$[y_1\ y_2]\begin{bmatrix}x_1\\x_2\end{bmatrix} = y_1x_1 + y_2x_2$

$\begin{bmatrix}x_1\\x_2\end{bmatrix}[y_1\ y_2] = \begin{bmatrix}x_1y_1 & x_1y_2\\x_2y_1 & x_2y_2\end{bmatrix}$

Lin dep. $\{\vec{v}_1, \dots, \vec{v}_n\}$ if there exists scalars $\alpha_1, \dots \alpha_n$ s.t. $\alpha_1\vec{v}_1 + \dots + \alpha_n\vec{v}_n = \vec{0}$ and not all α_i's are equal to zero

$\parallel$

an index i st. $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$

square matrices ↘
lin inde & → invertible

can't have
- non-trivial null space
- a zero eigen val
- det = 0
- is not full rank

__Vector Space ($\mathbb{V}, \mathbb{F}$)__

Vector addition

- associative : $\vec{u} + (\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$
- commutative : $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
- additive identity : $\exists\, \vec{0} \in \mathbb{V}$ s.t. $\vec{v}+\vec{0} = \vec{v}$
- additive inverse : for any $\vec{v} \in \mathbb{V}$, $\exists -\vec{v} \in V$ st. $\vec{v} + (-\vec{v}) = \vec{0}$

Scalar Mul

- associative = $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$
- multiplicative identity : $\exists\, 1 \in \mathbb{F}$ where $1 \cdot \vec{v} = \vec{v}$
- distributive in vec addition : $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$
- distributive in scalar addition : $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$

__Subspaces__

- contains $\vec{0}$
- closed under vec addition
- closed under scalar mul

has to be subset of $\mathbb{V}$ in the VS ($\mathbb{V}, \mathbb{F}$)

__Change of basis__

$\vec{u}_a = A^{-1}\vec{u}$

__Nullspace__

x s.t. $Ax = 0$

GE → write variables in terms of free variables

__Eigen__

$A\vec{v} = \lambda\vec{v}$

$\det(\lambda I - A) = 0 = \det(A - \lambda I)$

$A^{-1}\vec{v} = \lambda^{-1}\vec{v}$ and $A^n\vec{v} = \lambda^n\vec{v}$

Rotation Matrix	Reflect across x axis	Reflect across y axis	Reflect across y=x
$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

trace(A) = sum(diagonal(A))

col span = range = col space = pivot cols

rank(A) = dim(span(A)) = # of pivots

$\text{span}(A) = \{\vec{v} \mid \vec{v} = \sum_{i=1}^{m} x_i \vec{a}_i,\ x_i \to \text{scalars}\}$

basis → lin indep vectors

$m - \dim(\text{range}(A)) = \dim(N(A))$

__diagonalization__

- $n \times n$ matrix can be diagonalized if it has n lin inde eigen vec

$T = ADA^{-1}$ ← inverse of eig vec

$\det(A) = 0 \Rightarrow$ lin. dep.

Gaussian elim → upper triangular → find det(A) entries on diagonal are eig vals

$\det\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = ad - bc$

$\text{inv}\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

__Steady state__

use $\lambda = 1$ — negative λ → oscillation

solve for eig vec ⇒ importance score (want to scale to add up to 1)

eig val of 1 and other eig val < 1
↳ has a steady state
↳ blows up otherwise

__Inverses__

$A^{-1}A = AA^{-1} = I$

$(A^{-1})^{-1} = A$

$(kA)^{-1} = k^{-1}A^{-1}$, $k \neq 0$

$(A^T)^{-1} = (A^{-1})^T$

$(AB)^{-1} = B^{-1}A^{-1}$ — A, B invertible

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

Inverses share the same eigenvectors but not necessarily eig val

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ rotation by 180°

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotate 90° counterclockwise

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ rotate 90° clockwise

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ reflect across $y = -x$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Determinants

$\det(I_n) = 1$, $I_n = n \times n$ identity

$\det(A^T) = \det(A)$

$\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$

A, B equal size

$\det(AB) = \det(A)\det(B)$

$\det(cA) = c^n \det(A)$ for $n \times n$ matrix

$\det(A) = \text{prod}(\lambda_i)$

1.) switching rows negates the $\det(A)$

2.) adding a mul of one row to another doesn't affect $\det(A)$

3) Multiplying a row by a constant scales the $\det(A)$ by that constant

eigen val = 0

nontrivial soln in null space

$\det(A) = 0$

More change of basis

$A_B = B^{-1} A B$

$A = B A_B B^{-1}$

Diagonalization

needs to have n eig vals and n eig vectors

$$AB = V D_A D_B V^{-1} = V D_B D_A V^{-1} = BA$$

Diagonal

$$\begin{bmatrix} \lambda_{A,1}\lambda_{B,1} & 0 & \cdots \\ 0 & \lambda_{A,2}\lambda_{B_2} & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} \lambda_{B,1}\lambda_{A,1} & \cdots & 0 \\ 0 & \lambda_{B_2}\lambda_{A_2} & \\ 0 & & \ddots \end{bmatrix}$$

A is square, invertible, $n \times n$

Invertible Matrix Thm.

A is row and column equiv to I

A has n pivot and is full rank

$\det(A) \neq 0$

$Ax = 0$ is trivial, only nullspace $\{0\}$

There is unique soln to $Ax = b$

cols(A) are lin inde and span $\mathbb{R}^n$

A^T is invertible

0 is not an eigenval of A

$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax \\ y \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ ay \end{bmatrix}$

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax \\ by \end{bmatrix}$

Factoring

$ABAA^{-1} - \lambda I AA^{-1}$

$A(BA - \lambda I)A^{-1}$ ← keep relative order

Orthogonality

$\vec{z}$ is orthogonal to $\vec{x}$ if $\vec{z}^T \vec{x} = \vec{0}$

$x_0 = 0$

$x_1 = 1$

$x_t = -3x_{t-1} + 4x_{t-2}$

$$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}^{t-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$A = PDP^{-1}$

$$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = PD^{t-1}P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find λ, eigenvec

Proven

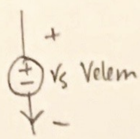
A and A^T have same eig vals

basics

$V = IR$

Ohm's Law

- wire: $V = 0$, $I = ?$
- resistor: $V = IR$, $I = \frac{V}{R}$
- open circuit: $V = ?$ $I = 0$
- voltage source $V = V_s$, $I = ?$
- current source $V = ?$ $I = I_s$



R in parallel

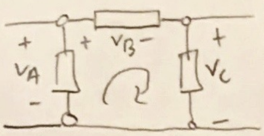
$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

Circuit analysis

KCL: current flowing in = current flowing out

KVL: sum of voltages across elements connected in a loop must = 0



$\boxed{V_A - V_B - V_C = 0}$

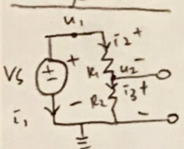
if arrow goes from − to +, add Velem

arrow + to −, add −Velem

Procedure:

1) Pick ground node
2) label all remaining nodes
3) label current thru each elem
4) add +/- labels
5) $A\vec{x} = \vec{b}$ $\vec{x}$ = unknown ckt vars A = nxn matrix, n = unknown var
6) Use KCL for each node to get eqns
7) Use IV relationships to fill in remaining eqns

Voltage divider



$i_1 + i_2 = 0$

$i_2 = i_3$

$u_1 - 0 = V_s$

$u_1 - u_2 = i_2 R_1$

$u_2 - 0 = i_3 R_2$

Voltage across R_2 = voltage drop $u_2 - 0$

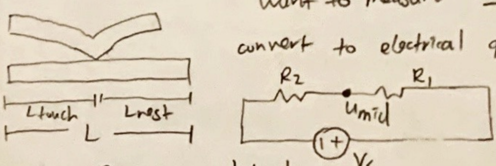
$$V_{out} = \frac{R_2}{R_1 + R_2} \cdot V_s = \frac{1}{1 + \frac{R_1}{R_2}} \cdot V_s$$

Resistive Touchscreen

$\boxed{I = \frac{dQ}{dt}} \Rightarrow 1A = \frac{C}{sec}$ $R = \rho \frac{L}{A}$ ← length, ← cross-section area

want to measure $\frac{L_{touch}}{L}$

convert to electrical quantities



$R_2 = \rho \frac{L_{touch}}{A}$

$R_1 = \rho \frac{L_{rest}}{A}$

$$U_{mid} = \frac{R_2}{R_1 + R_2} V_s = \frac{L_{touch}}{L} \cdot V_s$$

Power

$p = \frac{dE}{dt}$ $V = \frac{dE}{dQ}$ $\Rightarrow p = IV$

- pos: power dissipated
- neg: power generated (voltage source)

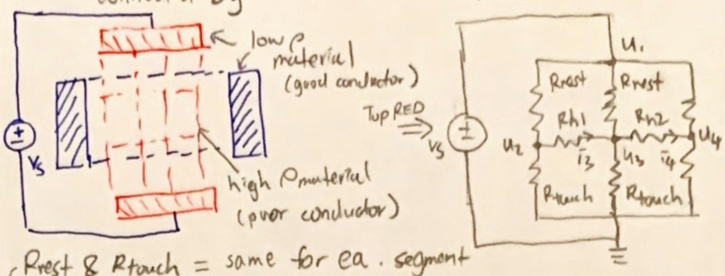
$p = \frac{V^2}{R}$ $p = I^2 R$

measuring voltage: Voltmeter must have 0 current → open ckt

measuring current: ammeter needs 0V across it → short ckt or wire

2D Touchscreen

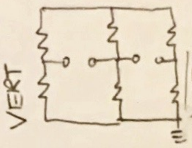
model: divide top plate into 3 equal segments → resistors connected by hor. resistors R_{h1}, R_{h2}



Rrest & Rtouch = same for ea. segment

Replace R_{h1} & R_{h2} w/ open ckts b/c $u_2 = u_3$, $u_3 = u_4$

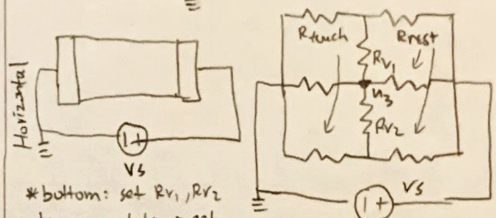
VERT



solve for $u_3 = \frac{R_{touch}}{R_{touch} + R_{rest}} \cdot V_s$, $R_t = \rho \frac{L_{touch}}{A}$

$\boxed{u_3 = \frac{L_{touch}}{L} V_s,\ L_{touch} = L_{touch,ver}}$ $R_r = \rho \frac{L_{rest}}{A}$

Horizontal



same analysis

$u_3 = \frac{L_{touch}}{L} \cdot V_s$

$L_{touch} = L_{touch,h}$

*bottom: set R_{v1}, R_{v2} to open ckts → get horizontal positions

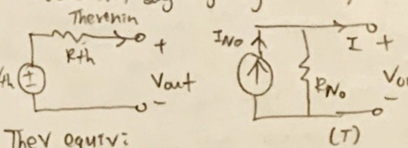
Superpositioning

- replace Vs w/ wire // Is → open ckt
- 0 Vs, 0 R → short ckt // 0 Is, ∞R → open ckt

Equivalence

2 ckts = equiv if IV relationships are identical

- does not say anything about power



In series ⇒ Use Itest

In parallel ⇒ Use Vtest

Thev equiv:

1) Find Vth/Voc across terminal // Find Isc(N)
2) Zero out indep. sources $R_{No} = R_{th} = \frac{V_{test}}{I_{test}}$
3) apply test current & meas Vout, or apply Vtest, meas I

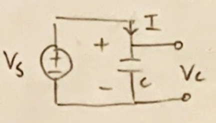
Capacitance

$\varepsilon = 8.85 \times 10^{-12}$ F/m

$Q = CV$ $C = \varepsilon \frac{A}{d}$

$I = C \cdot \frac{dV_c}{dt}$

$V_c(t) = \frac{I}{c} t + V_c(0)$



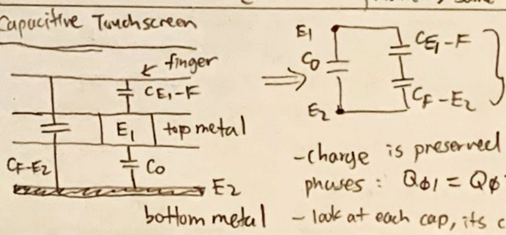
- once enough chg stored, $V_c = V_s$
- V can no longer increase past this point
- I thru cap = 0

Caps in parallell: $C_1 + C_2$ // Caps in series $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$

when a cap is charged, there is V across it → energy stored in cap. $E = \frac{1}{2} C V^2$

parallel ⇒ same V series ⇒ same Q, I

Capacitive Touchscreen



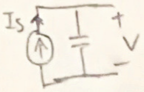
CΔ has some finite capacitance CΔ > 0

$C_\Delta = C_{eq} > C_0$

- charge is preserved between diff phases: $Q_{\phi 1} = Q_{\phi 2}$ // $Q_{\phi 1} = Q_{c1} + Q_{c2}$
- look at each cap, its chg, keep track of that

Measuring capacitance

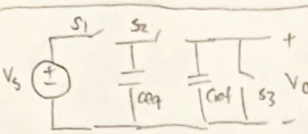
$I = C\cdot\frac{dv}{dt}$, $Q = CV$



what if you don't have a curr source?

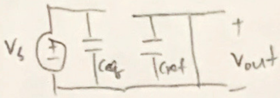
principle of charge sharing: between diff steady states for a ckt, Qtotal by all caps is same

need a ref (Cref) to dump charge from Ceq

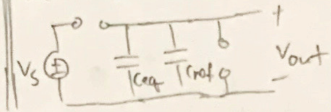


* meas cap w/ Vs

ϕ_1: S_1, S_3 on ; S_2 off



ϕ_2 S_1, S_3 OFF ; S_2 ON



$Q_{total,1} = C_{eq}\cdot V_s + \cancel{C_{ref}\cdot 0}$

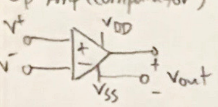
$Q_{total,2} = C_{eq}\cdot V_{out} + C_{ref}\cdot V_{out}$

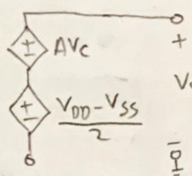
$C_{eq}\cdot V_s = (C_{eq} + C_{ref})\, V_{out}$

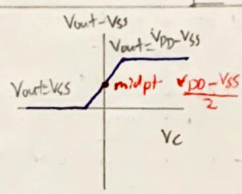
given Vout, solve for Ceq

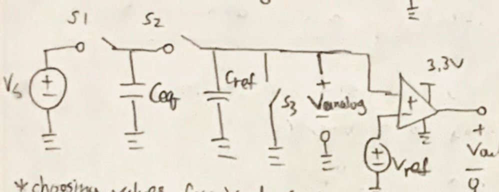
$$\boxed{V_{out} = \frac{C_{eq}}{C_{eq}+C_{ref}}\cdot V_s}$$

Op Amp (comparator)









Vana > Vref: Vout = 3.3V

Vana < Vref: Vout = 0

* choosing values for Vref, Cref

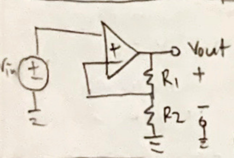
Cref → don't make it much larger than Ceq (look above)

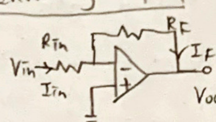
Vref → according to Vana, touch & Vana, no touch $V_{ref} = \frac{1}{2}(V_{ana,t} + V_{ana,nt})$

Op-Amp & NFB

Golden rules: $I^+ = I^- = 0$, $V^+ = V^-$

Non-inverting amp	Inverting Amp





* Don't forget negatives

$I_f = \frac{V^- - V_{out}}{R_f}$

$$\boxed{V_{out} = V_{in}\left(1 + \frac{R_1}{R_2}\right)}$$

$$\boxed{V_{out} = \frac{-R_f}{R_{in}}\cdot V_{in}}$$

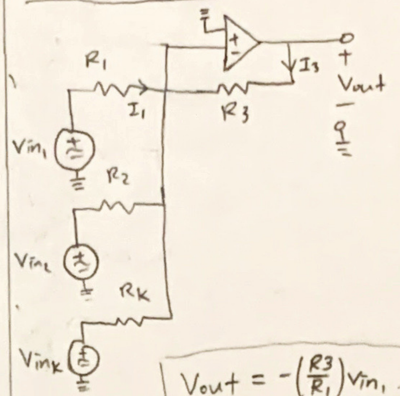
General Tips

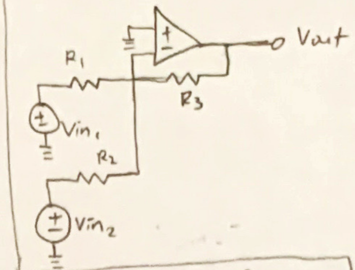
1) GR1 $I^+ = I^- = 0$
2) KCL
3) GR2
4) Ohm's Law @ each resistor (solve for I)
5) Plug (4) into (2), solve for Vout

Check NFB

1) Zero out indep sources
2) Dink output: is feedback from ckt opposite dir. of dinking output

Inverting summer





$$\boxed{V_{out} = -\frac{R_3}{R_1}\cdot V_{in_1} - \frac{R_3}{R_2} V_{in_2}}$$

$$\boxed{V_{out} = -\left(\frac{R_3}{R_1}\right)V_{in_1} - \left(\frac{R_3}{R_2}\right)V_{in_2} - \left(\frac{R_3}{R_k}\right)\cdot V_{ink}}$$

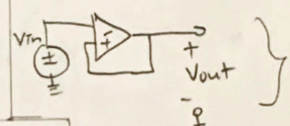
Designing circuits

→ [f()] → [g()] →

from perspective of f: see open ckt (g should not load f)

from perspective of g: see voltage source

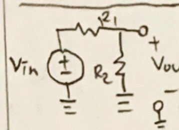
Unity Gain Buffer



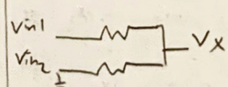
can add this between f and g to satisfy ideal isolation scenario

* when in doubt, add a buffer

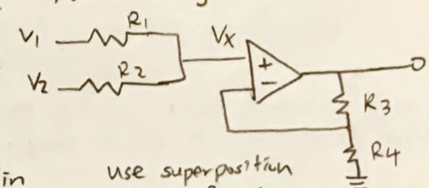
Resistive Divider



$V_{out} = \frac{R_2}{R_1+R_2}\cdot V_{in}$



Non-inverting summer



use superposition and VD for Vx

$$\boxed{V_x = \frac{R_2}{R_1+R_2} V_{in_1} + \frac{R_1}{R_1+R_2} V_{in_2}}$$

$$\boxed{V_{out} = V_x\left(1 + \frac{R_3}{R_4}\right)}$$

Design procedure

1) spec/rewrite goal
2) strategy/block diagram
3) implement/components
4) analyze/verify

$R * C$ = time

- If you're trying to build Is, do not connect ckt elem to which we want to supply I to ⏚
- ckt elem connected to current source must keep op-amp in NFB

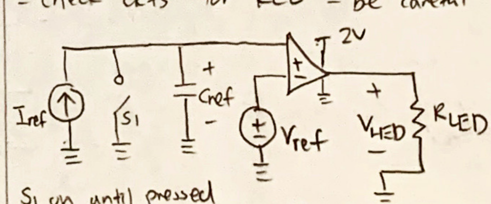
Countdown Timer

→ 2sec → LED (apply 2V to LED)

- want to relate current to time = $V_c(t) = \frac{I}{C}t + V_c(0)$
  ↳ capacitor
- want to compare voltage w the two seconds (or smth that means 2s)
  ↳ comparator $V_{ref} = V_{time}(2) = \frac{I_{ref}}{C_{ref}}(\text{want})$ ↳ do not neglect $V_c(0)$
- comparator needs pwr supply of 2V
- make sure you know initial cap voltage! Don't assume
- check ckts for KCL - be careful w/ switches

ϕ_1 : S_1 ON

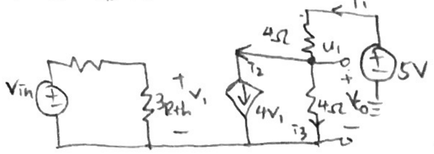
ϕ_2 : S_2 ON



S1 on until pressed

want to short ckt cap before button is pressed

Find V_{CD} in terms of V_{th}



$V_1 = V_{th} \cdot \frac{3 R_{th}}{4 R_{th}}$

$V_1 = \frac{3}{4} V_{th}$

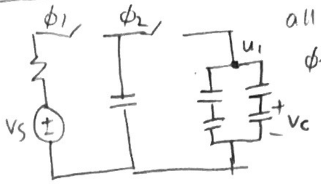
$4V_1 = \frac{3}{4} V_{th} \cdot 4$

$4V_1 = 3 V_{th}$

nodal @ u_1 = $i_1 = i_2 + i_3$

$\frac{5 - V_{CD}}{4} = 4V_1 + \frac{V_{CD} - 0}{4} \Rightarrow$ Solve

Charge sharing



@ steady state (ϕ_1)

all voltage goes to C_1

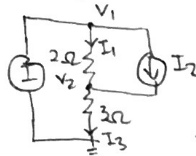
ϕ_2 : $\frac{1}{2}$ of the voltage goes to u_1

- caps in series → $\frac{1}{2}$ the voltage

$V_c = \frac{1}{4} V_S$

$Q_{total,1} = Q_{total,2}$

Null space



$I_3 = I_1 + I_2$

$I_3 = \frac{V_2}{3}$

$I_1 = \frac{V_1 - V_2}{2}$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/3 \\ 1 & 0 & 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↳ row reduce

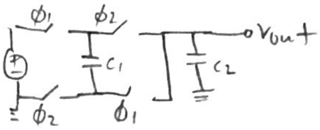
$I_1 = 2V_1 - 8V_2$

$I_2 = 2V_1 - 5V_2$

$I_3 = -3V_2$

$$\vec{x} = V_1 \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + V_2 \begin{bmatrix} -8 \\ -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

V_s & I_s can be any lin. combo of null space



cap in series

$C_{eff} = \frac{C_1 C_2}{C_1 + C_2}$ $\quad Q_{eff} = V_{in} \cdot C_{eff}$

$Q_{C_1} = V_{in} \cdot \frac{C_1 C_2}{C_1 + C_2} = C_1 \cdot V_{C_1} \Rightarrow$

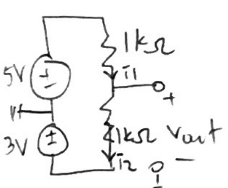
$$\boxed{\begin{aligned} V_{C_1} &= V_{in} \cdot \frac{C_2}{C_1 + C_2} \\ V_{C_2} &= V_{in} \cdot \frac{C_1}{C_1 + C_2} \end{aligned}}$$

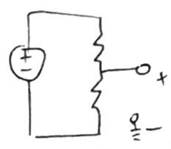
$Q_{C_1} = Q_{C_2} = V_{in} \cdot \frac{C_1 C_2}{C_1 + C_2}$

Phase 2 $\quad Q_{C_1} = C_1 V_{out}, \; Q_{C_2} = C_2 V_{out}$

Total $\quad 2V_{in} \cdot \frac{C_1 C_2}{C_1 + C_2} = (C_1 + C_2) V_{out}$ $\qquad V_{out} = \frac{2V_{in} \cdot C_1 C_2}{(C_1 + C_2)^2}$

Thevenin





$\boxed{V_{oc} = 1V}$

$V_{out_1} = \frac{5}{2} V$

$V_{out_2} = -\frac{3}{2} V$

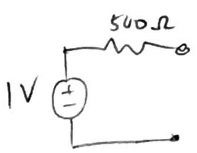
$I_{sc} = i_1 + i_2$

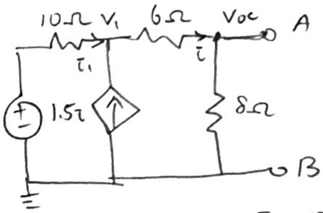
$= \frac{5}{1k\Omega} - \frac{3}{1k\Omega} = \frac{2V}{1000\Omega}$

$= \boxed{2mA}$

$R_{th} = R_{in} \parallel = \frac{1k\Omega}{2k\Omega}$

$= 500\Omega$





① Find V_{oc} w/ KCL

$i_1 + 1.5i = i$

$i_1 = \frac{12 - V_1}{10\Omega}$

In series, so

$i = \frac{V_1 - V_{oc}}{6\Omega} = \frac{V_{oc}}{8\Omega}$

$\frac{12 - V_1}{10\Omega} + 1.5 \frac{V_{oc}}{8\Omega} = \frac{V_{oc}}{8\Omega}$ (7)

$V_1 - V_{oc} = \frac{6V_{oc}}{8\Omega}$ $\qquad V_1 = \frac{14V_{oc}}{8\Omega}$

substitute V_1 in (7)

$\frac{12 - \frac{14V_{oc}}{8\Omega}}{10\Omega} + 1.5 \frac{V_{oc}}{8\Omega} = \frac{V_{oc}}{8\Omega} \Rightarrow V_{oc} = \frac{96}{9} V$

Thevenin

① Find V_{oc}

Norton

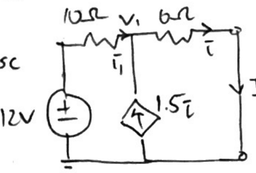
① Find I_{sc}

② Turn off all indep sources

③ Apply V_{test} to find I or apply I_{test} to find V

(I that flow thru V_{test}) (Voltage across I_{test})

② Find I_{sc}



KCL @ V_1

$i_1 + 1.5i = I_{sc}$

$\frac{12V - V_1}{10} + 1.5\frac{V_1}{6} = \frac{V_1}{6}$

$V_1 = 72V$

Solve for I_{sc}

$I_{sc} = i = \frac{V_1}{6}$

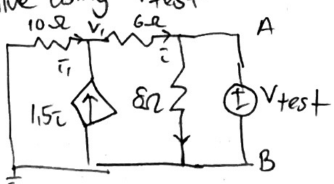
$I_{sc} = \frac{72V}{6} = 12A$

Solve for R_{th}

$V = IR$

$R_{th} = \frac{96}{9} V \cdot \frac{1}{12} = 0.889$

Solve using V_{test}



$R_{eq} = \frac{V_{test}}{I_{BA}}$

$= V_{test} \cdot \frac{8}{9V_{test}}$

$= \frac{8}{9} \Omega$

KCL @ node V_1

$\frac{0 - V_1}{10} + 1.5i = i$

$\frac{V_1 - V_{test}}{6} = i$

Solving : $\frac{V_1}{12} - \frac{V_1}{10} - \frac{V_{test}}{12} = 0$

$\Rightarrow \frac{-V_1}{60} = \frac{V_{test}}{12}$

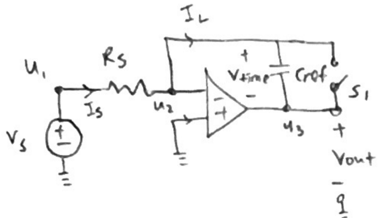
$\Rightarrow V_1 = -5V_{test}$

↑ use this to solve I_{BA}

$i = \frac{V_1 - V_{test}}{6} = -V_{test}$

$I_{BA} = \frac{V_{test} - 0}{8} - i = \frac{V_{test}}{8} + V_{test} = \frac{9}{8} V_{test}$

Alternate "current source"



$I_S = I_L$

ϕ_1 : S_1 on

ϕ_2 = S_1 OFF

$I_S = \frac{V_S}{R} = I_L$

$V_{time} = u_2 - u_3$

$I_S = C_{ref} \cdot \frac{d(\cancel{u_2} - u_3)}{dt}$

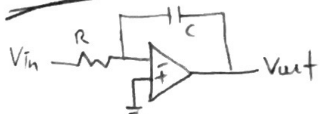
$u_3(t) = \frac{-I_S}{C_{ref}}(t) + \cancel{u_3(0)}^{0}$

$u_3(t) = \frac{-V_S \cdot t}{R_S C_{ref}} = -V_S$

unit-time

$V_{out} = -V_{time}$

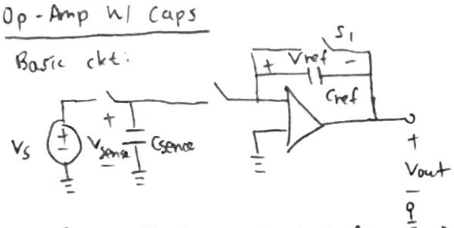
Integrator



$V_{out} = \frac{-1}{RC} \int V_{in}\, dt$

Op-Amp w/ caps

Basic ckt:



$C_{sense} \cdot V_S + 0 = 0 + C_{ref}(-V_{out})$

$$\boxed{V_{out} = -\frac{C_{sense}}{C_{ref}} \cdot V_S}$$

recall: touchscreen

ϕ_1 S_1 on, S_2 off

$Q_{sense} = C_{sense} \cdot \cancel{V_{sense_1}}^{V_S}$

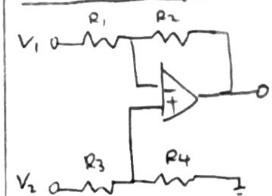
$Q_{ref} = C_{ref} \cdot 0$

ϕ_2 S_1 off, S_2 on

$Q_{sense} = C_{sense} \cdot \cancel{V_{sense_2}}^{0}$

$Q_{ref} = C_{ref} \cdot \cancel{V_{ref}}$ $(0 - V_{out})$

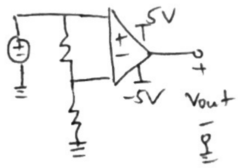
Differential Amp



$V_{out} = \left(1 + \frac{R_2}{R_1}\right)\left(\frac{R_4}{R_3 + R_4}\right) \cdot V_2 - \left(\frac{R_2}{R_1}\right) \cdot V_1$

If $R_1 = R_3$ and $R_2 = R_4$

$V_{out} = \left(\frac{R_2}{R_1}\right)(V_2 - V_1)$



$V^+ = V_{in}$

$V^- = \frac{2}{3} V_{in}$

$V^+ - V^- = \frac{1}{3} V_{in}$

$V_{out} = A \cdot \frac{1}{3} V_{in} + 0$

$V_{out} = \frac{1}{3} A V_{in}$

$V_{out} = A v_c + \frac{V_{DD} - V_{SS}}{2}$

how to choose V_{ref} for comparator

↳ something between threshold values

$(V^+ - V^-) > 0 : V_{DD}$

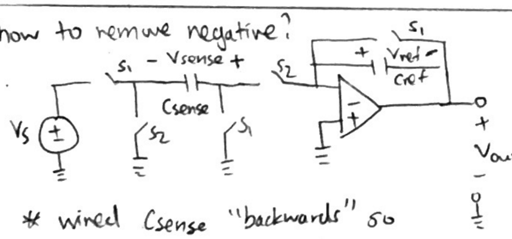
$(V^+ - V^-) < 0 : V_{SS}$

$V_{out_4} = 2.5V$

$V_{out_5} = 2.58V$

choose 2.54V for V_{ref}

how to remove negative?



* wired C_{sense} "backwards" so theres no - for chg sharing

* added some more switches

$Q_{total}, \phi_1 = C_S \cdot (-V_{sense}) + 0$

$= -C_S \cdot V_{sense}$

$Q_{total}, \phi_2 = C_S \cdot V_{sense} + C_{ref} \cdot V_{ref}$

$= C_{ref}(-V_{out})$

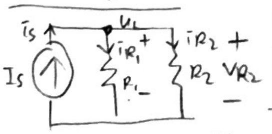
$= -C_{ref} \cdot V_{out}$

$$\boxed{V_{out} = \frac{C_S}{C_{ref}} \cdot V_S}$$

notes: use VD/CD, label nodes & redraw

$E = \frac{1}{2} C V^2$, efficiency $= \frac{E\phi_2}{E\phi_1}$ condense into R_{eq} & then break it apart

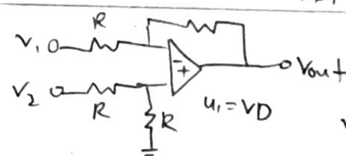
Current Divider



$$\boxed{\frac{R_2}{R_1 + R_2} \cdot I_S = i_1}$$

$$\boxed{\frac{R_1}{R_1 + R_2} \cdot I_S = i_2}$$

Find R_{eq} and V_{eq} // $V_{eq} = i_S \cdot R_{eq}$ // $i_1 = \frac{V_{eq}}{R_1} = \left(\frac{R_{eq}}{R_1}\right) i_S$



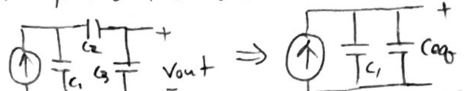
$V_{u_1} = V_2 \cdot \frac{R}{2R} = \frac{V_2}{2} = V_{u_2}$

@node u_1

$\frac{V_1 - \frac{V_2}{2}}{R} = \frac{\frac{V_2}{2} - V_{out}}{R}$

$$\boxed{V_{out} = V_2 - V_1}$$

ex: capacitors: find Vout



$C_{eq} = C_2 \| C_3$

$C_{eq} = \frac{C_2 C_3}{C_2 + C_3}$

$V_{C_{eq}}(t) = \int \frac{I_S}{C_1 + C_{eq}} dt = \frac{I_S(t)}{C_1 + C_{eq}} + \cancel{V_0} = \frac{I_S t}{C_1 + C_{eq}}$ $\boxed{I = C \frac{dV}{dt}}$

$I_{C_{eq}}(t) = C_{eq} \cdot \frac{dV_{C_{eq}}}{dt} = \boxed{\frac{C_{eq} \cdot I_S}{C_1 + C_{eq}}} \rightarrow I_{C_3}(t)$

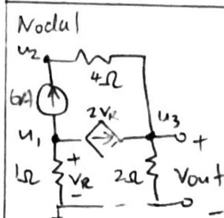
$* I_{C_{eq}} = I_{C_2} = I_{C_3}$

$I_{C_3}(t) = C_3 \frac{dV_{out}}{dt} \Rightarrow \frac{I_{C_3}(t)}{C_3} = \frac{dV_{out}(t)}{dt}$

$V_{out}(t) = \int \frac{I_{C_3}(t)}{dt} = \frac{C_{eq} \cdot I_S}{C_3(C_1 + C_{eq})} t + V_0 = \frac{C_2 I_S t}{C_1 C_2 + C_1 C_3 + C_2 C_3}$

- combine to C_{eq}
- find $V_{C_{eq}}$
- use $V_{C_{eq}}$ to find $I_{C_{eq}}$
- $I_{C_{eq}} = I_{C_3}$
- Find Vout using I_{C_3}

Nodal

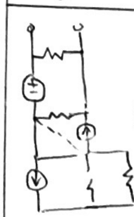


@u_1: $6A + 2V_R + \frac{V_R}{1} = 0$

$3V_R = -6A \Rightarrow V_R = -2A$

@u_3: $6 + 2V_R - \frac{V_{out}}{2} = 0$

$6 + (-4) = \frac{V_{out}}{2}$ $\boxed{V_{out} = 4V}$



① Find V_R w/ V_{S_1} on → VD $\frac{5}{25} V_S = 1V$

② Find V_R w/ I_{S_1} on → CD $\frac{20}{25} \cdot 1 = -0.8A$

③ Find V_R w/ I_{S_2} on ⇒ 0V, no current flowing thru R

④ Total V_R 1-4= (-3V)

$V = IR \rightarrow V_R = (-0.8)(5)$

$V_R = -4V$

cap @ steady state: I = 0 → OC

resistor in ||: have the same V across

series: same current

cap in ||: same voltage

in series: same charge (Q), current

Inner Product $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\|\|\vec{b}\|\cos(\theta_a - \theta_b)$

Symmetry $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

Linearity $\langle \vec{x}, \vec{y}+\vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

$\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$

positive definiteness $\langle \vec{x}, \vec{x} \rangle \geq 0$ with $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$

$\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle \Rightarrow \|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

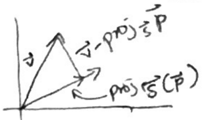
Cauchy-Schwarz

$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$

Norm length/dist of vector

$\hookrightarrow \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$\text{proj}_{\vec{a}} \vec{b}$ $\quad \hat{b} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2} \vec{a} = \frac{\langle \vec{b}, \vec{a} \rangle}{\langle \vec{a}, \vec{a} \rangle} \vec{a}$



Least squares

$\vec{x} = (A^TA)^{-1}A^T\vec{b}$

Underdetermined

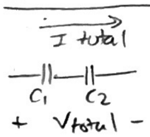
$\vec{x} = A^T(AA^T)^{-1}b$

Trilateration

$d_a = N_a \cdot T_s \cdot V$ → speed of sound

samples (shift) ↑ ↑ time/sample

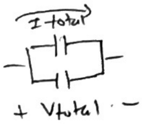
Current Divider



$I_1 = I_2 = I_{total}$

$V_1 = V_{total} C_2 / (C_1 + C_2)$

$V_2 = V_{total} C_1 / (C_1 + C_2)$

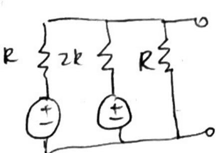


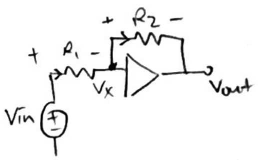
$V_1 = V_2 = V_{total}$

$I_1 = I_{total} C_1 / (C_1 + C_2)$

$I_2 = I_{total} C_2 / (C_1 + C_2)$

$V_{out} = \frac{2}{5}u_1 + \frac{1}{5}u_2$





$$\frac{V_{in} - V_x}{R_1} = \frac{V_x - V_{out}}{R_2}$$

MISC Notes

a diagonal matrix is non-invertible iff some $\lambda = 0$

$\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle$ (norm^2 = inner prod of x with itself)

$\langle \vec{a} - \vec{b}, \vec{c} \rangle = \langle \vec{a}, \vec{c} \rangle - \langle \vec{b}, \vec{c} \rangle$

least squares / projections $\vec{x} = \underbrace{(A^TA)^{-1}}_{(\|A\|^2)^{-1}} \underbrace{A^T\vec{b}}_{\text{Inner prod}} = \frac{\langle \vec{A}, \vec{b} \rangle}{\|A\|^2}$

if $\det(A) = 0$, A is not invertible

Resistance of O.C $= \infty$

OMP

- Inputs: set of m songs w/ length n: $S = \{\vec{s}_0, \vec{s}_1, \dots \vec{s}_{m-1}\}$
- n dim received signal
- sparsity level k
- threshold th to stop iter

Outputs: set of songs that were identified F, which has at most k elements

- vector, $\vec{x}$ containing song messages, length k or less
- n-dim residual $\vec{y}$

Procedure

$\vec{y} = \vec{r}$, $j = 1$, $k = k$, $A = [\,]$, $F = \{0\}$

while $((j < k) \,\&\&\, (\|\vec{r}\| \geq th))$:

1. cross corr $\vec{y}$ with shifted of all songs, find index i of song $\vec{s}_i^{(N_i)}$ w/ highest corr
2. Add i to the set of song indices, F
3. Column concatenate matrix A w/ the correct shifted version of the song $A = [A \,|\, \vec{s}_i^{(N_i)}]$
4. Use least squares to obtain $\vec{x} = (A^TA)^{-1}A^T\vec{r}_i$
5. Update the residual value $\vec{r}$ by subtracting $\vec{y} = \vec{r} - A\vec{x}$
6. update $j = j + 1$

Gram Schmidt

compute $\vec{q}_1$: $\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|}$

for $(i = 2 \dots n)$

1. Compute the vector $\vec{e}_i$
   $\vec{e}_i = \vec{s}_i - \sum_{j=1}^{i-1} (\vec{s}_i^T \vec{q}_j)\vec{q}_j$
2. Normalize to compute $\vec{q}_i$: $\vec{q}_i = \frac{\vec{e}_i}{\|\vec{e}_i\|}$

For an orthogonal matrix $Q^{-1}Q = I = Q^TQ$

FA15 FINAL - LEAST SQUARES

Found vectors $\vec{c}_a, \vec{c}_c$, s.t. $\vec{b} \approx x_a\vec{c}_a + x_c\vec{c}_c$

- Find LS estimate for $\hat{x}_a, \hat{x}_c$

$A = [\vec{c}_a \; \vec{c}_c]$ $\quad \begin{bmatrix} x_a \\ x_c \end{bmatrix} = (A^TA)^{-1}A^T\vec{b}$

$= \left(\begin{bmatrix} c_a \\ c_c \end{bmatrix} [c_a \; c_c] \right)^{-1} \left(\begin{bmatrix} c_a \\ c_c \end{bmatrix} \vec{b} \right)$

$$\begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix} = \begin{bmatrix} \langle c_a, c_a \rangle & \langle c_a, c_c \rangle \\ \langle c_c, c_c \rangle & \langle c_c, c_c \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle c_a, b \rangle \\ \langle c_c, b \rangle \end{bmatrix}$$

DIS 12B COST FN / LEAST SQUARES

apply various currents (I) & get output voltage (V) to try to measure R.

- ideally, Find R s.t. $V = IR$ but values not precise
- $\text{cost}(R) = \sum_{j=1}^{n} (V_j - I_jR)^2$ ★ want individual err to be minimized.
- $\text{cost}(R) = \langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \rangle$

def err $\vec{e} = \vec{V} - \vec{I}R$

$\sum e_j^2 \rightarrow \|\vec{e}\|^2 = \langle \vec{e}, \vec{e} \rangle$

Lin dep $\{\vec{v}_1, \dots, \vec{v}_n\}$ if there exists scalars $\alpha_1, \dots, \alpha_n$ s.t. $\alpha_1\vec{v}_1 + \dots + \alpha_n\vec{v}_n = \vec{0}$ and not all α_i's are equal to zero $\Rightarrow \vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$

square matrices + linear indep $\rightarrow$ Invertible

Can't have
- non-trivial null space
- a zero eigen val
- det = 0
- is not full rank

Vector Space $(\mathbb{V}, \mathbb{F})$

Vector addition
- associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
- commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- additive identity: $\exists \vec{0} \in \mathbb{V}$ s.t. $\vec{v} + \vec{0} = \vec{v}$
- additive inverse: for any $\vec{v} \in \mathbb{V}$, $\exists \vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = \vec{0}$

Scalar Mul
- associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$
- multiplicative identity: $\exists\, 1 \in \mathbb{F}$ where $1 \cdot \vec{v} = \vec{v}$
- distri in vec add: $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
- distri in scalar add: $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

Subspaces
- contains 0
- closed under vec addition
- closed under scalar mul

has to be subset of $\mathbb{V}$ in the VS $(\mathbb{V}, \mathbb{F})$

Change of basis
$\vec{u}_a = A^{-1}\vec{u}$

Nullspace
x st. $Ax = 0$

GE $\rightarrow$ write vars in terms of free vars

Eigen
$A\vec{v} = \lambda\vec{v}$

A and A^T have same eigen vals

$\det(\lambda I - A) = 0 = \det(A - \lambda I)$

$A^{-1}\vec{v} = \lambda^{-1}\vec{v}$ and $A^n\vec{v} = \lambda^n\vec{v}$

Rotation Matrix
$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Reflect across x-axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflect across y-axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

reflect across $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Rotation by $180°$ $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

rotate $90°$ counter $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

rotate 90 clockwise $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

reflect across $y = -x$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Determinants
- $\det(I_n) = 1$, I_n = $n \times n$ identity
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$
- A, B equal size
- $\det(AB) = \det(A)\det(B)$
- $\det(cA) = c^n \det(A)$ for $n \times n$ matrix
- $\det(A) = \text{prod}(\lambda_i)$

1) switching rows negate the det(A) $\quad \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} ax \\ y \end{bmatrix}$
2) adding a mul of one row to another doesn't affect det(A) $\quad \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ ay \end{bmatrix}$
3) Multiplying a row by a constant scales the det(A) by that constant. $\quad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} ax \\ by \end{bmatrix}$

(A is square, invertible, $n \times n$)

Invertible Matrix Thm.
- A is row and column equiv to I
- A has n pivot and is full rank
- $\det(A) \neq 0$
- $Ax = 0$ is trivial, only nullspace $\{0\}$
- There is unique soln to $Ax = b$
- cols(A) are lin indep and span $\mathbb{R}^n$
- A^T is invertible
- 0 is not an eigen val of A

Orthogonality
$\vec{z}$ is $\perp$ to $\vec{x}$ if $\vec{z}^T\vec{x} = \vec{0}$

$\text{trace}(A) = \sum(\text{diagonal}(A))$

col span = range = pivot cols

$\text{rank}(A) = \dim(\text{span}(A))$ = # of pivots

$\text{span}(A) = \{\vec{v} \mid \vec{v} = \sum_{i=1}^m x_i \vec{a}_i, x_i \rightarrow \text{scalars}\}$

basis = lin indep vectors

$m - \dim(\text{range}(A)) = \dim(\text{Null}(A))$

diagonalization
- $n \times n$ matrix can be diagonalized if it has n lin indep eigenvec

$T = ADA^{-1}$ ← inverse of eigen vec

$\det(A) = 0 \Rightarrow$ lin. dep.

Gaussian Elimination $\rightarrow$ upper triangular $\rightarrow$ find det(A)

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

$\text{inv}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

steady state
Use $\lambda = 1$ negative $\lambda \rightarrow$ oscillation

Solve for eigen vec $\Rightarrow$ important score (scale and add up to 1)

eigen val of 1 and other eigen val < 1
- ↳ has a steady state
- ↳ blows up otherwise

eigen val = 0
non-trivial soln in null space

$\det(A) = 0$

More change of basis
$A_B = B^{-1}AB$

$A = BA_BB^{-1}$

Inverses
$A^{-1}A = AA^{-1} = I$

$(A^{-1})^{-1} = A$

$(kA)^{-1} = k^{-1}A^{-1}, k \neq 0$

$(A^T)^{-1} = (A^{-1})^T$

$(AB)^{-1} = B^{-1}A^{-1}$, A, B invertible

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

Inverses share the same eig vec but not necessarily eig values.

$\rightarrow AB = VD_AD_BV^{-1} = VD_BD_AV^{-1} = BA$

$\begin{bmatrix} \lambda A_1 \lambda B_1 & & \\ & \ddots & \\ & & \lambda A_2 \lambda B_2 \end{bmatrix} = \begin{bmatrix} \lambda B_1 \lambda A_1 & & \\ & \ddots & \\ & & \lambda B_2 \lambda A_2 \end{bmatrix}$

Factoring
$ABAA^{-1} - \lambda IAA^{-1}$

$A(BA - \lambda I)A^{-1}$ ← keep relative order

$x_0 = 0$, $x_1 = 1$ $\quad \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = PD^{t-1}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_t = -3x_{t-1} + 4x_{t-2}$

$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 0 \end{bmatrix}^{t-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = PDP^{-1}$

Find eigen pair

$\vec{v}_1 \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$

① $\|v_1\| = \sqrt{2 + 2 + 0} = 2$

② $\vec{e}_2 = \vec{v}_2 - \text{Proj}_{\vec{q}_1}\vec{v}_2$

$\quad \vec{v}_2 - \langle \vec{v}_2, \vec{q}_1 \rangle \vec{q}_1$

$\vec{q}_2 = \frac{\vec{e}_2}{\|\vec{e}_2\|}$

$\vec{e}_3 = \vec{v}_3 - \text{Proj}_{\vec{q}_1}\vec{v}_3 - \text{Proj}_{\vec{q}_2}\vec{v}_3$

$\quad = \vec{v}_3 - \langle \vec{q}_1, \vec{v}_3 \rangle \vec{q}_1 - \langle q_2, \vec{v}_3 \rangle \vec{q}_2$

$q_3 = \frac{\vec{e}_3}{\|\vec{e}_3\|}$

$\vec{q}_1 \; [1/\sqrt{2} \; -1/\sqrt{2} \; 0]^T$

$\vec{q}_2 \; [0 \; 0 \; 1]^T$

$\vec{q}_3 \; [1/\sqrt{2} \; 1/\sqrt{2} \; 0]^T$