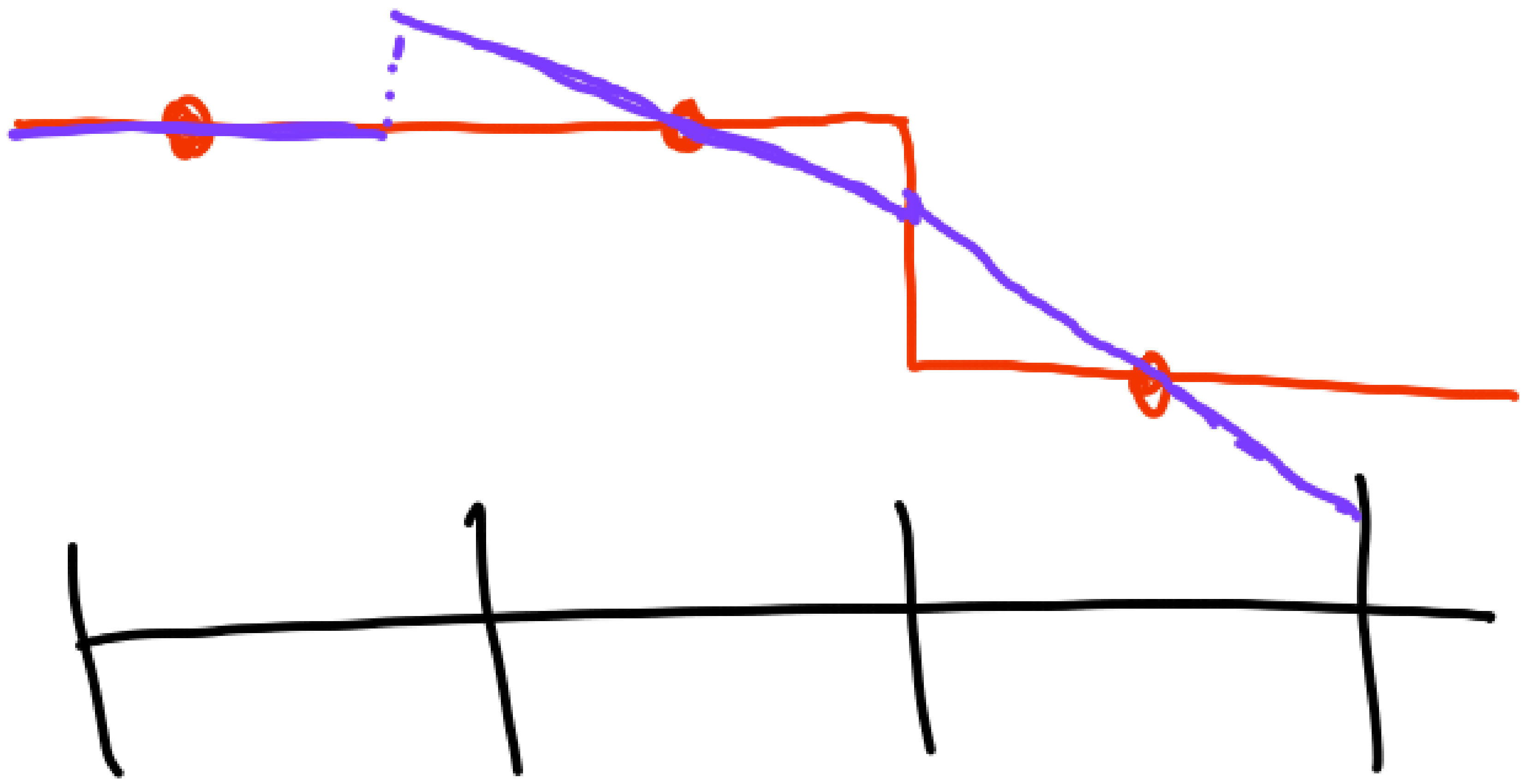
# Reconstruct-Evolve-Average (REA)

1. Approximate $q(x,t^n)$ by a piecewise polynomial — Might increase total variation
2. Solve Riemann problem at each interface — Don't increase the TV
3. Update solution/re-average



We want a TVD piecewise-linear reconstruction. Should be $\mathcal{O}(\Delta x^2)$.

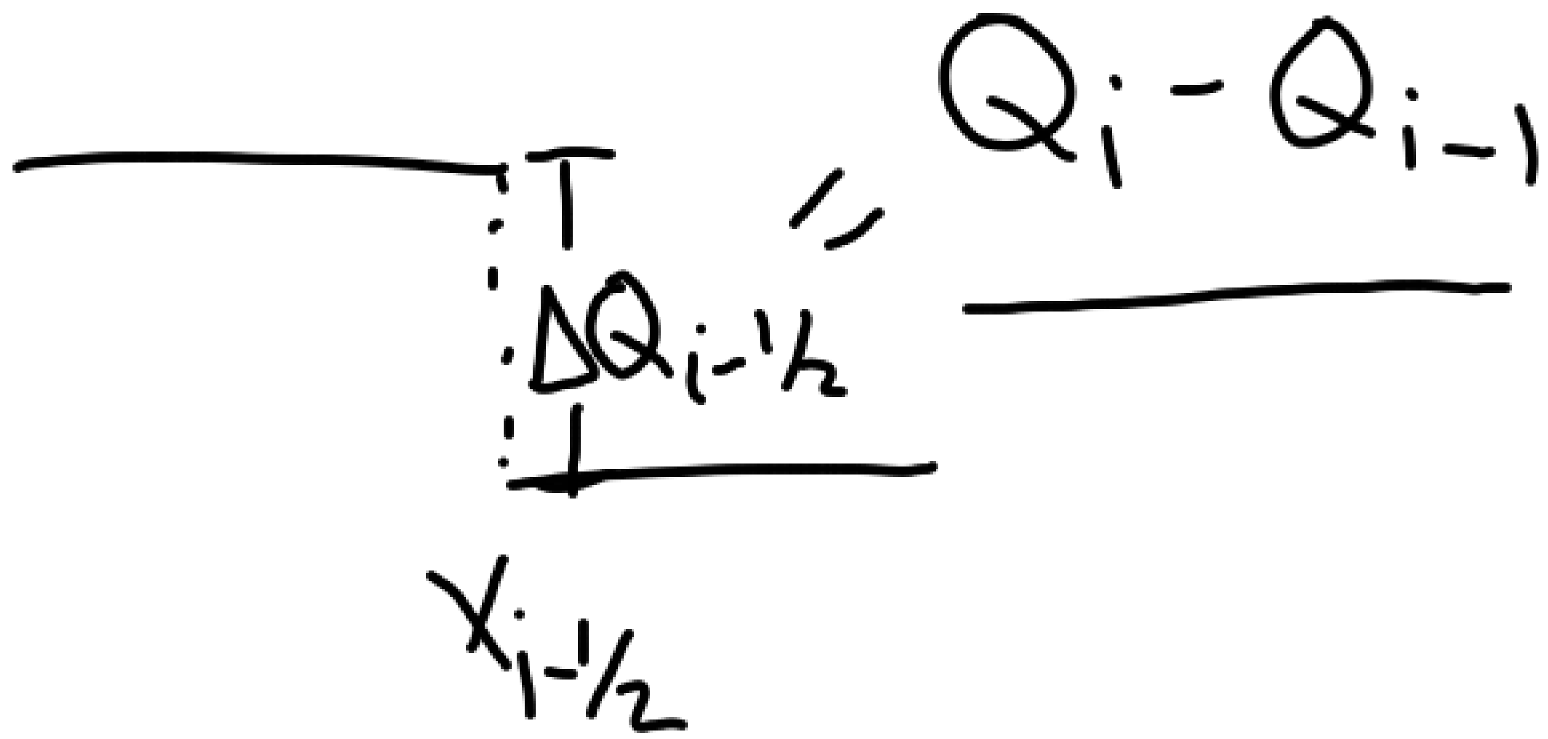
The idea is to "limit" the slope in each cell.

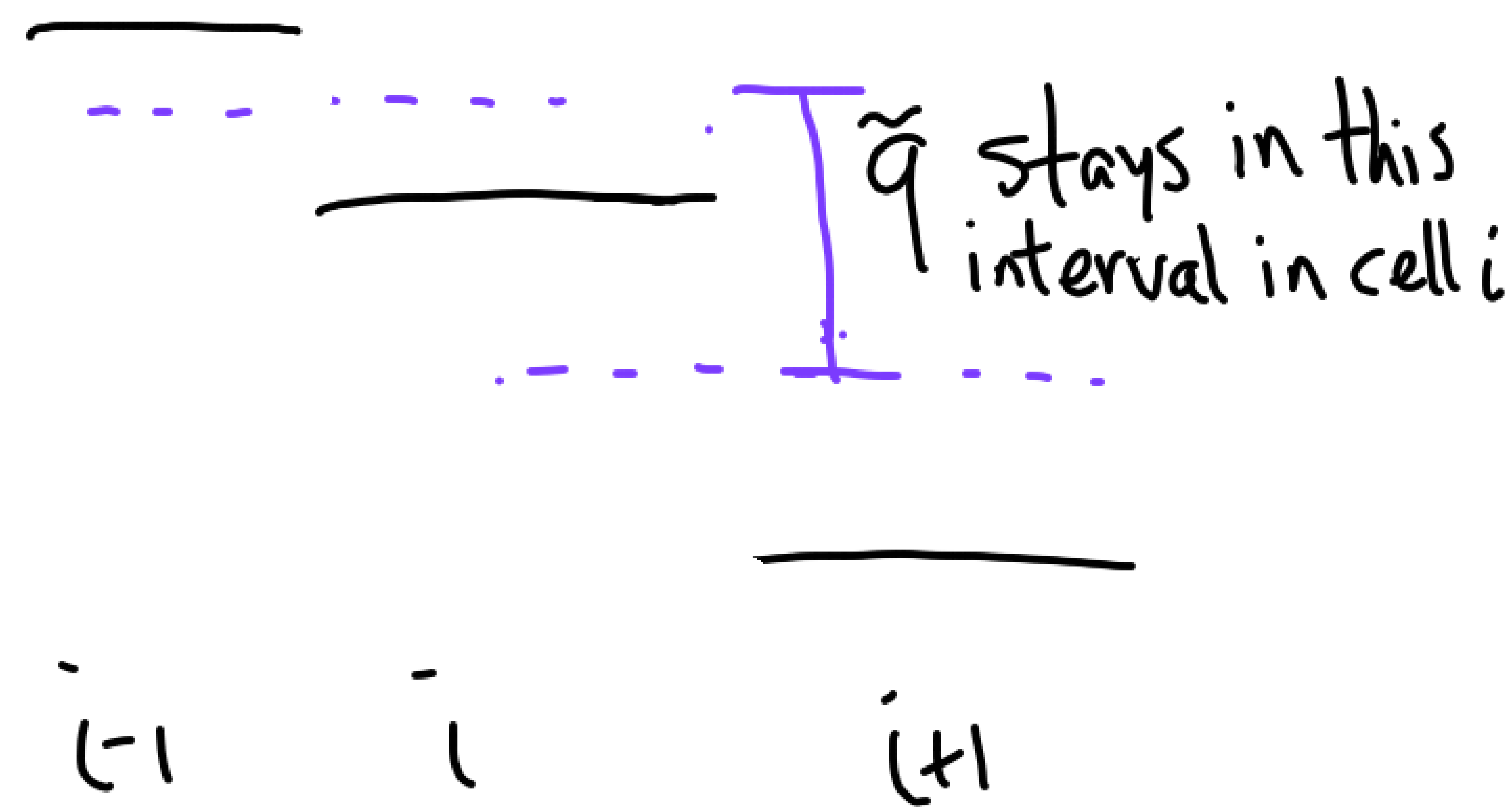
Minimum modulus ("minmod")

$$\tilde{q}(x,t) = Q_i + \underbrace{\sigma_i}_{\text{slope in cell } i.}(x - x_i) \quad \text{for } x \in (x_{i-1/2}, x_{i+1/2})$$

minmod:

$$\sigma_i = \begin{cases} 0 & \text{if } \Delta Q_{i-1/2} \cdot \Delta Q_{i+1/2} < 0 \\ \dfrac{\Delta Q_{i-1/2}}{\Delta x} & \text{if } |\Delta Q_{i-1/2}| < |\Delta Q_{i+1/2}| \\ \dfrac{\Delta Q_{i+1/2}}{\Delta x} & \text{if } |\Delta Q_{i-1/2}| > |\Delta Q_{i+1/2}| \end{cases}$$




$\ldots|$ and $\Delta Q_{i-1/2} \cdot \Delta Q_{i+1/2} > 0$

$\ldots_{i+1/2}|$ and $\Delta Q_{i-1/2} \cdot \Delta Q_{i+1/2} > 0$

So $TV(\tilde{q}) \le TV(Q)$

If we use this in the REA algorithm, the semi-discrete scheme

$$Q_i' = -\frac{\Delta t}{\Delta x}\left[\hat{f}_{i+1/2} - \hat{f}_{i-1/2}\right]$$

has the property that $\frac{d}{dt} TV(Q) \le 0$.

There are many other "slope limiters"; e.g.

van Leer

$$\sigma_i = \frac{\Delta Q_{i+1/2} \cdot \Delta Q_{i-1/2}}{\Delta Q_{i+1/2} + \Delta Q_{i-1/2}} \cdot \left(\operatorname{sgn}(\Delta Q_{i+1/2}) + \operatorname{sgn}(\Delta Q_{i-1/2})\right)$$

For systems of equations:

- First transform to characteristic variables $q \to w$
- Reconstruct $w$

ENO / WENO

Higher-order piecewise polynomial reconstruction

# Lax-Wendroff

(Cauchy-Kovalevsky)

Simultaneously discretize space and time

$$q_t + A q_x = 0 \Rightarrow q_{tt} = A^2 q_{xx}$$

$$q(x,t_{n+1}) = q(x,t_n) + \Delta t\, q_t(x,t_n) + \frac{\Delta t^2}{2} q_{tt}(x,t_n) + \mathcal{O}(\Delta t^3)$$

$$= q - \Delta t A q_x + \frac{\Delta t^2}{2} A^2 q_{xx} + \mathcal{O}(\Delta t^3)$$

Now use centered differences

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A\left(Q_{i+1}^n - Q_{i-1}^n\right)$$

$$\frac{\Delta t^2}{2\Delta x^2} A^2 \left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

2nd-order in time and space

We can write this as

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

$$F_{i-1/2} = \frac{1}{2}A\left(Q_{i-1}+Q_i\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}A^2\left(Q_i - Q_{i-1}\right)$$

$$= \underbrace{A^+Q_{i-1} + A^-Q_i}_{\text{Godunov's method}} + \underbrace{\frac{1}{2}|A|\left[I - \frac{\Delta t}{\Delta x}|A|\right](Q_i - Q_{i-1})}_{\text{2nd-order correction}}$$

$$|A| = A^+ - A^- = R|\Lambda|R^{-1}$$

$$A^{\pm} = R\Lambda^{\pm}R^{-1}$$

$$\Lambda^{\pm} = \begin{bmatrix} \lambda_1^{\pm} & & & \\ & \lambda_2^{\pm} & & \\ & & \ddots & \\ & & & \end{bmatrix}$$

We can make LW TVD by applying a limiter to the correction.

We could replace $Q_i - Q_{i-1}$ by $\text{minmod}(Q_i - Q_{i-1}, Q_{i+1} - Q_i)$

We can write the correction as

$$\frac{1}{2}|A|\left(I-\frac{\Delta t}{\Delta x}|A|\right)\sum_p \alpha^p_{i-1/2} r^p$$

apply limiter to this

$$\alpha^p_{i-1/2} \rightarrow \tilde{\alpha}^p_{i-1/2} = \text{minmod}\left(\alpha^p_{i-1/2}, \alpha^p_{I-1/2}\right)$$

where $I = \begin{cases} i+1 & \text{if } \lambda^p_{i-1/2} < 0 \\ i-1 & \text{if } \lambda^p_{i-1/2} > 0 \end{cases}$

This is TVD in characteristic variables