Total Variation

For a piecewise-constant function:

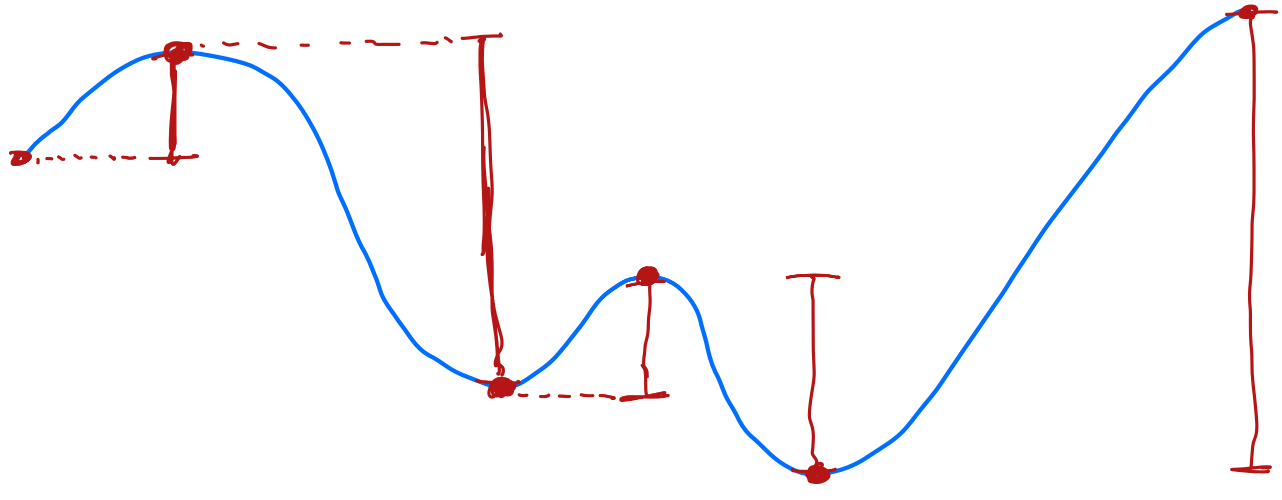
$$TV(Q) = \sum_i |Q_i - Q_{i-1}|$$

For a general function:

$$TV(q(x)) = \sup_{\{\xi_j\}} \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})|$$

For a differentiable function:

$$TV(q(x)) = \int |q'(x)|\,dx$$



The exact solution of

$$q_t + f(q)_x = 0 \qquad q(x,t) \in \mathbb{R}$$

satisfies

$$TV(q(x, t_0 + \Delta t)) \leq TV(q(x, t_0))$$

A numerical method is:

- Total-variation-diminishing (TVD) if

$$TV(Q^{n+1}) \le TV(Q^n)$$

- Monotonicity-preserving if

$$\underline{Q_i^n \ge Q_{i-1}^n \ \forall i} \Rightarrow \underline{Q_i^{n+1} \ge Q_{i-1}^{n+1} \ \forall i}$$

The exact solution of any scalar hyperbolic PDE is TVD and MP.

Godunov's Theorem

If a discretization of advection is linear

$$Q_i^{n+1} = \sum_j \alpha_j Q_{i+j}^n \qquad (1)$$

and MP, then it is at most 1st-order accurate.

Lemma: If (1) is MP, then $\alpha_j \geq 0$ $\forall j$.

Proof: Suppose $\alpha_p < 0$. Take

$$Q_i^n = \begin{cases} 1 & i < k \\ 0 & i \geq k \end{cases}$$

We have

$$Q_i^{n+1} - Q_{i-1}^{n+1} = \sum_j \alpha_j \big(\overbrace{Q_{i+j}^n - Q_{i+j-1}^n}^{=0 \text{ unless } i+j=k \Rightarrow j=k-i} \big)$$

$$= \alpha_{k-i} \left(Q_k^n - Q_{k-1}^n \right) = -\alpha_{k-i}$$

Take $k-i=p$. Then $Q_i^{n+1} - Q_{i-1}^{n+1} = -\alpha_p > 0$.

Now to prove Godunov's Theorem:
Suppose we have a method that is

MP, linear, and 2nd order for

$$q_t + a q_x = 0.$$

Let $q(x,0) = \left(\frac{x}{\Delta x} - \frac{1}{2}\right)^2 - \frac{1}{4}.$

Then $q(x,t) = \left(\frac{x - at}{\Delta x} - \frac{1}{2}\right)^2 - \frac{1}{4}$

$$Q_i^0 = q(x_i, t=0) = \left(i - \frac{1}{2}\right)^2 - \frac{1}{4} \geq 0 \quad \forall i \in \mathbb{Z}$$

Since the solution is quadratic, a 2nd-order method will solve it exactly:

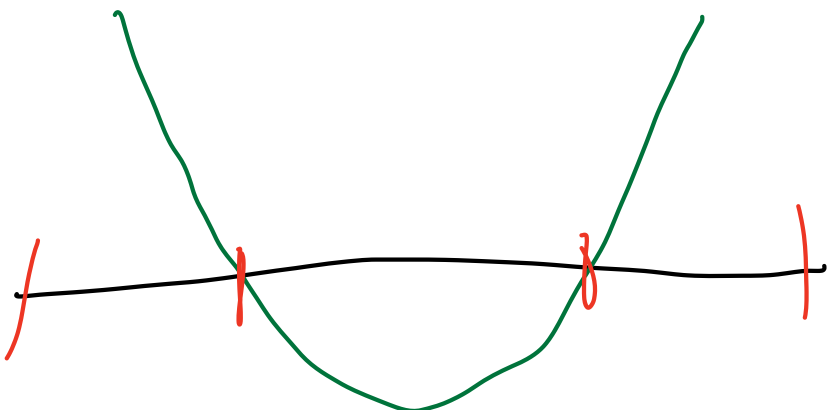
$$Q_i^1 = \left(i - \frac{a\Delta t}{\Delta x} - \frac{1}{2}\right)^2 - \frac{1}{4}$$

But

$$Q_i^1 = \sum_j \alpha_j Q^0_{j+i} = \sum_j \underset{\geq 0}{\alpha_j} \underbrace{\left[\left(j+i-\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right]}_{\substack{\geq 0 \text{ since} \\ i+j \in \mathbb{Z}}}$$

So $Q_i^1 \geq 0$. But we can choose i so that $\left(i - \frac{a\Delta t}{\Delta x} - \frac{1}{2}\right)^2 - \frac{1}{4} < 0$. Contradiction.

Essence of the proof:



All grid values are positive but exact solution goes negative.

Stronger version:

$$Q^{n+1} = LQ^n$$

$$Q_i^{n+1} = \sum_j l_{ij} Q_j^n \qquad (2)$$

Lemma: If (2) ~~is~~ MP, then $l_{ij} \geq 0 \quad \forall i,j.$

Proof: Take $Q_j^n = \begin{cases} 0 & j < J \\ 1 & j \geq J \end{cases}$

$$Q_J^{n+1} = \sum_j l_{Jj} Q_j^n = \sum_{j=J}^{m} l_{Jj}$$

$$Q_{J-1}^{n+1} = \sum_{j=J}^{m} l_{J-1,j} Q_j^n$$

$$Q_J^{n+1} - Q_{J-1}^{n+1} = \sum_{j=J}^{m} (l_{Jj} - l_{J-1,j})$$

Redux.

Dfn. We say a numerical scheme is monotone or MP if

$$Q_{i+1}^{n} \geq Q_i^n \ \forall i \Rightarrow Q_{i+1}^{n+1} \geq Q_i^{n+1} \ \forall i.$$

Lemma. Consider a discretization

$$Q_i^{n+1} = \sum_{j=-r}^{r} \alpha_j Q_{i+j}^{n} \qquad (1)$$

If the scheme is MP, then $\alpha_j \geq 0 \ \forall j$.

Proof: Let $Q_i^n = \begin{cases} 0 & i < k \\ 1 & i \geq k \end{cases}$

Then

$$Q_i^{n+1} - Q_{i-1}^{n+1} = \sum_{j=-r}^{r} \alpha_j \underbrace{\left(Q_{i+j}^{n} - Q_{i+j-1}^{n}\right)}_{= \begin{cases} 0 & i+j \neq k \\ 1 & i+j = k \end{cases}}$$

$$= \alpha_{k-i}$$

Suppose $\alpha_p < 0$. Let $i = k-p$.

Then $Q_{k-p}^{n+1} - Q_{k-p-1}^{n+1} = \alpha_p < 0$ so $Q_{k-p}^{n+1} < Q_{k-p-1}^{n+1}$.

Thm. Let a MP scheme (1) be given for the solution of

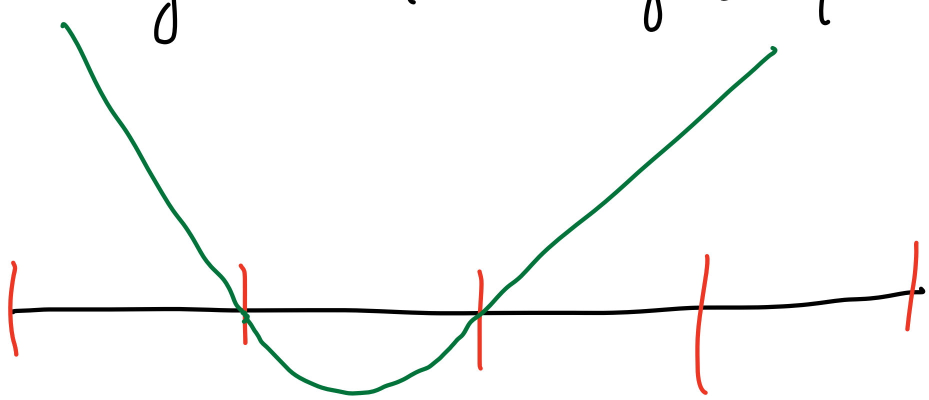
$$q_t + a q_x = 0.$$

Then either:

(i) $\frac{a\Delta t}{\Delta x} \in \mathbb{Z}$

or (ii) The scheme is at most 1st-order accurate.

Proof. Take Q_i^n to be the point values of a quadratic function whose minimum is negative but which is non-negative at all grid points:



If the method is 2nd-order accurate, it is exact for this problem.

If $\frac{a\Delta t}{\Delta x}$ is not an integer, then some solution value $Q_i^{n+1} < 0$. But this contradicts our Lemma.

Total Variation

$$TV(q(x)) = \sup_{\{\xi\}} \sum_{j=1}^{N} |q(\xi_j) - q(\xi_{j-1})|$$

The supremum is taken over all sets

$$-\infty = \xi_0 < \xi_1 < \cdots < \xi_N = \infty.$$

Differentiable case:

$$TV(q) = \int_{-\infty}^{\infty} |q'(x)|\,dx$$

For a grid function:

$$TV(Q) = \sum_{i=-\infty}^{\infty} |Q_i - Q_{i-1}|$$

For the advection equation:

$$q_t + a q_x = 0$$

$$q(x,0) = \mathring{q}(x)$$

we have $TV(q(x,t)) = TV(\mathring{q}(x))$.

In general, for the scalar conservation law $q_t + f(q)_x = 0$

we have $TV(q(x,t)) \leq TV(q(x))$.