# Problems with spatially-varying flux

$$q_t + f(q,x)_x = 0$$

For example: $\rho_t + (V(x)\rho(1-\rho))_x = 0$ $\quad 0 \le \rho \le 1$, $V > 0$

$V(x)$: max. speed (e.g. varying road surface or speed limit)

We can write this as $q_t + f(q)_x = 0$ by taking $q = \begin{bmatrix} \rho \\ V \end{bmatrix}$ and $V_t = 0$.

Then $f(q) = \begin{bmatrix} V\rho(1-\rho) \\ 0 \end{bmatrix}$ $\quad f'(q) = \begin{bmatrix} V(1-2\rho) & \rho(1-\rho) \\ 0 & 0 \end{bmatrix}$

$\lambda^1 = V(x)(1-2\rho)$ $\qquad -V \le \lambda^1 \le +V$

$\lambda^2 = 0$ $\qquad$ if $\rho = \frac{1}{2}$, $\lambda^1 = \lambda^2$

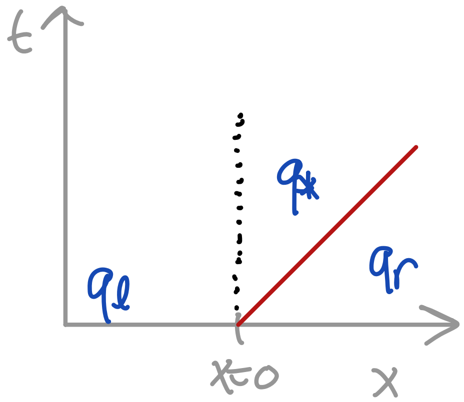
So this system is not strictly hyperbolic.

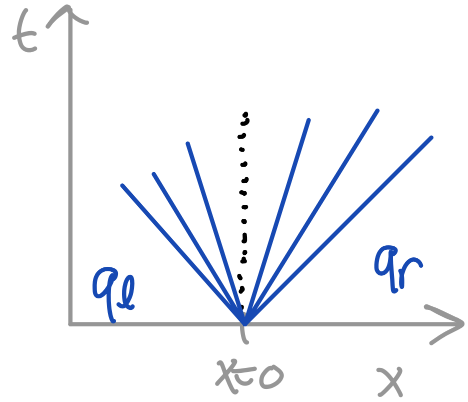
This suggests there should be a stationary wave in the Riemann solution.

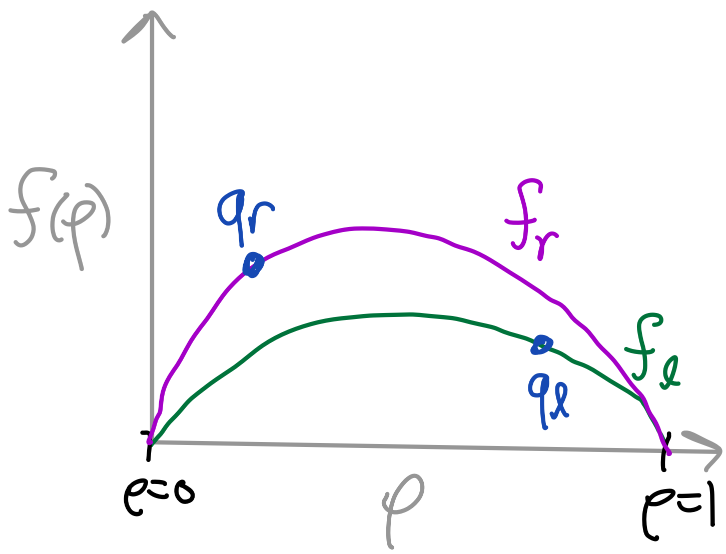
Notice that the waves will interact if there is a transonic 1-rarefaction. ("resonance")

# Riemann Problem

$$(\rho(x,0), v(x)) = \begin{cases} (\rho_l, v_l) & x<0 \\ (\rho_r, v_r) & x>0 \end{cases}$$




$f_r = V_r \rho(1-\rho)$

$f_l = V_l \rho(1-\rho)$

$\lambda' = V(1-2\rho)$

Useful observations:

1. $\rho<\frac{1}{2} \Rightarrow \lambda'>0 \Rightarrow$ no intermediate state with $\rho<\frac{1}{2}$ for $x<0$.

2. $\rho>\frac{1}{2} \Rightarrow \lambda'<0 \Rightarrow$ no intermediate state with $\rho>\frac{1}{2}$ for $x>0$

Lax entropy condition: $\lambda' = 1-2\rho$

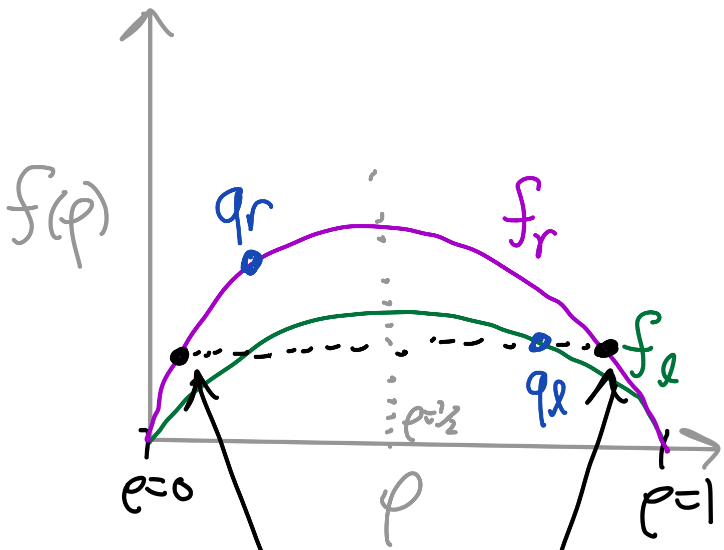
Shock adjacent $\rho_l$: $\lambda'(\rho_l) > s > \lambda'(\rho_*) \Rightarrow \rho_* > \rho_l$

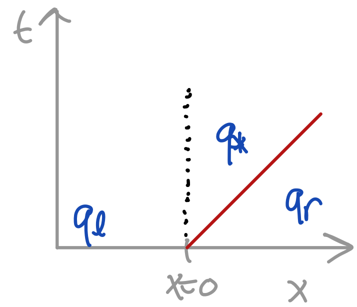
Shock adjacent $\rho_r$: $\lambda'(\rho_*) > s > \lambda'(\rho_r) \Rightarrow \rho_r > \rho_*$

Stationary 2-wave:

Flux must be continuous

$$f(\rho(0^-,t),V_l) = f(\rho(0^+,t),V_r)$$




$q_*$ is one of these

which one?