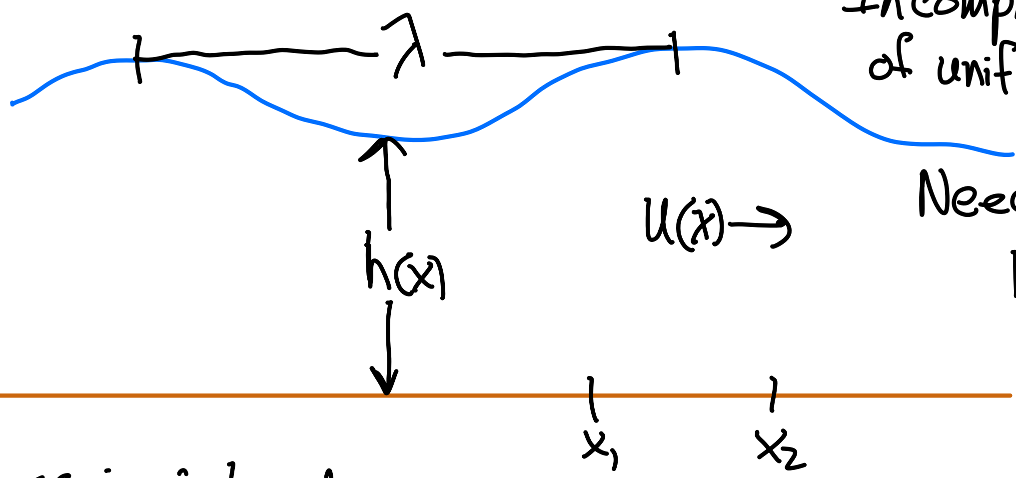
Shallow water equations

Depth-averaged
Incompressible fluid
of uniform density $\bar{\rho}$



Need to assume
$h \ll \lambda$

Mass in interval (x_1, x_2): $\int_{x_1}^{x_2} \bar{\rho} h(x)\,dx$

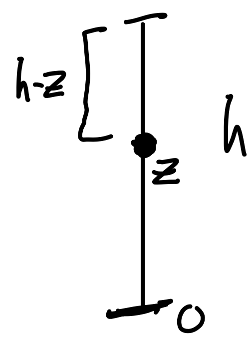
Flux: $\bar{\rho} h u$

Conservation of mass: $(\bar{\rho} h)_t + (\bar{\rho} h u)_x = 0 \Rightarrow h_t + (hu)_x = 0$

Conservation of momentum: $(\bar{\rho} h u)_t + (\bar{\rho} h u^2 + p)_x = 0$

What is the pressure?

Hydrostatic pressure: exactly enough to support weight of water.



$$p(x, z) = g \bar{\rho} (h - z)$$

Total pressure in a column:

$$\int_0^h g \bar{\rho} (h-z)\,dz = g\bar{\rho} \int_0^h (h-z)\,dz = g\bar{\rho}\left(h^2 - \frac{h^2}{2}\right)$$

$$= \frac{1}{2} g \bar{\rho} h^2$$

$$h_t + (hu)_x = 0$$
$$(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x = 0$$

1D Shallow water Model

Characteristic Structure

$$q_t + f(q)_x = 0 \Rightarrow \underline{q_t + f'(q)\,q_x = 0}$$

$$q = \begin{bmatrix} h \\ hu \end{bmatrix} \qquad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix} = \begin{bmatrix} q^2 \\ \frac{(q^2)^2}{q^1} + \frac{1}{2}g(q^1)^2 \end{bmatrix}$$

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gq^1 - \left(\frac{q^2}{q^1}\right)^2 & 2\frac{q^2}{q^1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}$$

Characteristic speeds are eigenvalues of $f'(q)$:

$$\det(\lambda I - f'(q)) = \det \begin{bmatrix} \lambda & -1 \\ u^2 - gh & \lambda - 2u \end{bmatrix} = \lambda(\lambda - 2u) + u^2 - gh$$

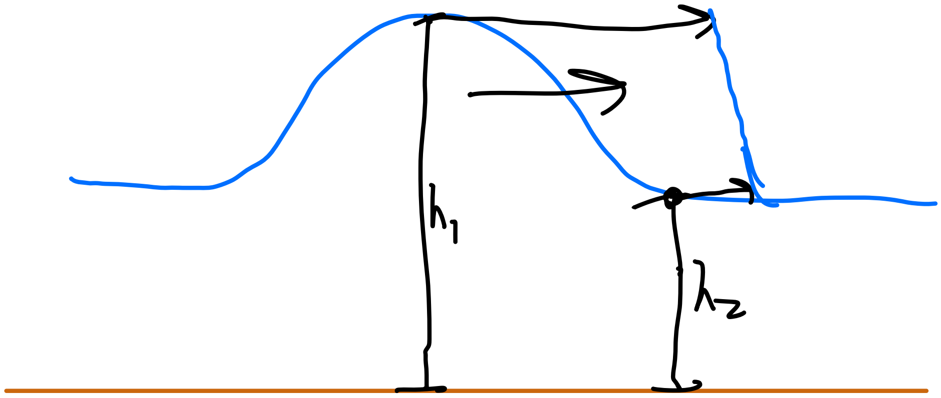
$$\lambda^2 - 2u\lambda + u^2 - gh = 0$$

$$\boxed{\begin{aligned} \lambda^1 &= u - \sqrt{gh} \\ \lambda^2 &= u + \sqrt{gh} \end{aligned}}$$

$$\boxed{\begin{aligned} r^1 &= \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix} \\ r^2 &= \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix} \end{aligned}}$$

Solution is <u>not</u> constant along characteristics, and they are not straight lines (in general)

Why do waves break in shallow water?



$h_1 > h_2$

$u + \sqrt{g h_1} > u + \sqrt{g h_2}$

In deep water

$\frac{h_1 - h_2}{h_1}$ is small.

Small perturbations move at the characteristic speeds.

$$q(x,t) = q_0 + \varepsilon \hat{q}(x,t) \qquad \varepsilon << 1$$

$$q_t + f'(q) q_x = (q_0 + \varepsilon \hat{q})_t + f'(q_0 + \varepsilon \hat{q})(q_0 + \varepsilon \hat{q})_x = 0$$

$$\varepsilon \hat{q}_t + \varepsilon f'(q_0) \hat{q}_x = \mathcal{O}(\varepsilon^2)$$

$$\hat{q}_t + f'(q_0) \hat{q}_x = \mathcal{O}(\varepsilon)$$

Now suppose $\hat{q}_x = \alpha(x) r^1(q_0)$. Then

$$\hat{q}_t + f'(q_0) \alpha(x) r^1(q_0) = \hat{q}_t + \alpha(x) \lambda^1(q_0) r^1(q_0) = 0.$$

R. H. Jump Condition

Let (h_*, u_*) be fixed and suppose we have a shock with (h, u) being the other state.

$$s(q - q_*) = f(q) - f(q_*)$$

$\alpha = h - h_*$

$$\rightarrow \left.\begin{aligned} s(h-h_*) &= hu - h_* u_* \\ s(hu - h_* u_*) &= hu^2 - h_* u_*^2 + \tfrac{1}{2} g (h^2 - h_*^2) \end{aligned}\right\} \begin{array}{l} \text{2 Eqns.} \\ \text{3 unknowns:} \\ s, h, u. \end{array}$$

Result: $s = \dfrac{hu - h_* u_*}{\alpha} = u_* \mp \sqrt{gh \dfrac{h + h_*}{2h_*}} \leftarrow$

$$hu = h_* u_* + \alpha \left[u_* \pm \sqrt{g h_* \left(1 + \frac{\alpha}{h_*}\right)\left(1 + \frac{\alpha}{2h_*}\right)} \right]$$

$$h = h_* + \alpha$$

Observe: $s^p \rightarrow \lambda^p$ as $\alpha \rightarrow 0$