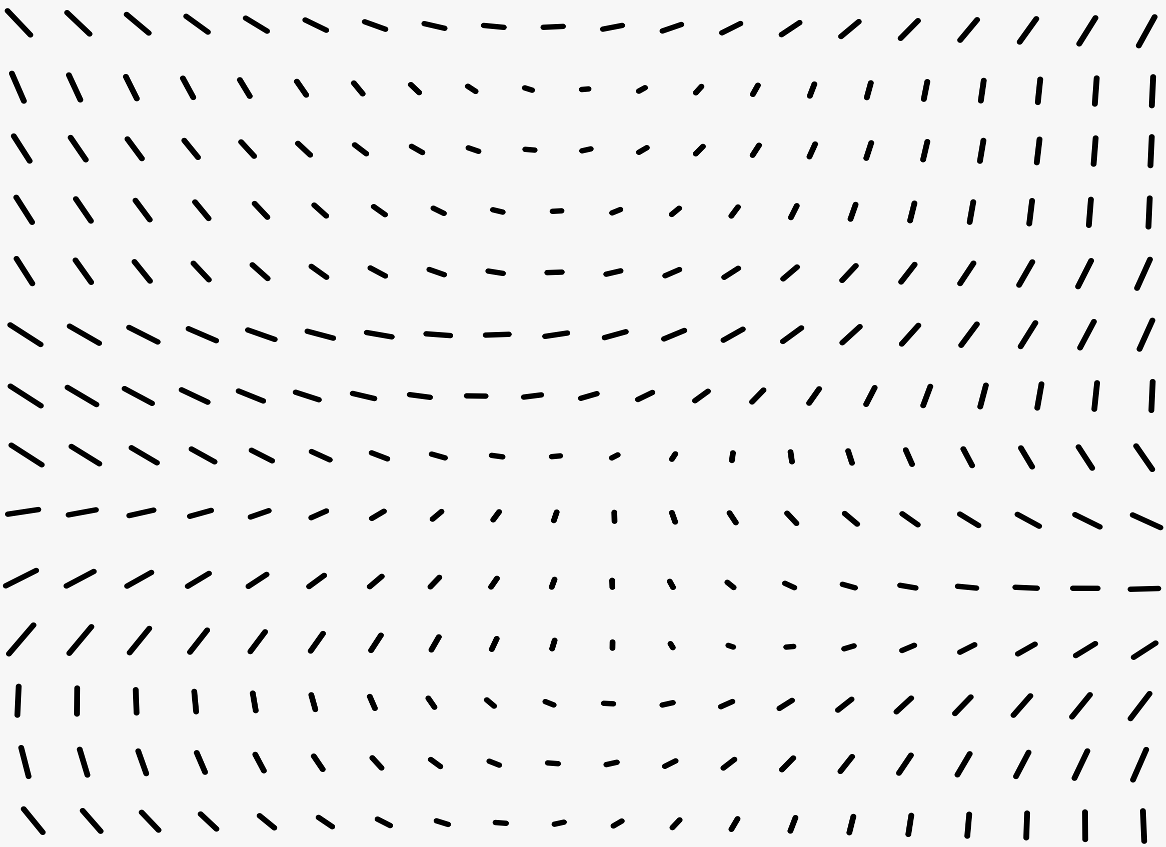


Bessel Functions

↳ Solutions of Bessel's differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

arbitrary complex number α
(called the order of the solution Bessel equations / functions)

→ Solutions do not have "closed form" solutions involving only elementary functions

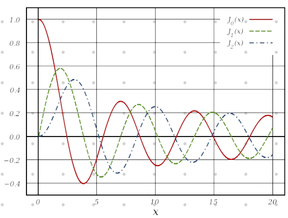
$\underline{\underline{\alpha}}$ integer ⟶ cylindrical Bessel functions

↳ appear in solution to Laplace's equation in cylindrical form

First Kind J_α ← order α

⟶ finite at the origin $x=0$ for integer or positive α
⟶ diverge as $x \to 0$ for negative, non-integer α

Bessel plots look like decaying sine/cosine functions ↳ proportionally to $x^{-0.5}$



(always well behaved)

plot for integer orders $\alpha = 0, 1, 2$

may be defined by series expansion around $x=0$:

$$\boxed{J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}}$$

gamma function

Bessel's integrals:

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i(x\sin\tau - n\tau)}\,d\tau \xrightarrow{\text{real}} J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\tau - x\sin\tau)\,d\tau$$

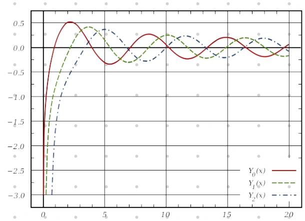
Second Kind Y_α (singular solution)

⟶ have a singularity at the origin $x=0$
↳ $Y_\alpha \to \infty$ as $x \to 0$

for non-integer α:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

(complex for $x < 0$)



plot of second kind for integer orders $\alpha = 0, 1, 2$

· for integer order n, function defined by taking limit as a non-integer $\alpha \to n$

$$\boxed{Y_n(x) = \lim_{\alpha \to n} Y_\alpha(x)}$$

Fourier Expansion / Series (watch 3b1b video on Fourier series)

we may write any periodic function as an infinite weighted sum of sinusoids

consider a real-valued function $s(x)$, integrable on an interval of length P, which is also the period of the function. The n^{th} harmonic frequency is given by: $\frac{n\pi}{P}$ n^{th} rotation frequency

each term in the sum:

$$c_n e^{\frac{i \cdot n 2\pi x}{P}} = \underbrace{\left(\begin{array}{c}\text{initial size + direction}\\ \text{of each rotating vector}\end{array}\right)}_{c_n} \exp\left[\underbrace{\left(\begin{array}{c}\text{rotation}\\ \text{frequency}\end{array}\right)}_{\frac{n\pi}{P} \equiv k} i x\right]$$

the Fourier series (in complex form) is given by:

$$\boxed{S_N(x) = \sum_{n=-N}^{+N} c_n e^{i \cdot 2\pi n x/P}}$$

and the coefficients are given by:

$$c_n = \begin{cases} A_0/2 = a_0/2 & n = 0 \\ \frac{A_n}{2} e^{-i\varphi_n} = \frac{1}{2}(a_n - i b_n) & n > 0 \\ c^*_{|n|} & n < 0 \end{cases}$$

we know $s(x)$ is real so

$$c_n = c^*_{-n} \leftarrow \text{complex conjugate}$$

$$\boxed{c_n = \frac{1}{P}\int_P s(x) \cdot e^{-i 2\pi n x/P} \, dx}$$

we also know that all coefficients will be real-valued, because the starting complex "vectors" lie in a one-dimentional line (real) in the complex (output) plane.

* need to figure out why the Fourier formulation of eq. 3.7 is missing a factor of 2 in the exponential

↓

domain is of size $2L$ $(-L \leq x \leq L)$
so the 2 cancels $\bar{w}$ the 2π

DISCRETE / FAST FOURIER TRASFORM (notes from §5.1 numerical methods book)

DISCRETE → see 3b1b video

N data points $\vec{y} = [\,y_0 \;\; y_1 \;\; \dots \;\; y_{N-1}\,]$ → evenly spaced in time, so $t_j = \tau j$ ← Sampling interval τ

where $\underbrace{j = 0, \dots, N-1}_{N \text{ points}}$

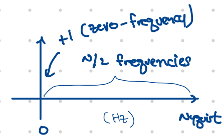
(spatial/time index)

DISCRETE TRANSFORM
$$\boxed{Y_k = \sum_{j=0}^{N-1} y_j \, e^{-2\pi i j k / N}}$$
(y_j: data points)

where $k = 0, \dots, N-1$
(frequency index)

(see p. 120 for coefficients)

each point of the transform has an associated frequency: $\boxed{f_k = \frac{k}{\tau N}}$



get a total of $N/2 + 1$ frequencies from transform

→ measure long time series to extract low frequencies
→ measure small τ for large frequencies

LOWEST → $f_1 = \frac{1}{\tau N} = \frac{1}{T}$ ← time series length

LARGEST → $f_{N-1} \approx \frac{1}{\tau}$

$\mathbb{R} \longrightarrow \mathbb{C}$

The Fourier transform, transforms a function

time space → frequency space

NOTE: Y_k is complex in frequency space.

ALIASING and NYQUIST

ALIASING → the effect when functions / signals of different frequencies become indistinguishable when sampled

due to the above, there is is maximum frequency for which we may resolve for a sampling interval τ

NYQUIST frequency → $f_{Ny} = \frac{1}{2\tau}$ ← the fourier transform may be truncated at this upper bound for accurate results

* one-half of sampling rate *

everything past the Nyquist frequency will be mirrored

SAMPLING RATE : $\frac{1}{\tau} \left[\frac{\text{samples}}{\text{sec}} \right] = [\text{sec}^{-1}]$ (for time series data)

NOTE:

SERIES	TRANSFORM
represents periodic function by a discrete sum of complex exponentials	represents a general non-periodic function by continuous superposition or integral of complex exponentials

FAST FOURIER TRANSFORM (FFT)

→ the number of operations for the discrete transform is $O(N^2)$ → $O(N)$ for one k → need to sum over all $k = 0, \dots, N-1$

FFT computes the transform in an efficient order such that the number of operations is reduced to $O(N\log_2 N)$ → most efficient for $N = 2^n$
(when the number of sampled points is a power of 2)

FFT - Based Differentiation (notes from Johnson, MIT 2011, writeup)

Consider a periodic function $y(x)$ w/ period L and write as Fourier series:

$$y(x) = \sum_{k=-\infty}^{\infty} Y_k e^{\frac{2\pi i}{L}kx} \quad \xrightarrow{\text{coefficients}} \quad Y_k = \frac{1}{L}\int_0^L e^{-\frac{2\pi i}{L}kx} y(x)\,dx$$

(continuous Fourier transform)

Differentiation is performed term by term in Fourier domain:

$$\frac{d}{dx}y(x) = y'(x) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{2\pi i}{L}k \cdot Y_k\right)}_{\substack{\text{simply multiplication of coefficients } Y_k \\ \text{by a } \frac{2\pi i}{L}k \text{ term}}} e^{\frac{2\pi i}{L}kx}$$

(note that is is not always valid, but is valid for a general $y(x)$ in PDE applications)

* approximate the Fourier series for Y_k by a discrete Fourier transform (or a fast Fourier transform)

the function $y(x)$ is replaced by N discrete samples $y_n = y\left(n\frac{L}{N}\right)$, $n \in \mathbb{Z}^+$

(n: sample number; N: # of samples)

$$Y_k = \frac{1}{N}\sum_{n=0}^{N-1} y_n e^{-\frac{2\pi i}{N}nk}$$

now, all we need to do is sample $y(x)$ to obtain y_n for a given number of n, N and the derivative may be computed

ZERO-PADDING Tecnique (notes from → https://www.bitweenie.com/listings/fft-zero-padding/)

great general resource
https://www.youtube.com/watch?v=NCzG-aK3FCs
(good overview of DFT and zero-padding) w/ examples

↪ adding zeros to the end of an input prior to computing transform

FFT Resolution

* these two are indentical w/ no zero-padding *

WAVEFORM FREQENCY RESOLUTION

minimum spacing between 2 frequencies that may be resolved

$$\Delta R = \frac{1}{T}$$

← time length of data signal

do not include zero padding time here! (only actual data signals)

FFT RESOLUTION

number of points in the spectrum
↪ directly proportional to # of points used in the FFT

$$\Delta R = \frac{f_s}{N_{fft}}$$

source and/or sample frequency

number of points in spectrum → increases w/ zero padding

how many Hz each DFT bin represents → decreases w/ zero-padding

FFT resolution increases when input fnction is zero-padded

the same effect is observed when input is sinc interpolated

(see zero-padding theorm)

NOTE: normalize with original N of the signal, not the number of points after zero padding

NOTE that when zero-padding an input, we are not adding any new information to the signal.
↪ therefore, we can increase the resolution of the FT but not the precision

resolution → # of frequencies resolved

not → be careful in interpreting the transform

* Essentially zero-padding is a computationally efficient method of interpolating a large number of points and mitigate against aliasing error and/or bring number of points to a power of 2 for FFT algorithm efficiency.

https://dsp.stackexchange.com/questions/741/why-should-i-zero-pad-a-signal-before-taking-the-fourier-transform

Trapezium rule (integration)

* notes from numerical methods book § 10.2 *

Consider the integral:

$$I = \int_a^b f(x)\,dx$$

evaluate $f(x)$ at few points → fit simple curve through points (eg. piecewise linear) → estimate area

1 2 3

1 Divide interval $[a,b]$ into $N-1$ intervals, define N points x_i:

$\rightarrow \quad f_i = f(x_i)$

2 Fit a simple linear piecewise function through the evaluated function points (see fig. 10.3, p.253)

3 True integral is estimated as the sum of the trapezoid areas T_i:

$$I \simeq I_T = T_1 + T_2 + \cdots T_{N-1}$$

$$\boxed{T_i = \frac{1}{2}(x_{i+1} - x_i)(f_{i+1} + f_i)}$$

single trapezoid area

note last area is indexed $N-1$, because there is one fewer panel than grid point

Equally spaced grid points → general formula simplifies

Spacing: $\boxed{h = \dfrac{b-a}{N-1}}$

$$\boxed{T_i = \frac{1}{2} h (f_{i+1} + f_i)}$$

single trap. area (simplified)

$$\frac{\text{domain length}}{\text{\# of panels}}$$

So: $x_i = a + \frac{1}{h}(i-1)$

trapezoid rule for evenly spaced points:

$$I_T(h) = \frac{1}{2} h f_1 + h f_2 + h f_3 + \ldots + h f_{N-1} + \frac{1}{2} h f_N$$

$$\boxed{I_T(h) = \frac{1}{2} h (f_1 + f_N) + h \sum_{i=2}^{N-1} f_i}$$

COEFFICIENTS

inter points h → each interior point appears in 2 trapezoids

exterior points $\frac{h}{2}$ → only appear in one trapezoid each

Periodic Trapezoidal Rule

→ the trapezoidal rule converges quickly for smooth periodic integrands

NOTES on AFM* paper

* THE NON-LOCAL AFM WATER WAVE METHOD FOR CYLINDRICAL GEOMETRY

The change of variables $(x,t) \mapsto (z,t)$; $z = x - ct$ is introduced to compute travelling wave solutions

constant wave speed, $c>0$

this is substituted into the Bernoulli eq'n, and many previous relations

The final travelling wave form is given by eq'n (2.18):

Bessel eq'n of first kind

Bessel of second kind

$$\int_{-L}^{L} kS\left[(1+S_z^2)(c^2-2\mathcal{F})\right]^{1/2}\Big(K_1(kb)I_1(kS) - I_1(kb)K_1(kS)\Big)e^{ikz}\,dz = 0, \qquad (2.18)$$

wave speed

α or $n=1$ (Bessel order)

where

$$\mathcal{F} \equiv \frac{\gamma\kappa}{\rho} - V - \mathcal{E}, \qquad \kappa = -\frac{S_{zz}}{(1+S_z^2)^{3/2}} + \frac{1}{S(1+S_z^2)^{1/2}}. \qquad (2.19)$$

potential field associated $\bar{c}$ body forces

Bernoulli constant

§ 3.1 NUMERICAL METHOD

Periodic travelling wave solutions are represented by a Fourier expansion:

$$S(z) \simeq S_N = \sum_{n=-N}^{N} a_n e^{in\pi z/L} \qquad (3.7)$$

$\boxed{k = \frac{n\pi}{L}}$

may replace $\exp(ikz) \longrightarrow \cos(kz)$

$S(z)$ real $\Downarrow$ $a_n = -a_n^*$ (a_n is real)

Substitute (3.7) into (2.18) $\longrightarrow$ also substitute (2.19) $\mathcal{F}$ and κ (take first and second $S(z)$ derivatives using FFT) } approximate integral using trapezoidal integration rule