Floating Point

Kai Zhang
Fudan University
zhangk@fudan.edu.cn

from https://csapp.cs.cmu.edu/3e/home.html

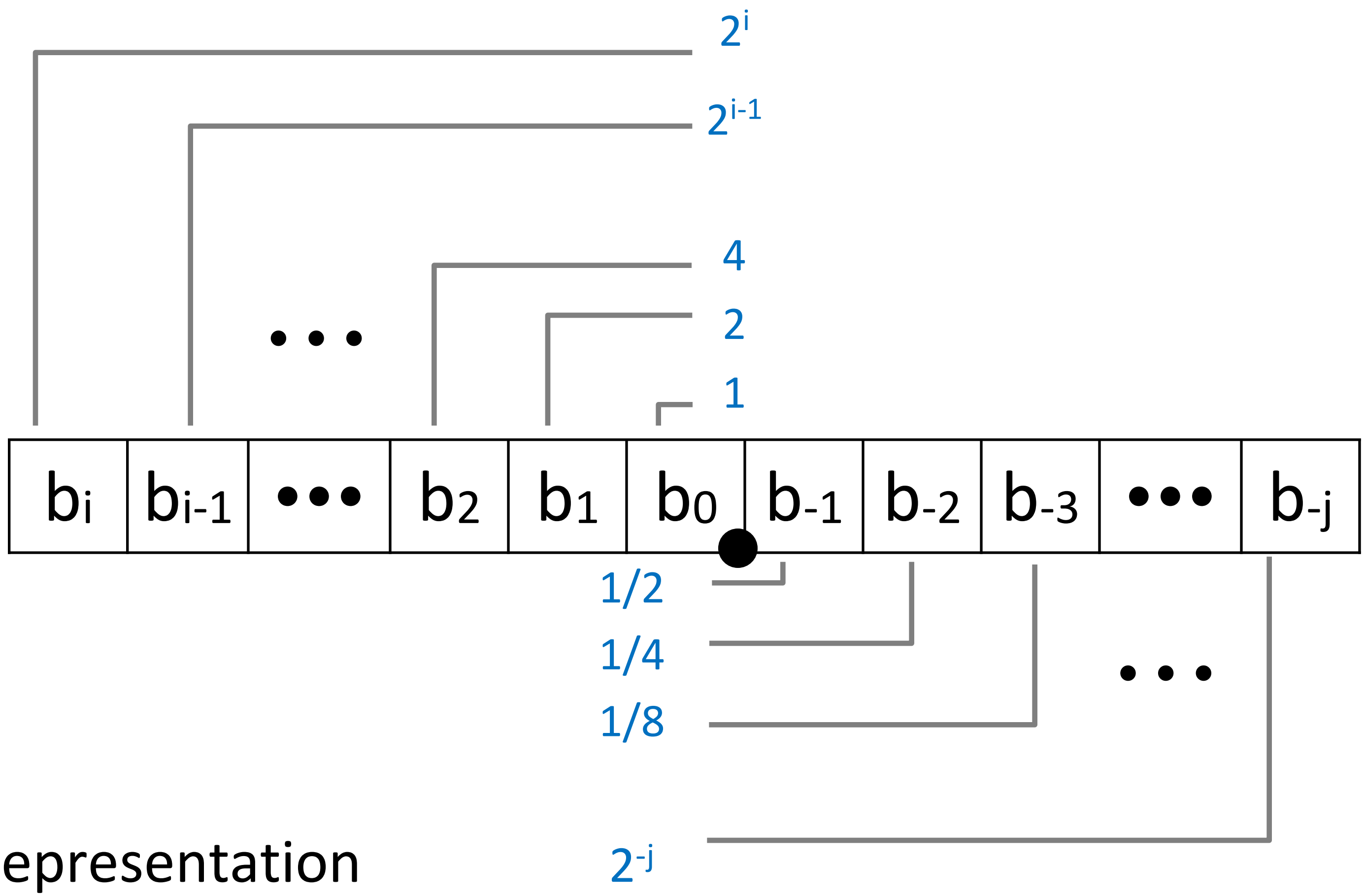
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional Binary Numbers

- What is 1011.101_2?

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- Value | Representation | $(10111 = 16+4+2+1 = 23)$

Value		Representation	
5 3/4	= 23/4	101.11_2	$= 4 + 1 + 1/2 + 1/4$
2 7/8	= 23/8	10.111_2	$= 2 + 1/2 + 1/4 + 1/8$
1 7/16	= 23/16	1.0111_2	$= 1 + 1/4 + 1/8 + 1/16$

- Observations
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Quiz Time!

Exercise 2.45

Representable Numbers

- Limitations?

Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 `0.0101010101[01]`$\ldots_2$
    - 1/5 `0.001100110011[0011]`$\ldots_2$
    - 1/10 `0.0001100110011[0011]`$\ldots_2$

- Limitation #2
  - Just one setting of binary point (二进制小数点) within the *w* bits
    - Limited range of numbers (very small values? very large?)

Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
  - Some CPUs don’t implement IEEE 754 in full
    e.g., early GPUs, Cell BE processor

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers
      in defining standard

Floating Point Representation

Example:
$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$

- How to represent very large or small float numbers?

Floating Point Representation

Example:
$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$

- Numerical Form:

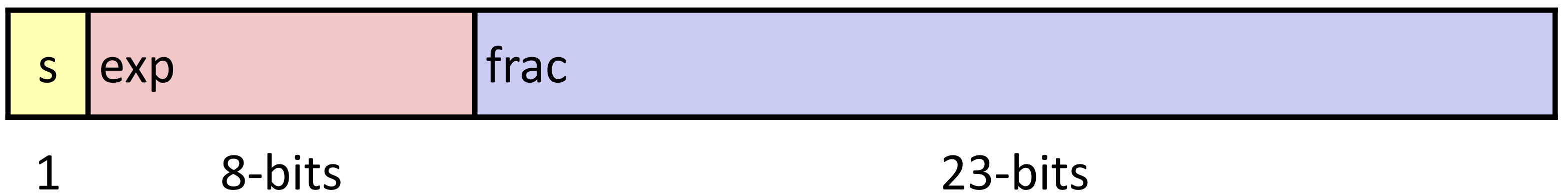
$$(-1)^s \, M \;\, 2^E$$

  - Sign bit **s** determines whether number is negative or positive
  - Significand **M** normally a fractional value in range [1.0,2.0)
  - Exponent **E** weights value by power of two

- Encoding
  - MSB s is sign bit s
  - exp field encodes E (but is not equal to E)
  - frac field encodes M (but is not equal to M)
  - Similar to Sign-Magnitude(原码, P47)

Precision options

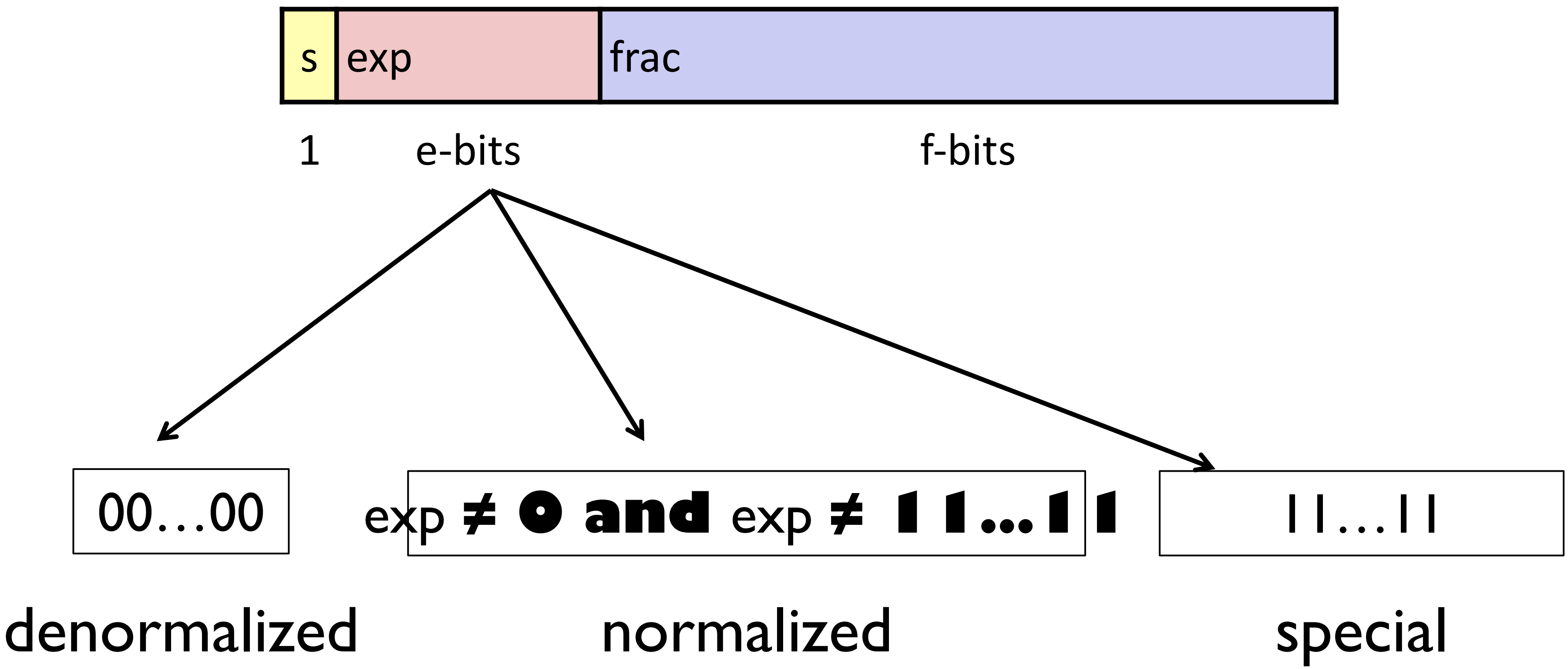
- Single precision: 32 bits
  $\approx$ 7 decimal digits, $10^{\pm 38}$



- Double precision: 64 bits
  $\approx$ 16 decimal digits, $10^{\pm 308}$

- Other formats: half precision, quad precision

Three “kinds” of floating point numbers



"Normalized" Values

$v = (-1)^s \, M \, 2^E$

- When: `exp` ≠ 000…0 and `exp` ≠ 111…1

- Exponent coded as a biased value: $E = \mathtt{exp} - \text{Bias}$
  - exp: unsigned value of exp field
  - Bias = $2^{k-1} - 1$, where k is number of exponent bits
    - Single precision: 127 (`exp`: 1…254, E: -126…127)
    - Double precision: 1023 (`exp`: 1…2046, E: -1022…1023)

- Significand coded with implied leading 1: $M = 1.\text{xxx…x}_2$
  - xxx…x: bits of frac field
  - Minimum when `frac`=000…0 (M = 1.0)
  - Maximum when `frac`=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s M \, 2^E$
$E = \mathbf{exp} - \text{Bias}$

Value: `float F = 15213.0;`

$$15213_{10} = 11101101101101_2$$
$$= 1.1101101101101_2 \times 2^{13}$$

Significand

$$M = 1.\underline{1101101101101}_2$$
$$\mathbf{frac} = \underline{1101101101101}0000000000_2$$

Exponent

$$E = 13$$
$$Bias = 2^{k-1} - 1 = 2^{8-1} - 1 = 127$$
$$\mathbf{exp} = E + \text{Bias} = 13 + 127 = 140 = 10001100_2$$

Result:

s	exp	frac
0	10001100	11011011011010000000000

C float Decoding Example

$$v = (-1)^s\ M\ 2^E$$
$$E = \mathbf{exp} - \text{Bias}$$

$\text{Bias} = 2^{k-1} - 1 = 127$

float: **`0xC0A00000`**

binary: ____ ____ ____ ____ ____ ____ ____ ____

E =

S =

M =

v = $(-1)^s$ M 2^E =

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example

$v = (-1)^s \, M \, 2^E$

$E = \textbf{exp} - \text{Bias}$

float: **0xC0A00000**

binary: 1100 0000 1010 0000 0000 0000 0000 0000

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

E =

S =

M = **1.**

$v = (-1)^s \, M \, 2^E$ =

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example

$$v = (-1)^s \, M \, 2^E$$
$$E = \mathbf{exp} - \text{Bias}$$

Bias = $2^{k-1} - 1 = 127$

float: **`0xC0A00000`**

binary: `1100 0000 1010 0000 0000 0000 0000 0000`

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

E = **`exp`** – Bias = 129 – 127 = 2 (decimal)

S = 1 -> negative number

M = `1.010 0000 0000 0000 0000 0000`

= `1 + 1/4 = 1.25`

v = $(-1)^s \, M \, 2^E = (-1)^1 * 1.25 * 2^2 = -5$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

How to represent 0 or numbers close to 0?

- Normalized numbers present 1.xxxx * 2^x

Denormalized Values

$v = (-1)^s \, M \, 2^E$
$E = 1 - \text{Bias}$

- Condition: exp = 000…0

- Exponent value: E = 1 – Bias (instead of **`exp`** – Bias, why?)
- Significand coded with implied leading 0: M = 0.xxx…x_2
  - **`xxx`**…**`x`**: bits of **`frac`**
- Cases
  - **`exp`** = `000…0`, **`frac`** = `000…0`
    - Represents zero value
    - Note distinct values: +0 and –0 (sign bit)
  - **`exp`** = `000…0`, **`frac`** ≠ `000…0`
    - Numbers closest to 0.0
    - Equispaced

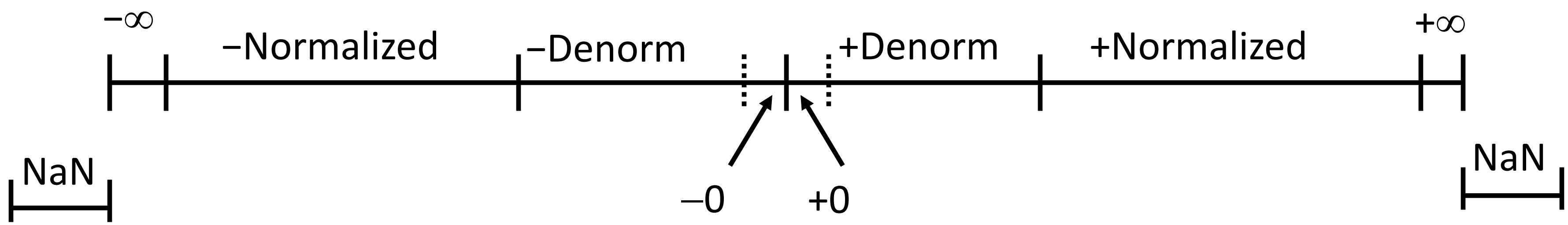
Special Values

- Condition: **`exp`** = **`111…1`**

- Case: **`exp`** = **`111…1`**, **`frac`** = **`000…0`**
  - **Represents value ∞ (infinity)**
  - Operation that overflows
  - Both positive and negative (+∞, -∞)
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞ (printf -> "inf")

- Case: **`exp`** = **`111…1`**, **`frac`** ≠ **`000…0`**
  - **Not-a-Number (NaN)**
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

Visualization: Floating Point Encodings



Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Tiny Floating Point Example

s	exp	frac
1	4-bits	3-bits

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the **`exp`**, with a bias of $2^{4-1} - 1 = 7$
  - the last three bits are the **`frac`**

- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (s=0 only)

$v = (-1)^s\, M\, 2^E$

norm: E = exp – Bias

denorm: E = 1 – Bias

	s	exp	frac	E	Value	
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	$(-1)^0(1+1/8)*2^{-6}$
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm

Dynamic Range (s=0 only)

$v = (-1)^s \, M \, 2^E$

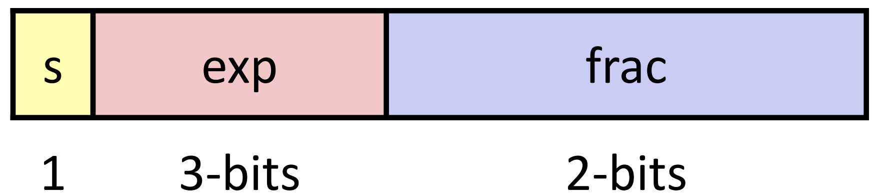
norm: E = exp – Bias

denorm: E = 1 – Bias

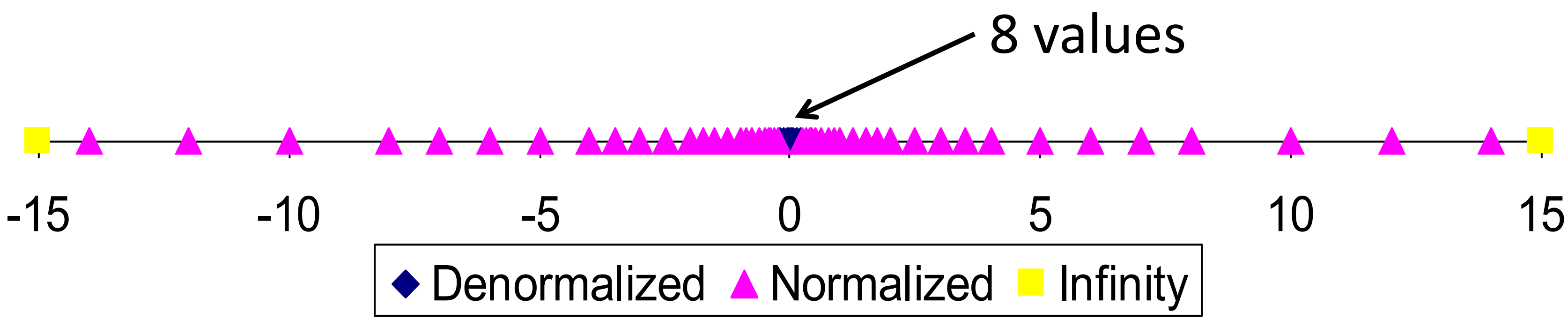
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	(-1)^0(0+1/4)*2^{-6}
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	(-1)^0(1+1/8)*2^{-6}
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

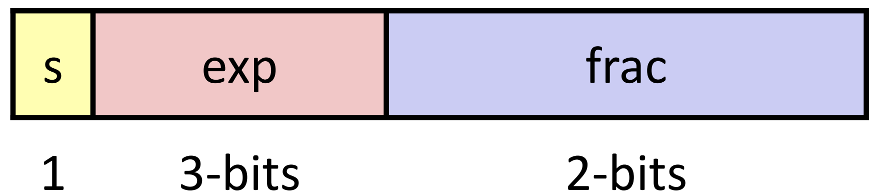


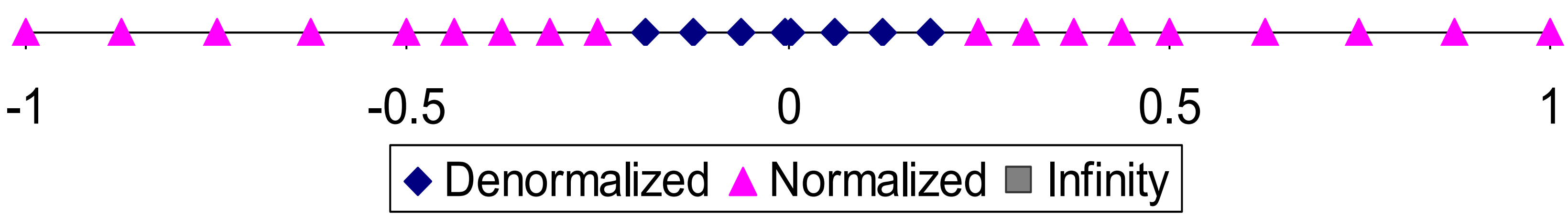
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Quiz Time!

Exercise 2.47

Special Properties of the IEEE Encoding

- FP Zero Same as Integer Zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield? The answer is complicated.
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Special Properties of the IEEE Encoding

- The smallest positive normalized value?
  - Exp = 1
  - Frac = 0
  - E = 1 – Bias = 1 – (2^(k-1) - 1) = -2^(k-1)+2
  - Value is 2^(-2^(k-1) + 2)

- The smallest positive denormalized value?
  - E = 1 – Bias = -2^(k-1)+2
  - Value is 2^(-2^(k-1) + 2) * 2^-n = 2^(-n-2^(k-1)+2)

- The largest denormalized value?
  - (1-2^(-n))*2*(-2*(k-1)+2)

- P82

Quiz Time!

Exercise 2.48

Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Floating Point Operations: Basic Idea

- $\mathtt{x} +_{\mathtt{f}} \mathtt{y} = \mathtt{Round}(\mathtt{x} + \mathtt{y})$

- $\mathtt{x} \times_{\mathtt{f}} \mathtt{y} = \mathtt{Round}(\mathtt{x} \times \mathtt{y})$

- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- Rounding Modes (illustrate with $ rounding)

	\$1.40	\$1.60	\$1.50	\$2.50	–\$1.50
Towards zero	\$1 ↓	\$1 ↓	\$1 ↓	\$2 ↓	–\$1 ↑
Round down (−∞)	\$1 ↓	\$1 ↓	\$1 ↓	\$2 ↓	–\$2 ↓
Round up (+∞)	\$2 ↑	\$2 ↑	\$2 ↑	\$3 ↑	–\$1 ↑
Nearest Even* (default)	\$1 ↓	\$2 ↑	\$2 ↑	\$2 ↓	–\$2 ↓

What is the statistic issue for Roundup (四舍五入) ?

*Round to nearest, but if half-way in-between then round to nearest even (偶数)

Closer Look at Round-To-Even

- Default Rounding Mode
  - **50% round up, 50% round down**
  - C99 has support for rounding mode management
  - All others are **statistically biased**
    - Sum of set of positive numbers will consistently be over- or under-estimated
- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - **Round** so that **least significant digit is even**
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100..._2$

- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- Round up conditions
  - Round = 1, Sticky = 1 → > 0.5
  - Round = 0 → < 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

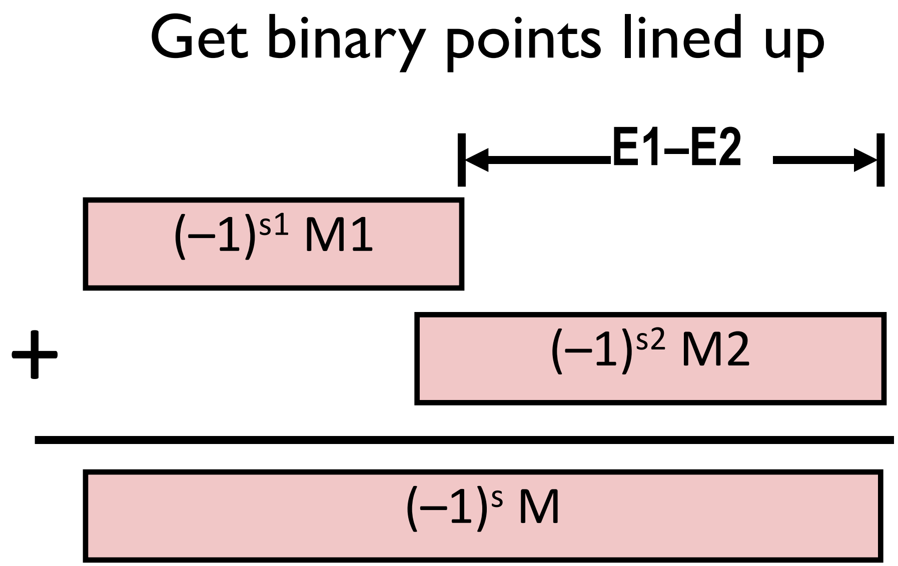
Fraction	GRS	Incr?	Rounded
1.0000000	000	N	1.000
1.1010000	100	N	1.101
1.0001000	010	N	1.000
1.0011000	110	Y	1.010
1.0001010	011	Y	1.001
1.1111100	111	Y	10.000

Quiz Time!

Exercise 2.50

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume E1 > E2
- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1
- Fixing
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit **`frac`** precision



$$1.010 * 2^2 + 1.110 * 2^3 = (0.1010 + 1.1100) * 2^3$$
$$= 10.0110 * 2^3 = 1.00110 * 2^4 = 1.010 * 2^4$$

Mathematical Properties of FP Add

Compare to those of Abelian Group（阿贝尔群，P62）

Closed under addition? Yes

But may generate infinity or NaN

Commutative? Yes

Associative? No

Overflow and inexactness of rounding

`(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`

0 is additive identity(加法单位元)? Yes

Every element has additive inverse? Almost

Yes, except for infinities & NaNs

Monotonicity

$a \geq b \Rightarrow a+c \geq b+c$? Almost

Except for infinities & NaNs

FP Multiplication

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \times\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2
- Fixing
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit **`frac`** precision
- Implementation
  - Biggest chore is multiplying significands（尾数）

4 bit significand: $1.010 * 2^2 \times 1.110 * 2^3 = 10.0011 * 2^5$
$= 1.00011 * 2^6 = 1.001 * 2^6$

Mathematical Properties of FP Mult

Compare to Commutative Ring (交换环)

Closed under multiplication? Yes

But may generate infinity or NaN

Multiplication Commutative? Yes

Multiplication is Associative? No

Possibility of overflow, inexactness of rounding

Ex: `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)= 1e20`

1 is multiplicative identity? Yes

Multiplication distributes over addition(a*(b+c) = a*b+a*c)? No

Possibility of overflow, inexactness of rounding

`1e20*(1e20-1e20)= 0.0,`

`1e20*1e20 - 1e20*1e20 = NaN`

Monotonicity

$a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? Almost

Except for infinities & NaNs

Today: Floating Point

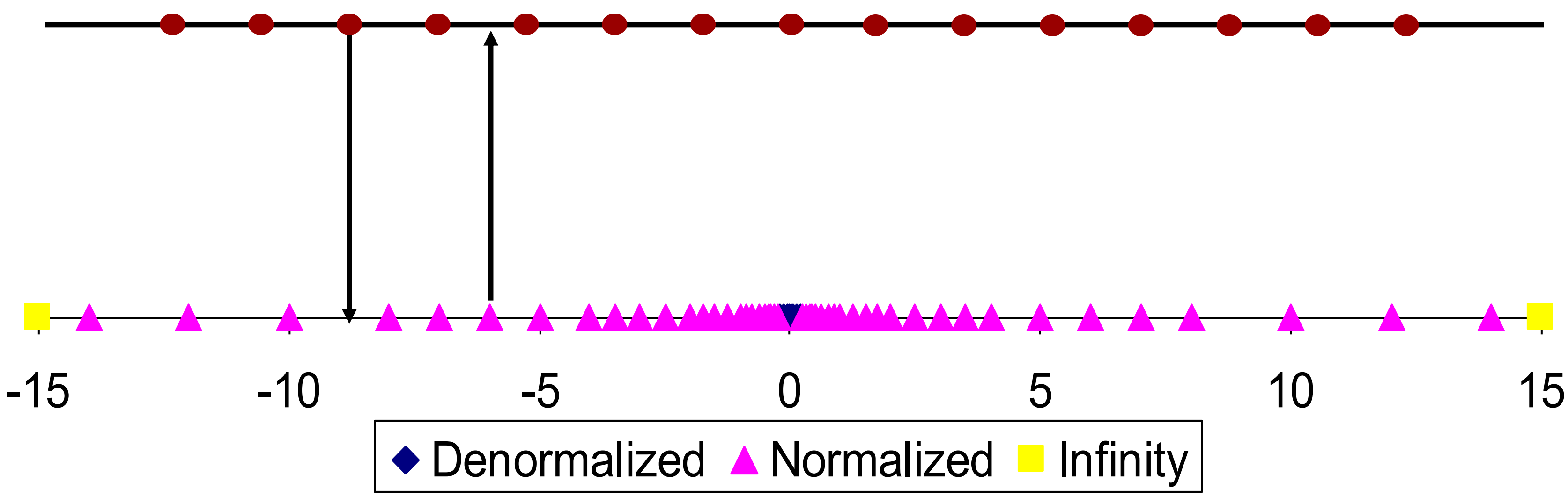
- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double (double has 64 bits, higher precision)`**
    - Exact conversion, as long as **`int`** has ≤ 53 (1 s + 52 frac) bit word size
  - **`int`** → **`float (no overflow, may rounding)`**
    - Will round according to rounding mode

int vs float

- There is no one-one mapping between int and float
  - int : uniform distributed in the space



Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

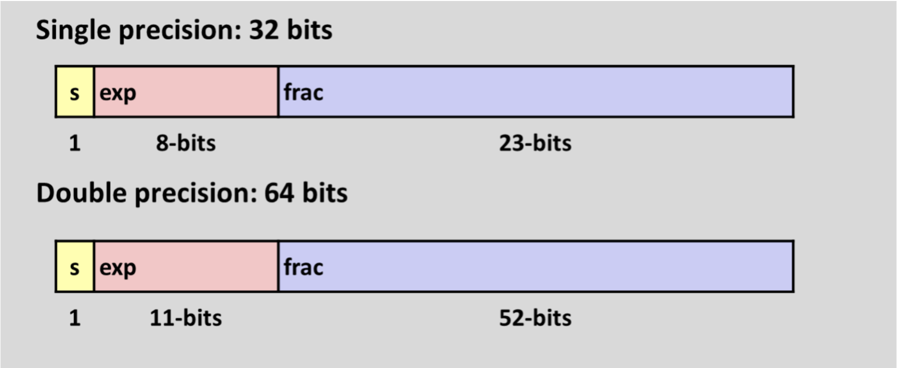
```
int x = …;
float f = …;
double d = …;
```

<u>Assume neither **d** nor **f** is NaN</u>

- `x == (int)(float) x` ✗
- `x == (int)(double) x` ✓
- `f == (float)(double) f` ✓
- `d == (double)(float) d` ✗
- `f == -(-f);` ✓
- `2/3 == 2/3.0` ✗
- `d < 0.0` ⇒ `((d*2) < 0.0)` ✓
- `d > f` ⇒ `-f > -d` ✓
- `d * d >= 0.0` ✓
- `(d+f)-d == f` ✗

Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form M x 2^E
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers



Additional Slides

Creating Floating Point Number

- Steps
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- Case Study
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

`128`	`10000000`
`15`	`00001101`
`33`	`00010001`
`35`	`00010011`
`138`	`10001010`
`63`	`00111111`

Normalize

- Requirement
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
`128`	`10000000`	`1.0000000`	`7`
`15`	`00001101`	`1.1010000`	`3`
`17`	`00010001`	`1.0001000`	`4`
`19`	`00010011`	`1.0011000`	`4`
`138`	`10001010`	`1.0001010`	`7`
`63`	`00111111`	`1.1111100`	`5`

Postnormalize

- Issue
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Numeric Result
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

Interesting Numbers `{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$

- Smallest Pos. Denorm.
  - Single ≈ 1.4×10^{-45}
  - Double ≈ 4.9×10^{-324}
- Largest Denormalized
  - Single ≈ 1.18×10^{-38}
  - Double ≈ 2.2×10^{-308}
- Smallest Pos. Normalized
  - Just larger than largest denormalized
- Largest Normalized
  - Single ≈ 3.4×10^{38}
  - Double ≈ 1.8×10^{308}