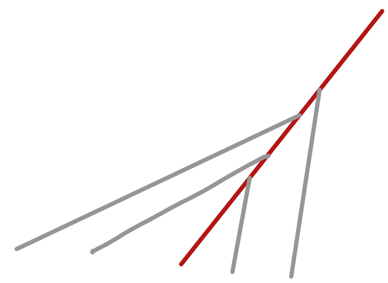
Lax entropy condition for systems

for a p-shock:

$$\lambda^p(q_l) > s > \lambda^p(q_r)$$



Consider a nonlinear hyperbolic system:

$$q_t + f(q)_x = 0 \qquad q(x,t) \in \mathbb{R}^m$$

We can apply Godunov's method:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right) \qquad (1)$$

where $\mathcal{F}_{i-1/2}(Q_{i-1}^n, Q_i^n)$ is the flux from the solution of the Riemann problem

$$q(x,t=0) = \begin{cases} Q_{i-1} & x < x_{i-1/2} \\ Q_i & x > x_{i-1/2} \end{cases}$$

We rewrote (1) in fluctuation form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) \qquad (2)$$

where $\mathcal{A}^{\pm}\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^{\pm} \mathcal{W}_{i-1/2}^p \qquad \mathcal{W}_{i-1/2}^p = \alpha_{i-1/2}^p r_{i-1/2}^p$

For the linear hyp. system:

$$q_t + A q_x = 0$$

we used

$$\Delta Q_{i-1/2} = Q_i^n - Q_{i-1}^n = \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^p$$

and $\quad A(Q_i - Q_{i-1}) = \sum_{p=1}^{m} \lambda^p \mathcal{W}_{i-1/2}^p = f(Q_i) - f(Q_{i-1})$

We used the exact solution of the R.P.

Now we will use approximate Riemann solvers.

Linearized solvers

We can linearize at each interface:

$$q_t + f(q)_x = 0 \Rightarrow \underline{q_t + f'(q) q_x = 0}$$

$$\Rightarrow q_t + \underbrace{\hat{A}_{i-1/2}(q_\ell, q_r)}_{\text{linearized flux Jacobian}} q_x = 0$$

Once $\hat{A}$ is chosen, we can just apply methods for linear hyp. systems.

We will refer to the eigenvalues of $\hat{A}_{i-1/2}$ as $s^p_{i-1/2}$, and the eigenvectors as $r^p_{i-1/2}$.

Roe solvers (Philip Roe, 1981)

What properties should $\hat{A}$ have?

① Consistency: $\hat{A}(q_\ell, q_r) \to f'(q)$ as $q_\ell, q_r \to q$

② Hyperbolicity: $\hat{A}$ diagonalizable w/real eigenvalues

③ Conservation: $\hat{A}(q_\ell, q_r)(q_r - q_\ell) = f(q_r) - f(q_\ell)$

Property ③ turns out to imply that the solver is exact when the Riemann solution consists of a single shock.

It is natural to take

$$\hat{A} = f'(\hat{q}) \text{ with } \hat{q} \text{ depending on } q_l, q_r.$$

Shallow water Roe solver

SW eqns. in quasilinear form

$$q_t + f'(\hat{q}) q_x = 0$$

$$h_t + (hu)_x = 0$$

$$(hu)_t + (g\hat{h} - \hat{u}^2) h_x + 2\hat{u}(hu)_x = 0$$

$$f'(\hat{q}) = \begin{bmatrix} 0 & 1 \\ g\hat{h} - \hat{u}^2 & 2\hat{u} \end{bmatrix}$$

Eigenvalues: $s^1 = \hat{u} - \sqrt{g\hat{h}}$ $\quad s^2 = \hat{u} + \sqrt{g\hat{h}}$

Eigenvectors: $r^1 = \begin{bmatrix} 1 \\ \hat{u} - \sqrt{g\hat{h}} \end{bmatrix}$ $\quad r^2 = \begin{bmatrix} 1 \\ \hat{u} + \sqrt{g\hat{h}} \end{bmatrix}$

We need to solve

$$\hat{A}(q_r - q_l) = \boxed{f'(\hat{q})(q_r - q_l) = f(q_r) - f(q_l)} \qquad (3)$$

i.e. $\quad q_r - q_l = \alpha^1 r^1 + \alpha^2 r^2$

Writing out (3): $h_r u_r - h_\ell u_\ell = h_r u_r - h_\ell u_\ell$ ✓

second component:

$$(g\hat{h} - \hat{u}^2)(h_r - h_\ell) + 2\hat{u}(h_r u_r - h_\ell u_\ell) = h_r u_r^2 - h_\ell u_\ell^2 + \tfrac{1}{2} g (h_r^2 - h_\ell^2)$$

Equate terms with and w/o g:

$$\hat{h}(h_r - h_\ell) = \tfrac{1}{2}(h_r^2 - h_\ell^2) \Rightarrow \hat{h} = \frac{h_r + h_\ell}{2}$$

$$\hat{u}^2 (h_r - h_\ell) - 2\hat{u}(h_r u_r - h_\ell u_\ell) + h_r u_r^2 - h_\ell u_\ell^2 = 0$$

$$\hat{u} = \frac{\cancel{2}(h_r u_r - h_\ell u_\ell)}{\cancel{2}(h_r - h_\ell)} \pm \frac{\sqrt{\cancel{4}(h_r u_r - h_\ell u_\ell)^2 - \cancel{4}(h_r - h_\ell)(h_r u_r^2 - h_\ell u_\ell^2)}}{2(h_r - h_\ell)}$$

The radical:

$$\cancel{h_r^2 u_r^2} + \cancel{h_\ell^2 u_\ell^2} - 2h_r u_r h_\ell u_\ell - \cancel{h_r^2 u_r^2} + h_r h_\ell u_\ell^2 + h_\ell h_r u_r^2 - \cancel{h_\ell^2 u_\ell^2}$$

$$= h_\ell h_r (u_r - u_\ell)^2$$

$$\hat{u} = \frac{h_r u_r - h_\ell u_\ell \pm (u_r - u_\ell)\sqrt{h_\ell h_r}}{h_r - h_\ell}$$

$$\Rightarrow \hat{u} = \frac{u_r\sqrt{h_r} \pm u_\ell \sqrt{h_\ell}}{\sqrt{h_r} \pm \sqrt{h_\ell}}$$ Roe takes "+".

This is called the "Roe average".

So we take $\hat{A}_{i-1/2} = f'\left(\begin{bmatrix}\hat{h}\\ \hat{h}\hat{u}\end{bmatrix}\right)$ where

$$\hat{h} = \frac{h_r + h_l}{2}$$

$$\hat{u} = \frac{u_r\sqrt{h_r} \pm u_l\sqrt{h_l}}{\sqrt{h_r} \pm \sqrt{h_l}}$$

We solve the resulting (linear) Riemann problem at each interface to determine $\mathcal{W}^p_{i-1/2}$, $\alpha^p_{i-1/2}$.

We can use these to implement Godunov's method or the Lax-Wendroff-LeVeque method.

Potential problems:

- We approximate rarefactions by discontinuities $\to$ entropy-violating shocks (solution: "entropy fix")
- The (approx.) Rieman solution can have $h_m < 0$.

Homework:

① 13.8 FVMHP

② Devise a Roe solver for the isothermal equations (2.38) in FVMHP.