Clawpack tutorial
Sunday 2–4 pm
Room: TBD

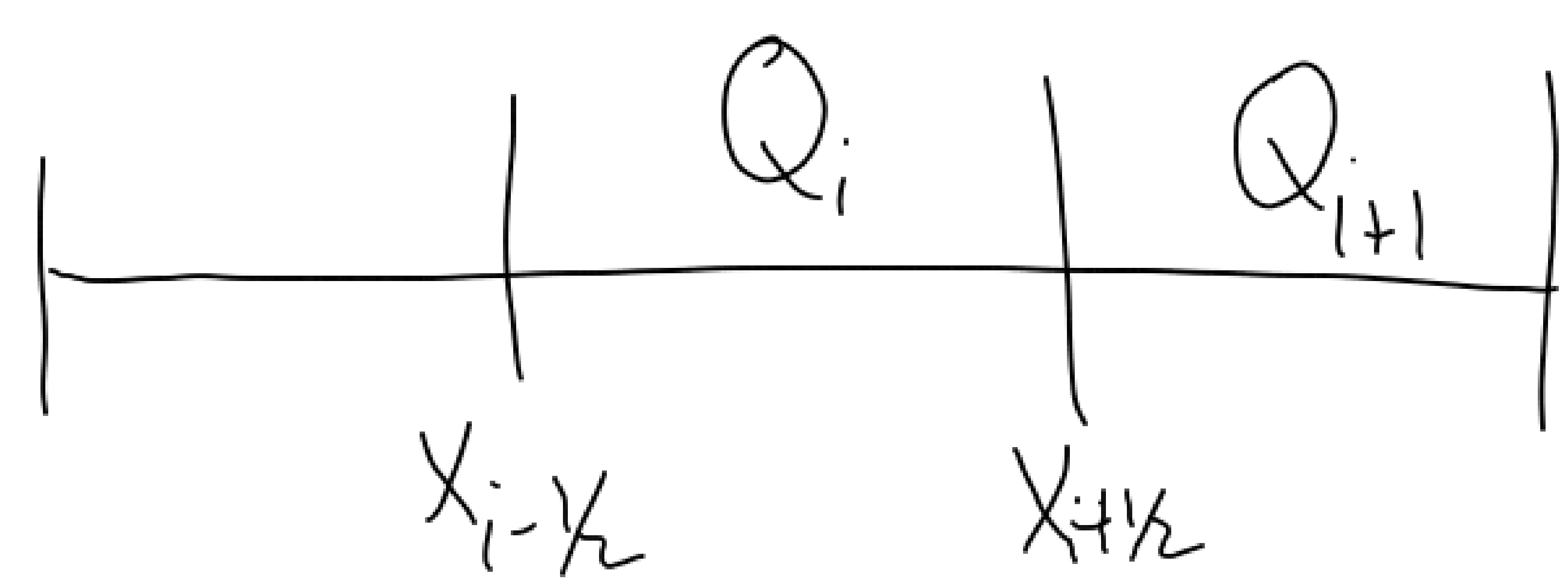
No class Feb. 18, 21

$$q_t + f(q)_x = 0$$

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta t} = -\frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x}$$

$$Q_i^n = \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t^n)\,dx$$

$$F_{i+1/2}^n = \mathcal{F}(Q_i^n, Q_{i+1}^n)$$



Take the limit as $\Delta t \to 0$:

$$Q_i'(t^n) = -\frac{F_{i+1/2} - F_{i-1/2}}{\Delta x}$$

Semi-discrete

Where $F_{i+1/2} = \mathcal{F}(Q_i(t), Q_{i+1}(t))$

This is a useful starting point to design high-order methods (via the method of lines).

Consistency + Stability $\Rightarrow$ convergence

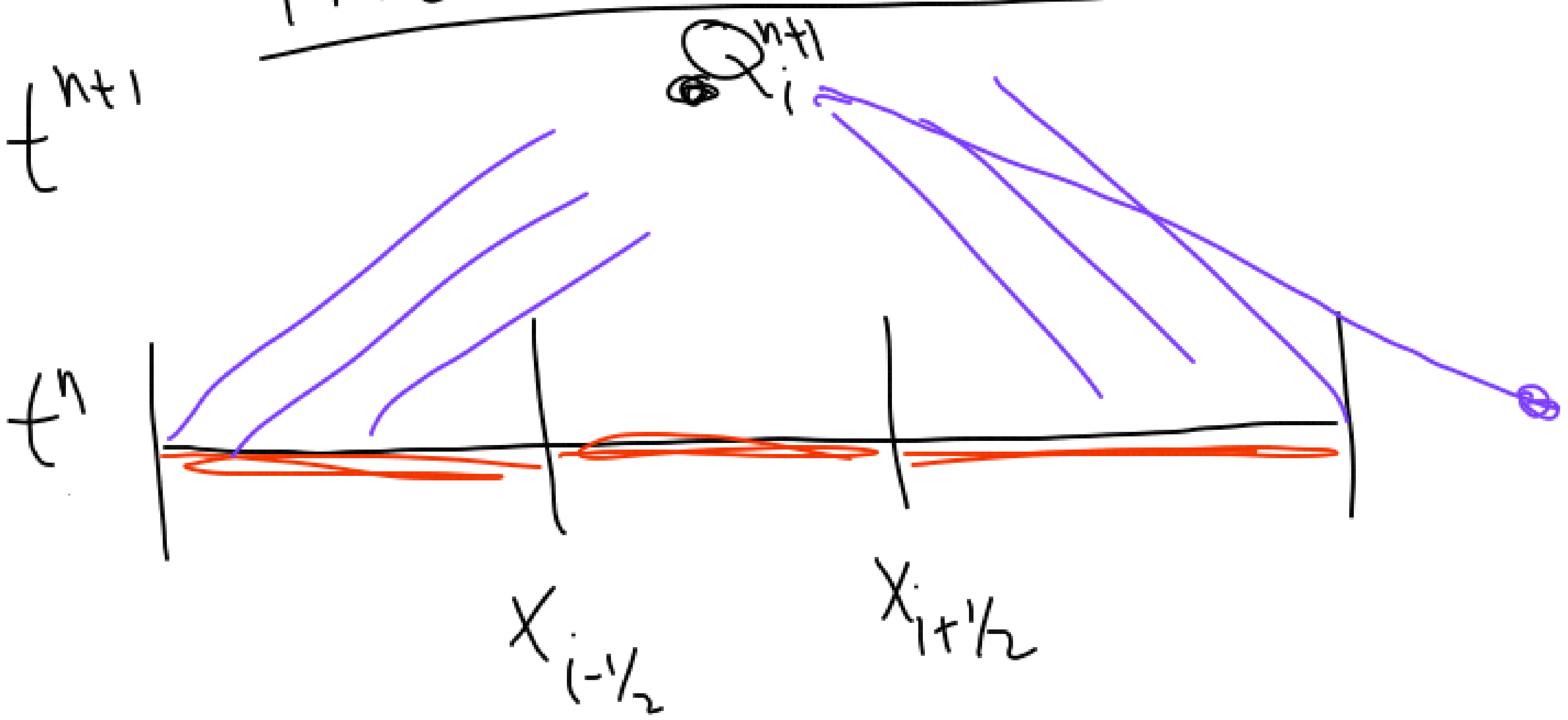
Consistency:

1. $\mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$

2. Lipschitz continuity:

$$|\mathcal{F}(Q_{i-1}, Q_i) - f(\bar{q})| < L \max(|Q_i - \bar{q}|, |Q_{i-1}, \bar{q}|)$$

The CFL condition



Q_i^{n+1} depends (numerically)

on: $Q_{i-1}^n, Q_i^n, Q_{i+1}^n$

What does $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t^{n+1})\,dx$

depend on?

For acoustics: it depends on

$q(x,t^n)$ for

$x \in [x_{i-1/2} - c\Delta t,\ x_{i+1/2} + c\Delta t]$

CFL says: the numerical D.o.D. must contain the true D.o.D.
(in the limit as $\Delta t, \Delta x \to 0$)

In this case, we have the condition

$$c\Delta t \le \Delta x \quad \text{or} \quad \underbrace{\frac{c\Delta t}{\Delta x}}_{\text{Courant number}} \le 1$$

Characteristics: slope $\frac{1}{c}$

In general for a hyperbolic system with

$$\sigma(f'(q)) = \{\lambda^1, \lambda^2, \dots, \lambda^m\}$$

We need $\max_{1 \le p \le m} |\lambda^p| \frac{\Delta t}{\Delta x} \le 1$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2}^n - F_{i-1/2}^n\right]$$

Average flux: $F_{i+1/2}^n = \dfrac{f(Q_{i+1}) + f(Q_i)}{2}$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}\left(f(Q_{i+1}) - f(Q_{i-1})\right)$$

$\Rightarrow$ unstable

(modi

approximates $q_t + f(q)_x = -q_{xx}$

Consider advection: $f(q) = aq$

$$Q_i^{n+1} = Q_i^n - \frac{a\Delta t}{2\Delta x}\left(Q_{i+1}^n - Q_{i-1}^n\right)$$

Suppose there exists $\tilde{q}(x,t)$ that exactly satisfies this eqn.

$$Q_i^n \to \tilde{q}(x_i, t_n)$$

$$Q_i^{n+1} \to \tilde{q}(x_i, t_{n+1}) = \tilde{q}(x_i, t_n) + \Delta t\, \tilde{q}_t(x_i, t_n) + \frac{\Delta t^2}{2}\tilde{q}_{tt}(x_i, t_n) + \mathcal{O}(\Delta t^3)$$

$$Q_{i\pm 1}^n \to \tilde{q}(x_i \pm \Delta x, t_n) = \tilde{q} \pm \Delta x\, \tilde{q}_x + \frac{\Delta x^2}{2}\tilde{q}_{xx} + \mathcal{O}(\Delta x^3)$$

We find

$$\mathcal{O}(\Delta t^3) + \cancel{\tilde{q}} + \Delta t\, \tilde{q}_t + \frac{\Delta t^2}{2}\tilde{q}_{tt} = \cancel{\tilde{q}} - \frac{a\Delta t}{2\Delta x}\left(2\Delta x\, \tilde{q}_x + \mathcal{O}(\Delta x^3)\right)$$

$$\mathcal{O}(\Delta t^3) + \Delta t\, \tilde{q}_t + \frac{\Delta t^2}{2}\tilde{q}_{tt} = -a\Delta t\, \tilde{q}_x + \mathcal{O}(\Delta x^2 \Delta t)$$

$$\tilde{q}_t + a\tilde{q}_x = \underbrace{-\frac{\Delta t}{2}\tilde{q}_{tt}} + \mathcal{O}(\Delta t^2, \Delta x^2 \Delta t)$$

$$\tilde{q}_t + a\tilde{q}_x = \mathcal{O}(\Delta t)$$

$$\tilde{q}_{tt} = -a\tilde{q}_{xt} + \mathcal{O}(\Delta t)$$

$$\tilde{q}_{tx} = -a\tilde{q}_{xx} + \mathcal{O}(\Delta t)$$

$$\tilde{q}_{tt} = a^2\tilde{q}_{xx} + \mathcal{O}(\Delta t)$$

$$\tilde{q}_t + a\tilde{q}_x = \underbrace{-\frac{\Delta t}{2} a^2 \tilde{q}_{xx}}_{\text{anti-diffusive}} + O(\Delta t^2)$$

Let's add diffusion!

$$\mathcal{F}(Q^n_{i-1}, Q^n_i) = \frac{1}{2}\left[f(Q^n_{i-1}) + f(Q^n_i)\right] - \frac{\Delta x}{2\Delta t}(Q^n_i - Q^n_{i-1})$$

This gives:

$$Q^{n+1}_i = \frac{1}{2}(Q^n_{i+1} + Q^n_{i-1}) - \frac{\Delta t}{2\Delta x}\left(f(Q_{i+1}) - f(Q_{i-1})\right)$$

Lax-Friedrichs 1st-order

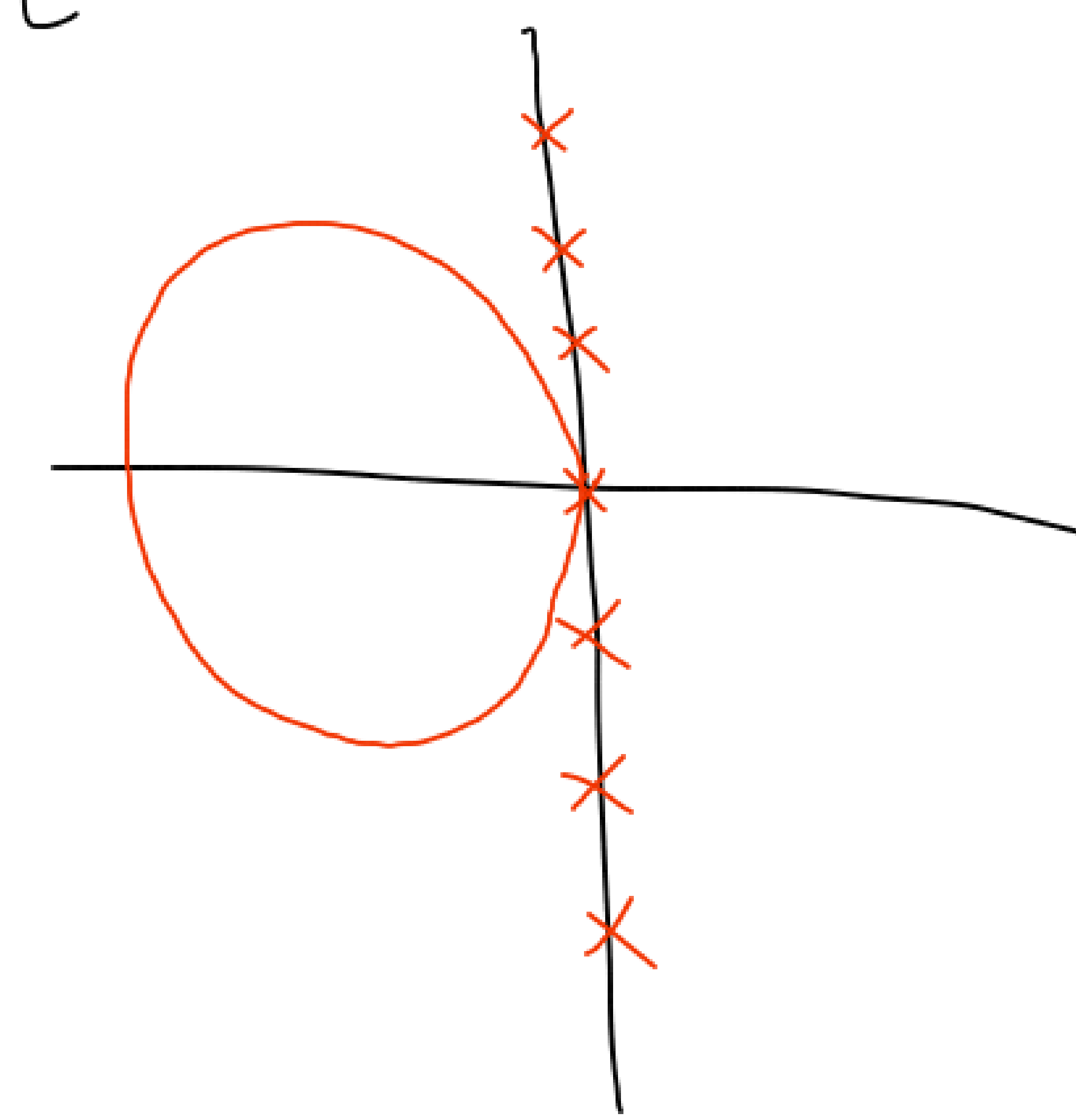
Stable for $\frac{a\Delta t}{\Delta x} \leq 1$.

Method-of-lines analysis

Centered:

$$Q_i'(t) = -\frac{1}{2\Delta x}(Q_{i+1} - Q_{i-1})$$

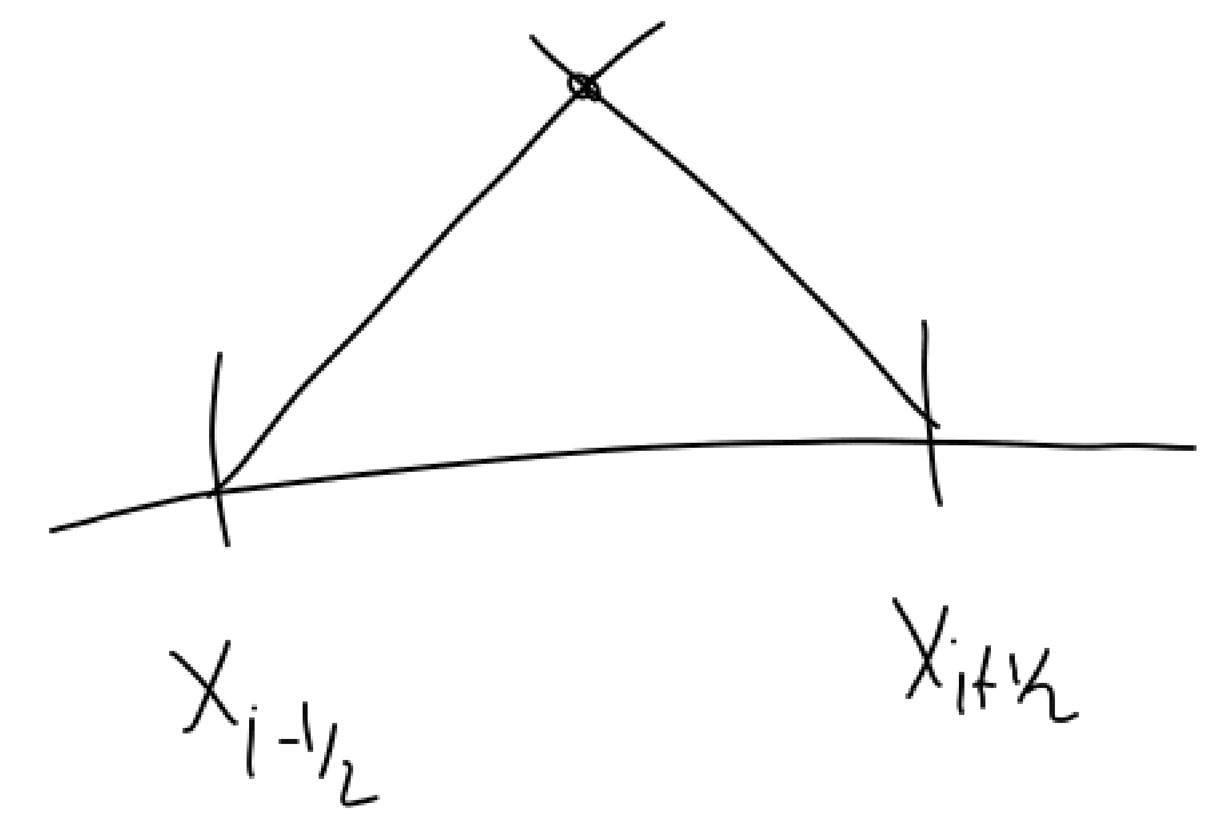
Apply FE



Godunov's method

① Approximate q by cell averages:

$$\hat{q} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx \quad \text{for } x \in [x_{i-1/2}, x_{i+1/2}]$$



② Evolve the solution exactly in time by solving a Riemann problem at $x_{i+1/2}$

③ Re-average

Exercise 4.2 from FVMHP
(Next Wed.)