Homework_10

November 13, 2019

1 Homework 10

1.0.1 Instructions

Your homeworks have two components: a written portion and a portion that also involves code. Written work should be completed on paper, and coding questions should be done in the notebook. You are welcome to LaTeX your answers to the written portions, but staff will not be able to assist you with LaTeX related issues. It is your responsibility to ensure that both components of the homework are submitted completely and properly to Gradescope. Refer to the bottom of the notebook for submission instructions.

```
In [2]: # HIDDEN
        from datascience import *
        from prob140 import *
        import numpy as np
        import matplotlib.pyplot as plt
        plt.style.use('fivethirtyeight')
        %matplotlib inline
        import math
        from scipy import stats
```

1.0.2 1. Peter Meets Paul

Peter and Paul agree to meet at a restaurant at noon. Peter arrives at time normally distributed with mean 12:00 noon and SD 5 minutes. Paul arrives at a time normally distributed with mean 12:02 P.M. and SD 3 minutes.

Find the chances below assuming that the two arrival times are independent. First, write a formula for the chance in terms of the standard normal cdf Φ. Then use a code cell to find the numerical value. You do not have to turn in any coding work for this question.

a) $P(\text{Peter arrives before Paul})$

b) $P(\text{both men arrive within 3 minutes of noon})$

c) $P(\text{the two men arrive within 3 minutes of each other})$

1.1 Solutions

1a) Let 12:00 $= 0$

Peter's arrival time = X: normal $(0, 25)$

Paul's arrival time = Y: normal $(2, 9)$

$X - Y$: normal $(-2, 34)$
$P(\text{Peter arrives before Paul}) = P(X < Y) = P(X - Y < 0) = \Phi(\frac{0-(-2)}{\sqrt{34}}) = \Phi(0.343) = 0.634$
1b) $P(\text{both men arrive within 3 minutes of noon}) = P(-3 < X < 3) \cdot P(-3 < Y < 3)$
$= (\Phi(\frac{3-0}{5}) - \Phi(\frac{-3-0}{5})) \cdot (\Phi(\frac{3-2}{3}) - \Phi(\frac{-3-2}{3}))$
$= (\Phi(\frac{3}{5}) - \Phi(\frac{-3}{5})) \cdot (\Phi(\frac{1}{3}) - \Phi(\frac{-5}{3})) = 0.263$
1c) $P(|X - Y| < 3) = P(-3 < X - Y < 3) = \Phi(\frac{3-(-2)}{\sqrt{34}}) - \Phi(\frac{-3-(-2)}{\sqrt{34}}) = \Phi(\frac{5}{\sqrt{34}}) - \Phi(\frac{-1}{\sqrt{34}})$
$= 0.3725$

In [13]:
```
# Calculation for 1a
stats.norm.cdf(0.343)
```

Out[13]: 0.63420076992237573

In [14]:
```
# Calculation for 1b
(stats.norm.cdf(3/5) - stats.norm.cdf(-3/5))* \
    (stats.norm.cdf(1/3) - stats.norm.cdf(-5/3))
```

Out[14]: 0.26311625700489039

In [16]:
```
# Calculation for 1c
stats.norm.cdf(0.8575) - stats.norm.cdf(-0.1715)
```

Out[16]: 0.37250037737314079

1.1.1 2. Slices of a Normal Cake

Let X and Y be independent standard normal random variables.
a) Find $P(X > 0, Y > 0)$.
Yes, it's easy. But get a piece of paper and draw the event on the plane anyway. Imagine the joint density surface over the plane, and try to imagine the relevant volume under the joint density surface as a quadrant-shaped slice of a bell-shaped cake.
b) Find $P(X > 0, Y > X)$.
c) Find $P(X > 0, Y > \sqrt{3}X)$.
2a) $P(X > 0, Y > 0) = P(X > 0) \cdot P(Y > 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
$P(X > 0, Y > 0)$ is one fourth of the volume under the joint density surface.
2b) Drawing the event on the plane, we see that $P(X > 0, Y > X)$ is one eighth of the volume under the joint density surface.
$P(X > 0, Y > X) = \frac{1}{8}$
2c) Drawing the event on the plane, we see the line $y = \sqrt{3}x$ makes a 60 degree angle with the x axis which means that the probability of this event is $\frac{30}{90} \cdot \frac{1}{4} = \frac{1}{12}$ of the volume under the joint density surface.
$P(X > 0, Y > \sqrt{3}X) = \frac{1}{12}$

1.1.2 3. Distance Between Two Normal Points

Suppose two shots are fired at a target. Assume each shot hits with independent normally distributed coordinates, with the same means and equal unit variances. Let D be the distance between the point where the two shots strike.
a) Find $E(D)$. Your calculation will go faster if you remember that a normal $(0, \sigma^2)$ variable can be written as σZ where Z is standard normal.
b) Find $Var(D)$.

1.2 Solutions

3a) First point = (X_1, Y_1)
Second point = (X_2, Y_2)
$X_1 : N(\mu, 1)$
$Y_1 : N(\mu, 1)$
$X_2 : N(\mu, 1)$
$Y_2 : N(\mu, 1)$
$D = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$
$X_1 - X_2 \stackrel{d}{=} Y_1 - Y_2 = \sqrt{2}Z$ where Z is standard normal, because the distribution of $Y_1 - Y_2$ is normal $(0, 2)$.
$(X_1 - X_2)^2 \stackrel{d}{=} (Y_1 - Y_2)^2 = 2Z^2$
$D = \sqrt{2Z_1^2 + 2Z_2^2}$ where $Z_i = N(0, 1)$ are iid
$D = \sqrt{2}\sqrt{Z_1^2 + Z_2^2}$
$\sqrt{Z_1^2 + Z_2^2}$ has the Rayleigh distribution (see section 20.1)
$E(D) = \sqrt{2}\sqrt{\frac{\pi}{2}} = \sqrt{\pi}$
3b) $E(D^2) = E((\sqrt{2(Z_1^2 + Z_2^2)})^2) = 2E(Z_1^2 + Z_2^2)$
$Z_1^2 + Z_2^2 = exp(\frac{1}{2})$ (See section 20.1)
$E(Z_1^2 + Z_2^2) = 2$
$E(D^2) = 2 \cdot 2 = 4$
$Var(D) = 4 - (\sqrt{\pi})^2 = 4 - \pi$

1.2.1 4. Min and Max of IID Uniforms

Let $U_1, U_2, \ldots, U_n$ be i.i.d. uniform on $(0, 1)$. Let $U_{(1)}$ and $U_{(n)}$ be the minimum and maximum of $U_1, U_2, \ldots, U_n$.

a) Find the joint density of $U_{(1)}$ and $U_{(n)}$.

b) Find the density of $U_{(1)}$.

c) Fix $x \in (0, 1)$ and find the conditional density of $U_{(n)}$ given $U_{(1)} = x$.

d) For fixed $x \in (0, 1)$, let $X_1, X_2, \ldots, X_{n-1}$ be $n - 1$ i.i.d. uniform $(x, 1)$ random variables. Find the density of $M = \max\{X_1, X_2, \ldots, X_{n-1}\}$ and compare it to your answer to Part **c**.

e) The random variable $R_n = U_{(n)} - U_{(1)}$ is called the *range* of the sample $U_1, U_2, \ldots, U_n$. Find $E(R_n)$.

4a) Apply the multinomial formula. For $0 < x < y < 1$,

$$\begin{aligned}
P(U_{(1)} \in dx, U_{(n)} \in dy) &= P(\text{none in } (0, x),\ \text{one in } dx,\ n-2 \text{ in } (x, y),\ \text{one in } dy,\ \text{none in } (y, 1)) \\
&= \frac{n!}{0!1!(n-2)!1!0!} x^0 (1dx)^1 (y-x)^{n-2} (1dy)^1 (1-y)^0 \\
&= n(n-1)(y-x)^{n-2} dxdy
\end{aligned}$$

So the joint density f is given by $f(x, y) = n(n-1)(y-x)^{n-2}$ on $0 < x < y < 1$.
4b) beta $(1, n)$ by text/lecture
4c) For $y \in (x, 1)$,

$$f_{U_{(n)}|U_{(1)}=x}(y) \;=\; \frac{n(n-1)(y-x)^{n-2}}{n(1-x)^{n-1}} \;=\; (n-1)\frac{(y-x)^{n-2}}{(1-x)^{n-1}}$$

by **a** and **b**

4d) For $y \in (x,1)$,

$$\begin{aligned} P(M \in dy) &= P(n-2 \text{ in } (x,y), \text{ one in } dy, \text{ none in } (y,1) \\ &= \frac{(n-1)!}{(n-2)!1!0!}\left(\frac{y-x}{1-x}\right)^{n-2}\left(\frac{1}{1-x}dy\right)^{1}\left(\frac{1-y}{1-x}\right)^{0} \\ &= (n-1)\frac{(y-x)^{n-2}}{(1-x)^{n-1}} \end{aligned}$$

which is the same as the answer to **c**. This indicates that given $U_{(1)} = x$, the other $n-1$ uniforms behave like $n-1$ iid uniforms on $(x,1)$.

4e) $E(R_n) = E(U_{(n)}) - E(U_{(1)}) = \frac{n}{n+1} - \frac{1}{n+1}$. The individual expectations follow either by using the expectations of the beta $(n,1)$ and beta $(1,n)$ distributions or by noting that the n uniform points create $n+1$ identically distributed segments of the unit interval.

1.2.2 5. Poisson MGF

Let X have Poisson(μ) distribution, and let Y independent of X have Poisson (λ) distribution.

a) Find the mgf of X.

b) Use the result of (a) to show that the distribution of $X+Y$ is Poisson.

5a)

Recall the taylor series expansion for e^x:

$\sum_0^\infty \frac{x^n}{n!} = e^x$

$M_X(t) = E(e^{tX}) = \sum_{k=0}^\infty \frac{e^{-\mu}\mu^k}{k!}e^{tk} = e^{-\mu}\sum_{k=0}^\infty \frac{(\mu e^t)^k}{k!} = e^{-\mu}e^{\mu e^t} = e^{\mu e^t - \mu} = e^{\mu(e^t-1)}$

5b)

If two distributions have the same mgf, then they must be the same distribution.

Let $X = Pois(\mu), Y = Pois(\lambda)$ and let X and Y be independent. Let $Z = X+Y$

$M_Z(t) = M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\mu(e^t-1)}e^{\lambda(e^t-1)} = e^{(\mu+\lambda)(e^t-1)}$

Let $W = Pois(\mu+\lambda)$

By part (a), we know that $M_W(t) = e^{(\mu+\lambda)(e^t-1)}$

Therefore, since Z and W have the same mgf, they must have the same distribution, which means $Z = X+Y$ has the $Pois(\mu+\lambda)$ distribution.

1.3 Submission Instructions

Many assignments throughout the course will have a written portion and a code portion. Please follow the directions below to properly submit both portions.

1.3.1 Written Portion

- Scan all the pages into a PDF. You can use any scanner or a phone using an application. Please **DO NOT** simply take pictures using your phone.

- Please start a new page for each question. If you have already written multiple questions on the same page, you can crop the image or fold your page over (the old-fashioned way). This helps expedite grading.
- It is your responsibility to check that all the work on all the scanned pages is legible.

1.3.2 Code Portion

- Save your notebook using File > Save and Checkpoint.
- Generate a PDF file using File > Download as > PDF via LaTeX. This might take a few seconds and will automatically download a PDF version of this notebook.
  - If you have issues, please make a follow-up post on the general HW 10 Piazza thread.

1.3.3 Submitting

- Combine the PDFs from the written and code portions into one PDF. Here is a useful tool for doing so.
- Submit the assignment to Homework 10 on Gradescope.
- **Make sure to assign each page of your pdf to the correct question.**
- **It is your responsibility to verify that all of your work shows up in your final PDF submission.**

1.3.4 We will not grade assignments which do not have pages selected for each question.

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In [ ]:
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