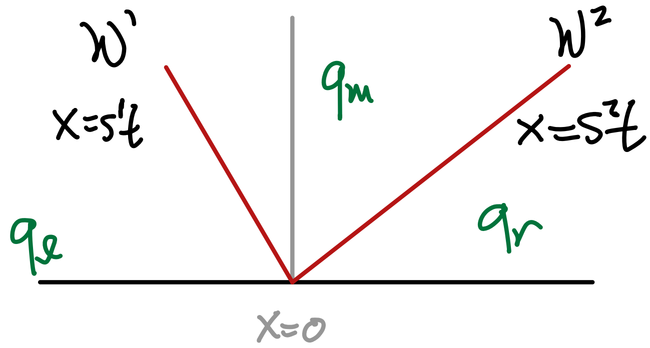
Two-wave Riemann Solvers

Fictitious Riemann solution:



Lump the effect of all waves into just 2.

(A) $\mathcal{W}^1 = q_m - q_\ell$

(B) $\mathcal{W}^2 = q_r - q_m$

Conservation:

(C) $s^1\mathcal{W}^1 + s^2\mathcal{W}^2 = f(q_r) - f(q_\ell)$

Substitute (A), (B) into (C) and solve for q_m:

$$q_m = \frac{f(q_r) - f(q_\ell) + s^1 q_\ell - s^2 q_r}{s^1 - s^2} \qquad (*)$$

For system of M conservation laws, we have

M+2 unknowns

M constraints

We will determine s^1, s^2 and then use $(*)$ to find q_m.

For stability we need s^1 and s^2 to bound the true wave speeds appearing in the Riemann solution.

Note that we don't need eigenvectors.

Lax-Friedrichs / Rusanov

We choose

$$-s^1 = s^2 = \max(|f'(q)|)$$

Lax-Friedrichs: Take max over whole grid (more diffusive)

Local-Lax-Friedrichs: Take max over $q \in [\min(q_\ell, q_r), \max(q_\ell, q_r)]$
(Rusanov)

Harten-Lax-van Leer (HLL)

$$\left.\begin{aligned} s^1 &= \min(f'(q)) \\ s^2 &= \max(f'(q)) \end{aligned}\right\}$$ These always bound the true wave speeds.

For systems, we could look at the eigenvalues:

$$s^1 = \min_q \min_p (\lambda^p(q))$$

where $\lambda^p(q)$ are the eigenvalues of $f'(q)$

$$s^2 = \max_q \max_p (\lambda^p(q))$$

Two-wave solvers for the SW equations

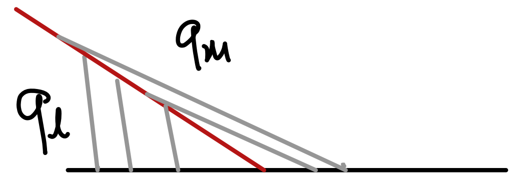
We have

$$\lambda^1 = u - \sqrt{gh} \qquad \lambda^2 = u + \sqrt{gh} \qquad \lambda^1 < \lambda^2$$

So for an HLL solver we could take

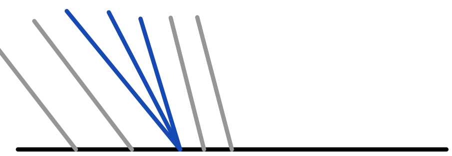
$$s^1 = \min\left(\lambda^1(q_\ell), \lambda^1(q_m)\right)$$

$$s^2 = \max\left(\lambda^2(q_m), \lambda^2(q_r)\right)$$



but we don't know q_m.

If $\lambda^1(q_m) < \lambda^1(q_\ell)$ then we have a 1-shock, so we really want to estimate the shock speed.



We can do this using the Roe average:

$$\hat{h} = \frac{h_\ell + h_r}{2}$$

$$\hat{u} = \frac{u_\ell\sqrt{h_\ell} + u_r\sqrt{h_r}}{\sqrt{h_\ell} + \sqrt{h_r}}$$

$$\begin{cases} s^1 = \min\left(\lambda^1(q_\ell), \hat{u} - \sqrt{g\hat{h}}\right) \\ \\ s^2 = \max\left(\lambda^2(q_r), \hat{u} - \sqrt{g\hat{h}}\right) \end{cases}$$

This choice was proposed by Einfeldt, and the resulting solver is HLLE.

Entropy Glitch and fix

For Burgers equation

$$q_t + \left(\tfrac{1}{2}q^2\right)_x = 0$$

Roe Solver: $\quad q_t + \hat{q} q_x = 0$

To satisfy conservation:

$$\hat{q}(q_r - q_l) = f(q_r) - f(q_l) = \tfrac{1}{2}(q_r + q_l)(q_r - q_l)$$

this gives $\quad \hat{q} = \dfrac{q_r + q_l}{2}$ (the shock speed)

This can lead to a entropy-violating shock.

Entropy fix

If a Roe solver is used, we should check for a transonic rarefaction.
In that case, we split the wave into two:

$$W^1 = q_m - q_l \qquad s^1 = \frac{q_l + q_m}{2}$$

$$W^2 = q_r - q_m \qquad s^2 = \frac{q_m + q_r}{2}$$

Here q_m should be the rarefaction state along $x=0$.

For Burgers, $q_m=0$ and we have:

$$W^1 = -q_\ell \qquad s^1 = \frac{q_\ell}{2}$$

$$W^2 = q_r \qquad s^2 \; \frac{q_r}{2}$$