Scalar nonlinear conservation laws

LWR Traffic flow

$\rho(x,t)$: density of cars

$u(x,t)$: velocity

$\rho = 1$: road is full

speed limit $= 1$

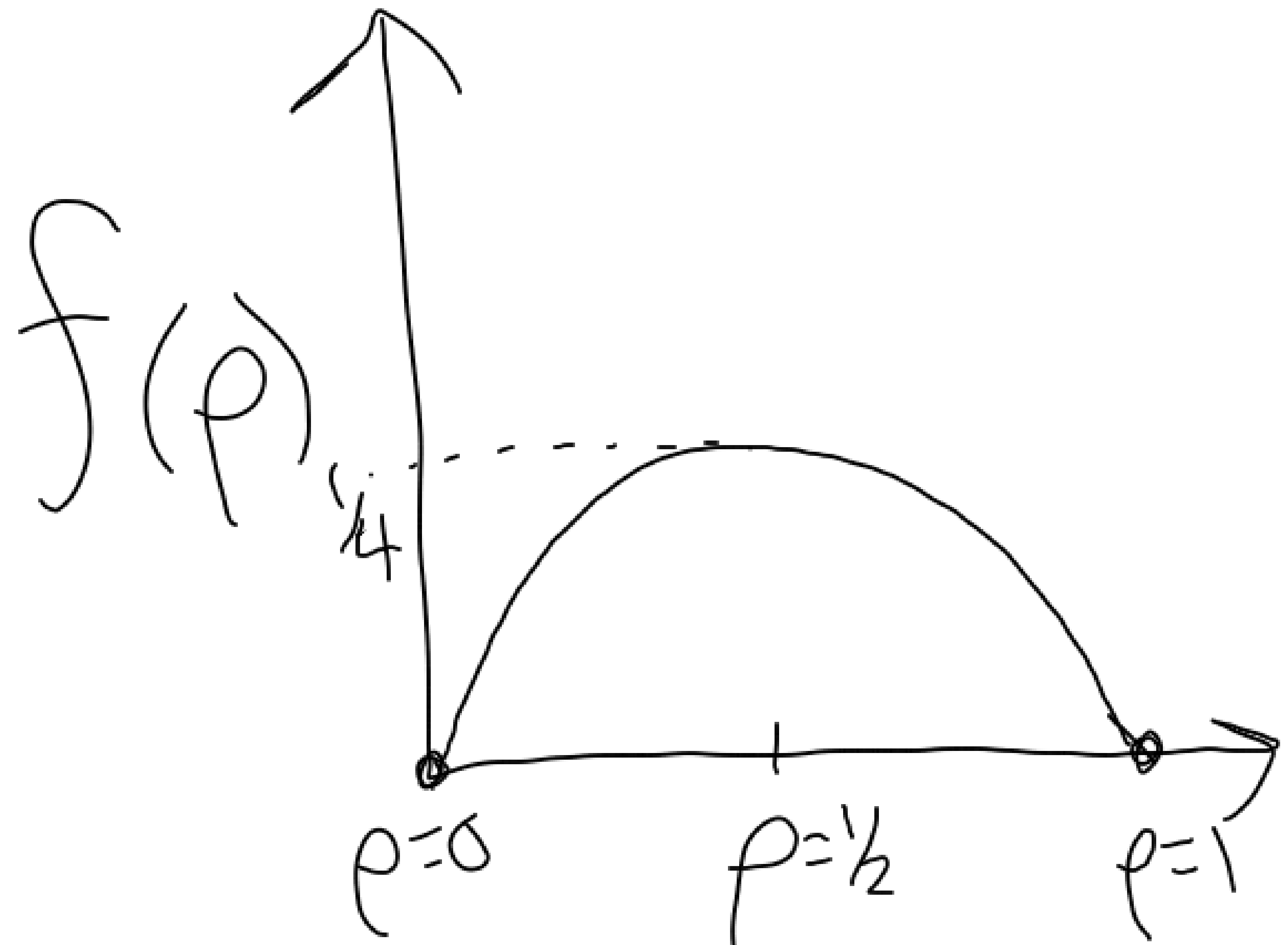
$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u(\rho) = 1 - \rho \end{cases}$$

$$\rho_t + (\underbrace{\rho - \rho^2}_{f(\rho)})_x = 0 \qquad f'(\rho) = 1 - 2\rho$$

$$\rho_t + (1 - 2\rho)\rho_x = 0$$

Quasilinear form

Characteristic speed: $1 - 2\rho = c(\rho)$



Cauchy IVP

$$\rho_t + f(\rho)_x = 0$$

$$\rho(x, t=0) = \rho_0(x)$$

$$X(t) = x_0 + ct$$

$$= x_0 + (1 - 2\rho(X(t), t))t$$

$$\frac{d}{dt}\rho(X(t), t) = \frac{\partial}{\partial x}\rho\, X'(t) + \frac{\partial}{\partial t}\rho$$

$$\rho_t + (1 - 2\rho(X(t), t))\rho_x = 0$$

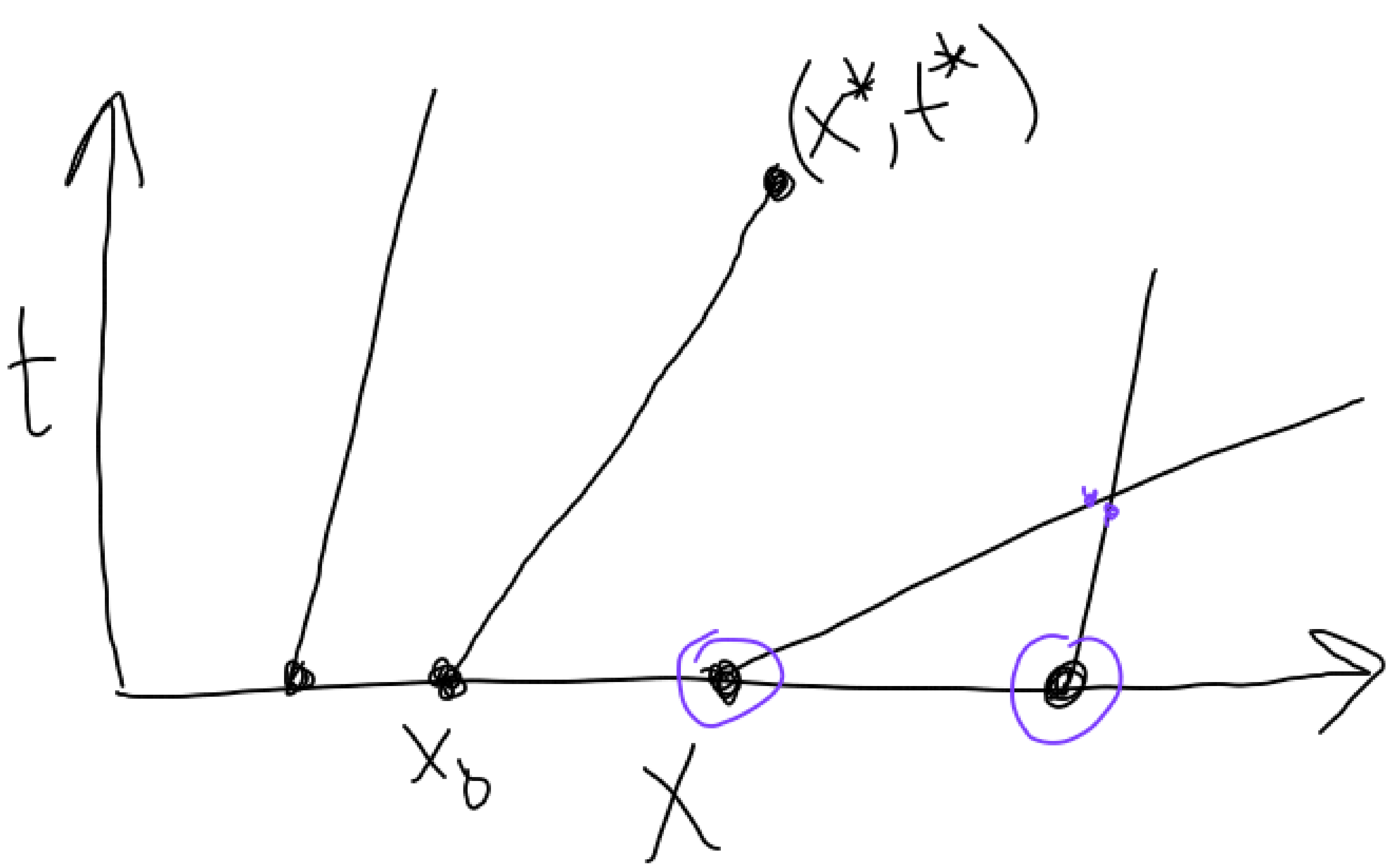
Along each characteristic, ρ is constant.

So characteristics are straight lines.

We can solve the IVP using characteristics.

Given x^*, t^* we need to find x_0 such that

$$x^* = x_0 + (1 - 2\rho_0(x_0))t^*$$

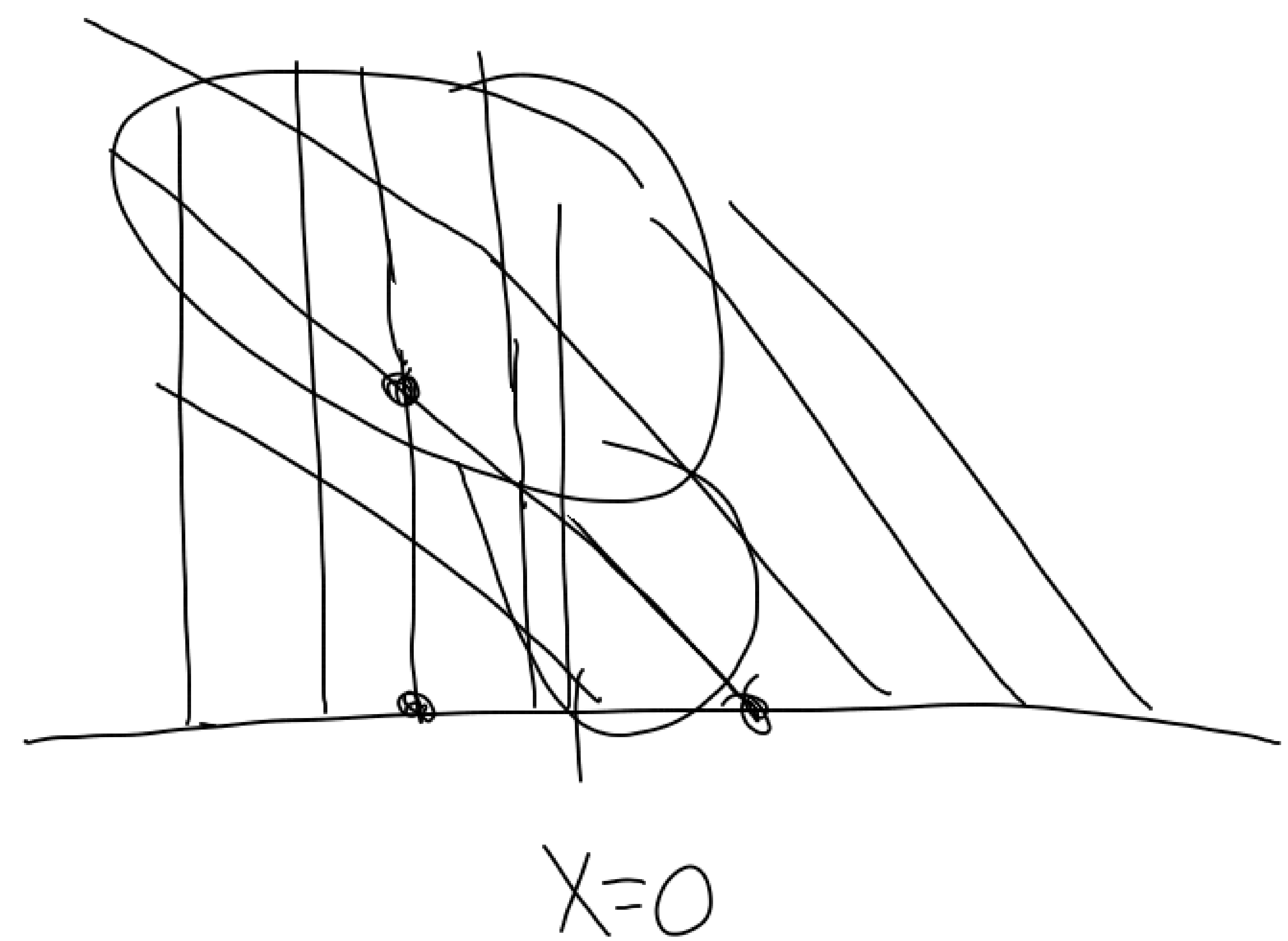


Traffic jam

$$\rho_0(x) = \begin{cases} \frac{1}{2} & x < 0 \\ 1 = \rho_r & x > 0 \end{cases}$$

$c(\rho_\ell) = 0$

$c(\rho_r) = -1$



Starting from $\rho_0(x)$ continuous, we can solve by characteristics up to some time (when they first cross). This solution satisfies

$$(1) \quad \rho_t + f(\rho)_x = 0.$$

(strong solution)

After chars. cross, there is no strong solution so we must consider weak solutions.

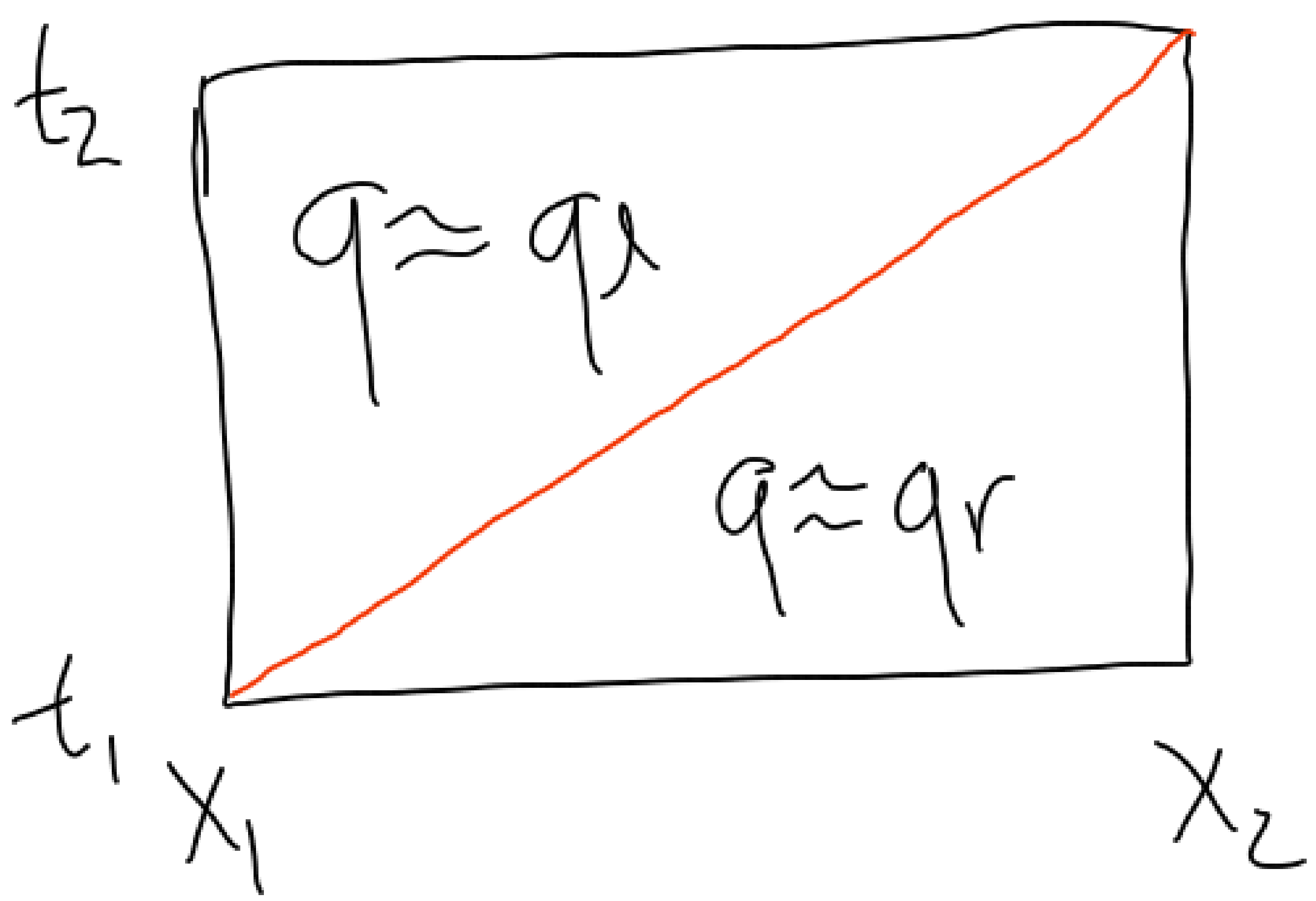
We say $\rho(x,t)$ is a weak solution of the IVP if it satisfies (1) almost everywhere and satisfies the Rankine-Hugoniot jump conditions at each point of discontinuity.

Suppose that our solution contains discontinuities moving along continuous paths in the x-t plane.

R-H Jump Conditions

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} \left(q_t + f(q)_x\right) dt\,dx = 0$$

(x_1,t_1) and (x_2,t_2) are two nearby pts. along the path of a shock.



$$\Delta x = x_2 - x_1$$

$$\Delta t = t_2 - t_1$$

$$s = \frac{\Delta x}{\Delta t}$$

$$\int_{x_1}^{x_2} \left(q(x,t_2) - q(x,t_1)\right) dx + \int_{t_1}^{t_2} \left(f(q(x_2,t)) - f(q(x_1,t))\right) dt$$

$$\Delta x \left(q_l - q_r\right) + \Delta t \left(f(q_r) - f(q_l)\right) = 0$$

$$f(q_r) - f(q_l) = \frac{\Delta x}{\Delta t}\left(q_r - q_l\right)$$

$$\left. f(q_r) - f(q_l) = s\left(q_r - q_l\right) \right\} \text{RH Jump Condition}$$

$$[f] = s[q]$$

Green light Problem

$$\rho_0(x) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}$$

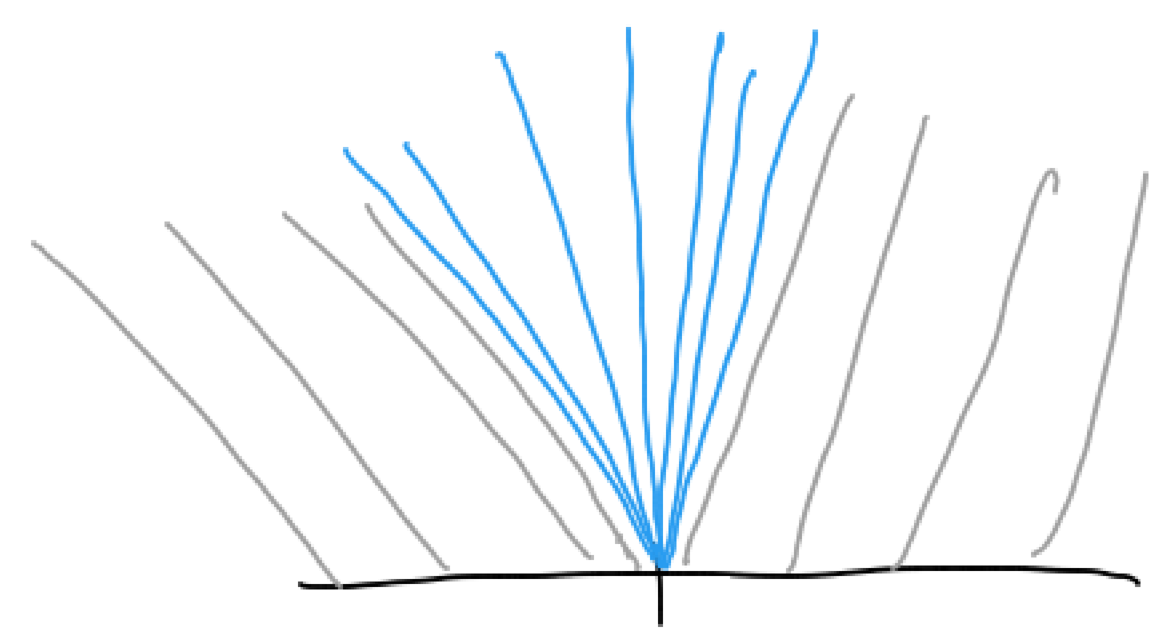
One weak solution:

$$\rho(x,t) = \rho_0(x).$$

Characteristics: $c = 1-2\rho$

$x=0$

$x=0$



$x=0$

Homework:
11.1, 11.5
FVMHP