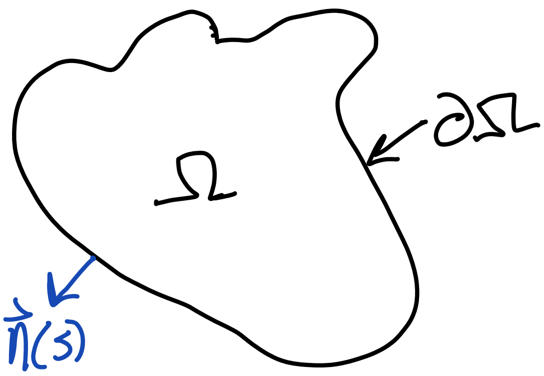
Multidimensional Conservation Laws

$q(x,y,t)$



$\begin{bmatrix} n^x \\ n^y \end{bmatrix} = \vec{n}(s)$ outward unit normal

Total mass: $\iint_\Omega q(x,y,t)\,dx\,dy$ $\qquad \vec{f}(q) = \begin{bmatrix} f(q) \\ g(q) \end{bmatrix}$

Rate of change: $\frac{d}{dt}\iint q\,dx\,dy = -\int_{\partial\Omega} \vec{n}(s)\cdot\vec{f}(q(x(s),y(s),t))\,ds$

$$\vec{n}(s)\cdot\vec{f}(q) = n^x f(q) + n^y g(q)$$

With enough smoothness: (and using div. thm.)

$$\iint q_t\,dx\,dy = -\iint_\Omega \nabla\cdot\vec{f}(q(x,y,t))\,dx\,dy$$

$\iint_\Omega (q_t + \nabla\cdot\vec{f}(q))\,dx\,dy = 0 \Rightarrow$ integrand must vanish pointwise.

$q_t + \nabla\cdot\vec{f}(q) = 0$ or $q_t + f(q)_x + g(q)_y = 0.$

Defn. of hyperbolicity

Linear 2D cons. law: $q_t + Aq_x + Bq_y = 0$ (1)

$$A = R_A \Lambda_A R_A^{-1} \quad B = R_B \Lambda_B R_B^{-1}$$

A and B must be diagonalizable with real eigenvalues.
Is this enough?

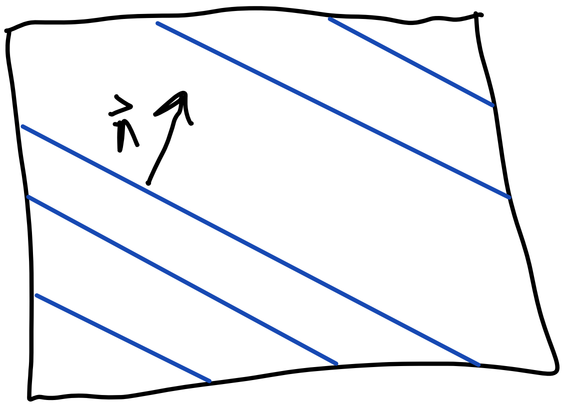
$$q_t + R_A \Lambda_A R_A^{-1} q_x + R_B \Lambda_B R_B^{-1} q_y = 0$$

We can't simultaneously diagonalize A and B.
(in general)

Suppose we choose plane-wave initial data:

$$q(x,y,0) = \mathring{q}(\vec{n}\cdot\vec{x}) \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \mathring{q}(n^x x + n^y y) = \mathring{q}(\xi)$$



Solution is constant along these lines.

We want

$$q(x,y,t) = \mathring{q}(\vec{n}\cdot\vec{x} - st)$$

Then $q_t = -s\mathring{q}'(\xi)$

$$q_x = n^x \mathring{q}'(\xi)$$

$$q_y = n^y \mathring{q}'(\xi)$$

$$-s\mathring{q}'(\xi) + n^x A\mathring{q}'(\xi) + n^y B\mathring{q}'(\xi) = 0$$

$$(n^x A + n^y B)\, q'(\xi) = s\, q'(\xi)$$

$q'(\xi)$ must an eigenvector of $n^x A + n^y B$ with eigenvalue s

So we need

$$n^x A + n^y B$$

to be diagonalizable with real eigenvalues, In this case we say (1) is hyperbolic.

For the nonlinear system $q_t + f(q)_x + g(q)_y = 0$

i.e.

$$q_t + f'(q) q_x + g'(q) q_y = 0$$

we say it is hyperbolic for $q \in D$ if

$$n^x f'(q) + n^y g'(q)$$

is diagonalizable for all (n^x, n^y) and all $q \in D$.

Dispersion relation approach:

Assume $q = v e^{i\vec{k}\cdot\vec{x}} e^{i\omega(\vec{k})t}$ and solve for $\omega(k)$

we should find that $\omega(\vec{k})$ is real-valued for all $\vec{k}$ iff (1) is hyperbolic.

$$\underbrace{q_t + \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}}\begin{bmatrix} q_x \\ q_y \end{bmatrix} \qquad \begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

$$q_t + \begin{bmatrix} Aq_x \\ Bq_y \end{bmatrix} = 0 \quad \not\Leftrightarrow \quad q_t + Aq_x + Bq_y = 0$$

Example: 2D Acoustics

$$p_t + K\nabla\cdot\vec{u} = 0 \qquad \vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$u_t + \frac{1}{\rho}\nabla p = 0$$

$$p_t + Ku_x + Kv_y = 0$$

$$u_t + \frac{1}{\rho}p_x = 0$$

$$v_t + \frac{1}{\rho}p_y = 0$$

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix} \qquad A = \begin{bmatrix} 0 & K & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 1/\rho & 0 & 0 \end{bmatrix}$$

Notice: $A = PBP^T$ where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

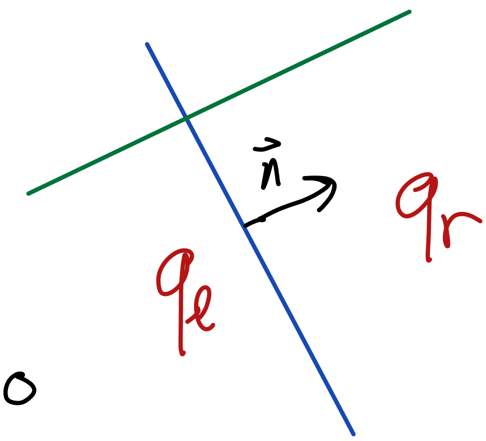
$$\Lambda_A = \begin{bmatrix} -c & & \\ & 0 & \\ & & +c \end{bmatrix} \qquad c = \sqrt{K/\rho} \qquad Z = \sqrt{K\rho}$$

$$R_A = \begin{bmatrix} -Z & 0 & Z \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_B = \begin{bmatrix} -Z & 0 & Z \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \Lambda_B = \Lambda_A$$

(IIRC)

$n^x A + n^y B$ has eigenvectors $\begin{bmatrix} -Z & 0 & Z \\ n^x & n^y & n^x \\ n^y & n^x & n^y \end{bmatrix}$

Planar Riemann Problem



$$q_t + f(q)_x + g(q)_y = 0$$

$$q(x,y,t=0) = \begin{cases} q_\ell & \vec{n}\cdot\vec{x} < 0 \\ q_r & \vec{n}\cdot\vec{x} > 0 \end{cases}$$

The solution is again self-similar. For the linear system (1), the solution of this problem can be obtained from that of

$$q_t + \underline{(n^x A + n^y B)} q_\mu = 0$$

$$q(\mu, t=0) = \begin{cases} q_\ell & \mu < 0 \\ q_r & \mu > 0 \end{cases}$$

2D shallow water equations

$$h_t + \nabla\cdot(h\vec{u}) = 0$$

$$(hu)_t + \nabla\left(\tfrac{1}{2}gh^2\right) + \nabla\cdot(h\vec{u}\vec{u}^T) = 0$$

Written out fully:

$$h_t + (hu)_x + (hv)_y = 0$$

$$(hu)_t + \left(\tfrac{1}{2}gh^2 + hu^2\right)_x + (huv)_y = 0$$

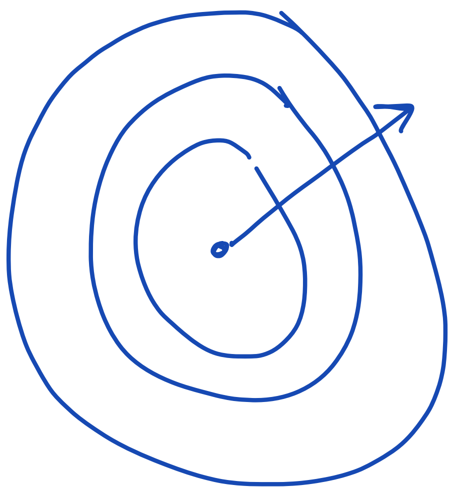
$$(hv)_t + (huv)_x + \left(\tfrac{1}{2}gh^2 + hv^2\right)_y = 0$$

Notice that these are invariant under rotation. e.g.

$u \leftrightarrow v$

$x \leftrightarrow y$

Axisymmetric data:



$\mathring{q}(x,y,0) = \tilde{q}(x^2+y^2)$

is preserved