Turtlebot Kinematics Derivation
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Here I will derive the forward and inverse kinematics for the differential drive robot parameterized by track width d and wheel radius r. The configuration of the robot is x, y, ϕ and the wheel speeds are u_l, u_r.

First we need to obtain the body frame twist in the frame of each wheel. The left and right wheels are offset from the origin of the body frame by a y distance of $+d, -d$, respectively.

$$A_{bl} = \begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A_{lb} = \begin{bmatrix} 1 & 0 & 0 \\ -d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$A_{br} = \begin{bmatrix} 1 & 0 & 0 \\ -d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A_{rb} = \begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Then, for $i = l, r$, we can convert from body frame twists to twists in the frame of the wheels via the following equation.

$$V_i = A_{ib} V_b \tag{3}$$

Next I will expand these for the two wheels.

$$\begin{bmatrix} \dot{\theta} \\ v_{xl} \\ v_{y1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ v_x - d\dot{\theta} \\ v_y \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} \dot{\theta} \\ v_{xr} \\ v_{yr} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ v_x + d\dot{\theta} \\ v_y \end{bmatrix} \tag{5}$$

Next, we need to consider how the rotational velocity of the wheel $\dot{\phi}$ corresponds to the wheel's velocity. These are conventional wheels, so for $i = l, r$, we have

$$\begin{bmatrix} v_{xi} \\ v_{yi} \end{bmatrix} = \begin{bmatrix} r\dot{\phi}_i \\ 0 \end{bmatrix} \tag{6}$$

Next, we go from body twist to wheel motion. We substitute in equation 6 into equations 4 and 5 and obtain the inverse kinematics equations that take us from body twist to wheel velocity.

$$\begin{bmatrix} \dot{\theta} \\ r\dot{\phi}_l \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ v_x - d\dot{\theta} \\ v_y \end{bmatrix} \implies \dot{\phi}_l = \frac{v_x - d\dot{\theta}}{r} \tag{7}$$

$$\begin{bmatrix} \dot{\theta} \\ r\dot{\phi}_r \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ v_x + d\dot{\theta} \\ v_y \end{bmatrix} \implies \dot{\phi}_r = \frac{v_x + d\dot{\theta}}{r} \tag{8}$$

Note that these equations imply that the body velocity v_y must be 0. This is because we assume the robot cannot slide.

The forward kinematics of a diff drive robot are given in equation 13.15 of the Modern Robotics textbook (`http://hades.mech.northwestern.edu/images/7/7f/MR.pdf`).

$$\dot{q} = \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\frac{r}{2d} & \frac{r}{2d} \\ \frac{r}{2}\cos\theta & \frac{r}{2}\cos\theta \\ \frac{r}{2}\sin\theta & \frac{r}{2}\sin\theta \end{bmatrix} \begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix} \tag{9}$$

$$\dot{\theta} = \frac{r}{2d}\left(\phi_r - \phi_l\right) \tag{10}$$

$$\dot{x} = \frac{r}{2}\cos\theta\left(\phi_r + \phi_l\right) \tag{11}$$

$$\dot{y} = \frac{r}{2}\sin\theta\left(\phi_r + \phi_l\right) \tag{12}$$