Convergence and Stability for PDE Discretizations (Sections 9.3-9.5)

Last time: $u_t = u_{xx}$ "infinitely stiff"

Semi-Discretization $U'(t) = AU$ finite stiffness $|\lambda_{max}| \approx \frac{4}{h^2}$

As $h \to 0$, this becomes larger →

Convergence for ODEs:

Zero-stability (limit as $k \to 0$)	bound on E for any fixed final time T	necessary + suff. for convergence
absolute stability	bound on propagated errors for all t.	not necessary for convergence

For PDEs, Absolute stability is necessary for convergence.

Why? Because we need both $k, h \to 0$

System of ODEs grows as $h \to 0$

Recall our full discretizations of the heat equation:

$$\text{FE:} \quad U^{n+1} = kAU^n$$

$$\text{Trap. rule:} \quad U^{n+1} = \frac{k}{2}A(U^n + U^{n+1})$$

$$A_h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

For both methods, we have

$$U^{n+1} = R(kA_h)U^n$$

where $R(z) = 1+z$ for FE

$$R(z) = \frac{1+z/2}{1-z/2} \quad \text{for trapezoidal}$$

$$\downarrow$$

$$R(kA_h) = \left(I - \frac{k}{2}A_h\right)^{-1}\left(I + \frac{k}{2}A_h\right)$$

We'll write $B_{k,h} = R(kA_h)$.

So $\boxed{U^{n+1} = B_{k,h}U^n} \longrightarrow U^n = B_{k,h}^n U^0$

True solution: $U^n = [u(x_1,t^n), u(x_2,t^n), \ldots, u(x_M,t^n)]^T$

Substitute: $\boxed{U^{n+1} = B_{k,h} U^n + k\tau^n}$ ← local truncation error

$$U^{n+1} - u^{n+1} = E^{n+1} = B_{k,h} E^n - k\tau^n$$

$$E^n = B_{k,h}^n E^0 - kB^{n-1}\tau^0 - kB^{n-2}\tau^1 - \cdots - k\tau^{n-1}$$

$$E^n = B^n E^0 - k\sum_{j=0}^{n-1} B^{n-1-j}\tau^j \qquad N = \frac{T}{k}$$

$$\|E^N\| \le \|B^N\|\cdot\|E^0\| + k\sum_{j=0}^{N-1}\|B^{N-1-j}\|\,\|\tau^j\|$$

Assume: $\|B_{k,h}^n\| < C(T)$ for all n, where C does not depend on k, h.

$$\|E^N\| \le C\|E^0\| + kNC\max_{0\le j\le N-1}\|\tau^j\| = C\|E^0\| + TC\max_j\|\tau_j\|$$

If our method is consistent: $\|E^0\| \to 0$ as $k, h \to 0$

$\|\tau_j\| \to 0$

So $\|E^N\| \to 0$ as $k, h \to 0$. (convergence)

This is known as Lax-Richtmeyer stability.

Let's prove stability: $\|(B_{k,h})^n\|_2 \leq C(T)$

FE: $B_{k,h} = I + kA$

$\boxed{\begin{array}{l}\text{Eigenvalues of } A: \\ \lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1) \\ p = 1, 2, \dots, m \\ h = \frac{1}{m+1}\end{array}}$

$\|B_{k,h}\|_2 = \max_p |1 + k\lambda_p|$

we need $|1 + k\lambda_p|^n < C(\underline{T})$

$\Rightarrow |1 + k\lambda_p| \leq 1$

$-\frac{4}{h^2} \leq \lambda_p \leq 0$

$1 - \frac{4k}{h^2} \leq 1 + k\lambda_p \leq 1$

$\Rightarrow 1 - \frac{4k}{h^2} \geq -1$

$-\frac{4k}{h^2} \geq -2$

$k \leq \frac{h^2}{2}$ (Absolute stability)

Proof of stability for Trapezoidal method:

$$U^{n+1} = \underbrace{\left(I - \tfrac{k}{2}A\right)^{-1}\left(I + \tfrac{k}{2}A\right)}_{B_{k,h}} U^n$$

$$\|B_{k,h}\| = \max_p \left| \frac{1 + \frac{k}{2}\lambda_p}{1 - \frac{k}{2}\lambda_p} \right| \qquad \lambda_p \le 0$$

$\|B_{k,h}\| \le 1$ for all k, h (A-stable)