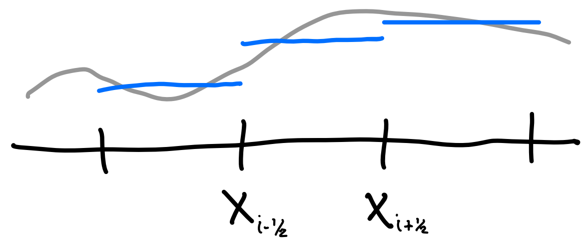
Godunov's method:

① Approximate $q(x,t_n)$ by cell averages:

$$Q_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)\,dx$$



② Solve Riemann problem at each interface to determine numerical flux

$$F_{i-1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}),t)\,dt = f(\check{q}(x_{i-1/2}))$$

③ Evolve to t_{n+1}.

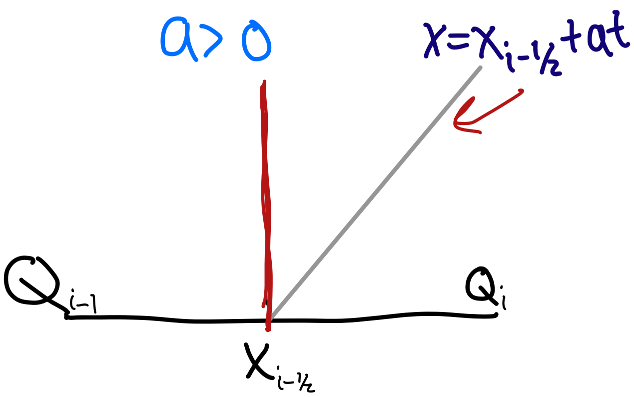
Let's work through it for the advection equation:

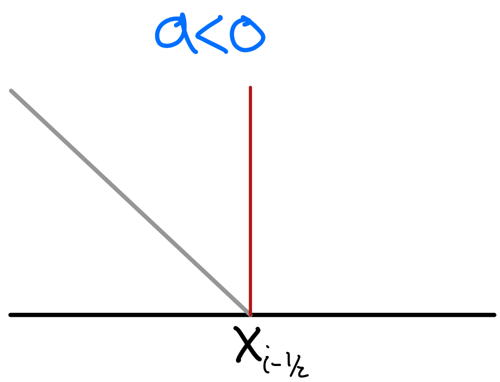
$$q_t + a q_x = 0$$

Riemann problem:

$$q(x,t_n) = \begin{cases} Q_{i-1} & x < x_{i-1/2} \\ Q_i & x > x_{i-1/2} \end{cases}$$

Solution:





So $q^{\downarrow}_{i-1/2} = \begin{cases} Q_{i-1} & \text{if } a>0 \\ Q_i & \text{if } a<0. \end{cases}$

We get $Q_i^{n+1} = Q_i^n - \underbrace{\frac{a\Delta t}{\Delta x}}_{\text{CFL \# } \nu}\left(q^{\downarrow}_{i+1/2} - q^{\downarrow}_{i-1/2}\right)$

$\nu = 1: \; Q_i^{n+1} = Q_{i-1}^n$

For $a>0$:

$$Q_i^{n+1} = Q_i^n - \nu\left(Q_i^n - Q_{i-1}^n\right) = (1-\nu)Q_i^n + \nu Q_{i-1}^n.$$

This is called the <u>upwind method</u>.

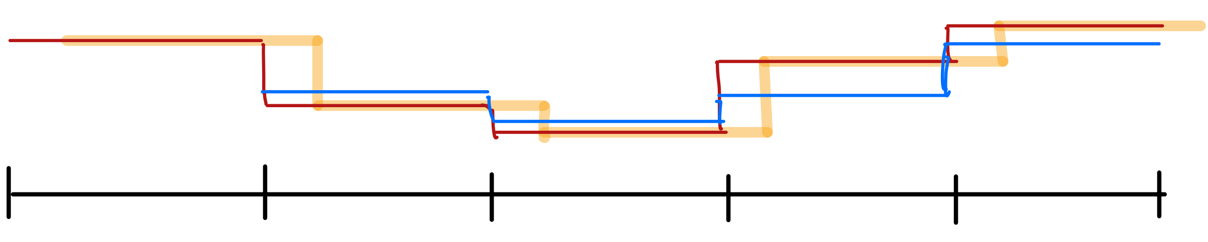
What does the CFL condition require?

$$0 \le \nu \le 1$$

- We could use modified equation analysis to show that this method is stable (but dissipative). I.e. it approximately satisfies

$$q_t + a q_x = \varepsilon q_{xx}$$

- It's also equivalent to the piecewise-constant solution shifted and re-averaged.

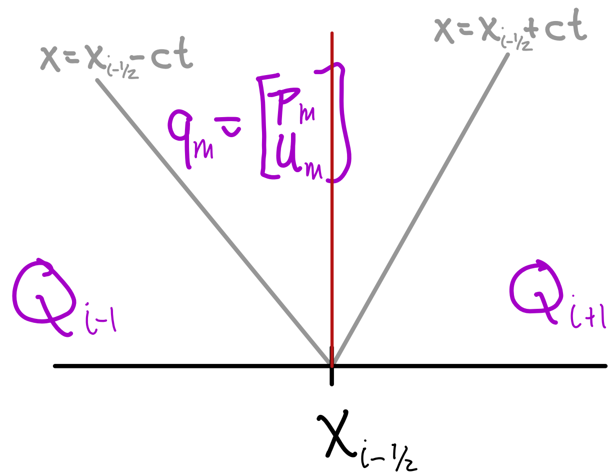


Godunov's method for acoustics

$q_t + A q_x = 0$

Riemann problem at $x_{i-1/2}$:

$$\begin{bmatrix} p \\ u \end{bmatrix} = \begin{cases} \begin{bmatrix} p_{i-1} \\ u_{i-1} \end{bmatrix} & x < x_{i-1/2} \\ \begin{bmatrix} p_i \\ u_i \end{bmatrix} & x > x_{i-1/2} \end{cases}$$



$q^{\vee}_{i-1/2} = q_m$

Define: $q_m - Q_{i-1} = \mathcal{W}^1_{i-1/2} = \alpha^1_{i-1/2} r^1$

$Q_{i+1} - q_m = \mathcal{W}^2_{i-1/2} = \alpha^2_{i-1/2} r^2$

$\lambda^1 = -c \quad \lambda^2 = +c$

$A r^1 = -c r^1 \quad A r^2 = +c r^2$

Flux:

$\quad\quad // \; A q_m$

$$F^n_{i-1/2} = A q^{\vee}_{i-1/2} = A Q_{i-1} - \alpha^1_{i-1/2} c r^1 = A Q_{i-1} + \lambda^1 \mathcal{W}^1_{i-1/2}$$

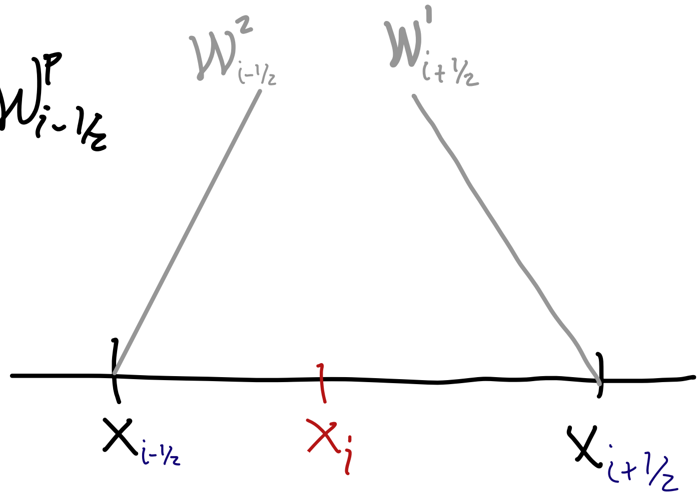
$$= A Q_i - \alpha^2_{i-1/2} c r^2 = A Q_i - \lambda^2 \mathcal{W}^2_{i-1/2}$$

Define: $(x)^+ = \max(x, 0)$ $\quad \lambda^1 = -c$

$(x)^- = \min(x, 0)$ $\quad \lambda^2 = +c$

Then $$F^n_{i-1/2} = AQ_{i-1} + \sum_{p=1}^{2} (\lambda^p)^- \mathcal{W}^p_{i-1/2}$$

$$= AQ_i - \sum_{p=1}^{2} (\lambda^p)^+ \mathcal{W}^p_{i-1/2}$$



$$F^n_{i+1/2} =$$

CFL condition: $\max_p |\lambda^p| \frac{\Delta t}{\Delta x} \leq 1$

We get

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$ ← Conservative

$$= Q_i^n - \frac{\Delta t}{\Delta x}\left(\cancel{AQ_i} + \sum_p (\lambda^p)^- \mathcal{W}_{i+1/2}^p - (\cancel{AQ_i} - \sum_p (\lambda^p)^+ \mathcal{W}_{i-1/2}^p\right)$$

$$= Q_i^n - \frac{\Delta t}{\Delta x}\left(\underbrace{\sum_p (\lambda^p)^- \mathcal{W}_{i+1/2}^p}_{A^-\Delta Q_{i+1/2}} + \underbrace{\sum_p (\lambda^p)^+ \mathcal{W}_{i-1/2}^p}_{A^+\Delta Q_{i-1/2}}\right)$$

"Fluctuations"

Explanation of notation:

$A = R\Lambda R^{-1}$ $\quad R = \left[r^1 | r^2 \right]$ $\quad \Lambda = \begin{bmatrix} -c & \\ & +c \end{bmatrix}$

$\Lambda^- = \begin{bmatrix} -c & \\ & 0 \end{bmatrix}$ $\quad \Lambda^+ = \begin{bmatrix} 0 & \\ & +c \end{bmatrix}$ $\quad A^\pm = R\Lambda^\pm R^{-1}$

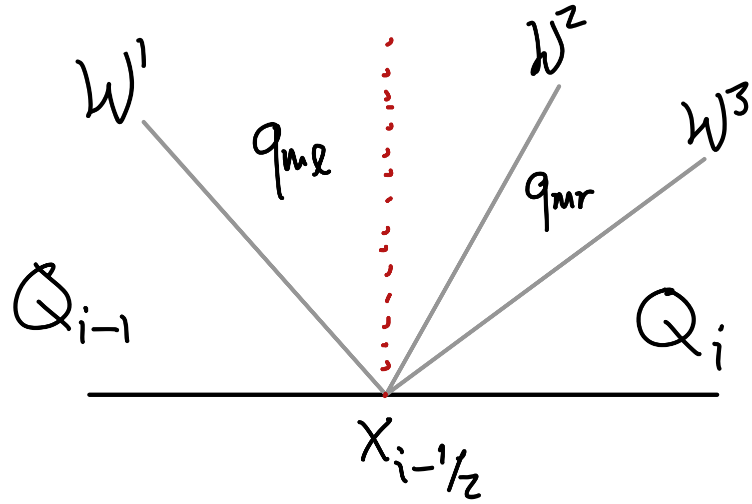
Then

$$A^+(Q_i - Q_{i-1}) = \sum_p (\lambda^p)^+ \mathcal{W}_{i-1/2}^p = A^+\Delta Q_{i-1/2}$$

$$A^-(Q_i - Q_{i-1}) = \sum_p (\lambda^p)^- \mathcal{W}_{i-1/2}^p = A^-\Delta Q_{i-1/2}$$

$$\boxed{Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right)}$$

Homework: 4.2 of FVMHP



$q^{\Downarrow} = q_{ml}$

$F(q^{\Downarrow}) = f(q_{ml})$

$$A^-\Delta Q_{i-1/2} = \sum_{p=1}^{3} (\lambda^p)^- W^p_{i-1/2} = \lambda^1 W^1_{i-1/2}$$

$$A^+\Delta Q_{i-1/2} = \sum_{p=1}^{3} (\lambda^p)^+ W^p_{i-1/2} = \lambda^2 W^2_{i-1/2} + \lambda^3 W^3_{i-1/2}$$