Minority Games

El Farol Bar Problem

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1 Minority Game Introduction

The minority game is a concept where an agent benefits from being in the minority of the total set of the agents. The El Farol Bar Problem introduced by Brian Arthur is an example of minority games where agents can decide to go to the bar on a Friday night or stay at home. The dilemma arises from the fact that a visit to the bar is more enjoyable when it is not overcrowded, making it preferable to go out. However, if the bar is too crowded, one would prefer staying at home. This scenario presents individuals with two options: to go to the bar or to stay home, each leading to varying outcomes depending on the choice made by the majority.

For clarity and simplicity in our theoretical framework, we define the following variables:

- **n = number of agents (default: 101)**
- **c = capacity (default: 60)**

Decision outcomes are in binary terms:

- **0: staying home;**
- **1: attending the bar.**

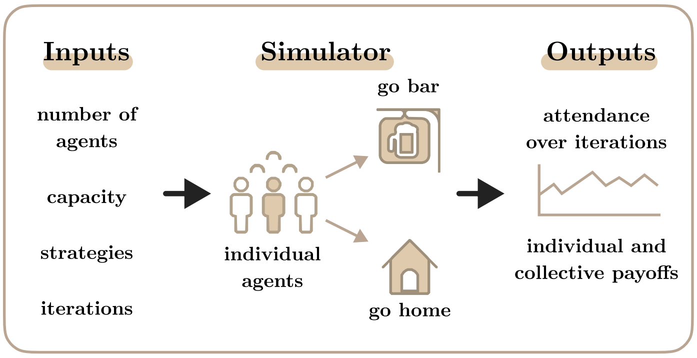

Figure 1: El Farol Bar Game Visual

This report delves into the mechanisms of optimally navigating such games, exploring the intricacies of different strategic approaches and contexts. These include ***static games***, where decisions are made without prior information; ***repeated games***, which involve applying static strategies over multiple weeks; and repeated games with ***inductive strategies***, where players adapt and learn to optimise their decisions. Through this analysis, we are aiming to shed light on the dynamics of minority games and the strategies that lead to successful outcomes.

2 Single-Shot Static Games

2.1 Game Description

In a static game, all agents make decisions simultaneously without communication, memory, or consideration of past actions. The challenge of static games, particularly in scenarios like the minority games, is the absence of a universally best option. If all players adopt the same strategy (either to go or not to go), the bar in the El Farol bar problem would always be either completely empty or overcrowded. Thus, players must adopt either distinct deterministic strategies or **utilise symmetric strategies with some randomness** to avoid adopting a single strategy, which would prevent achieving the minority status necessary for success.

2.2 Theoretical Analysis

Let us begin by defining the agents and their payoffs. In a static two-player game framework, we consider one individual, for whom we aim to optimise the strategy, and aggregate the rest as a single ‘collective player’ representing the (N-1) others.

		N-1 Players	
	Payoff	***GO***	***Not GO***
Player 1	***GO***	*(-1, -1)*	*(1, 0)*
	Not GO	*(0, 1)*	*(0,0)*

Figure 2: Payoff Table for Each Agent

We identify two Nash equilibrium, (Not Go, Go) and (Go, Not Go), indicating that Player 1's optimal strategy is to stay home if the collective goes, and to go if the collective stays home. In a static game, since Player 1 cannot predict the actions of the collective in advance, adopting mixed strategies, where decisions are based on a probabilities becomes necessary. Assuming a symmetric mixed strategy, where all players adopt the optimal strategy, there are ***N* agents**, each with a **probability *p*** of going to the bar, with **capacity *C***. The expected number of attendees is then ***p* × *N***.

Define the following payoffs:

U_{go} **= 1: Going when it's not overcrowded**
$U_{crowded}$ **= -1: Going when it's overcrowded**
U_{home} **= 0: Stay at home**

For a symmetric based mixed strategy to be a Nash equilibrium, the expected payoff of an agent going to the bar must equal the payoff of not going, making a player indifferent between going and not going. This equilibrium occurs when the probability of the bar not being overcrowded (C) makes the player neither gain nor lose by changing their decision. The agent will go to the bar if they believe ≤ *C - 1* people will go.

Let X be the random variable defining the number of other agents going to the bar. This gives a binomial distribution ***X = Bin(N-1, p)***. This represents that N-1 other players decides independently with probability p. Therefore, we have the following:

$$P(X = x) = \binom{N-1}{x} p^x (1-p)^{N-1-x}$$

The expected utility of going to the bar E[Ugo] is calculated as follows:

$$E[U_{go}] = \sum_{x=0}^{C-1} P(X = x) \cdot U_{go} + \sum_{x=C}^{N-1} P(X = x) \cdot U_{overcrowded}$$

Now, we want $E[U_{go}] = E[U_{home}]$. Since the payoff of $U_{home} = 0$, we have $E[U_{home}] = 0$. Therefore, we want to define p such that $E[U_{go}] = 0$. This gives us the following:

$$\sum_{x=0}^{C-1} P(X = x) \cdot U_{go} + \sum_{x=C}^{N-1} P(X = x) \cdot U_{overcrowded} = 0$$

Solving this equation gives us $\boldsymbol{p = C \,/\, N}$ (the optimal probability p is therefore the maximum capacity over the number of agents), achieving an equilibrium where no single agent has an incentive to deviate unilaterally from their chosen strategy. The symmetric mixed strategy Nash equilibrium ensures that on average, the bar reaches its optimal capacity without becoming overcrowded and while balancing the desire of all agents involved.

2.3 Discussion

This section delves into the practical implications of the symmetric mixed strategy assumption in the context of the El Farol Bar problem. By positing a scenario where the bar's capacity is 60% of the total number of agents, it provides a tangible example of how varying participation probabilities can impact overall payoffs. Specifically, it illustrates that if the majority of agents choose to go to the bar 40% of the time, those who choose to attend more frequently could potentially enjoy higher payoffs. Conversely, if attendance rates increases to 80%, the appeal of going decreases, underscoring the dynamic nature of strategic decision-making in this context. There is stability in the probability of agents going to the bar only if p as seen in the Nash equilibria is equal to C / N.

3 Repeated Static Games

3.1 Game Description

The extension of our analysis to repeated static games explores how agents' payoffs evolve over repeated instances. The game is therefore a static game, where agents cannot communicate with one another nor can they learn from their past experiences, however they play the game numerous times. The best strategy to maximise individual payoff over a long set of repetitions is to follow the same strategy each round, as agents do not adapt for repeated static games.

3.2 Theoretical Analysis

Using a Python-based program, we simulated the static game numerous times to study how agents behave over multiple iterations, and their individual and collective payoffs. The first simulation is designed to study the individual payoff, with the second simulation designed to understand the collective utility, measuring the mean, sum, and distribution.

While the parameter is adjustable within the code, we use the default parameters of N = 101 and C = 60 over 50 iterations:

To calculate the expected collective payoff, we compute the expected payoff at each value of k (actual number of people going). This gives us:

$$E = \sum_{k=0}^{60} P(X = k) \cdot k - \sum_{k=61}^{101} P(X = k) \cdot k$$

This formula returns expected collective payoff of approximately **0.609**.

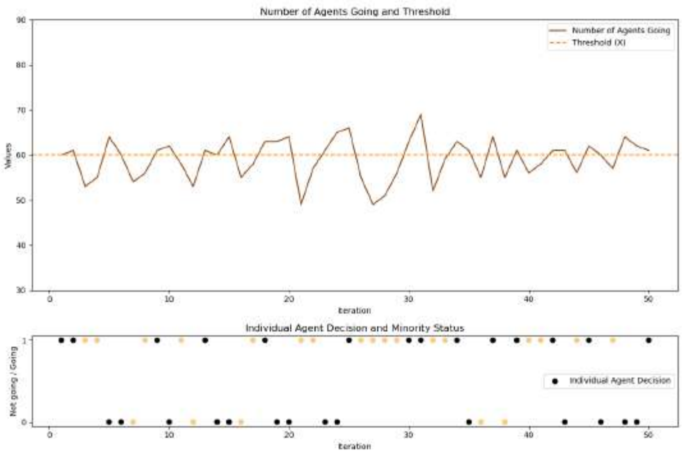

Figure 3: Results of Repeated Static Games (Individual Agent Payoff)

In this example, we observed an individual payoff of **0.06** per week, falling in the distribution of an expected payoff.

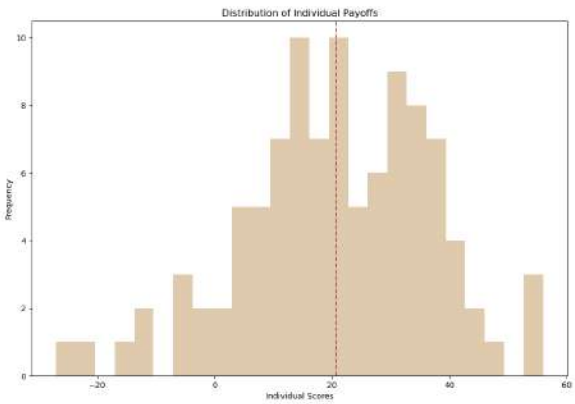

Figure 4: Collective Payoff Distribution

Although the collective payoff provides a comprehensive overview of the simulation's performance, it doesn't fully capture individual agents. To address this, we analysed all individual payoffs as a histogram. This analysis was conducted over 1,000 iterations, resulting in a cumulative payoff of 2,094 and average individual payoff of **0.02** per week.

While some agents outperformed others, the distribution of payoffs is observed to be normal, suggesting a degree of fairness.

4 Repeated Inductive Games

4.1 Game Description

The repeated static game is simulated with one strategy where all agents make their choices independently and probabilistically each week. Hence, relying on leveraging the best probability with a degree of randomness. Repeated inductive games, however, acknowledge that agents may adopt varying strategies, reflecting the diverse decision-making processes observed in real-life scenarios. For example, if the bar has been crowded for consecutive weeks, some might choose not to go, anticipating a continuation of this trend. Conversely, others might choose to go, predicting a break in the pattern.

We want to simulate these agent behaviours with each agents making adaptive decisions to **maximise their own individual utility**. Repeated inductive games are designed to capture this variability by providing agents with a portfolio of strategies and a value to determine which are active. This enables agents to dynamically select and adjust their strategies based on its performance over iterations. The simulation assesses how agents' strategies evolve over time, analysing the outcomes for individual players as well as the collective results.

4.2 Theoretical Analysis

In the examination of repeated inductive games, we explore and refine our optimal strategy through iterative testing of multiple approaches. This section is structured as follows:

1. **brute force method for multiple strategies;**
2. **predictive strategies (25) for variability;**
3. **weighted strategy evaluation to fine-tune performance;**
4. **2 strategies with intuitive decision-making.**

This methodology allows a systematic evaluation to assess the effectiveness of each approach in achieving optimal outcomes.

4.2.1 Approach 1: Rigid Strategies

4.2.1.1 Methodology

The initial strategy is a brute force method, equipping agents with strategies to make choices influenced by a history of outcomes extending back m (3) weeks with strategy profile ($\sigma = 5$), indicating how many of the 256 strategies each agents will select from. This method emphasises the agents' ability to learn from previous weeks.

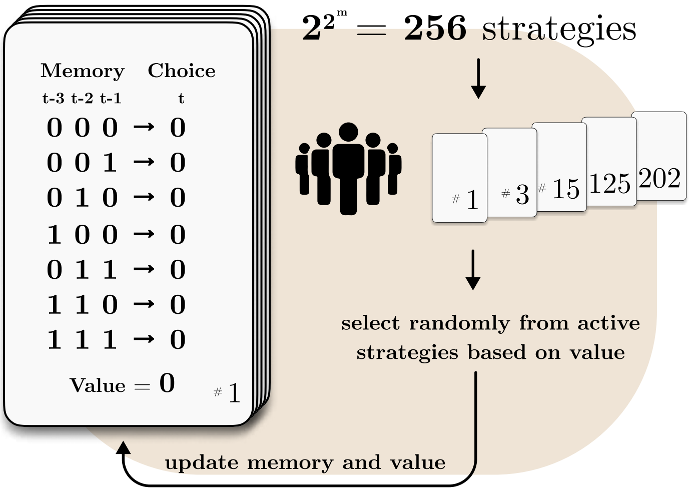

Figure 5: Repeated Inductive Games with $m = 3$, $n = 101$, $\sigma = 5$

There are 256 potential strategy combinations when $m = 3$, derived from 2^m potential memory sets, and binary choices for each. The active strategy is the one that holds the greatest value; when there are equal active strategies, the strategy is chosen at random. Value increments by 1 following a successful outcome, with further refinement discussed in Section 4.2.3.

4.2.1.2 Results and Discussion

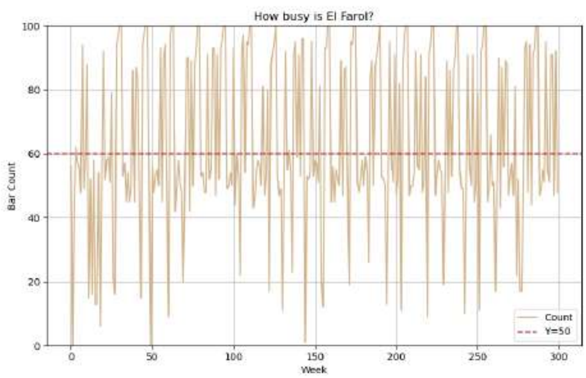

Figure 6: Approach 1 Simulation Results

Despite the wide array of strategies (256 in total), the lack of behavioral diversity and the deterministic nature of these strategies limited the system from effectively converging to 50. Considering the limitations observed, we introduce predictive strategies to pivot towards a more flexible decision-making model. By reducing the number of strategies and focusing on their distinctiveness and adaptability, this approach better mimics real-world decision-making processes, achieving more realistic and efficient simulation outcomes.

4.2.2 Approach 2: Predictive Strategies

4.2.2.1 Methodology

The Intuitive Strategies approach introduces a shift from the rigid strategy selection to a more dynamic, forecast method. This revised approach incorporates a set of 25 predictive strategies and strategy profile ($\sigma = 3$), tailored to reflect a broader variance of human-like decision-making:

Example strategies include:

- Previous Week's Outcome
- Minimum/Maximum over X weeks
- Mean over X weeks
- Median over X weeks
- Standard Deviation over X weeks
- Mirror Image Strategy
- Weighted Trend
- Inverse Trend

The remaining methodology is the same as the rigid strategies in Section 4.2.1.

4.2.2.2 Results and Discussion

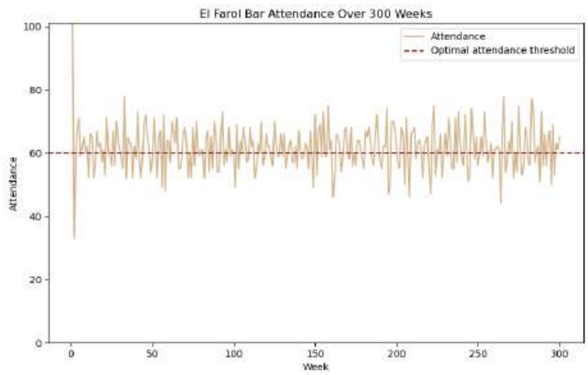

Figure 7: Approach 2 Simulation Results

Figure 7 reveal oscillation around the target convergence point of 60 (C) attendees, indicating that agents are iteratively refining their strategies. The progression toward this equilibrium is observed to be unstable. We hypothesise that this may be attributed to the current methodology of defining 'active strategies', where a strategy's value increments by 1 following a successful outcome. This linear approach to strategy valuation might not sufficiently capture the dynamic and often non-linear nature of decision-making processes.

4.2.3 Approach 3: Weighted Valuation

4.2.3.1 Methodology

To assess the strategy evaluation method, the third approach introduces a weighted strategy valuation method. This method diverges from the binary evaluation (1 or 0) of minority outcomes, favouring forecasts that are closer to the actual attendance. In addition to adding weighted values to successful strategies, a weighted penalty mechanism for unsuccessful strategies is introduced to influence agents to evaluate their optimal strategies. When doing so, we must consider the value vs quality trade-off:

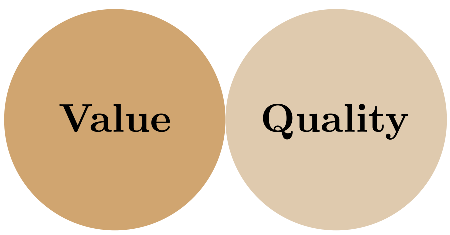

Figure 8: Value vs Quality Trade-off

The concept of 'value' refers to making the correct choice, while 'quality' refers to the precision of the prediction. For example, if the prediction is correct, the error should not matter for that particular case. While 'quality' is important to achieve the correct 'value', the primary concern is having the correct 'value'. This suggests that a strategy should not be penalised when deviating from its predicted value as long as it leads to the correct decision. However, 'quality' should be accounted for to improve decision-making in future rounds. Therefore, we incorporated the following formula:

$$E = \begin{cases} |\text{predicted - actual}| & : \text{value is True} \\ 0 & : \text{value is False} \end{cases}$$

$$score_{new} = score_{current} + \alpha \cdot (1 - E)$$

After multiple iterations, the default α value is set to 0.1, defining the magnitude of quality. In this scenario, a penalty is only applied when the prediction is incorrect, therefore not penalising a strategy if it leads to the correct outcome.

4.2.3.2 Results and Discussion

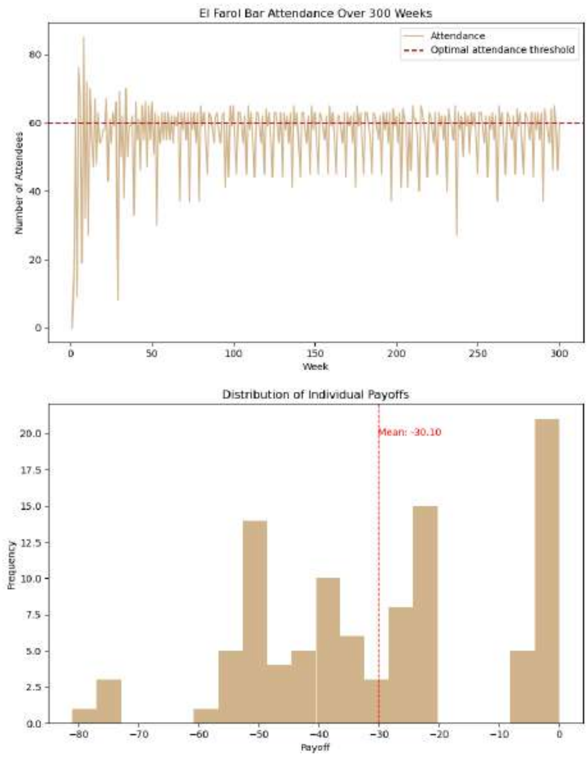

Figure 9: Approach 3 Simulation Results

Notably, the overall payoffs are negative. This is because the attendance oscillates around the target capacity. For instance, if the bar is near capacity and an additional person enters, the payoff for all attendees switches from +1 to -1, significantly reducing the cumulative payoff. This results in an overall negative payoff over multiple weeks. To address this issue, we considered two approaches: 1) artificially lowering the capacity to reduce the likelihood of overcrowding, and **2) refining predictive strategies so that agents are less likely to attend if they forecast higher attendance**. We opted for the latter method, as it better reflects the individual agents' self-interested and educated decisions, thereby reflecting a more nuanced approach to managing attendance relative to the capacity.

4.2.4 Approach 4: Adaptive Agents

4.2.4.1 Methodology

To improve the individual and collective payoff, we introduce additional complexity based on forecast values. For the final approach, we employed **2 strategy categories** to effectively compare static and adaptive strategies' effectiveness in optimising attendance and maximizing payoffs.

- **strategy 1:** 50 agents will use the Repeated Static Games strategy, with $p = \mathrm{C/N}$.
- **strategy 2:** 51 agents will use the predictive strategy and proposed complexity.

4.2.4.2 Results and Discussion

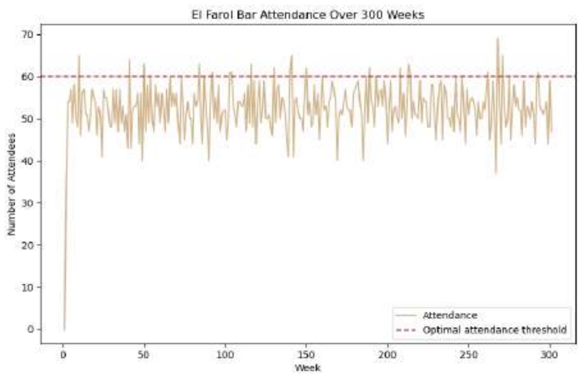

Figure 10: Final Approach Simulation Results

This graph highlights the model's success in approaching the desired capacity without frequently exceeding it, while ensuring that all agents act for their individual payoff. The collective payoff increased to 12,413 over 300 weeks (+41 per week), representing a significant improvement over prior negative outcomes.

Figure 11: Performance of Strategy Types

Figure 11 indicates that static agents outperformed adaptive agents, possibly because adaptive agents exhibited greater caution in not exceeding capacity, in contrast to static agents who based their decisions solely on probabilities for personal gain. Additionally, the box plot reveals an equitable distribution of payoffs among agents, with certain adaptive agents achieving superior results, possibly attributed to their forecast quality.

4.3 Discussion

Inductive strategies have presented an effective alternative to symmetric strategies, leading to improved individual and collective payoffs for all agents. Through four distinct approaches, we refined an inductive method to optimise collective payoff while ensuring that each agent prioritises their own payoff. Ideally, consistently achieving a Nash Equilibrium would yield the most desirable outcomes; however, this goal is challenging to achieve without communication. Our results, achieved with limited resources and two strategies, indicates substantial scope for further improvements and refinements moving forward.

5 Real-World Applications

Minority games, with their essence in predicting and being part of the minority choice, find intriguing applications in the real world, especially in areas with competition and resource limitations. For instance, in financial markets, traders aim to buy low and sell high, often striving to be in the minority that sells before a downturn or buys before an upturn; in energy consumption patterns, consumers plan to use electricity during off-peak times to benefit from lower rates; in traffic management, driver's will choose the least congested route to minimise travel time. However, in all cases, if too many agents opt for what is believed to be the minority option, it becomes the majority, negating any advantage.

The application of minority games extends beyond these examples. It encompasses any game that requires strategic decision-making against collective behavior. The underlying strategies—whether probabilistic, predictive, or unexplored advanced approaches like reinforcement learning—offer a framework for navigating and influencing complex systems.

Appendix

How to navigate our repository!

- **minority-game-theory-and-mechanism**
  - **Static**
    - singleshot_static.py → **Page 3 (Section 2.2)**
    - payoff_static.py → **Page 4 (Section 3.2)**
    - repeated_static.py → **Page 4 (Section 3.2)**
  - **Repeated**
    - 4.2.1.py → **Page 5 (Section 4.2.1)**
    - 4.2.2.py → **Page 5 (Section 4.2.2)**
    - 4.2.3.py → **Page 6 (Section 4.2.3)**
    - 4.2.4.py → **Page 7 (Section 4.2.4)**
  - **Future Exploration**

Work Distribution

We allocated the workload equitably and collaborated throughout the entire project, with each member assuming leadership roles for various work packages.

Work Packages	Ken	Felix
Game Definition	50%	50%
Theoretical Analysis	50%	50%
Implementation & Simulation of Static Game	45%	55%
Implementation & Simulation of Repeated Game	45%	55%
Proposal of Inductive Strategies	55%	45%
Implementation & Simulation of Inductive Strategies	55%	45%
Analysis of Simulation Outcomes	50%	50%
Minority Game in the Real World	45%	55%
Report Writing	50%	50%