# "Shocking" Wave Phenomena in Periodic Materials

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# Outline

1. Nonlinear Waves in Periodic Media
2. Godunov-Type Methods and Limiters
3. Computational Investigation of Shock Formation
4. Generalizations

# Elasticity in 1D

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Velocity: $u(x, t)$

Density: $\rho$

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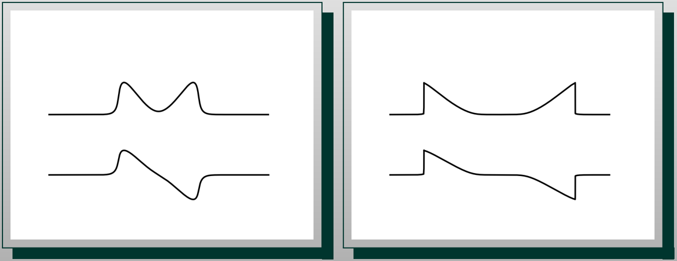
$$\sigma(\epsilon) = K\epsilon \implies \text{linear waves}$$
$$\sigma(\epsilon) = e^{K\epsilon} - 1 \implies \text{nonlinear waves}$$

# Time-Reversibility of Nonlinear Waves



- Waves are time-reversible as long as they remain smooth
- Time-reversibility is lost after shocks form

# Entropy

$$\eta(u,\epsilon) = \frac{1}{2}\rho u^2 + \int_0^\epsilon \sigma(s)ds.$$

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$$\int \eta dx.$$

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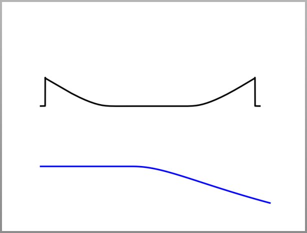

# Dispersive Waves

$$\epsilon_t - u_x = \epsilon u_{xxx}$$
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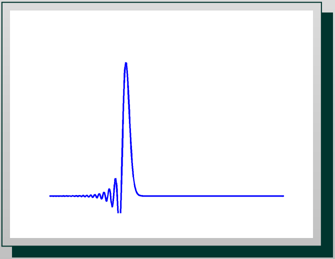
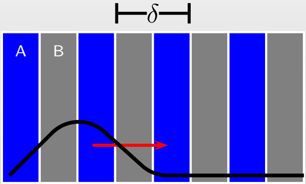


# Linear Elasticity in Periodic Media




$$
\begin{aligned}
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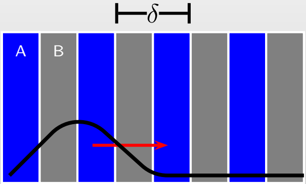
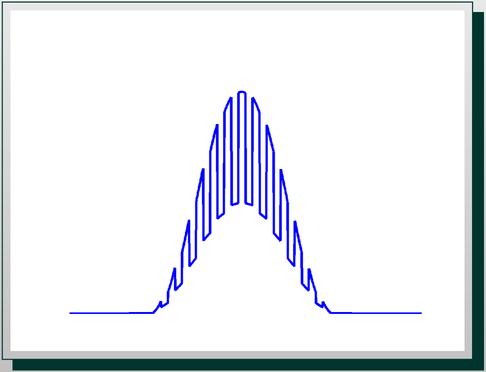



$$\begin{aligned} \epsilon_t - u_x &= 0 \\ \rho(x)u_t - K(x)\epsilon_x &= 0 \end{aligned}$$

$K(x)$ and $\rho(x)$ are piecewise-constant periodic functions
Impedance: $Z(x) = \sqrt{\rho(x)K(x)}$

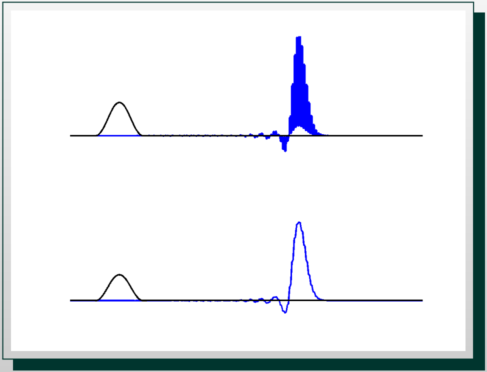
# Impedance Matched Materials

$$\rho_A K_A = \rho_B K_B$$



In this case there are no reflections.

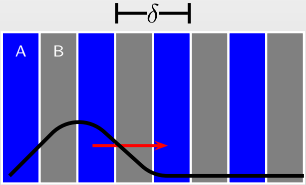
# Impedance mismatch: Effective Dispersion



Animation

- Due to reflections, the pulse travels more slowly than it would in either component material
- The medium introduces **effective** dispersive behavior

# Nonlinear Elasticity in Periodic Media

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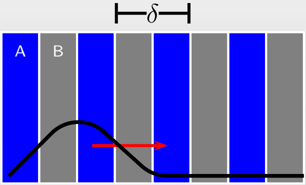


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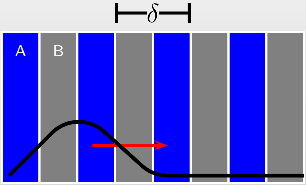


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Impedance: $Z(\epsilon, x) = \sqrt{\rho(x)\sigma_\epsilon(\epsilon, x)}$

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Animation

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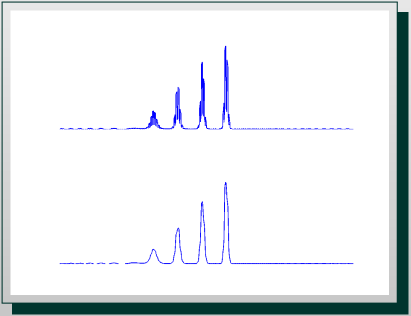
Animation

Nonlinearity dominates $\rightarrow$ shocks form

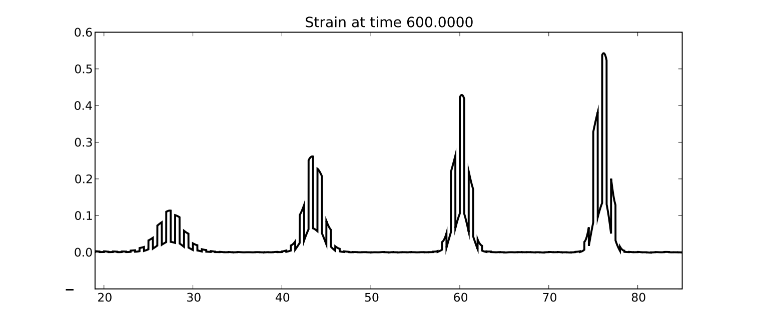
# Higher Impedance Contrast

$$Z_{0,A}/Z_{0,B} \approx 4$$



Close-up

# Stegotons



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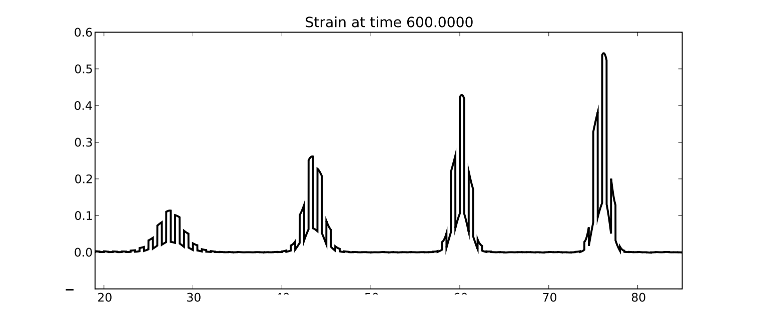
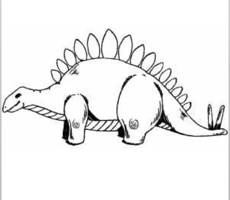


LeVeque & Yong, 2003

# Soliton-like Properties

- All stegotons have same shape (in time-profile) under a rescaling
- Stegotons appear to interact only through a phase-shift

Two stegotons

Close-up

# Outline

# Conservation Laws

Defining

$$\mathbf{q} = \begin{bmatrix} \epsilon \\ \rho u \end{bmatrix}$$

and

$$f(\mathbf{q}, x) = \begin{bmatrix} -u \\ -\sigma(\epsilon, x) \end{bmatrix}$$

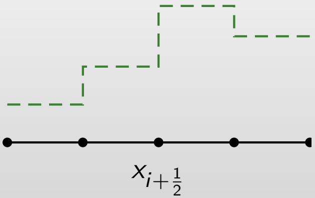
we can write the elasticity equations as a pair of conservation laws:

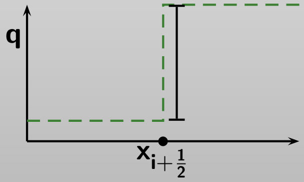
$$\mathbf{q}_t + f(\mathbf{q}, x)_x = 0.$$

# Godunov's Method

- Solution represented by cell averages $Q_i$
- Each cell interface poses a **Riemann problem**
- Riemann problem at $x_{i-\frac{1}{2}}$ is solved to find flux $F_{i-\frac{1}{2}}$
- Flux-differencing:

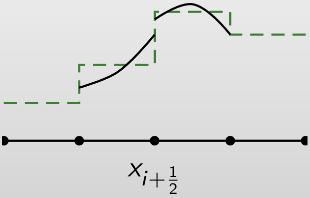
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)$$

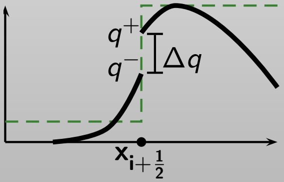


# High order wave propagation

Piecewise-polynomial Reconstruction

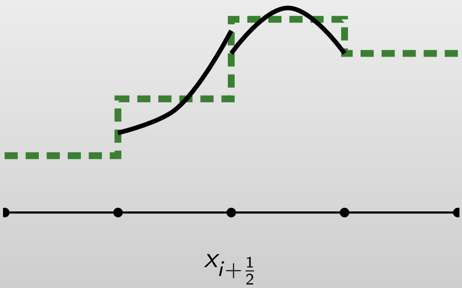



$$\frac{\partial Q_i}{\partial t} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)$$

- High order reconstruction in space
- High order ODE solver in time
- The fluxes are determined from the reconstructed states at each interface.
- Reconstruction must be done in a non-oscillatory manner

# Non-oscillatory Reconstruction




- Limiters are employed to avoid overshoots
- Limiters try to maintain "true" discontinuities while avoiding "spurious" oscillations
- Difficult to achieve the right balance between compression and dissipation

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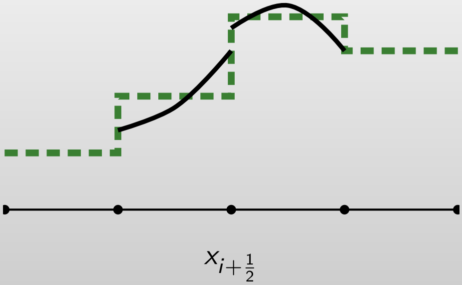


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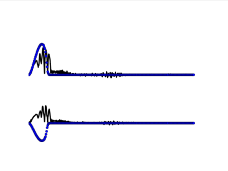
# Outline

# Time-reversibility of Stegotons

$$Z_{0,A}/Z_{0,B} = 2$$



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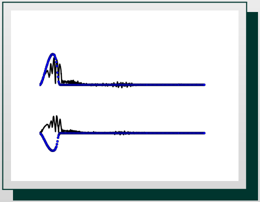

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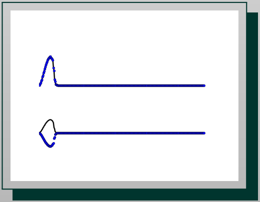

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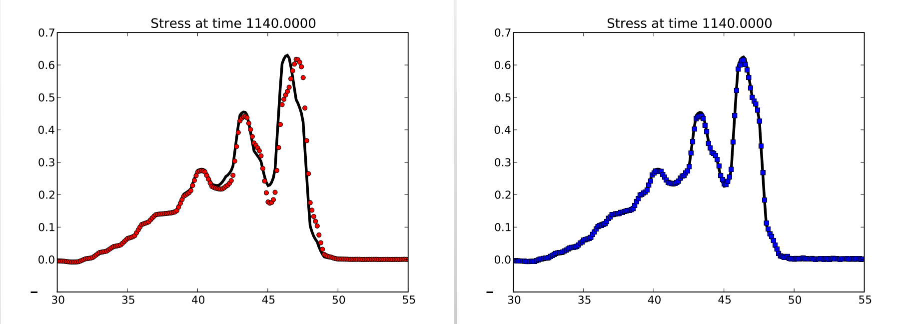



This is a useful test for high order numerical methods:

- Nonlinear, yet solution remains smooth for long times
- Known exact solution (apparently)
- Highlights ability of schemes to deal with spatially varying flux

# Entropy Evolution

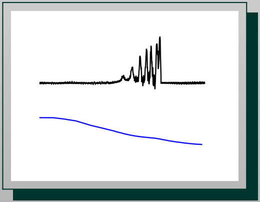
$$\eta(u, \epsilon) = \frac{1}{2}\rho u^2 + \int_0^\epsilon \sigma(s)ds.$$

Total entropy:

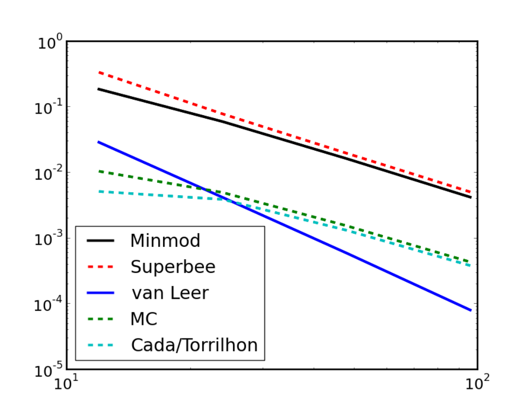
$$\int \eta dx.$$

- Conserved for smooth solutions
- Decreases when shocks are present
- Solutions are time-reversible until shocks form

How does the entropy behave for nonlinear waves in periodic media?



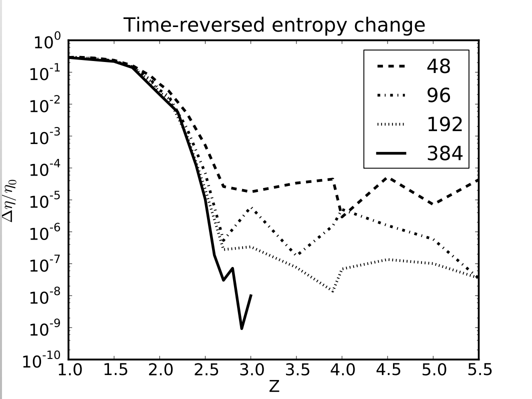
# Entropy and Limiters




- Entropy change seems to be converging to zero with $\Delta x$
- Minmod, van Leer are dissipative (entropy decreases)
- Superbee, monotonized centered, and Cada/Torrilhon are compressive (entropy increases)

# Probing the "phase transition"

- Entropy loss in high-order WENO simulations is even much smaller
- Numerical entropy errors are nearly time-reversible
- This allows us to probe shock formation very precisely



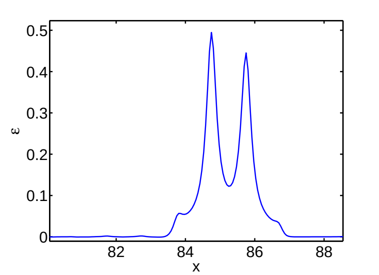
# Outline

# Generalizations

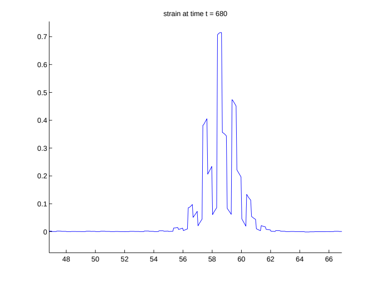
How general is this phenomenon? Is it generic for

- Different nonlinearities?
- Different materials?
- More general hyperbolic systems?

# Different Media




Sinusoidally varying medium

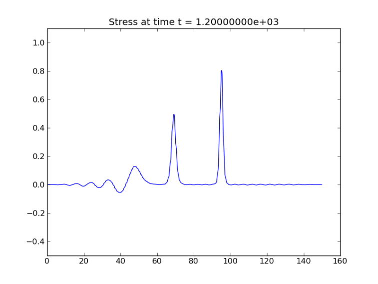



Three-layer medium

# Different Nonlinearities

Cubic nonlinearity:

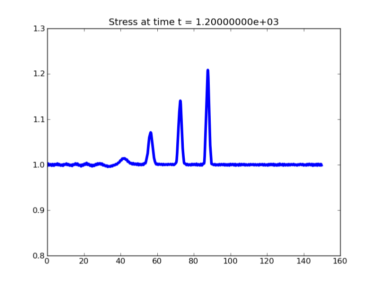
$$\sigma(\epsilon, x) = K_1(x)\epsilon + K_3(x)\epsilon^3$$




Isothermal gas:

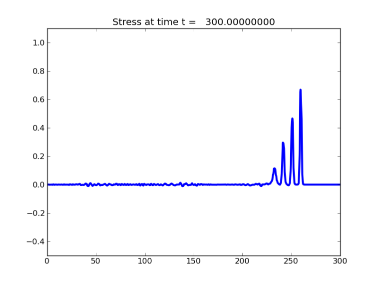
$$\sigma(\epsilon, x) = -\kappa(x)/\epsilon^2$$

$$\rho(x) = 1$$

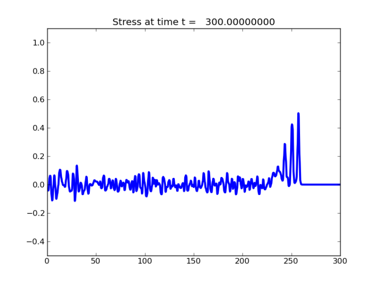


# Randomly perturbed media

Near-periodic medium with randomly perturbed interface locations



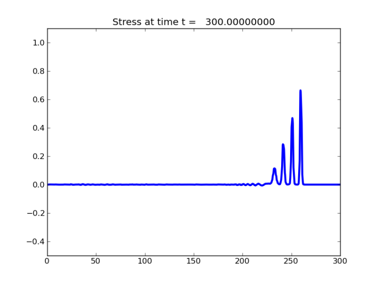

Perturbed by 1%




Perturbed by 10%

# Randomly perturbed media

Near-periodic medium with randomly perturbed piecewise constant coefficients



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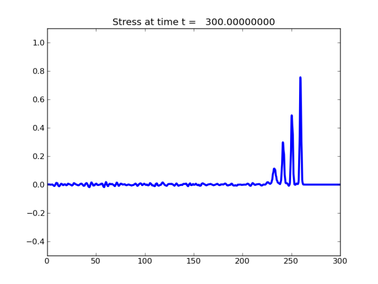
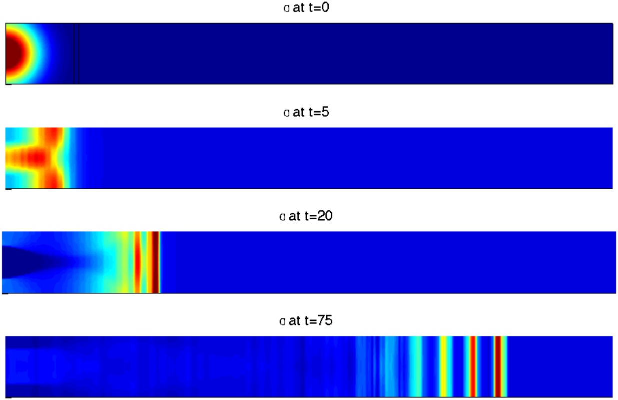


Perturbed by 25%

# Stability in higher dimensions



# Summary of Main Points

- Entropy evolution and time-reversibility allow us to measure the formation of shocks in nonlinear wave simulations
- Periodic materials with high impedance contrast appear to suppress the formation of shocks
- Large-amplitude signals in such materials can travel long distances without information loss
- Studying this phenomenon can also tell us new things about our numerical methods

# Experimental Realization: Optical Stegotons?

- Medium consists of alternating layers of air and some material with high index of refraction and low loss
- High intensity laser pulse
- Currently we are designing simulations to test feasible design parameters

Joint work with Boon Ooi (KAUST).