Fourier Analysis of Linear PDEs and discretizations

Appendix E
Section 9.6

$$\|v(x)\|_2 = \left(\int_{-\infty}^{\infty} |v(x)|^2 dx\right)^{1/2} < \infty$$

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x) e^{-i\xi x} dx$$ Fourier transform

$$\hat{v}(\xi) = \mathcal{F}(v(x)) \qquad \|\hat{v}(\xi)\|_2 = \|v(x)\|_2$$

$$v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{v}(\xi) e^{i\xi x} d\xi$$ Inverse Fourier Transform

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$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\xi x} dx$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi,t) e^{i\xi x} d\xi$$

$$u_t(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_t(\xi,t) e^{i\xi x} d\xi$$

$$u_{xx}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\xi^2 \hat{u}(\xi,t) e^{i\xi x} d\xi$$

$$u_t - u_{xx} = 0$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u_t - u_{xx}) e^{-i\xi x} dx$$

$$(\hat{u}_t + \xi^2 \hat{u}) e^{i\xi x} = 0$$

$$\boxed{\hat{u}_t = -\xi^2 \hat{u}}$$ ODE for $\hat{u}(\xi,t)$

$\hat{u}_t = -\xi^2 \hat{u}$

Initial condition: Fourier transform of $u(x,0)$

$$\hat{u}_0(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-i\xi x} dx$$

$$\hat{u}(\xi,t) = e^{-\xi^2 t} \hat{u}_0(\xi)$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi,t) e^{i\xi x} d\xi$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{e^{i\xi x - \xi^2 t}} \hat{u}_0(\xi) d\xi}$$

$u_t = u_{xx}$ Ansatz: $u(x,t) = e^{i(\xi x - \omega t)}$ $\omega = \omega(\xi)$

$$-i\omega e^{i(\xi x - \omega t)} = -\xi^2 e^{i(\xi x - \omega t)}$$

$$\boxed{\omega(\xi) = -i\xi^2} \Rightarrow u(x,t) = e^{i(\xi x + i\xi^2 t)} = \boxed{e^{i\xi x - \xi^2 t}}.$$

Dispersion relation

Advection: $u_t = u_x$

$$-i\omega e^{i(\xi x - \omega t)} = i\xi e^{i(\xi x - \omega t)}$$

$$\omega(\xi) = -\xi \Rightarrow u(x,t) = e^{i(\xi x + \xi t)} = \boxed{e^{i\xi(x+t)}}.$$

Von Neumann analysis:

$$U_j^{n+1} = U_j^n + \frac{K}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$ ← FE + CD

Ansatz: $U_j^n = (g(\xi))^n e^{i\xi x_j} = \boxed{(g(\xi))^n e^{i\xi hj}}$

$$g(\xi) e^{i\xi jh} = e^{i\xi jh} + \frac{K}{h^2}\left(e^{i\xi h(j+1)} - 2e^{i\xi hj} + e^{i\xi h(j-1)}\right)$$

$$g(\xi) = 1 + \frac{K}{h^2}\left(e^{i\xi h} - 2 + e^{-i\xi h}\right)$$

$$g(\xi) = 1 + \frac{K}{h^2}\left(2\cos(\xi h) - 2\right) = \boxed{1 + 2\frac{K}{h^2}\left(\cos(\xi h) - 1\right)}$$

We need $|g(\xi)| \le 1$ for stability.

$$\left|1 + 2\frac{K}{h^2}(-1-1)\right| = \left|1 - 4\frac{K}{h^2}\right| \le 1 \iff 1 - 4\frac{K}{h^2} \ge -1$$

$$\boxed{K \le \frac{h^2}{2}}$$

$$U^{n+1} = U^n + \frac{K}{h^2} A U^n$$

$$A = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

Periodic BCs.

Circulant matrix

$$U^{n+1} = \left(I + \frac{K}{h^2} A\right) U^n$$

$g(\xi)$ is eigenvalue of this

Textbook E.3.1–E.3.3 E.3.9