Burgers

$q_t + \frac{1}{2}(q^2)_x = 0$

Riemann solution is a shock if

$q_l > q_r$

since $f'(q) = q$

Traffic

$\rho_t + (\rho - \rho^2)_x = 0$

Riemann solution is a shock if

$\rho_l < \rho_r$

since $f'(\rho) = 1 - 2\rho$

Entropy function

We say $\eta(q)$ is an entropy fcn. for

$$q_t + f(q)_x = 0$$

if:

- $\int \eta \, dx$ is conserved for strong solutions
- $\int \eta \, dx$ is decreasing in the presence of shocks

For a scalar cons. law $q_t + f(q)_x = 0$, $\eta(q) = q^2$ is an entropy.

Consider the viscously-regularized C.L.

$$\rho_t + (\rho - \rho^2)_x = \varepsilon \rho_{xx}$$

$$2\rho\rho_t + 2\rho(1-2\rho)\rho_x = 2\varepsilon\rho\rho_{xx}$$

$$(\rho^2)_t + (\rho^2)_x - \frac{4}{3}(\rho^3)_x = 2\varepsilon\left[(\rho\rho_x)_x - (\rho_x)^2\right]$$

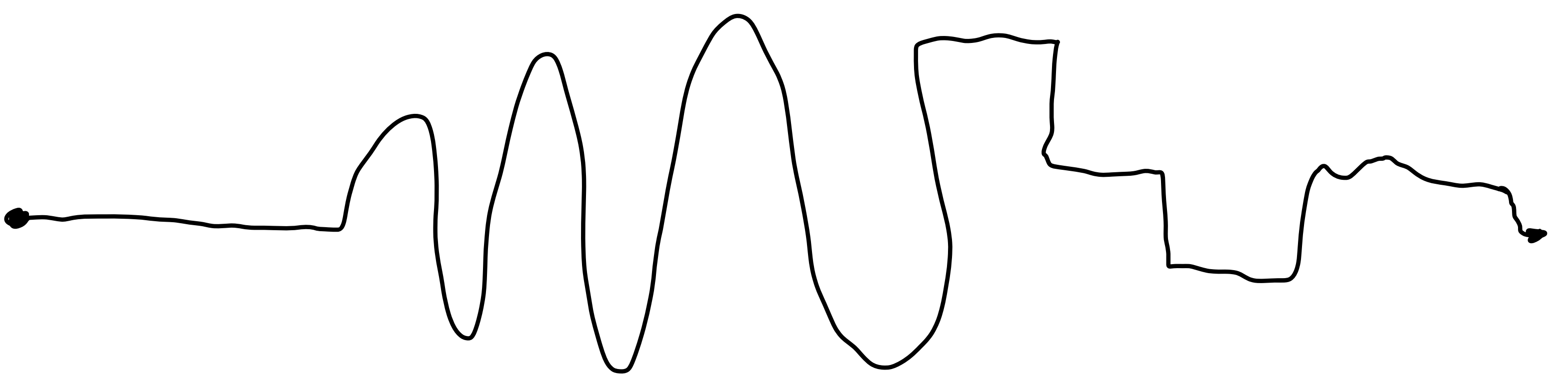
Assume $\lim\limits_{x\to\infty} \rho = \lim\limits_{x\to-\infty} \rho$ and $\lim\limits_{x\to\infty} \rho_x = \lim\limits_{x\to-\infty} \rho_x$

$$\frac{d}{dt}\int_{-\infty}^{\infty} \rho^2 dx + \cancel{\rho^2\Big|_{-\infty}^{\infty}} - \frac{4}{3}\cancel{\rho^3\Big|_{-\infty}^{\infty}} = 2\varepsilon\left[\cancel{\rho\rho_x\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty}(\rho_x)^2 dx\right]$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} \rho^2 dx = -2\varepsilon\int_{-\infty}^{\infty}(\rho_x)^2 dx$$

If ρ is smooth, then as $\varepsilon \to 0$ the RHS vanishes, so $\int \eta(\rho) = \int \rho^2$ is conserved.

If ρ is discontinuous, $\rho_x \to \delta$ so the RHS will be negative.



Godunov flux for LWR

$$\rho(x,0)=\begin{cases}\rho_l & x<0\\ \rho_r & x>0\end{cases}$$

What is $f(\rho(x=0,t>0))$
$=f^{\downarrow}$?

If $\rho_r>\rho_l$:

$$\rho(x,t)=\begin{cases}\rho_l & x<st\\ \rho_r & x>st\end{cases}$$

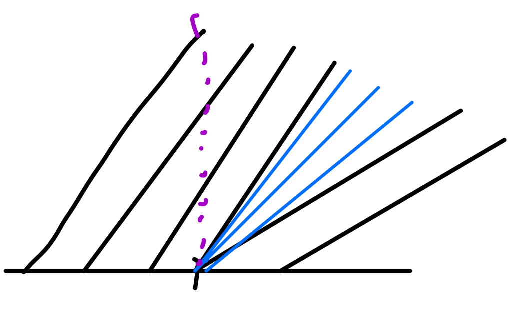
$$s=\frac{f(\rho_r)-f(\rho_l)}{\rho_r-\rho_l}$$

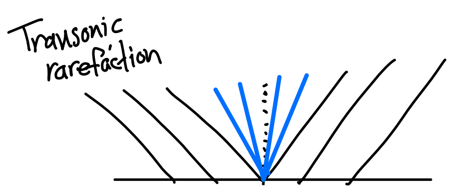
so $f^{\downarrow}=\begin{cases}\rho_l & \text{if } s>0\\ \rho_r & \text{if } s<0\end{cases}$

If $\rho_l>\rho_r$:

$$\rho(x,t)=\begin{cases}\rho_l & x<f'(\rho_l)t\\ \frac{1-\frac{x}{t}}{2} & f'(\rho_l)t<x<f'(\rho_r)t\\ \rho_r & x>f'(\rho_r)t\end{cases}$$

so $f^{\downarrow}=\begin{cases}\rho_l & \text{if } f'(\rho_l)>0\\ \rho_r & \text{if } f'(\rho_r)<0\\ \frac{1}{2} & \text{otherwise}\end{cases}$





Dfn. of weak solution

Let $\phi(x,t)$ be continuously diff'ble with compact support.

Writing

$$\int_0^\infty \int_{-\infty}^\infty (q_t + f(q)_x)\,\phi \, dx\,dt$$

and integrating by parts:

$$(*) \quad \int_0^\infty \int_{-\infty}^\infty \left[q\phi_t + f(q)\,\phi_x \right] dx\,dt = -\int_{-\infty}^\infty q(x,0)\,\phi(x,0)\,dx$$

We say $q(x,t)$ is a weak solution if $(*)$ is satisfied for all such $\phi(x,t)$.

Lax-Wendroff

If we apply a consistent and conservative numerical method, and if a sequence of numerical solutions converges to $q(x,t)$ as $\Delta x, \Delta t \to 0$, then q is a weak solution.