1 Inclusion-Exclusion

1. For a group of 7 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.

2. Alice attends a small college in which each class meets only once a week. She is deciding between 30 non-overlapping classes. There are 6 classes to choose from for each day of the week, Monday through Friday. Trusting in the benevolence of randomness, Alice decides to register for 7 randomly selected classes out of the 30, with all choices equally likely. What is the probability that she will have classes every day, Monday through Friday? (This problem can be done either directly using the naive definition of probability, or using inclusion-exclusion.)

1) Let S be the event that the group contains a birthday in spring.
Let B be the event that the group contains a birthday in summer
Let F be the event that the group contains a birthday in fall.
Let W be the event that the group contains a birthday in winter.

What is $Pr(S \cap B \cap F \cap W)$?

$$Pr(S) = Pr(B) = Pr(F) = Pr(W) = 1 - \left(\tfrac{3}{4}\right)^n$$

$$Pr(S \cap B) = \ldots = Pr(F \cap W)$$

$$Pr(S \cup B) = Pr(S) + Pr(B) - Pr(S \cap B)$$

$$Pr(S \cup B) = 1 - \left(\tfrac{2}{4}\right)^n$$
$$\Rightarrow Pr(S \cap B) = Pr(S) + Pr(B) - Pr(S \cup B)$$
$$= 2\left(1 - \left(\tfrac{3}{4}\right)^n\right) - \left(1 - \left(\tfrac{2}{4}\right)^n\right)$$
$$= 2 - 2\left(\tfrac{3}{4}\right)^n - 1 + \left(\tfrac{2}{4}\right)^n$$
$$= 1 - 2\left(\tfrac{3}{4}\right)^n + \left(\tfrac{2}{4}\right)^n$$

$$Pr(S \cap B \cap F) = \ldots = Pr(B \cap F \cap W)$$

$$Pr(S \cup B \cup F) = Pr(S) + Pr(B) + Pr(F) - Pr(S \cap B) - Pr(S \cap F) - Pr(B \cap F) + Pr(S \cap B \cap F)$$
$$= 3\left(1 - \left(\tfrac{3}{4}\right)^n\right) - 3\left(1 - 2\left(\tfrac{3}{4}\right)^n + \left(\tfrac{2}{4}\right)^n\right) + Pr(S \cap B \cap F)$$
$$= -3\left(\tfrac{3}{4}\right)^n + 6\left(\tfrac{3}{4}\right)^n - 3\left(\tfrac{2}{4}\right)^n + Pr(S \cap B \cap F)$$
$$= 3\left(\tfrac{3}{4}\right)^n - 3\left(\tfrac{2}{4}\right)^n + Pr(S \cap B \cap F)$$

$$Pr(S \cup B \cup F) = 1 - \left(\tfrac{1}{4}\right)^n \Rightarrow Pr(S \cap B \cap F) = 1 - \left(\tfrac{1}{4}\right)^n + 3\left(\tfrac{2}{4}\right)^n - 3\left(\tfrac{3}{4}\right)^n$$

$$Pr(S \cup B \cup F \cup W) = 4\left(1 - \left(\tfrac{3}{4}\right)^n\right) - \binom{4}{2}\left(1 - 2\left(\tfrac{3}{4}\right)^n + \left(\tfrac{2}{4}\right)^n\right) + \binom{4}{3}\left(1 - \left(\tfrac{1}{4}\right)^n + 3\left(\tfrac{2}{4}\right)^n - 3\left(\tfrac{3}{4}\right)^n\right) - Pr(S \cap B \cap F \cap W)$$

$$1 = 4 - 4\left(\tfrac{3}{4}\right)^n - 6 + 12\left(\tfrac{3}{4}\right)^n - 6\left(\tfrac{2}{4}\right)^n + 4 - 4\left(\tfrac{1}{4}\right)^n + 12\left(\tfrac{2}{4}\right)^n - 12\left(\tfrac{3}{4}\right)^n - Pr(S \cap B \cap F \cap W)$$
$$1 = 2 - 4\left(\tfrac{3}{4}\right)^n + 6\left(\tfrac{2}{4}\right)^n - 4\left(\tfrac{1}{4}\right)^n - Pr(S \cap B \cap F \cap W)$$

$$Pr(S \cap B \cap F \cap W) = 1 - 4\left(\tfrac{3}{4}\right)^n + 6\left(\tfrac{2}{4}\right)^n - 4\left(\tfrac{1}{4}\right)^n$$

For $n=7$, $Pr(S \cap B \cap F \cap W) \approx \boxed{0.513}$

2)

Let M be event that chosen classes have no Monday
Let T be event that chosen classes have no Tuesday
Let W be event that chosen classes have no Wednesday
Let H be event that chosen classes have no Thursday
Let F be event that chosen classes have no Friday

$$\Pr(\text{chosen classes have all days}) = 1 - \Pr(M \cup T \cup W \cup H \cup F)$$

$$\Pr(M \cup T \cup W \cup H \cup F) = 5\Pr(M) - \binom{5}{2}\Pr(M \cap T) + \binom{5}{3}\Pr(M \cap T \cap W) - \binom{5}{4}\Pr(M \cap T \cap W \cap H) + \Pr(M \cap T \cap W \cap H \cap F)$$

$$\Pr(M) = \Pr(T) = \ldots = \Pr(F) = \frac{24}{30} \cdot \frac{23}{29} \cdot \ldots \cdot \frac{18}{24} = \frac{24!\,23!}{30!\,17!}$$

$$\Pr(M \cap T) = \frac{18}{30} \cdot \frac{17}{29} \cdot \ldots \cdot \frac{12}{24} = \frac{18!\,23!}{30!\,11!}$$

$$\frac{\binom{24}{7}}{\binom{30}{7}}$$

$$\Pr(M \cap T \cap W) = \frac{12!\,23!}{30!\,5!}$$

$$\Pr(M \cap T \cap W \cap H) = 0, \quad \Pr(M \cap T \cap W \cap H \cap F) = 0$$

$$\Pr(M \cup T \cup W \cup H \cup F) = 5\left(\frac{24!\,23!}{30!\,17!}\right) - 10\left(\frac{18!\,23!}{30!\,11!}\right) + 10\left(\frac{12!\,23!}{30!\,5!}\right)$$

$$= 0.6976$$

$$\Pr(\text{chosen classes have all days}) = 1 - 0.6976 = \boxed{0.302}$$

1. Is it possible that an event is independent of itself? If so, when?
2. Is it always true that if A and B are independent events, then A^c and B^c are independent events? Show that it is, or give a counterexample.
3. Give an example of 3 events A, B, C which are pairwise independent but not independent. Hint: find an example where whether C occurs is completely determined if we know whether A occurred and whether B occurred, but completely undetermined if we know only one of these things.
4. Give an example of 3 events A, B, C which are not independent, yet satisfy $P(A \cap B \cap C) = P(A)P(B)P(C)$. Hint: consider simple and extreme cases.

2.1) For A to be independent of B, then $P(A|B) = P(A)$.
For A to be independent of itself, then

$$P(A|A) = P(A)$$

$$\frac{P(A \cap A)}{P(A)} = P(A)$$

$$P(A \cap A) = [P(A)]^2$$

Since $P(A \cap A) = P(A)$

$$P(A) = (P(A))^2$$

$$[P(A)]^2 - P(A) = 0$$

$$P(A)[P(A) - 1] = 0$$

$$\Rightarrow P(A) = 1 \text{ or } P(A) = 0$$

$\square$

2.2) If A and B are independent, then

$$P(A|B) = P(A)$$

$$P(A \cap B) = P(A) \cdot P(B)$$

For A^c and B^c to be independent,

$$P(A^c \cap B^c) = P(A^c) P(B^c)$$

$$P(A^c) = 1 - P(A)$$

$$P(B^c) = 1 - P(B)$$

$$(1 - P(A))(1 - P(B))$$

$$= 1 - P(B) - P(A) + P(A)P(B)$$

$$= 1 - P(A \cup B) \quad (\text{inclusion-exclusion})$$

$$= P[(A \cup B)^c]$$

$$= P(A^c \cap B^c)$$

$$\Rightarrow P(A^c)P(B^c) = P(A^c \cap B^c)$$

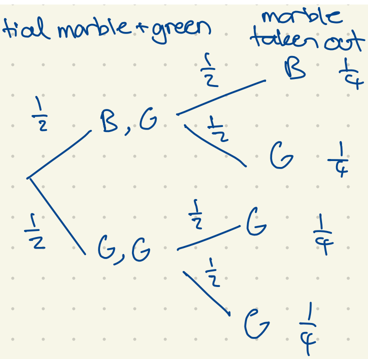
$\square$.

2.3) $P(A \cap B) = P(A)P(B)$
$P(B \cap C) = P(B)P(C)$
$P(A \cap C) = P(A)P(C)$

But,
$P(A \cap B \cap C) \neq P(A)P(B)P(C)$

Let A be the event you flip heads
Let B be the event you flip tails
Let C be the event of you winning a bet where your two tosses are heads and tails

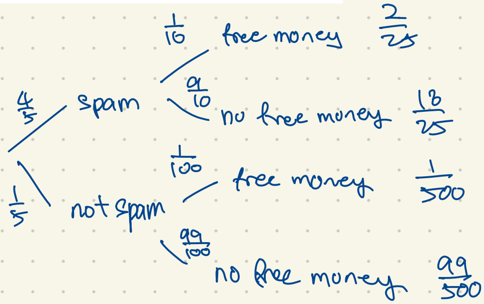
1. A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green?



$$\Pr(\text{remaning green} \mid \text{taken green}) = \frac{\Pr(\text{remaning green} \cap \text{taken green})}{\Pr(\text{taken green})}$$

$$= \frac{\left(\frac{1}{2}\right)}{\frac{3}{4}} = \boxed{\frac{2}{3}}$$

2) 2. A spam filter is designed by looking at commonly occurring phrases in spam. Suppose that 80% of email is spam. In 10% of the spam emails, the phrase "free money" is used, whereas this phrase is only used in 1% of non-spam emails. A new email has just arrived, which does mention "free money". What is the probability that it is spam?



$$\Pr(\text{spam} \mid \text{free money}) = \frac{\Pr(\text{spam} \cap \text{free money})}{\Pr(\text{free money})}$$

$$= \frac{\left(\frac{2}{25}\right)}{\left(\frac{2}{25} + \frac{1}{500}\right)}$$

$$\boxed{= \frac{40}{41}}$$

3. Let G be the event that a certain individual is guilty of a certain robbery. In gathering evidence, it is learned that an event E_1 occurred, and a little later it is also learned that another event E_2 also occurred.

(a) Is it possible that individually, these pieces of evidence increase the chance of guilt (so $P(G|E_1) > P(G)$ and $P(G|E_2) > P(G)$), but together they decrease the chance of guilt (so $P(G|E_1, E_2) < P(G)$)?

(b) Show that the probability of guilt given the evidence is the same regardless of whether we update our probabilities all at once, or in two steps (after getting the first piece of evidence, and again after getting the second piece of evidence). That is, we can either update all at once (computing $P(G|E_1, E_2)$ in one step), or we can first update based on E_1, so that our new probability function is $P_{\text{new}}(A) = P(A|E_1)$, and then update based on E_2 by computing $P_{\text{new}}(G|E_2)$.

a) $P(G|E_1 \cap E_2) = \dfrac{P(G \cap E_1 \cap E_2)}{P(E_1 \cap E_2)}$

$P(G|E_1) = \dfrac{P(G \cap E_1)}{P(E_1)}$

Yes. If its common for E_1 and E_2 to occur together, but uncommon for E_1 and E_1 to occur alone.

b) $P(G|E_1 \cap E_2) = \dfrac{P(G \cap E_1 \cap E_2)}{P(E_1 \cap E_2)}$

$P(G|E_1) = \dfrac{P(G \cap E_1)}{P(E_1)}$

$P_{new}(G|E_2) = \dfrac{P_{new}(G \cap E_2)}{P_{new}(E_2)} = \dfrac{P(G \cap E_2 | E_1)}{P(E_2|E_1)} = \dfrac{\frac{P(G \cap E_2 \cap E_1)}{\cancel{P(E_1)}}}{\left(\frac{P(E_2 \cap E_1)}{\cancel{P(E_1)}}\right)}$

$= \dfrac{P(G \cap E_2 \cap E_1)}{P(E_2 \cap E_1)}$

$= P(G|E_1 \cap E_2)$

D.

4. A crime is committed by one of two suspects, A and B. Initially, there is equal evidence against both of them. In further investigation at the crime scene, it is found that the guilty party had a blood type found in 10% of the population. Suspect A does match this blood type, whereas the blood type of Suspect B is unknown.

(a) Given this new information, what is the probability that A is the guilty party?

(b) Given this new information, what is the probability that B's blood type matches that found at the crime scene?

a) Let G_A be event that A is guilty. Let B_A be event that A matches guilty blood type

Let G_B be event that B is guilty. Let B_B be event that B matches guilty blood type

$$P(G_A) = 0.5$$
$$P(G_B) = 0.5$$

$$P(G_A \mid B_A) = \frac{P(G_A \cap B_A)}{P(B_A)} = \frac{P(B_A \mid G_A) \cdot P(G_A)}{P(B_A)}$$
$$= \frac{P(B_A \mid G_A) \cdot P(G_A)}{P(B_A \mid G_A) \cdot P(G_A) + P(B_A \mid G_B) \cdot P(G_B)}$$
$$= \frac{0.5}{0.5 + 0.1 \cdot 0.5} \boxed{= \frac{10}{11}}$$

b) $$P(B_B \mid B_A) = P(B_B \mid B_A \cap G_A) \cdot P(G_A \mid B_A) + P(B_B \mid B_A \cap G_B) \cdot P(G_B \mid B_A)$$
$$= \frac{1}{10} \cdot \frac{10}{11} + 1 \cdot \frac{1}{11} = \boxed{\frac{2}{11}}$$

5)

5. You are going to play 2 games of chess with an opponent whom you have never played against before (for the sake of this problem). Your opponent is equally likely to be a beginner, intermediate, or a master. Depending on which, your chances of winning an individual game are 90%, 50%, or 30%, respectively.

(a) What is your probability of winning the first game?

Let event where first game is won be F
Let event where player is beginner be B
Let event where player is intermediate be I
Let event where player is expert be E

$$P(F) = P(F|B)\cdot P(B) + P(F|I)\cdot P(I) + P(F|E)\cdot P(E)$$
$$= \frac{9}{10}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + \frac{3}{10}\cdot\frac{1}{3}$$
$$= 17/30 \approx \boxed{0.567}$$

b)

(b) Congratulations: you won the first game! Given this information, what is the probability that you will also win the second game (assume that, given the skill level of your opponent, the outcomes of the games are independent)?

(c) Explain the distinction between assuming that the outcomes of the games are independent and assuming that they are conditionally independent given the opponent's skill level. Which of these assumptions seems more reasonable, and why?

$$\frac{P(F\cap B)}{P(B)} = P(F|B)\cdot P(B)$$

$P(F\cap B)$

Let event where second game is won be S

$$P(S|F) = \frac{P(S\cap F)}{P(F)}$$

$$P(S\cap F) = P(F|B)^2\cdot P(B) + P(F|I)^2\cdot P(I) + P(F|E)^2\cdot P(E)$$
$$= \frac{81}{100}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} + \frac{9}{100}\cdot\frac{1}{3} = \frac{23}{60}$$

$$P(S|F) = \frac{23}{60}\Big/\frac{17}{30} = \frac{23}{34} = 0.676$$

c) outcomes are independent v.s conditionally independent given skill level.

Outcomes are independent $\Rightarrow$ Probability that outcome of one game does not affect outcome of second game.
$\Rightarrow P(S\cap F) = P(F)P(F)$

Conditionally independent given skill level $\Rightarrow$ Given that we know an opponents skill level, the outcome of one game does not ~~affect~~ the outcome of the second game.
$\Rightarrow P(S\cap F|E) = P(S|E)\cdot P(F|E)$

2nd makes more sense. Probability that you win first match depends on the opponent skill level. Winning the first makes gives info about the skill level of the opponent.

1. Arby has a belief system assigning a number $P_{\text{Arby}}(A)$ between 0 and 1 to every event A (for some sample space). This represents Arby's subjective degree of belief about how likely A is to occur. For any event A, Arby is willing to pay a price of $1000 \cdot P_{\text{Arby}}(A)$ dollars to buy a certificate such as the one shown below:

> **Certificate**
>
> The owner of this certificate can redeem it for \$1000 if A occurs. No value if A does not occur, except as required by federal, state, or local law. No expiration date.

Likewise, Arby is willing to sell such a certificate at the same price. Indeed, Arby is willing to buy or sell any number of certificates at this price, as Arby considers it the "fair" price.

Arby, not having taken Stat 110, stubbornly refuses to accept the axioms of probability. In particular, suppose that there are two *disjoint* events A and B with

$$P_{\text{Arby}}(A \cup B) \neq P_{\text{Arby}}(A) + P_{\text{Arby}}(B).$$

Show how to make Arby go bankrupt, by giving a list of transactions Arby is willing to make that will *guarantee* that Arby will lose money (you can assume it will be known whether A occurred and whether B occurred the day after any certificates are bought/sold).

1) For Arby to lose money,
$P(A) > P_{arby}(A)$

$P(A \cup B) = P(A) + P(B)$
$P_{Arby}(A \cup B) \neq P(A) + P(B)$

$\Rightarrow P_{Arby}(A \cup B) > P(A) + P(B)$ or
$P_{Arby}(A \cup B) < P(A) + P(B)$

Suppose $P_{Arby}(A \cup B) < P_{Arby}(A) + P_{Arby}(B)$
then buy all $P_{Arby}(A \cup B)$

Suppose $P_{Arby}(A \cup B) > P_{Arby}(A) + P_{Arby}(B)$
then $P_{Arby}[(A \cup B)^c]$
$= 1 - P_{Arby}(A \cup B)$
$< 1 - P_{Arby}(A) - P_{Arby}(B)$

$\Rightarrow$ By all $P_{Arby}[(A \cup B)^c]$

2)

2. A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit ("void in a suit" means having no cards of that suit)?

Let S, H, C, D be events when suits are missing from hand

$P(S) = \dfrac{\binom{39}{13}}{\binom{52}{13}}$ $\quad P(S \cap H) = \dfrac{\binom{26}{13}}{\binom{52}{13}}$ $\quad P(S \cap H \cap C) = \dfrac{1}{\binom{52}{13}}$

$P(S \cup H \cup C \cup D) = 4P(S) - \binom{4}{2} P(S \cap H) + \binom{4}{3} P(S \cap H \cap C)$

≈ 0.051

3. A family has 3 children, creatively named A, B, and C.

(a) Discuss intuitively (but clearly) whether the event "A is older than B" is independent of the event "A is older than C."

(b) Find the probability that A is older than B, given that A is older than C.

a) Let A_B be the event that A is older than B. Same applies to all children.
No. Suppose A_C, then by birth order, the sequences possible are:

C, A, B
C, B, A ✓
B, C, A ✓

Hence A_B given A_C only has 2 possible sequences.
However A_B independently has 3 options.

C, B, A
B, C, A
B, A, C

Thus, knowing A_C narrows down possible sequences for A_B from 3 to 2.
Therefore they are not independent.

b) $P(A_B | A_C) = \dfrac{P(A_B \cap A_C)}{P(A_C)}$

$P(A_B \cap A_C) = \dfrac{2}{3!}$ $\quad P(A_C) = \dfrac{3}{3!}$

$\therefore P(A_B | A_C) = \left(\dfrac{2}{3!}\right) / \left(\dfrac{3}{3!}\right) = \dfrac{2}{3}$ //

4)

4. Two coins are in a hat. The coins look alike, but one coin is fair (with probability 1/2 of Heads), while the other coin is biased, with probability 1/4 of Heads. One of the coins is randomly pulled from the hat, without knowing which of the two it is. Call the chosen coin "Coin C".

(a) Coin C is tossed twice, showing Heads both times. Given this information, what is the probability that Coin C is the fair coin?

(b) Are the events "first toss of Coin C is Heads" and "second toss of Coin C is Heads" independent? Explain.

(c) Find the probability that in 10 flips of Coin C, there will be exactly 3 Heads. (The coin is equally likely to be either of the 2 coins; do not assume it already landed Heads twice as in (a).)

$$P(H) = P(H|U) \cdot P(U) + P(H|F) \cdot P(F) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

a) $P(C \text{ is fair} \mid HH) = \dfrac{P(C \text{ is fair} \cap HH)}{P(HH)}$

$= \dfrac{1/4 \cdot \frac{1}{2}}{P(HH|F) \cdot P(F) + P(HH|U) \cdot P(U)}$

$= \dfrac{1/4 \cdot \frac{1}{2}}{\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{1}{2}} = \dfrac{4}{5}$ //

b) No. Knowing that first toss is heads provides information on the type of coin being used.

c) $P(H \times 3 \cap T \times 7 \mid F) \cdot P(F) + P(H \times 3 \cap T \times 7 \mid U) \cdot P(U)$

$= \binom{10}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7 \cdot \left(\frac{1}{2}\right) + \binom{10}{3}\left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)$

$= \binom{10}{3}\left(\frac{1}{2}\right)\left(\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7 + \left(\frac{1}{2}\right)^{10}\right)$

$= \binom{10}{3}\left(\frac{1}{2}\right)\left(\frac{3^7}{4^{10}} + \frac{1}{2^{10}}\right)$

5. A woman has been murdered, and her husband is accused of having committed the murder. It is known that the man abused his wife repeatedly in the past, and the prosecution argues that this is important evidence pointing towards the man's guilt. The defense attorney says that the history of abuse is irrelevant, as only 1 in 1000 men who beat their wives end up murdering them.

Assume that the defense attorney's 1 in 1000 figure is correct, and that half of men who murder their wives previously abused them. Also assume that 20% of murdered women were killed by their husbands, and that if a woman is murdered and the husband is *not* guilty, then there is only a 10% chance that the husband abused her. What is the probability that the man is guilty? Is the prosecution right that the abuse is important evidence in favor of guilt?

Let event where husband murder wife be H
event where husband abuses wife be A
event where wife is murdered be M

$P(A|M) = P(A|M \cap H^c) \cdot P(H^c|M) + P(A|M \cap H) \cdot P(H|M)$

$P(H|A) = 1/1000$ $\quad P(H|M) = 1/5$

$P(A|H) = 1/2$ $\quad P(A|M \cap H^c) = 1/10$

$P(H|M \cap A) = \dfrac{P(M \cap A|H)\, P(H)}{P(M \cap A)} = \dfrac{P(A|H) \cdot P(H)}{P(M \cap A)}$

$P(M \cap A) = P(A|M) \cdot P(M)$

$= [P(A|M \cap H^c) \cdot P(M \cap H^c) + P(A|H) \cdot P(H)] \cdot P(M)$

$= [1/10 \cdot P(M \cap H^c) + 1/2\, P(H)] \cdot P(M)$

$P(H^c|M) = \frac{4}{5}$

$P(H^c \cap M) = \frac{4}{5} \cdot P(M) = 4P(H)$

$P(M \cap H^c) = P(H^c|M) \cdot P(M)$

$P(M \cap A) = [1/10 \cdot 4/5\, P(M) + \frac{1}{2} P(H)] \cdot P(M)$

$= \frac{2}{25} P(M)^2 + \frac{1}{2} P(H) \cdot P(M)$

$$P(H \mid M \cap A) = \frac{\frac{1}{2} \cdot P(H)}{\frac{2}{25} P(M)^2 + \frac{1}{2} P(H) \cdot P(M)} = \frac{1}{P(M)} \left[\frac{P(H)}{\frac{4}{25} P(M) + P(H)} \right]$$

$$P(H \mid M) = \frac{P(H \cap M)}{P(M)} = \frac{P(H)}{P(M)} = \frac{1}{5}$$

$$\Rightarrow \quad P(M) = 5P(H)$$

$$\Rightarrow \quad P(H \mid M \cap A) = \frac{1}{5P(H)} \cdot \frac{P(H)}{\frac{4}{5} P(H) + P(H)}$$

$$= \frac{1}{5} \cdot \frac{1}{\frac{9}{5} P(H)} = \frac{1}{9P(H)}$$

$$P(H \mid M \cap A) = \frac{1}{9P(H)}$$

$$P(H \mid M) = \frac{P(M \mid H) \cdot P(H)}{P(M)}$$

$$P(H \mid M \cap A) = \frac{500}{9} P(A)$$

$$= \frac{5}{9P(M)}$$

$$P(H \mid M \cap A) = \frac{P(H \cap M \cap A)}{P(M \cap A)} = \frac{P(H \cap A)}{P(M \cap A)} = \frac{P(H)}{-}$$

$$P(H \cap A) = P(H \mid A) \cdot P(A) = P(A \mid H) \cdot P(H)$$

$$P(H \mid M \cap A) = \frac{P(H \cap M \cap A)}{P(M \cap A)}$$

$$= \frac{P(A \mid H \cap M) \cdot P(H \cap M)}{P(M \cap A)}$$

$$P(A \mid M) = \frac{P(M \cap A)}{P(M)}$$

$$= \frac{P(A \mid H) \cdot P(H)}{P(A \mid M) \cdot P(M)} = \frac{1}{5} \frac{P(A \mid H)}{P(A \mid M)}$$

$$= \frac{1}{10} \frac{1}{P(A \mid M)} = \frac{1}{10} \cdot \frac{50}{9} = \frac{5}{9} \;//$$

b) $P(H \mid M) = 1/5$ while $P(H \mid M \cap A) = \frac{5}{9}$
so yes.

D.

6. A family has two children. Assume that birth month is independent of gender, with boys and girls equally likely and all months equally likely, and assume that the elder child's characteristics are independent of the younger child's characteristics).

(a) Find the probability that both are girls, given that the elder child is a girl who was born in March.

(b) Find the probability that both are girls, given that at least one is a girl who was born in March.

6) Let M_E be event that eldest is born in March
Let M_Y be event that youngest is born in March

Let G_E be event that eldest is girl
Let G_Y be event that youngest is girl.

$$P(G_E \cap G_Y \mid M_E) = \tilde{P}(G_E \mid G_Y) \cdot \tilde{P}(G_Y) \quad \text{where } \tilde{P}(x) = P(x \mid M_E)$$
$$= \tilde{P}(G_E) \cdot \tilde{P}(G_Y)$$
$$= P(G_E \mid M_E) \cdot P(G_Y \mid M_E)$$
$$= P(G_E \mid M_E) \cdot P(G_Y)$$
$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(G_E \cap G_Y \mid (M_E \cap G_E) \cup (M_Y \cap G_Y))$$
$$= \frac{P[(M_E \cap G_E) \cup (M_Y \cap G_Y) \mid G_E \cap G_Y] \cdot P(G_E \cap G_Y)}{P[(M_E \cap G_E) \cup (M_Y \cap G_Y)]}$$

$$P[(M_E \cap G_E) \cup (M_Y \cap G_Y)] = P(M_E \cap G_E) + P(M_Y \cap G_Y) - P(M_E \cap G_E \cap M_Y \cap G_Y)$$
$$= \frac{1}{12} \cdot \frac{1}{2} + \frac{1}{12} \cdot \frac{1}{2} - \frac{1}{576}$$
$$= \frac{47}{576}$$

$$P(G_E \cap G_Y) = \frac{1}{4}$$

$$P[(M_E \cap G_E) \cup (M_Y \cap G_Y) \mid G_E \cap G_Y]$$
$$= P[M_E \cup M_Y] = \frac{1}{6} - P(M_E \cap M_Y)$$
$$= \frac{1}{6} - \frac{1}{144}$$
$$= \frac{23}{144}$$

$$\frac{\frac{23}{144} \cdot \frac{1}{4}}{\frac{47}{576}} = \frac{23}{47} \approx 0.489 \,//$$