Boundary Conditions

Initial-boundary value problem

$$q_t + f(q)_x = 0 \quad a \le x \le b$$

$$q(x, t=0) = \mathring{q}(x)$$

Boundary conditions at $x=a, x=b$

Advection

$$q_t + \bar{u} q_x = 0$$

$$\left.\begin{aligned} &\text{if } \bar{u} > 0: \quad q(a,t) = f_a(t) \\ &\text{if } \bar{u} < 0: \quad q(b,t) = f_b(t) \end{aligned}\right\} \text{upwind boundary}$$

All characteristics passing through the downwind bdy. have value defined by the initial data or upwind boundary.

Linear hyp. system

$$q_t + A q_x = 0$$

$$A = R \Lambda R^{-1}$$

$$\lambda^1, \lambda^2, \ldots, \lambda^n < 0$$

$$\lambda^{n+1}, \ldots \lambda^m > 0$$

$$w_t + \Lambda w_x = 0$$

Boundary cond. at $x=b$ for $w^1, w^2, \ldots, w^n$

B.C. at $x=a$ for $w^{n+1}, \ldots, w^m$

Physical BCs

1. Acoustics with a solid wall (reflecting)

Physical condition: $u(a,t)=0$

What is this in terms of w^1, w^2?

$$w^1 = \frac{Zu - p}{2Z} \qquad w^2 = \frac{Zu + p}{2Z}$$

$$u = w^1 + w^2$$

At $x=a$, we can only specify w^2.

$$w^2(a,t) = -w^1(a,t)$$

Similarly, at $x=b$ we would impose

$$w^1(b,t) = -w^2(b,t)$$

What is this in terms of the physical values p, u?

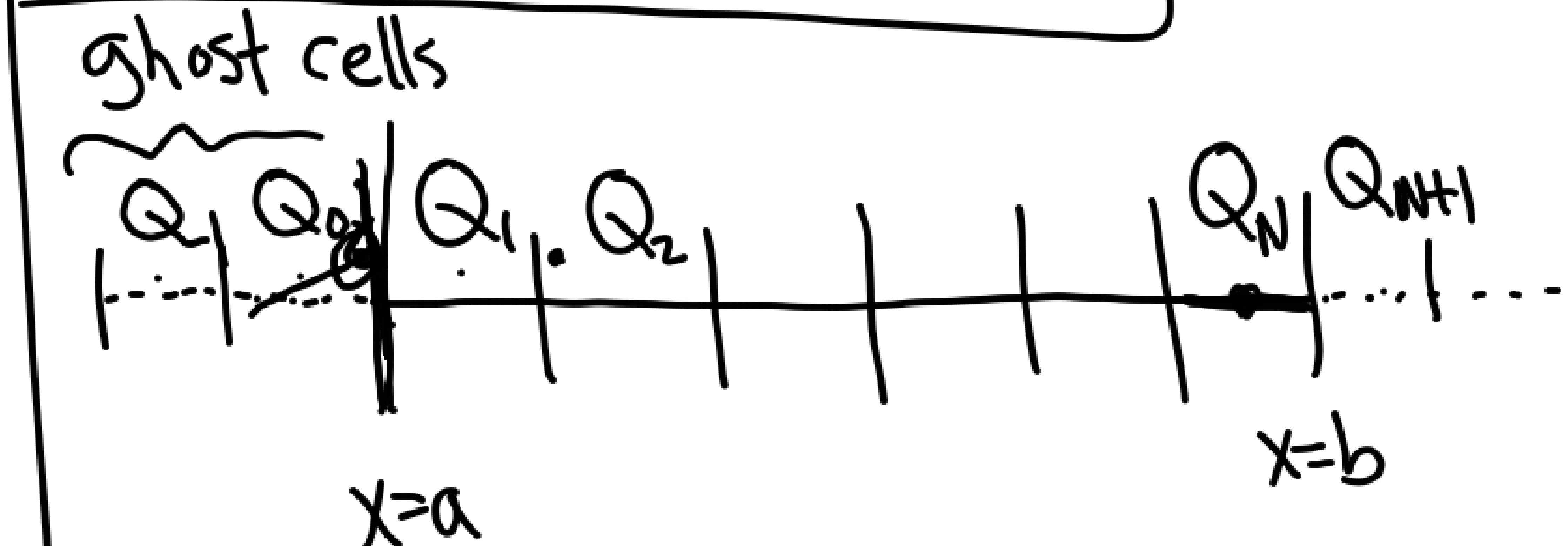
$$u(a,t) = 0$$

$$p(a,t) = Z(w^2 - w^1) = -2Z w^1(a,t)$$

$$u(b,t) = 0$$

$$p(b,t) = 2Z w^2$$

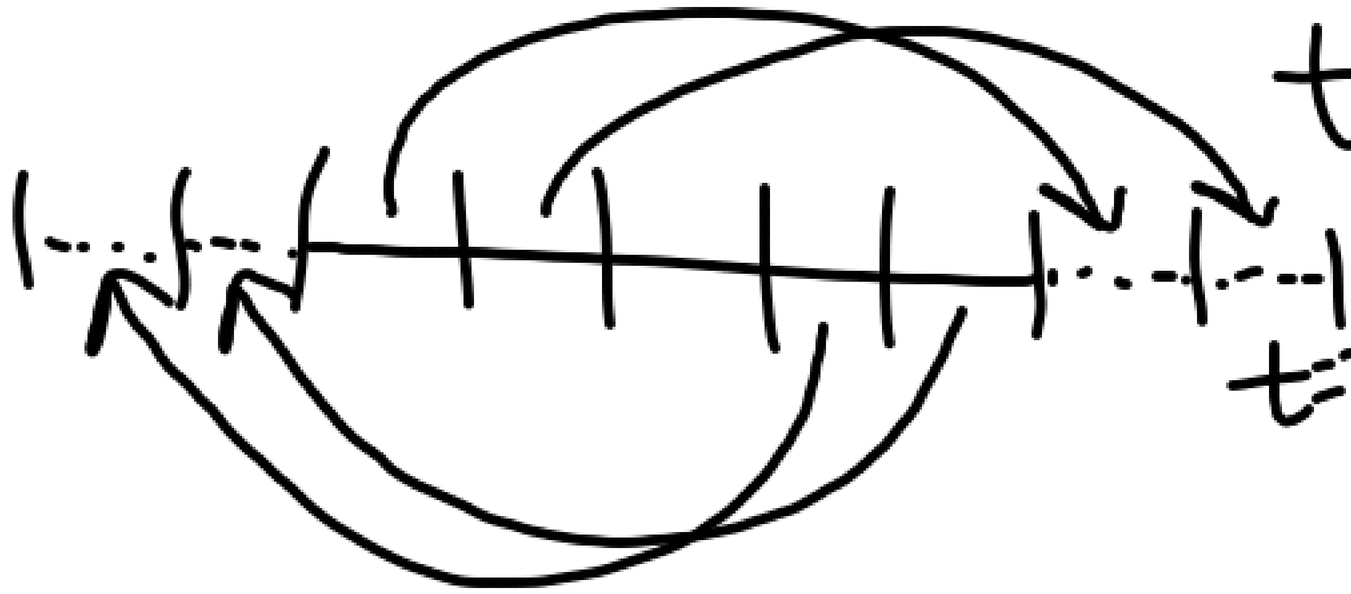
Numerical B.C.s Using ghost cells

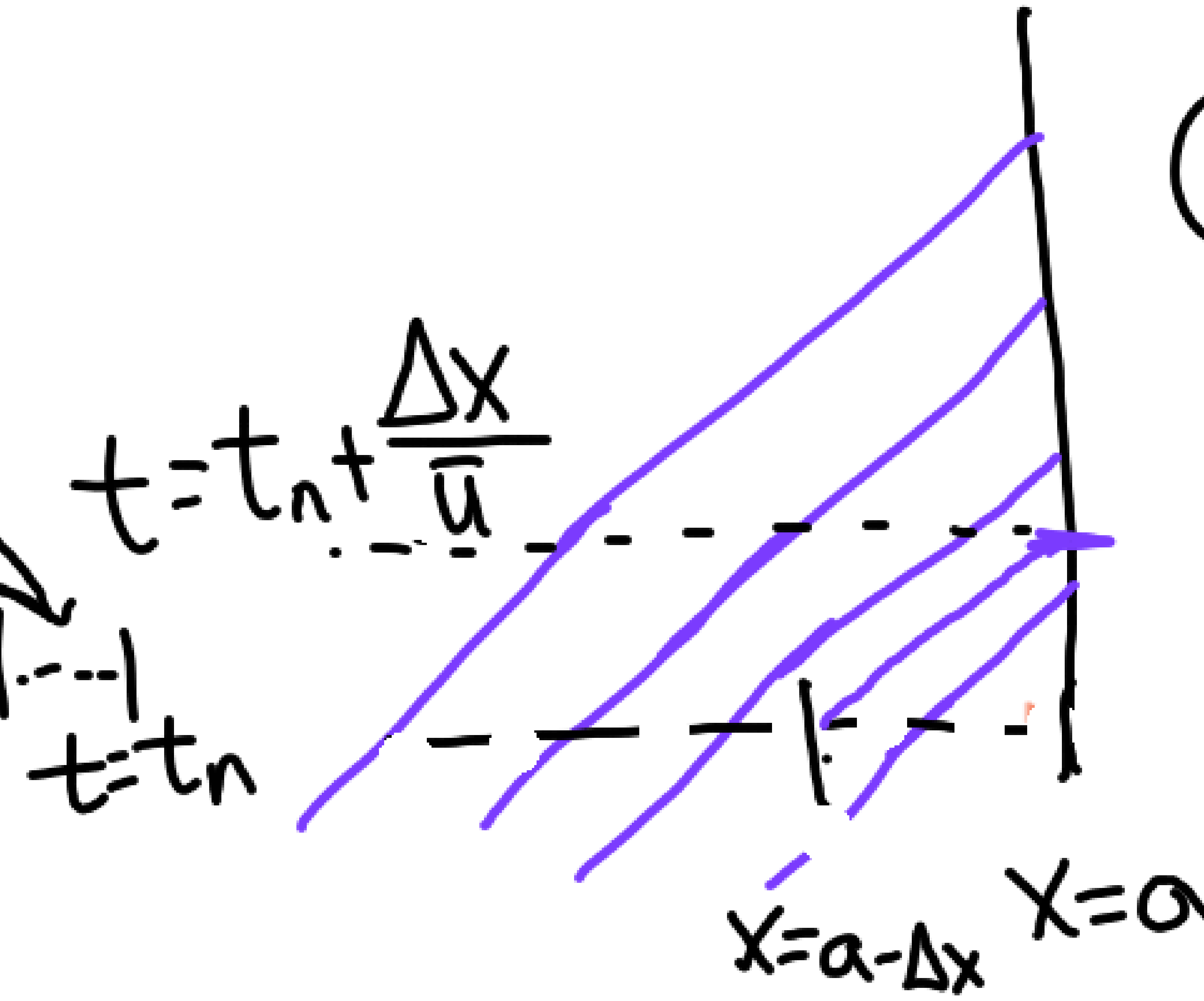


Each time step:

1. We start knowing $Q_1^n, \ldots, Q_N^n$
2. Use BCs to set ghost cell values
3. Use numerical method to compute $Q_1^{n+1}, \ldots, Q_N^{n+1}$

Periodic BCs





$$Q_0^n = \frac{\bar{u}}{\Delta x}\int_{t_n}^{t_n + \frac{\Delta x}{\bar{u}}} f_a(t)\,dt \Bigg\}\text{ High order accurate}$$

Advection

$\bar{u} > 0$ $\quad q(a,t) = f_a(t)$

$$\left.\begin{aligned} Q_0^n &= f_a(t_n) \\ Q_{-1}^n &= f_a(t_n) \end{aligned}\right\}\text{simple}$$

MOL

$$Q_i'(t) = -\frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$

Outflow

At the downwind we have no mathematical BC but still need to fill ghost cells.

We want to choose these values so there are no incoming waves.

Acoustics

Pure outflow: e.g. $w^2 = 0$
w^1: outflow

Standard approach: zero-order extrapolation

$x=a$

Outflow + inflow:

$$w^2 = f_a(t)$$

w^1: outflow

Use zero-order extrap. for w^1

Dirichlet for w^2