Stochastic Simulation
Introduction

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Practicalities

- Notes will handed out next week, some will be available online
- Course evaluation is: passed/not passed - based on lab reports, report over final project, and possibly oral presentation of project.
- Teachers:
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  - Flavia Dalia Frumosa fdal@dtu.dk, Jesper Fink Andersen jfan@dtu.dk, Oscar Peralta Gutierrez osgu@dtu.dk

Significance

- One of the most (The most?)important Operations Research techniques
- Many modern statistical techniques rely on simulation

What is simulation?

- (From *Concise Oxford Dictionary*): To simulate: To pretend, to act like, to mimic, to imitate.
- Here: *Computer experiments with mathematical model*
- **Stochastic simulation**

  To (have a computer) simulate a system which is affected by randomness.

  Narrow sense: To generate (pseudo)random numbers from a prescribed distribution (e.g. Gaussian)
- Computer experiments with mathematical model
- General engineering technique
- Analytical/numerical solutions

Why simulate?

- Real system expensive
- Mathematical model to complex
- Get idea of dynamic behaviour

Related areas

- Statistics
- Computer science
- Operations research

Target group

- Methodology course of general interest
- Of special importance for students specialising in
  - Computer science
  - Statistics
  - Operations Research
  - Planning and management

Course goal

- Topics related to scientific computer experimentation
- Specialised techniques
  - Variance reduction methods
  - Random number generation
  - Random variable generation
  - The event-by-event principle
- Simulation based statistical techniques
  - Markov chain Monte Carlo
  - Bootstrap
- Validition and verification of models
- Model building

Recommended reading

- Sheldon M. Ross: Simulation, fifth edition, Academic Press 2013
- Averill M. Law: Simulation Modeling and Analysis, McGraw-Hill 2015, ISBN 0-07-340132-3
- Jerry Banks, John S. Carson II, Barry L. Nelson, David M. Nicol: Discrete-Event System Simulation, Prentice and Hall 1999, ISBN 0-13-088702-1
- Brian Ripley: Stochastic Simulation, John Wiley & Sons 1987, ISBN 0-471-818884-4.
- Reuven Y. Rubinstein and Benjamin Melamed: Modern Simulation and Modelling, John Wiley & Sons 1998, ISBN 0-471-17077-1
- Jack P. C. Kleijnen: Statistical Tools for Simulation Practitioners, Marcel Dekker 1987, ISBN 0-8247-7333-0

Knowledge/science in simulation

- Modelling skill
- Statistical methods - it is necessary to understand statistical methodology
- OR - Stochastic Processes
- Technical skills
  - Random number generations
  - Sampling from distributions
  - Variance reduction techniques
  - Statistical techniques bootstrap/MCMC
- General purpose/and specialised simulation software

Discrete versus continuous

- Discrete event simulation
- as opposed to continuous simulation
- mixed models

Probability basics

- $0 \leq P(A) \leq 1 \qquad P(\Omega) = 1 \qquad P(\emptyset) = 0$
- $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$
- Complement rule $P(A^c) = 1 - P(A)$
- Difference rule for $A \subset B$: $P(B \cap A^c) = P(B) - P(A)$
- Inclusion, exclusion for 2 events

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Conditional probability: for A given B (partial information): $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Multiplication rule: $P(A \cap B) = P(B)P(A|B)$
- Law of total probability: (B_i is a partitioning), $P(A) = \sum_i P(B_i)P(A|B_i)$
- Bayes theorem: (B_i is a partitioning): $P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)}$
- independence: $P(A|B) = P(A|B^c) \qquad (P(A \cap B) = P(A)P(B))$

Random variables

- Mapping from sample space to the real line
- Probabilities defined in terms of the preimage
- Most probabilitistic calculations are performed with only a slight reference to the underlying sample space

Random variables

- Random variables: *maps outcomes* to real values
  - ◇ Distribution $P(X = x)$ $\quad \sum_x P(X = x) = 1$
  - ◇ Joint distribution
    $P(X = x, Y = y)$ $\quad \sum_{x,y} P(X = x, Y = y) = 1$
  - ◇ Marginal distribution $P_X(X = x) = \sum_y P(X = x, Y = y)$
  - ◇ Conditional distribution $P(Y = y|X = x) = \frac{P(X=x,Y=y)}{P_X(X=x)}$
  - ◇ independence $P(Y = y, X = x) = P_X(X = x)P_Y(Y = y)$
- Mean value $E(X) = \sum x \cdot P(X = x)$
- General expectation $E(g(X)) = \sum g(x) \cdot P(X = x)$
- Linearity $E(aX + bY + c) = aE(X) + bE(Y) + c$

Continuous random variables

- Uniform distribution of two variables: $P\left((x,y)\in C\right) = \frac{A(C)}{A(D)}$
- Continuous random variables
  - ◇ Density: $f(x) \geq 0, \quad \int f(x)\mathrm{d}x = 1, \quad P(X \in \mathrm{d}x) = f(x)\mathrm{d}x$
  - ◇ Mean, variance (moments): $\mathsf{E}(X) = \int x f(x)\mathrm{d}x$
    $\mathsf{E}(g(X)) = \int g(x)f(x)\mathrm{d}x$, $\mathsf{E}\left(X^k\right) = \int x^k f(x)\mathrm{d}x$
- Normal distribuion: $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad Z = \frac{X-\mu}{\sigma}$
- Joint densities

$$f(x,y)\mathrm{d}x\mathrm{d}y = P(x \leq X \leq x+\mathrm{d}x, y \leq Y \leq y+\mathrm{d}y), f(x,y) \geq 0$$

- Joint distribution

$$F(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\mathrm{d}u\mathrm{d}v$$

Continuous random variables continued

- Conditional continous distributions $f(y|X=x) = \frac{f(x,y)}{f_X(x)}$
- Integral version of law of total probability
  $P(A) = \int P(A|X=x) f_X(x) \mathrm{d}x$
- Conditional expectation $E(Y) = E(E(Y|X))$
- Covariance/corellation

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{SD(X)SD(Y)}$$

- (X,Y) independent $\Rightarrow Corr(X,Y) = 0$
- Variance of sum of variables
  $Var\left(\sum_{k=1}^{N} X_k\right) = \sum_{k=1}^{n} Var(X_k) + 2\sum_{1\leq j<k\leq n} Cov(X_j, X_k)$
- Bilinearity of covariance
  $Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$