Investigation of Model and Premium Uncertainty in Longevity-Linked Securities

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Mathematical Finance

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Table of Contents

Acceptance Page **3**

Acknowledgments **4**

Summary of the Thesis **5**

Anti-Plagiarism Declaration **6**

1 Introduction **7**
1.1 Background of the Study 7
1.2 Statement of the Problem 10
1.3 Pricing Survivor Contracts 11
1.4 Mortality Model and Premium Principle Uncertainty 12
1.5 Framework Overview 15
1.6 Scope and Limitations 15
1.7 Organization of Thesis 17

2 Preliminaries **18**
2.1 Exposed to Risk, Force of Mortality, and Survival Probabilities 18
2.2 Mortality Models 20
2.3 Premium Principles 26
2.3.1 Desirable Properties of Premium Principles 29
2.4 Longevity-Linked Instruments 30
2.4.1 Survivor Forward 30
2.4.2 Survivor Swaps 31

3 Methodology **33**
3.1 Data and Assumptions 33
3.2 Model Parameters 33
3.3 Residual Analysis 37
3.4 Pricing Parameter Calibration 39
3.5 Pricing Longevity Linked Instruments 42

4 Results and Discussion **44**

5 Conclusions and Recommendations **51**
5.1 Summary and Conclusion 51
5.2 Recommendations and Limitations 53

A Code and Implementation **54**

References **55**

Acceptance Page

The Faculty of the Department of Mathematics of Ateneo de Manila University accepts the undergraduate thesis entitled

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Summary of the Thesis

Longevity risk poses significant financial challenges for pension and annuity providers as people live longer than expected. Transferring this risk to capital markets through standardized longevity-linked derivatives like survivor forwards and survivor swaps has potential to mitigate this risk. Still, more consensus is needed on appropriate pricing models. This thesis investigates model and premium uncertainty by examining the uncertainty in S-forward and S-swap valuations across different mortality models, including Lee-Carter, Renshaw-Haberman, Cairns-Blake-Dowd, and M6. Pricing principles considered include risk-neutral transforms like Wang, proportional hazard, dual power, Gini, exponential transform, and real-world principles like standard deviation, variance, and mean absolute deviation. Results reveal that more significant uncertainty stems from the choice of mortality model rather than the premium principle. This analysis highlights uncertainty in longevity derivative pricing and the need for further mortality and market data to develop standards. It also suggests insurers should focus on refining mortality models over pricing techniques for now in managing longevity risk through capital markets.

Anti-Plagiarism Declaration

I declare that I have authored this thesis independently, that I have not used materials other than the declared sources or resources, and that I have explicitly marked all materials which have been quoted either literally or by content from the used sources.

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Chapter 1

Introduction

1.1 Background of the Study

On a global scale, life expectancy has been generally increasing since the 19th century. Advancements in public health, medical technology, and curing diseases have decreased mortality rates across age and time [24]. While increased longevity is welcome from a social perspective, pension funds and life insurance companies providing annuities are increasingly exposed to longevity risk.

Longevity risk refers to the risk that people live longer relative to expectation or the lifespan assumed in the specification and valuation of insurance policies. Longevity risk poses a significant financial risk to pension and life annuity providers as they are at risk of paying out pensions and annuities for longer than anticipated. In 2013, the estimated potential size of the global longevity risk market for pension liabilities is around USD 60 trillion to USD 80 trillion [6]. These calculations were based on the accumulated assets of private pension systems, the US social security system, the aggregate liability of the US State Retirement system, and the public social security systems in 170 countries.

This longevity risk exposure can strain the finances of providers. Providers seek risk transfer solutions to mitigate this risk, which involve transferring some of the longevity risk to a third party in exchange for a premium payment. Longevity reinsurance is the most common type of longevity risk transfer. However, there is increasing interest in transferring longevity risk to capital markets as reinsurers become concentrated and need some place to lay off their longevity risk exposure. Securitization and longevity risk transfer to the financial market from pension funds to the financial market are becoming increasingly attractive solutions. Pension funds can offload some of their longevity risk exposure by transferring longevity risk through longevity-linked securities to capital markets. Transferring longevity risk to capital markets allows for greater diversification of longevity risk across a broader range of investors rather than concentrating it within a few reinsurers. Securitization also improves liquidity by enabling providers to hedge their exposure through trading standardized longevity-linked securities. An active market for longevity-linked securities increases price transparency and reduces counterparty credit risk compared to reinsurance contracts. Compared to a customized longevity risk transfer, a standardized longevity-linked derivative is more desirable due to cheaper costs and significant liquidity potential [11, 20]. These benefits have led to an emerging market for standardized longevity-linked derivatives. However, the market for longevity-linked derivatives is smaller than typical financial markets, resulting in a slow development for standardized longevity-linked derivatives.

Two primary longevity-linked securities investigated in the literature are the survivor forward and survivor swap [14]. A survivor forward (S-forward) is a contract between two parties to exchange an amount proportional to the realized survival rate of a given population for an amount proportional to the fixed survival rate agreed upon by both parties at inception to be payable at a future date. For example, consider a 10-year S-forward referencing a cohort of 65-year-old males. The contract could specify a fixed 10-year survival rate of 80%, meaning 80% of the cohort is expected to survive to age 75. At maturity in 10 years, if the actual proportion alive at 75 is higher than 80%, the seller of the fixed rate pays the difference. However, if the realized rate is below 80%, the buyer of the fixed rate compensates the seller.

On the other hand, a survivor swap (S-swap) involves a buyer of the swap paying a pre-arranged fixed level of cash flows to the swap provider in exchange for cash flows linked to the realized mortality experience. A pension of life insurance fund will purchase an S-swap to exchange cash flows linked to a floating mortality rate for a fixed cash flow payment to hedge longevity risk. S-swaps can remove longevity risk without either party needing an upfront payment, allowing pension plans to retain control of the asset allocation [6]. The first publicly announced S-swap occurred in April 2007 between Swiss Re and UK life office Friend's Provident [21]. Swiss Re paid for an undisclosed premium in exchange for assuming the longevity risk based on Friend's Provident $\pounds 1.7B$ book of 78,000 pension annuity contracts written from July 2001 to December 2006. In essence, while a S-forward is a single survival rate exchange at one future time, a S-swap can be described as a portfolio or series of many S-forwards covering all the relevant payment periods. A S-swap specifies a fixed survival rate that the pension fund paying the fixed leg expects for each year. S-swaps provide low transaction costs, great flexibility, and do not require the existence of a liquid life market [32]. To illustrate this, consider a 10-year S-swap referencing a cohort of 65-year-old males. The contract could specify a fixed rate of 80% for each year. If the actual proportion alive for a given year is higher than 80%, then the seller of the fixed rate pays the difference. This is repeated for each year within the term of the contract and both parties settle at the end of the contract term.

The primary problem researchers and practitioners face in standardizing longevity-linked securities is determining appropriate prices ([5], [13], [20], [25], [32]). Pricing longevity-linked securities involves two key components: projecting future mortality rates and selecting an appropriate valuation principle. Projecting future mortality rates for the reference population cohort requires fitting historical mortality data and trends to model and forecast future mortality trends. The projected mortality rates are then used to calculate expected future survival rates that form the basis for valuing the security's cash flows. The second key component in pricing longevity-linked securities is selecting an appropriate premium principle to calculate risk-adjusted premiums. The choice of premium principle, whether risk-neutral or pricing by market expectations (real-world), impacts the valuation. The premium principle dictates the framework for transforming the projected future survival rates and cash flows to arrive at a valuation for longevity-linked securities.

The lack of a universally accepted mortality model and standard premium principle contributes to the difficulty in pricing longevity-linked securities ([5]; [25]; [30]). There is currently no single prevailing approach that is universally accepted by market participants for mortality models or premium principles. On the mortality projection front, various models incorporate multiple factors ranging from discrete to continuous stochastic models. The diversity of modeling techniques leads to a need for more consensus on projected mortality rates. Furthermore, there is no standard, consistent premium principle that has been adopted across the industry and agreed upon as the methodology to value longevity-linked cash flows. Practitioners utilize different valuation techniques like risk-neutral pricing, pricing by market expectations using different discounting approaches, or proprietary pricing models. This absence of standardized models and pricing methods results in a lack of transparency and makes pricing processes opaque. It becomes difficult to coherently compare longevity instrument valuations and ensure consistent pricing relationships since the core inputs and pricing frameworks differ.

1.2 Statement of the Problem

Our work extends Tang & Li [25] by considering additional mortality models, and by adopting a simulation-based approach to pricing the contracts. While some of the findings by Tang & Li [25] are consistent with our results, some findings deviate, highlighting the sensitivity of longevity-linked derivative valuation to the choice of mortality models. Given the lack of literature systematically analyzing how mortality model and premium principle selection affect these securities, our work helps replicate and validate Tang & Li's [25] approach on an expanded scope. The objectives of the thesis are as follows:

- To investigate the impact of using different mortality models (LC, RH, CBD, and M6) on the valuation of survivor-linked contracts (S-forwards and S-swaps).
- To analyze the effect of various premium principles (risk-neutral and real-world) on the pricing of S-forwards and S-swaps.
- To quantify the relative impact of mortality model selection versus premium principle selection in determining the valuations of these longevity-linked contracts.
- To compare and contrast the findings with the previous work by Tang & Li [25] on the impact of mortality model and premium principle uncertainty.
- To provide insights into the sensitivity of longevity-linked security valuations to the choices of mortality models and premium principles

1.3 Pricing Survivor Contracts

To examine the impact of mortality model uncertainty, we examine four mortality models. The Lee-Carter (LC), the Renshaw-Haberman (RH) model, the Cairns, Blake, Dowd model (CBD), and the M6 model. The LC model [18] incorporates historical age-specific mortality rates and produces forecasted mortality rates by age and projection year. The LC does not incorporate information about external factors such as medical, behavioral, or social influences on mortality rates. The accuracy of the LC model relies on the continuation of historical patterns. Despite this, Lee & Carter [18] found a linear decline and relatively constant variance in the trend parameters from 1900-1989, indicating that such a stable long-term trend will continue. Since the LC is a time-series model, creating a statistical distribution to measure uncertainty is possible, implying that different sample paths of future mortality rates can be generated. The potential variability of the forecasted mortality rates is calibrated from historical data, and the underlying model is used for the age index while ignoring parameter uncertainty. Therefore, the LC model can lead to narrower prediction intervals than other statistical models (such as the Cairns, Blake, and Dowd model), which is undesirable given that the model will be used to forecast mortality rates. The RH model [23] is an extension of the LC model that captures age, period, and cohort effects. A cohort refers to a group of people born in the same time period, such as a particular year or range of years. a cohort effect refers to differences in mortality rates between different birth cohorts that are not explained by standard factors like age and period effects. The change from the original LC model is the addition of a variable to capture the change in mortality between successive cohorts. It is important to note that the period and cohort parameters are assumed to be independent of the RH model, which may not be realistic.

The CBD model [8] assumes that age, period, and cohort effects are each different and that there exists randomness between each year. This thesis uses the original formulation of the CBD model with two parameter indices. The indices represent the varying age pattern of mortality improvements rather than the overall level of mortality alone. The two CBD mortality indices represent the logit-transformed mortality curve with any slope and level. It is important to note that while each mortality index has its interpretation, it is crucial to consider the two mortality indices jointly because the correlation between them significantly impacts a portfolio's overall longevity risk exposure. The M6 model extends the CBD model, incorporating two parametric mortality indices and a cohort effect term. It is important to note that CBD and M6 models allow for a nontrivial correlation structure between the year-on-year changes in mortality rates at different ages because they all have more than one underlying period risk factor. Meanwhile, the LC and RH models have a trivial correlation structure because there is a perfect correlation between changes in mortality rates at different ages from one year to the next since there is a single time-series process in the models.

Premium principles can be broadly categorized into two categories: risk-neutral and real-world. Under the risk-neutral measure, the price of a contract is equal to the expected present value of the cash flows under a distorted version of the real-world probability measure. Meanwhile, real-world valuation principles use the historical probability measure and assume that mortality rates and prices will repeat historical trends. This thesis examines five risk-neutral premium principles: the Wang transform, proportional hazard transform, dual power transform, Gini transform, and exponential transform. The risk-neutral premium principles apply a distortion function to the cumulative distribution function of the risk to produce a risk-adjusted fair value. Risk-neutral pricing relies on risk replication, which is only possible for highly liquid and deeply traded assets. Since the longevity market is immature, there is a lack of liquidity, and risk-neutral pricing methods cannot be used carelessly [3]. Hence, under the real-world measure, the price of an instrument is determined using real-world probabilities derived from historical data. This thesis examines three real-world premium principles: the standard deviation, the variance principle, and the median absolute deviation principle.

1.4 Mortality Model and Premium Principle Uncertainty

Mortality modeling plays a crucial role in the insurance industry, directly impacting the pricing and reserving of life insurance products. Accurately predicting the likelihood of death at different ages is essential for insurance companies to set premiums appropriately and maintain profitability and solvency. However, there is inherent uncertainty in the choice of mortality models and the underlying assumptions, which can lead to mispriced premiums and potential financial risks for insurers.

The primary work in this area by Tang & Li [25] investigated the impact of different mortality models and premium principles on the pricing of S-forwards and S-swaps using UK mortality data. Tang & Li [25] compared risk premiums from twelve premium principles calibrated under the Lee-Carter model with cohort effect and Cairns-Blake-Dowd model with cohort effect and quadratic term mortality models. Tang & Li [25] found that the choice of mortality model has greater influence on risk premiums than the premium principle. Atance et al. [2] compared different models to predict mortality rates, highlighting the importance of selecting the most effective model for a given location. They found that the Lee-Carter model performed well in predicting life expectancy in European countries and suggested a method to choose the best model for any region. Li et al. [16] decomposed historical US mortality improvements into age, period, and cohort components, experimenting with various models to find the most robust and explanatory ones. Cairns et al. [10] compared eight stochastic mortality models to explain improvements in mortality rates in England and Wales and the United States. For higher ages, an extension of the CBD model with a cohort effect fit the England and Wales males' data best, while the RH extension to the LC model with a cohort effect provided the best fit for US males' data. Renshaw & Haberman [15] compared different mortality models to forecast life expectancy and annuity values for different age groups using mortality data from England & Wales and the USA. The paper investigated how recent model improvements address the shortcomings in [10]. Dowd et al. [17] evaluated the forecasting performance of six different stochastic mortality models applied to English & Welsh male mortality data using a backtesting framework. Results indicated that most models perform adequately in backtests, with wider prediction intervals for parameter uncertainty, and show little difference in performance except for the RH model displaying forecast instability. Yang et al. [31] discussed the importance of accurately predicting mortality rates for longevity-linked products. The authors introduced a method that combines process, parameter, and model errors to improve mortality projections, highlighting the impact of model selection on risk-neutral valuation results. [9] discussed various stochastic mortality models proposed in the past 15-20 years, focusing on discrete-time models for statistical modeling and forecasting. The authors also mentioned a discrete-time market model for valuing mortality-linked contracts with embedded options and simpler continuous-time models for dynamic hedging of mortality risk. Additionally, the text reviewed different financial instruments, including S-swaps, that can be used to hedge against mortality risk in the market. The illiquid and emerging nature of the longevity risk market further complicates determining an appropriate longevity risk premium. Pelsser [22] argues that the Wang transform is not suitable for pricing such risks in a consistent manner with the market dynamics. Leung et al. [19] delved into this issue by exploring various pricing approaches using a Bayesian state-space mortality model. Their study aims to address parameter uncertainties and obtain a distribution of the longevity risk-premium, offering insights into analyzing pricing methods in an illiquid and incomplete longevity market.

The slow development of the longevity-risk transfer market, combined with the market's illiquidity and lack of transparency, needs to be improved in determining what premium principle should be used in pricing longevity-linked securities. Barrieu & Veraart [4] examined how different factors affect the pricing of a financial instrument called a q-forward contract based on longevity risk. A q-forward contract involves exchanging the realized mortality rate of a population for a predetermined fixed rate. The study focuses on analyzing the impact of model choice for mortality rates, the time window used for estimation, and the pricing method on determining the fixed rate for q-forward contracts. Barrieu et al. [3] explored advancements in longevity-risk modeling and the challenges faced by the financial and insurance sectors. It discusses critical concepts for understanding longevity risk and highlights the need for improved risk assessment and management practices in response to increasing life expectancy. The article also addressed capital markets, such as insurance-linked securities, to transfer longevity risk and emphasized the importance of evolving industry regulations for effective risk management.

1.5 Framework Overview

This thesis seeks to understand the impact of uncertainty surrounding the choice of mortality model and premium principles on the longevity-linked securities by the pricing of S-forwards and S-swaps. We aim not to find the best model and principle, as this varies for every data set. Instead, we aim to understand the impact of model and principle uncertainty.

This thesis follows the framework laid out by Tang & Li [25]. This thesis pays particular attention to the sensitivity of the price of the contracts under different mortality models. This thesis builds upon the literature as follows. We investigate the impact of the choice of three other mortality models on the valuation of S-forward and S-swaps. While the thesis by Tang & Li [25] investigated the LC model with cohort effect and CBD model with quadratic terms and cohort effect, we analyzed the LC and CBD model with and without the cohort effect term. We analyzed the residuals and goodness-of-fit of the mortality models. Since Tang & Li [25] mentioned that the choice of mortality model has a more significant effect on the valuation of the S-forward and S-swap, it is essential to look into how the mortality model fits the dataset under consideration. We analyze the mortality models and conclude that our dataset has a significant cohort effect. We discuss in-depth a replicable procedure for calibrating the pricing parameter λ necessary for each premium principle. While most of the literature omits details on the calibration procedure or uses information that is not publicly available, we discuss a procedure to calibrate the pricing parameter using publicly available data. We value the contracts using the simulation-based procedure in Boyer et al. [7]. The simulation-based approach is more flexible and allows pricing a wide range of longevity-linked securities. Extending the framework used in this thesis to price other longevity-linked contracts is possible because of the simulation-based approach.

1.6 Scope and Limitations

While this thesis investigates the impact of uncertainty in selecting mortality models and premium premium principles on the valuation of longevity-linked securities like (S-forwards and S-swaps), it does not aim to identify the single best mortality model or premium principle conclusively. The optimal choice may vary across different datasets, populations, and contexts. The primary goal is to understand and quantify the sensitivity of the contract valuations to the choices of mortality models and premium principles rather than to provide definitive recommendations on which specific models or principles should be universally adopted.

The scope of this study is confined to the analysis of S-forwards and S-swaps, two common types of longevity-linked securities. However, the findings and conclusions drawn from this work may not directly apply to other longevity risk transfer instruments with different underlying cash flow structures, assumptions, or mechanisms. The valuation framework and results presented here are specific to the structures of S-forwards and S-swaps, and caution should be exercised when attempting to extrapolate or generalize the insights to other longevity-linked products.

It is important to note that the methodological framework employed in this paper is based on the approach proposed by Tang & Li [25]. While the current study expands upon their work by considering additional mortality models and adopting a simulation-based pricing technique, it may still inherit inherent limitations, assumptions, or potential biases in their original methodology. One such limitation is we assume a fixed floating leg throughout the contract term rather than a different floating leg value each year. Consequently, the conclusions drawn from this analysis should be interpreted within the context of the specific set of mortality models, premium principles, and securities examined.

Furthermore, the study focuses on valuing these securities using publicly available data and replicable procedures for calibrating the pricing parameters required by the various premium principles. While this approach enhances transparency and reproducibility, it may also introduce model error.

1.7 Organization of Thesis

The thesis is organized as follows: Chapter 2 presents preliminary concepts related to actuarial science. This chapter includes four central concepts that serve as preliminary material: exposed to risk, force of mortality, survival probabilities, mortality models, premium principles and the survivor-linked contracts. Chapter 3 presents the data and assumptions used in the analysis. This chapter analyzes and discusses the fitted parameter values for the mortality models used. It also includes an analysis of mortality model residuals. In this chapter, we include the calibration of the pricing parameter and a discussion of the formulas used in pricing the contracts. Chapter 4 contains the results of the thesis. This chapter presents and discusses the key results obtained for the risk-adjustment terms. Finally, Chapter 5 is the discussion and conclusion. It presents a recap of the results. The main result of the thesis is that the choice of mortality model has a more significant impact on premiums than the choice of premium principle.

Chapter 2

Preliminaries

This chapter discusses exposed to risk, force of mortality, and survival probabilities, the mortality models considered, the premium principles used, and the longevity-linked contracts. Throughout the thesis, let the random variable $D_{x,t}$ denote the number of deaths in a population at age x during the year t and $d_{x,t}$ the corresponding empirical estimate for the observed number of deaths.

2.1 Exposed to Risk, Force of Mortality, and Survival Probabilities

In the analysis of mortality rates, the *exposed to risk* for a population aged x refers to the number of survivors in a population who are alive at age x. There are two main ways to calculate exposed to risk. The initial exposed to risk counts the population at the start of the period. In contrast, the central exposed to risk counts the average number of people at risk over a period.

Define the estimated number of deaths between age x and $x+1$ at time t as $d_{x,t}$. The number of people exposed to risk are the eligible population members generating the deaths. Define $P_x(0)$ as the size of the population between x and $x+1$ at the beginning of the year and $P_x(1)$ at the end of the year. The central exposed to risk defines the number of individuals exposed to risk as follows:

$$E^c_{x,t} = \frac{P_x(0) + P_x(1)}{2}. \tag{2.1.1}$$

Meanwhile, the initial exposed to risk defines the number of individuals exposed to risk as follows:

$$E^0_{x,t} = P_x(0). \tag{2.1.2}$$

Given the central exposed to risk, an approximation to the initial exposed to risk is given by,

$$E^0_{x,t} \approx E^c_{x,t} + \frac{d_{x,t}}{2}. \tag{2.1.3}$$

This is the average between the population at the start and end of the year, added to half the number of people dying in the year, resulting in an approximation to the initial size of the population at risk. This approximation assumes that deaths occur on average in the middle of the year.

In this thesis, we assume that death rates are modelled by a binomial or a Poisson random variable. For the latter models, the force of mortality for the Poisson model is calculated at each instant to the exposed population at the specific instant. The exposed population can be interpreted as an average for the period, which can be considered the exposure in the middle of the period; hence, the Poisson model uses the central exposed to risk. The force of mortality under the Poisson model is given by,

$$\hat{m}_{x,t} = \frac{d_{x,t}}{E^c_{x,t}}. \tag{2.1.4}$$

The standard practice in the literature is to model the natural logarithm of the central force of mortality $\ln(m_{x,t})$ to reduce the impact of outliers and the skewness of the distribution.

On the other hand, the force of mortality for the binomial model is calculated by counting the number of people exposed at the start of a period and counting how many have died by the end of the period. A member of the population dies or does not die in the period, and it does not matter when the member has died. The exposed population is only the number of people exposed at the start of the period; hence, the binomial model uses the initial exposed to risk. The force of mortality under the binomial model is given by,

$$\hat{q}_{x,t} = \frac{d_{x,t}}{E^0_{x,t}}. \tag{2.1.5}$$

The standard practice in the literature is to model the logit transform of the initial force of mortality $\ln(\frac{q_{x,t}}{1-q_{x,t}})$. The main difference between $m_{x,t}$ and $q_{x,t}$ is the definition of exposed to risk. The central exposed to risk uses the central exposed to risk, while the initial force of mortality uses the initial exposed to risk.

The central force of mortality $\hat{m}_{x,t}$ and initial force of mortality $\hat{q}_{x,t}$ are also referred to as the central mortality rate and initial mortality rate respectively. Both quantities relate the expected number $d_{x,t}$ of people dying between age x and $x+1$ to an expected number of people 'exposed to risk.' While the former relates the number of deaths to the average number of people in an age interval, the latter relates the number of deaths to the initial number of people in the same age interval.

Define the one year probability of survival for a person aged x at time t as $p_{x,t}$. The quantity $p_{x,t}$ is the probability of an individual aged x at time t to be alive at the age $x+1$ at time $t+1$. For convenience, we omit different notations for the initial and central survival rates. That is, $p_{x,t} := 1 - \hat{m}_{x,t}$ if the mortality model uses the central force of mortality and $p_{x,t} := 1 - \hat{q}_{x,t}$ if the mortality model uses the initial force of mortality.

2.2 Mortality Models

Numerous individual mortality models have been used in the literature on modeling and forecasting mortality rates. We focus on the family of Generalized Age-Period-Cohort (GAPC) discrete-time series mortality models available in the StMoMo package by [26]. GAPC models decompose the force of mortality $m_{x,t}$ and $q_{x,t}$ across the age x, at time t across the dimensions of age x, period t and cohort $t-x$. Define the number of deaths as $D_{x,t}$, and its empirical estimate $d_{x,t}$. We assume that the number of deaths $D_{x,t}$ follow a Poisson distribution of a binomial distribution such that

$$D_{x,t} \sim \text{Poisson}(E^c_{x,t} m_{x,t}) \tag{2.2.1}$$

or

$$D_{x,t} \sim \text{Binomial}(E^0_{x,t}, q_{x,t}) \tag{2.2.2}$$

where, $\mathbb{E}\left(\frac{D_{x,t}}{E^c_{x,t}}\right) = m_{x,t}$ and $\mathbb{E}\left(\frac{D_{x,t}}{E^0_{x,t}}\right) = q_{x,t}$.

The mortality models target either the natural logarithm of the force of mortality or a logit of the force of mortality. The canonical link pairs the Poisson distribution with the log link function and the binomial distribution with the logit link function. We follow Cairns et al. [10] and target the log link for the Lee-Carter and Renshaw Haberman model, and target the logit link for the Cairns-Blake-Dowd and M6 model.

Furthermore, each of the following models exhibits an identifiability problem. That is, knowledge of the underlying or theoretical death rate does not uniquely identify the values of the age, period, and cohort effects because there are too many unknown variables. We impose restrictions on the models to ensure the models are identifiable.

The Lee-Carter model

We first consider the Lee-Carter model [18], which expresses the natural logarithm of the central death rate $m_{x,t}$ is given by:

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t \tag{2.2.3}$$

where α_x is the average level of mortality at age x, κ_t is the time-index of mortality, and β_x represents the age sensitivity of mortality to changes in κ_t. We model the mortality index κ_t as a random walk with drift to predict future mortality values.

$$\kappa_t = \kappa_{t-1} + \theta + u_t \tag{2.2.4}$$

where θ is an estimated drift term, and u_t is a sequence of independent and identically distributed random variables following the standard Gaussian distribution.

Lee & Carter [18] suggest the following parameter constraints to ensure model identifiability:

$$\sum_x \beta_x = 1, \quad \sum_t \kappa_t = 0 \tag{2.2.5}$$

The LC model is one of the most widely used mortality model for predicting mortality rates. However, the model falls short of capturing non-linear patterns in mortality data.

Renshaw Haberman model

Renshaw & Haberman [23] extend the Lee-Carter model to include the cohort effect. The cohort effect captures the long-term impact of events on people born in different periods and does not change with one's age. The RH model produces a better fit than the LC for mortality data with a prominent cohort effect. The natural logarithm of the central death rate $m_{x,t}$ is given by,

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \gamma_{t-x} \tag{2.2.6}$$

To predict future mortality rates, we model the cohort parameter γ_{t-x} as an AR(1) process,

$$\gamma_{t-x} = a_0 + a_1 \gamma_{t-x-1} + e_t \tag{2.2.7}$$

where a_0 is an estimated drift term, a_1 is the estimated sensitivity of the previous cohort step, and the standard Gaussian error term e_t is assumed to be independent of u_t.

Model identifiability can be ensured using the following parameter constraints:

$$\sum_x \beta_x = 1, \quad \sum_t \kappa_t = 0, \quad \sum_{c=t_1-x_k}^{t_n-x_1} \gamma_c = 0 \tag{2.2.8}$$

Cairns-Blake-Dowd model

The CBD model [8] is a two-factor parametric mortality model. In contrast to the non-parametric age structure in the LC and RH model, the CBD treats age as a continuous variable that varies linearly with the logit of the force of mortality. The CBD model has two latent factors $\kappa_t^{(1)}$, $\kappa_t^{(2)}$ that allows for more flexibility in capturing the dynamics of mortality changes. Furthermore, the CBD model has the advantage of modeling mortality at higher ages.

The CBD model expresses the logit transform of one-year mortality rates $q_{x,t}$ of a life aged x in year t is given by,

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) \tag{2.2.9}$$

Here, $\kappa_t^{(1)}, \kappa_t^{(2)}$ represent the estimated level and gradient of the mortality curve in year t, and $\bar{x}$ is the mean across the sample age range. The two indices $\kappa_t^{(1)}, \kappa_t^{(2)}$ are modelled by a multivariate random walk with drift,

$$\boldsymbol{K_t} = \boldsymbol{K_{t-1}} + \boldsymbol{\Theta} + \boldsymbol{\epsilon_t}, \tag{2.2.10}$$

where $\boldsymbol{K_t} = (\kappa_t^{(1)}, \kappa_t^{(2)})'$, and $\boldsymbol{\Theta}$ is a 2×1 vector which contains two estimated drift coefficients, and the 2×1 error vector ϵ_t is assumed to follow the standard multivariate Gaussian distribution.

The CBD model does not have identifiability issues; hence, there are no parameter constraints.

M6 model

The M6 model incorporates the cohort parameter into the CBD model. The logit transform of one-year mortality rates $q_{x,t}$ of a life aged x in year t is given by,

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \gamma_{t-x} \tag{2.2.11}$$

Like the Renshaw-Haberman model, the cohort parameter is modeled as an AR(1) process.

To ensure model identifiability, the following parameter constraints must be imposed:

$$\sum_{c=t_1-x_k}^{t_n-x_1} \gamma_c = 0, \quad \sum_{c=t_1-x_k}^{t_n-x_1} c\gamma_c = 0, \quad \sum_{c=t_1-x_k}^{t_n-x_1} c^2\gamma_c = 0. \tag{2.2.12}$$

This constraint ensures that the residuals cohort effect reverts about the mean zero and has no linear or quadratic trend.

Goodness of Fit

We analyze the residuals of the fitted mortality model to evaluate the goodness-of-fit. Residual analysis involves examining the discrepancies between the observed mortality data and the mortality predictions made by the model. Define the scaled deviance residual as,

$$r_{x,t} = \text{sign}\left(d_{x,t} - \hat{d}_{x,t}\right)\sqrt{\frac{\text{dev}(x,t)}{\hat{\phi}}}, \quad \hat{\phi} = \frac{D\left(d_{x,t}, \hat{d}_{x,t}\right)}{K - \nu} \tag{2.2.13}$$

where,

$$D\left(d_{x,t}, \hat{d}_{x,t}\right) = \sum_x \sum_t \omega_{x,t}\, \text{dev}(x,t). \tag{2.2.14}$$

is the total model deviance, $K = \sum_x \sum_t \omega_{x,t}$ is the number of observations, and ν is the number of model parameters. The weights $\omega_{x,t}$ are given as:

$$\omega_{x,t} = \begin{cases} 0, & \text{if the } (x,t) \text{ cell from the data frame is excluded in the estimation} \\ 1, & \text{if the } (x,t) \text{ cell from the data frame is included in the estimation} \end{cases} \tag{2.2.15}$$

Define $\hat{d}_{x,t}$ to be the model-implied death rates. If the model involves a Poisson random variable, then:

$$\text{dev}(x,t) = 2\left[d_{x,t}\log\left(\frac{d_{x,t}}{\hat{d}_{x,t}}\right) - \left(d_{x,t} - \hat{d}_{x,t}\right)\right]. \tag{2.2.16}$$

If the model involves a binomial random variable, then:

$$\text{dev}(x,t) = 2\left[d_{x,t}\log\left(\frac{d_{x,t}}{\hat{d}_{x,t}}\right) + \left(E^0_{x,t} - d_{x,t}\right)\log\left(\frac{E^0_{x,t} - d_{x,t}}{E^0_{x,t} - \hat{d}_{x,t}}\right)\right]. \tag{2.2.17}$$

In general, models with more parameters capture more variation in the data at the expense of complexity. For example, if we solely compare the maximum likelihoods attained by each model, then it is natural for models with more parameters to fit the data 'better.' To avoid this problem, we need to penalize models that are over-parameterized. We need to see a 'significant' improvement in the maximum likelihood for each additional model parameter instead of continuously increasing more parameters. To avoid over-parameterization and have a uniform standard of comparison, we apply information criteria that modify the maximum likelihood function by penalizing more parameters. Define the maximum value of the log-likelihood function as $\mathcal{L}$. The Akaike Information Criteria (AIC) is given by,

$$\text{AIC} = 2\nu - 2\mathcal{L}. \tag{2.2.18}$$

The Bayesian Information Criteria (BIC) is given by,

$$\text{BIC} = \nu log(K) - 2\mathcal{L}. \tag{2.2.19}$$

A lower AIC and BIC are desirable in a model. A comparison of the AIC and BIC indicates that the BIC penalty term is larger than the AIC penalty term when $\ln(n) > 2$. Therefore, the BIC has a stronger penalty effect than the AIC when $\ln(n) > 2$. It is important to note that the two criteria measure the goodness of fit of the models on historical data but do not guarantee accurate forecasts.

Forecasting

Since the rates of mortality are driven by the period indices $\kappa_t^{(i)}, i = 1, 2$ and the cohort index γ_{t-x}, forecasting mortality rates require the modeling of these indices using time-series techniques.

The period indices are assumed to follow a univariate random walk with drift for the LC and RH models; meanwhile, the period indices are assumed to follow a multivariate random walk with drift for the CBD and M6 models. The dynamics of the period indices under a multivariate random walk with drift are given by,

$$\kappa_t = \kappa_{t-1} + \theta + u_t, \quad u_t \sim N(0, \Sigma) \tag{2.2.20}$$

where κ_t and θ is a one-dimensional vector for the period index and drift term, and Σ is the estimated variance matrix of the white noise u_t under the LC and RH models. Meanwhile, $\kappa_=(\kappa_t^{(1)}, \kappa_t^{(2)})'$ is a two-dimensional vector for the period index and $\theta = (\theta^{(1)}, \theta^{(2)})'$ is a two-dimensional drift vector, and Σ is a 2×2 variance-covariance matrix of the white noise process under the CBD and M6 models.

The cohort parameter is assumed to follow an AR(1) process for the RH and M6 models. The dynamics of the cohort parameter under an AR(1) process is given by,

$$\gamma_{t-x} = a_0 + a_1 \gamma_{t-x-1} + e_t, \quad e_t \sim N(0, \xi) \tag{2.2.21}$$

where a_0 is a drift term, a_1 is the sensitivity of the previous cohort step, and ξ is the variance of the white noise process. Each of these terms must be estimated.

2.3 Premium Principles

In actuarial applications, the primary interest is modeling the loss distribution for insurance products. For example, in property and casualty (P&C) or life insurance, an insurer may develop a compound Poisson model for the losses on a portfolio of policies.

Define $V_0[X]$ as the valuation at time 0 of a future liability or cash flow given by the random variable X. Assuming that the loss random variable X is nonnegative in insurance contexts is usually appropriate. The choice of V_0 is equivalent to choosing a valuation principle.

Define the probability density function (pdf) as $f(x)$ and the cumulative distribution function (cdf) as $F(x)$. Define the de-cumulative distribution function $S(x) = 1 - F(x)$. While the cdf gives the probability that the random variable X takes on a value less than or equal to x, the de-cumulative distribution function gives the probability that X is greater than x.

The risk premium is the expectation of the loss random variable X

$$\mathbb{E}[X] = \int_0^\infty x f(x) dx = \int_0^\infty [1 - F(x)] dx = \int_0^\infty S(x) dx. \tag{2.3.1}$$

Define $f^*(x), F^*(x), S^*(x), \mathbb{E}^*(x)$ as the risk-neutral pdf, cdf, decumulative function, and expectation of the risk respectively.

This thesis considers eight premium principles; the first five are risk-neutral, and the last three are real-world premium principles. Risk-neutral valuation principles take the expected present value of the cash flows under a distorted version of the real-world prices. The distortion of the real-world valuation into the risk-neutral valuation is parameterized by the pricing parameter λ. To be specific, the following premium principles are considered: Wang, proportional hazard, Gini, dual-power, exponential transform, standard deviation, variance, and MAD principles.

The Wang transform embeds a Gaussian distortion function that returns a distorted cdf [27],

$$F^*(x) = \Phi\left(\Phi^{-1}(F(x)) - \lambda\right), \quad \lambda \geq 0. \tag{2.3.2}$$

Here, $\Phi(\cdot)$ represents the cdf of a standard Gaussian distribution, and $\Phi^{-1}(\cdot)$ is the inverse standard Gaussian cdf. For a given risk X with cdf $F(X)$, the Wang transform produces a risk-adjusted cdf $F^*(X)$. The mean value under $F^*(X)$, denoted as $\mathbb{E}^*[X]$ is the risk-adjusted fair value of X at time T, which will be further discounted to time zero using the risk-free interest rate. One advantage of the Wang transform is that it is reasonably quick to evaluate numerically. [20] used the Wang transform to price mortality bonds by distorting the distribution of mortality rates.

The proportional hazard transform has the advantage of having a simple distortion function of the following form [28]:

$$F^*(x) = 1 - (1 - F(x))^{1/\lambda}, \quad \lambda \geq 1. \tag{2.3.3}$$

The proportional hazard transform is quite sensitive to the choice of λ. Wang [28] used the proportional hazard transform to price insurance risk.

The dual-power transform [29] is given by,

$$F^*(x) = F(x)^\lambda, \quad \lambda \geq 1. \tag{2.3.4}$$

The distortion function of the dual power transform is similar to the proportional hazard transform, the difference being the distortion of the de-cumulative distribution function instead of the cumulative distribution function.

The Gini principle [12] has a risk-adjusted de-cumulative function given by,

$$F^*(x) = 1 - \left((1+\lambda)(1-F(x)) - \lambda(1-F(x))^2\right), \quad 0 \leq \lambda \leq 1. \tag{2.3.5}$$

The exponential transform uses weighted probabilities to map the liability denoted by the random variable X from $[0,1]$ onto $[0,1]$. The risk-adjusted de-cumulative function is given by,

$$F^*(x) = 1 - \frac{1-e^{-\lambda(1-F(x))}}{1-e^{-\lambda}}, \quad \lambda > 0. \tag{2.3.6}$$

The immaturity of the longevity risk transfer market results in low liquidity for survivor contracts, making it difficult to apply risk-neutral premium principles. Real world premium principles that use historical mortality rates offer an alternative methodology to price survivor contracts.

The price under the standard deviation principle is given by,

$$V_0[X] = \mathbb{E}[X] + \left[\lambda \times \mathrm{SD}[X]\right], \quad \lambda > 0. \tag{2.3.7}$$

A pure premium is defined as $V_0[X] = \mathbb{E}[X]$. Hence, the standard deviation principle is equal to the pure premium plus a risk-loading term proportional to the standard deviation of the liability.

The price under the variance principle is given by,

$$V_0[X] = \mathbb{E}[X] + \left[\lambda \times \mathrm{VAR}[X]\right], \quad \lambda > 0. \tag{2.3.8}$$

Like the standard deviation principle, the variance principle is a pure premium plus a risk-loading term proportional to the liability variance.

The price under the median absolute deviation principle is given by,

$$V_0[X] = S^{-1}(0.5) + \left[\lambda \times MAD[X]\right], \quad \lambda > 0. \tag{2.3.9}$$

Here, $\text{MAD}[X] = \text{MAD}(|X - S^{-1}(0.5)|)$. Since mean-variance statistics tend to be sensitive to outliers, we include a premium principle that uses the median. The MAD principle is better suited to datasets with small sample sizes and potential outliers.

2.3.1 Desirable Properties of Premium Principles

It is essential to consider the fundamental properties that one might expect from something that will be used to measure risk. These represent a basic set of common sense rules, the failure to comply with which must put into question a method's suitability for measuring or allocating risk. A premium principle is coherent if it possesses four properties: sub-additivity, monotonicity, positive homogeneity, and translation invariance [1]. Among the eight premium principles considered in this thesis, the five risk-neutral principles: Wang transform, proportional hazard transform, dual-power transform, Gini transform, and exponential transform are coherent.

- (Sub-Additivity): Define A and B as two future cash flows or liabilities. A combination of two portfolios must generate less risk. When risk portfolios are brought together, a risk diversification benefit may exist.

$$V_0[A + B] \leq V_0[A] + V_0[B]. \tag{2.3.1}$$

- (Monotonicity): If a portfolio is worth more than another, it must not be riskier.

$$A \leq B \rightarrow V_0[A] \leq V_0[B]. \tag{2.3.2}$$

- (Positive Homogeneity): Scaling a portfolio by a constant will change the portfolio's risk by the same proportion.

$$V_0[kA] = k \times V_0[A], \qquad k \text{ is a constant.} \tag{2.3.3}$$

- Adding a constant amount (positive or negative) to an existing risk adds the same amount to the risk measure.

$$V_0[A + k] = V_0[A] + k, \quad k \text{ is a constant .} \tag{2.3.4}$$

- (Translation Invariance): Translation invariance also implies that the risk measure for a non-random loss variable with known value k is just the amount of the loss K,

$$V_0[k] = k, \quad \text{k is a known constant .} \tag{2.3.5}$$

A summary of the premium principles and their properties is given in table 2.1

Premium Principle	**Sub Additivity**	**Monotonicity**	**Positive Homogeneity**	**Translation Invariance**
Wang	Y	Y	Y	Y
Proportional	Y	Y	Y	Y
Dual Power	Y	Y	Y	Y
Gini	Y	Y	Y	Y
Exponential	Y	Y	Y	Y
Std. Dev.	Y	N	Y	Y
Variance	N	N	N	Y
MAD	N	N	Y	Y

Table 2.1: Properties of various premium principles

2.4 Longevity-Linked Instruments

We assume a pension/annuity fund enters a long position in a survivor contract to hedge longevity risk. In a long position, the fund pays a fixed amount K for floating cash flows linked to a future survival rate S. If survivors exceed expectations, the contract payouts hedge the fund's larger liabilities. If fewer survivors occur, the negative payouts are offset by reduced liabilities. As previously mentioned, this paper considers two survivor contracts, the S-forward and S-swap.

2.4.1 Survivor Forward

A survivor forward (S-forward) is an agreement between two counterparties to exchange a payment linked to the number of survivors in a reference population at a predetermined future date T. The buyer of an S-forward pays a fixed forward rate K to the seller and receives a floating rate $S(T)$. The forward rate is specified at the contract's start and reflects the expected future longevity level. Assume a notional amount equal to one. The fixed leg K must be determined such that the fair value of the survivor forward at $t = 0$ is zero. Mathematically, this is given by

$$V_0[S(T) - K] = 0. \tag{2.4.1}$$

Here, the $V_0[\cdot]$ is a value function, which refers to a valuation principle, and the fixed forward rate is determined such that the S-forward has zero value at the start of the contract.

We follow Boyer et al. [7] and assume that the fixed leg is the known anticipated one-year survival probability scaled to some unknown constant linear risk-adjustment term π. The concrete formulation for the survival forward is given by

$$V_0[s_{x,T}^{\text{realized}} - (1+\pi)s_{x,T}^{\text{anticipated}}] = 0. \tag{2.4.2}$$

Based on the above formulation, the risk-adjustment term under risk-neutral measure is given by

$$\pi = \frac{s_{x,T}^{\text{realized}}}{s_{x,T}^{\text{anticipated}}} - 1. \tag{2.4.3}$$

Here, $s_{x,t}^{\text{realized}}$ is the average of simulated one-year survival probability that an individual aged x survives from time $t-1$ to time t under the premium principle considered. The denominator, $s_{x,t}^{\text{anticipated}}$ is obtained by setting the pricing parameter λ equal to zero.

2.4.2 Survivor Swaps

A survivor swap (S-swap) involves two counterparties exchanging a stream of future cash flow linked to the difference between the floating and fixed rates periodically (i.e., for every $t = 1, 2, \ldots, T$). We assume that the forward rate K is constant for all periods. A S-swap consists of a series of S-forwards with different maturities and can be interpreted as a portfolio of S-forwards. Assume a notional principal is equal to one. Analogous to the S-forward, the fixed leg of a S-swap K is determined such that the S-swap has zero value at the onset of the contract,

$$V_0 \left[\sum_{t=1}^{T} S(t) - K \right] = 0. \tag{2.4.1}$$

Similar to the S-forward, we assume that the fixed leg for a S-swap is the sum of known anticipated one-year survival probability scaled to some unknown constant linear risk-adjustment term π. The concrete formulation for the survival forward is given by

$$V_0 \left[\sum_{t=1}^{T} s_{x,t}^{\text{realized}} - (1+\pi) \sum_{t=1}^{T} s_{x,t}^{\text{anticipated}} \right] = 0. \tag{2.4.2}$$

Based on the above formulation, the risk-adjustment term under risk-neutral measure is given by

$$\pi = \frac{\sum_{t=1}^{T} s_{x,t}^{\text{realized}}}{\sum_{t=1}^{T} s_{x,t}^{\text{anticipated}}} - 1. \tag{2.4.3}$$

Chapter 3

Methodology

3.1 Data and Assumptions

We utilize mortality data from the population of England and Wales for the empirical analysis from the Human Mortality Database[1]. We use male mortality data for the age range 60-89 cohorts born between 1961 and 2011. We chose the starting age of 60 since we are interested in managing longevity risk for pensioners. We chose the ending age of 89 to avoid the increased volatility in mortality data at higher ages. In pricing the contracts, we assume that the interest rate used in discounting is compounded continuously.

3.2 Model Parameters

Mortality models are used to understand mortality trends over time. This section presents fitted model parameter values for the LC, RH, CBD, and M6 models. Parameter estimates were obtained by maximizing the model log-likelihood. These models will then be assessed using AIC and BIC.

The fitted parameter values for the models are available in Figures 3.1 to 3.4. Notice that the fitted model parameters for the LC model α_x presents an increasing linear trend. Meanwhile, β and κ_t present a decreasing drift. Like the LC model, the fitted parameters of the RH model show that α_x has an increasing linear trend. Contrary to the LC model, the RH model has a positive drift. The constant trend seen in $\beta_x^{(0)}$ is because of the assumption that the coefficient of the cohort term has a coefficient equal to 1. Lastly, the cohort effect term γ_{t-x} has a downward trend. For the CBD model, κ_1 has a decreasing quadratic trend. β_2 presents an increasing linear trend, and κ_2 has an increasing trend. Lastly for the M6 model, κ_1 has a decreasing linear trend. β_2 has an increasing linear drift. β_0 has a constant value of 1. The cohort effect term γ_{t-x} increases until 1900, then decreases until 1935 before increasing again.

[1] www.mortality.org/

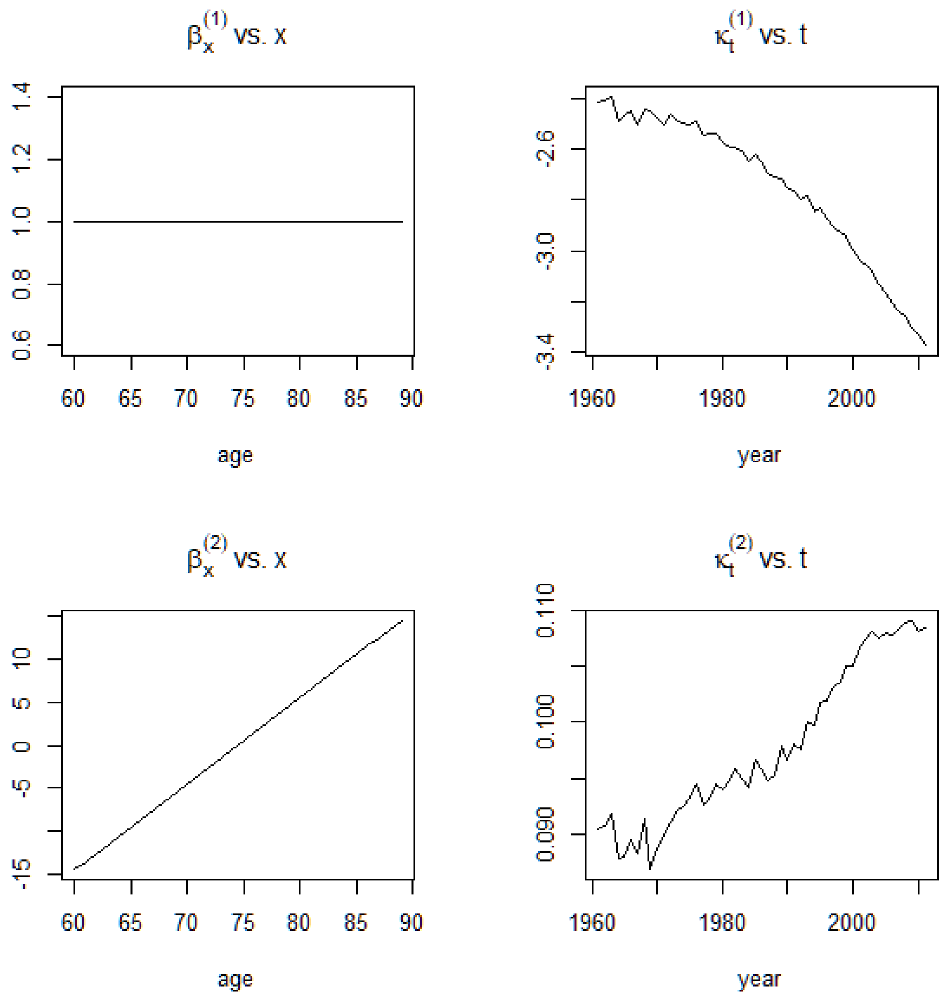

Figure 3.1: Fitted parameter values under the Lee-Carter model

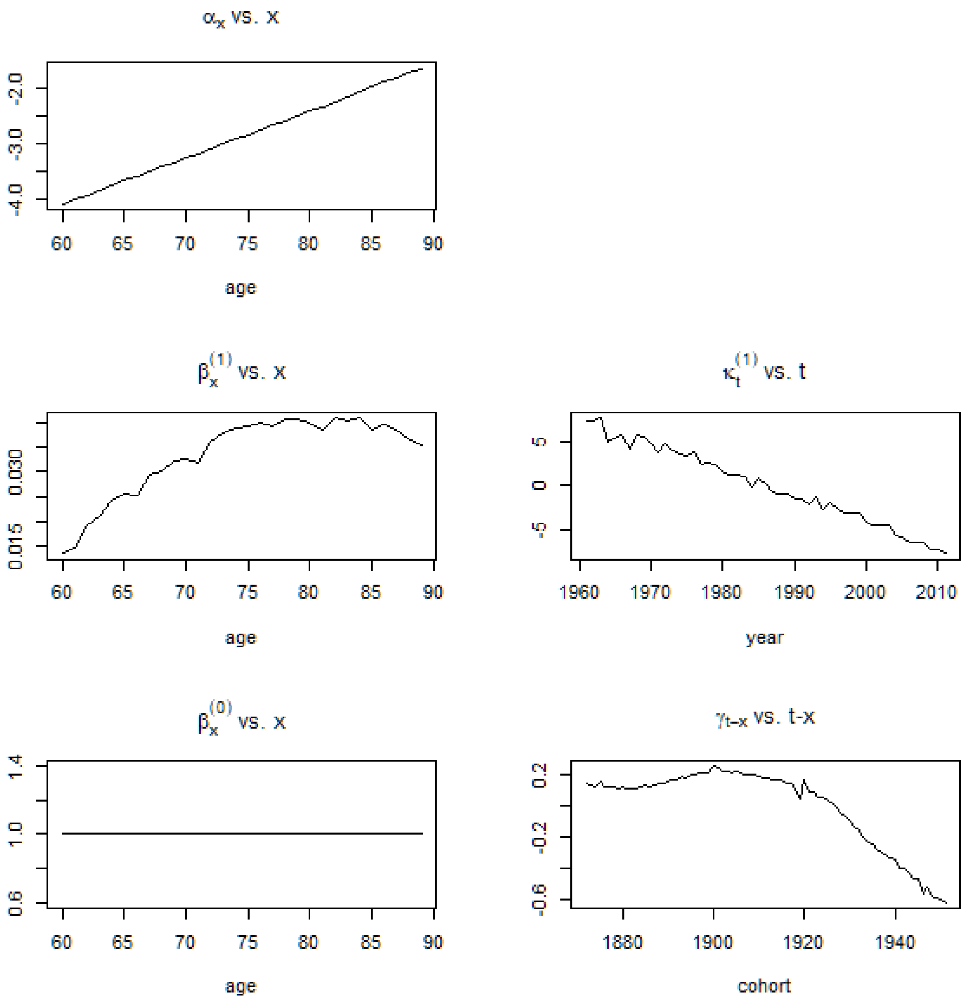

Figure 3.2: Fitted parameter values under the Renshaw-Haberman model

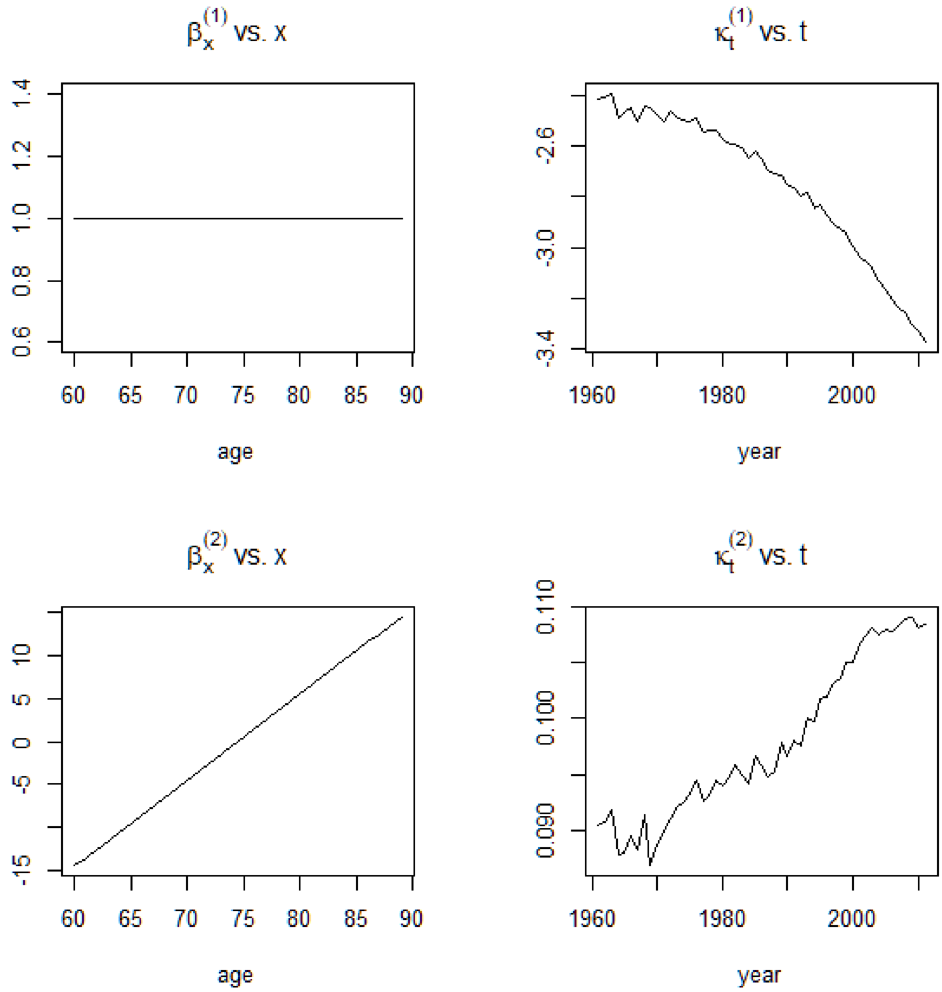

Figure 3.3: Fitted parameter values under the CBD model

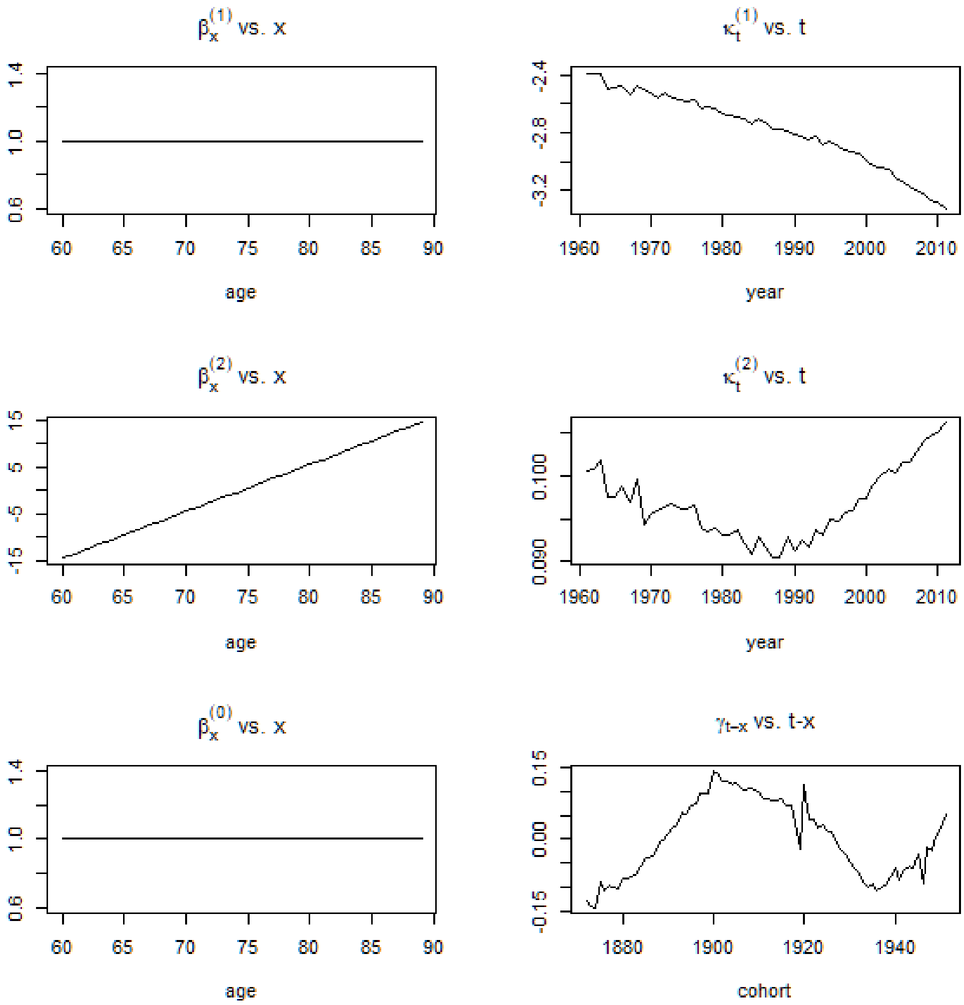

Figure 3.4: Fitted parameter values under the M6 model

3.3 Residual Analysis

The resulting AIC and BIC for each mortality model is shown in Table (3.1). The RH model has the lowest AIC and BIC among the four mortality models for the England and Wales dataset. It is important to note that the RH and M6 models have a smaller AIC and BIC than their corresponding counterparts without the cohort term. This suggests that including a cohort effect term in the mortality models produces a better fit.

The heatmaps for the fitted model parameters are given by Figures 3.5 to 3.8. The heatmaps for the model residuals under the LC and CBD exhibit a strong linear pattern. This suggests that the LC and CBD models, which do not incorporate a cohort effect term, cannot capture the dynamics of the cohort effect in the England and Wales dataset.

Model	AIC	BIC
LC	25,442	26,208
RH	19,118	19,081
CBD	26,024	26,752
M6	20,121	20,041

Table 3.1: Values for AIC and BIC for the fitted mortality models

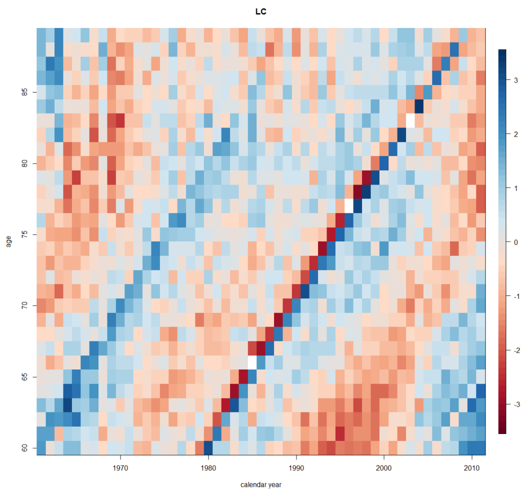

Figure 3.5: Heatmap of the residuals under the LC model with calendar year in the x-axis and age in the y-axis

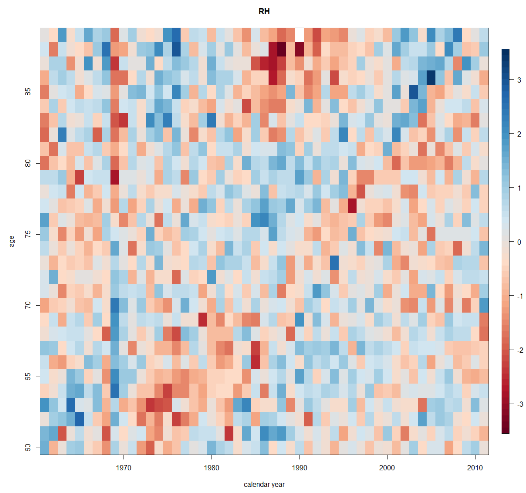

Figure 3.6: Heatmap of the residuals under the RH model with calendar year in the x-axis and age in the y-axis

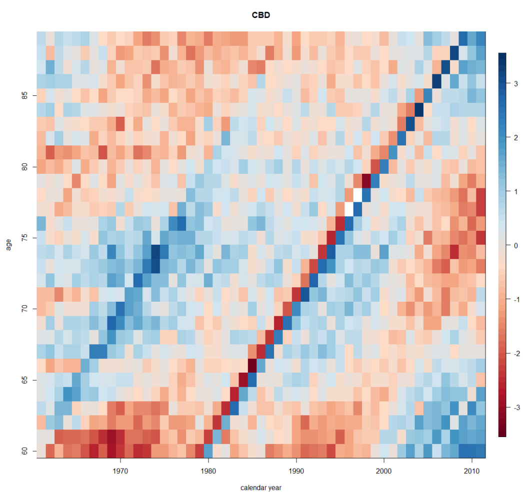

Figure 3.7: Heatmap of the residuals under the CBD model with calendar year in the x-axis and age in the y-axis

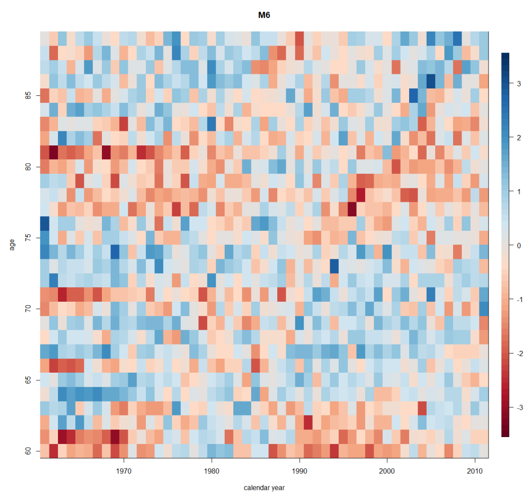

Figure 3.8: Heatmap of the residuals under the M6 model with calendar year in the x-axis and age in the y-axis

3.4 Pricing Parameter Calibration

Since mortality-linked derivatives are not publicly traded, there is scant information regarding transaction details. This thesis overcomes the model calibration problem by linking the S-forwards and S-swaps to annuity rates.

The fitted historical mortality rates end in 2011; hence, we use 2012 annuity rates. As of the first quarter of 2011, the annuity rate is level payments of £6,000 per £100,000 funds for a single life aged 65 with level payments. The risk-free rate is assumed to be the 15-year Gilt rate quoted at 2.04% for the first quarter of 2011.

There are no closed-form solutions to the pricing parameter λ for risk-neutral premium measures. Hence, we resort to numerical root-finding algorithms. We apply a Newton-Raphson type of algorithm to obtain the value of λ.

Define the following parameters and notation:

- $\psi := 6,000$ is the annuity payment
- $\xi := 100,000$ is annuity value
- $r := 2.04\%$ is the risk-free rate
- DF_t is the discount factor to take the present value.

Suppose the premium principle is the Wang transform, then we solve the following equation for λ:

$$\xi = \sum_{t=1}^{T} \psi DF_t[1 - \Phi(\Phi^{-1}(1 - p_{x,t}) - \lambda)]. \tag{3.4.1}$$

To pose the equation above as a root-finding problem, we find λ using a Newton-Raphson-type algorithm such that the sum of the squared errors is minimized:

$$\left\{ \sum_{t=1}^{T} \psi DF_t[1 - \Phi(\Phi^{-1}(1 - p_{x,t}) - \lambda)] - \xi \right\}^2 = 0. \tag{3.4.2}$$

A summary of the minization problem for each premium principle is given in Table 3.2

Premium	Minimize
Wang	$\left\{\sum_{t=1}^{T}\psi DF_t[1-\Phi(\Phi^{-1}(1-p_{x,t})-\lambda)]-\xi\right\}^2=0$
Proportional	$\left\{\sum_{t=1}^{T}\psi DF_t(p_{x,t})^{\lambda}-\xi\right\}^2=0$
Dual	$\left\{\sum_{t=1}^{T}\psi DF_t[1-(1-(p_{x,t})^{\lambda})]-\xi\right\}^2=0$
Gini	$\left\{\sum_{t=1}^{T}\psi DF_t[(1+\lambda)p_{x,t}-\lambda(p_{x,t})^2]-\xi\right\}^2=0$
Exponential	$\left\{\sum_{t=1}^{T}\psi DF_t\left[\frac{1-e^{-\lambda p_{x,t}}}{1-e^{-\lambda}}\right]-\xi\right\}^2=0$

Table 3.2: Risk-neutral premium principles and their associated minimization problem to obtain pricing parameter λ

For the real-world premium principles, standard deviation, variance, MAD principle, closed-form expressions for λ can be derived. Suppose that the liability X is a Bernoulli random variable such that

$$X=\begin{cases}\sum_{t=1}^{T}DF_t\psi, & \text{person is alive}\\ 0, & \text{person is not alive.}\end{cases} \tag{3.4.3}$$

In other words, the liability of the pension/ life annuity fund is the present value of the payment stream if the person is alive and zero if the person is not.

Since the probability that a person aged x is alive at time t is given by $p_{x,t}$, taking the expectation and variance of the Bernoulli random variable yields,

$$\mathbb{E}[X]=\sum_{t=1}^{T}DF_t\psi p_{x,t} \tag{3.4.4}$$

$$\mathrm{VAR}[X]=\sum_{t=1}^{T}(DF_t\psi p_{x,t})[DF_t\psi(1-p_{x,t})] \tag{3.4.5}$$

Suppose the premium is the standard deviation principle, then

$$\xi=\sum_{t=1}^{T}DF_t\psi p_{x,t}+\lambda\sqrt{\sum_{t=1}^{T}(DF_t\psi p_{x,t})[DF_t\psi(1-p_{x,t})]} \tag{3.4.6}$$

Solving for λ yields,

$$\lambda = \frac{\xi - \sum_{t=1}^{T} DF_t \psi p_{x,t}}{\sqrt{\sum_{t=1}^{T} (DF_t \psi p_{x,t})[DF_t \psi (1 - p_{x,t})]}} \tag{3.4.7}$$

Analogous to the standard deviation principle, if the premium is the variance principle, then λ is given by,

$$\lambda = \frac{\xi - \sum_{t=1}^{T} DF_t \psi p_{x,t}}{\sum_{t=1}^{T} (DF_t \psi p_{x,t})[DF_t \psi (1 - p_{x,t})]} \tag{3.4.8}$$

If the premium is the MAD principle, then λ is given by,

$$\lambda = \frac{\xi - \sum_{t=1}^{T} DF_t \psi \mathrm{MEDIAN}(p_{x,t})}{\sum_{t=1}^{T} DF_t \psi \mathrm{MAD}(p_{x,t})} \tag{3.4.9}$$

Here, $\mathrm{MAD}(p_{x,t}) = \mathrm{MEDIAN}(|p_{x,t} - S^{-1}(0.5)|)$.

The results obtained for the pricing parameter λ are given in Table 3.3. The pricing parameters for models of the LC type are larger than those of the RH type. Based on the Table 3.3, it is implied that the survival rates implied from the LC and RH models require a greater return for the pension fund to take on the risk associated with the survival rate of the reference population of the pension fund. Furthermore, a higher pricing parameter implies that the model assumptions are not aligning well with market expectations, hence the need for a greater correction term. A higher value for the pricing parameter implies that the model might not be as reliable as models with a smaller pricing parameter value, which could lead to greater model uncertainty.

Table 3.3: Values obtained for the pricing parameter λ

Premium	LC	RH	CBD	M6
Wang	4.373e-01	4.346e-01	3.993e-01	3.906e-01
Proportional	2.300e+00	2.290e+00	2.155e+00	2.125e+00
Dual	1.386e+00	1.383e+00	1.344e+00	1.334e+00
Gini	6.344e-01	6.317e-01	5.951e-01	5.858e-01
Exponential	1.602e+00	1.593e+00	1.479e+00	1.451e+00
Std. Dev.	9.804e-01	9.746e-01	8.971e-01	8.786e-01
Variance	1.586e-04	1.579e-04	1.487e-04	1.460e-04
MAD	7.516e-01	7.491e-01	7.418e-01	7.964e-01

3.5 Pricing Longevity Linked Instruments

Consider a S-forward contract with maturity T. We begin by simulating future mortality scenarios using the fitted mortality model. Define the average simulated value at time t for an individual aged x as $\bar{p}_{x,T}$. We solve for the risk-adjustment term π such that the S-forward has zero value at inception. Mathematically, this is given by Equation 2.4.2. The variable of interest is π, which can be evaluated using Equation 2.4.3. Expressions for $s_{x,T}^{\text{realized}}$ and $s_{x,T}^{\text{anticipated}}$ can be found in Table 3.4. On the other hand,

Premium Principle	$s_{x,T}^{\text{realized}}$	$s_{x,T}^{\text{anticipated}}$
Wang	$\mathrm{DF}_T \left\{ 1 - \Phi \left[\Phi^{-1} \left(1 - \bar{p}_{x,T} \right) - \lambda \right] \right\}$	$\mathrm{DF}_T \left\{ 1 - \Phi \left[\Phi^{-1} \left(1 - \bar{p}_{x,T} \right) \right] \right\}$
Proportional	$\mathrm{DF}_T \left(\bar{p}_{x,T} \right)^{\frac{1}{\lambda}}$	$\mathrm{DF}_T \bar{p}_{x,T}$
Dual	$\mathrm{DF}_T \left[1 - \left(1 - \bar{p}_{x,T} \right)^{\lambda} \right]$	$\mathrm{DF}_T \left[1 - \left(1 - \bar{p}_{x,T} \right) \right]$
Gini	$\mathrm{DF}_T \left[(1+\lambda) \bar{p}_{x,T} - \lambda \left(\bar{p}_{x,T} \right)^2 \right]$	$\mathrm{DF}_T (1+\lambda) \bar{p}_{x,T}$
Exponential	$\mathrm{DF}_T \left[\frac{1-e^{-\lambda \bar{p}_{x,t}}}{1-e^{-\lambda}} \right]$	$\mathrm{DF}_T \left[\frac{1-e^{-\bar{p}_{x,t}}}{1-e^{-1}} \right]$
Std. Dev.	$\mathrm{DF}_T \left[\mathbb{E} \left(\bar{p}_{x,T} \right) + \mathrm{STDEV} \left(\bar{p}_{x,T} \right) \right]$	$\mathrm{DF}_T \mathbb{E} \left(\bar{p}_{x,T} \right)$
Variance	$\mathrm{DF}_T \left[\mathbb{E} \left(\bar{p}_{x,T} \right) + \mathrm{VAR} \left(\bar{p}_{x,T} \right) \right]$	$\mathrm{DF}_T \mathbb{E} \left(\bar{p}_{x,T} \right)$
MAD	$\mathrm{DF}_T \left[\mathrm{MEDIAN} \left(\bar{p}_{x,T} \right) + \lambda \mathrm{MAD} \left(\bar{p}_{x,T} \right) \right]$	$\mathrm{DF}_T \mathrm{MEDIAN} \left(\bar{p}_{x,T} \right)$

Table 3.4: Expressions for $s_{x,T}^{\text{realized}}$ and $s_{x,T}^{\text{anticipated}}$ used in pricing S-forwards.

consider a S-swap with maturity T that exchanges cash flows annually. Similar to an

S-forward, the risk-adjustment term π of the S-swap is determined such that the contract is fair at inception. Mathematically, this is given by Equation 2.4.2. The variable of interest is π, which can be evaluated using Equation 2.4.3. Expressions for $\sum_{t=1}^{T} s_{x,T}^{\text{realized}}$ and $\sum_{t=1}^{T} s_{x,T}^{\text{anticipated}}$ can be found in Table 3.5.

Premium Principle	$\sum_{t=1}^{T} s_{x,T}^{\text{realized}}$	$\sum_{t=1}^{T} s_{x,T}^{\text{anticipated}}$
Wang	$\sum_{t=1}^{T} \mathrm{DF}_t \left\{1 - \Phi\left[\Phi^{-1}\left(1 - \bar{p}_{x,t}\right) - \lambda\right]\right\}$	$\sum_{t=1}^{T} \mathrm{DF}_t \left\{1 - \Phi\left[\Phi^{-1}\left(1 - \bar{p}_{x,t}\right)\right]\right\}$
Proportional	$\sum_{t=1}^{T} \mathrm{DF}_t \left(\bar{p}_{x,t}\right)^{\frac{1}{\lambda}}$	$\sum_{t=1}^{T} \mathrm{DF}_t \bar{p}_{x,t}$
Dual	$\mathrm{DF}_T \left[1 - \left(1 - \bar{p}_{x,t}\right)^{\lambda}\right]$	$\mathrm{DF}_T \left[1 - \left(1 - \bar{p}_{x,t}\right)\right]$
Gini	$\sum_{t=1}^{T} \mathrm{DF}_t \left(1 + \lambda\right) \bar{p}_{x,t} - \lambda \left(\bar{p}_{x,t}\right)^2$	$\sum_{t=1}^{T} \mathrm{DF}_t \left(1 + \lambda\right) \bar{p}_{x,t}$
Exponential	$\sum_{t=1}^{T} \mathrm{DF}_t \frac{1 - e^{-\lambda \bar{p}_{x,t}}}{1 - e^{-\lambda}}$	$\sum_{t=1}^{T} \mathrm{DF}_t \frac{1 - e^{-\bar{p}_{x,t}}}{1 - e^{-1}}$
Std. Dev.	$\sum_{t=1}^{T} \mathrm{DF}_t \left[\mathbb{E}\left(\bar{p}_{x,t}\right) + \mathrm{SD}\left(\bar{p}_{x,t}\right)\right]$	$\sum_{t=1}^{T} \mathrm{DF}_t \mathbb{E}\left(\bar{p}_{x,t}\right)$
Variance	$\sum_{t=1}^{T} \mathrm{DF}_t \left[\mathbb{E}\left(\bar{p}_{x,t}\right) + \mathrm{VAR}\left(\bar{p}_{x,t}\right)\right]$	$\sum_{t=1}^{T} \mathrm{DF}_t \mathbb{E}\left(\bar{p}_{x,t}\right)$
MAD	$\sum_{t=1}^{T} \mathrm{DF}_t \left[\mathrm{MEDIAN}\left(\bar{p}_{x,t}\right) + \lambda \mathrm{MAD}\left(\bar{p}_{x,t}\right)\right]$	$\sum_{t=1}^{T} \mathrm{DF}_t \mathrm{MEDIAN}\left(\bar{p}_{x,t}\right)$

Table 3.5: Expressions for $s_{x,T}^{\text{realized}}$ and $\sum_{t=1}^{T} s_{x,T}^{\text{anticipated}}$ used in pricing S-swaps.

Chapter 4

Results and Discussion

Figures 4.1 and 4.2 present the values obtained for the risk-adjustment term π under different maturities for S-forwards and S-swaps respectively. Several interesting observations can be made from these figures. Firstly, S-forwards generally have a higher risk-adjustment term than S-swaps of the same maturity. For example, under the LC mortality model, the 10-year S-forward risk risk-adjustment term is 0.52% higher than the 10-year S-swap. This general pattern holds across the various models tested. One reason the S-swaps have a smaller risk-adjustment term is that an S-swap involves an annual exchange of cash flows linked to the number of survivors from the reference population at each time period. In contrast, a S-forward involves only a single exchange of cash flows. Second, note that the values of the risk-adjustment term increase as the contract term length increases. This phenomenon is expected since there is more significant uncertainty in longevity levels when comparing more extended periods to shorter periods. Notice that the risk-adjustment term values for the RH model shows that the premium principles are more concentrated compared to the risk-adjustment term values for the other mortality models. If an insurance company aims to have similar risk-adjustment term values for different premium principles, then the RH model would be a good fit for this dataset.

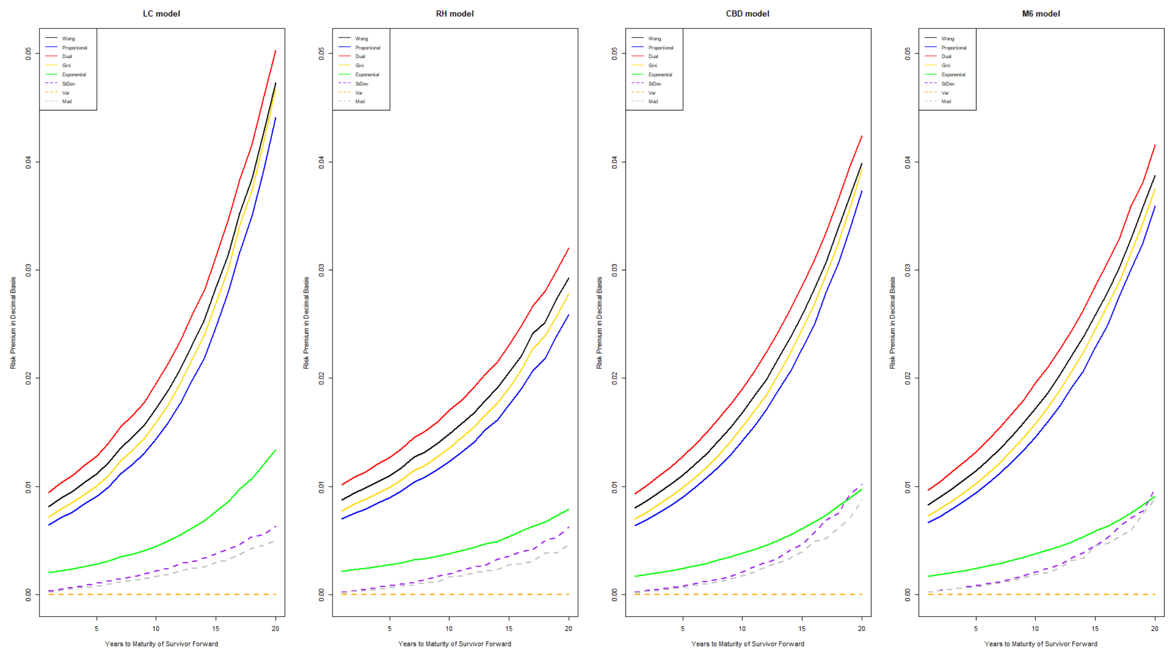

Figure 4.1: From left to right, the plot of values obtained for the risk-adjustment term π for S-forward under fixed mortality model over different maturities. (**a**) The plot under LC model. (**b**) The plot under the RH model. (**c**) The plot under the CBD model. (**d**) The plot under the M6 model.

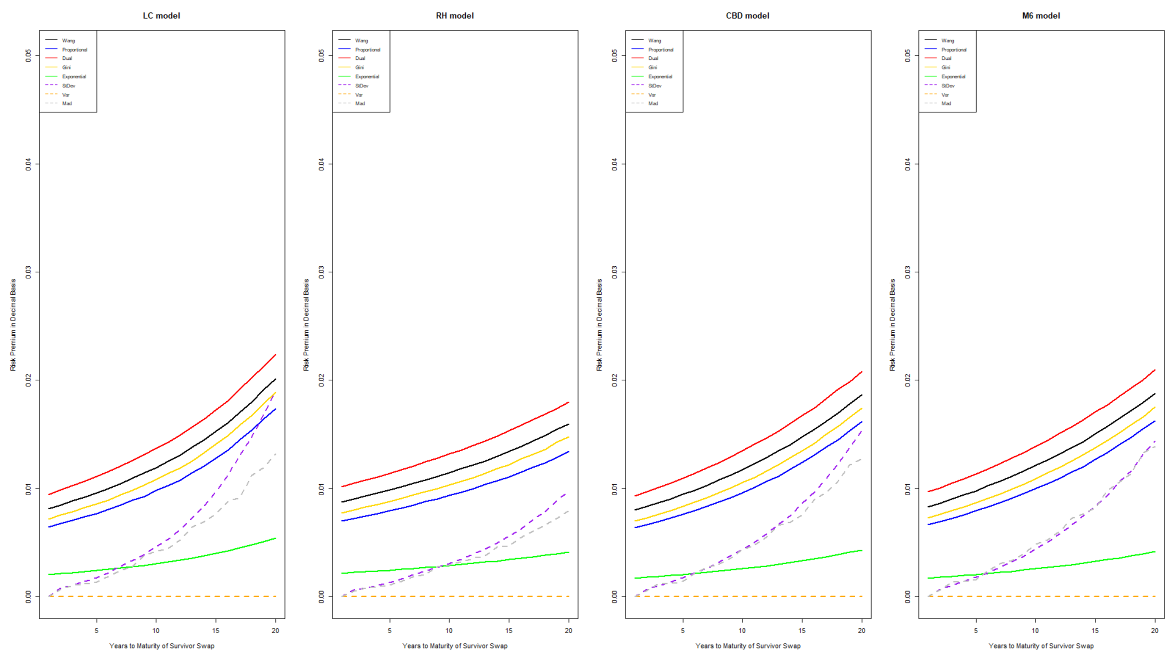

Figure 4.2: From left to right, the plot of values obtained for the risk-adjustment term π for S-swap under fixed mortality model over different maturities. (**a**) The plot under LC model. (**b**) The plot under the RH model. (**c**) The plot under the CBD model. (**d**) The plot under the M6 model.

The risk-adjustment term values generated under the real-world measure are generally lower than those generated from risk-neutral measures. The exponential transform generally produced the lowest risk-adjustment term among the risk-neutral measures. Moreover, A general pattern exists among the risk-adjustment term values. The dual-hazard transform has the largest risk-adjustment term. On the other hand, the variance principle produced the lowest risk-adjustment term. A generalization can be made that the risk-adjustment term generated follows a trend. The highest risk-adjustment term is generated by the dual hazard transform, followed by the Wang transform, the Gini transform, the proportional hazard transform, the standard deviation principle, then finally, the variance principle.

On the other hand, Table 4.1 and Table 4.2 show that the LC model generates the highest variance and range of risk-adjustment term values across both instruments. For instance, the 10-year S-forward risk-adjustment term range is 3.92% under LC, compared to just 2.06% under RH. This statistic indicates that the LC approach introduces greater uncertainty into longevity projections. This translates into higher and more variable risk-adjustment term values. For a 10-year contract, the S-forward risk-adjustment term is 0.52% larger than the S-swap risk-adjustment term. This gap is the largest across the mortality models, highlighting the LC model's propensity to amplify single-payment instrument risk. Across the premium principles, the variance principle still produces the lowest and least dispersed risk-adjustment term values. However, the dual hazard transform generates the highest risk-adjustment term values with the widest spread.

Moreover, the RH model generated the lowest mean, range, and variance among the risk-neutral risk-adjustment term values. The RH model having the smallest mean implies that the RH model generates the smallest risk-adjustment term values among the four mortality models. Furthermore, having the smallest range and variance suggests that the RH model has the least uncertainty in providing a contract risk-adjustment term. The variance in risk premiums across different pricing principles is smallest under RH model.

Table 4.1: Values for the risk-adjustment parameter π for the LC and RH model for a survivor forward

	LC			RH		
	Range	Mean	Variance	Range	Mean	Variance
Wang	3.923e-02	2.165e-02	1.433e-04	2.056e-02	1.671e-02	3.953e-05
Proportional	3.764e-02	1.886e-02	1.299e-04	1.892e-02	1.410e-02	3.293e-05
Dual	4.090e-02	2.397e-02	1.580e-04	2.190e-02	1.884e-02	4.525e-05
Gini	3.955e-02	2.049e-02	1.444e-04	2.006e-02	1.544e-02	3.773e-05
Exponential	1.137e-02	5.833e-03	1.202e-05	5.756e-03	4.344e-03	3.059e-06
Std. Dev.	5.992e-03	2.687e-03	3.285e-06	6.046e-03	2.440e-03	3.204e-06
Variance	6.026e-09	1.634e-09	3.365e-18	5.697e-09	1.438e-09	2.851e-18
MAD	4.737e-03	2.109e-03	2.097e-06	4.442e-03	1.848e-03	1.743e-06

Table 4.2: Values for the risk-adjustment parameter π for the LC and RH model for a survivor swap

	LC			RH		
	Range	Mean	Variance	Range	Mean	Variance
Wang	1.205e-02	1.293e-02	1.360e-05	7.203e-03	1.190e-02	4.908e-06
Proportional	1.092e-02	1.073e-02	1.122e-05	6.453e-03	9.755e-03	3.860e-06
Dual	1.299e-02	1.471e-02	1.585e-05	7.893e-03	1.363e-02	5.878e-06
Gini	1.174e-02	1.181e-02	1.291e-05	6.996e-03	1.078e-02	4.599e-06
Exponential	3.400e-03	3.348e-03	1.074e-06	1.972e-03	3.018e-03	3.641e-07
Std. Dev.	1.903e-02	6.698e-03	3.252e-05	9.747e-03	3.891e-03	8.324e-06
Variance	5.860e-08	1.225e-08	2.882e-16	1.527e-08	3.767e-09	2.085e-17
MAD	1.320e-02	5.233e-03	1.625e-05	7.960e-03	3.372e-03	5.651e-06

Finally, Table 4.3 and Table 4.4 show that the CBD model generates higher variance and range of S-forward premiums compared to the M6 model. For example, the 10-year S-forward variance is 9.686x10-5 under CBD versus 8.764x10-5 under M6. However, for S-swaps, the premium variance is more similar between CBD and M6. When comparing the differences between S-forward and S-swap premiums, the CBD and M6 models behave more similarly than LC or RH. For 10-year contracts, the S-forward premium exceeds the S-swap by around 0.51% for both CBD and M6. Under the CBD model, the standard deviation premium principle produces the highest variance premiums for S-forwards, contrasting with other models where the dual hazard is highest.

Meanwhile, the M6 model sees the lowest mean premiums under the mean absolute deviation principle. Generally, the CBD and M6 models exhibit a middle-ground behavior between the extremes of LC and RH for variance, range, and internal consistency of longevity risk pricing. They strike a balance in the magnitude and dispersion of premiums generated.

Regarding the risk-adjustment term statistics for the S-forward (Tables 4.1 and 4.3), the results suggest that the LC model produces the highest risk-adjustment term values, followed by the M6, the CBD, and the RH model. Furthermore, the risk-adjustment term values obtained for the real-world risk-adjustment term values are less dispersed than those obtained from the risk-neutral risk-adjustment term values.

On the other hand, the risk-adjustment statistics for the S-swap (Tables 4.2 and 4.4) suggest that the LC model generates the highest risk-adjustment term values, followed by the M6 model, CBD model, and the RH model. Among the real-world risk-adjustment term values for the S-swap, the dispersion between the CBD and M6 models is less pronounced. However, the difference between the risk-adjustment term values generated by the LC and RH models is more pronounced. This suggests that for a S-swap, adding the cohort effect term to models of the CBD type does not seem to affect the risk-adjustment term.

Table 4.3: Values for the risk-adjustment parameter π for the CBD and M6 model for a survivor forward

	CBD			M6		
	Range	Mean	Variance	Range	Mean	Variance
Wang	3.189e-02	1.991e-02	9.686e-05	3.043e-02	1.995e-02	8.764e-05
Proportional	3.097e-02	1.737e-02	8.902e-05	2.926e-02	1.743e-02	8.032e-05
Dual	3.312e-02	2.209e-02	1.056e-04	3.189e-02	2.215e-02	9.586e-05
Gini	3.230e-02	1.868e-02	9.733e-05	3.023e-02	1.867e-02	8.664e-05
Exponential	8.020e-03	4.635e-03	6.128e-06	7.370e-03	4.454e-03	5.069e-06
Std. Dev	9.967e-03	3.381e-03	9.359e-06	9.579e-03	3.172e-03	7.736e-06
Variance	1.938e-08	3.544e-09	2.812e-17	1.799e-08	3.333e-09	2.482e-17
MAD	8.572e-03	2.786e-03	6.182e-06	8.591e-03	2.862e-03	6.341e-06

Table 4.4: Values for the risk-adjustment parameter π for the CBD and M6 model for a survivor swap

	CBD			M6		
	Range	Mean	Variance	Range	Mean	Variance
Wang	1.069e-02	1.250e-02	1.092e-05	1.048e-02	1.281e-02	1.058e-05
Proportional	9.850e-03	1.039e-02	9.240e-06	9.614e-03	1.068e-02	8.925e-06
Dual	1.150e-02	1.425e-02	1.279e-05	1.129e-02	1.459e-02	1.222e-05
Gini	1.043e-02	1.133e-02	1.050e-05	1.029e-02	1.160e-02	1.004e-05
Exponential	2.592e-03	2.794e-03	6.540e-07	2.469e-03	2.755e-03	5.815e-07
Std. Dev.	1.537e-02	5.833e-03	2.141e-05	1.441e-02	5.685e-03	1.896e-05
Variance	4.174e-08	9.742e-09	1.529e-16	3.847e-08	9.286e-09	1.302e-16
MAD	1.273e-02	5.282e-03	1.570e-05	1.388e-02	5.796e-03	1.842e-05

Comparison with Previous Works

We compare our results to that of Tang & Li [25]. In their paper, the authors found that S-forwards have higher risk premiums than S-swaps of the same maturity. We obtained similar results, which can be seen in Figures 4.1 and 4.2. One possible explanation for this is that S-swaps involve multiple cash flow exchanges over its horizon compared to the single payment structure of an S-forward. Second, the authors found that within the risk-neutral premium principles, the risk premiums obtained are very similar across the nine premiums. In contrast, real-world premiums produce somewhat higher risk premiums than risk-neutral ones. Our results show that four of the five risk-neutral premiums are close to one another, except the exponential transform with a lower risk-adjustment term value than other risk-neutral premiums. Furthermore, our results generally show that real-world premiums generate a lower risk adjustment term compared to risk-neutral premiums. Lastly, the authors found that the choice of mortality model has a bigger impact on risk premiums than the choice of premium principle. This results of show that the RH model tends to produce higher premiums than the M6 model with quadratic terms. Our results indicate that the LC model produced the largest values for the risk-adjustment term, while the RH model generated the smallest values. We obtained similar observations that the magnitude of the risk-adjustment term values generated from one mortality model versus another is substantially different compared to fixing a mortality model and varying the premium principle. This suggests that the choice of mortality model has a more significant impact on the implied risk-adjustment term when compared to the premium principle.

Chapter 5

Conclusions and Recommendations

5.1 Summary and Conclusion

In this thesis, we investigated the impact of the mortality model and premium principle uncertainty on pricing S-forwards and S-swaps by comparing the risk-adjustment terms calculated from eight premium principles (Wang, proportional hazard, dual power, Gini, exponential transforms, standard deviation, variance, and median absolute deviation principles) under four mortality models (LC, RH, CBD, and M6). The mortality models are fitted to England & Wales (E&W) mortality data, and the premium principles are calibrated using market quotations of UK standard pension annuities.

We adopted a flexible simulation-based framework to overcome restrictive assumptions and enable the future extension to price other longevity-linked securities. The analysis for pricing showed that S-forwards generally have a higher risk-adjustment term than S-swaps of the same maturity across models, with swaps involving annual cash flow exchanges versus a single exchange for forwards. Risk-adjustment values increase with longer contract terms due to greater longevity uncertainty. The RH model produces a smaller spread between values across principles. Real-world measures generate lower values than risk-neutral measures, with exponential transform being the lowest overall. A trend exists from dual-hazard transform having the highest values to variance principle as the lowest. The LC model produces the greatest variance and range of values, introducing greater uncertainty into projections and amplifying risk, especially for single-payment instruments. In contrast, the RH model generates the lowest mean, range, and variance of risk-adjustment values across principles, implying less uncertainty in estimating contract risk premiums.

The CBD model produced a higher variance and range of risk-adjustment term values for S-forwards than the M6, but the models behave more similarly for S-swaps. The CBD and M6 models show closer alignment in premium differences between S-forwards and S-swaps than the LC or RH. Overall, CBD and M6 exhibit a middle ground between LC and

RH in the magnitude and dispersion of risk premiums generated. Regarding models, LC gives the highest risk premium values, followed by M6, CBD, and RH for both contracts. We notice that real-world premium principle values are less dispersed than risk-neutral ones. The choice of mortality model has a bigger impact than the premium principle regarding the value for the risk-adjustment term.

In summary, Our results indicate that the uncertainty arising from the choice of mortality model is larger than the uncertainty arising from the choice of premium principle. Among the four mortality models considered, the LC model had the largest risk-adjustment term values for both the S-forward and S-swap whereas the RH model has the lowest values. Between the premium principles, those based on risk-neutral measures produced higher risk-adjustment terms when compared to real-world measures. The dual-power transform produced the largest risk-adjustment term and so may be selected as the premium principle for a conservative pricing estimate.

We address each of the thesis objectives as follows:

- Impact of Mortality Models: We analyzed four mortality models (LC, RH, CBD, and M6) and observed variations in the risk-adjustment terms generated. The LC model consistently produced the highest risk-adjustment terms, while the RH model yielded the lowest values. This suggests that the choice of mortality model substantially impacts the valuation of longevity-linked contracts.

- Effect of Premium Principles: We examined eight premium principles and found that the choice of premium principle also influences the risk-adjustment terms. Real-world premium principles generally resulted in lower risk-adjustment terms compared to risk-neutral principles. The dual-power transform generated the highest risk-adjustment terms, indicating a conservative pricing estimate.

- Relative Impact of Model and Principle Selection: Our analysis quantified the relative impact of mortality model selection versus premium principle selection on contract valuations. We found that the uncertainty arising from the choice of mortality model was larger than the uncertainty arising from the choice of premium principle.

- Our comparison with Tang & Li [25] revealed consistent findings, such as S-forwards consistently carrying higher risk premiums than S-swaps. However, our study offered greater insights into how mortality models and premium principles impact contract valuations. While Tang & Li provided valuable insights into general trends, our analysis allowed for a more detailed exploration of the combined effects of mortality models and premium principles.

5.2 Recommendations and Limitations

One possible extension to the thesis would include a broader range of mortality models, such as the PLAT model. This thesis did not include more mortality models due to time constraints. The PLAT model combines the CBD model with some features of the LC model. The quadratic CBD model with cohort effect (M7 model) that extends the original CBD model by adding a cohort effect and a quadratic age effect is also worth analyzing. Incorporating a wide range of mortality models would be accessible using the framework and code described in this paper. Furthermore, performing a multi-country analysis of mortality models would be interesting.

Another possible extension would be to include more longevity-linked derivatives such as mortality options. Moreover, it would be interesting to calibrate the pricing parameter to actual products instead of annuity quotations when the longevity market becomes more mature and details of contracts become publicly available.

Appendix A

Code and Implementation

The code used for this thesis is available at https://github.com/kenrickraymond/Longevity-Instrument-Pricing, allowing the reader to replicate and extend this thesis by looking into other contracts, mortality models, and premium principles.

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