Advection (Hyperbolic PDEs)

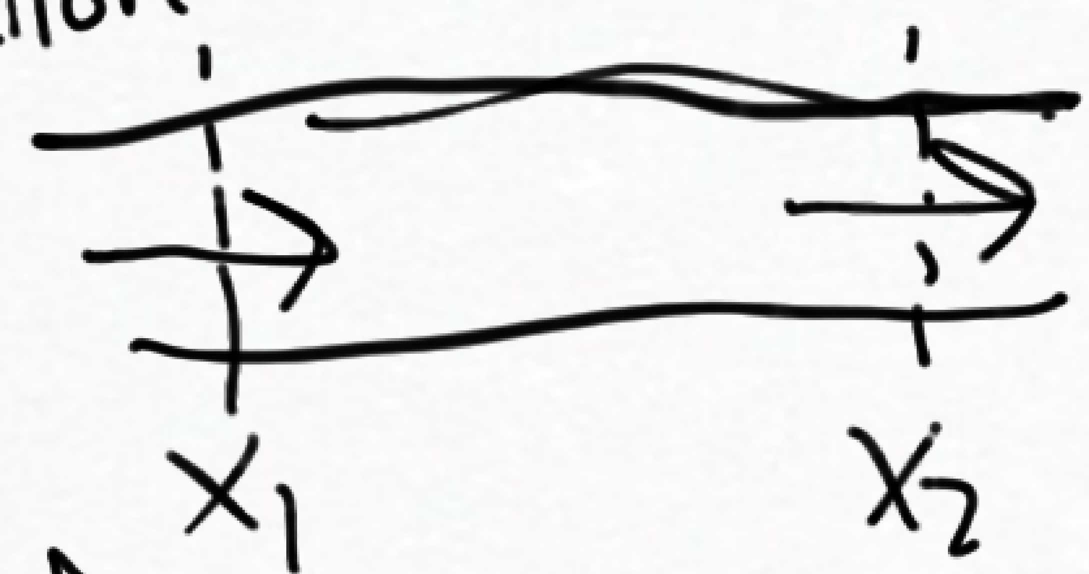
- Hyp. PDEs describe:
  - Water waves
  - Sound waves
  - Light
  - Pressure waves
    - Motion of fluids

Simplest example: advection equation

Derivation of a conservation law:

$u(x,t)$: concentration

Narrow channel



$$U = \int_{x_1}^{x_2} u(x,t)\,dx$$

flux: $f(u,x,t)$

$$\boxed{\frac{dU}{dt} = f(u(x_1,t),x_1,t) - f(u(x_2,t),x_2,t)}$$

$$\boxed{\int_{x_1}^{x_2} \frac{\partial u}{\partial t}\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(u(x,t),x,t)\,dx}$$

$$\boxed{u_t + f(u)_x = 0}$$

$$\int_{x_1}^{x_2} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}\right) dx = 0 \Rightarrow \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

Assume constant flow with velocity a: $f(u) = au$

$$u_t + a u_x = 0 \quad \text{(Advection)}$$

$$\boxed{u(x,t) = u_0(x - at)} \quad \text{where } u_0 = u(x,0)$$

Check: $u_t = -a u_0'(x-at)$, $u_x = u_0'(x-at)$ $\Big\}$ Substitute into PDE: $-au_0' + au_0' = 0$

$u_t + au_x = 0 \rightarrow$ Forward diff. in time
Centered diff. in space

$$\frac{U_j^{n+1} - U_j^n}{k} = -a \frac{U_{j+1}^n - U_{j-1}^n}{2h}$$

$$U_j^{n+1} = U_j^n - \frac{1}{2}\frac{ka}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Stability:

Von Neumann: $U_j^n = g^n e^{ijh\xi}$

$$g^{n+1} e^{ijh\xi} = g^n\left(e^{ijh\xi} - \frac{1}{2}\frac{ka}{h}\left(e^{ih\xi(j+1)} - e^{ih\xi(j-1)}\right)\right)$$

$$g = 1 - \frac{1}{2}\frac{ka}{h}\left(e^{ih\xi} - e^{-ih\xi}\right)$$

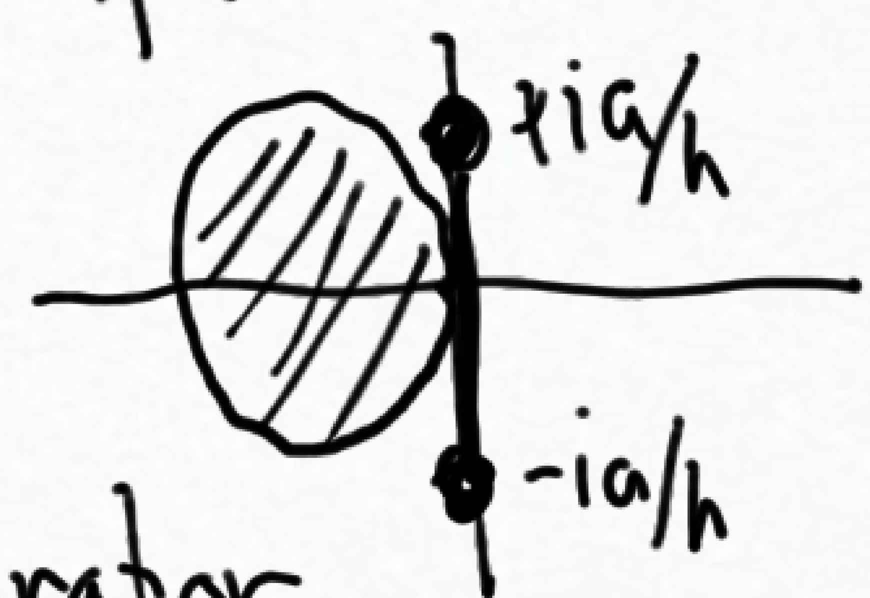
$$g = 1 - \frac{1}{2}\frac{ka}{h}\left(2i\sin(h\xi)\right)$$

$$g = 1 - \frac{ka}{h} i \sin(h\xi) \quad |g| > 1 \Rightarrow \text{unstable!}$$

MOL analysis

$$U'(t) = AU \quad A = \frac{-a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

$$\lambda_p = \frac{-ia}{h}\sin(2\pi p h)$$



Need a time integrator that is stable on the imaginary axis:

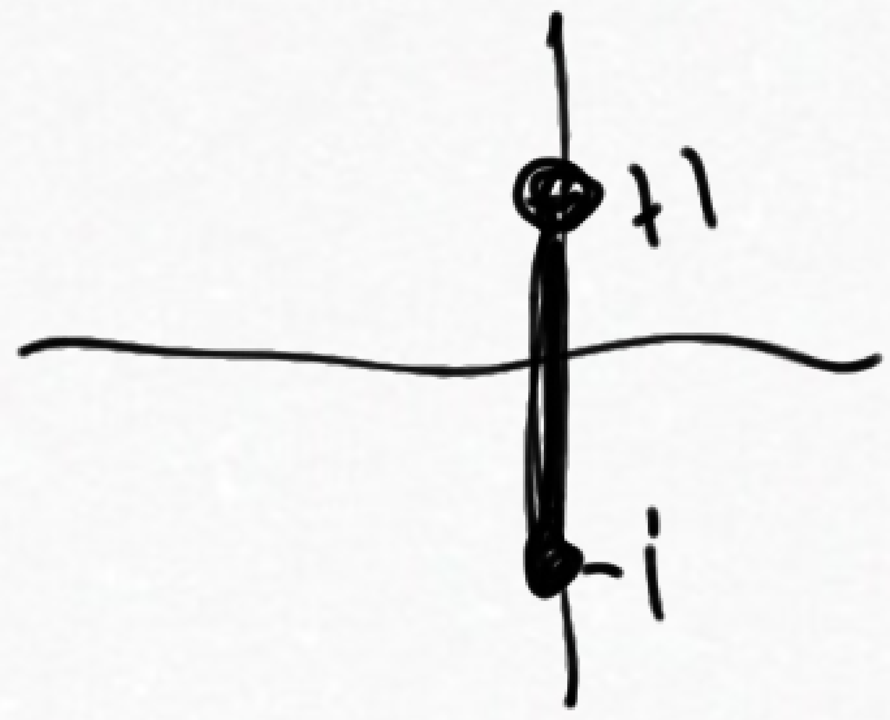
- Leapfrog
- RK4

Problem is not stiff, so we'll use explicit methods.

Leapfrog

$U^{n+1} = U^{n-1} + 2kAU^n$ 2-step linear multistep method

$$U_j^{n+1} = U_j^{n-1} - \frac{ka}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$



Stability: $\left|\frac{ka}{h}\right| \le 1$

$\frac{ka}{h}$: Dimensionless number
Distance solution travels in one time step, measured in grid units.

"CFL number" or "Courant number"

Courant-Friedrichs-Lewy (1927 paper)

Lax-Friedrichs

$$U_j^{n+1} = \underbrace{\frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right)}_{\text{Average}} - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$\frac{1}{2}\left(U_{j+1}^n - U_{j-1}^n\right) = U_j^n + \frac{1}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$$U_j^{n+1} = U_j^n + \frac{1}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$\frac{U^{n+1} - U_j^n}{k} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} = \frac{h^2}{2k} \cdot \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

Looks like a disc. of:

$$u_t + au_x = \frac{h^2}{2k} u_{xx}$$

if $h = \mathcal{O}(k)$ then diffusion term vanishes as $k, h \to 0$