Advection — Scalar, linear

$\downarrow$

Acoustics — Linear const.-coeff. system

$\downarrow$

Variable-coefficient acoustics — Linear var.-coeff. system

Acoustics Riemann Problem

$$(p(x,0), u(x,0)) = \begin{cases} \begin{bmatrix} p_l \\ u_l \end{bmatrix} & x<0 \\ \begin{bmatrix} p_r \\ u_r \end{bmatrix} & x>0 \end{cases}$$

$$q_t + Aq_x = 0$$

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad A = \begin{bmatrix} 0 & K \\ 1/\rho_0 & 0 \end{bmatrix}$$

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}(q_t + Aq_x)\,dx\,dt = 0$$

$x_2 - x_1 = \Delta x$

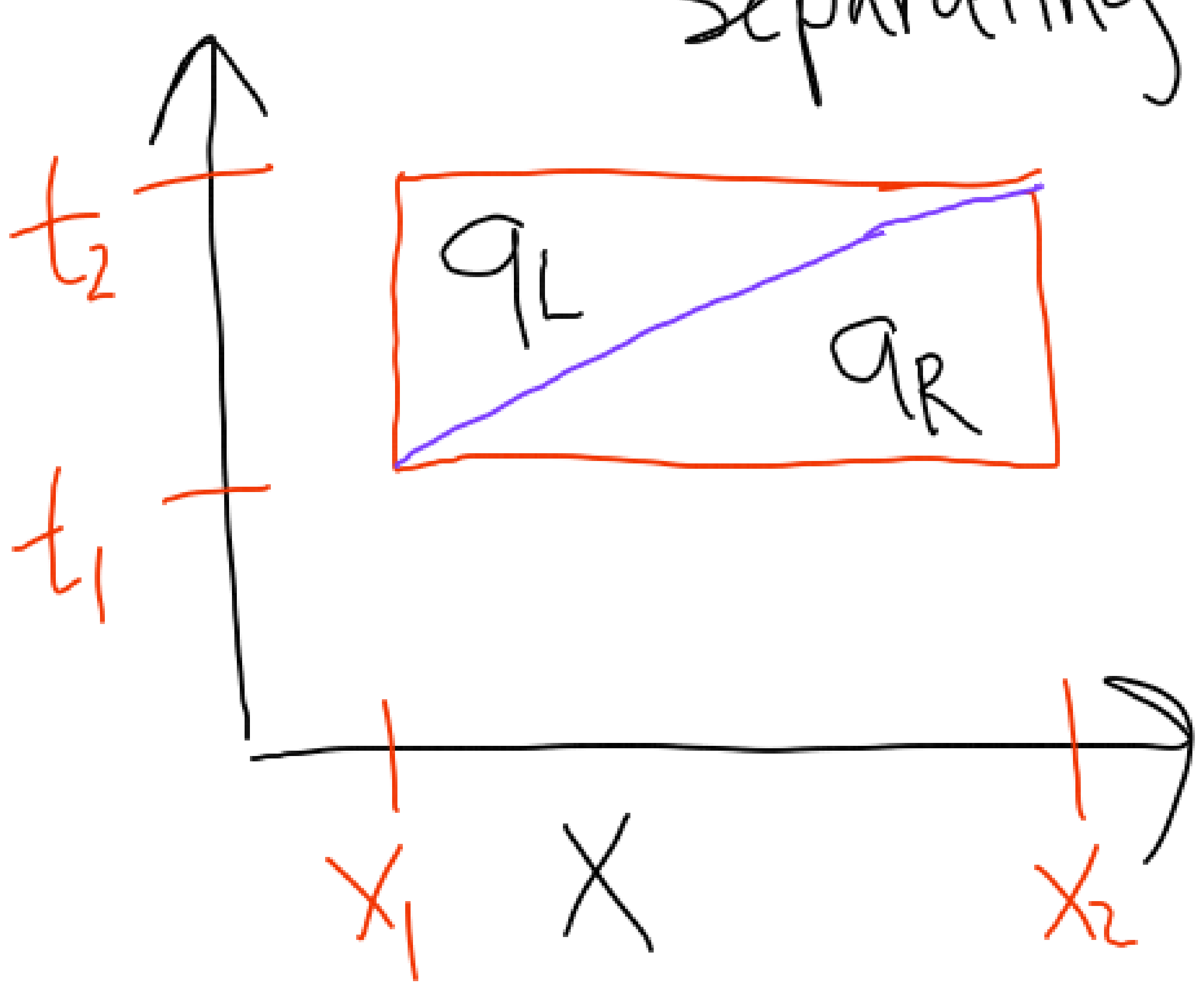
$t_2 - t_1 = \Delta t$

$q_R - q_L = \Delta q$

$[q]$

$$\int_{x_1}^{x_2}\big(q(x,t_2) - q(x,t_1)\big)\,dx + \int_{t_1}^{t_2}\big(Aq(x_2,t) - Aq(x_1,t)\big)\,dt = 0$$

Consider a traveling discontinuity separating states q_L, q_R



We have

$$\Delta x q_L - \Delta x q_R + \Delta t A q_R - \Delta t A q_L = 0$$

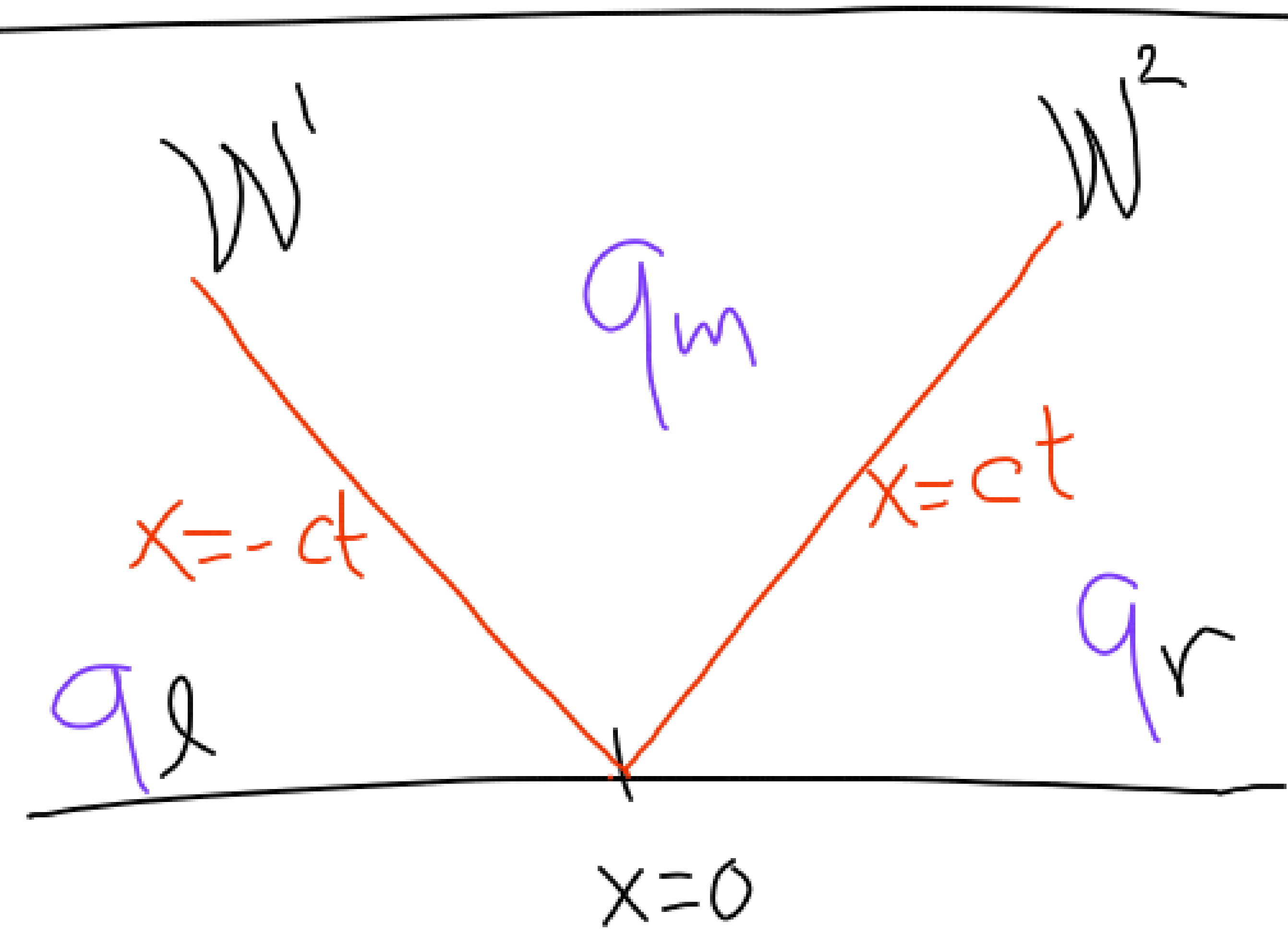
$$-\Delta x \Delta q + \Delta t A \Delta q = 0$$

$$A\Delta q = \frac{\Delta x}{\Delta t}\Delta q$$

Δq is an eigenvector of A

$s = \frac{\Delta x}{\Delta t}$ is the e-value

Solution to the Riemann Problem



We must have

$$q_m - q_\ell = \alpha^1 \begin{bmatrix} -Z \\ 1 \end{bmatrix} = \alpha^1 r^1$$

$$q_r - q_m = \alpha^2 \begin{bmatrix} Z \\ 1 \end{bmatrix} = \alpha^2 r^2$$

Decompose $q_r - q_\ell = \alpha^1 r^1 + \alpha^2 r^2$

Then

$$q(x,t) = \begin{cases} q_\ell & x < -ct \\ q_m = q_\ell + \alpha^1 r^1 = q_r - \alpha^2 r^2 & -ct < x < ct \\ q_r & x > ct \end{cases}$$

Variable-coefficient acoustics

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho_0(x) & 0 \end{bmatrix} \quad q = \begin{bmatrix} p \\ u \end{bmatrix}$$

$$q_t + A(x) q_x = 0$$

Assuming $K(x) > 0$, $\rho_0(x) > 0$ $\forall x$, this is hyperbolic.

$$A(x) = R(x) \Lambda(x) R^{-1}(x)$$

$$\Lambda(x) = \begin{bmatrix} -c(x) & \\ & c(x) \end{bmatrix}$$

$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}$$

$$c(x) = \sqrt{\frac{K(x)}{\rho(x)}} \quad Z = \sqrt{K(x)\rho(x)}$$

$$q_t + R(x)\Lambda(x)R^{-1}(x)q_x = 0$$

$$R^{-1}(x)q_t + \Lambda(x)R^{-1}(x)q_x = 0$$

$$w = R^{-1}(x)q \qquad q = Rw$$

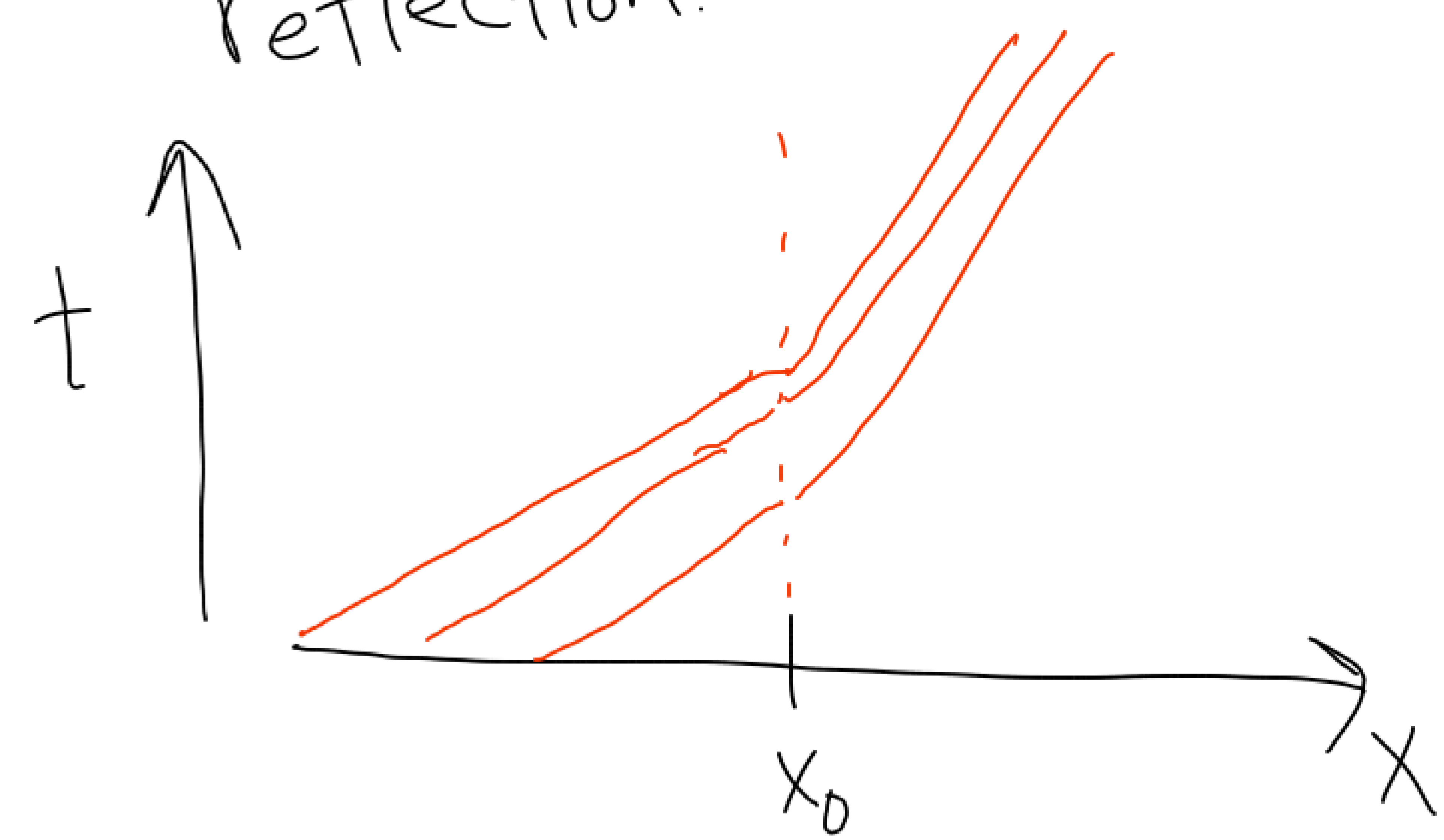
$$w_t = R^{-1}(x)q_t$$

$$w_x = R^{-1}_x(x)q + R^{-1}(x)q_x$$

$$w_t + \Lambda(x)w_x = \Lambda R^{-1}_x(x)R(x)w$$

The left- and right-going components are coupled whenever $R(x)$ is not constant.

If $Z(x)$ is constant then so is $R(x)$ and there is no reflection.



$\rho_l = 1$

$c_l = 1 = \sqrt{\frac{K}{\rho}} \Rightarrow K_l = 1$

$\rho_r = 2$

$c_r = \frac{1}{2}$

$Z = \sqrt{K\rho} = \rho c$

$\rho_l = 1$

$c_l = 1$

$Z_l = 1$

$\rho_r = 4$

$c_r = \frac{1}{2}$

$Z_r = 2$

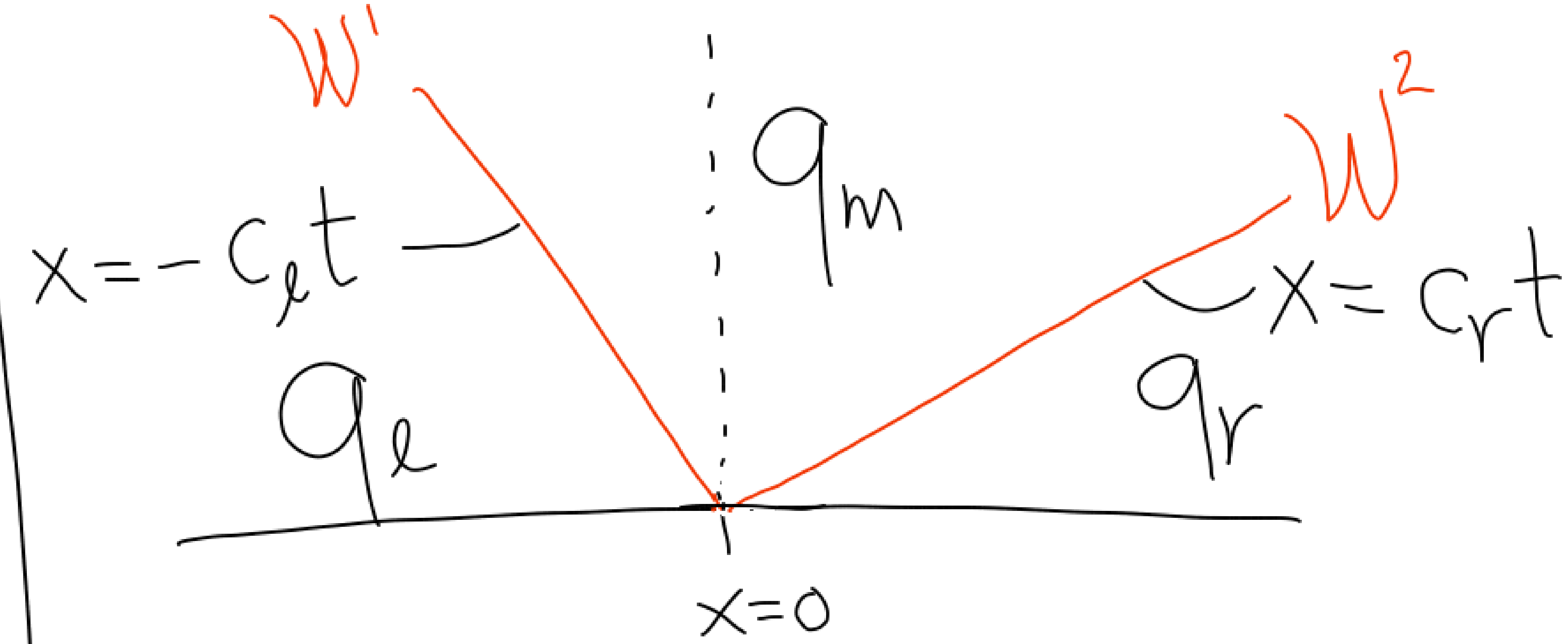
RP for V.C. Acoustics

$$q(x,t=0) = \begin{cases} \begin{bmatrix} p_\ell \\ u_\ell \end{bmatrix} & x<0 \\[2ex] \begin{bmatrix} p_r \\ u_r \end{bmatrix} & x>0 \end{cases}$$

$$(\rho, K) = \begin{cases} (\rho_\ell, K_\ell) & x<0 \\ (\rho_r, K_r) & x>0 \end{cases}$$

$$c_\ell = \sqrt{\frac{K_\ell}{\rho_\ell}} \qquad c_r = \sqrt{\frac{K_r}{\rho_r}}$$

$$Z_\ell = \sqrt{K_\ell \rho_\ell} \qquad Z_r = \sqrt{K_r \rho_r}$$



$$q_m - q_\ell = \alpha^1 \begin{bmatrix} -Z_\ell \\ 1 \end{bmatrix}$$

$$q_r - q_m = \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix}$$

Just need to find α^1, α^2 s.t.

$$q_r - q_\ell = \alpha^1 \begin{bmatrix} -Z_\ell \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix}$$

$$\Delta q = \underbrace{\begin{bmatrix} -Z_\ell & Z_r \\ 1 & 1 \end{bmatrix}}_{R_{\ell r}} \underbrace{\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix}}_{\alpha}$$

$$R_{\ell r}^{-1} \Delta q = \alpha$$

$$R_{\ell r}^{-1} = \frac{1}{Z_\ell + Z_r} \begin{bmatrix} -1 & Z_r \\ 1 & Z_\ell \end{bmatrix}$$

Suppose

$$q_r - q_\ell = \beta \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix}$$

The reflected and transmitted wave amplitudes are given by

$$R_{\ell r}^{-1} \beta \begin{bmatrix} Z_\ell \\ 1 \end{bmatrix} = \beta \left(\frac{1}{Z_\ell + Z_r} \right) \begin{bmatrix} Z_r - Z_\ell \\ 2Z_\ell \end{bmatrix}$$

$$= \beta \begin{bmatrix} \dfrac{Z_r - Z_\ell}{Z_r + Z_\ell} \\[2ex] \dfrac{2Z_\ell}{Z_r + Z_\ell} \end{bmatrix}$$

Reflection Coeff.

Transmission Coeff.