Lax Entropy Condition (for systems)

For a shock in the pth char. family, we must

$$\lambda^p(q_\ell) > s^p > \lambda^p(q_r)$$

For SW eqns: (1-shock)

$$u_\ell - \sqrt{gh_\ell} > u_\ell - \sqrt{gh_r \frac{h_\ell + h_r}{2h_\ell}} = u_r - \sqrt{gh_\ell \frac{h_\ell + h_r}{2h_r}} > u_r - \sqrt{gh_r}$$

$$-\sqrt{gh_\ell} > -\sqrt{gh_r \frac{h_\ell + h_r}{2h_\ell}} \Rightarrow h_\ell < \frac{h_r}{h_\ell} \frac{h_r + h_\ell}{2}$$

$$\frac{h_r}{h_\ell} \cdot \frac{h_r + h_\ell}{2h_\ell} > 1 \iff \boxed{h_\ell < h_r}$$

Admissibility for a 1-shock

For 2-shock: $h_r > h_\ell$

Simple waves

Ansatz $q(x,t) = \tilde{q}(\xi(x,t))$ $\qquad \xi : \mathbb{R}^2 \to \mathbb{R}$

$$\Rightarrow \xi_t \tilde{q}'(\xi) + \xi_x f'(q)\tilde{q}'(\xi) = 0$$

Suppose also that $\underline{\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q})}$ $\qquad f'(q) r^p(\tilde{q}) = \lambda^p r^p$

$$\xi_t \cancel{\alpha(\xi)} r^p(\tilde{q}) + \xi_x \cancel{\alpha(\xi)} \lambda^p(\tilde{q}) r^p(\tilde{q}) \qquad \alpha(\xi) \neq 0$$

$$\left(\xi_t + \xi_x \lambda^p(\tilde{q})\right) r^p(\tilde{q}) = 0 \qquad r^p(\tilde{q}) \neq 0 \qquad \lambda^p(\tilde{q}(\xi))$$

$$\xi_t + \xi_x \lambda^p(\tilde{q}) = 0$$ ← scalar hyp. PDE.

Behaves like a scalar problem.

$$\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q})$$

$\alpha = 1$

For SWE: $h'(\xi) = \alpha(\xi)(1) \Rightarrow h = \xi$

$$q^{2\prime}(\xi) = u(\xi) \mp \sqrt{g\xi}$$

Given a state (h_*, u_*), at every ξ, $\tilde{q}(\xi)$ must satisfy

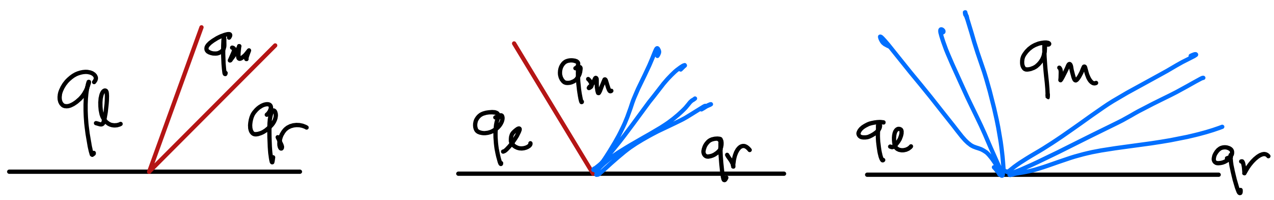
$$\underline{\tilde{u} \pm 2\sqrt{g\tilde{h}} = u_* \pm 2\sqrt{g h_*}}$$

Centered Rarefactions

Consider the Riemann problem

$$q(x,t=0) = \begin{cases} q_\ell & x<0 \\ q_r & x>0 \end{cases}$$

We know the solution depends only on x/t.
The solution will consist of 3 constant states q_ℓ, q_m, q_r separated by 2 waves, each of which is a shock or rarefaction.



We need to find how q varies inside a rarefaction. These rarefactions are simple waves.

$q(x,t) = \tilde{q}(x/t)$ Take $p=1$

$$\frac{x}{t} = \underline{\xi} = \lambda^1(\tilde{q}(\xi)) = \underline{u - \sqrt{gh}} \Rightarrow \boxed{h = \frac{(u-\xi)^2}{g}}$$

$$\boxed{\underbrace{u_l + 2\sqrt{gh_l}}_{w_l'} = u + 2\sqrt{gh}}$$

$$w_l' = u + 2(u - \xi) = 3u - 2\xi \Rightarrow u = \frac{w_l' - 2\xi}{3}$$

$$\Rightarrow h = \frac{(w_l' - \xi)^2}{9g}$$

Homework: 11.8, 13.7