

Bits, Bytes, and Integers – Part 2

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Summary From Last Lecture

- Representing information as bits
- Bit-level manipulations
- Integers
 - Representation: unsigned and signed
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
- Representations in memory, pointers, strings
- Summary

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit



Two's Complement Examples (w = 5)

	-16	8	4	2	1
10 =	0	1	0	1	0

$$8+2 = 10$$

	-16	8	4	2	1
-10 =	1	0	1	1	0

$$-16+4+2 = -10$$

Unsigned & Signed Numeric Values

X	$B2U(X)$	$B2T(X)$
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

■ Equivalence

- Same encodings for nonnegative values

■ Uniqueness

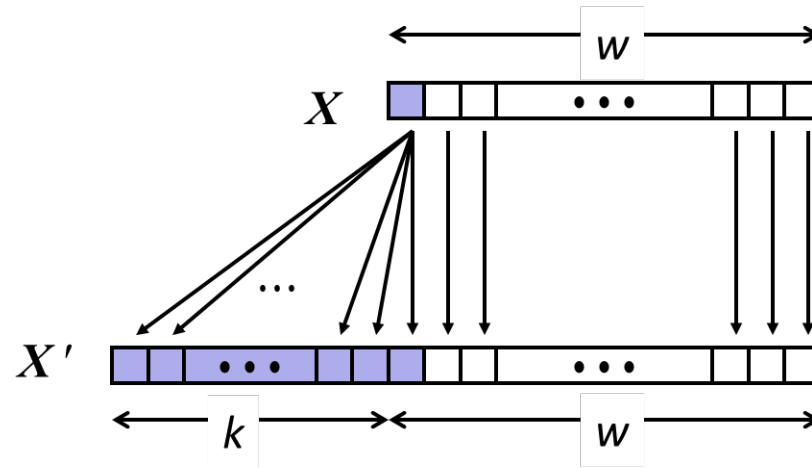
- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

■ Expression containing signed and unsigned int:

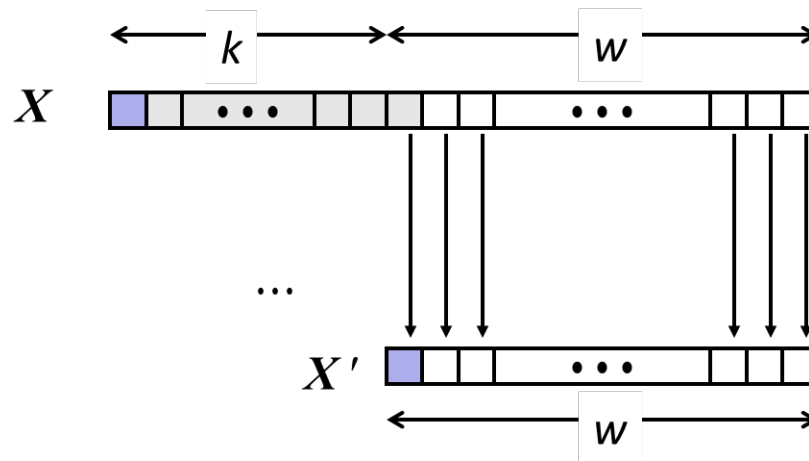
`int` is cast to `unsigned`

Sign Extension and Truncation

■ Sign Extension



■ Truncation



Today: Bits, Bytes, and Integers

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- Bit-level manipulations
- **Integers**
 - Representation: unsigned and signed
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 - Expanding, truncating
 - **Addition, negation, multiplication, shifting**
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Unsigned Addition

Operands: w bits


u 

$+ v$ 

True Sum: $w+1$ bits

$u + v$ 

Discard Carry: w bits

$\text{UAdd}_w(u, v)$ 

■ Standard Addition Function

- Ignores carry output

■ Implements Modular Arithmetic

$$s = \text{UAdd}_w(u, v) = u + v \bmod 2^w$$

unsigned char

```

      1110 1001
    + 1101 0101
    -----
  1 1011 1110
    -----
      1011 1110
  
```

```

      E9
    + D5
    -----
    1BE
    -----
      BE
  
```

```

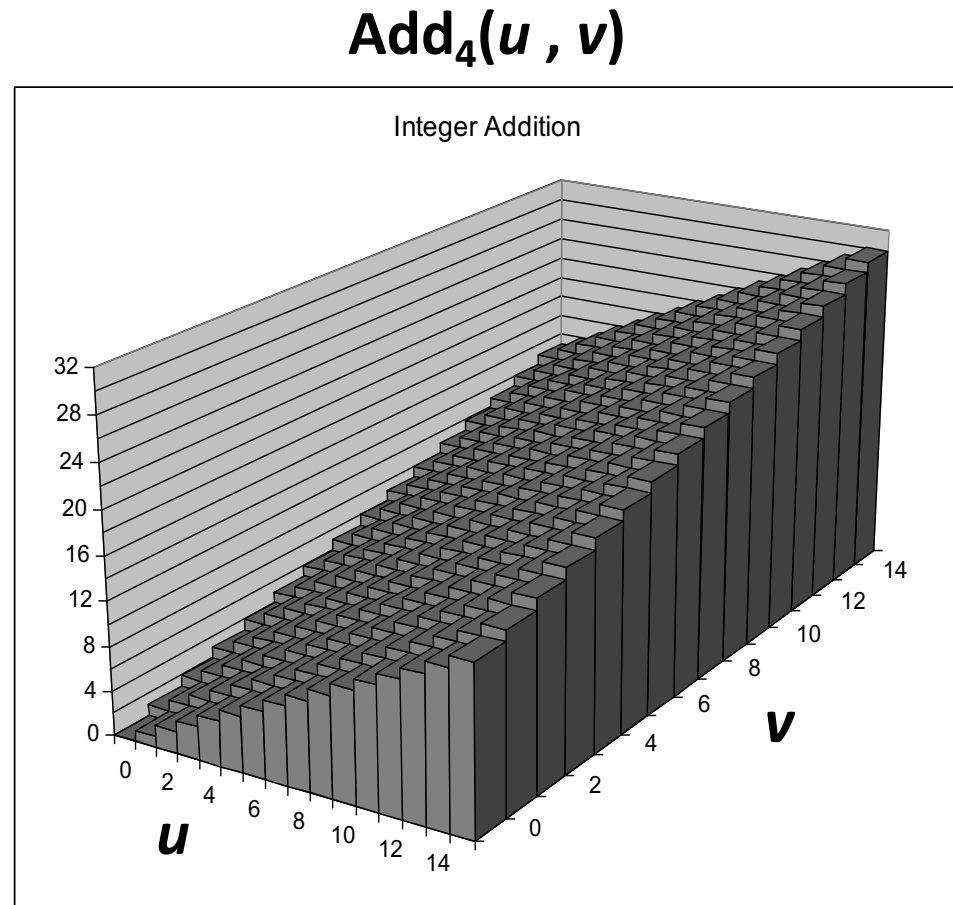
      223
    + 213
    -----
    446
    -----
      190
  
```

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Visualizing (Mathematical) Integer Addition

■ Integer Addition

- 4-bit integers u, v
- Compute true sum $\text{Add}_4(u, v)$
- Values increase linearly with u and v
- Forms planar surface

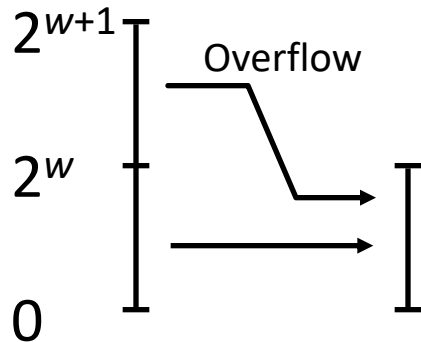


Visualizing Unsigned Addition

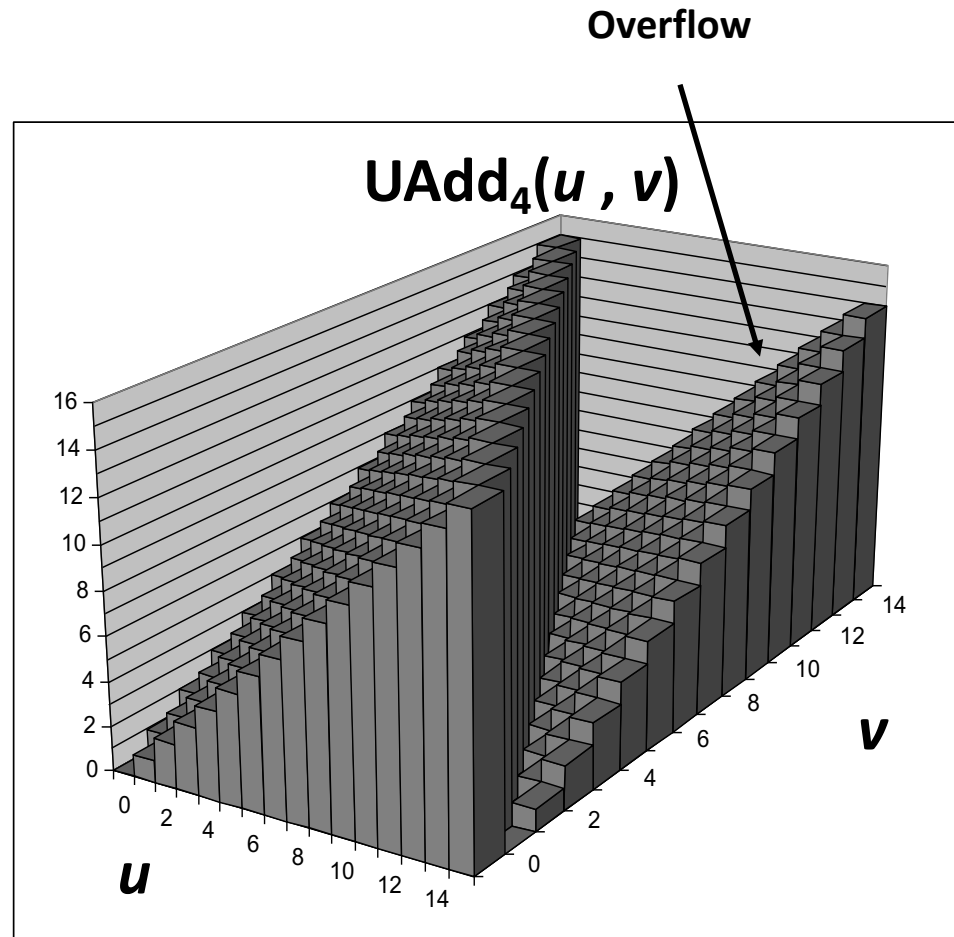
■ Wraps Around

- If true sum $\geq 2^w$
- At most once

True Sum



Modular Sum



Two's Complement Addition

Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ TAdd and UAdd have Identical Bit-Level Behavior

- Signed vs. unsigned addition in C:

```
int s, t, u, v;
```

```
s = (int) ((unsigned) u + (unsigned) v);
```

```
t = u + v
```

- Will give $s == t$

$$\begin{array}{r}
 1110 \ 1001 \\
 + \ 1101 \ 0101 \\
 \hline
 1 \ 1011 \ 1110 \\
 \hline
 1011 \ 1110
 \end{array}$$

$$\begin{array}{r}
 \text{E9} \\
 + \ \text{D5} \\
 \hline
 1\text{BE} \\
 \hline
 \text{BE}
 \end{array}$$

$$\begin{array}{r}
 -23 \\
 + \ -43 \\
 \hline
 -66 \\
 \hline
 -66
 \end{array}$$

TAdd Overflow (Signed)

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer

0 111...1

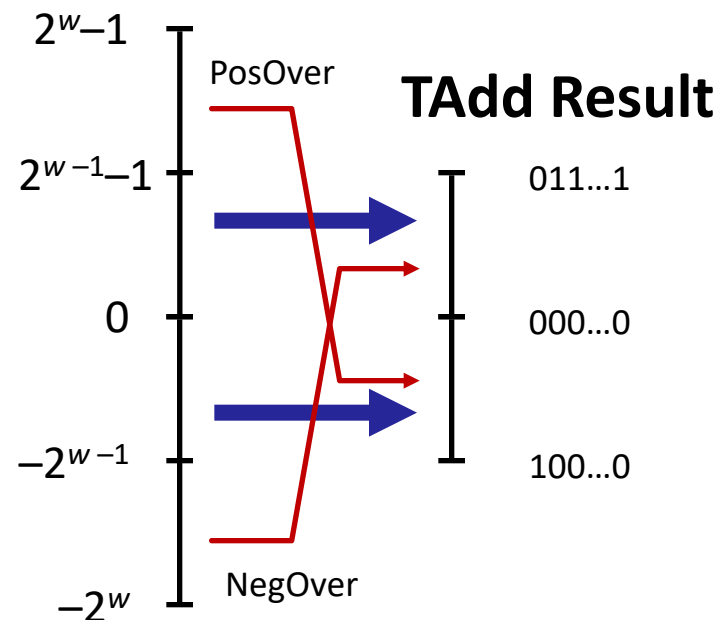
0 100...0

0 000...0

1 011...1

1 000...0

True Sum



Visualizing 2's Complement Addition

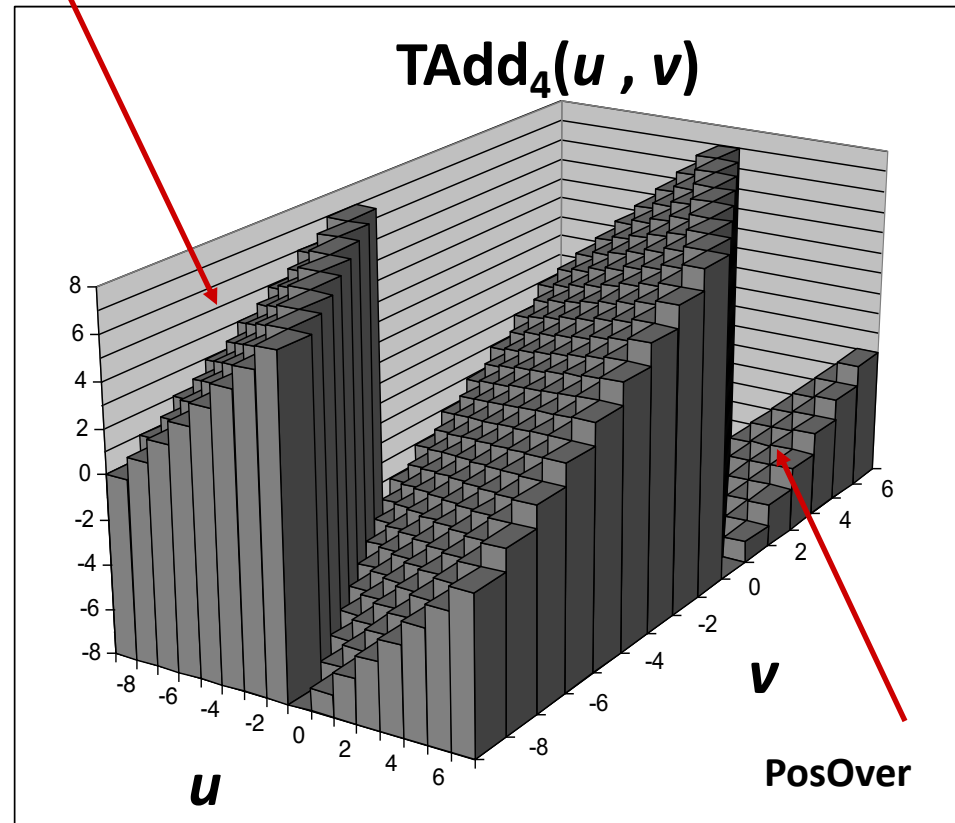
■ Values

- 4-bit two's comp.
- Range from -8 to +7

■ Wraps Around

- If $\text{sum} \geq 2^{w-1}$
 - Becomes negative
 - At most once
- If $\text{sum} < -2^{w-1}$
 - Becomes positive
 - At most once

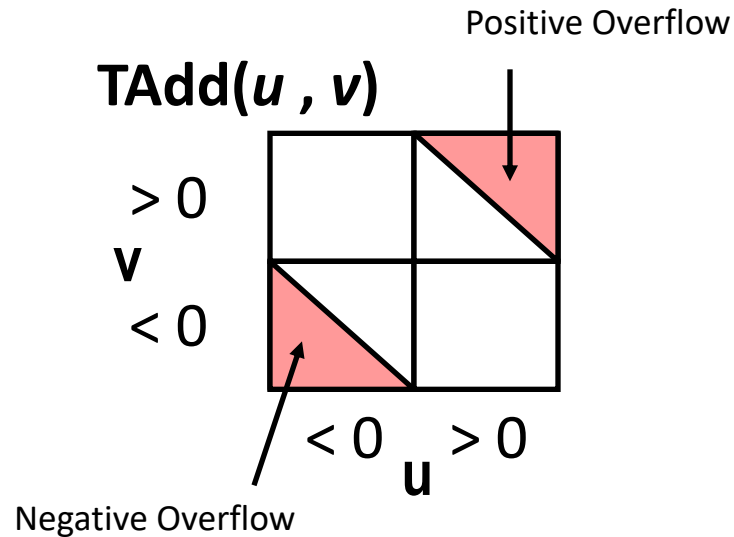
NegOver



Characterizing TAdd

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer



$$TAdd_w(u, v) = \begin{cases} u + v + 2^w & u + v < TMin_w \text{ (NegOver)} \\ u + v & TMin_w \leq u + v \leq TMax_w \\ u + v - 2^w & TMax_w < u + v \text{ (PosOver)} \end{cases}$$

Quiz Time!

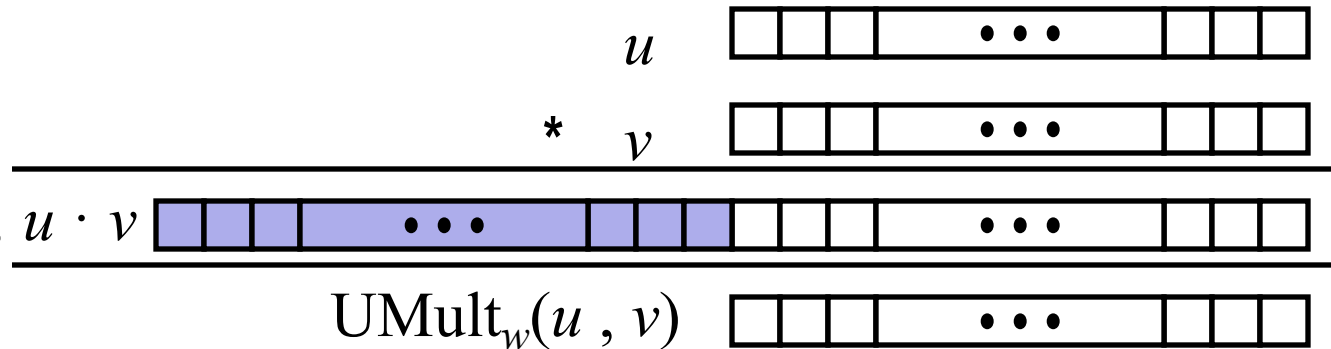
Exercise 2.29

Multiplication

- **Goal: Computing Product of w -bit numbers x, y**
 - Either signed or unsigned
- **But, exact results can be bigger than w bits**
 - Unsigned: up to $2w$ bits
 - Result range: $0 \leq x * y \leq (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
 - Two's complement min (negative): Up to $2w-1$ bits
 - Result range: $x * y \geq (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
 - Two's complement max (positive): Up to $2w$ bits, but only for $(TMin_w)^2$
 - Result range: $x * y \leq (-2^{w-1})^2 = 2^{2w-2}$
- **So, maintaining exact results...**
 - would need to keep expanding word size with each product computed
 - is done in software, if needed
 - e.g., by “arbitrary precision” arithmetic packages

Unsigned Multiplication in C

Operands: w bits



Discard w bits: w bits

■ Standard Multiplication Function

- Ignores high order w bits

- **Implements Modular Arithmetic**

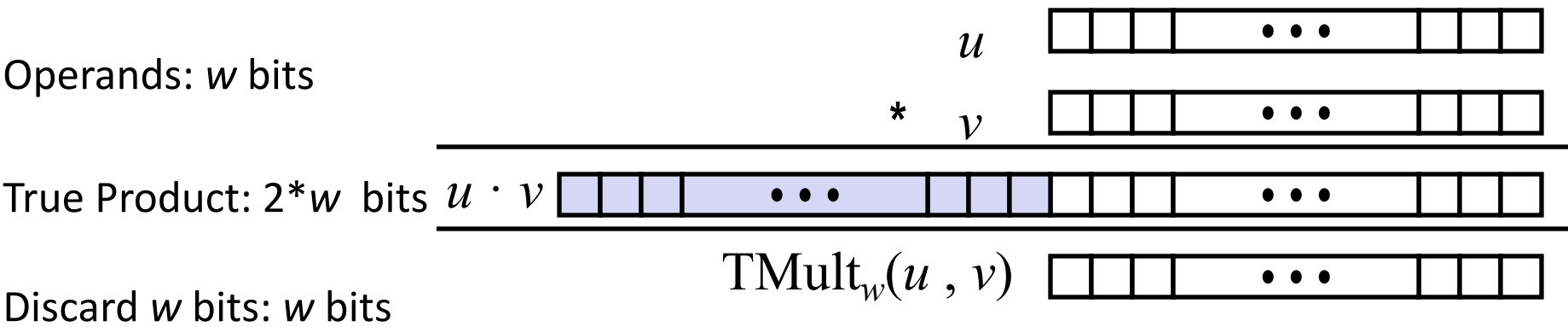
$$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$$

		1110	1001
*		1101	0101
<hr/>			
1100	0001	1101	1101
<hr/>			
		1101	1101

$$\begin{array}{r} \text{E9} \\ * \text{D5} \\ \hline \text{C1DD} \\ \hline \text{DD} \end{array}$$

$$\begin{array}{r} 223 \\ * 213 \\ \hline 47499 \\ \hline 221 \end{array}$$

Signed Multiplication in C



■ Standard Multiplication Function

- Ignores high order w bits
- Some of which are different for signed vs. unsigned multiplication
- Lower bits are the same

	1110 1001	E9	-23
*	1101 0101	* D5	* -43
	0000 0011 1101 1101	03DD	989
	1101 1101	DD	-35

Quiz Time!

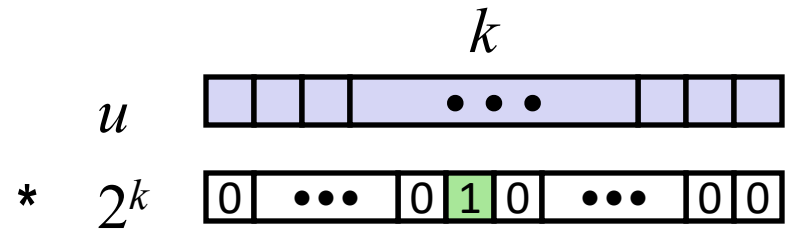
Exercise 2.34

Power-of-2 Multiply with Shift

■ Operation

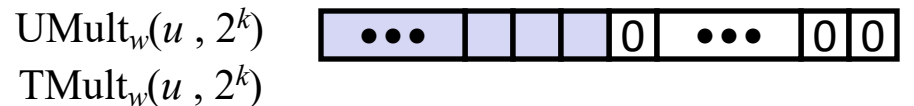
- $u \ll k$ gives $u * 2^k$
- Both signed and unsigned

Operands: w bits



True Product: $w+k$ bits $u \cdot 2^k$

Discard k bits: w bits



■ Examples

- $u \ll 3 == u * 8$
- $(u \ll 5) - (u \ll 3) == u * 24$
- Most machines shift and add **faster** than multiply
 - Compiler generates this code automatically

```
int8_t a = 127;
printf(a<<1);
```

-2

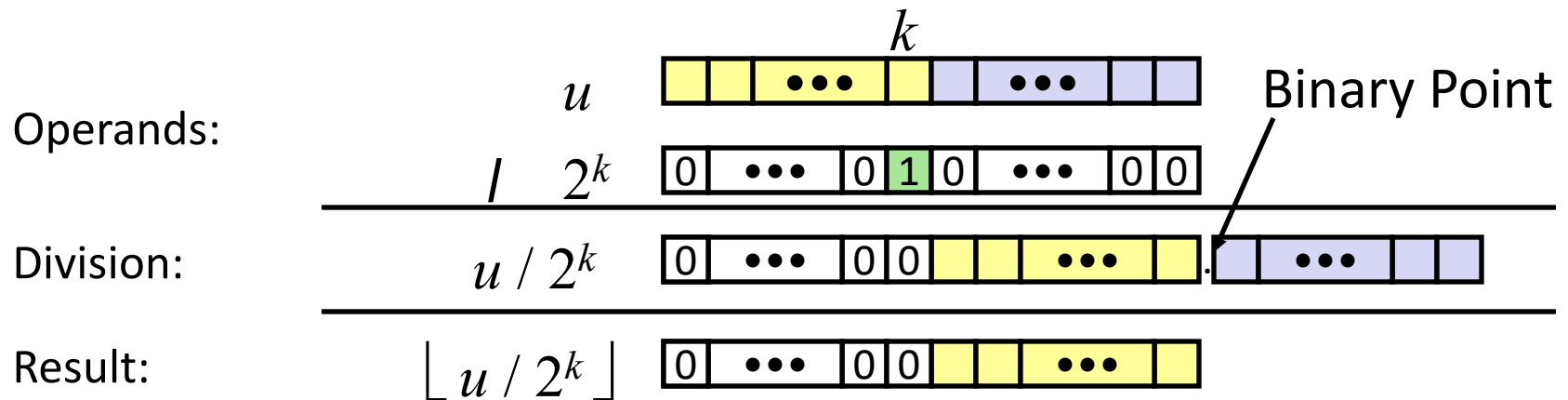
Quiz Time!

Exercise 2.40

Unsigned Power-of-2 Divide with Shift

■ Quotient of **Unsigned** by Power of 2

- $u \gg k$ gives $\lfloor u / 2^k \rfloor$
- Uses logical shift

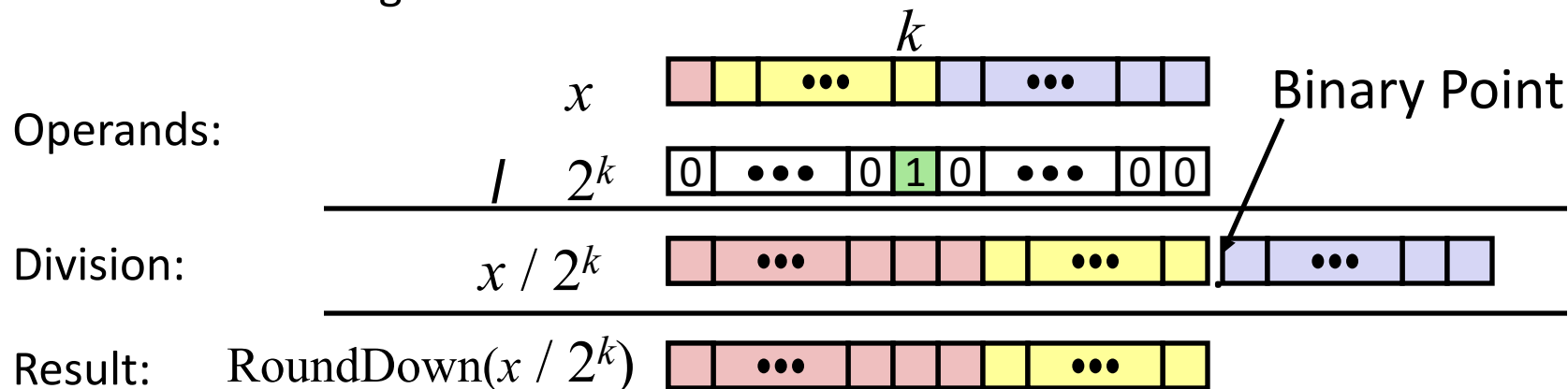


	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	00011101 10110110
x >> 4	950.8125	950	03 B6	00000011 10110110
x >> 8	59.4257813	59	00 3B	00000000 00111011

Signed Power-of-2 Divide with Shift

■ Quotient of **Signed** by Power of 2

- $x \gg k$ gives $\lfloor x / 2^k \rfloor$
- Uses arithmetic shift
- Rounds wrong direction when $u < 0$



	Division	Computed	Hex	Binary
y	-15213	-15213	C4 93	11000100 10010011
y >> 1	-7606.5	-7607	E2 49	11100010 01001001
y >> 4	-950.8125	-951	FC 49	11111100 01001001
y >> 8	-59.4257813	-60	FF C4	11111111 11000100

Correct Power-of-2 Divide

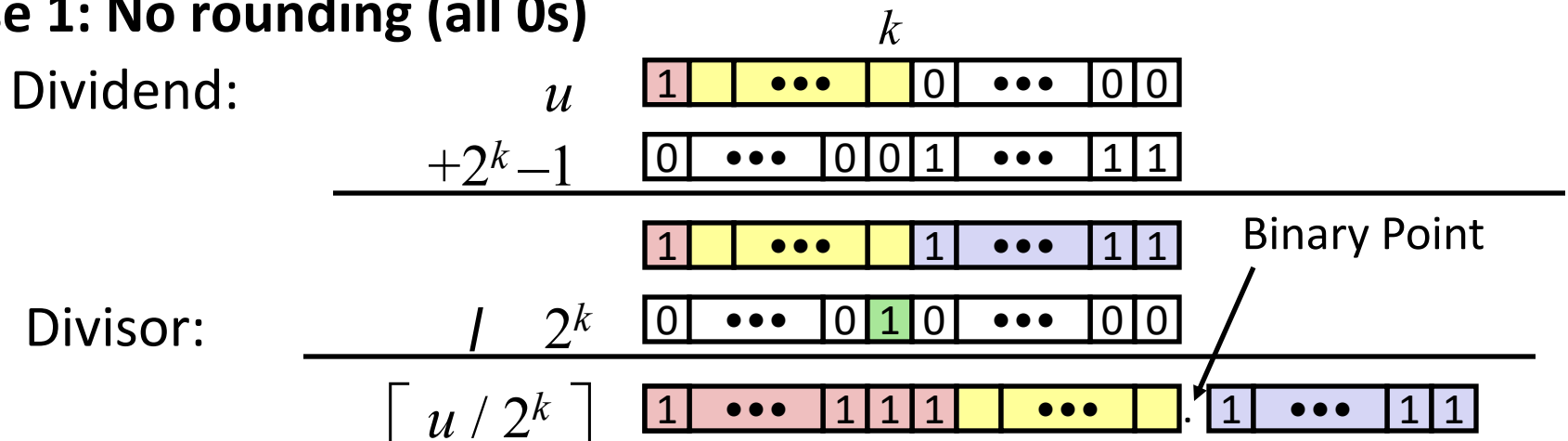
- Quotient of Negative Number by Power of 2
 - Want $\lceil x / 2^k \rceil$ (Round Toward 0 , 向0舍入)

Correct Power-of-2 Divide

■ Quotient of Negative Number by Power of 2

- Want $\lceil x / 2^k \rceil$ (Round Toward 0 , 向0舍入)
- Compute as $\lfloor (x+2^k-1) / 2^k \rfloor$
 - In C: $(x + (1 \ll k) - 1) \gg k$
 - Biases dividend toward 0

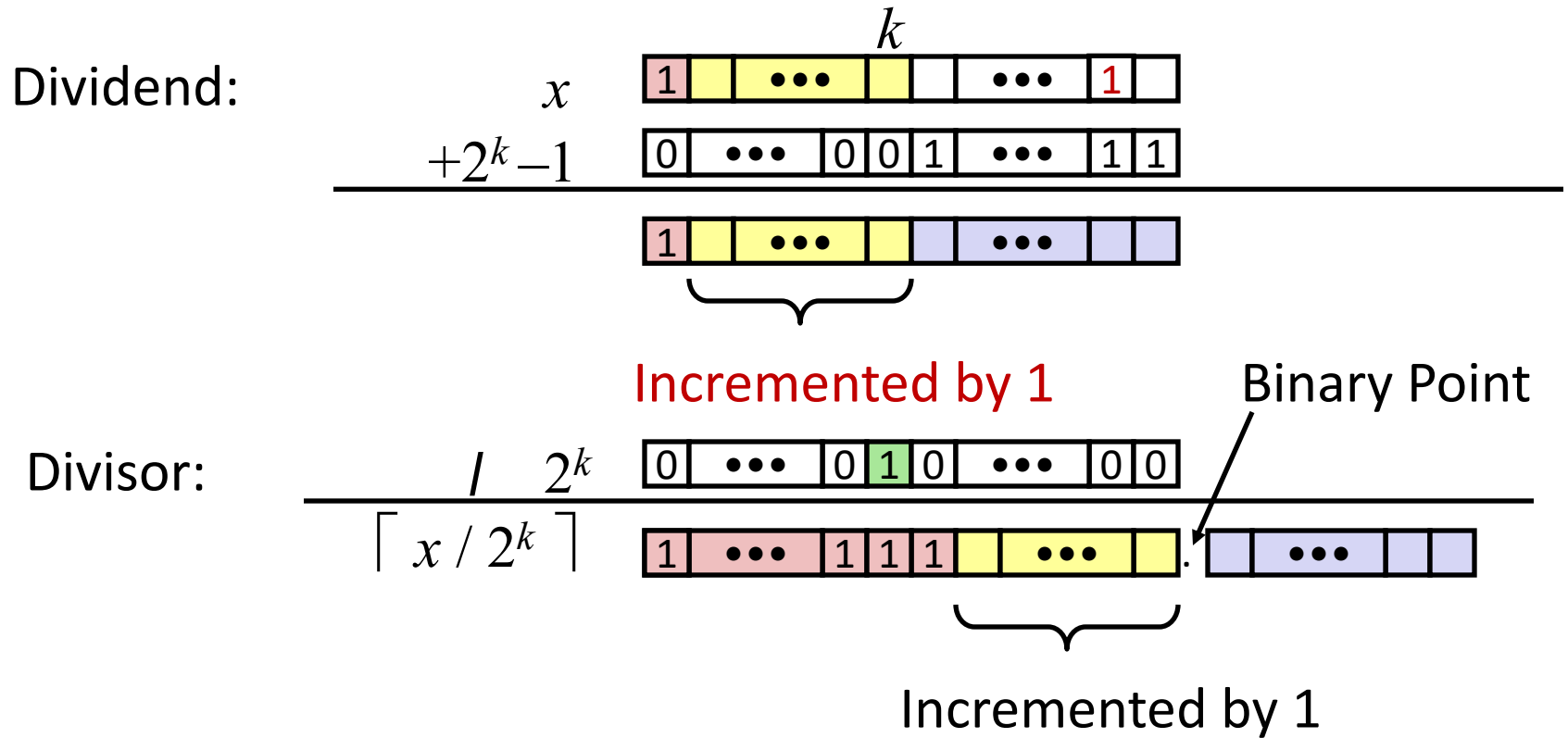
Case 1: No rounding (all 0s)



Biasing has no effect

Correct Power-of-2 Divide

Case 2: Rounding



Biasing adds 1 to final result

Negation: Complement & Increment

- How to negate

- $-x = ?$

Negation: Complement & Increment

- Negate through complement and increase

$$\sim x + 1 == -x$$

- Example

- Observation: $\sim x + x == 1111\dots111 == -1$

$$\begin{array}{r} x \quad 10011101 \\ + \quad \sim x \quad 01100010 \\ \hline -1 \quad 11111111 \end{array}$$

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

Complement & Increment Examples

x = 0

	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

x = TMin

	Decimal	Hex	Binary
x	-32768	80 00	10000000 00000000
~x	32767	7F FF	01111111 11111111
~x+1	-32768	80 00	10000000 00000000

Quiz Time!

Exercise 2.43

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
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 - **Summary**
- Representations in memory, pointers, strings

Arithmetic: Basic Rules

■ Addition:

- Unsigned/signed: Normal addition followed by **truncate**, same operation on bit level
- Unsigned: addition mod 2^w
 - Mathematical addition + possible subtraction of 2^w
- Signed: modified addition mod 2^w (result in proper range)
 - Mathematical addition + possible addition or subtraction of 2^w

■ Multiplication:

- Unsigned/signed: Normal multiplication followed by **truncate**, same operation on bit level
- Unsigned: multiplication mod 2^w
- Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

- ***Don't* use without understanding implications**

- Easy to make mistakes

```
unsigned i;  
for (i = cnt-2; i >= 0; i--)  
    a[i] += a[i+1];
```

- Can be very subtle

```
#define DELTA sizeof(int)  
int i;  
for (i = CNT; i-DELTA >= 0; i-= DELTA)  
    . . .
```


Counting Down with Unsigned

■ Proper way to use **unsigned** as loop index

```
unsigned i;  
for (i = cnt-2; i < cnt; i--)  
    a[i] += a[i+1];
```

■ See Robert Seacord, *Secure Coding in C and C++*

- C Standard guarantees that **unsigned addition** will behave **like modular arithmetic**
 - $0 - 1 \rightarrow UMax$

■ Even better

```
size_t i;  
for (i = cnt-2; i < cnt; i--)  
    a[i] += a[i+1];
```

- Data type **size_t** defined as unsigned value with length = word size
- Code will work even if **cnt** = *UMax*
- What if **cnt** is signed and < 0? ***signed values implicitly cast to unsigned***

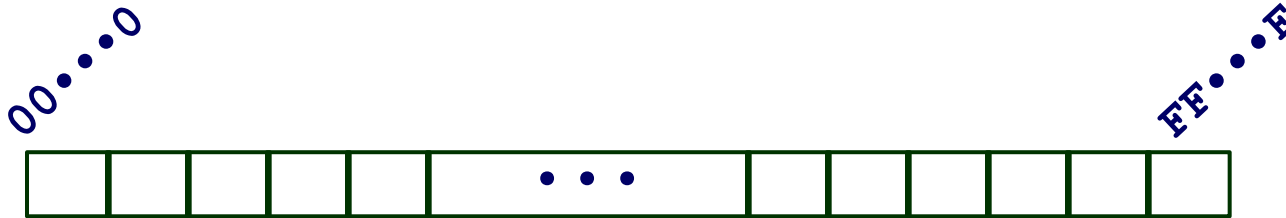
Why Should I Use Unsigned? (cont.)

- **Do Use When Performing Modular Arithmetic**
 - Multiprecision arithmetic
- **Do Use When Using Bits to Represent Sets**
 - Logical right shift, no sign extension
- **Do Use In System Programming**
 - Bit masks, device commands,...

Today: Bits, Bytes, and Integers

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Byte-Oriented Memory Organization



- **Programs refer to data by address**
 - Conceptually, envision it as a very **large array of bytes**
 - In reality, it's not, but can think of it that way
 - An address is like an **index** into that array
 - and, a pointer variable stores an address

- **Note: system provides private address spaces to each “process”**
 - Think of a process as a program being executed
 - So, a program can clobber its own data, but not that of others

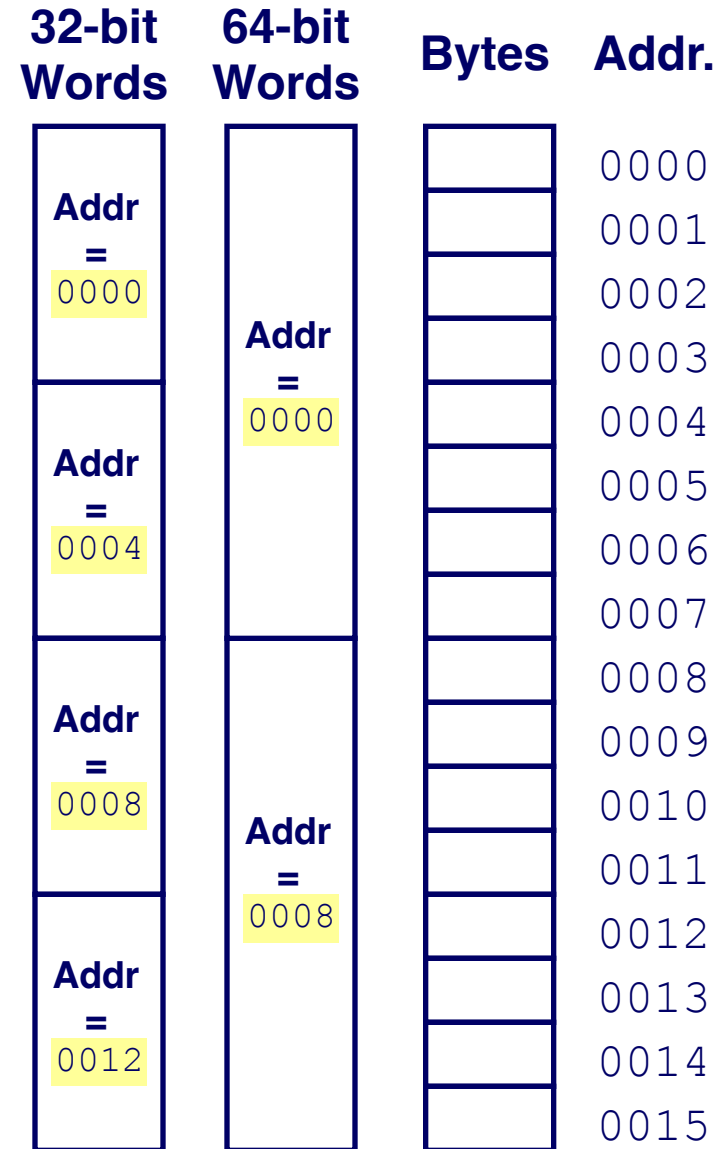
Machine Words

- **Any given computer has a “Word Size”**
 - Nominal size of integer-valued data
 - and of addresses
 - Until recently, most machines used 32 bits (4 bytes) as word size
 - Limits addresses to 4GB (2^{32} bytes)
 - Increasingly, machines have 64-bit word size
 - Potentially, could have 18 EB (exabytes) of addressable memory
 - That's 18.4×10^{18}
 - Machines still support multiple data formats
 - Fractions or multiples of word size
 - Always integral number of bytes

Word-Oriented Memory Organization

■ Addresses Specify Byte Locations

- Address of first byte in word
- Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
<code>char</code>	1	1	1
<code>short</code>	2	2	2
<code>int</code>	4	4	4
<code>long</code>	4	8	8
<code>float</code>	4	4	4
<code>double</code>	8	8	8
<code>pointer</code>	4	8	8

Byte Ordering

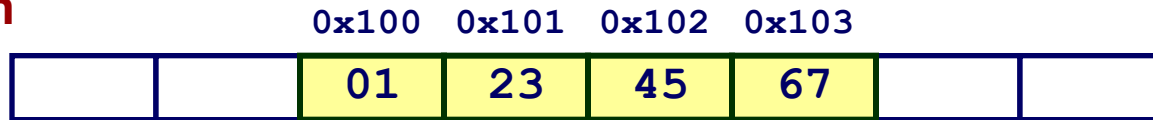
- So, how are the bytes within a multi-byte word ordered in memory?
- Conventions
 - **Big Endian**: Sun (Oracle SPARC), PPC Mac, *Internet*
 - Least significant byte has highest address
 - **Little Endian**: *x86*, ARM processors running Android, iOS, and Linux
 - Least significant byte has lowest address

Byte Ordering Example

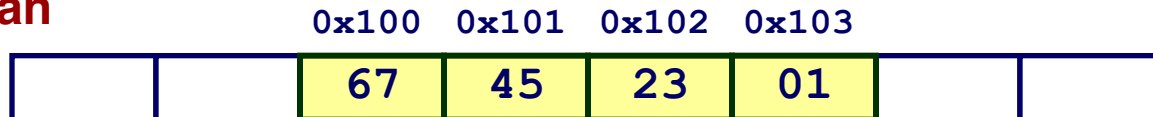
■ Example

- Variable x has 4-byte value of 0x01234567
- Address given by &x is 0x100

Big Endian



Little Endian



Representing Integers

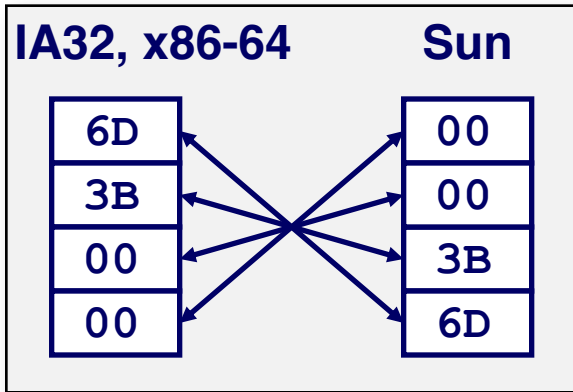
Decimal: 15213

Binary: 0011 1011 0110 1101

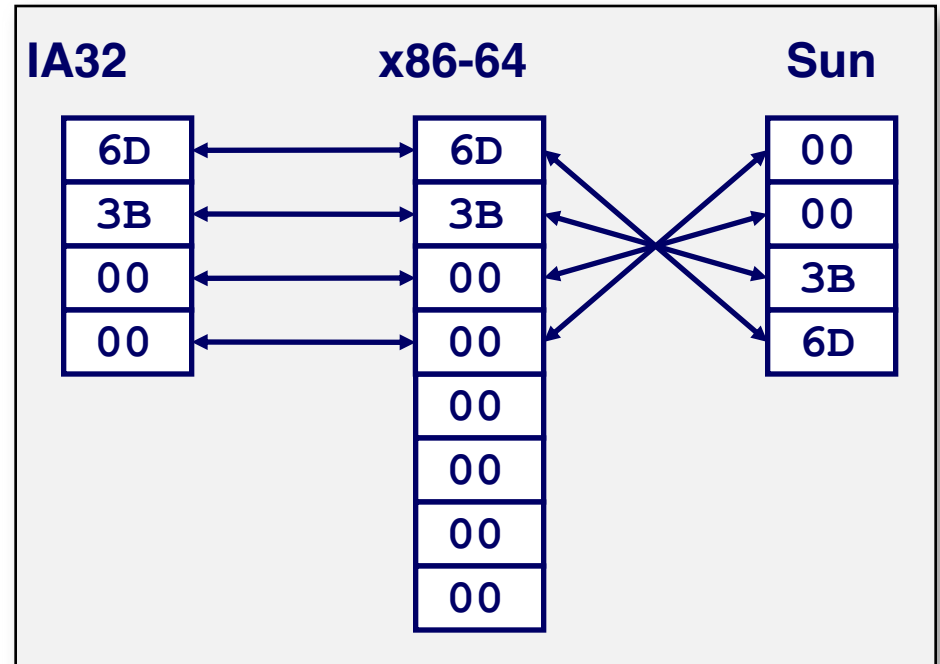
Hex: 3 B 6 D

`int A = 15213;`

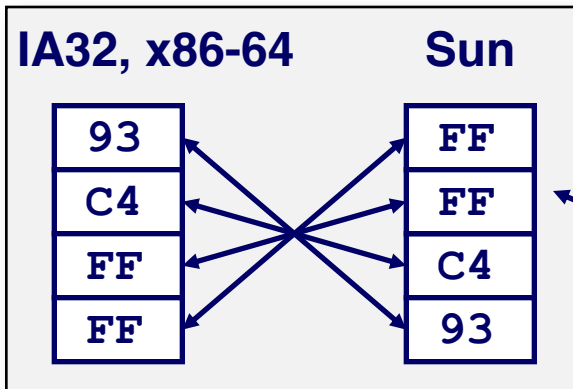
Increasing addresses
↓



`long int C = 15213;`



`int B = -15213;`



Two's complement representation

Examining Data Representations

■ Code to Print Byte Representation of Data

- Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
    size_t i;
    for (i = 0; i < len; i++)
        printf("%p\t0x%.2x\n", start+i, start[i]);
    printf("\n");
}
```

Printf directives:

%p: Print pointer
%x: Print Hexadecimal

show_bytes Execution Example

```
int a = 15213;  
printf("int a = 15213;\n");  
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux x86-64):

```
int a = 15213;  
0x7ffffb7f71dbc    6d  
0x7ffffb7f71dbd    3b  
0x7ffffb7f71dbe    00  
0x7ffffb7f71dbf    00
```

Representing Pointers

```
int B = -15213;  
int *P = &B;
```

Sun	IA32	x86-64
EF	AC	3C
FF	28	1B
FB	F5	FE
2C	FF	82
		FD
		7F
		00
		00

Different compilers & machines assign different locations to objects

Even get different results each time run program

Representing Strings

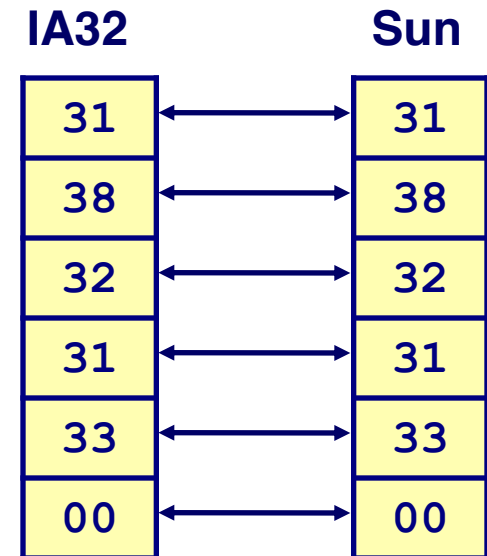
```
char S[6] = "18213";
```

■ Strings in C

- Represented by **array of characters**
- Each character encoded in ASCII format
 - Standard 7-bit encoding of character set
 - Character "0" has code 0x30
 - Digit i has code $0x30+i$
- String should be null-terminated
 - Final character = 0

■ Compatibility

- Byte ordering not an issue



Reading Byte-Reversed Listings

■ Disassembly

- Text representation of binary machine code
- Generated by program that reads the machine code

■ Example Fragment

Address	Instruction Code	Assembly Rendition
8048365:	5b	pop %ebx
8048366:	81 c3 ab 12 00 00	add \$0x12ab,%ebx
804836c:	83 bb 28 00 00 00 00	cmpl \$0x0,0x28(%ebx)

■ Deciphering Numbers

- Value: 0x12ab
- Pad to 32 bits: 0x000012ab
- Split into bytes: 00 00 12 ab
- Reverse: ab 12 00 00

Integer C Puzzles

Initialization

```
int x = foo();  
int y = bar();  
unsigned ux = x;  
unsigned uy = y;
```

<code>x < 0</code>	$\Rightarrow ((x*2) < 0)$	✗
<code>ux >= 0</code>		✓
<code>x & 7 == 7</code>	$\Rightarrow (x \ll 30) < 0$	✓
<code>ux > -1</code>		✗
<code>x > y</code>	$\Rightarrow -x < -y$	✗
<code>x * x >= 0</code>		✗
<code>x > 0 && y > 0</code>	$\Rightarrow x + y > 0$	✗
<code>x >= 0</code>	$\Rightarrow -x \leq 0$	✓
<code>x <= 0</code>	$\Rightarrow -x \geq 0$	✗
<code>(x -x)>>31 == -1</code>		✗
<code>ux >> 3 == ux/8</code>		✓
<code>x >> 3 == x/8</code>		✗
<code>x & (x-1) != 0</code>		✗

Quiz Time!

Exercise 2.44

Summary

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
 - Representation: unsigned and signed
 - Conversion, casting
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