```
In[•]:= n = 4; (* Number of strands *)
r = 3; (* Order of root of unity / 2) *)
q = Exp[Pi * I / r];
s = q ^ (r - 1);

(* LKB action on W_{4,2}^D*)
br[i_, w_{j_,k_}] :=
  {  s ^ (-4) q ^ 2 w_{i,i+1}                                                          j == i && k == i + 1
     s ^ (-1) w_{i+1,k} + (1 - s ^ (-2)) w_{i,k} - s ^ (i - k - 1) (1 - s ^ (-2)) q ^ 2 w_{i,i+1}   j == i && k ≠ i + 1
     s ^ (-1) w_{i,k}                                                                  j == i + 1
     s ^ (-1) w_{j,i+1} + (1 - s ^ (-2)) w_{j,i} - s ^ (i - j - 1) (1 - s ^ (-2)) w_{i,i+1}        k == i
     s ^ (-1) w_{j,i}                                                                  j ≠ i && k == i + 1
     w_{j,k}                                                                           True

(* Action of B_n on N_{4,2}^D for n = -2 mod r *)
br[i_, b_{j_}] := {  s ^ (i - n) t w_{i,i+1} + (1 - s ^ (-2)) b_i + s ^ (-1) b_{i+1}   j == i && i < n - 1
                     s ^ (-1) b_i                                                   j == i + 1 && i < n - 1
                     b_j - s ^ (n - j - 1) b_{n-1}                                  i == n - 1 && j < n - 1
                     s ^ (-1) t w_{n-1,n} - s ^ (-2) b_{n-1}                        i == n - 1 && j == n - 1
                     b_j                                                            True

(* Additivity and scalar multiplication rules *)
br[i_, c_ * w_{j_,k_}] := Expand[c * br[i, w_{j,k}]];
br[i_, w_{j_,k_} * c_] := Expand[c * br[i, w_{j,k}]];
br[i_, c_ * b_{j_}] := Expand[c * br[i, b_j]];
br[i_, b_{j_} * c_] := Expand[c * br[i, b_j]];
br[i_, x_ + y_] := br[i, x] + br[i, y];

In[•]:= (* Computing the matrices representing the braid group generators *)
dim = n (n - 1) / 2 + n - 1;
vecs = Flatten[Table[b_j, {j, 1, n - 1}] ~Join~ Table[w_{j,k}, {j, 1, n - 1}, {k, j + 1, n}]];
S = Table[Table[CoefficientArrays[br[i, vecs[[j]]], vecs][[2]] // Normal,
      {j, 1, dim}] // Transpose, {i, 1, n - 1}] // Simplify;

In[•]:= (* Functions for checking the braid group relations *)
CheckBraidComm[M_] := Table[Simplify[M[[i]].M[[j]] - M[[j]].M[[i]]] ==
      Table[0, Length[M[[1]]], Length[M[[1]]]], {i, 3, n - 1}, {j, 1, i - 2}] ~Join~
    Table[Simplify[M[[i]].M[[j]] - M[[j]].M[[i]]] == Table[0, Length[M[[1]]],
      Length[M[[1]]]], {i, 1, n - 3}, {j, i + 2, n - 1}] // Flatten;
CheckBraidRel[M_] := Table[Simplify[M[[i]].M[[i + 1]].M[[i]] - M[[i + 1]].M[[i]].
      M[[i + 1]]] == Table[0, Length[M[[1]]], Length[M[[1]]]], {i, 1, n - 2}];
```

In[•]:=
```
(* Verification of the braid group relations for N4,2^D *)
CheckBraidComm[S]
CheckBraidRel[S]
```
Out[•]= {True, True}

Out[•]= {True, True}

In[•]:=
```
(* Define the vectors ϕ c̄i *)
For[i = 1, i < n, i++, fi = Sum[ s^(-(i-1)) q^2 s^(-j) wi,i+j, {j, 1, n-i}] +
    Sum[ s^(-(j-1)) wj,i, {j, 1, i-1}] - Sum[ s^(n-i) s^(-(j-1)) wj,n, {j, 1, n-1}]];
```

In[•]:=
```
(* Verify the calculations *)
W = Table[wi,j, {i, 1, n-1}, {j, i+1, n}] // Flatten;
({f1 - {1, q^4, q^2-1, 0, q, -q^2}.W,
    f2 - {1, 0, q, q^4, 0, -1}.W,
    f3 - {0, 1, -q^2, q^4, -1, q^2-q^4}.W} // Simplify) == {0, 0, 0}
```
Out[•]= True

In[•]:=
```
(* Verify that the vectors fi span a representation
 isomorphic to Burau (using the relations of Jackson-Kerler) *)
Join[Table[br[i, fj] == fj, {i, 1, n-2}, {j, 1, i-1}],
   Table[br[i, fj] == fj, {i, 1, n-2}, {j, i+2, n-1}],
   Table[br[i, fi] == s^(-1) fi+1 + (1 - s^(-2)) fi, {i, 1, n-2}],
   Table[br[i, fi+1] == s^(-1) fi, {i, 1, n-2}],
   Table[br[n-1, fj] == fj - s^(n-j-1) fn-1, {j, 1, n-2}],
   {br[n-1, fn-1] == -s^(-2) fn-1}] // Flatten // Simplify
```
Out[•]= {True, True, True, True, True, True, True, True, True}

In[•]:=
```
(* Define the vectors gi of the complement of S4,2^D in W4,2^D *)
g1 = {1, -q^2, 0, 0, q, 1}.W // Simplify;
g2 = -(1/4) q^2 {1, 2, 1, 1, 2, 1}.W // Simplify;
g3 = {0, -q, -1, -1, q-1, 0}.W // Simplify;
```

In[•]:=
```
(* Verify that the six vectors are linearly independent *)
Solve[
 CoefficientArrays[Sum[ci fi, {i, 1, n-1}] + Sum[cn-1+i gi, {i, 1, n-1}] == 0, W][[2]] ==
  Table[0, 2 (n-1), 2 (n-1)]]
```
Out[•]= {{c1 → 0, c2 → 0, c3 → 0, c4 → 0, c5 → 0, c6 → 0}}

```
In[•]:= (* Write the action of B4 on gi wrt fi and gi *)
eqsBraid =
  Table[br[i, gj // Expand] == Sum[ai,j,k gk, {k, 1, n - 1}] + Sum[bi,j,k fk, {k, 1, n - 1}],
    {i, 1, n - 1}, {j, 1, n - 1}] // Flatten;
(* Solve the linear system *)
coeffsBraid = CoefficientArrays[eqsBraid, W] // Normal;
solsBraid =
  Solve[coeffsBraid[[2]] == Table[0, Length[eqsBraid], Length[W]]] // Simplify;

(* B is the matrix of the Burau representation *)
B = Transpose /@ Join[
    Which[n < 3, {},
     n == 3, {{{1 - s^(-2), s^(-1)}, {s^(-1), 0}}},
     n > 3, {ArrayFlatten[
        {{{{1 - s^(-2), s^(-1)}, {s^(-1), 0}}, 0}, {0, IdentityMatrix[n - 3]}}]}],
    Table[ArrayFlatten[{{IdentityMatrix[i - 1], 0, 0}, {0, {{1 - s^(-2), s^(-1)},
          {s^(-1), 0}}, 0}, {0, 0, IdentityMatrix[n - i - 2]}}], {i, 2, n - 3}],
    If[n > 3, {ArrayFlatten[{{IdentityMatrix[n - 3], 0},
         {0, {{1 - s^(-2), s^(-1)}, {s^(-1), 0}}}}]}, {}],
    If[n > 2, {ArrayFlatten[{{IdentityMatrix[n - 2],
          Table[{-s^(n - i - 1)}, {i, 1, n - 2}]}, {0, -s^(-2)}}]}, {-s^(-2)}]];

(* Creating the matrix for the Bn action on W4,2^D in the new basis *)
G1 = Table[Table[ai,j,k, {k, 1, n - 1}, {j, 1, n - 1}], {i, 1, n - 1}] /. solsBraid[[1]];
G2 = Table[Table[bi,j,k, {k, 1, n - 1}, {j, 1, n - 1}], {i, 1, n - 1}] /. solsBraid[[1]];
G = Table[ArrayFlatten[{{G1[[i]], 0}, {G2[[i]], B[[i]]}}], {i, 1, n - 1}];

(* Verify that the matrices of G and G1 satisfy the braid group relations *)
CheckBraidComm[G]
CheckBraidRel[G]
CheckBraidComm[G1]
CheckBraidRel[G1]

Out[•]= {True, True}

Out[•]= {True, True}

Out[•]= {True, True}

Out[•]= {True, True}

In[•]:= (* 3-dim irreducible representation of the cubic Hecke algebra on 4 strands *)
```

$$H = \left\{ \begin{pmatrix} z & 0 & 0 \\ x z + y^2 & y & 0 \\ y & 1 & x \end{pmatrix}, \begin{pmatrix} x & -1 & y \\ 0 & y & -x z - y^2 \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} z & 0 & 0 \\ x z + y^2 & y & 0 \\ y & 1 & x \end{pmatrix} \right\};$$

In[•]:=
```
(* Find a, b,c such that the matrices of G1 and H are the same *)
Solve[Table[G1[[i]] - H [[i]] == Table[0, 3, 3], {i, 1, 3}]]
```

Out[•]= $\left\{\left\{x \to \frac{-i + \sqrt{3}}{i + \sqrt{3}}, y \to 1, z \to 1\right\}\right\}$

In[•]:=
```
(* Writing the complex number in the solutions as a power of q *)
```

In[•]:= $\frac{-i + \sqrt{3}}{i + \sqrt{3}}$ `- q^5 // Simplify`

Out[•]= 0

In[•]:=
```
(* Verify the equality between the matrices *)
Table[G1[[i]] - H[[i]] /. {x → q^5, y → 1, z → 1} // Simplify // MatrixForm, {i, 1, 3}]
```

Out[•]= $\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$