$$p = P(\rho) \Rightarrow \rho = P^{-1}(p)$$

$$\rho_t = (P^{-1})'(p)\, p_t \qquad 0 = \rho_t + (\rho u)_x = (P^{-1})'(p)\, p_t + (P^{-1})'(p)\, p_x u + \rho u_x$$

$$\rho_x = (P^{-1})'(p)\, p_x$$

$$\Rightarrow p_t + p_x u + \frac{\rho}{(P^{-1})'(p)} u_x = 0$$

$$P^{-1}(P(\rho)) = \rho$$

$$\Rightarrow (P^{-1})'(\underbrace{P(\rho)}_{p})\, P'(\rho) = 1$$

$$\Rightarrow (P^{-1})'(p) = \frac{1}{P'(\rho)}$$

$$0 = (\rho u)_t + (\rho u^2 + p)_x$$

$$= \underline{\cancel{\rho_t} u} + \rho u_t + \underline{\cancel{(\rho u)_x} u} + \rho u u_x + p_x$$

$$= \rho u_t + \rho u u_x + p_x$$

$$u_t + u u_x + \frac{1}{\rho} p_x = 0$$

22 b) $\rho_t + u\rho_x + \rho P'(\rho) u_x = 0$

$u_t + \frac{1}{\rho} P_x + u u_x = 0$

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \underbrace{\begin{bmatrix} u & \rho P'(\rho) \\ \frac{1}{\rho} & u \end{bmatrix}}_{A} \begin{bmatrix} u \\ \rho \end{bmatrix}_x = 0 \Rightarrow \det(A - \lambda I) = 0$$

$$\hookrightarrow \begin{vmatrix} u-\lambda & \rho P'(\rho) \\ 1/\rho & u-\lambda \end{vmatrix} = 0$$

$$(u-\lambda)^2 - P'(\rho) = 0$$

$$(u-\lambda) = \pm\sqrt{P'(\rho)}$$

$$u \pm \sqrt{P'(\rho)} = \lambda_{1,2}$$

$$\boxed{P'(\rho) > 0}$$

$$\begin{bmatrix} u & 0 \\ 1/\rho & u \end{bmatrix}$$

$$\lambda = u$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$u v_1 = \lambda v_1$$

$$\frac{1}{\rho} v_1 + u v_2 = \lambda v_2$$

$$\frac{1}{\rho} v_1 = 0$$ Not hyperbolic

Let $P'(\rho)=0$ but $p=p(x,t)$

Then we get

$$p_t + u p_x = 0$$

$$u_t + \frac{1}{\rho} p_x + u u_x = 0$$

$$\lambda = u_0 \pm \sqrt{\frac{K_0}{\rho_0}}$$

$$q_t + \underline{f(q)}_x = 0$$

$$f(q) = \begin{bmatrix} -v \\ p(v) \end{bmatrix}$$

$$P'(q_0) = \begin{bmatrix} 0 & -1 \\ p'(V_0) & 0 \end{bmatrix}$$

$$p(V) = a^2/V$$

$$p'(V) = -\frac{a^2}{V^2}$$

$$\lambda_1 + \lambda_2 = 0$$

$$\lambda_{1,2} = \pm \frac{a}{V_0}$$

$$X_t(\xi, t)$$

$$\lambda_1 \lambda_2 = p'(V_0)$$

$$\tilde{\lambda}_{1,2} = u_0 \pm \frac{a}{V_0}$$

$$= U(\xi, t)$$

$$\lambda_{1,2} = \pm\sqrt{-p'(V_0)}$$

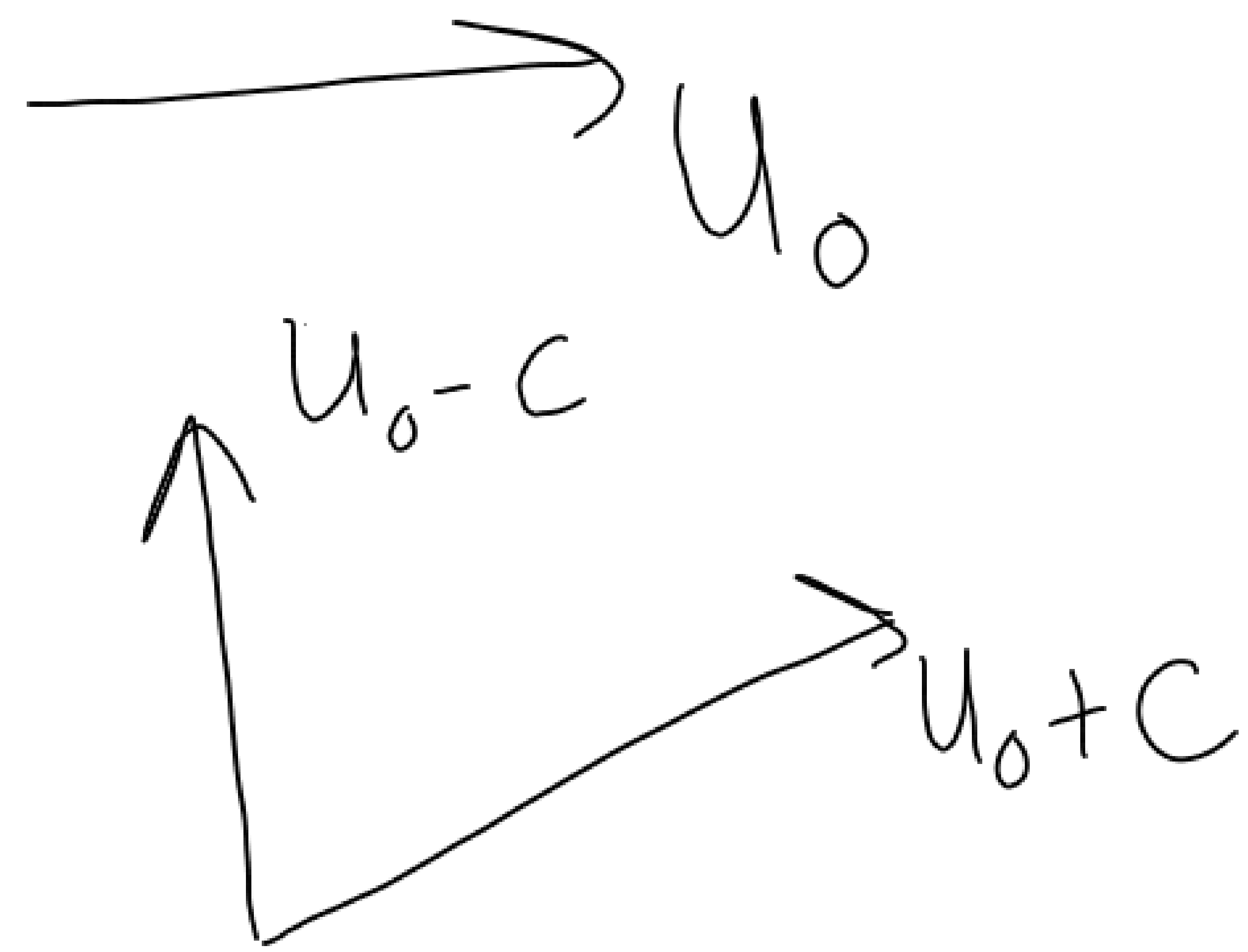
$$U(\xi, t) = u(X(\xi, t), t)$$

$$P = \frac{a^2}{V} \qquad V = \frac{1}{\rho}$$

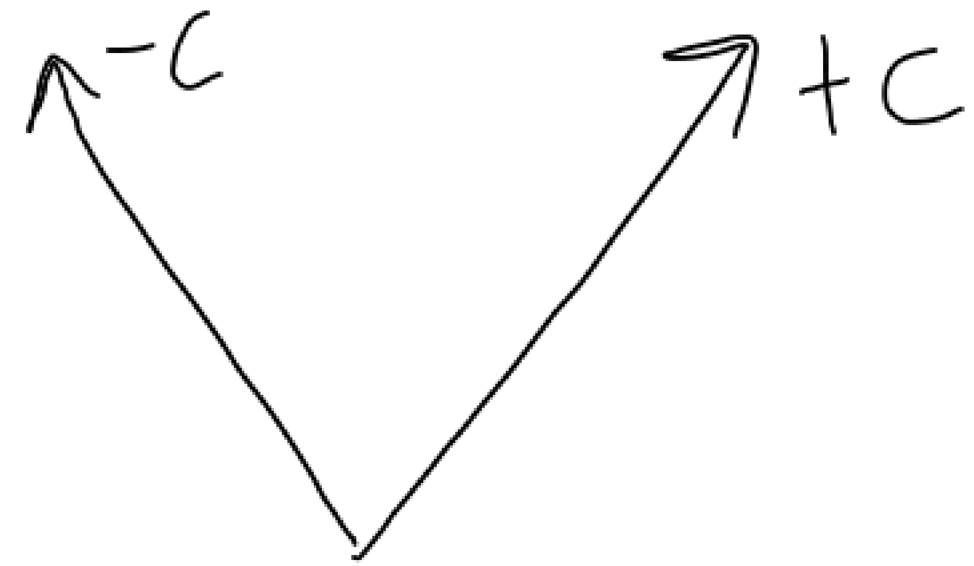
$$P = a^2 \rho$$

$$P'(\rho) = a^2$$

Eulerian



Lagrangian



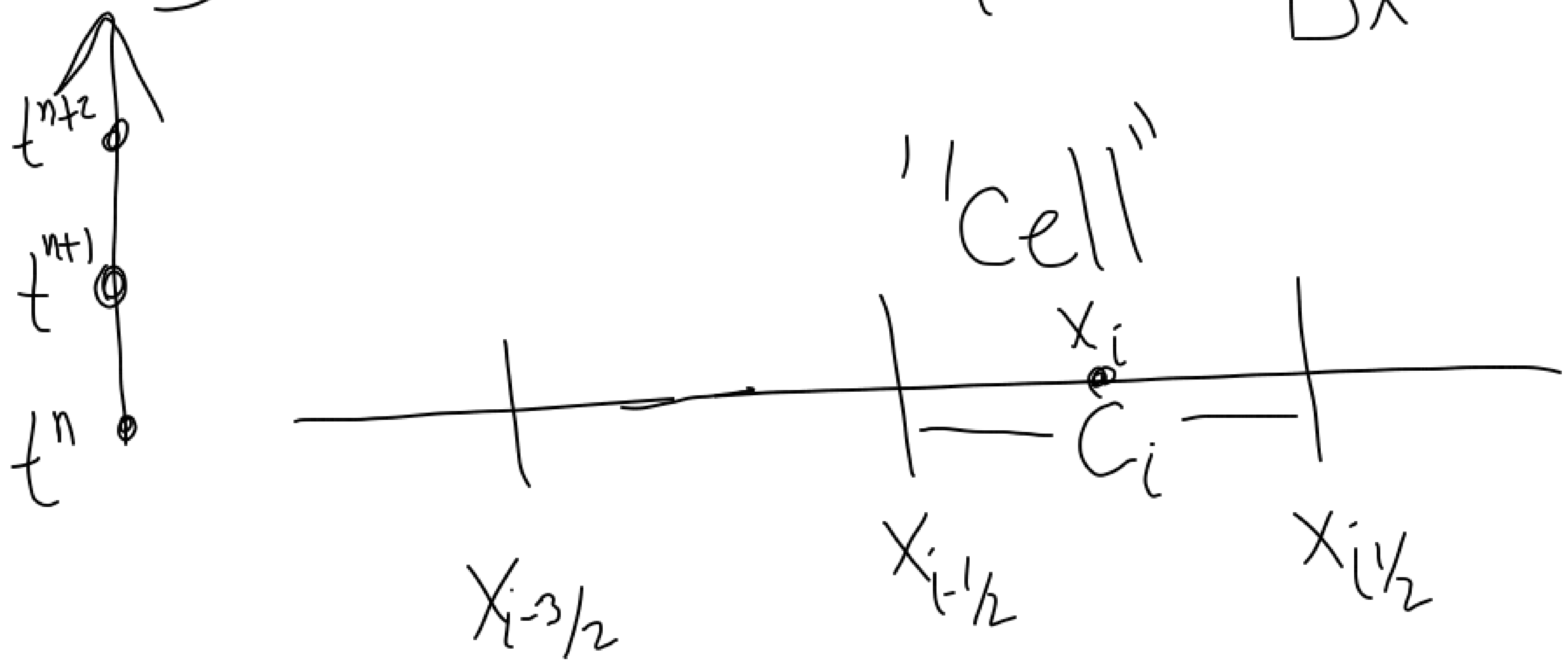
$$\frac{d\xi}{dx} = \rho$$

Finite Volume Methods

FD FV FE
DG

All make use of:
- Riemann solvers
- Limiters

Discretization: $\Delta x = x_{i+1/2} - x_{i-1/2}$

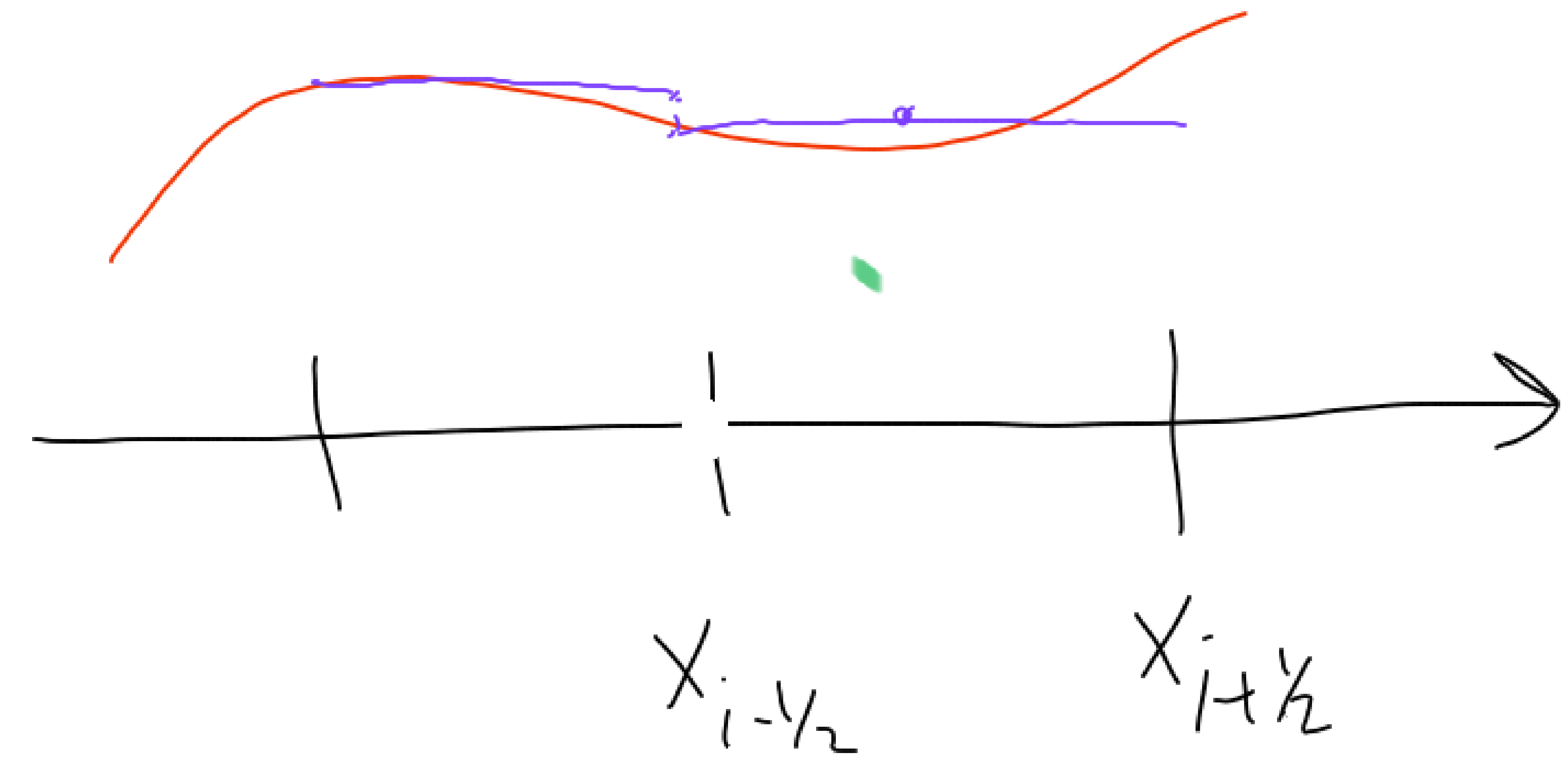


Degrees of freedom:

Cell averages $\displaystyle \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t^n)\,dx = Q_i^n$

$$q_t + f(q)_x = 0$$

Integrate conservation law over C_i and $[t^n, t^{n+1}]$.



$$\frac{Q_i^{n+1}-Q_i^{n}}{\Delta t} = -\frac{F^n_{i+1/2}-F^n_{i-1/2}}{\Delta x}$$

$$\longrightarrow \quad Q_i^{n+1} = Q_i^{n} - \frac{\Delta t}{\Delta x}\left[F^n_{i+1/2}-F^n_{i-1/2}\right]$$

$$F^n_{i+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(q(x_{i+1/2},t))\,dt$$

Notice that

$$\sum_i Q_i^{n+1} = \sum_i Q_i^{n} + F^n_{Left} + F_{Right}$$

Conservative discretization

Fluxes at domain boundary

In practice we need to approximate these

$$F^n_{i+1/2} \approx \mathcal{F}(Q^n_{i-1}, Q^n_i)$$

Numerical flux

Similarity solution

$$q(x,t) = \tilde{q}\left(\frac{x}{t}\right)$$