3B.2) If $(ST)^2 = 0$, then for all $v \in V$, $(ST)^2 v = 0$.

$$(ST)^2 v = (ST)(ST)v$$
$$= STS(Tv)$$
$$= STSu, \text{ where } u \in \text{range } T, u \in V.$$
$$= STw, \text{ where } w \in \text{range } S$$
$$\Rightarrow w \in \text{null } T$$
$$\Rightarrow STw = S(Tw)$$
$$= S0$$
$$= 0.$$
$$\therefore (ST)^2 = 0.$$

D.

3B.12) Suppose V is finite dimensional, and $T \in \mathcal{L}(V, W)$, there exists a subspace U of V s.t

$$U \cap \text{null } T = \{0\} \text{ and range } T = \{Tu : u \in U\}$$

Pf: First we prove that there exists a subspace U such that

$$U \cap \text{null } T = \{0\}$$

Let $v_1, \ldots, v_n$ be a basis of null T.

Let U be the set $\{0\}$, the empty subspace.

Now we pick a $v \in V$ s.t $v \notin \text{null } T$.

$\Rightarrow v$ is not a linear combination of $v_1, \ldots, v_n$.

If $\dim \text{null } T = \dim V$, then $\nexists\, v \in V$ s.t $v \notin \text{null } T$

$\Rightarrow U = \{0\}$.

$\Rightarrow$ such a subspace exists that satisfy $U \cap \text{null } T = \{0\}$.

If $\dim \text{null } T < \dim V$,

Since $v_1, \ldots, v_n$ are linearly independent, we can extend $v_1, \ldots, v_n$ into a basis for V

Let $s_1, \ldots, s_m$ be a linearly independent list s.t

$v_1, \ldots, v_n, s_1, \ldots, s_m$ forms a basis of V

$\Rightarrow \dim V = n + m$

Then we can pick $s_1, \ldots, s_m$ to be a basis of the subspace U.

Since $s_1, \ldots, s_m \notin \text{null } T$,

there is no vector in $\text{span}(s_1, \ldots, s_m)$ that can be in null T,

$\Rightarrow U \cap \text{null } T = \{0\}$

$\therefore$ We can build a subspace U that satisfies $U \cap \text{null } T = \{0\}$

Now we have $\dim U = m$, $\dim \text{null } T = n$, and $\dim V = n + m$.

using the fundamental theorem of linear maps,

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

$$n + m = n + \dim \text{range } T$$
$$m = \dim \text{range } T.$$
$$\dim \text{range } T = \dim U.$$

...

Let $w_1, \ldots, w_m$ be a basis of range T.

Since $v_1, \ldots, v_n$ is a basis of null T,

$$T(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = 0.$$

T can be any map in $\mathcal{L}(V, W)$. Cannot prove for specific T.

Let $Ts_k = w_k$ for $k = 1, \ldots, m$ and $Tv_p = \cancel{0}$ for $p = 1, \ldots, n$.

Then $T(c_1s_1 + \ldots c_ms_m + d_1v_1 + \ldots + d_nv_n) = c_1w_1 + \ldots + c_mw_m$ for $c_k, d_k \in \mathbb{F}$,

$\Rightarrow T(V) = \operatorname{span}(w_1, \ldots, w_m) = \dim U$

$\Rightarrow$ range $T = \{ Ts : s \in U \}$

$\therefore$ We built a subspace U in V such that $U \cap \text{null } T = 0$ and range $T = \{ Tu : u \in U \}$.

$\square$.

Show that range $T = \{ Tu : u \in U \}$. given that $U \cap \text{null } T = \{0\}$.

First we show that

3B-20) Suppose W is finite dimensional and $T \in \mathcal{L}(V, W)$.
Prove that T is injective iff $S \in \mathcal{L}(W, V)$ s.t. ST is identity map on V.

Let S be a function s.t. for all $v \in V$, $STv = v$. Since T is injective, all mappings between V and W through T are one to one.
Hence we can define such a mapping $S \in \mathcal{L}(W, V)$ that can have the property
$$STv = v$$
To prove $S \in \mathcal{L}(W, V)$, we prove that S satisfies additivity & homogeneity properties.

Additivity Let $v_1, v_2 \in V$.
$$ST(v_1 + v_2) = S[Tv_1 + Tv_2] = v_1 + v_2$$
$$= STv_1 + STv_2$$
$\Rightarrow$ additivity.

Homogeneity Let $v \in V$, $\lambda \in \mathbb{F}$.
$$ST(\lambda v) = \lambda v$$
$$= \lambda (STv)$$
$\Rightarrow$ homogeneity
$\Rightarrow S \in \mathcal{L}(V, W)$.

Now for the other direction. Suppose $\exists\, S \in \mathcal{L}(W, V)$ such that for some $T \in \mathcal{L}(V, W)$,
$$STv = v.$$

Let $u, v \in V$. s.t $Tu = Tv = w \in W$.
$$Sw = S(Tu) = STu = u.$$
$$Sw = S(Tv) = STv = v.$$
$\Rightarrow u = v$
$\Rightarrow Tu = Tv \Rightarrow u = v$
$\Rightarrow$ injective

D.

3B 21) $T \in \mathcal{L}(V, W)$

Suppose T is surjective, then each element in W is mapped at least once to an element to V.
Hence we can define a function such that Sw is the unique element in ~~V~~ such that $TSw = w$.

We prove that S is linear.

Additivity: Let $w_1, w_2 \in W$.

$$TS(w_1 + w_2) = w_1 + w_2$$
$$= TSw_1 + TSw_2$$
$$= T(Sw_1 + Sw_2)$$

$\Rightarrow$ S is closed under addition.

Homogeneity: Let $w_1 \in W$

$$TS\lambda w = \lambda w$$
$$= \lambda TSw$$
$$= T(\lambda Sw)$$
$$\Rightarrow \quad S\lambda w = \lambda Sw$$

$\Rightarrow$ homogeneity.

$\Rightarrow$ S is a linear map $\Rightarrow$ $S \in \mathcal{L}(W, V)$

Assuming $S \in \mathcal{L}(W, V)$ and $T \in \mathcal{L}(V, W)$ s.t. $TSw = w$,
we now prove that T must be surjective
For T to be surjective, range $T = W$

Since $T(Sw) = w$ for all $w \in W$,
T must be able to output all elements in W
$\Rightarrow$ range $T = W$
$\Rightarrow$ surjective.

$\square$.

3B.2a) Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in null φ. Prove that

$$V = \text{null}\,\varphi \oplus \{au : a \in \mathbb{F}\}.$$

$\{au : a \in \mathbb{F}\} \cap \text{null}\,\varphi = \{0\}$ since $u \notin \text{null}\,\varphi$.

$\{au : a \in \mathbb{F}\}$ is a subspace of V since it is
closed under addition: $a, b \in \mathbb{F}$ $\quad au + bu = (a+b)u$
closed under multiplication: $a, b \in \mathbb{F}$ $\quad b(au) = (ab)u$
Additive Identity: $\quad 0 \cdot u = 0$

let $S \in \ldots$

For $\varphi \in \mathcal{L}(V, \mathbb{F})$, $\varphi \in \mathcal{L}(\{au : a \in \mathbb{F}\}, \mathbb{F})$.

confusing. Define a separate mapping for $\mathcal{L}(\{au : a \in \mathbb{F}\}, \mathbb{F})$

$\Rightarrow \varphi$ is injective over $\mathcal{L}(\{au : a \in \mathbb{F}\}, \mathbb{F})$
since over $\{au : a \in \mathbb{F}\} \to \mathbb{F}$, null $\varphi = \{0\}$.
null $S = \{0\}$

$\dim \{au : a \in \mathbb{F}\} = \dim \text{null}\,\varphi + \dim \text{range}\,\varphi$
$1 = 0 + \dim \text{range}\,\varphi$ $\quad = \dim \text{null}\,S + \dim \text{range}\,S$
$\Rightarrow \dim \text{range}\,\varphi = 1$ $\quad 1 = 0 + \dim \text{range}\,S$
$\Rightarrow \text{range}\,\varphi = \mathbb{F}$. $\quad \Rightarrow \text{range}\,S = \mathbb{F}$.
$\Rightarrow \varphi \in \mathcal{L}(\{au : a \in \mathbb{F}\}, \mathbb{F})$ is bijective. $\Rightarrow S$ is bijective.

Since $\text{null}\,\varphi \cap \{au : a \in \mathbb{F}\} = \{0\}$,
$\text{null}\,\varphi \oplus \{au : a \in \mathbb{F}\}$.

Now we prove that $\text{null}\,\varphi \oplus \{au : a \in \mathbb{F}\} = V$.

Let $v \in V$ s.t $\varphi v = a$ where $a \in \mathbb{F}$

If $a = 0$, $v \in \text{null}\,\varphi \Rightarrow v = \text{null}\,\varphi \oplus \{au : a \in \mathbb{F}\}$ since only $0 \in \{au : a \in \mathbb{F}\}$ can satisfy the direct sum given the condition $a = 0$.

If $a \neq 0$, $v \in \{au : a \in \mathbb{F}\} \Rightarrow v = \text{null}\,\varphi \oplus \{au : a \in \mathbb{F}\}$ since null φ will not affect the result of φ. as $a \neq 0$.

Hence we proved that for any vector $v \in V$, it will be an element of
$\text{null}\,\varphi \oplus \{au : a \in \mathbb{F}\}$

Thus $\quad V = \text{null}\,\varphi \oplus \{au : a \in \mathbb{F}\}$

$\square$.

Linalg Homework 3.

Date No.

3.C3 Let $u_1, \dots, u_n$ be a basis for null T.

We can extend $u_1, \dots, u_n$ to a basis for V since null T is a subspace of V.

Let $u_1, \dots, u_n, v_1, \dots, v_m$ be a basis for V.

$v_1, \dots v_m \notin$ null T.

Let $w_1, \dots, w_p$ be a basis for range T.

Let $S_1, \dots, S_q$ be ~~the~~ a linearly independent list in W such that $w_1, \dots, w_p, S_1, \dots S_q$ forms a basis of W

Since $v_1, \dots, v_m \notin$ null T, then in T,

$Tv_1, \dots, Tv_m$ must form a spanning list for range T.

Suppose $\exists\ x, y \in \text{span}(v_1, \dots, v_m)$ s.t

$$Tx = Ty$$

$$Tx - Ty = 0$$

$$T(x-y) = 0.$$

$\Rightarrow x = y$ since span$(v_1, \dots, v_m)$ is not in the nullspace of T.

Thus $Tv_1, \dots, Tv_m$ forms a basis for range T.

Hence for $w_1, \dots, w_p$ that is a basis of range T,

$$p = m.$$

Let $Tv_1 = w_1$, $Tv_2 = w_2, \dots, Tv_m = w_m$.

Then $Tv_1 = 1 \cdot w_1, \dots, Tv_m = 1 \cdot w_m$

$\Rightarrow A_{j,j} = 1$ for $1 \le j \le \dim \text{range } T$.

$\therefore$ such a basis exists.

$\square$.

Date No.

3C.4 Let $v_1, \ldots, v_m$ be a basis of V

Since $v_1, \ldots, v_m$ form a basis,

there exists $u_1, \ldots, u_m \in W$ such that

$$Tv_1 = u_1$$
$$\vdots$$
$$Tv_m = u_m$$

For $Tv_1 = u_1$, if $u_1 = 0$, then

$Tv_1 = 0$

$\Rightarrow Tv_1 = 0 \cdot w_1 + 0\, w_2 + \ldots + 0\, w_n$

$\Rightarrow A_{1,1} = A_{1,2} = \ldots = A_{1,n} = 0$

$\Rightarrow$ column 1 is 0.

Now if $u_1 \neq 0$, then we can let u_1 be ~~as~~ part of a basis of W

Let $u_1 = w_1$. We can extend w_1 to form a basis of W

Since u_1 is a basis of W,

$Tv_1 = 1 \cdot u_1 + 0 \cdot w_2 + \ldots + 0 \cdot w_n$

$\Rightarrow$ column 1 is all 0 except for $A_{1,1} = 1$.

$\therefore$ such a basis $w_1, \ldots, w_n$ of W exists

$\square$.

Date No.

3D.7) Suppose V, W are finite dimensional.

Let $v \in V$.

Let $E = \{T \in \mathcal{L}(V,W) : Tv = 0\}$.

a) For E to be a subspace, E must satisfy

- closed under addition
- closed under scalar multiplication.
- Additive Identity.

- Addition: Let $T, S \in E$.

$$(T+S)v = Tv + Sv$$
$$= 0$$
$$\Rightarrow T+S \in E. \checkmark$$

Multiplication: Let $T \in E$

$$\lambda Tv = \lambda(Tv) = \lambda \cdot 0 = 0$$
$$\Rightarrow \lambda T \in E \checkmark$$

Additive identity: Let $T \in E$ s.t $Tv = 0$ for all $v \in V$.

For some $S \in E$

$$(S+T)v = Sv + Tv = Sv + 0$$
$$\Rightarrow (S+T)v = Sv$$
$$\Rightarrow T \text{ is additive identity. } \checkmark$$

$\therefore$ E is a subspace of $\mathcal{L}(V,W)$ $\square$.

b) Suppose $v \neq 0$, then E contains all linear maps in $\mathcal{L}(V,W)$ such that v is in their nullspace.

Let A be the matrix of $T \in \mathcal{L}(V,W)$ such that $Tv = 0$ and $v \neq 0$.

Since $v \neq 0$, let $v_1, \ldots, v_n$ be a basis of V such that $v_1 = v$.

Then $A_{\cdot,1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$\Rightarrow \dim A = (\dim V - 1)(\dim W)$$
$$= \dim V \dim W - \dim W.$$

//

Date No.

3D.16) Suppose V is finite dimensional, and $T \in \mathcal{L}(V)$.
Prove that $T = \lambda I$ i.f.f $ST = TS$ for every $S \in \mathcal{L}(V)$.

Suppose $T = \lambda I$, then for some $S \in \mathcal{L}(V)$

$$ST = S(\lambda I) \qquad TS = \lambda I(S)$$
$$= \lambda S I$$

For some $v \in V$,

$$\lambda S I v = \lambda S(Iv) \qquad \lambda I S v = \lambda S v.$$
$$= \lambda S v$$
$$\therefore \quad TSv = STv = \lambda S v$$
$$\Rightarrow \quad TS = ST$$

Now for the other direction.

Suppose there exists $T \in \mathcal{L}(V)$ such that $TS = ST$ for all $S \in \mathcal{L}(V)$.

Let $v \in V$. $TSv = STv$.

$$(TS - ST)v = 0$$

Let $u \in V$. Let $S \in \mathcal{L}(V)$ such that

$$Sv = 0 \text{ i.f.f } u = v.$$
$$TSu = T0 = 0$$
$$STu = 0$$
$$\Rightarrow Tu = \lambda u$$

since $S\lambda u = \lambda S u = \lambda 0 = 0$.

$\therefore$ For T to satisfy the declared S,
T must be the scalar multiple of the identity.

$\therefore T = \lambda I \Leftrightarrow ST = TS$ for all $S \in \mathcal{L}(V)$

D.

1) $T \in \mathcal{L}(V)$.

a) Example of V and $T \in \mathcal{L}(V)$ s.t $T^2 = T$.

$$TTv = Tv$$

Let $V = \mathcal{P}^1(\mathbb{R})$, $T \in \mathcal{L}(\mathcal{P}^1(\mathbb{R}))$ s.t for $v \in V$,

$$Tv = \begin{cases} v' & \text{if degree } v = 1 \\ v & \text{if degree } v = 0 \end{cases}$$

We show that T is a linear map.

<u>Additivity</u>: Let $u, v \in V$. By cases on the degree of u, v,

degree u = degree $v = 1$.

$$T(u+v) = u' + v' = T(u) + T(v) \quad \checkmark$$

degree $u = 1$, degree $v = 0$.

$$T(u+v) = u' + v = T(u) + T(v) \quad \checkmark.$$

degree u = degree $v = 0$.

$$T(u+v) = u+v = T(u) + T(v) \quad \checkmark.$$

$\Rightarrow$ T is additive.

<u>Homogeneity</u>: Let $u \in V$.

$$T(\lambda u) = \lambda u' = \lambda T(u) \quad \checkmark.$$

$\therefore$ T is a linear map.

Now we show that $TTv = Tv$. Let $v \in V$. Suppose degree $v = 1$.

$$TTv = T(Tv) = Tv' \text{ where degree } v' = 0.$$

$\Rightarrow Tv' = v'$

$\therefore TTv = Tv = v' \quad \checkmark$

Suppose degree $v = 0$. $TTv = Tv = v \quad \checkmark$

$\therefore$ T is a linear map such that $TTv = Tv$.

$\square$.

b) <u>Proposition</u>: If $T^2 = T$, then $V = \text{null}\, T \oplus \text{null}\,(T-I)$.

<u>Pf</u>: If $T^2 = T$, then for all $v \in V$,

$$T^2 v = Tv$$
$$T^2 v - Tv = 0$$
$$(T^2 - T)v = 0$$
$$T(T-I)v = 0.$$

Suppose $v \in \text{null}\,(T-I)$, then
$T(T-I)v = T0 = 0$.
Suppose $v \notin \text{null}\,(T-I)$, then
$T(T-I)v = Tu$, where $u = (T-I)v$.

$$Tu = 0 \Rightarrow u \in \text{null}\, T.$$

$\therefore$ For $T(T-I)v = 0$ for all $v \in V$,
v must either be in $\text{null}\, T$ or $\text{null}\,(T-I)$.

$$\Rightarrow V = \text{null}\, T + \text{null}\,(T-I)$$

Now we prove that $\text{null}\, T \cap \text{null}\,(T-I) = \{0\}$ for $\text{null}\, T \oplus \text{null}\,(T-I)$.

$\text{null}\,(T-I) = 0 \Rightarrow$ for $v \in \text{null}\,(T-I)$

$$(T-I)v = Tv - Iv = 0$$
$$\Rightarrow Tv = Iv$$
$$Tv = v.$$

$\Rightarrow \text{null}\,(T-I)$ are all vectors such that
$Tv = v$.

$\text{null}\, T$ are all vectors $v \in \text{null}\, T$ s.t
$Tv = 0$.

$\therefore \text{null}\, T \cap \text{null}\,(T-I)$
$\Rightarrow Tv = v = 0.$
$\Rightarrow v = 0$
$\Rightarrow \text{null}\, T \cap \text{null}\,(T-I) = \{0\}.$

$\Rightarrow V = \text{null}\, T \oplus \text{null}\,(T-I)$

D.

c) <u>Proposition</u>: If $V = \text{null}\, T + \text{null}(T-I)$, then $T^2 = T$.

<u>Pf</u>: Let $V = \text{null}\, T + \text{null}(T-I)$.

First, we can show that $\text{null}\, T \cap \text{null}(T-I) = \{0\}$

$\text{null}\, T = 0 \Rightarrow v \in V$ such that $Tv = 0$.

$\text{null}(T-I) = 0 \Rightarrow v \in V$ such that

$$(T-I)v = 0$$
$$Tv - v = 0$$
$$Tv = v.$$

$\therefore\ \text{null}\, T \cap \text{null}(T-I)$

$\Rightarrow Tv = v = 0$

$\Rightarrow v = 0$

$\Rightarrow \text{null}\, T \cap \text{null}(T-I) = \{0\}$

$\Rightarrow V = \text{null}\, T \oplus \text{null}(T-I)$.

Let $v \in V$ such that $v \in \text{null}\, T$. Then $Tv = 0$.
Since $v \in \text{null}\, T$, $v \notin \text{null}(T-I)$.

$\Rightarrow (T-I)v = u$, where $u \neq 0$.

$$u = Tv - v.$$
$$= 0 - v$$
$$= -v.$$

Since $(T-I)v = -v$.

$$T(T-I)v = T(-v) = -T(v)$$
$$= 0.$$

Alternatively, if $v \in \text{null}(T-I)$, $v \notin \text{null}\, T$,

$$(T-I)v = 0$$
$$Tv - v = 0$$
$$Tv = v.$$

$$\Rightarrow (T-I)Tv$$
$$= (T-I)v$$
$$= 0.$$

$\therefore$ For all $v \in V$, $(T-I)Tv = 0$

$$\Rightarrow (T^2 - T)v = 0$$
$$T^2 v - Tv = 0$$
$$T^2 v = Tv$$

$$\Rightarrow T^2 = T$$

$\square$.

d) $T^2 = -I$.

$T^2 + I = 0$.

Let $V = \mathbb{C}$, $Tv = iv$.

For some $c \in \mathbb{C}$, let $c = a + bi$.

$$Tc = i(a+bi)$$
$$= ai - b.$$

$$T(Tc) = T(ai-b)$$
$$= -a - bi$$
$$= -c$$
$$\Rightarrow T^2 c = -c$$
$$\Rightarrow T^2 = -I.$$

$\square$.

2) V, W are finite dimensional.

$T \in \mathcal{L}(V,W)$ and $S \in \mathcal{L}(W,U)$.

dim range $S \leq$ dim W via the same reasoning as above.

$Sw \in$ range S.

dim V = dim range T + dim null T.
dim range T = dim V − dim null T
$\Rightarrow$ dim range $T \leq$ dim V.

Let $A \in \mathcal{L}$(range T, W).

$AT = ST$ since for the map ST, let $v \in V$.

$STv = Sw$, where $Tv = w$.
$\Rightarrow w \in$ range T.

$\therefore$ For the map A,

dim range T = dim range A + dim null A
dim range A = dim range T − dim null A.

$\Rightarrow$ dim range $A \leq$ dim range T.

dim range A = dim range AT since
$ATv = Aw$, where $w \in$ range T

dim range A = dim range ST since $ST = AT$

$\therefore$ dim range $A \leq$ dim range T
$\Rightarrow$ dim range $ST \leq$ dim range T //

b) <u>Proposition:</u>

$\dim(\text{range } ST) = \dim(\text{range } T) \Leftrightarrow \text{range } T + \text{null } S = \text{range } T \oplus \text{null } S$

<u>Pf</u>: We first prove the forward implication.

Suppose $\dim(\text{range } ST) = \dim(\text{range } T)$

Let $A \in \mathcal{L}(\text{range } T, U)$. s.t for all $w \in \text{range } T$,

$$Aw = Sw.$$

$$\dim \text{range } T = \dim \text{range } A + \dim \text{null } A.$$

$$\dim \text{range } A = \dim \text{range } ST$$

since domain of A is range T and $Aw = Sw$ for all $w \in \text{range } T$.

Hence $\dim \text{range } A = \dim \text{range } ST$
$= \dim \text{range } T.$

$\therefore \dim \text{null } A = \dim \text{range } T - \dim \text{range } A$
$= \dim \text{range } T - \dim \text{range } T$
$= 0.$

$\Rightarrow$ For all $w \in \text{range } T$, $w \notin \text{null } A$ except $w = 0$.

Since $w \notin \text{null } A$, $w \notin \text{null } S$ by definition of A.

$\therefore \text{range } T \cap \text{null } S = \{0\}$. since no element of T is in null S other than 0.

$\therefore \text{range } T \oplus \text{null } S.$

Now, we will prove the backwards implication.

Suppose $\text{range } T \oplus \text{null } S$, then

$$\text{range } T \cap \text{null } S = \{0\}$$

Then all values in range T map to a non-zero value in U through S, other than 0.

For all $v \in V$, $STv = Sw$, where $w \in \text{range } T$,
$= u$, where $u \in \text{range } ST$.

Since $Sw = u$, where $u \in U$, $u \neq 0$, we must now show that

$$\dim \text{range } T = \dim \text{range } ST.$$

For $\dim \text{range } T = \dim \text{range } ST$, we must show that for
$A \in \mathcal{L}(\text{range } T, U)$ s.t $Aw = Sw$,

$\dim \text{range } A = \dim \text{range } ST$ by definition

$\dim \text{null } A = 0$ since all values in range T map to non-zero $u \in U$ through S, and
$Aw = Sw$ for $w \in \text{range } T$.

$\therefore \dim \text{range } T = \dim \text{range } A + \dim \text{null } A$

$\dim \operatorname{range} T = \dim \operatorname{range} ST + 0$

$\Rightarrow \dim \operatorname{range} T = \dim \operatorname{range} ST$

$\therefore \dim \operatorname{range} ST = \dim \operatorname{range} T \iff \operatorname{range} T + \operatorname{null} S = \operatorname{range} T \oplus \operatorname{null} S.$ $\square$.

c) <u>Proposition</u>: $\dim \operatorname{null} ST \leq \dim(\operatorname{null} S) + \dim(\operatorname{null} T)$.

$$\dim V = \dim \operatorname{range} ST + \dim \operatorname{null} ST$$
$$\dim \operatorname{null} ST = \dim V - \dim \operatorname{range} ST$$

Let $A \in \mathcal{L}(\operatorname{range} T, U)$ s.t. $\operatorname{range} T$

$$Aw = Sw \text{ for } w \in \operatorname{range} T.$$

$$\begin{aligned} \dim \operatorname{range} T &= \dim \operatorname{range} A + \dim \operatorname{null} A. \\ &= \dim \operatorname{range} ST + \dim \operatorname{null} A. \end{aligned}$$

$\dim V - \dim \operatorname{range} ST \geq$
$\dim V - \dim \operatorname{range} T + \dim \operatorname{null} S$

$$\dim \operatorname{null} A \leq \dim \operatorname{null} S.$$

$$\dim \operatorname{range} T \leq \dim \operatorname{range} ST + \dim \operatorname{null} S$$

$$\dim \operatorname{range} T - \dim \operatorname{null} S \leq \dim \operatorname{range} ST$$

$$-\dim \operatorname{range} ST \leq \dim \operatorname{null} S - \dim \operatorname{range} T$$

$$\begin{aligned} \dim \operatorname{null} ST &= \dim V - \dim \operatorname{range} ST \\ &= \dim V + (-\dim \operatorname{range} ST) \\ &\leq \dim V + \dim \operatorname{null} S - \dim \operatorname{range} T \\ &= \dim \operatorname{range} T + \dim \operatorname{null} T + \dim \operatorname{null} S - \dim \operatorname{range} T \\ &= \dim \operatorname{null} T + \dim \operatorname{null} S \end{aligned}$$

$$\Rightarrow \dim \operatorname{null} ST \leq \dim \operatorname{null} T + \dim \operatorname{null} S$$ $\square$.

d) $\dim(\text{null}\, ST) = \dim(\text{null}\, S) + \dim(\text{null}\, T)$

Let $v \in V$ such that $STv = 0$.

$\Rightarrow$ either $Tv = 0$ or $Tv \in \text{null}\, S$.

$\Rightarrow$ v must be in either null T or

v must be such that $Tv \in \text{null}\, S$.

Let $A = \{v : Tv \in \text{null}\, S \text{ and } v \notin \text{null}\, T\}$

$\text{null}\, ST = \text{null}\, T \oplus A$

$\Rightarrow \dim \text{null}\, ST = \dim \text{null}\, T + \dim A - \dim(\text{null}\, T \cap A)$

$= \dim \text{null}\, T + \dim A$

Since we want $\dim \text{null}\, ST = \dim(\text{null}\, S) + \dim(\text{null}\, T)$

$\dim \text{null}\, S + \dim \text{null}\, T = \dim \text{null}\, T + \dim A$

$\dim \text{null}\, S = \dim A.$

Every vecto in A mapped to a vector in null S through T.

Hence di