same suit

$= \dfrac{4}{\binom{52}{5}}$ → possible hands

$= \dfrac{4}{2598960}$

- $P(\text{4 of a kind}) = \dfrac{|A|}{|\Omega|}$

Ex: 7777K

$= \dfrac{\binom{13}{1}\underbrace{\binom{12}{1}\binom{4}{1}}_{\text{other card } \binom{48}{1}}}{\binom{52}{5}}$

→ no. of 4 of a kind

- $P(\text{straight flush}) = \dfrac{|A|}{|\Omega|}$

Ex: all same suits
A 2 3 4 5
2 3 4 5 6
3 4 5 6 7
- - -
9 10 J Q K
10 J Q K A

$= \dfrac{\binom{9}{1}\binom{4}{1}}{\binom{52}{5}}$ → the suit

→ beginning number

- $P(\text{full house}) = \dfrac{|A|}{|\Omega|}$

Ex: $\underbrace{777}_{\text{3 of the same rank}}\underbrace{QQ}_{\text{2 of the same rank}}$

$= \dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$

3 suits

suits

→ rank of the 3 of a kind

→ rank of the 2 of a kind

- $P(\text{flush}) = \dfrac{\binom{4}{1}\binom{13}{5} - \binom{9}{1}\binom{4}{1} - 4}{\binom{52}{5}}$

Ex: all same suit
but not straight

- $P(\text{straight}) = \dfrac{\binom{10}{1}\binom{4}{1}^5 - \binom{9}{1}\binom{4}{1} - 4}{\binom{52}{5}}$

- $P(\text{3 of a kind}) = \dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$

Ex: 777Q9

7♡ 7♢ 7♣

7♡ 7♢ 7♠

7♢ 7♣ 7♠

7♡ 7♣ 7♠

- $P(\text{2 pair}) = \dfrac{\binom{13}{2}\binom{4}{2}^2 \overbrace{\binom{11}{1}\binom{4}{1}}^{5^{th}\text{ card}}}{\binom{52}{2}}$

Ex: 77QQ3

** $\binom{13}{1}\binom{12}{1} \stackrel{?}{=} \binom{13}{2}$ } for "P" order matters. for "C" order doesnot matter

$13P2 \neq \binom{13}{2}$

- Why in P(full house) we wrote $\binom{13}{1}$ and why in P(flush) we wrote $\binom{13}{5}$?

77QQ $\stackrel{?}{=}$ QQ77

777QQ $\neq$ QQQ77

x

* **Revisit the "Working Def. of Prob.":-**

$\therefore P(A) = \dfrac{|A|}{|\Omega|}$ "the classic def." in us through the 1800's.

Consider the random experiment of spinning:

$A = \{R\}$

$\therefore P(A) \neq \dfrac{|\{R\}|}{|\Omega|} = \boxed{\dfrac{1}{3}}$

** There is a hidden assumption for Ω:

$$\forall \omega \in \Omega . \; P(\{\omega\}) = \frac{1}{|\Omega|}$$

This is "equally likely" outcome.

Ex: flipping coins, rolling die, seating people, drawing cards.

* **New Def. of Prob.:-**

(1) Limiting Frequency Def.:

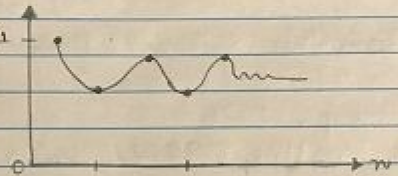
indicator func.: First define

$$1_{\omega \in A} := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$$\therefore P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{\infty} 1_{\omega_i \in A}}{n}$$

$$= \frac{\#\{\omega_i \in A\}}{n}$$

Von Mises, 1928 : as n gets larger $P(A)$ becomes more stable.



In 1654, Chevalier de Mere who wrote a letter to Pascal and Fermat, and said "I think $P(\{\geq 1 \text{ double} - 6 \text{ in } 24 \text{ rolls of two die}\}) < 1/2$

$\therefore$ True Prob. $= 0.4914$

- Problems :-

(1) Requires experimentation, infinite experiments which is impossible.

$\Rightarrow$ We can only know an approximation which is always wrong and could be very wrong.

(ii) not general

$P(\text{OJ Simpson guilty}) \neq P(\text{Irma hits Miami})$