2A 11
B 5 7
C 1, 7, 11, 12
3A 11, 14.

2A. 11) Given linearly independent $v_1, \dots, v_n \in V$ and $w \in V$,
Suppose $v_1, \dots, v_m, w$ is linearly independent in V.

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n + c_w w = 0$$
$$c_w w = -c_1 v_1 - c_2 v_2 - \dots - c_n v_n$$

Since the list is linearly independent,
$$c_1 = c_2 = \dots = c_n = c_w = 0$$

Hence there is no representation of w in $\mathrm{span}(v_1, \dots, v_n)$
Thus $w \notin \mathrm{span}(v_1, \dots, v_n)$

Suppose $w \notin \mathrm{span}(v_1, \dots, v_n)$, and $v_1, \dots, v_n$ is linearly independent.

$\Rightarrow$ no linear combination of $v_1, \dots, v_n$ exists that equals w.

Let $c_1 v_1 + c_2 v_2 + \dots + c_n v_n + c_w w = 0$
$$c_w \cdot w = -c_1 v_1 - c_2 v_2 - \dots - c_n v_n$$
Since $w \notin \mathrm{span}(v_1, \dots, v_n)$
There is no non zero value of $c_1, c_2, \dots c_n$ to form $c_w \cdot w$.
Hence $c_w = c_1 = c_2 = \dots = c_n = 0$
$\therefore$ $v_1, \dots, v_n, w$ is linearly independent.

$\square$.

B 5 7
$P_3(\mathbb{F})$ is the set of all polynomials with degree 3 and coefficients in $\mathbb{F}$.

$$c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$p_0 = 1,\ p_1 = x,\ p_2 = x^2 + x^3,\ p_3 = x^3.$$

Basis $\Rightarrow$ spans & linear independent.

Prove that $p_0, \dots, p_3$ spans $P_3(\mathbb{F})$:
$$\mathrm{span}(p_0, p_1, p_2, p_3) = c_0 + c_1 x + c_2 x^2 + (c_2 + c_3) x^3$$
$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3$$
$\Rightarrow$ spans $P_3(\mathbb{F})$.

Prove linear independence:

Let $c_0 + c_1 x + c_2 x^2 + (c_2 + c_3) x^3 = 0$.
$\Rightarrow c_0 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0$
$\Rightarrow c_0 + c_1 x + c_2 x^2 + (c_2 + c_3) x^3 = 0$ iff $c_1 = c_2 = c_3 = c_0 = 0$
$\therefore$ linearly independent.

$\therefore$ Such a basis exists

$\square$.

Linear Algebra Week 2.

Date No.

2B.7. $\mathrm{span}(v_1, v_2, v_3, v_4) = V$. $v_1, \dots, v_4$ is linearly independent.

U ~~is~~ subspace of V s.t

$v_1, v_2 \in U$ and $v_3, v_4 \notin U$

Basis $\Rightarrow$ span covers the space and is linearly independent.

~~St~~ Proposition is true.

Lemma 1: ~~\\\\\\~~ $\mathrm{span}(v_1, v_2) = U$

Pf: By contradiction, suppose that $\mathrm{span}(v_1, v_2) \subset U$, then

$\exists v \in U$ ~~and~~ : $v \notin \mathrm{span}(v_1, v_2)$

Since $v \in V$, ~~\\\\\\~~ $v \in \mathrm{span}(v_1, v_2, v_3, v_4)$.

For $v \in U$ and $v \in \mathrm{span}(v_1, v_2, v_3, v_4)$,

either $v_3 \in U$ and/or $v_4 \in U$ Easier to spot mistake if we let

However, this contradicts the definition of U. $v = c_3 v_3 + c_4 v_4$

Since a contradiction is derived, lemma 1 must be true

$\square$.

Since $\mathrm{span}(v_1, v_2) = U$ and v_1, v_2 are linearly independent by definition,

~~so~~ v_1, v_2 forms a basis of U.

2C.1 Suppose V is finite dimensional, and U is a subspace of V, such that $\dim U = \dim V$, then $U = V$.

Pf: Since U is a subspace of V,

~~$\dim U \leq \dim V$.~~

By definition in proposition, $\dim U = \dim V$.

$\Rightarrow \exists\, u_1, \dots, u_n \in U$ that is linearly independent, and

$\exists\, v_1, \dots, v_n \in V$ that is linearly independent.

since $u_1, \dots, u_n \in U$,

$u_1, \dots, u_n \in V$.

$\Rightarrow u_1, \dots, u_n$ ~~is~~ is a linearly independent list of vectors in V.

Since $u_1, \dots, u_n$ has same length ~~as~~ as the basis of V,

$u_1, \dots, u_n$ is also a basis of V.

Since $u_1, \dots, u_n$ is a basis of V and a basis of U,

$U = V$.

$\square$.

2C.7 a) Let $U = \{ p \in P_4(\mathbb{F}) : p(2) = p(5) = p(6) \}$

~~p(2)~~

$p = (x-2)(x-5)(x-6)^2 + c$ or

$(x-2)(x-5)^2(x-6) + c$ or

$(x-2)^2(x-5)(x-6) + c$.

Basis of U: $(x-2)(x-5)(x-6)^2 + 1$,

$(x-2)(x-5)^2(x-6) + 1$,

$(x-2)^2(x-5)(x-6) + 1$.

b) ~~span~~ span of U: $(x-2)(x-5)(x-6)[c_1(x-6) + c_2(x-5) + c_3(x-2)] + c_1 + c_2 + c_3$.

$= (x-2)(x-5)(x-6)[x(c_1+c_2+c_3) - 6c_1 - 5c_2 - 2c_3] + c_1 + c_2 + c_3$.

$= x(x-2)(x-5)(x-6)(c_1+c_2+c_3)$
$- (x-2)(x-5)(x-6)(6c_1 + 5c_2 + 2c_3)$
$+ c_1 + c_2 + c_3$.

Coefficient of x^4: $c_1 + c_2 + c_3$.

Coefficient of constant: $c_1 + c_2 + c_3$.

$\Rightarrow$ in U, all polynomials have same coefficients for x^4 and constant.

For a basis of $P_4(\mathbb{F})$ to be built from the span of U, we must ~~enable~~ ~~P4~~ ~~to~~ add vectors in the list to allow x^4 and constant to vary.

$\therefore$ $(x-2)(x-5)(x-6)^2 + 1$,

$(x-2)(x-5)^2(x-6) + 1$,

$(x-2)^2(x-5)(x-6) + 1$,

1,

x^4 //

Date No.

c) ~~XX~~ Let $W = \text{span}(1, x^4)$ //

2C.11) Suppose U and W are subspaces of $\mathbb{R}^8$.

$\dim U = 3$, $\dim W = 5$.

$U + W = \mathbb{R}^8$.

Since $U + W = \mathbb{R}^8$,

$\mathbb{R}^8 = \{u + w \mid u \in U, w \in W\}$.

For $\mathbb{R}^8 = U \oplus W$, ~~X~~ $U \cap W = \{0\}$, or

for $u \in U$, $w \in W$, $u + w = 0$ i.f.f $u = 0$ and $w = 0$.

$\Rightarrow \nexists\, u \in U$ and $w \in W$ s.t $u = w$ and $u = w \neq 0$.

Here to show that $\mathbb{R}^8 = U \oplus W$, we must show that the basis of U is linearly independent to the basis of W.

Let $u_1, u_2, u_3 \in U$ be a basis of U.

Let $w_1, w_2, w_3, w_4, w_5 \in W$ be a basis of W.

Then $\mathbb{R}^8 = \{c_1u_1 + c_2u_2 + c_3u_3 + c_4w_1 + c_5w_2 + c_6w_3 + c_7w_4 + c_8w_5 \mid c_1, \ldots c_8 \in \mathbb{F}\}$

Since $u_1, \ldots, u_3, w_1, \ldots, w_5$ spans $\mathbb{R}^8$, $\dim \mathbb{R}^8 = 8$, and the spanning list has length 8, $u_1, \ldots, u_3, w_1, \ldots, w_5$ is a basis of $\mathbb{R}^8$.

Since $u_1, \ldots, u_3, w_1, \ldots, w_5$ is a basis of $\mathbb{R}^8$, all vectors in the list are linearly independent to one another.

Then the only way to build $0 \in \mathbb{R}^8$ ~~is~~ using that as a basis is if

~~$0 = c_1$~~, $0 \cdot u_1 + \ldots + 0 \cdot u_3 + 0 \cdot w_1 + \ldots + 0 \cdot w_5 = 0$

Hence suppose $\exists\, u \in U, w \in W$ s.t $u = w$,

$c_1u_1 + c_2u_2 + c_3u_3 = c_4w_1 + c_5w_2 + c_6w_3 + c_7w_4 + c_8w_5$

$c_1u_1 + \ldots + c_3u_3 + c_4w_1 + \ldots + c_8w_5 = 0$.

The only solution is if $c_1 = c_2 = \ldots = c_8 = 0$

$\therefore U \cap W = \{0\}$

$\Rightarrow U + W$ is a direct sum.

$\square$.

Besform

Date No.

2C.12 Let U, W be subspaces of $\mathbb{R}^9$. $\dim U = \dim W = 5$.

~~We start with the basis of U and attempt to build a basis of $\mathbb{R}^9$ from U using the basis of W.~~

Let $u_1, \dots, u_5$ be a basis of U.

Let $w_1, \dots, w_5$ be a basis of W.

Let $v_1, \dots, v_9$ be a basis of $\mathbb{R}^9$.

The dimension of $\mathbb{R}^9$ ~~cannot be~~ is 9.

~~Using the pigeonhole principle, there must~~

Suppose $U + W = \mathbb{R}^9$, then the list $u_1, \dots, u_5, w_1, \dots, w_5$ is a spanning list of $\mathbb{R}^9$.

However the length of the spanning list is 10, while the dimension of $\mathbb{R}^9$ is 9.

~~Hence one vector in $u_1, \dots, u_5, w_1, \dots, w_5$ is at~~

~~$v_i \in \operatorname{span}(u_1, \dots, u_5, w_1, \dots, w_5)$~~

~~Hence there exists j~~

Let $u_1 = x_1, u_2 = x_2, \dots u_5 = x_5, w_1 = x_6, \dots, w_5 = x_{10}$.

$\exists j \in \{0, 1, \dots, 10\}$ s.t $x_j \in \operatorname{span}(x_1, \dots, x_{j-1})$.

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$
$$\dim \mathbb{R}^9 = \dim U + \dim W - \dim(U \cap W)$$
$$9 = 5 + 5 - \dim(U \cap W)$$
$$\dim(U \cap W) = 1.$$
$$\Rightarrow U \cap W \neq \{0\} \text{ since } \dim\{0\} = 0.$$

$\square$.

3A.11) Let U be a subspace of V.
Let $S \in \mathcal{L}(U,W)$ Let $T \in \mathcal{L}(V,W)$.
$Su_i = w_i$ for all $u_i \in U$, $w_i \in W$.

Since $T \in \mathcal{L}(V,W)$ and U is a subspace of V,
$T \in \mathcal{L}(U,W)$ as $u \in U \Rightarrow u \in V$.
$\Rightarrow \mathcal{L}(U,W)$ is a subspace of $\mathcal{L}(V,W)$.
Since $\mathcal{L}(U,W)$ is a subspace of $\mathcal{L}(V,W)$
For all $S \in \mathcal{L}(U,W)$, there must exist an equivalent $T \in (V,W)$ such that $S_U = T_U$.

$\square$.

3A.14) $\dim V \geq 2$. Let $S, T \in \mathcal{L}(V,V)$.
Let $S: V \to V$, where $Sv_i = \pi(v_i)$
Let $T: V \to V$ where $Tv_i = \sigma(v_i)$.

$TSv_i = T\pi(v_i) = \sigma(\pi(v_i))$.
$STv_i = S\sigma(v_i) = \pi(\sigma(v_i))$.

Now let $\pi(v_i) = 0$ for all $v_i \in V$. ~~Let $\sigma(v_i) = \langle 1,1 \rangle$ i~~
~~$\sigma(v_i) = \langle 1,1 \rangle$ iff $v_i = 0$ else $\sigma(v_i) = v_i$.~~
Then $\sigma(\pi(v_i)) = \sigma(0) = \cancel{\langle 1,1 \rangle}, 0$.
$\pi(\sigma(v_i)) = \cancel{\pi(v_i) = 0}$ ~~or~~ $\cancel{= \pi(\langle 1,1 \rangle) = 0}$.
$\cancel{= \pi(\langle 1,1 \rangle) = 0}$.
$= \cancel{\pi(v_i) \neq}$

Let $\sigma(v_i) = \langle 1,1,\dots,1 \rangle$ iff $v_i = 0$ else $\sigma(v_i) = 0$.
$\sigma(\pi(v_i)) = \sigma(0) = \langle 1,1,\dots,1 \rangle$.
$\pi(\sigma(v_i)) = \pi(v_i)$ or $\pi(\langle 1,1,\dots,1 \rangle)$
$= 0$.
$\therefore \sigma(\pi(v_i)) \neq \pi(\sigma(v_i))$
$\Rightarrow TSv_i \neq STv_i$

$\square$.

Besform

Suppose $\text{span}(v_1, \ldots, v_n) = V$. For some $v \in V$,
let $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$.

Let $f_1, \ldots, f_n \in V^*$ be maps such that $f_i(v) = a_i$.
We will prove that $f_1, \ldots, f_n$ form a basis of V^*.

First, we will show that $f_i \in V^*$. Hence f_i must satisfy additivity and homogeneity properties.

Additivity: $f_i(v+u) = f_i(a_1 v_1 + \ldots + a_i v_i + \ldots + a_n v_n + b_1 v_1 + \ldots + b_i v_i + \ldots + b_n v_n)$
$= f_i((a_1+b_1)v_1 + \ldots + (a_i+b_i)v_i + \ldots + (a_n+b_n)v_n)$
$= a_i + b_i$
$= f_i(v) + f_i(u)$ ✓

Homogeneity: $f_i(\lambda v) = f_i(\lambda a_1 v_1 + \ldots + \lambda a_i v_i + \ldots + \lambda a_n v_n)$
$= \lambda a_i$
$= \lambda f_i(v)$ ✓.

Since f_i satisfies additivity and homogeneity, f_i is in V^*.

Now, to prove that $f_1, \ldots, f_n$ is a basis for V^*, we must prove that
$f_1, \ldots, f_n$ is a spanning list of V^*, and
$f_1, \ldots, f_n$ is linearly independent.

Spanning list: Let $g \in V^*$ s.t $g(v_i) = c_i$ for some $v_i \in V$, $c_i \in \mathbb{F}$.

$g(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) = c_i$
$g(a_1 v_1) + \ldots + g(a_n v_n) = c_i$
$a_1 g(v_1) + \ldots + a_n g(v_n) = c_i$
$a_1 c_1 + a_2 c_2 + \ldots a_n c_n = c_i$
Since $f_i(v) = a_i$
$c_1 f_1(v) + \ldots + c_n f_n(v) = c_i$
$\Rightarrow g(v) = c_1 f_1(v) + c_2 f_2(v) + \ldots + c_n f_n(v)$.
$\Rightarrow$ all elements in V^* are in the span $(f_1, \ldots, f_n)$
$\Rightarrow f_1, \ldots, f_n$ spans V^*.

Independence: Let $c_1, \ldots, c_n \in \mathbb{F}$ s.t
$c_1 f_1 + \ldots + c_n f_n = 0$ where 0 is the additive identity in V^*.

Then $c_1 f_1 v + \ldots + c_n f_n v = 0 v = 0$ for all $v \in V$.

$c_i f_i v = -c_1 f_1 v - \ldots - c_n f_n v$
$c_i a_i = -c_1 a_1 - \ldots - c_n a_n$

If we let $v = v_i$, $c_1 f_1 v_i + \ldots + c_i f_i v_i + \ldots + c_n f_n v_i = 0$
$\Rightarrow c_i f_i v_i = 0$
$\Rightarrow c_i = 0$
$\Rightarrow c_i = 0$ for all $i = 1, 2, \ldots, n$

$\Rightarrow$ linearly independent
$\therefore f_1, \ldots, f_n$ forms a basis of V^*. $\text{Dim } V^* = n$.
$\square$.

2) Let v_1, v_2 be a basis for V. Let w_1, w_2, w_3 be a basis for W.

Find a basis for $\mathcal{L}(V, W)$.

Let $f_{i,j} \in \mathcal{L}(V, W)$ where $i = 1, 2$, $j = 1, 2, 3$ such that for some $v \in V$, where $v = a_1 v_1 + a_2 v_2$,

$$f_{i,j}(v) = a_i w_j \quad \text{e.g. } f_{1,2}(v) = a_1 w_2.$$
$$f_{2,3}(v) = a_2 w_3.$$

We will prove that $f_{1,1}, f_{1,2}, f_{1,3}, f_{2,1}, f_{2,2}, f_{2,3}$ forms a basis of $\mathcal{L}(V, W)$.

First, we will show that $f_{i,j} \in \mathcal{L}(V, W)$. We must prove additivity and homogeneity in $f_{i,j}$.

Let $v, u \in V$ s.t $v = a_1 v_1 + a_2 v_2$ and $u = b_1 v_1 + b_2 v_2$.

We will prove the the case for $f_{1,2}$ since the properties of the remaining are identical.

$$\begin{aligned} f_{1,2}(v+u) &= f_{1,2}((a_1+b_1)v_1 + (a_2+b_2)v_2) \\ &= (a_1+b_1)w_2 \\ &= a_1 w_2 + b_1 w_2 \\ &= f_{1,2}(v) + f_{1,2}(u) \\ &\Rightarrow \text{additivity.} \end{aligned}$$

$$\begin{aligned} f_{1,2}(\lambda v) &= f_{1,2}(\lambda a_1 v_1 + \lambda a_2 v_2) \\ &= \lambda a_1 w_1 \\ &= \lambda f_{1,2}(\lambda v) \Rightarrow \text{homogeneity.} \end{aligned}$$

Now we prove that $f_{1,1}, \dots f_{2,3}$ spans $\mathcal{L}(V, W)$.

Let $g \in \mathcal{L}(V, W)$ s.t $g v_i = w_i$ for some $v_i \in V$, $w_i \in W$. where $v_i = c_1 v_1 + c_2 v_2$; $w_i = b_1 w_1 + b_2 w_2 + b_3 w_3$

$$\begin{aligned} g(v_i) = g(c_1 v_1 + c_2 v_2) &= c_1 g(v_1) + c_2 g(v_2) \\ &= c_1 w_i + c_2 w_j \\ &= c_1(b_1 w_1 + b_2 w_2 + b_3 w_3) + \\ &\quad c_2(b_1' w_1 + b_2' w_2 + b_3' w_3) \\ &= b_1 c_1 w_1 + b_2 c_1 w_2 + b_3 c_1 w_3 + \\ &\quad b_1' c_2 w_1 + b_2' c_2 w_2 + b_3' c_2 w_3 \\ &= b_1 f_{1,1}(v) + b_2 f_{1,2}(v) + b_3 f_{1,3}(v) + \\ &\quad b_1' f_{2,1}(v) + b_2' f_{2,2}(v) + b_3' f_{2,3}(v) \end{aligned}$$

$\Rightarrow g$ is a linear combination of $f_{1,1}, \dots, f_{2,3}$.

Now, we prove that $f_{1,1}, \dots, f_{2,3}$ are linearly independent.

Suppose there are $c_1, \dots, c_6 \in \mathbb{F}$ s.t

$$c_1 f_{1,1} + c_2 f_{1,2} + \dots + c_6 f_{2,3} = 0$$

Then $c_1 f_{1,1} v + c_2 f_{1,2} v + \dots + c_6 f_{2,3} v = 0$. for some $v \in V$.

Let $v = b_1 v_1 + b_2 v_2$.

$\Rightarrow c_1 b_1 w_1 + c_2 b_1 w_2 + c_3 b_1 w_3 +$
$c_4 b_2 w_1 + c_5 b_2 w_2 + c_6 b_2 w_3 = 0$.

Since $0 \in W$ and w_1, w_2, w_3 is a basis for W, the only linear combination of w_1, w_2, w_3 is for

$c_1 b_1 = c_2 b_1 = \dots = c_6 b_2 = 0$

Since $b_1, b_2 \neq 0$, then

$c_1 = c_2 = \dots = c_6 = 0$

$\Rightarrow f_{1,1}, \dots, f_{2,3}$ are linearly independent.

Since $f_{1,1}, \dots, f_{2,3}$ spans $\mathcal{L}(V,W)$ and are linearly independent, they form a basis for $\mathcal{L}(V,W)$. $\dim \mathcal{L}(V,W) = 6$.

D.

3) a) For U to be a subspace, U must satisfy the properties of
- closed under addition
- closed under multiplication,
- additive identity.

First, we will find a closed form for the recurrence $v_i + v_{i+2} = v_{i+1}$
$\Rightarrow v_{i+2} = v_{i+1} - v_i$.

$$\begin{aligned} v_{i+2} &= v_{i+1} - v_i \\ &= v_i - v_{i-1} - v_{i-1} + v_{i-2} \\ &= v_i - 2v_{i-1} + v_{i-2} \\ &= v_{i-1} - v_{i-2} - 2v_{i-1} + v_{i-2} \\ &= -v_{i-1} \end{aligned}$$

$$\Rightarrow v_i = -v_{i-3} = v_{i-6} = (-1)^k v_{i-3k}$$

Suppose $i = 3n$, $v_{3n} = (-1)^k v_{3n-3k} = (-1)^n v_0$
$i = 3n+1$, $v_{3n+1} = (-1)^k v_{3n+1-3k} = (-1)^n v_1$
$i = 3n+2$, $v_{3n+2} = (-1)^n v_2$
$= (-1)^n (v_1 - v_0)$
$= (-1)^n v_1 - (-1)^n v_0$

<u>Lemma 1</u>: For $n \geq 0$ $v_{3n+2} = v_{3n+1} - v_{3n}$
$\Rightarrow v_{3n} = (-1)^n v_0$. $v_{3n+1} = (-1)^n v_1$ $v_{3n+2} = (-1)^n v_1 - (-1)^n v_0$ for $v_0, v_1 \in \mathbb{R}$.

<u>Pf</u>: By induction,

Base case: $n = 0$. $v_2 = v_1 - v_0$. $v_2 = v_1 - v_0$
$= (-1)^0 v_1 - (-1)^0 v_0$
$\Rightarrow$ lemma is true for $n = 0$.

<u>Inductive step</u> Assuming the lemma is true for some $0 \leq k \leq n$, then for $k+1$,

Case 1: $v_{3(k+1)} = v_{3k+3} = v_{3k+2} - v_{3k+1}$ (by the recurrence definition)
$= (-1)^k v_1 - (-1)^k v_0 - (-1)^k v_1$ (by I.H)
$= -(-1)^k v_0$
$= (-1)^{k+1} v_0$
$\Rightarrow v_{3(k+1)} = (-1)^{k+1} v_0$ ✓.

Case 2: $v_{3(k+1)+1} = v_{3k+4} = v_{3k+3} - v_{3k+2}$ (by recurrence definition)
$= (-1)^{k+1} v_0 - (-1)^k v_1 + (-1)^k v_0$ (by I.H)
$= -(-1)^k v_0 + (-1)^{k+1} v_1 + (-1)^k v_0$
$= (-1)^{k+1} v_1$ ✓.

Case 3: $v_{3(k+1)+2} = v_{3k+5} = v_{3k+4} - v_{3k+3}$
$= (-1)^{k+1} v_1 - (-1)^{k+1} v_0$ ✓.

$\Rightarrow$ lemma is true.

D.

U is the subset of vectors in $\mathbb{R}^\infty$ such that for all $u \in U$, where
$$u = (x_1, x_2, \ldots, x_{3n}, x_{3n+1}, x_{3n+2}, \ldots)$$
$$x_{3n} = (-1)^n x_1, \quad x_{3n+1} = (-1)^n x_2, \quad x_{3n+2} = (-1)^n x_2 - (-1)^n x_1$$
for some $x_1, x_2 \in \mathbb{R}$ as shown by lemma 1.

let $u, v \in U$. where $u = (x_1, x_2, \ldots)$ and $v = (y_1, y_2, \ldots)$

$\ldots)$

Closed under addition

$$u+v = (x_1+y_1, x_2+y_2, \ldots, (-1)^n x_1 + (-1)^n y_1, (-1)^n x_2 + (-1)^n y_2, (-1)^n x_2 - (-1)^n x_1 + (-1)^n y_2 - (-1)^n y_1, \ldots)$$

Let $a_1 = x_1 + y_1$. $a_2 = x_2 + y_2$

$$u+v = (a_1, a_2, \ldots, (-1)^n a_1, (-1)^n a_2, (-1)^n a_2 - (-1)^n a_1, \ldots)$$

$\Rightarrow$ $u+v \in U$ as $u+v$ satisfies the properties of U.

Closed under multiplication

$$\begin{aligned} \lambda u &= \lambda(x_1, x_2, \ldots) \\ &= (\lambda x_1, \lambda x_2, \ldots) \\ &= (a_1, a_2, \ldots) \text{ where } (a_1, a_2, \ldots) \in U \end{aligned}$$

$\Rightarrow \lambda u \in U$.

Additive identity

For $u \in U$, where $u = (x_1, x_2, \ldots)$, Let $x_1 = x_2 = 0$

$\Rightarrow x_{3n} = 0, \ x_{3n+1} = 0, \ x_{3n+2} = 0$

$\Rightarrow$ For $\forall v \in U$, $v + u = v$

$\Rightarrow u = 0$.

Since U has the properties of a vector space, U is a subspace of $\mathbb{R}^\infty$.

b,c) Since $\forall u \in U$, where $u = (x_1, x_2, \dots, x_k, \dots)$, $x_1, x_2, \dots \in \mathbb{R}$.

$\forall x_k$ where $k > 2$ is defined by the values x_1, x_2 as shown in lemma 1.

Hence there is a bijection between vectors in $\mathbb{R}^2$ and U as x_1, x_2 are the values that define the values of all x_k where $k > 2$.

Thus to prove x and y form a basis, we just have to prove that $(0,1)$ and $(1,0)$ are a basis of $\mathbb{R}^2$. since the first two values of x, y are $(0,1)$ and $(1,0)$.

Suppose some $c_1, c_2 \in \mathbb{R}$ s.t

$c_1(0,1) + c_2(1,0) = (0,0)$

$(c_1, c_2) = (0,0)$

$\Rightarrow c_1 = 0,\ c_2 = 0$

$\Rightarrow (1,0), (0,1)$ are linearly independent.

Since $((1,0),(0,1))$ has length of 2, and $\dim \mathbb{R}^2$ is 2, and $(0,1), (1,0)$ is linearly independent,

$(1,0), (0,1)$ form a basis in $\mathbb{R}^2$

$\Rightarrow x, y$ form a basis in U. $\square$.

d) We first prove that $W + U = \mathbb{R}^\infty$.

Let $u \in U$ s.t $u = (x_1, x_2, \dots, x_{3n}, x_{3n+1}, x_{3n+2})$ according to the rules in lemma 1.

Let $w \in W$ where $w = (0, 0, y_3, y_4, \dots)$, $y_i \in \mathbb{R}$.

$u + w = (x_1, x_2, x_3 + y_3, x_4 + y_4, \dots, x_{3n} + y_{3n}, x_{3n+1} + y_{3n+1}, x_{3n+2} + y_{3n+2}, \dots)$

$= (x_1, x_2, z_3, z_4, \dots, z_n, \dots)$

$= \mathbb{R}^\infty$.

For $u \in U$, where $u = (x_1, x_2, \dots)$, if $x_1 = x_2 = 0$, then $u = 0$ as defined by lemma 1.

Then for $v \in W$, where $v = (0, 0, v_3, v_4, \dots)$, there does not exist $v_i \in W$ and $v_i \in U$ other then the additive identity

Since $W + U = \mathbb{R}^\infty$ and $W \cap U = \{0\}$, $W \oplus U$

$\square$.