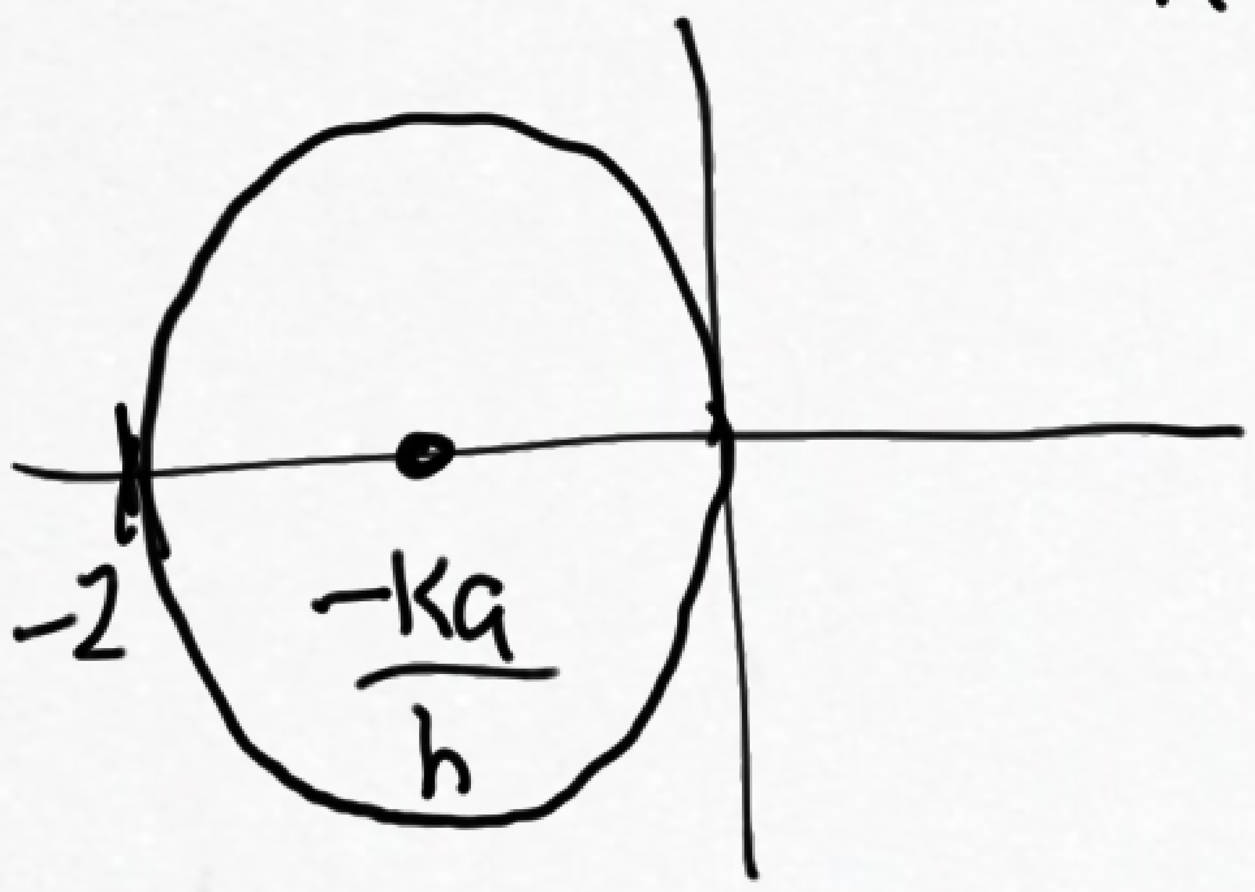
The upwind method

$u_t + a u_x = 0 \qquad a > 0$

$$U_j^{n+1} = U_j^n - \frac{ka}{h}\left(U_j^n - U_{j-1}^n\right)$$

Semi-discrete: $U'(t) = \underbrace{-\frac{a}{h}\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}}_{A \,\leftarrow\, \text{non-normal}} U(t)$

Eigenvalues of A: $-\frac{a}{h}$



$$-\frac{ka}{h} \geq -2 \Leftrightarrow \boxed{\frac{ka}{h} \leq 2} \qquad k \leq \frac{2h}{a}$$

Von Neumann analysis (Periodic BCs)

$U_j^n \to g^n e^{ijh\xi}$

$$U_j^{n+1} = U_j^n - \frac{ka}{h}\left(U_j^n - U_{j-1}^n\right)$$

$$g^{n+1} e^{ijh\xi} = g^n e^{ijh\xi} - \frac{ka}{h}\left(g^n e^{ijh\xi} - g^n e^{i(j-1)h\xi}\right) \qquad \text{Divide by } g^n e^{ijh\xi}$$

$$g = 1 - \frac{ka}{h}\left(1 - e^{-ih\xi}\right) = 1 - \frac{ka}{h}\left(1 - \cos(h\xi) + i\sin(h\xi)\right) \qquad \frac{ka}{h} = \nu$$

$$g = \underline{1 - \nu + \nu\cos(h\xi)} - \underline{i\nu\sin(h\xi)}$$

$$|g|^2 = 1 - 2\nu + \nu^2 + 2\nu\cos(h\xi) - 2\nu^2\cos(h\xi) + \overbrace{\nu^2\cos^2(h\xi) + \nu^2\sin^2(h\xi)}^{\nu^2}$$

$$= 1 - 2\nu + 2\nu^2 + 2\nu\cos(h\xi) - 2\nu^2\cos(h\xi)$$

$$= 1 - 2\nu(1-\nu)(1 - \cos(h\xi))$$

$|g|^2 = 1 - 2\nu(1-\nu)(1-\cos(h\xi))$ $\boxed{\text{If } 0 \le \nu \le 1 \text{ then } |g|^2 \le 1}$

Observe: $0 \le \nu(1-\nu) \le \frac{1}{4}$ $\quad 0 \le 1-\cos(h\xi) \le 2$

$$0 \le 2\nu(1-\nu) \le \frac{1}{2}$$

So $|g|^2 \le 1$.

We need $0 \le \frac{ka}{h} \le 1$ $\rightarrow$ unstable if $a < 0$.

Let $A \in \mathbb{C}^{m\times m}$ be diagonalizable: $A = R\Lambda R^{-1}$

$AR = R\Lambda$

$$A^n = R\Lambda R^{-1} R\Lambda R^{-1} \cdots R\Lambda R^{-1}$$

$$A^n = R\Lambda^n R^{-1}$$

spectral radius $\quad$ condition #

$$\|A^n\|_2 \le \|R\|_2 \|\Lambda^n\|_2 \|R^{-1}\|_2 = \|\Lambda\|_2^n K(R) = \rho(A)^n K(R)$$

If A is normal, then R is unitary $\rightarrow K(R) = 1 \rightarrow \|A^n\|_2 = \rho(A)^n$

Dfn.: Matrix $A \in \mathbb{C}^{m\times m}$ is normal if $A^*A = AA^*$.

Symm., skew-symm. and circulant matrices are all normal.

For a normal matrix with $\rho(A) < 1$: $\|A^n\| \le \rho(A)^n < 1$

$$\lim_{n\to\infty} \|A^n\| = 0$$

Non-normal case: we can have $\|A^n\| > 1$ even though $\rho(A) < 1$.

but still $\lim_{n\to\infty} \|A^n\| = 0$

Moral of the story: For non-normal matrices, eigenvalues don't fully characterize the behavior.

Example: $A_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}$ $\lambda = 0.8, 0.9$ $\|A^n\| \le \|A\| \ \forall n$

$A_2 = \begin{bmatrix} 0.8 & 100 \\ 0 & 0.9 \end{bmatrix}$ $\lambda = 0.8, 0.9$ $\|A^n\| > \|A\|$ for $n = 2, 3, \ldots$

$R = \begin{bmatrix} 1 & 1 \\ 0 & 10^{-3} \end{bmatrix}$ $\kappa(R) = 2000$

Modified Equation Analysis

Goal: Find a PDE that our numerical discretization satisfies exactly.

$$V(x_j,t_n) = U_j^n \qquad \text{Upwind: } \frac{U_j^{n+1}-U_j^n}{k} = -a\frac{U_j^n - U_{j-1}^n}{h}$$

Substitute v into the discretization:

$$\boxed{\frac{V(x_j,t_{n+1}) - V(x_j,t_n)}{k} = -a\,\frac{V(x_j,t_n) - V(x_{j-1},t_n)}{h}}$$

$$V(x_j,t_{n+1}) = V(x_j,t_n) + kV_t(x_j,t_n) + \frac{k^2}{2}V_{tt}(x_j,t_n) + \mathcal{O}(k^3)$$

$$V(x_{j-1},t_n) = V(x_j,t_n) - hV_x(x_j,t_n) + \frac{h^2}{2}V_{xx}(x_j,t_n) + \mathcal{O}(h^3)$$

$$V_t + \frac{k}{2}V_{tt} + \mathcal{O}(k^2) = -aV_x + a\frac{h}{2}V_{xx} + \mathcal{O}(h^2)$$

$$\boxed{V_t + aV_x = \frac{1}{2}(ahV_{xx} - kV_{tt})} + \mathcal{O}(h^2,k^2)$$

$$V_t \approx -aV_x \Rightarrow \left.\begin{array}{l} V_{tt} \approx -aV_{xt} \\ V_{tx} \approx -aV_{xx} \end{array}\right\} \quad V_{tt} = +a^2V_{xx} + \mathcal{O}(h,k)$$

$$V_t + aV_x = \frac{1}{2}(ahV_{xx} - a^2kV_{xx}) + \mathcal{O}(h^2,k^2,kh)$$

$$V_t + aV_x = \frac{a}{2}V_{xx}(h-ka) \longrightarrow V_t + aV_x = \frac{1}{2}ah\left(1-\frac{ka}{h}\right)V_{xx} + \cdots$$

$$V_t + aV_x = \underline{\frac{ah}{2}(1-\nu)}V_{xx} + \mathcal{O}(k^2, h^2, kh)$$

V satisfies an advection-diffusion equation

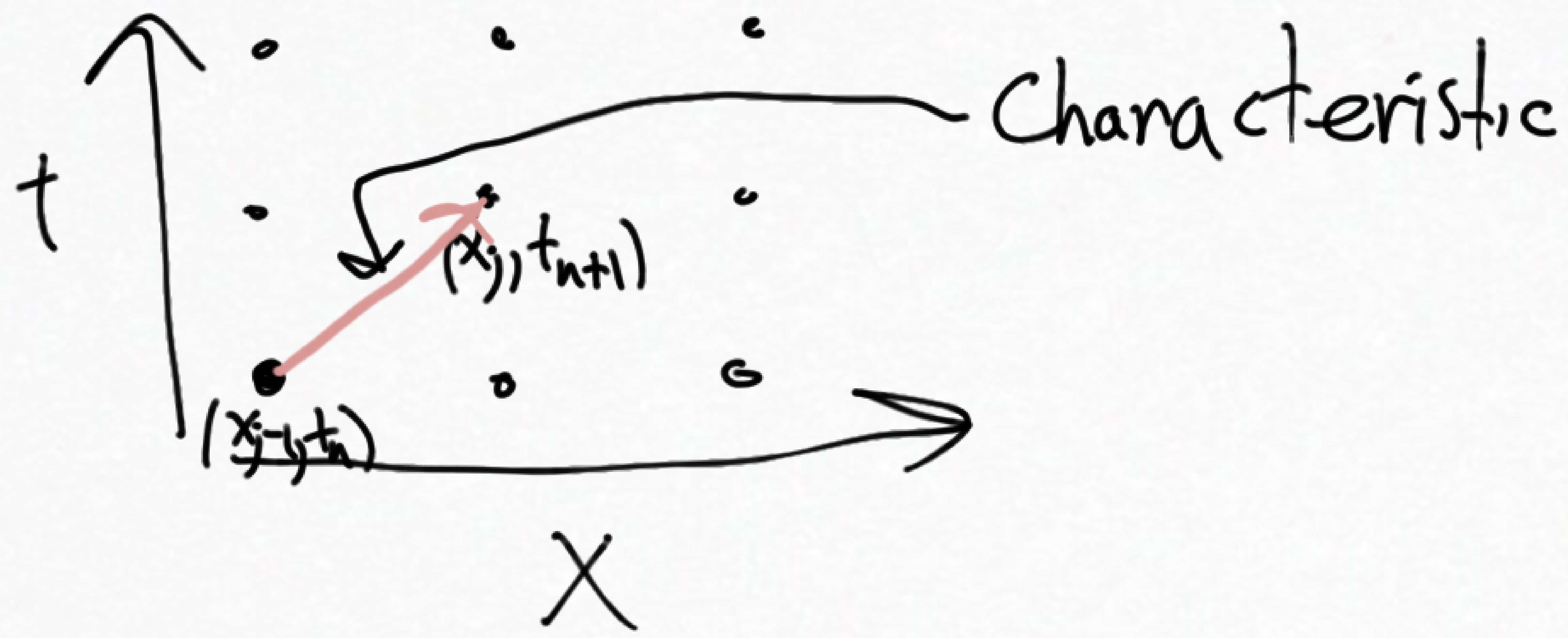
The diffusion term is proportional to h

If $\nu = 1$ (if $ka = h$) $\rightarrow$ no diffusion.

$$U_j^{n+1} = U_j^n - \frac{ka}{h}(U_j^n - U_{j-1}^n) \qquad \text{Let } \frac{ka}{h} = 1.$$

$$\underbrace{U_j^{n+1} = U_{j-1}^n}$$

Exact solution!

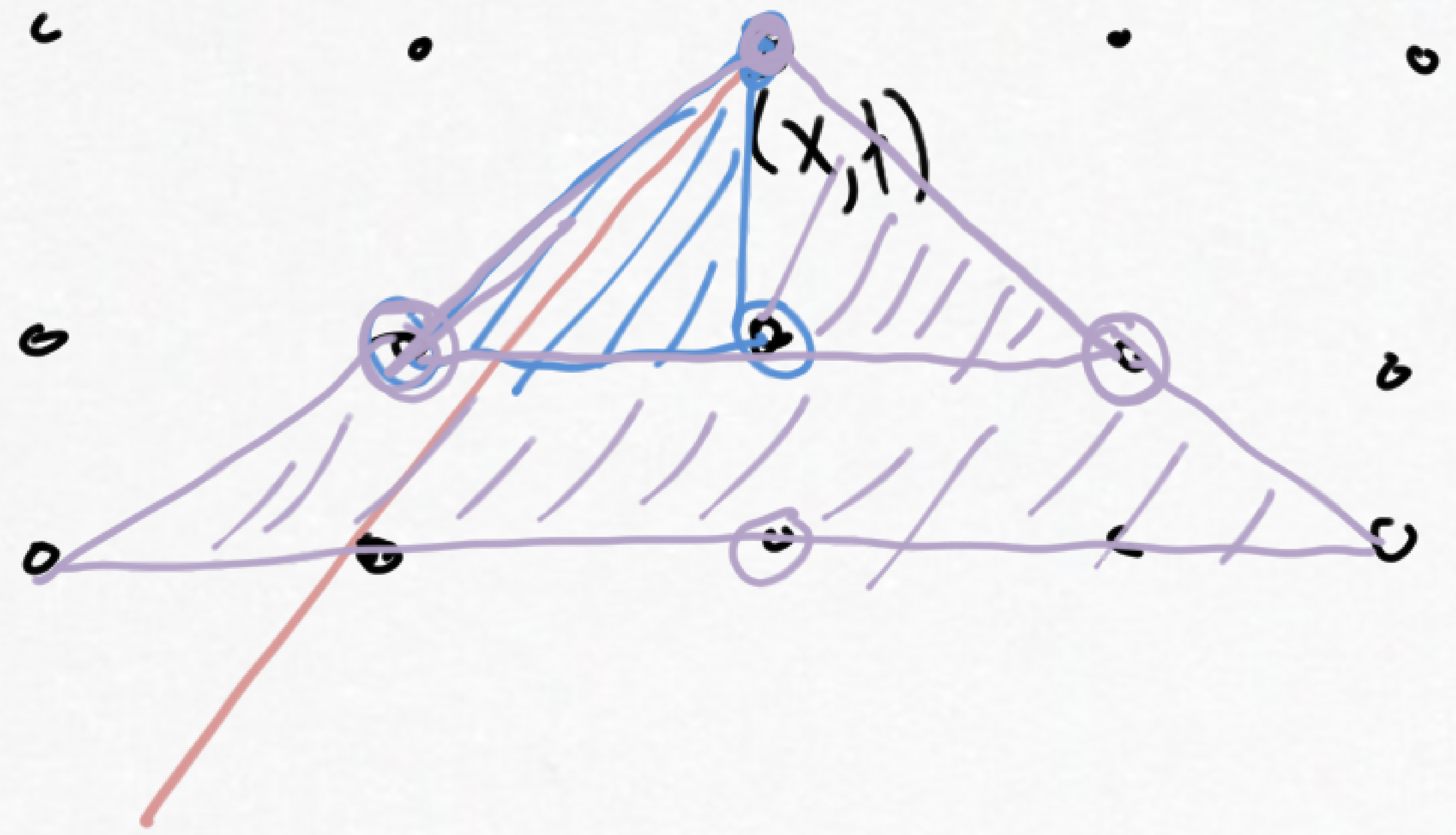


The CFL condition

The domain of dependence of $u(x,t)$ is the set of points at earlier times on which $u(x,t)$ depends.

The numerical domain of dependence of U_j^n is the set of points on the computational grid that affect U_j^n.

For any convergent method, the numerical d. of. d. must contain the (PDE d. of. d.) in the limit $h, k \to 0$.



Upwind

Leapfrog

$$-1 \le \frac{ka}{h} \le 1$$

$$0 \le \frac{ka}{h} \le 1$$

(upwind)

For hyp. PDEs: Finite speed of propagation

For parabolic PDEs: Infinite speed of propagation.

Text sections:
10.4-10.7, 10.9, D.4