Balance Laws

$$\underbrace{q_t + f(q)_x}_{\text{Hyperbolic system}} = \underbrace{\psi(q,x,t)}_{\text{source terms}}$$

Examples:

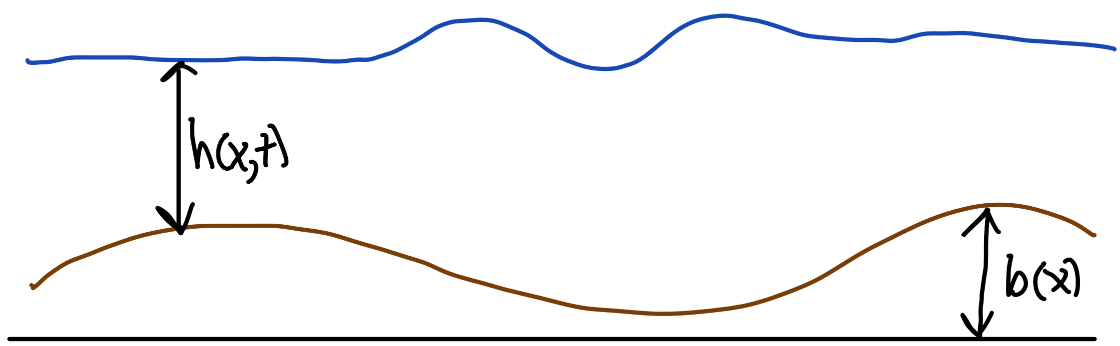
① Navier-Stokes (second-derivative terms)

② Atmospheric flow (Euler + Gravity)

③ Shallow water with bathymetry $b(x)$:

$$h_t + (hu)_x = 0$$

$$(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x = -ghb_x$$



④ Reactive flow (convection + chemistry)

etc.

Simple example:

$$q_t + a q_x = -\beta q \qquad \beta > 0$$

A simple first-order method:

$$Q_i^{n+1} = Q_i^n - \Delta t\left[a\,\frac{Q_i^n - Q_{i-1}^n}{\Delta x} + \beta Q_i^n\right]$$

Lax-Wendroff approach to 2nd-order

$$q_t = \underbrace{(-a\partial_x - \beta)}_{\mathcal{L}}\, q = \mathcal{L} q$$

$$q(x,t) = e^{t\mathcal{L}} q(x,0)$$

$$q(x,t+\Delta t) = e^{\Delta t\mathcal{L}} q(x,t)$$

$$\boxed{\partial_x^2 q = q_{xx}}$$

$$e^{\Delta t\mathcal{L}} = I - \Delta t(a\partial_x + \beta) + \frac{\Delta t^2}{2}(a\partial_x + \beta)^2 + \mathcal{O}(\Delta t^3)$$

$$= I - \Delta t(\underline{a\partial_x} + \beta) + \frac{\Delta t^2}{2}\left(a^2\partial_x^2 + \beta^2 + \underbrace{2a\beta\partial_x}\right) + \mathcal{O}(\Delta t^3)$$

Interaction of advection and reaction

So a 2nd-order discretization is

$$Q_i^{n+1} = Q_i^n - \Delta t\left[a\,\frac{Q_{i+1} - Q_{i-1}}{2\Delta x} + \beta Q_i^n\right] + \frac{\Delta t^2}{2}\left[a^2\,\frac{Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n}{(\Delta x)^2} + \beta^2 Q_i^n + 2a\beta\,\frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x}\right]$$

Operator Splitting

To solve (approximately)

$$q_t + f(q)_x = \psi(q,x,t)$$

we can alternate between solving

$$\left.\begin{array}{l} (1)\quad q_t + f(q)_x = 0 \\ \text{and} \\ (2)\quad q_t = \psi(q) \end{array}\right\}$$

over small time steps.

Lie-Trotter splitting
or
Godunov splitting

Even if we solve (1) and (2) exactly, we have a 1st-order "splitting error".

For our advection-diffusion problem:

$$\hat{Q}_i^n = Q_i^n - \frac{\Delta t}{\Delta x} a\left(Q_i^n - Q_{i-1}^n\right)$$

$$Q_i^{n+1} = \hat{Q}_i^n - \Delta t B \hat{Q}_i^n$$

could instead use

$$Q_i^{n+1} = e^{-\Delta t B}\hat{Q}_i^n$$

(exact)

Consider a linear problem

$$q_t = Aq + Bq$$

Lie-Trotter splitting gives: (with exact time evolution)

$$\hat{Q}^n = e^{\Delta t A} Q^n$$

$$Q^{n+1} = e^{\Delta t B}\hat{Q}^n = e^{\Delta t B}\, e^{\Delta t A} Q^n$$

i.e.

$$Q^{n+1} = \left(I + \Delta t B + \frac{\Delta t^2}{2} B^2 + \cdots\right)\left(I + \Delta t A + \frac{\Delta t^2}{2} A^2 + \cdots\right) Q^n$$

$$= \left(I + \Delta t (A+B) + \frac{\Delta t^2}{2}\left(A^2 + B^2 + 2BA\right) + \cdots\right) Q^n$$

If A, B do not commute then

$$e^{\Delta t (A+B)} = \left(I + \Delta t (A+B) + \frac{\Delta t^2}{2}\left(A^2 + B^2 + \underline{BA} + \underline{AB}\right) + \mathcal{O}(\Delta t^3)\right)$$

so in general, this method is 1st-order accurate.

<u>Strang splitting</u>

1. Solve $q_t = Aq$ with step $\frac{\Delta t}{2}$
2. Solve $q_t = Bq$ with step Δt
3. Solve $q_t = Aq$ with step $\frac{\Delta t}{2}$

This has a 2nd-order splitting error.

Method of lines

Just discretize in space and apply a standard time integrator.

$$q_t = F(q) + G(q)$$

Often F and G have different properties.
For instance:

F: non-stiff, nonlinear

G: stiff, linear

For instance:

$$q_t + \underbrace{\left(\tfrac{1}{2}q^2\right)_x}_{F} = \underbrace{q_{xx}}_{G}.$$

In this setting you may wish to use:

① ImEx methods: treat F explicitly and G implicitly.

② Exponential methods: directly compute and use $\exp(\Delta t G)$.

Well-balanced Discretizations

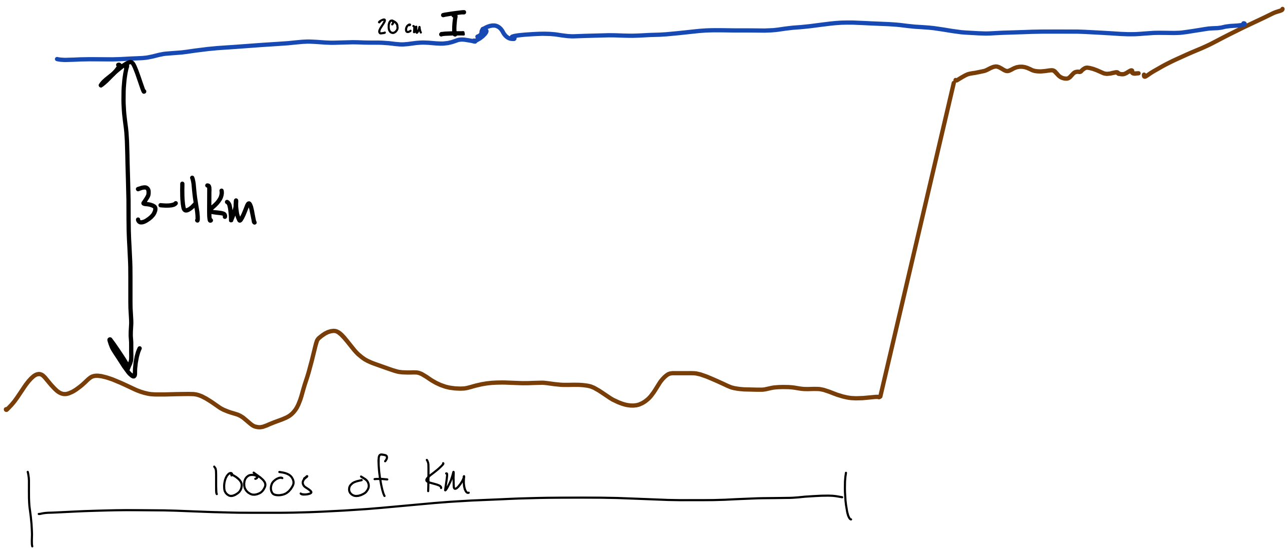
We often need to solve

$$q_t + f(q)_x = \psi(q)$$

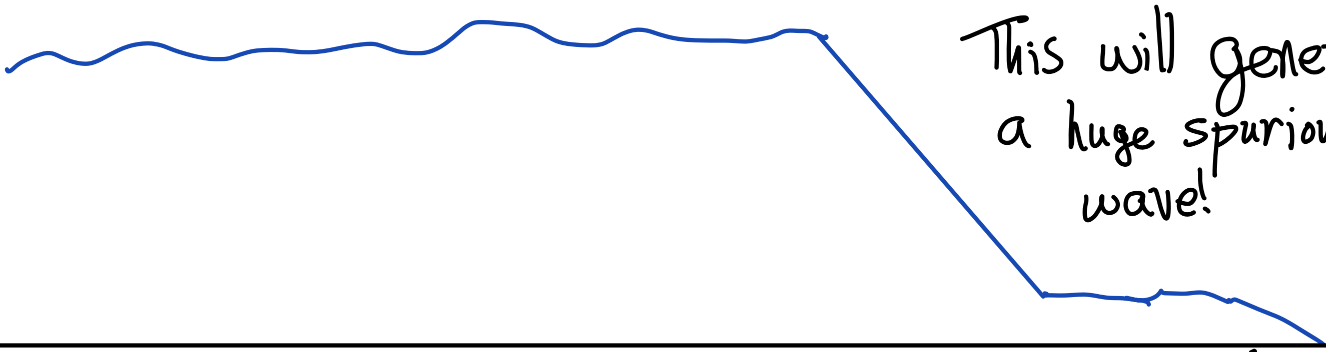
with

$$f(q)_x \approx \psi(q) \quad \text{(near equilibrium)}$$

For example shallow water (with non-flat bottom)



If we use a fractional step method:



This will generate a huge spurious wave!

We say a scheme is well-balanced if it exactly preserves some set of equilibrium solutions (steady states).

One approach to well-balancing (f-wave method)

Shallow water "lake at rest"

$$u(x,t)=0 \qquad \eta(x,t)=h+b=\text{constant}$$

$$h_t + (hu)_x = 0 \quad \Rightarrow \quad h_t = 0$$

$$(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x = -ghb_x \Rightarrow ghh_x = -ghb_x$$

$$\Rightarrow gh(h+b)_x = 0$$

- Approximate $b(x)$ by a constant in each cell
- In the Riemann solver, take the difference

$$\Delta f_{i-1/2} = f(Q_i) - f(Q_{i-1}) - \Psi_{i-1/2}$$

where $\Psi_{i-1/2} \approx -ghb_x\big|_{x=x_{i-1/2}}$

We will then decompose $\Delta f_{i-1/2}$:

$$A^-\Delta Q_{i-1/2} + A^+\Delta Q_{i-1/2} = \Delta f_{i-1/2}$$

How to choose $\Psi_{i-1/2}$?

We want $\Delta f_{i-1/2} = 0$ if $u_i = u_{i-1} = 0$ and

$$h_{i-1} + b_{i-1} = h_i + b_i \Rightarrow h_i - h_{i-1} = -(b_i - b_{i-1})$$

Taking $u_i = u_{i-1} = 0$, we have

$$f(Q_i) - f(Q_{i-1}) = \frac{1}{2} g (h_i^2 - h_{i-1}^2) = g \overbrace{\frac{h_i + h_{i-1}}{2}}^{\bar{h}} (h_i - h_{i-1})$$

$$= \underbrace{\boxed{-g\bar{h}(b_i - b_{i-1}) = \Psi_{i-1/2}}}_{\approx -ghb_x}$$

This choice produces $\Delta f_{i-1/2} = 0$,

so no waves will be generated.