Today:

Compressible gas dynamics

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Acoustics

$\downarrow$

Solution of linear hyperbolic systems

$$\rho_t + (\rho u)_x = 0$$

$\rho(x,t)$: density

$u(x,t)$: Velocity

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$p(\rho)$: pressure e.g. $p = K\rho^\gamma$

$$q_t + f(q)_x = 0$$

Quasilinear form:

$$q_t + f'(q) q_x = 0$$

$f'(q)$: flux Jacobian

$$q = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}$$

$$q^1_t + q^2_x = 0$$

$$q^2_t + \left(\frac{(q^2)^2}{q^1} + p(q^1) \right)_x = 0$$

$$f'(q) = \begin{bmatrix} 0 & 1 \\ -\left(\frac{q^2}{q^1}\right)^2 + p'(q^1) & 2\frac{q^2}{q^1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -u^2 + p'(\rho) & 2u \end{bmatrix}$$

Small perturbations (linearization)

$$\rho(x,t) = \rho_0 + \tilde{\rho}(x,t)$$

$$(\rho u)(x,t) = (\rho u)_0 + \widetilde{\rho u}(x,t)$$

$$q_0 = q_0 + \tilde{q}(x,t)$$

$$q_t + f(q)_x = 0$$

$$f(q) = f(q_0 + \tilde{q}(x,t)) = f(q_0) + f'(q_0)\tilde{q} + \dots$$

$$\tilde{q}_t + f'(q_0)\tilde{q}_x = 0$$

$$\tilde{\rho}_t + \widetilde{\rho u}_x = 0$$

Taking $u_0 = 0$

$$\underline{\widetilde{\rho u}_t} + \underline{p'(\rho_0)\tilde{\rho}_x} = 0$$

$$p(\rho) = p(\rho_0 + \tilde{\rho}) = \overbrace{p(\rho_0)}^{P_0} + p'(\rho_0)\tilde{\rho} + \dots$$

$$P_0 + \tilde{p} = p(\rho)$$

$$\tilde{p} \approx p'(\rho_0)\tilde{\rho}$$

$$\tilde{\rho} \approx \frac{\tilde{p}}{p'(\rho_0)}$$

$$\underline{\widetilde{\rho u} = \rho_0\tilde{u} + \tilde{\rho}u_0 + \rho_0 u_0 = \rho_0\tilde{u}}$$

$$\widetilde{\rho u} = (\rho_0 + \tilde{\rho})(u_0 + \tilde{u})$$

$$\tilde{\rho}_t + \rho_0\tilde{u}_x = 0$$

$$\rho_0\tilde{u}_t + p'(\rho_0)\tilde{\rho}_x = 0$$

$$\frac{\tilde{P}_t}{P'(\rho_0)} + \rho_0 \hat{u}_x = 0$$

$$\rho_0 \hat{u}_t + \tilde{P}_x = 0$$

$$\hat{P}_t + \underbrace{P'(\rho_0)\rho_0}_{K} \hat{u}_x = 0$$

$$\hat{u}_t + \frac{1}{\rho_0}\tilde{P}_x = 0$$

From here on we drop tildes

$$q_t + A q_x = 0$$

$$q = \begin{bmatrix} P \\ u \end{bmatrix} \quad A = \begin{bmatrix} 0 & K \\ 1/\rho_0 & 0 \end{bmatrix}$$

Acoustics 1st-order linear PDE system

$$P_t + K u_x = 0$$

$$u_t + \frac{1}{\rho_0} P_x = 0$$

$$P_{tt} + K u_{xt} = 0 \qquad u_{xt} = u_{tx}$$

$$u_{tx} + \frac{1}{\rho_0} P_{xx} = 0$$

$$P_{tt} - \frac{K}{\rho_0} P_{xx} = 0$$

Wave equation

$$c = \sqrt{\frac{K}{\rho_0}}$$

Traveling wave solutions

$q(x,t) = \hat{q}(x - st)$

$-s\hat{q}'(\xi) + A\hat{q}'(\xi) = 0$

$A\hat{q}' = s\hat{q}'$

Eigenvalue equation

$\hat{q}'(\xi)$ is an eigenvector of A

s is the eigenvalue

$A = \begin{bmatrix} 0 & K \\ 1/\rho_0 & 0 \end{bmatrix}$ $\quad \lambda^2 - \frac{K}{\rho_0} = 0 \quad \lambda = \pm\sqrt{\frac{K}{\rho_0}}$

$= \pm c$

$Ar = \lambda r$

$Kr_2 = \pm\sqrt{\frac{K}{\rho_0}}\, r_1$

$A = R\Lambda R^{-1}$

$r_1 = \pm\underbrace{\sqrt{K\rho_0}}_{\substack{\text{Impedance} \\ Z}}\, r_2$

$r = \begin{bmatrix} \pm Z \\ 1 \end{bmatrix} \quad R = \begin{array}{cc} \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix} \\ \;\; r^1 \quad\; r^2 \end{array} \quad \Lambda = \begin{bmatrix} -c & \\ & +c \end{bmatrix}$

If we take

$$q_0(x) = \alpha(x) r^1$$

$$q(x,t) = q_0(x+ct)$$

$$q_t + R\Lambda R^{-1} q_x = 0$$

$$R^{-1} q_t + \Lambda R^{-1} q_x = 0 \qquad w = R^{-1} q$$

$$w_t + \Lambda w_x = 0$$

$$w^1_t - c w^1_x = 0$$

$$w^2_t + c w^2_x = 0$$

$q_t + A q_x = 0$ is hyperbolic

if A is diagonalizable

with real eigenvalues.

$q_t + f(q)_x = 0$ is hyperbolic

if $f'(q)$ is diagonalizable

with real eigenvalues.

General solution of a linear hyperbolic system in 1D

Given $q_t + Aq_x = 0$

$$q(x,0) = q_0(x) \quad x \in \mathbb{R}$$

$$A \in \mathbb{R}^{m \times m} \qquad A = R \Lambda R^{-1}$$

with real eigenvalues (Λ is real diag. matrix)

① Write $q_0(x)$ in terms of eigenvectors of A:

$$q_0(x) = \sum_{j=1}^{m} w_0^j(x) r^j$$

② We have

$$w^j(x,t) = w_0^j(x - \lambda^j t)$$

Since $w_t^j + \lambda^j w_x^j = 0$

③ $q = Rw$

$$q(x,t) = \sum_{j=1}^{m} w^j(x,t) r^j$$

$$= \sum_{j=1}^{m} w_0^j(x - \lambda^j t) r^j$$

Homework: Exercises 2.2, 2.8