<u>Problem Set 1</u>

<u>Probability</u>

a. Bishop 1.6

Two random variables are independent iff $cov(x,y)=0$.

$$cov(x,y) = E(xy) - E(x)E(y)$$

If x, y are independent:

$$E(xy) = E(x) \cdot E(y)$$

Therefore:

$$cov(x,y) = E(x)E(y) - E(x)E(y)$$
$$= 0 \qquad \blacksquare$$

b. HIV Test

(i). $P(\text{Positive} \mid \text{HIV}) = 72/75$
$= 0.96$ (96%)

(ii). $P(\text{Negative} \mid \text{HIV}) = 3/75$
$= 0.04$ (4%)

(iii). $P(\text{HIV} \mid \text{Positive}) = (P(\text{Positive} \mid \text{HIV}) \cdot P(\text{HIV})) / P(\text{Positive})$
$= (72/75 \cdot 12/100) / 84/158$
$= 0.22$ (22%)

$P(\text{HIV} \mid \text{Negative}) = (P(\text{Negative} \mid \text{HIV}) \cdot P(\text{HIV})) / P(\text{Negative})$
$= (3/75 \cdot 12/100) / 74/158$
$= 0.01$ (1%)

Bayes Theorem

a. Monty Hall problem

$P(A) = 0.20$
$P(B) = 0.20$
$P(C) = 0.40$
$P(D) = 0.20$

Figure out how to do this with Bayes

b. Cheating with Chin

$$P(\text{Cheating} \mid \text{Confronted}) = \frac{P(\text{Confronted} \mid \text{Cheating})}{P(\text{Confronted})} \cdot P(\text{Cheating})$$

$$\begin{aligned} P(\text{Confronted}) &= P(\text{Confronted} \mid \text{Cheating}) \cdot P(\text{Cheating}) + P(\text{Confronted} \mid !\text{Cheating}) \cdot P(!\text{Cheating}) \\ &= (0.90 \times 2/75) + (0.20 \times 73/75) \\ &= 0.2187 \end{aligned}$$

$$\therefore P(\text{Cheating} \mid \text{Confronted}) = \frac{0.90}{0.2187} \cdot 2/75 \approx 0.11 \ (10.97\%)$$

Linear Algebra

a. Bishop 2.22

If A is a symmetric matrix:

$$A \cdot A^{-1} = I$$

Taking the transpose of both sides:

$$A^T \cdot (A^{-1})^T = I^T$$
$$= I$$

Since A is symmetric $\rightarrow A^T = A$:

$$A \cdot (A^{-1})^T = I$$

Therefore:

$$(A^{-1})^T = A^{-1}$$

A^{-1} is symmetric $\blacksquare$

b. Eigenvalues and Eigenvectors

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 4 & -1 \\ -1 & -2-\lambda & 1 \\ 3 & 9 & 0-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 3-\lambda \begin{vmatrix} -1 & -2-\lambda \\ 3 & 9 \end{vmatrix} - 4 \begin{vmatrix} -1 & 1 \\ 3 & 0-\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & -2-\lambda \\ 3 & 9 \end{vmatrix}$$

$$\lambda = 2, -3$$

$\lambda = 2 : (A - 2I)x = 0$

$$\begin{bmatrix} 1 & 4 & -1 & 0 \\ -1 & -4 & 1 & 0 \\ 3 & 9 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$\lambda = 3 : (A + 3I)x = 0$

$$\begin{bmatrix} 6 & 4 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

A symmetric matrix is positive definite iff its eigenvalues are positive.

A is $\underbrace{\text{!symmetric}}_{A^T \neq A}$ and $\lambda = -3 < 0 \rightarrow$ A is not positive definite

Probability Distributions

a. Bishop 2.2

$$P(x=1|\mu) + P(x=-1|\mu) = \left(\frac{1+\mu}{2}\right) + \left(\frac{1-\mu}{2}\right)$$

$$= 1$$

$\rightarrow$ (2.261) is normalized

$$E(x) = \left(\frac{1+\mu}{2}\right) - \left(\frac{1-\mu}{2}\right)$$

$$= \mu$$

$$E(x^2) = \left(\frac{1+\mu}{2}\right) + \left(\frac{1-\mu}{2}\right)$$

$$= 1$$

$$\mathrm{var}(x) = E(x^2) - (E(x))^2$$

$$= 1-\mu^2$$

$$H(x) = -\sum_{x=-1}^{x=1} P(x|\mu) \ln P(x|\mu)$$

$$= -\left(1-\mu/2\right) \ln\left(1-\mu/2\right) - \left(1+\mu/2\right) \ln\left(1+\mu/2\right)$$

b. Bishop 2.10

$$\int \prod_{k=1}^{M} \mu_k^{\alpha_k - 1} \, d\mu = \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_M)}{\Gamma(\alpha_0)}$$

$$E(\mu_j) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_M)} \int \mu_j \prod_{k=1}^{M} \mu_k^{\alpha_k - 1} \, d\mu$$

$$= \dots$$

$$= \alpha_j / \alpha_m$$

$$\mathrm{Var}(\mu_j) = E(\mu_j^2) - (E(\mu_j))^2$$

$$= \frac{\alpha_j(\alpha_0 - \alpha_j)}{\alpha_0^2(\alpha_0 + 1)}$$

$$\mathrm{cov}(\mu_j, \mu_k) = E(\mu_j \mu_k) - E(\mu_j)E(\mu_k)$$ for $j \neq k$

$$= \frac{\alpha_j \alpha_k}{\alpha_0(\alpha_0 + 1)} - \frac{\alpha_j}{\alpha_0} \frac{\alpha_k}{\alpha_0}$$

$$= -\frac{\alpha_j \alpha_k}{\alpha_0^2(\alpha_0 + 1)}$$

c. Bishop 2.12

$$\int_a^b \frac{1}{b-a}\,dx = \frac{b-a}{b-a}$$
$$= 1$$
$\to$ normalized

$$E(x) = \int_a^b \frac{1}{b-a}\,x\,dx$$
$$= \left[\frac{x^2}{2(b-a)}\right]_a^b$$
$$= \frac{b^2-a^2}{2(b-a)}$$
$$= \frac{a+b}{2}$$

$$E(x^2) = (\ldots)$$
$$= \frac{a^2+ab+b^2}{3}$$

$$\mathrm{var}(x) = E(x^2) - (E(x))^2$$
$$= \frac{(b-a)^2}{12}$$

2. Bishop 2.15

$$H(x) = -\int \mathcal{N}(x|\mu,\Sigma) \ln \mathcal{N}(x|\mu,\Sigma)\, dx$$
$$= \int \mathcal{N}(x|\mu,\Sigma)\, 1/2 \left(D \ln(2\pi) + \ln|\Sigma| + (x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$
$$= 1/2 \left(D \ln(2\pi) + \ln|\Sigma| + \mathrm{Tr}[\Sigma^{-1}\Sigma]\right)$$
$$= 1/2 \left(D \ln(2\pi) + \ln|\Sigma| + D\right)$$