- Solution of the Riemann problem
- Centered rarefaction wave
- Lax entropy condition
- Vanishing viscosity solutions
- Weak solution
- Lax-Wendroff Thm.

$u = 1 - \rho$

LWR model

$$\rho_t + (\rho - \rho^2)_x = 0$$

$$c(\rho) = f'(\rho) = 1 - 2\rho$$

We really want to find the solution of

$$\rho_t + f(\rho)_x = \varepsilon \rho_{xx}$$

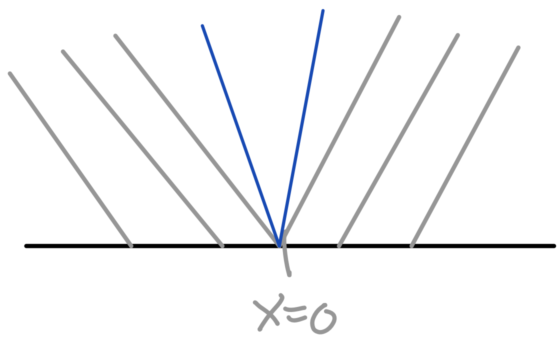
in the limit $\varepsilon \to 0$.

ρ_ε

$$\lim_{\varepsilon \to 0} \rho_\varepsilon = \rho \quad \text{a.e.}$$

Green light problem

$$\rho_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$



Entropy-violating weak solution: $\rho(x,t) = \rho_0(x)$

$f'(\rho_l) = -1 \qquad f'(\rho_r) = 1$

Lax Entropy Condition

In the vicinity of a shock, characteristics must be converging, and impinging on the shock

$$f'(\rho_l) > s > f'(\rho_r)$$

Here $\quad s = \dfrac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l}$.

We say a weak solution is **entropy-satisfying** if each shock satisfies the entropy condition.

Centered rarefaction waves

Ansatz: $\rho(x,t) = \tilde{\rho}\left(\frac{x}{t}\right) = \tilde{\rho}(\xi)$ (similarity solution)

$$\rho_t = \tilde{\rho}'(\xi)\,\xi_t = -\frac{x}{t^2}\tilde{\rho}'(\xi)$$

$$f(\rho)_x = f'(\tilde{\rho})\tilde{\rho}'(\xi)\,\xi_x = \frac{1}{t}f'(\tilde{\rho})\tilde{\rho}'(\xi)$$

$$\rho_t + f(\rho)_x = 0 \Rightarrow -\frac{x}{t^2}\tilde{\rho}'(\xi) + \frac{1}{t}f'(\tilde{\rho})\tilde{\rho}'(\xi) = 0$$

If $\tilde{\rho}'(\xi) \neq 0$: $f'(\tilde{\rho}) = \frac{x}{t}$

For LWR: $1 - 2\tilde{\rho}(\xi) = \xi \Rightarrow \tilde{\rho} = \frac{1-\xi}{2} = \frac{1-\frac{x}{t}}{2}$

Green light solution:

$$\rho(x,t) = \begin{cases} \rho_l & x < f'(\rho_l)\,t \quad \left(\frac{x}{t} < f'(\rho_l)\right) \\ \frac{1-\frac{x}{t}}{2} & f'(\rho_l) < \frac{x}{t} < f'(\rho_r) \\ \rho_r & \frac{x}{t} > f'(\rho_r) \end{cases}$$

General solution of the Riemann problem

$$\rho_t + f(\rho)_x = 0$$

$$\rho_0(x) = \begin{cases} \rho_\ell & x<0 \\ \rho_r & x>0 \end{cases}$$

solution is either

shock

$(\rho_\ell < \rho_r)$

$$\rho(x,t) = \begin{cases} \rho_\ell & x<st \\ \rho_r & x>st \end{cases}$$

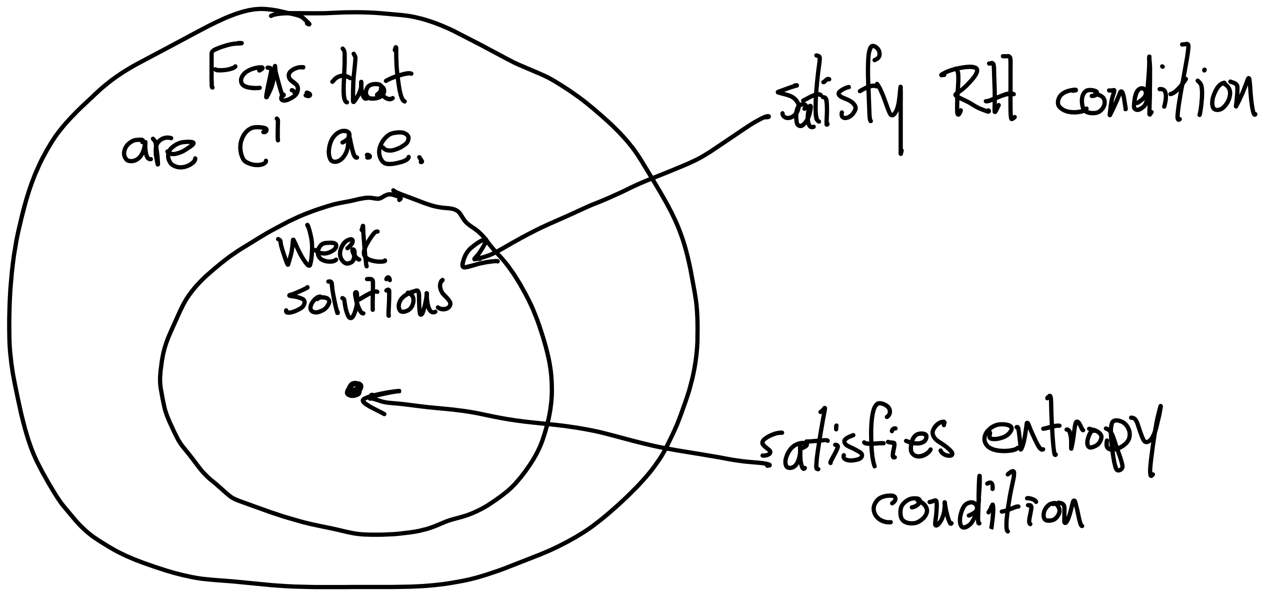
$$s = \frac{f(\rho_r)-f(\rho_\ell)}{\rho_r - \rho_\ell}$$

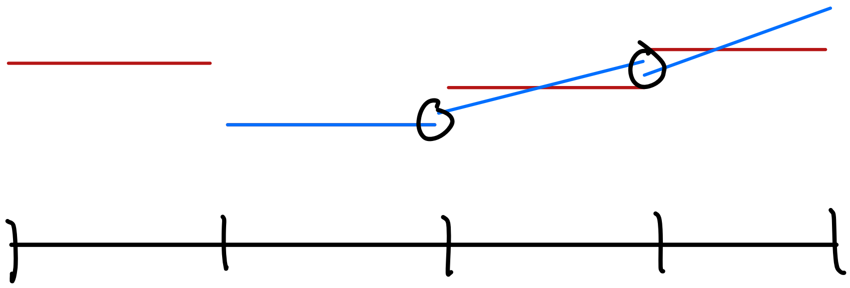
centered rarefaction

$(\rho_\ell > \rho_r)$

$$\rho(x,t) = \begin{cases} \rho_\ell & x < f'(\rho_\ell)\, t \\ \frac{1-\frac{x}{t}}{2} & f'(\rho_\ell) < \frac{x}{t} < f'(\rho_r) \\ \rho_r & \frac{x}{t} > f'(\rho_r) \end{cases}$$

Solutions of a HCL





$$TV(\hat{q}) = \sum_i \Delta x \, |\sigma_i| + \sum_i \left| q^-_{i-1/2} - q^+_{i-1/2} \right|$$