Divide-and-Conquer: Polynomial Multiplication

Neil Rhodes

Department of Computer Science and Engineering
University of California, San Diego

Algorithmic Design and Techniques
Algorithms and Data Structures

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

Multiplying Polynomials

Example

Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$

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$$B(x) = 5x^2 + x + 2$$

Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$
$$B(x) = 5x^2 + x + 2$$
$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

Multiplying polynomials

Input: Two $n-1$ degree polynomials:

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

$$b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$$

Output:

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Output: The product polynomial:

$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1 x + c_0$$

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Multiplying Polynomials

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Output: $C = (15, 13, 33, 9, 10)$

Outline

MultPoly(A, B, n)

```
product ← Array[2n − 1]
for i from 0 to 2n − 2:
    product[i] ← 0
```

$\texttt{MultPoly}(A, B, n)$

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product ← Array[2n − 1]
for i from 0 to 2n − 2:
   product[i] ← 0
for i from 0 to n − 1:
   for j from 0 to n − 1:
      product[i + j] ← product[i + j] + A[i] × B[j]
```

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Runtime: $O(n^2)$

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- $AB = (D_1x^{\frac{n}{2}} + D_0)(E_1x^{\frac{n}{2}} + E_0)$
  $= (D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$

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- Calculate D_1E_1, D_1E_0, D_0E_1, and D_0E_0

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Recurrence: $T(n) = 4T(\frac{n}{2}) + kn.$

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

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$$D_1(x) = 4x + 3$$

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$$\begin{aligned} AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 + \\ & 6x^2 + 11x + 4 \\ = {} & 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4 \end{aligned}$$

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$R\,[n\,.\,.\,2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

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  $R[0] = A[a_l] * B[b_l]$; `return` R

$R[0..n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

D_1E_0 = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

$R[\frac{n}{2} \ldots n + \frac{n}{2} - 2]$ += $D_1E0 + D_0E_1$

Function Mult2(A, B, n, a_l, b_l)

```
R = array[0..2n - 1]
if n = 1:
   R[0] = A[a_l] * B[b_l] ; return R
R[0..n-2] = Mult2(A, B, n/2, a_l, b_l)
R[n..2n-2] = Mult2(A, B, n/2, a_l + n/2, b_l + n/2)
D_0E_1 = Mult2(A, B, n/2, a_l, b_l + n/2)
D_1E_0 = Mult2(A, B, n/2, a_l + n/2, b_l)
R[n/2 ... n + n/2 - 2] += D_1E0 + D_0E_1
return R
```

$R = \texttt{array}[0..2n-1]$

if $n = 1$:

$\quad R[0] = A[a_l] * B[b_l]$; return R

$R[0..n-2]$ = Mult2$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = Mult2$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = Mult2$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

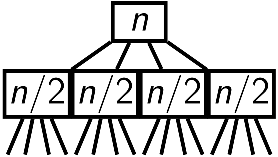
D_1E_0 = Mult2$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

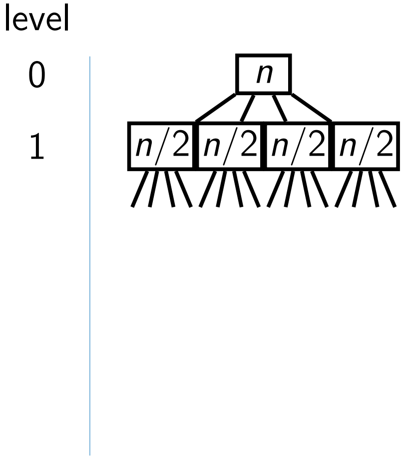
$R[\frac{n}{2} \ldots n + \frac{n}{2} - 2]$ += $D_1E0 + D_0E_1$

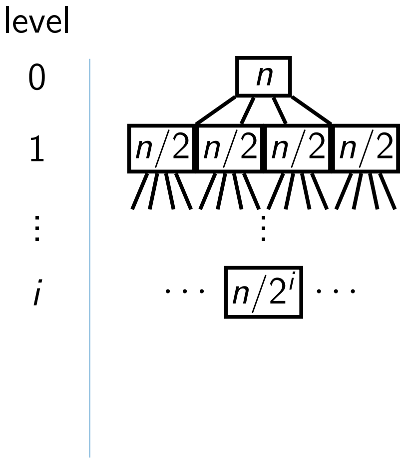
return R

$\boxed{n}$

level




level
0
1
n
n/2
n/2
n/2
n/2


level
0
1
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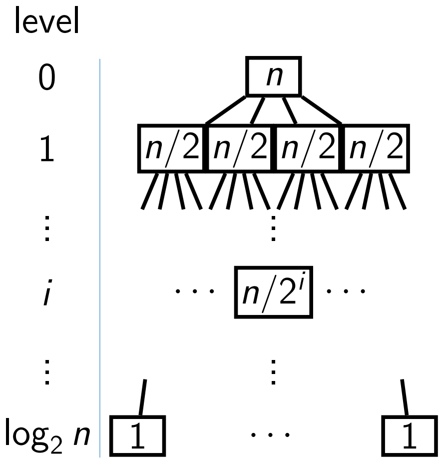
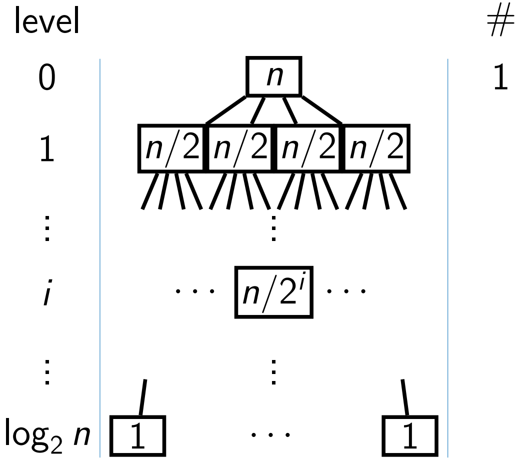
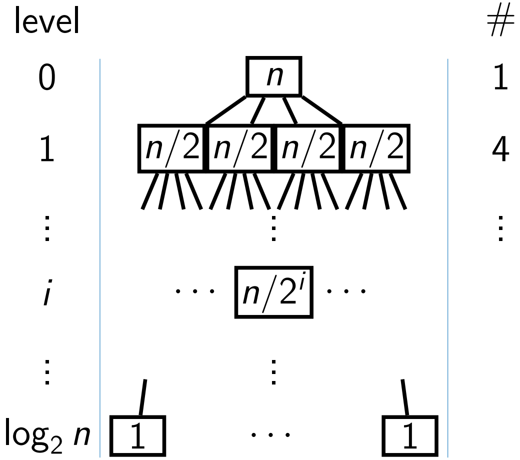
i
n
n/2 n/2 n/2 n/2
⋮
· · · n/2^i · · ·


level
0
1
i
log2 n
n
n/2
n/2
n/2
n/2
n/2^i
1
1


level
#
0
1
n
1
$n/2$
$n/2$
$n/2$
$n/2$
i
$n/2^i$
$\log_2 n$
1
1


level
#
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$n/2$
$n/2$
$n/2$
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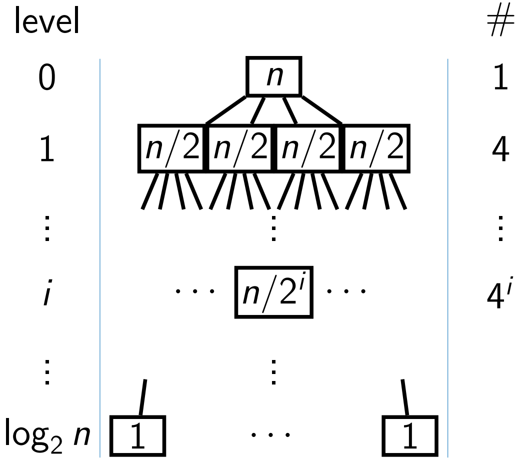
i
$\cdots$ $n/2^i$ $\cdots$
$\log_2 n$
1
$\cdots$
1


level
#
0
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4
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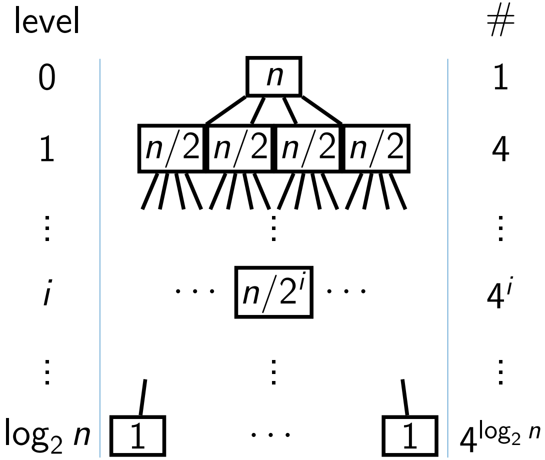
4^i
log_2 n
1
1


level
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n
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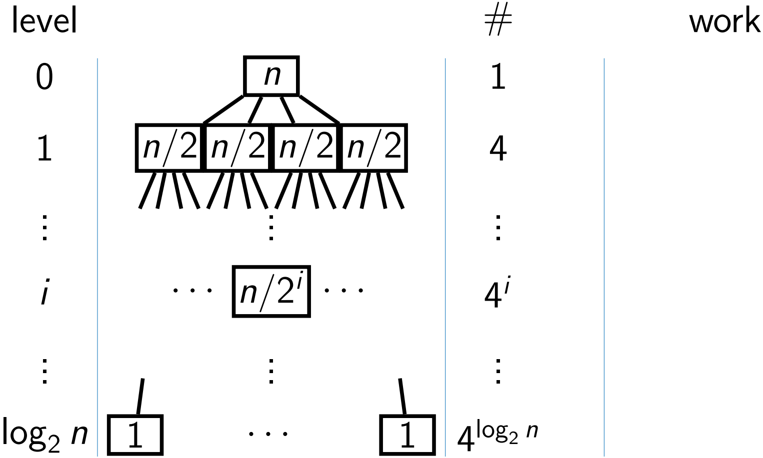
1
n/2
n/2
n/2
n/2
4
i
n/2^i
4^i
log_2 n
1
1
4^{log_2 n}


level
#
work
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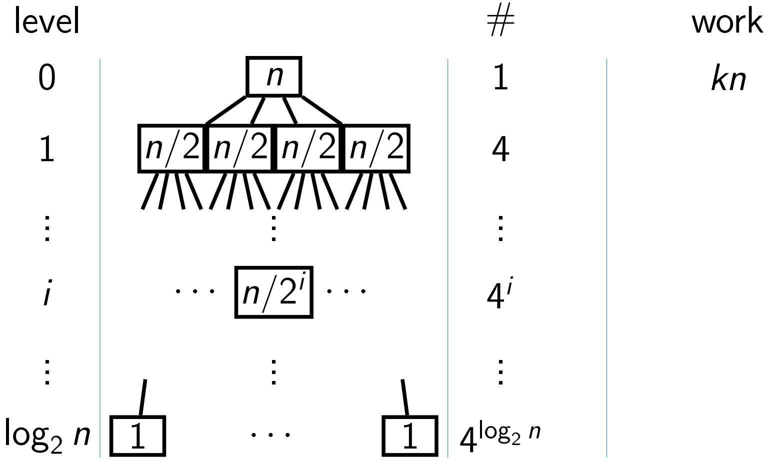
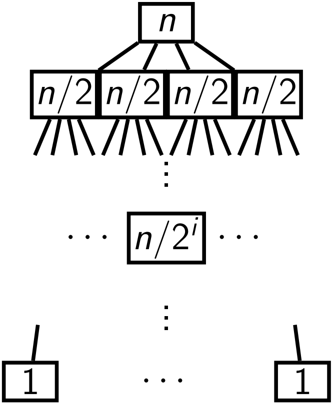
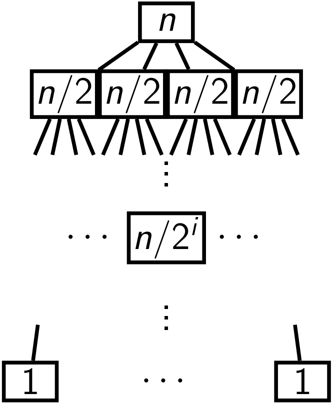
n
1
1
$n/2$ $n/2$ $n/2$ $n/2$
4
i
$\cdots$ $n/2^i$ $\cdots$
4^i
$\log_2 n$
1 $\cdots$ 1
$4^{\log_2 n}$


level
#
work
0
n
1
kn
1
n/2
n/2
n/2
n/2
4
i
n/2^i
4^i
log_2 n
1
1
4^{log_2 n}



level	#	work
0	1	kn
1	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	
i	4^i	
$\vdots$	$\vdots$	
$\log_2 n$	$4^{\log_2 n}$	



level	#	work
0	1	kn
1	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$
i	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	
$\log_2 n$	$4^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\,\boxed{n/2}\,\boxed{n/2}\,\boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots\ \boxed{n/2^i}\ \cdots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots\ \boxed{n/2^i}\ \cdots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

Total: $\sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \Theta(n^2)$

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

Karatsuba approach

Karatsuba approach

$$A(x) = a_1 x + a_0$$

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$$B(x) = b_1 x + b_0$$

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$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Karatsuba approach

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Needs 4 multiplications

Rewrite as:

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1 b_1 x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + a_0 b_0$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$
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$$A(x) = a_1 x + a_0$$

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Rewrite as:

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Needs 3 multiplications

Karatsuba Example

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

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$D_1(x) = 4x + 3$ $\qquad$ $D_0(x) = 2x + 1$

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$$D_1 E_1 = 4x^2 + 11x + 6$$

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$$(D_1 + D_0)(E_1 + E_0) =$$

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$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)$$

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$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

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$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6)\\ &= 24x^2 + 52x + 24\end{aligned}$$

Karatsuba Example

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$$AB = (4x^2 + 11x + 6)x^4 +$$

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$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (\\ & \qquad)x^2 +\end{aligned}$$

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$$\begin{aligned}AB = {}& (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad)x^2 +\end{aligned}$$

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$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad - (6x^2 + 11x + 4))x^2 +\end{aligned}$$

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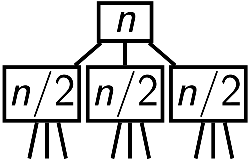
$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad - (6x^2 + 11x + 4))x^2 + \\ & 6x^2 + 11x + 4 \\ = {} & 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4\end{aligned}$$

$\boxed{n}$

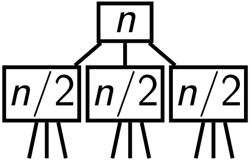
level

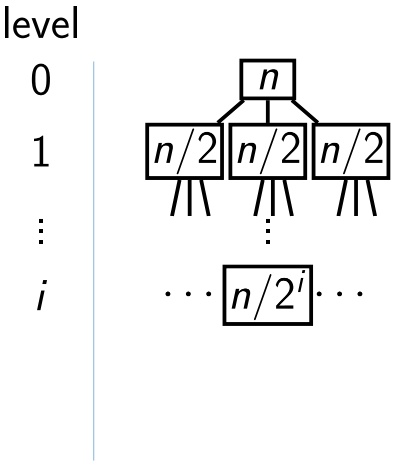


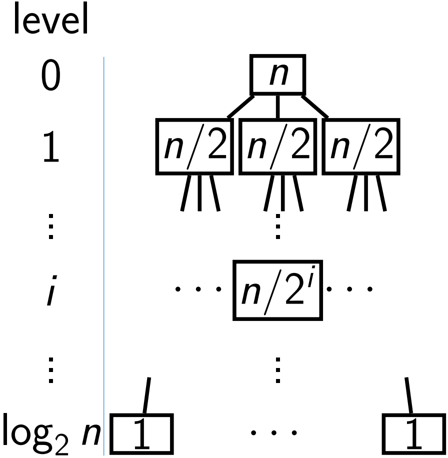
level

0

1




level
0
1
⋮
i
n
n/2
n/2
n/2
n/2^i


level
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1
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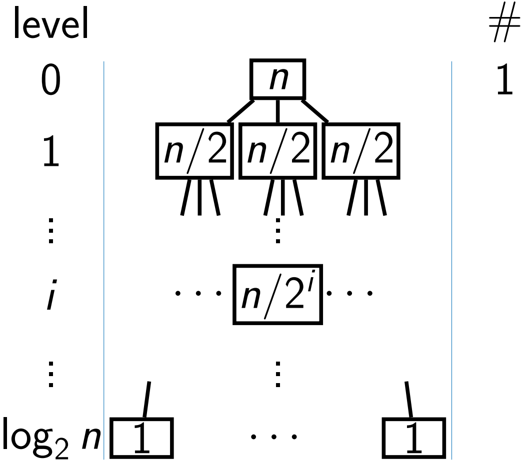
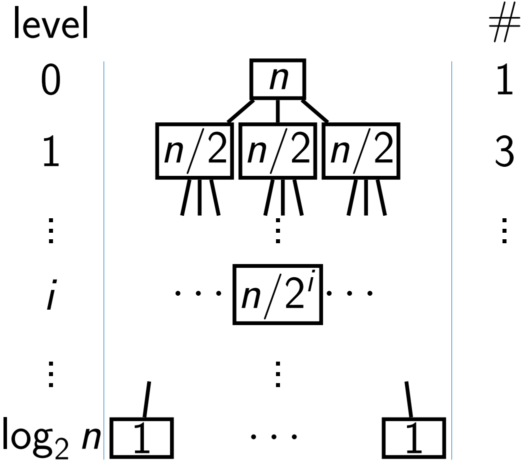
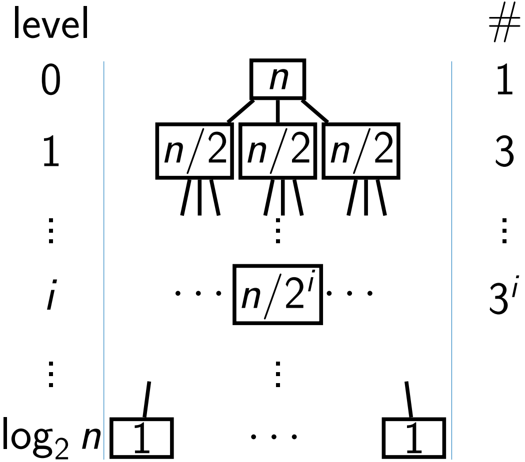
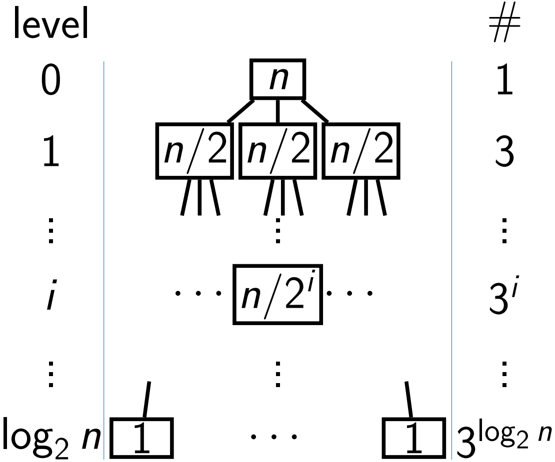
⋮
log2 n
n
n/2
n/2
n/2
⋮
· · · n/2^i · · ·
⋮
1
· · ·
1

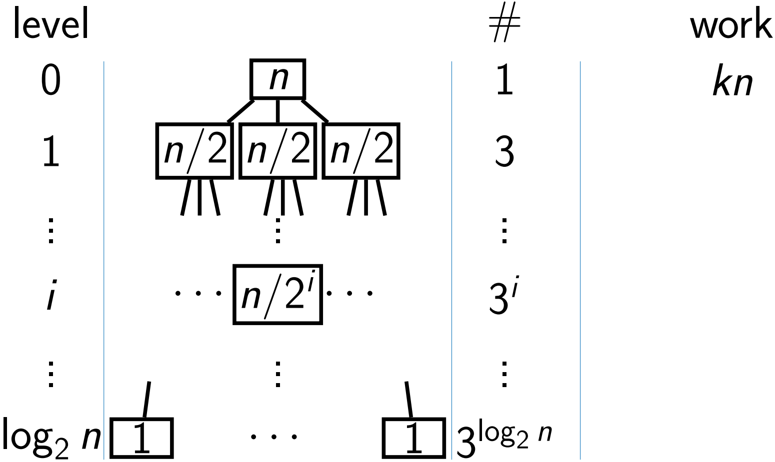

level
#
0
1
n
1
$n/2$
$n/2$
$n/2$
i
$n/2^i$
$\log_2 n$
1
1

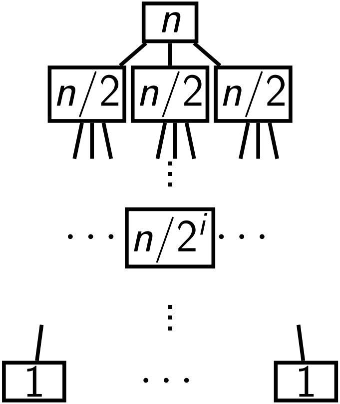

level
#
0
n
1
1
n/2
n/2
n/2
3
i
n/2^i
log2 n
1
1


level
#
0
n
1
1
$n/2$
$n/2$
$n/2$
3
i
$\cdots$ $n/2^i$ $\cdots$
3^i
$\log_2 n$
1
$\cdots$
1


level
#
0
1
i
log₂ n
n
n/2
n/2
n/2
n/2^i
1
1
3
3^i
3^log₂ n

level		#	work
0	$\boxed{n}$	1	
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	
$\vdots$	$\vdots$	$\vdots$	
i	$\cdots \boxed{n/2^i} \cdots$	3^i	
$\vdots$	$\vdots$	$\vdots$	
$\log_2 n$	$\boxed{1} \quad \cdots \quad \boxed{1}$	$3^{\log_2 n}$	


level
#
work
0
n
1
kn
1
n/2
n/2
n/2
3
i
n/2^i
3^i
log_2 n
1
1
3^(log_2 n)



level	#	work
0	1	kn
1	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	
i	3^i	
$\vdots$	$\vdots$	
$\log_2 n$	$3^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	3^i	$3^i k\frac{n}{2^i} = k\left(\frac{3}{2}\right)^i n$
$\vdots$	$\vdots$	$\vdots$	
$\log_2 n$	$\boxed{1} \quad \cdots \quad \boxed{1}$	$3^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	3^i	$3^i k \frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	3^i	$3^i k \frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

$$\text{Total: } \sum_{i=0}^{\log_2 n} 3^i k \frac{n}{2^i} = \Theta(n^{\log_2 3})$$

$$= \Theta(n^{1.58})$$