10 cards 4R, 6B

$P(2R \text{ in } 3 \text{ cards}) = \dfrac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$

$P(xR \text{ in } 3 \text{ cards}) = \dfrac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$

$P(xR \text{ in } n) = \dfrac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$

K reds, 10 − K blues

$P(xR \text{ in } n) = \dfrac{\binom{K}{x}\binom{10-K}{n-x}}{\binom{10}{n}}$

$X \sim \text{HyperGeometric}(n, K, N) := \underbrace{\dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}}_{p(x)}$

r.v. distributed

$X \sim \text{Bern}(p) := \overbrace{p^x(1-p)^{1-x}}^{p(x)\text{ PMF}}$

$\text{Supp}[X] = \{0,1\}$ $p \in (0,1)$ param. space

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100 students, 53 are female, pick 8 at random.
Provide X, a random variable model which counts the # of females

$X \sim \text{Hypergeometric}(8, 53, 100) = \dfrac{\binom{53}{x}\binom{47}{8-x}}{\binom{100}{8}} = p(x)$

Parameter Space of the Hypergeometric random variable

$N = 0$? No!

$n = 0$? No!

$N = 1 \Rightarrow K \leftarrow \{0,1\} \Rightarrow n = 1$

if $K = 0$ Degenerate! $X \sim \text{Deg}(0)$

if $K = 1$ Degenerate! $X \sim \text{Deg}(1)$

$N = 2$ $n \neq 0$ $n = 2$? $X \sim \text{Deg}(K)$

$n = 1$? $K = 0 \Rightarrow X \sim \text{Deg}(0)$

$K = 2 \Rightarrow X \sim \text{Deg}(1)$

$K = 1 \Rightarrow X \sim \text{Hyper}(1,1,2)$

$= \dfrac{\binom{1}{x}\binom{1}{1-x}}{\binom{2}{1}} = \text{Bern}\left(\frac{1}{2}\right)$

$p(0) = \dfrac{1*1}{2} = \dfrac{1}{2}$

$p(1) = \dfrac{1*1}{2} = \dfrac{1}{2}$

$$X \sim Hyper(1, K, N) = \frac{\binom{K}{x}\binom{N-K}{1-x}}{\underbrace{\binom{N}{1}}_{N}} = Bern\left(\frac{K}{N}\right)$$

$Supp[X] = \{0, 1\}$

$$p(0) = \frac{\binom{K}{0}\binom{N-K}{1}}{N} = \frac{N-K}{N} = 1 - \frac{K}{N} = \text{"}1-p\text{"}$$

$$p(1) = \frac{\binom{K}{1}\binom{N-K}{0}}{N} = \frac{K}{N} = \text{"}p\text{"}$$

$N \in \{2, 3, 4, \ldots\}$

$K \in \{1, 2, \ldots, N-1\}$

$n \in \{1, 2, \ldots, N-1\}$

Consider the following cases

Self note

$n-(N-K) < Supp[X] < n$

a) $X \sim Hyper(2, 4, 10)$, $Supp[X] = \{0, 1, 2\}$

"there's a bag of 10 marbles
4 are considered "success"
you pull out 2
Of the 2 you pulled out how
many successess can you get?

Question: counting the number of successes

b) $X \sim Hyper(5, 4, 10)$, $Supp[X] = \{0, 1, 2, 3, 4\}$

c) $X \sim Hyper(8, 4, 10)$, $Supp[X] = \{2, 3, 4\}$ (0 & 1 impossible, only 6 "failures")

d) $X \sim Hyper(5, 7, 10)$, $Supp[X] = \{2, 3, 4, 5\}$

Generalize cases a-d

a) $n < K$, $n < N-K$

$Supp[X] = \{0, 1, \ldots, n\}$

b) $n \geq K$, $n < N-K$

$Supp[X] = \{0, 1, \ldots, K\}$

c) $n \geq K$, $n \geq N-K$

$Supp[X] = \{n-(N-K), \ldots, K\}$

d) $n < K$, $n \geq N-K$

$Supp[X] = \{n-(N-K), \ldots, n\}$

	$n < K$	$n \geq K$
$n < N-K$	$\{0, \dots, n\}$	$\{0, \dots, K\}$
$n \geq N-K$	$\{n-(N-K), \dots, n\}$	$\{n-(N-K), \dots, K\}$

$\Rightarrow Supp[X] =$

$= \{ \max\{0, n-(N-K)\}, \dots, \min\{n, K\} \}$

$$\sum_{x \in Supp[X]} \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} = 1$$

Equivalent parameterization

let $p = \frac{K}{N} \Rightarrow K = pN$

$p \in \{ \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N} \}$

$$X \sim Hyper(n, p, N) = \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

$p = 0.5 \quad n = 6, N = 100$

$$p(3) = ? = \frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = 0.3223$$

$N = 1000$

$$p(3) = \frac{\binom{500}{3}\binom{500}{3}}{\binom{1000}{6}} = 0.3134$$

$N = 10{,}000$

$$p(3) = \frac{\binom{5000}{3}\binom{5000}{3}}{\binom{10000}{6}} = 0.3126$$

$\lim_{N \to \infty} p(x) = \dots$ as N increases it becomes similiar to "without replacement" such that choosing permutation and combination doesn't matter.