An Introduction to Stochastic Calculus with Applications to Finance

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Preface

The goal of this work is to introduce elementary Stochastic Calculus to senior undergraduate as well as to master students with Mathematics, Economics and Business majors. The author's goal was to capture as much as possible of the spirit of elementary Calculus, at which the students have been already exposed in the beginning of their majors. This assumes a presentation that mimics similar properties of deterministic Calculus, which facilitates the understanding of more complicated concepts of Stochastic Calculus. Since deterministic Calculus books usually start with a brief presentation of elementary functions, and then continue with limits, and other properties of functions, we employed here a similar approach, starting with elementary stochastic processes, different types of limits and pursuing with properties of stochastic processes. The chapters regarding differentiation and integration follow the same pattern. For instance, there is a product rule, a chain-type rule and an integration by parts in Stochastic Calculus, which are modifications of the well-known rules from the elementary Calculus.

Since deterministic Calculus can be used for modeling regular business problems, in the second part of the book we deal with stochastic modeling of business applications, such as Financial Derivatives, whose modeling are solely based on Stochastic Calculus.

In order to make the book available to a wider audience, the rigor was sacrificed for clarity. Most of the time we assumed maximal regularity conditions for which the computations hold and the statements are valid. This will be found attractive by both Business and Economics students, who might get lost otherwise in a very profound mathematical textbook where the forest's scenary is obscured by the sight of the trees.

An important feature of this textbook is the large number of solved problems and examples from which will benefit both the beginner as well as the advanced student.

This book grew from a series of lectures and courses given by the author at the Eastern Michigan University (USA), Kuwait University (Kuwait) and Fu-Jen University (Taiwan). Several students read the first draft of these notes and provided valuable feedback, supplying a list of corrections, which is by far exhaustive. Any typos or comments regarding the present material are welcome.

The Author,
Ann Arbor, October 2012

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Part I

Stochastic Calculus

Chapter 1

Basic Notions

1.1 Probability Space

The modern theory of probability stems from the work of A. N. Kolmogorov published in 1933. Kolmogorov associates a random experiment with a probability space, which is a triplet, $(\Omega, \mathcal{F}, P)$, consisting of the set of outcomes, Ω, a σ-field, $\mathcal{F}$, with Boolean algebra properties, and a probability measure, P. In the following sections, each of these elements will be discussed in more detail.

1.2 Sample Space

A *random experiment* in the theory of probability is an experiment whose outcomes cannot be determined in advance. These experiments are done mentally most of the time.

When an experiment is performed, the set of all possible outcomes is called the *sample space*, and we shall denote it by Ω. In financial markets one can regard this also as the *states of the world*, understanding by this all possible states the world might have. The number of states the world that affect the stock market is huge. These would contain all possible values for the vector parameters that describe the world, and is practically infinite.

For some simple experiments the sample space is much smaller. For instance, flipping a coin will produce the sample space with two states $\{H, T\}$, while rolling a die yields a sample space with six states. Choosing randomly a number between 0 and 1 corresponds to a sample space which is the entire segment $(0, 1)$.

All subsets of the sample space Ω form a set denoted by 2^Ω. The reason for this notation is that the set of parts of Ω can be put into a bijective correspondence with the set of binary functions $f : \Omega \to \{0, 1\}$. The number of elements of this set is $2^{|\Omega|}$, where $|\Omega|$ denotes the cardinal of Ω. If the set is finite, $|\Omega| = n$, then 2^Ω has 2^n elements. If Ω is infinitely countable (i.e. can be put into a bijective correspondence with the set of natural numbers), then $2^{|\Omega|}$ is infinite and its cardinal is the same as that of the real number set $\mathbb{R}$. As a matter of fact, if Ω represents all possible states of the financial world, then 2^Ω describes all possible events, which might happen in the market; this is a supposed to be a fully description of the total information of the financial world.

The following couple of examples provide instances of sets 2^Ω in the finite and infinite cases.

Example 1.2.1 *Flip a coin and measure the occurrence of outcomes by 0 and 1: associate a*

0 if the outcome does not occur and a 1 if the outcome occurs. We obtain the following four possible assignments:

$$\{H \to 0, T \to 0\}, \ \{H \to 0, T \to 1\}, \ \{H \to 1, T \to 0\}, \ \{H \to 1, T \to 1\},$$

so the set of subsets of $\{H, T\}$ can be represented as 4 sequences of length 2 formed with 0 and 1: $\{0,0\}$, $\{0,1\}$, $\{1,0\}$, $\{1,1\}$. These correspond in order to the sets $\emptyset, \{T\}, \{H\}, \{H,T\}$, which is the set $2^{\{H,T\}}$.

Example 1.2.2 *Pick a natural number at random. Any subset of the sample space corresponds to a sequence formed with 0 and 1. For instance, the subset $\{1,3,5,6\}$ corresponds to the sequence $10101100000\dots$ having 1 on the 1st, 3rd, 5th and 6th places and 0 in rest. It is known that the number of these sequences is infinite and can be put into a bijective correspondence with the real number set $\mathbb{R}$. This can be also written as $|2^{\mathbb{N}}| = |\mathbb{R}|$, and stated by saying that the set of all subsets of natural numbers $\mathbb{N}$ has the same cardinal as the real numbers set $\mathbb{R}$.*

1.3 Events and Probability

The set of parts 2^Ω satisfies the following properties:

1. It contains the empty set $\emptyset$;
2. If it contains a set A, then it also contains its complement $\bar{A} = \Omega \backslash A$;
3. It is closed with regard to unions, i.e., if $A_1, A_2, \dots$ is a sequence of sets, then their union $A_1 \cup A_2 \cup \cdots$ also belongs to 2^Ω.

Any subset $\mathcal{F}$ of 2^Ω that satisfies the previous three properties is called a *σ-field*. The sets belonging to $\mathcal{F}$ are called *events*. This way, the complement of an event, or the union of events is also an event. We say that an event occurs if the outcome of the experiment is an element of that subset.

The chance of occurrence of an event is measured by a probability function $P : \mathcal{F} \to [0,1]$ which satisfies the following two properties:

1. $P(\Omega) = 1$;
2. For any mutually disjoint events $A_1, A_2, \dots \in \mathcal{F}$,
$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots .$$

The triplet $(\Omega, \mathcal{F}, P)$ is called a *probability space*. This is the main setup in which the probability theory works.

Example 1.3.1 *In the case of flipping a coin, the probability space has the following elements: $\Omega = \{H, T\}$, $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$ and P defined by $P(\emptyset) = 0$, $P(\{H\}) = \frac{1}{2}$, $P(\{T\}) = \frac{1}{2}$, $P(\{H,T\}) = 1$.*

Example 1.3.2 *Consider a finite sample space $\Omega = \{s_1, \dots, s_n\}$, with the σ-field $\mathcal{F} = 2^\Omega$, and probability given by $P(A) = |A|/n$, $\forall A \in \mathcal{F}$. Then $(\Omega, 2^\Omega, P)$ is called the classical probability space.*

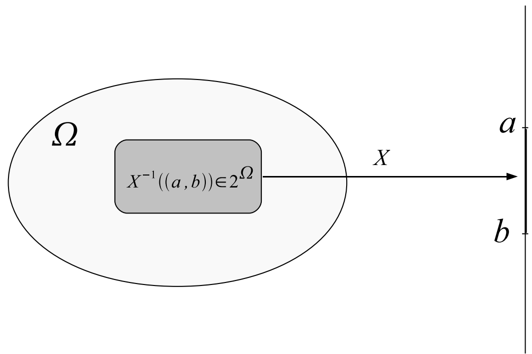


Figure 1.1: *If any pullback $X^{-1}\big((a,b)\big)$ is known, then the random variable $X : \Omega \to \mathbb{R}$ is 2^{Ω}-measurable.*

1.4 Random Variables

Since the σ-field $\mathcal{F}$ provides the knowledge about which events are possible on the considered probability space, then $\mathcal{F}$ can be regarded as the information component of the probability space $(\Omega, \mathcal{F}, P)$. A *random variable* X is a function that assigns a numerical value to each state of the world, $X : \Omega \to \mathbb{R}$, such that the values taken by X are known to someone who has access to the information $\mathcal{F}$. More precisely, given any two numbers $a, b \in \mathbb{R}$, then all the states of the world for which X takes values between a and b forms a set that is an event (an element of $\mathcal{F}$), i.e.

$$\{\omega \in \Omega; a < X(\omega) < b\} \in \mathcal{F}.$$

Another way of saying this is that X is an *$\mathcal{F}$-measurable function.* It is worth noting that in the case of the classical field of probability the knowledge is maximal since $\mathcal{F} = 2^{\Omega}$, and hence the measurability of random variables is automatically satisfied. From now on instead of *measurable* terminology we shall use the more suggestive word *predictable.* This will make more sense in a future section when we shall introduce conditional expectations.

Example 1.4.1 *Let $X(\omega)$ be the number of people who want to buy houses, given the state of the market ω. Is X predictable? This would mean that given two numbers, say $a = 10,000$ and $b = 50,000$, we know all the market situations ω for which there are at least $10,000$ and at most $50,000$ people willing to purchase houses. Many times, in theory, it makes sense to assume that we have enough knowledge to assume X predictable.*

Example 1.4.2 *Consider the experiment of flipping three coins. In this case Ω is the set of all possible triplets. Consider the random variable X which gives the number of tails obtained. For instance $X(HHH) = 0$, $X(HHT) = 1$, etc. The sets*

$$\begin{aligned} &\{\omega; X(\omega) = 0\} = \{HHH\}, \quad &&\{\omega; X(\omega) = 1\} = \{HHT, HTH, THH\}, \\ &\{\omega; X(\omega) = 3\} = \{TTT\}, &&\{\omega; X(\omega) = 2\} = \{HTT, THT, TTH\} \end{aligned}$$

belong to 2^{Ω}, and hence X is a random variable.

Example 1.4.3 *A graph is a set of elements, called nodes, and a set of unordered pairs of nodes, called edges. Consider the set of nodes* $\mathcal{N} = \{n_1, n_2, \dots, n_k\}$ *and the set of edges* $\mathcal{E} = \{(n_i, n_j), 1 \le i, j \le n, i \ne j\}$. *Define the probability space* $(\Omega, \mathcal{F}, P)$, *where*

- *the sample space is the the complete graph,* $\Omega = \mathcal{N} \cup \mathcal{E}$*;*
- *the* σ*-field* $\mathcal{F}$ *is the set of all subgraphs of* Ω*;*
- *the probability is given by* $P(G) = n(G)/k$, *where* $n(G)$ *is the number of nodes of the graph* G.

As an example of a random variable we consider $Y : \mathcal{F} \to \mathbb{R}$, $Y(G) =$ *the total number of edges of the graph* G. *Since given* $\mathcal{F}$, *one can count the total number of edges of each subgraph, it follows that* Y *is* $\mathcal{F}$*-measurable, and hence it is a random variable.*

1.5 Distribution Functions

Let X be a random variable on the probability space $(\Omega, \mathcal{F}, P)$. The *distribution function* of X is the function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(x) = P(\omega; X(\omega) \le x).$$

It is worth observing that since X is a random variable, then the set $\{\omega; X(\omega) \le x\}$ belongs to the information set $\mathcal{F}$.

The distribution function is non-decreasing and satisfies the limits

$$\lim_{x \to -\infty} F_X(x) = 0, \qquad \lim_{x \to +\infty} F_X(x) = 1.$$

If we have

$$\frac{d}{dx} F_X(x) = p(x),$$

then we say that $p(x)$ is the *probability density function* of X. A useful property which follows from the Fundamental Theorem of Calculus is

$$P(a < X < b) = P(\omega; a < X(\omega) < b) = \int_a^b p(x)\, dx.$$

In the case of discrete random variables the aforementioned integral is replaced by the following sum

$$P(a < X < b) = \sum_{a < x < b} P(X = x).$$

For more details the reader is referred to a traditional probability book, such as Wackerly et. al. [**?**].

1.6 Basic Distributions

We shall recall a few basic distributions, which are most often seen in applications.

Normal distribution A random variable X is said to have a *normal distribution* if its probability density function is given by

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)},$$

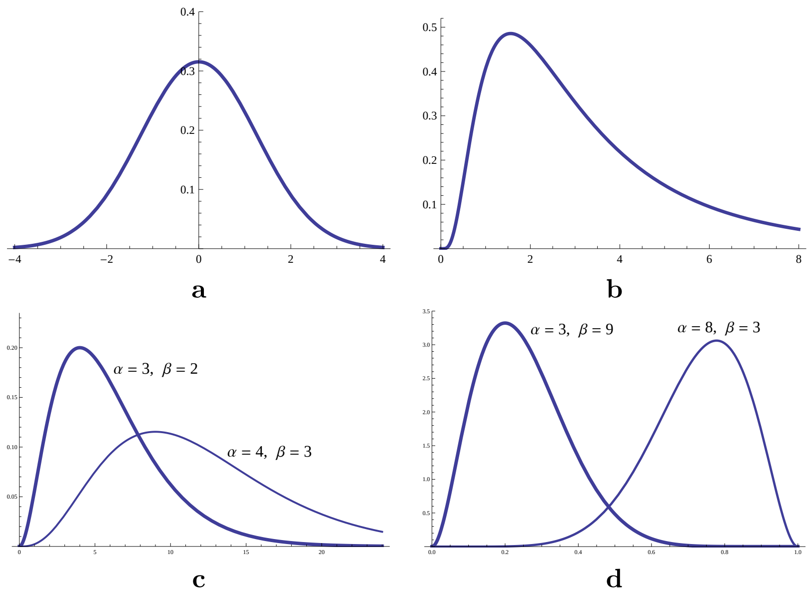


Figure 1.2: **a** *Normal distribution;* **b** *Log-normal distribution;* **c** *Gamma distributions;* **d** *Beta distributions.*

with μ and $\sigma > 0$ constant parameters, see Fig.1.2a. The mean and variance are given by

$$E[X] = \mu, \qquad Var[X] = \sigma^2.$$

If X has a normal distribution with mean μ and variance σ^2, we shall write

$$X \sim N(\mu, \sigma^2).$$

Exercise 1.6.1 *Let $\alpha, \beta \in \mathbb{R}$. Show that if X is normal distributed, with $X \sim N(\mu, \sigma^2)$, then $Y = \alpha X + \beta$ is also normal distributed, with $Y \sim N(\alpha\mu + \beta, \alpha^2\sigma^2)$.*

Log-normal distribution Let X be normally distributed with mean μ and variance σ^2. Then the random variable $Y = e^X$ is said to be *log-normal distributed*. The mean and variance of Y are given by

$$\begin{aligned} E[Y] &= e^{\mu + \frac{\sigma^2}{2}} \\ Var[Y] &= e^{2\mu + \sigma^2}(e^{\sigma^2} - 1). \end{aligned}$$

The density function of the log-normal distributed random variable Y is given by

$$p(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0,$$

see Fig.1.2b.

Exercise 1.6.2 *Given that the moment generating function of a normally distributed random variable $X \sim N(\mu, \sigma^2)$ is $m(t) = E[e^{tX}] = e^{\mu t + t^2\sigma^2/2}$, show that* (*a*) $E[Y^n] = e^{n\mu + n^2\sigma^2/2}$, *where* $Y = e^X$.

(*b*) *Show that the mean and variance of the log-normal random variable* $Y = e^X$ *are*

$$E[Y] = e^{\mu+\sigma^2/2}, \quad Var[Y] = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

Gamma distribution A random variable X is said to have a *gamma distribution* with parameters $\alpha > 0$, $\beta > 0$ if its density function is given by

$$p(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad x \geq 0,$$

where $\Gamma(\alpha)$ denotes the gamma function,[1] see Fig.1.2c. The mean and variance are

$$E[X] = \alpha\beta, \qquad Var[X] = \alpha\beta^2.$$

The case $\alpha = 1$ is known as the *exponential distribution*, see Fig.1.3a. In this case

$$p(x) = \frac{1}{\beta}e^{-x/\beta}, \quad x \geq 0.$$

The particular case when $\alpha = n/2$ and $\beta = 2$ becomes the χ^2-*distribution* with n degrees of freedom. This characterizes also a sum of n independent standard normal distributions.

Beta distribution A random variable X is said to have a *beta distribution* with parameters $\alpha > 0$, $\beta > 0$ if its probability density function is of the form

$$p(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 \leq x \leq 1,$$

where $B(\alpha, \beta)$ denotes the beta function.[2] See see Fig.1.2d for two particular density functions. In this case

$$E[X] = \frac{\alpha}{\alpha+\beta}, \quad Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Poisson distribution A discrete random variable X is said to have a *Poisson probability distribution* if

$$P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}, \qquad k = 0, 1, 2, \dots,$$

with $\lambda > 0$ parameter, see Fig.1.3b. In this case $E[X] = \lambda$ and $Var[X] = \lambda$.

Pearson 5 distribution Let $\alpha, \beta > 0$. A random variable X with the density function

$$p(x) = \frac{1}{\beta\Gamma(\alpha)}\frac{e^{-\beta/x}}{(x/\beta)^{\alpha+1}}, \qquad x \geq 0$$

is said to have a Pearson 5 distribution[3] with positive parameters α and β. It can be shown that

$$E[X] = \begin{cases} \dfrac{\beta}{\alpha-1}, & \text{if } \alpha > 1 \\ \infty, & \text{otherwise,} \end{cases} \qquad Var(X) = \begin{cases} \dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)}, & \text{if } \alpha > 2 \\ \infty, & \text{otherwise.} \end{cases}$$

[1]Recall the definition of the gamma function $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$; if $\alpha = n$, integer, then $\Gamma(n) = (n-1)!$

[2]Two definition formulas for the beta functions are $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy$.

[3]The Pearson family of distributions was designed by Pearson between 1890 and 1895. There are several Pearson distributions, this one being distinguished by the number 5.

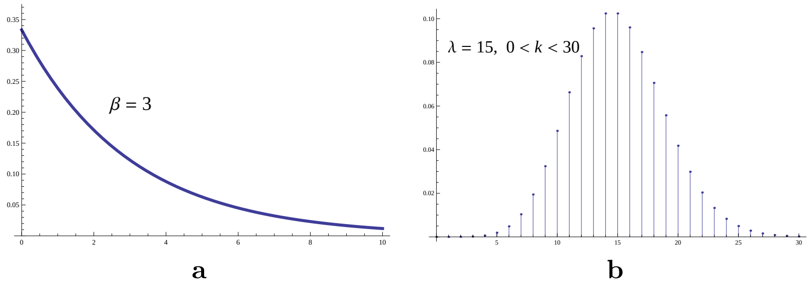


Figure 1.3: **a** *Exponential distribution;* **b** *Poisson distribution.*

The mode of this distribution is equal to $\dfrac{\beta}{\alpha+1}$.

The Inverse Gaussian distribution Let $\mu, \lambda > 0$. A random variable X has an inverse Gaussian distribution with parameters μ and λ if its density function is given by

$$p(x) = \frac{\lambda}{2\pi x^3} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}, \qquad x > 0. \tag{1.6.1}$$

We shall write $X \sim IG(\mu, \lambda)$. Its mean, variance and mode are given by

$$E[X] = \mu, \qquad Var(X) = \frac{\mu^3}{\lambda}, \qquad Mode(X) = \mu\Big(\sqrt{1 + \frac{9\mu^2}{4\lambda^2}} - \frac{3\mu}{2\lambda}\Big).$$

This distribution will be used to model the time instance when a Brownian motion with drift exceeds a certain barrier for the first time.

1.7 Independent Random Variables

Roughly speaking, two random variables X and Y are independent if the occurrence of one of them does not change the probability density of the other. More precisely, if for any sets $A, B \subset \mathbb{R}$, the events

$$\{\omega; X(\omega) \in A\}, \qquad \{\omega; Y(\omega) \in B\}$$

are independent,[4] then X and Y are called *independent* random variables.

Proposition 1.7.1 *Let X and Y be independent random variables with probability density functions $p_X(x)$ and $p_Y(y)$. Then the joint probability density function of (X, Y) is given by $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$.*

Proof: Using the independence of sets, we have[5]

$$\begin{aligned} p_{X,Y}(x,y)\, dxdy &= P(x < X < x+dx,\, y < Y < y+dy) \\ &= P(x < X < x+dx)P(y < Y < y+dy) \\ &= p_X(x)\, dx\; p_Y(y)\, dy \\ &= p_X(x)p_Y(y)\, dxdy. \end{aligned}$$

[4] In Probability Theory two events A_1 and A_2 are called independent if $P(A_1 \cap A_2) = P(A_1)P(A_2)$.

[5] We are using the useful approximation $P(x < X < x+dx) = \int_x^{x+dx} p(u)\, du = p(x)dx$.

Dropping the factor $dxdy$ yields the desired result. We note that the converse holds true. ■

1.8 Integration in Probability Measure

The notion of expectation is based on integration on measure spaces. In this section we recall briefly the definition of an integral with respect to the probability measure P.

Let $X : \Omega \to \mathbb{R}$ be a random variable on the probability space $(\Omega, \mathcal{F}, P)$. A partition $(\Omega_i)_{1\leq i\leq n}$ of Ω is a family of subsets $\Omega_i \subset \Omega$ satisfying

1. $\Omega_i \cap \Omega_j = \emptyset$, for $i \neq j$;
2. $\displaystyle\bigcup_i^n \Omega_i = \Omega$.

Each Ω_i is an event with the associated probability $P(\Omega_i)$. A *simple function* is a sum of characteristic functions $f = \sum_i^n c_i\chi_{\Omega_i}$. This means $f(\omega) = c_k$ for $\omega \in \Omega_k$. The integral of the simple function f is defined by

$$\boxed{\int_\Omega f\,dP = \sum_i^n c_i P(\Omega_i).}$$

If $X : \Omega \to \mathbb{R}$ is a random variable such that there is a sequence of simple functions $(f_n)_{n\geq 1}$ satisfying:

1. f_n *is fundamental in probability*: $\forall\epsilon > 0$ $\lim_{n,m\to\infty} P(\omega; |f_n(\omega) - f_m(\omega)| \geq \epsilon) \to 0$,
2. f_n *converges to* X *in probability*: $\forall\epsilon > 0$ $\lim_{n\to\infty} P(\omega; |f_n(\omega) - X(\omega)| \geq \epsilon) \to 0$

then the integral of X is defined as the following limit of integrals

$$\boxed{\int_\Omega X\,dP = \lim_{n\to\infty}\int_\Omega f_n\,dP.}$$

From now on, the integral notations $\int_\Omega X\,dP$ or $\int_\Omega X(\omega)\,dP(\omega)$ will be used interchangeably. In the rest of the chapter the integral notation will be used formally, without requiring a direct use of the previous definition.

1.9 Expectation

A random variable $X : \Omega \to \mathbb{R}$ is called *integrable* if

$$\int_\Omega |X(\omega)|\,dP(\omega) = \int_\mathbb{R} |x|p(x)\,dx < \infty,$$

where $p(x)$ denotes the probability density function of X. The previous identity is based on changing the domain of integration from Ω to $\mathbb{R}$.

The *expectation* of an integrable random variable X is defined by

$$E[X] = \int_\Omega X(\omega)\,dP(\omega) = \int_\mathbb{R} x\,p(x)\,dx.$$

Customarily, the expectation of X is denoted by μ and it is also called the *mean*. In general, for any continuous[6] function $h : \mathbb{R} \to \mathbb{R}$, we have

$$E[h(X)] = \int_\Omega h\big(X(\omega)\big)\, dP(\omega) = \int_\mathbb{R} h(x)p(x)\, dx.$$

Proposition 1.9.1 *The expectation operator E is linear, i.e. for any integrable random variables X and Y*

1. $E[cX] = cE[X], \qquad \forall c \in \mathbb{R}$;
2. $E[X + Y] = E[X] + E[Y]$.

Proof: It follows from the fact that the integral is a linear operator. ∎

Proposition 1.9.2 *Let X and Y be two independent integrable random variables. Then*

$$E[XY] = E[X]E[Y].$$

Proof: This is a variant of Fubini's theorem, which in this case states that a double integral is a product of two simple integrals. Let p_X, p_Y, $p_{X,Y}$ denote the probability densities of X, Y and (X, Y), respectively. Since X and Y are independent, by Proposition 1.7.1 we have

$$E[XY] = \iint xyp_{X,Y}(x, y)\, dxdy = \int xp_X(x)\, dx \int yp_Y(y)\, dy = E[X]E[Y].$$

∎

1.10 Radon-Nikodym's Theorem

This section is concerned with existence and uniqueness results that will be useful later in defining conditional expectations. Since this section is rather theoretical, it can be skipped at a first reading.

Proposition 1.10.1 *Consider the probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a σ-field included in $\mathcal{F}$. If X is a $\mathcal{G}$-predictable random variable such that*

$$\int_A X\, dP = 0 \qquad \forall A \in \mathcal{G},$$

then $X = 0$ a.s.

Proof: In order to show that $X = 0$ almost surely, it suffices to prove that $P\big(\omega; X(\omega) = 0\big) = 1$. We shall show first that X takes values as small as possible with probability one, i.e. $\forall \epsilon > 0$ we have $P(|X| < \epsilon) = 1$. To do this, let $A = \{\omega; X(\omega) \geq \epsilon\}$. Then

$$0 \leq P(X \geq \epsilon) = \int_A dP = \frac{1}{\epsilon}\int_A \epsilon\, dP \leq \frac{1}{\epsilon}\int_A X\, dP = 0,$$

and hence $P(X \geq \epsilon) = 0$. Similarly $P(X \leq -\epsilon) = 0$. Therefore

$$P(|X| < \epsilon) = 1 - P(X \geq \epsilon) - P(X \leq -\epsilon) = 1 - 0 - 0 = 1.$$

[6]in general, measurable

Taking $\epsilon \to 0$ leads to $P(|X| = 0) = 1$. This can be formalized as follows. Let $\epsilon = \frac{1}{n}$ and consider $B_n = \{\omega; |X(\omega)| \le \epsilon\}$, with $P(B_n) = 1$. Then

$$P(X = 0) = P(|X| = 0) = P(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n) = 1.$$

■

Corollary 1.10.2 *If X and Y are $\mathcal{G}$-predictable random variables such that*

$$\int_A X\,dP = \int_A Y\,dP \qquad \forall A \in \mathcal{G},$$

then $X = Y$ a.s.

Proof: Since $\int_A (X - Y)\,dP = 0$, $\forall A \in \mathcal{G}$, by Proposition 1.10.1 we have $X - Y = 0$ a.s. ■

Theorem 1.10.3 (Radon-Nikodym) *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ be a σ-field included in $\mathcal{F}$. Then for any random variable X there is a $\mathcal{G}$-predictable random variable Y such that*

$$\int_A X\,dP = \int_A Y\,dP, \qquad \forall A \in \mathcal{G}. \tag{1.10.2}$$

We shall omit the proof but discuss a few aspects.

1. All σ-fields $\mathcal{G} \subset \mathcal{F}$ contain impossible and certain events $\emptyset, \Omega \in \mathcal{G}$. Making $A = \Omega$ yields

$$\int_\Omega X\,dP = \int_\Omega Y\,dP,$$

which is $E[X] = E[Y]$.

2. Radon-Nikodym's theorem states the existence of Y. In fact this is unique almost surely. In order to show that, assume there are two $\mathcal{G}$-predictable random variables Y_1 and Y_2 with the aforementioned property. Then from (1.10.2) yields

$$\int_A Y_1\,dP = \int_A Y_2\,dP, \qquad \forall A \in \mathcal{G}.$$

Applying Corollary (1.10.2) yields $Y_1 = Y_2$ a.s.

3. Since $E[X] = \int_\Omega X\,dP$ is the expectation of the random variable X, given the full knowledge $\mathcal{F}$, then the random variable Y plays the role of the expectation of X given the partial information $\mathcal{G}$. The next section will deal with this concept in detail.

1.11 Conditional Expectation

Let X be a random variable on the probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. Since X is $\mathcal{F}$-predictable, the expectation of X, given the information $\mathcal{F}$ must be X itself. This shall be written as $E[X|\mathcal{F}] = X$ (for details see Example 1.11.3). It is natural to ask what is the expectation of X, given the information $\mathcal{G}$. This is a random variable denoted by $E[X|\mathcal{G}]$ satisfying the following properties:

1. $E[X|\mathcal{G}]$ is $\mathcal{G}$-predictable;
2. $\int_A E[X|\mathcal{G}]\,dP = \int_A X\,dP, \qquad \forall A \in \mathcal{G}$.

$E[X|\mathcal{G}]$ is called the *conditional expectation* of X given $\mathcal{G}$.

We owe a few explanations regarding the correctness of the aforementioned definition. The existence of the $\mathcal{G}$-predictable random variable $E[X|\mathcal{G}]$ is assured by the Radon-Nikodym theorem. The almost surely uniqueness is an application of Proposition (1.10.1) (see the discussion point 2 of section 1.10).

It is worth noting that the expectation of X, denoted by $E[X]$ is a number, while the conditional expectation $E[X|\mathcal{G}]$ is a random variable. When are they equal and what is their relationship? The answer is inferred by the following solved exercises.

Example 1.11.1 *Show that if $\mathcal{G} = \{\emptyset, \Omega\}$, then $E[X|\mathcal{G}] = E[X]$.*

Proof: We need to show that $E[X]$ satisfies conditions 1 and 2. The first one is obviously satisfied since any constant is $\mathcal{G}$-predictable. The latter condition is checked on each set of $\mathcal{G}$. We have

$$\begin{aligned} \int_\Omega X\,dP &= E[X] = E[X]\int_\Omega dP = \int_\Omega E[X]dP \\ \int_\emptyset X\,dP &= \int_\emptyset E[X]dP. \end{aligned}$$

■

Example 1.11.2 *Show that $E[E[X|\mathcal{G}]] = E[X]$, i.e. all conditional expectations have the same mean, which is the mean of X.*

Proof: Using the definition of expectation and taking $A = \Omega$ in the second relation of the aforementioned definition, yields

$$E[E[X|\mathcal{G}]] = \int_\Omega E[X|\mathcal{G}]\,dP = \int_\Omega X dP = E[X],$$

which ends the proof. ■

Example 1.11.3 *The conditional expectation of X given the total information $\mathcal{F}$ is the random variable X itself, i.e.*

$$E[X|\mathcal{F}] = X.$$

Proof: The random variables X and $E[X|\mathcal{F}]$ are both $\mathcal{F}$-predictable (from the definition of the random variable). From the definition of the conditional expectation we have

$$\int_A E[X|\mathcal{F}]\,dP = \int_A X\,dP, \quad \forall A \in \mathcal{F}.$$

Corollary (1.10.2) implies that $E[X|\mathcal{F}] = X$ almost surely. ■

General properties of the conditional expectation are stated below without proof. The proof involves more or less simple manipulations of integrals and can be taken as an exercise for the reader.

Proposition 1.11.4 *Let X and Y be two random variables on the probability space $(\Omega, \mathcal{F}, P)$. We have*

1. Linearity:

$$E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}], \qquad \forall a, b \in \mathbb{R};$$

2. Factoring out the predictable part:

$$E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$$

if X is $\mathcal{G}$-predictable. In particular, $E[X|\mathcal{G}] = X$.

3. Tower property:

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}], \textit{ if } \mathcal{H} \subset \mathcal{G};$$

4. Positivity:

$$E[X|\mathcal{G}] \geq 0, \textit{ if } X \geq 0;$$

5. Expectation of a constant is a constant:

$$E[c|\mathcal{G}] = c.$$

6. An independent condition drops out:

$$E[X|\mathcal{G}] = E[X],$$

if X is independent of $\mathcal{G}$.

Exercise 1.11.5 *Prove the property 3 (tower property) given in the previous proposition.*

Exercise 1.11.6 *Let X be a random variable on the probability space $(\Omega, \mathcal{F}, P)$, which is independent of the σ-field $\mathcal{G} \subset \mathcal{F}$. Consider the characteristic function of a set $A \subset \Omega$ defined by $\chi_A(\omega) = \begin{cases} 1, & \textit{if } \omega \in A \\ 0, & \textit{if } \omega \notin A \end{cases}$ Show the following:*

(a) *χ_A is $\mathcal{G}$-predictable for any $A \in \mathcal{G}$;*

(b) *$P(A) = E[\chi_A]$;*

(c) *X and χ_A are independent random variables;*

(d) *$E[\chi_A X] = E[X]P(A)$ for any $A \in \mathcal{G}$;*

(e) *$E[X|\mathcal{G}] = E[X]$.*

1.12 Inequalities of Random Variables

This section prepares the reader for the limits of sequences of random variables and limits of stochastic processes. We shall start with a classical inequality result regarding expectations:

Theorem 1.12.1 (Jensen's inequality) *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function and let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, P)$. If $\varphi(X)$ is integrable, then*

$$\varphi(E[X]) \leq E[\varphi(X)]$$

almost surely (i.e. the inequality might fail on a set of probability zero).

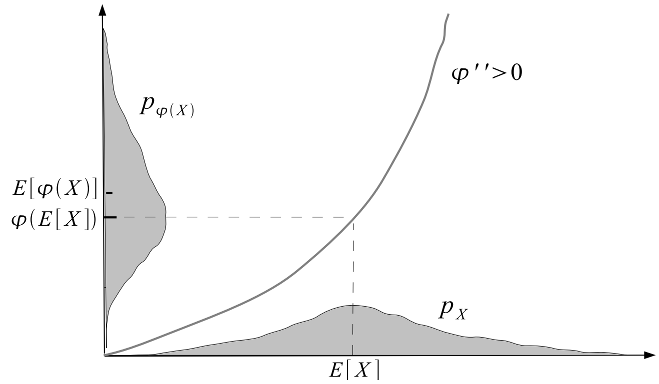


Figure 1.4: *Jensen's inequality $\varphi(E[X]) < E[\varphi(X)]$ for a convex function φ.*

Proof: We shall assume φ twice differentiable with φ'' continuous. Let $\mu = E[X]$. Expand φ in a Taylor series about μ and get

$$\varphi(x) = \varphi(\mu) + \varphi'(\mu)(x-\mu) + \frac{1}{2}\varphi''(\xi)(\xi-\mu)^2,$$

with ξ in between x and μ. Since φ is convex, $\varphi'' \geq 0$, and hence

$$\varphi(x) \geq \varphi(\mu) + \varphi'(\mu)(x-\mu),$$

which means the graph of $\varphi(x)$ is above the tangent line at $\big(x, \varphi(x)\big)$. Replacing x by the random variable X, and taking the expectation yields

$$\begin{aligned} E[\varphi(X)] &\geq E[\varphi(\mu) + \varphi'(\mu)(X-\mu)] = \varphi(\mu) + \varphi'(\mu)(E[X]-\mu) \\ &= \varphi(\mu) = \varphi(E[X]), \end{aligned}$$

which proves the result. ■

Fig.1.4 provides a graphical interpretation of Jensen's inequality. If the distribution of X is symmetric, then the distribution of $\varphi(X)$ is skewed, with $\varphi(E[X]) < E[\varphi(X)]$.

It is worth noting that the inequality is reversed for φ concave. We shall next present a couple of applications.

A random variable $X : \Omega \to \mathbb{R}$ is called *square integrable* if

$$E[X^2] = \int_\Omega |X(\omega)|^2\, dP(\omega) = \int_{\mathbb{R}} x^2 p(x)\, dx < \infty.$$

Application 1.12.2 *If X is a square integrable random variable, then it is integrable.*

Proof: Jensen's inequality with $\varphi(x) = x^2$ becomes

$$E[X]^2 \leq E[X^2].$$

Since the right side is finite, it follows that $E[X] < \infty$, so X is integrable. ■

Application 1.12.3 *If $m_X(t)$ denotes the moment generating function of the random variable X with mean μ, then*

$$m_X(t) \geq e^{t\mu}.$$

Proof: Applying Jensen inequality with the convex function $\varphi(x) = e^x$ yields

$$e^{E[X]} \leq E[e^X].$$

Substituting tX for X yields

$$e^{E[tX]} \leq E[e^{tX}]. \tag{1.12.3}$$

Using the definition of the moment generating function $m_X(t) = E[e^{tX}]$ and that $E[tX] = tE[X] = t\mu$, then (1.12.3) leads to the desired inequality. ∎

The *variance* of a square integrable random variable X is defined by

$$Var(X) = E[X^2] - E[X]^2.$$

By Application 1.12.2 we have $Var(X) \geq 0$, so there is a constant $\sigma_X > 0$, called *standard deviation*, such that

$$\sigma_X^2 = Var(X).$$

Exercise 1.12.4 *Prove the following identity:*

$$Var[X] = E[(X - E[X])^2].$$

Exercise 1.12.5 *Prove that a non-constant random variable has a non-zero standard deviation.*

Exercise 1.12.6 *Prove the following extension of Jensen's inequality: If φ is a convex function, then for any σ-field $\mathcal{G} \subset \mathcal{F}$ we have*

$$\varphi(E[X|\mathcal{G}]) \leq E[\varphi(X)|\mathcal{G}].$$

Exercise 1.12.7 *Show the following:*

(*a*) $|E[X]| \leq E[|X|]$;

(*b*) $|E[X|\mathcal{G}]| \leq E[|X|\,|\mathcal{G}]$, *for any σ-field $\mathcal{G} \subset \mathcal{F}$;*

(*c*) $|E[X]|^r \leq E[|X|^r]$, *for $r \geq 1$;*

(*d*) $|E[X|\mathcal{G}]|^r \leq E[|X|^r\,|\mathcal{G}]$, *for any σ-field $\mathcal{G} \subset \mathcal{F}$ and $r \geq 1$.*

Theorem 1.12.8 (Markov's inequality) *For any $\lambda, p > 0$, we have the following inequality:*

$$P(\omega; |X(\omega)| \geq \lambda) \leq \frac{1}{\lambda^p} E[|X|^p].$$

Proof: Let $A = \{\omega; |X(\omega)| \geq \lambda\}$. Then

$$\begin{aligned} E[|X|^p] &= \int_\Omega |X(\omega)|^p \, dP(\omega) \geq \int_A |X(\omega)|^p \, dP(\omega) \geq \int_A \lambda^p \, dP(\omega) \\ &= \lambda^p \int_A dP(\omega) = \lambda^p P(A) = \lambda^p P(|X| \geq \lambda). \end{aligned}$$

Dividing by λ^p leads to the desired result. ∎

Theorem 1.12.9 (Tchebychev's inequality) *If X is a random variable with mean μ and variance σ^2, then*

$$P(\omega; |X(\omega) - \mu| \geq \lambda) \leq \frac{\sigma^2}{\lambda^2}.$$

Proof: Let $A = \{\omega; |X(\omega) - \mu| \geq \lambda\}$. Then

$$\begin{aligned} \sigma^2 &= Var(X) = E[(X-\mu)^2] = \int_\Omega (X-\mu)^2 \, dP \geq \int_A (X-\mu)^2 \, dP \\ &\geq \lambda^2 \int_A dP = \lambda^2 P(A) = \lambda^2 P(\omega; |X(\omega) - \mu| \geq \lambda). \end{aligned}$$

Dividing by λ^2 leads to the desired inequality. ■

The next result deals with exponentially decreasing bounds on tail distributions.

Theorem 1.12.10 (Chernoff bounds) *Let X be a random variable. Then for any $\lambda > 0$ we have*

1. $P(X \geq \lambda) \leq \dfrac{E[e^{tX}]}{e^{\lambda t}}$, $\forall t > 0$;
2. $P(X \leq \lambda) \leq \dfrac{E[e^{tX}]}{e^{\lambda t}}$, $\forall t < 0$.

Proof: 1. Let $t > 0$ and denote $Y = e^{tX}$. By Markov's inequality

$$P(Y \geq e^{\lambda t}) \leq \frac{E[Y]}{e^{\lambda t}}.$$

Then we have

$$\begin{aligned} P(X \geq \lambda) &= P(tX \geq \lambda t) = P(e^{tX} \geq e^{\lambda t}) \\ &= P(Y \geq e^{\lambda t}) \leq \frac{E[Y]}{e^{\lambda t}} = \frac{E[e^{tX}]}{e^{\lambda t}}. \end{aligned}$$

2. The case $t < 0$ is similar. ■

In the following we shall present an application of the Chernoff bounds for the normal distributed random variables.

Let X be a random variable normally distributed with mean μ and variance σ^2. It is known that its moment generating function is given by

$$m(t) = E[e^{tX}] = e^{\mu t + \frac{1}{2}t^2\sigma^2}.$$

Using the first Chernoff bound we obtain

$$P(X \geq \lambda) \leq \frac{m(t)}{e^{\lambda t}} = e^{(\mu - \lambda)t + \frac{1}{2}t^2\sigma^2}, \forall t > 0,$$

which implies

$$P(X \geq \lambda) \leq e^{\min\limits_{t>0}[(\mu-\lambda)t + \frac{1}{2}t^2\sigma^2]}.$$

It is easy to see that the quadratic function $f(t) = (\mu - \lambda)t + \frac{1}{2}t^2\sigma^2$ has the minimum value reached for $t = \dfrac{\lambda - \mu}{\sigma^2}$. Since $t > 0$, λ needs to satisfy $\lambda > \mu$. Then

$$\min_{t>0} f(t) = f\Big(\frac{\lambda - \mu}{\sigma^2}\Big) = -\frac{(\lambda - \mu)^2}{2\sigma^2}.$$

Substituting into the previous formula, we obtain the following result:

Proposition 1.12.11 *If X is a normally distributed variable, with $X \sim N(\mu, \sigma^2)$, then for any $\lambda > \mu$*

$$P(X \ge \lambda) \le e^{-\dfrac{(\lambda - \mu)^2}{2\sigma^2}}.$$

Exercise 1.12.12 *Let X be a Poisson random variable with mean $\lambda > 0$.*
(a) Show that the moment generating function of X is $m(t) = e^{\lambda(e^t - 1)}$;
(b) Use a Chernoff bound to show that

$$P(X \ge k) \le e^{\lambda(e^t - 1) - tk}, \qquad t > 0.$$

Markov's, Tchebychev's and Chernoff's inequalities will be useful later when computing limits of random variables.

The next inequality is called Tchebychev's inequality for monotone sequences of numbers.

Lemma 1.12.13 *Let (a_i) and (b_i) be two sequences of real numbers such that either*

$$a_1 \le a_2 \le \cdots \le a_n, \qquad b_1 \le b_2 \le \cdots \le b_n$$

or

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad b_1 \ge b_2 \ge \cdots \ge b_n$$

If (λ_i) is a sequence of non-negative numbers such that $\sum_{i=1}^{n} \lambda_i = 1$, then

$$\Big(\sum_{i=1}^{n} \lambda_i a_i\Big)\Big(\sum_{i=1}^{n} \lambda_i b_i\Big) \le \sum_{i=1}^{n} \lambda_i a_i b_i.$$

Proof: Since the sequences (a_i) and (b_i) are either both increasing or both decreasing

$$(a_i - a_j)(b_i - b_j) \ge 0.$$

Multiplying by the positive quantity $\lambda_i\lambda_j$ and summing over i and j we get

$$\sum_{i,j} \lambda_i\lambda_j(a_i - a_j)(b_i - b_j) \ge 0.$$

Expanding yields

$$\Big(\sum_j \lambda_j\Big)\Big(\sum_i \lambda_i a_i b_i\Big) - \Big(\sum_i \lambda_i a_i\Big)\Big(\sum_j \lambda_j b_j\Big) - \Big(\sum_j \lambda_j a_j\Big)\Big(\sum_i \lambda_i b_i\Big)$$
$$+\Big(\sum_i \lambda_i\Big)\Big(\sum_j \lambda_j a_j b_j\Big) \ge 0.$$

Using $\sum_j \lambda_j = 1$ the expression becomes

$$\sum_i \lambda_i a_i b_i \geq \Big(\sum_i \lambda_i a_i\Big)\Big(\sum_j \lambda_j b_j\Big),$$

which ends the proof. ■

Next we present a meaningful application of the previous inequality.

Proposition 1.12.14 *Let X be a random variable and f and g be two functions, both increasing or both decreasing. Then*

$$E[f(X)g(X)] \geq E[f(X)]E[g(X)]. \tag{1.12.4}$$

Proof: If X is a discrete random variable, with outcomes $\{x_1, \cdots, x_n\}$, inequality (1.12.4) becomes

$$\sum_j f(x_j)g(x_j)p(x_j) \geq \sum_j f(x_j)p(x_j) \sum_j g(x_j)p(x_j),$$

where $p(x_j) = P(X = x_j)$. Denoting $a_j = f(x_j)$, $b_j = g(x_j)$, and $\lambda_j = p(x_j)$, the inequality transforms into

$$\sum_j a_j b_j \lambda_j \geq \sum_j a_j \lambda_j \sum_j b_j \lambda_j,$$

which holds true by Lemma 1.12.13.

If X is a continuous random variable with the density function $p : I \to \mathbb{R}$, the inequality (1.12.4) can be written in the integral form

$$\int_I f(x)g(x)p(x)\,dx \geq \int_I f(x)p(x)\,dx \ \int_I g(x)p(x)\,dx. \tag{1.12.5}$$

Let $x_0 < x_1 < \cdots < x_n$ be a partition of the interval I, with $\Delta x = x_{k+1} - x_k$. Using Lemma 1.12.13 we obtain the following inequality between Riemann sums

$$\sum_j f(x_j)g(x_j)p(x_j)\Delta x \geq \Big(\sum_j f(x_j)p(x_j)\Delta x\Big)\Big(\sum_j g(x_j)p(x_j)\Delta x\Big),$$

where $a_j = f(x_j)$, $b_j = g(x_j)$, and $\lambda_j = p(x_j)\Delta x$. Taking the limit $\|\Delta x\| \to 0$ we obtain (1.12.5), which leads to the desired result. ■

Exercise 1.12.15 *Show the following inequalities:*

(*a*) $E[X^2] \geq E[X]^2$;

(*b*) $E[X\sinh(X)] \geq E[X]E[\sinh(X)]$;

(*c*) $E[X^6] \geq E[X]E[X^5]$;

(*d*) $E[X^6] \geq E[X^3]^2$.

Exercise 1.12.16 *For any $n, k \geq 1$, show that*

$$E[X^{2(n+k+1)}] \geq E[X^{2k+1}]E[X^{2n+1}].$$

1.13 Limits of Sequences of Random Variables

Consider a sequence $(X_n)_{n\geq 1}$ of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. There are several ways of making sense of the limit expression $X = \lim\limits_{n\to\infty} X_n$, and they will be discussed in the following sections.

Almost Certain Limit

The sequence X_n converges *almost certainly* to X, if for all states of the world ω, except a set of probability zero, we have

$$\lim_{n\to\infty} X_n(\omega) = X(\omega).$$

More precisely, this means

$$P\Big(\omega; \lim_{n\to\infty} X_n(\omega) = X(\omega)\Big) = 1,$$

and we shall write $\text{ac-}\lim\limits_{n\to\infty} X_n = X$. An important example where this type of limit occurs is the Strong Law of Large Numbers:

If X_n is a sequence of independent and identically distributed random variables with the same mean μ, then $ac\text{-}\lim\limits_{n\to\infty} \dfrac{X_1 + \cdots + X_n}{n} = \mu$.

It is worth noting that this type of convergence is also known under the name of *strong convergence*. This is the reason that the aforementioned theorem bares its name.

Example 1.13.1 *Let $\Omega = \{H, T\}$ be the sample space obtained when a coin is flipped. Consider the random variables $X_n : \Omega \to \{0, 1\}$, where X_n denotes the number of heads obtained at the n-th flip. Obviously, X_n are i.i.d., with the distribution given by $P(X_n = 0) = P(X_n = 1) = 1/2$, and the mean $E[X_n] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$. Then $X_1 + \cdots + X_n$ is the number of heads obtained after n flips of the coin. By the law of large numbers, $\frac{1}{n}(X_1 + \cdots + X_n)$ tends to $1/2$ strongly, as $n \to \infty$.*

Mean Square Limit

Another possibility of convergence is to look at the mean square deviation of X_n from X. We say that X_n converges to X in the *mean square* if

$$\lim_{n\to\infty} E[(X_n - X)^2] = 0.$$

More precisely, this should be interpreted as

$$\lim_{n\to\infty} \int_\Omega \big(X_n(\omega) - X(\omega)\big)^2 \, dP(\omega) = 0.$$

This limit will be abbreviated by $\text{ms-}\lim\limits_{n\to\infty} X_n = X$. The mean square convergence is useful when defining the Ito integral.

Example 1.13.1 *Consider a sequence X_n of random variables such that there is a constant k with $E[X_n] \to k$ and $Var(X_n) \to 0$ as $n \to \infty$. Show that $ms\text{-}\lim\limits_{n\to\infty} X_n = k$.*

Proof: Since we have

$$\begin{aligned} E[|X_n - k|^2] &= E[X_n^2 - 2kX_n + k^2] = E[X_n^2] - 2kE[X_n] + k^2 \\ &= \big(E[X_n^2] - E[X_n]^2\big) + \big(E[X_n]^2 - 2kE[X_n] + k^2\big) \\ &= Var(X_n) + \big(E[X_n] - k\big)^2, \end{aligned}$$

the right side tends to 0 when taking the limit $n \to \infty$. ∎

Exercise 1.13.2 *Show the following relation*

$$E[(X - Y)^2] = Var[X] + Var[Y] + \big(E[X] - E[Y]\big)^2 - 2Cov(X, Y).$$

Exercise 1.13.3 *If X_n tends to X in mean square, with $E[X^2] < \infty$, show that:*

(a) $E[X_n] \to E[X]$ *as* $n \to \infty$;

(b) $E[X_n^2] \to E[X^2]$ *as* $n \to \infty$;

(c) $Var[X_n] \to Var[X]$ *as* $n \to \infty$;

(d) $Cov(X_n, X) \to Var[X]$ *as* $n \to \infty$.

Exercise 1.13.4 *If X_n tends to X in mean square, show that $E[X_n|\mathcal{H}]$ tends to $E[X|\mathcal{H}]$ in mean square.*

Limit in Probability or Stochastic Limit

The random variable X is the *stochastic limit* of X_n if for n large enough the probability of deviation from X can be made smaller than any arbitrary ϵ. More precisely, for any $\epsilon > 0$

$$\lim_{n\to\infty} P\big(\omega; |X_n(\omega) - X(\omega)| \le \epsilon\big) = 1.$$

This can be written also as

$$\lim_{n\to\infty} P\big(\omega; |X_n(\omega) - X(\omega)| > \epsilon\big) = 0.$$

This limit is denoted by $\text{st-}\lim_{n\to\infty} X_n = X$.

It is worth noting that both almost certain convergence and convergence in mean square imply the stochastic convergence. Hence, the stochastic convergence is weaker than the aforementioned two convergence cases. This is the reason that it is also called the *weak convergence*. One application is the Weak Law of Large Numbers:

If $X_1, X_2, \ldots$ are identically distributed with expected value μ and if any finite number of them are independent, then $st\text{-}\lim_{n\to\infty} \dfrac{X_1 + \cdots + X_n}{n} = \mu$.

Proposition 1.13.5 *The convergence in the mean square implies the stochastic convergence.*

Proof: Let $\text{ms-}\lim_{n\to\infty} Y_n = Y$. Let $\epsilon > 0$ be arbitrarily fixed. Applying Markov's inequality with $X = Y_n - Y$, $p = 2$ and $\lambda = \epsilon$, yields

$$0 \le P(|Y_n - Y| \ge \epsilon) \le \frac{1}{\epsilon^2} E[|Y_n - Y|^2].$$

The right side tends to 0 as $n \to \infty$. Applying the Squeeze Theorem we obtain

$$\lim_{n\to\infty} P(|Y_n - Y| \geq \epsilon) = 0,$$

which means that Y_n converges stochastically to Y. ■

Example 1.13.6 *Let X_n be a sequence of random variables such that $E[|X_n|] \to 0$ as $n \to \infty$. Prove that $st\text{-}\lim_{n\to\infty} X_n = 0$.*

Proof: Let $\epsilon > 0$ be arbitrarily fixed. We need to show

$$\lim_{n\to\infty} P\big(\omega; |X_n(\omega)| \geq \epsilon\big) = 0. \tag{1.13.6}$$

From Markov's inequality (see Exercise 1.12.8) we have

$$0 \leq P\big(\omega; |X_n(\omega)| \geq \epsilon\big) \leq \frac{E[|X_n|]}{\epsilon}.$$

Using Squeeze Theorem we obtain (1.13.6). ■

Remark 1.13.7 *The conclusion still holds true even in the case when there is a $p > 0$ such that $E[|X_n|^p] \to 0$ as $n \to \infty$.*

Limit in Distribution

We say the sequence X_n converges *in distribution* to X if for any continuous bounded function $\varphi(x)$ we have

$$\lim_{n\to\infty} \varphi(X_n) = \varphi(X).$$

This type of limit is even weaker than the stochastic convergence, i.e. it is implied by it.

An application of the limit in distribution is obtained if we consider $\varphi(x) = e^{itx}$. In this case, if X_n converges in distribution to X, then the characteristic function of X_n converges to the characteristic function of X. In particular, the probability density of X_n approaches the probability density of X.

It can be shown that the convergence in distribution is equivalent with

$$\lim_{n\to\infty} F_n(x) = F(x),$$

whenever F is continuous at x, where F_n and F denote the distribution functions of X_n and X, respectively. This is the reason that this convergence bares its name.

Remark 1.13.8 *The almost certain convergence implies the stochastic convergence, and the stochastic convergence implies the limit in distribution. The proof of these statements is beyound the goal of this book. The interested reader can consult a graduate text in probability theory.*

1.14 Properties of Limits

Lemma 1.14.1 *If* $ms\text{-}\lim_{n\to\infty} X_n = 0$ *and* $ms\text{-}\lim_{n\to\infty} Y_n = 0$*, then*

1. $$ms\text{-}\lim_{n\to\infty}(X_n + Y_n) = 0$$

2. $$ms\text{-}\lim_{n\to\infty}(X_n Y_n) = 0.$$

Proof: Since $\text{ms-}\lim_{n\to\infty} X_n = 0$, then $\lim_{n\to\infty} E[X_n^2] = 0$. Applying the Squeeze Theorem to the inequality[7]

$$0 \le E[X_n]^2 \le E[X_n^2]$$

yields $\lim_{n\to\infty} E[X_n] = 0$. Then

$$\begin{aligned}
\lim_{n\to\infty} Var[X_n] &= \lim_{n\to\infty}\Big(E[X_n^2] - \lim_{n\to\infty} E[X_n]^2\Big) \\
&= \lim_{n\to\infty} E[X_n^2] - \lim_{n\to\infty} E[X_n]^2 \\
&= 0.
\end{aligned}$$

Similarly, we have $\lim_{n\to\infty} E[Y_n^2] = 0$, $\lim_{n\to\infty} E[Y_n] = 0$ and $\lim_{n\to\infty} Var[Y_n] = 0$. Then $\lim_{n\to\infty} \sigma_{X_n} = \lim_{n\to\infty} \sigma_{Y_n} = 0$. Using the correlation definition formula of two random variables X_n and Y_n

$$Corr(X_n, Y_n) = \frac{Cov(X_n, Y_n)}{\sigma_{X_n}\sigma_{Y_n}},$$

and the fact that $|Corr(X_n, Y_n)| \le 1$, yields

$$0 \le |Cov(X_n, Y_n)| \le \sigma_{X_n}\sigma_{Y_n}.$$

Since $\lim_{n\to\infty} \sigma_{X_n}\sigma_{X_n} = 0$, from the Squeeze Theorem it follows that

$$\lim_{n\to\infty} Cov(X_n, Y_n) = 0.$$

Taking $n \to \infty$ in the relation

$$Cov(X_n, Y_n) = E[X_n Y_n] - E[X_n]E[Y_n]$$

yields $\lim_{n\to\infty} E[X_n Y_n] = 0$. Using the previous relations, we have

$$\begin{aligned}
\lim_{n\to\infty} E[(X_n + Y_n)^2] &= \lim_{n\to\infty} E[X_n^2 + 2X_nY_n + Y_n^2] \\
&= \lim_{n\to\infty} E[X_n^2] + 2\lim_{n\to\infty} E[X_nY_n] + \lim_{n\to\infty} E[Y_n^2] \\
&= 0,
\end{aligned}$$

which means $\text{ms-}\lim_{n\to\infty}(X_n + Y_n) = 0$. ■

[7]This follows from the fact that $Var[X_n] \ge 0$.

Proposition 1.14.2 *If the sequences of random variables X_n and Y_n converge in the mean square, then*

$$\begin{aligned} &1. \qquad ms\text{-}\lim_{n\to\infty}(X_n+Y_n) = ms\text{-}\lim_{n\to\infty} X_n + ms\text{-}\lim_{n\to\infty} Y_n \\ &2. \qquad ms\text{-}\lim_{n\to\infty}(cX_n) = c\cdot ms\text{-}\lim_{n\to\infty} X_n, \quad \forall c\in\mathbb{R}. \end{aligned}$$

Proof: 1. Let $\text{ms-}\lim\limits_{n\to\infty} X_n = L$ and $\text{ms-}\lim\limits_{n\to\infty} Y_n = M$. Consider the sequences $X'_n = X_n - L$ and $Y'_n = Y_n - M$. Then $\text{ms-}\lim\limits_{n\to\infty} X'_n = 0$ and $\text{ms-}\lim\limits_{n\to\infty} Y'_n = 0$. Applying Lemma 1.14.1 yields

$$\text{ms-}\lim_{n\to\infty}(X'_n + Y'_n) = 0.$$

This is equivalent with

$$\text{ms-}\lim_{n\to\infty}(X_n - L + Y_n - M) = 0,$$

which becomes

$$\text{ms-}\lim_{n\to\infty}(X_n + Y_n) = L + M.$$

■

1.15 Stochastic Processes

A *stochastic process* on the probability space $(\Omega, \mathcal{F}, P)$ is a family of random variables X_t parameterized by $t \in \mathbf{T}$, where $\mathbf{T} \subset \mathbb{R}$. If $\mathbf{T}$ is an interval we say that X_t is a stochastic process in *continuous time*. If $\mathbf{T} = \{1, 2, 3, \dots\}$ we shall say that X_t is a stochastic process in *discrete time*. The latter case describes a sequence of random variables. The aforementioned types of convergence can be easily extended to continuous time. For instance, X_t converges in the strong sense to X as $t \to \infty$ if

$$P\Big(\omega; \lim_{t\to\infty} X_t(\omega) = X(\omega)\Big) = 1.$$

The evolution in time of a given state of the world $\omega \in \Omega$ given by the function $t \longmapsto X_t(\omega)$ is called a *path* or *realization* of X_t. The study of stochastic processes using computer simulations is based on retrieving information about the process X_t given a large number of it realizations.

Consider that all the information accumulated until time t is contained by the σ-field $\mathcal{F}_t$. This means that $\mathcal{F}_t$ contains the information of which events have already occurred until time t, and which did not. Since the information is growing in time, we have

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$$

for any $s, t \in \mathbf{T}$ with $s \le t$. The family $\mathcal{F}_t$ is called a *filtration.*

A stochastic process X_t is called *adapted* to the filtration $\mathcal{F}_t$ if X_t is $\mathcal{F}_t$- predictable, for any $t \in \mathbf{T}$.

Example 1.15.1 *Here there are a few examples of filtrations:*

1. $\mathcal{F}_t$ represents the information about the evolution of a stock until time t, with $t > 0$.

2. $\mathcal{F}_t$ represents the information about the evolution of a Black-Jack game until time t, with $t > 0$.

Example 1.15.2 *If X is a random variable, consider the conditional expectation*

$$X_t = E[X|\mathcal{F}_t].$$

From the definition of conditional expectation, the random variable X_t is $\mathcal{F}_t$-predictable, and can be regarded as the measurement of X at time t using the information $\mathcal{F}_t$. If the accumulated knowledge $\mathcal{F}_t$ increases and eventually equals the σ-field $\mathcal{F}$, then $X = E[X|\mathcal{F}]$, i.e. we obtain the entire random variable. The process X_t is adapted to $\mathcal{F}_t$.

Example 1.15.3 *Don Joe is asking a doctor how long he still has to live. The age at which he will pass away is a random variable, denoted by X. Given his medical condition today, which is contained in $\mathcal{F}_t$, the doctor infers that Mr. Joe will die at the age of $X_t = E[X|\mathcal{F}_t]$. The stochastic process X_t is adapted to the medical knowledge $\mathcal{F}_t$.*

We shall define next an important type of stochastic process.[8]

Definition 1.15.4 *A process X_t, $t \in \mathbf{T}$, is called a martingale with respect to the filtration $\mathcal{F}_t$ if*

1. *X_t is integrable for each $t \in \mathbf{T}$;*
2. *X_t is adapted to the filtration $\mathcal{F}_t$;*
3. *$X_s = E[X_t|\mathcal{F}_s]$, $\forall s < t$.*

Remark 1.15.5 *The first condition states that the unconditional forecast is finite $E[|X_t|] = \int_\Omega |X_t|\, dP < \infty$. Condition 2 says that the value X_t is known, given the information set $\mathcal{F}_t$. This can be also stated by saying that X_t is $\mathcal{F}_t$-predictable. The third relation asserts that the best forecast of unobserved future values is the last observation on X_t.*

Remark 1.15.6 *If the third condition is replaced by*

$3'$. $\quad X_s \leq E[X_t|\mathcal{F}_s]$, $\forall s \leq t$

then X_t is called a submartingale; and if it is replaced by

$3''$. $\quad X_s \geq E[X_t|\mathcal{F}_s]$, $\forall s \leq t$

then X_t is called a supermartingale.

It is worth noting that X_t is a submartingale if and only if $-X_t$ is a supermartingale.

Example 1.15.1 *Let X_t denote Mr. Li Zhu's salary after t years of work at the same company. Since X_t is known at time t and it is bounded above, as all salaries are, then the first two conditions hold. Being honest, Mr. Zhu expects today that his future salary will be the same as today's, i.e. $X_s = E[X_t|\mathcal{F}_s]$, for $s < t$. This means that X_t is a martingale.*

If Mr. Zhu is optimistic and believes as of today that his future salary will increase, then X_t is a submartingale.

Exercise 1.15.7 *If X is an integrable random variable on $(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_t$ is a filtration. Prove that $X_t = E[X|\mathcal{F}_t]$ is a martingale.*

Exercise 1.15.8 *Let X_t and Y_t be martingales with respect to the filtration $\mathcal{F}_t$. Show that for any $a, b, c \in \mathbb{R}$ the process $Z_t = aX_t + bY_t + c$ is a $\mathcal{F}_t$-martingale.*

Exercise 1.15.9 *Let X_t and Y_t be martingales with respect to the filtration $\mathcal{F}_t$.*

(a) *Is the process X_tY_t always a martingale with respect to $\mathcal{F}_t$?*

(b) *What about the processes X_t^2 and Y_t^2?*

[8] The concept of martingale was introduced by Lévy in 1934.

Exercise 1.15.10 *Two processes X_t and Y_t are called conditionally uncorrelated, given $\mathcal{F}_t$, if*

$$E[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = 0, \qquad \forall\, 0 \leq s < t < \infty.$$

Let X_t and Y_t be martingale processes. Show that the process $Z_t = X_t Y_t$ is a martingale if and only if X_t and Y_t are conditionally uncorrelated. Assume that X_t, Y_t and Z_t are integrable.

In the following, if X_t is a stochastic process, the minimum amount of information resulted from knowing the process X_t until time t is denoted by $\mathcal{F}_t = \sigma(X_s; s \leq t)$. In the case of a discrete process, we have $\mathcal{F}_n = \sigma(X_k; k \leq n)$.

Exercise 1.15.11 *Let X_n, $n \geq 0$ be a sequence of integrable independent random variables, with $E[X_n] < \infty$, for all $n \geq 0$. Let $S_0 = X_0$, $S_n = X_0 + \cdots + X_n$. Show the following:*

(*a*) *$S_n - E[S_n]$ is an $\mathcal{F}_n$-martingale.*

(*b*) *If $E[X_n] = 0$ and $E[X_n^2] < \infty$, $\forall n \geq 0$, then $S_n^2 - Var(S_n)$ is an $\mathcal{F}_n$-martingale.*

(*c*) *If $E[X_n] \geq 0$, then S_n is an $\mathcal{F}_n$-submartingale.*

Exercise 1.15.12 *Let X_n, $n \geq 0$ be a sequence of independent, integrable random variables such that $E[X_n] = 1$ for $n \geq 0$. Prove that $P_n = X_0 \cdot X_1 \cdots X_n$ is an $\mathcal{F}_n$-martingale.*

Exercise 1.15.13 (*a*) *Let X be a normally distributed random variable with mean $\mu \neq 0$ and variance σ^2. Prove that there is a unique $\theta \neq 0$ such that $E[e^{\theta X}] = 1$.*

(*b*) *Let $(X_i)_{i \geq 0}$ be a sequence of identically normally distributed random variables with mean $\mu \neq 0$. Consider the sum $S_n = \sum_{j=0}^{n} X_j$. Show that $Z_n = e^{\theta S_n}$ is a martingale, with θ defined in part* (*a*).

In section 9.1 we shall encounter several processes which are martingales.

Chapter 2

Useful Stochastic Processes

This chapter deals with the most common used stochastic processes and their basic properties. The two main basic processes are the Brownian motion and the Poisson process. The other processes described in this chapter are derived from the previous two.

2.1 The Brownian Motion

The observation made first by the botanist Robert Brown in 1827, that small pollen grains suspended in water have a very irregular and unpredictable state of motion, led to the definition of the Brownian motion, which is formalized in the following:

Definition 2.1.1 *A Brownian motion process is a stochastic process B_t, $t \geq 0$, which satisfies*

1. *The process starts at the origin, $B_0 = 0$;*
2. *B_t has stationary, independent increments;*
3. *The process B_t is continuous in t;*
4. *The increments $B_t - B_s$ are normally distributed with mean zero and variance $|t-s|$,*

$$B_t - B_s \sim N(0, |t-s|).$$

The process $X_t = x + B_t$ has all the properties of a Brownian motion that starts at x. Since $B_t - B_s$ is stationary, its distribution function depends only on the time interval $t - s$, i.e.

$$P(B_{t+s} - B_s \leq a) = P(B_t - B_0 \leq a) = P(B_t \leq a).$$

It is worth noting that even if B_t is continuous, it is nowhere differentiable. From condition 4 we get that B_t is normally distributed with mean $E[B_t] = 0$ and $Var[B_t] = t$

$$B_t \sim N(0, t).$$

This implies also that the second moment is $E[B_t^2] = t$. Let $0 < s < t$. Since the increments are independent, we can write

$$E[B_s B_t] = E[(B_s - B_0)(B_t - B_s) + B_s^2] = E[B_s - B_0]E[B_t - B_s] + E[B_s^2] = s.$$

Consequently, B_s and B_t are not independent.

Condition 4 has also a physical explanation. A pollen grain suspended in water is kicked by a very large numbers of water molecules. The influence of each molecule on the grain is independent of the other molecules. These effects are average out into a resultant increment of the grain coordinate. According to the Central Limit Theorem, this increment has to be normal distributed.

Proposition 2.1.2 *A Brownian motion process B_t is a martingale with respect to the information set $\mathcal{F}_t = \sigma(B_s; s \leq t)$.*

Proof: The integrability of B_t follows from Jensen's inequality

$$E[|B_t|]^2 \leq E[B_t^2] = Var(B_t) = |t| < \infty.$$

B_t is obviously $\mathcal{F}_t$-predictable. Let $s < t$ and write $B_t = B_s + (B_t - B_s)$. Then

$$\begin{aligned} E[B_t|\mathcal{F}_s] &= E[B_s + (B_t - B_s)|\mathcal{F}_s] \\ &= E[B_s|\mathcal{F}_s] + E[B_t - B_s|\mathcal{F}_s] \\ &= B_s + E[B_t - B_s] = B_s + E[B_{t-s} - B_0] = B_s, \end{aligned}$$

where we used that B_s is $\mathcal{F}_s$-predictable (from where $E[B_s|\mathcal{F}_s] = B_s$) and that the increment $B_t - B_s$ is independent of previous values of B_t contained in the information set $\mathcal{F}_t = \sigma(B_s; s \leq t)$. ■

A process with similar properties as the Brownian motion was introduced by Wiener.

Definition 2.1.3 *A Wiener process W_t is a process adapted to a filtration $\mathcal{F}_t$ such that*

1. *The process starts at the origin, $W_0 = 0$;*
2. *W_t is an $\mathcal{F}_t$-martingale with $E[W_t^2] < \infty$ for all $t \geq 0$ and*

$$E[(W_t - W_s)^2] = t - s, \qquad s \leq t;$$

3. *The process W_t is continuous in t.*

Since W_t is a martingale, its increments are unpredictable[1] and hence $E[W_t - W_s] = 0$; in particular $E[W_t] = 0$. It is easy to show that

$$Var[W_t - W_s] = |t - s|, \qquad Var[W_t] = t.$$

Exercise 2.1.4 *Show that a Brownian process B_t is a Winer process.*

The only property B_t has and W_t seems not to have is that the increments are normally distributed. However, there is no distinction between these two processes, as the following result states.

Theorem 2.1.5 (Lévy) *A Wiener process is a Brownian motion process.*

In stochastic calculus we often need to use infinitesimal notation and its properties. If dW_t denotes the infinitesimal increment of a Wiener process in the time interval dt, the aforementioned properties become $dW_t \sim N(0, dt)$, $E[dW_t] = 0$, and $E[(dW_t)^2] = dt$.

[1] This follows from $E[W_t - W_s] = E[W_t - W_s|\mathcal{F}_s] = E[W_t|\mathcal{F}_s] - W_s = W_s - W_s = 0$.

Proposition 2.1.6 *If W_t is a Wiener process with respect to the information set $\mathcal{F}_t$, then $Y_t = W_t^2 - t$ is a martingale.*

Proof: Y_t is integrable since

$$E[|Y_t|] \le E[W_t^2 + t] = 2t < \infty, \qquad t > 0.$$

Let $s < t$. Using that the increments $W_t - W_s$ and $(W_t - W_s)^2$ are independent of the information set $\mathcal{F}_s$ and applying Proposition 1.11.4 yields

$$\begin{aligned}
E[W_t^2|\mathcal{F}_s] &= E[(W_s + W_t - W_s)^2|\mathcal{F}_s] \\
&= E[W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2|\mathcal{F}_s] \\
&= E[W_s^2|\mathcal{F}_s] + E[2W_s(W_t - W_s)|\mathcal{F}_s] + E[(W_t - W_s)^2|\mathcal{F}_s] \\
&= W_s^2 + 2W_sE[W_t - W_s|\mathcal{F}_s] + E[(W_t - W_s)^2|\mathcal{F}_s] \\
&= W_s^2 + 2W_sE[W_t - W_s] + E[(W_t - W_s)^2] \\
&= W_s^2 + t - s,
\end{aligned}$$

and hence $E[W_t^2 - t|\mathcal{F}_s] = W_s^2 - s$, for $s < t$. ■

The following result states the memoryless property of Brownian motion[2] W_t.

Proposition 2.1.7 *The conditional distribution of W_{t+s}, given the present W_t and the past W_u, $0 \le u < t$, depends only on the present.*

Proof: Using the independent increment assumption, we have

$$\begin{aligned}
&P(W_{t+s} \le c|W_t = x, W_u, 0 \le u < t) \\
&= P(W_{t+s} - W_t \le c - x|W_t = x, W_u, 0 \le u < t) \\
&= P(W_{t+s} - W_t \le c - x) \\
&= P(W_{t+s} \le c|W_t = x).
\end{aligned}$$

■

Since W_t is normally distributed with mean 0 and variance t, its density function is

$$\phi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Then its distribution function is

$$F_t(x) = P(W_t \le x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{u^2}{2t}}\, du$$

The probability that W_t is between the values a and b is given by

$$P(a \le W_t \le b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{u^2}{2t}}\, du, \qquad a < b.$$

Even if the increments of a Brownian motion are independent, their values are still correlated.

[2]These type of processes are called Marcov processes.

Proposition 2.1.8 *Let $0 \leq s \leq t$. Then*

1. $Cov(W_s, W_t) = s$;
2. $Corr(W_s, W_t) = \sqrt{\dfrac{s}{t}}$.

Proof: 1. Using the properties of covariance

$$\begin{aligned} Cov(W_s, W_t) &= Cov(W_s, W_s + W_t - W_s) \\ &= Cov(W_s, W_s) + Cov(W_s, W_t - W_s) \\ &= Var(W_s) + E[W_s(W_t - W_s)] - E[W_s]E[W_t - W_s] \\ &= s + E[W_s]E[W_t - W_s] \\ &= s, \end{aligned}$$

since $E[W_s] = 0$.

We can also arrive at the same result starting from the formula

$$Cov(W_s, W_t) = E[W_s W_t] - E[W_s]E[W_t] = E[W_s W_t].$$

Using that conditional expectations have the same expectation, factoring the predictable part out, and using that W_t is a martingale, we have

$$\begin{aligned} E[W_s W_t] &= E[E[W_s W_t | \mathcal{F}_s]] = E[W_s E[W_t | \mathcal{F}_s]] \\ &= E[W_s W_s] = E[W_s^2] = s, \end{aligned}$$

so $Cov(W_s, W_t) = s$.

2. The correlation formula yields

$$Corr(W_s, W_t) = \frac{Cov(W_s, W_t)}{\sigma(W_t)\sigma(W_s)} = \frac{s}{\sqrt{s}\sqrt{t}} = \sqrt{\frac{s}{t}}.$$

■

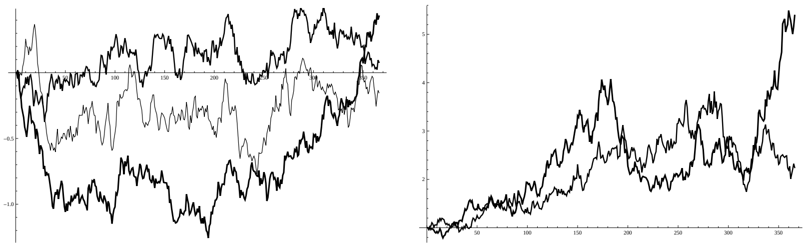
Remark 2.1.9 *Removing the order relation between s and t, the previous relations can also be stated as*

$$\begin{aligned} Cov(W_s, W_t) &= \min\{s, t\}; \\ Corr(W_s, W_t) &= \sqrt{\frac{\min\{s, t\}}{\max\{s, t\}}}. \end{aligned}$$

The following exercises state the translation and the scaling invariance of the Brownian motion.

Exercise 2.1.10 *For any $t_0 \geq 0$, show that the process $X_t = W_{t+t_0} - W_{t_0}$ is a Brownian motion. This can be also stated as saying that the Brownian motion is translation invariant.*

Exercise 2.1.11 *For any $\lambda > 0$, show that the process $X_t = \frac{1}{\sqrt{\lambda}} W_{\lambda t}$ is a Brownian motion. This says that the Brownian motion is invariant by scaling.*



a **b**

Figure 2.1: **a** *Three simulations of the Brownian motion process* W_t; **b** *Two simulations of the geometric Brownian motion process* e^{W_t}.

Exercise 2.1.12 *Let* $0 < s < t < u$. *Show the following multiplicative property*

$$Corr(W_s, W_t)Corr(W_t, W_u) = Corr(W_s, W_u).$$

Exercise 2.1.13 *Find the expectations* $E[W_t^3]$ *and* $E[W_t^4]$.

Exercise 2.1.14 (a) *Use the martingale property of* $W_t^2 - t$ *to find* $E[(W_t^2 - t)(W_s^2 - s)]$;
(b) *Evaluate* $E[W_t^2 W_s^2]$;
(c) *Compute* $Cov(W_t^2, W_s^2)$;
(d) *Find* $Corr(W_t^2, W_s^2)$.

Exercise 2.1.15 *Consider the process* $Y_t = tW_{\frac{1}{t}}$, $t > 0$, *and define* $Y_0 = 0$.
(a) *Find the distribution of* Y_t;
(b) *Find the probability density of* Y_t;
(c) *Find* $Cov(Y_s, Y_t)$;
(d) *Find* $E[Y_t - Y_s]$ *and* $Var(Y_t - Y_s)$ *for* $s < t$.

Exercise 2.1.16 *The process* $X_t = |W_t|$ *is called Brownian motion reflected at the origin. Show that*
(a) $E[|W_t|] = \sqrt{2t/\pi}$;
(b) $Var(|W_t|) = (1 - \frac{2}{\pi})t$.

Exercise 2.1.17 *Let* $0 < s < t$. *Find* $E[W_t^2|\mathcal{F}_s]$.

Exercise 2.1.18 *Let* $0 < s < t$. *Show that*

(a) $E[W_t^3|\mathcal{F}_s] = 3(t-s)W_s + W_s^3$;

(b) $E[W_t^4|\mathcal{F}_s] = 3(t-s)^2 + 6(t-s)W_s^2 + W_s^4$.

Exercise 2.1.19 *Show that the following processes are Brownian motions*

(a) $X_t = W_T - W_{T-t}$, $0 \le t \le T$;

(b) $Y_t = -W_t$, $t \ge 0$.

2.2 Geometric Brownian Motion

The process $X_t = e^{W_t}$, $t \geq 0$ is called *geometric Brownian motion.* A few simulations of this process are contained in Fig.2.1 b. The following result will be useful in the following.

Lemma 2.2.1 *$E[e^{\alpha W_t}] = e^{\alpha^2 t/2}$, for $\alpha \geq 0$.*

Proof: Using the definition of expectation

$$\begin{aligned} E[e^{\alpha W_t}] &= \int e^{\alpha x}\phi_t(x)\,dx = \frac{1}{\sqrt{2\pi t}}\int e^{-\frac{x^2}{2t}+\alpha x}\,dx \\ &= e^{\alpha^2 t/2}, \end{aligned}$$

where we have used the integral formula

$$\int e^{-ax^2+bx}\,dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}, \qquad a > 0$$

with $a = \frac{1}{2t}$ and $b = \alpha$. ∎

Proposition 2.2.2 *The geometric Brownian motion $X_t = e^{W_t}$ is log-normally distributed with mean $e^{t/2}$ and variance $e^{2t} - e^t$.*

Proof: Since W_t is normally distributed, then $X_t = e^{W_t}$ will have a log-normal distribution. Using Lemma 2.2.1 we have

$$\begin{aligned} E[X_t] &= E[e^{W_t}] = e^{t/2} \\ E[X_t^2] &= E[e^{2W_t}] = e^{2t}, \end{aligned}$$

and hence the variance is

$$Var[X_t] = E[X_t^2] - E[X_t]^2 = e^{2t} - (e^{t/2})^2 = e^{2t} - e^t.$$

∎

The distribution function of $X_t = e^{W_t}$ can be obtained by reducing it to the distribution function of a Brownian motion.

$$\begin{aligned} F_{X_t}(x) &= P(X_t \leq x) = P(e^{W_t} \leq x) \\ &= P(W_t \leq \ln x) = F_{W_t}(\ln x) \\ &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\ln x} e^{-\frac{u^2}{2t}}\,du. \end{aligned}$$

The density function of the geometric Brownian motion $X_t = e^{W_t}$ is given by

$$p(x) = \frac{d}{dx}F_{X_t}(x) = \begin{cases} \dfrac{1}{x\sqrt{2\pi t}}e^{-(\ln x)^2/(2t)}, & \text{if } x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Exercise 2.2.3 *Show that*

$$E[e^{W_t - W_s}] = e^{\frac{t-s}{2}}, \qquad s < t.$$

Exercise 2.2.4 *Let* $X_t = e^{W_t}$.

(*a*) *Show that* X_t *is not a martingale.*

(*b*) *Show that* $e^{-\frac{t}{2}} X_t$ *is a martingale.*

(*c*) *Show that for any constant* $c \in \mathbb{R}$, *the process* $Y_t = e^{cW_t - \frac{1}{2}c^2 t}$ *is a martingale.*

Exercise 2.2.5 *If* $X_t = e^{W_t}$, *find* $Cov(X_s, X_t)$

(*a*) *by direct computation;*

(*b*) *by using Exercise* 2.2.4 (*b*).

Exercise 2.2.6 *Show that*

$$E[e^{2W_t^2}] = \begin{cases} (1-4t)^{-1/2}, & 0 \le t < 1/4 \\ \infty, & \textit{otherwise.} \end{cases}$$

2.3 Integrated Brownian Motion

The stochastic process

$$Z_t = \int_0^t W_s \, ds, \qquad t \ge 0$$

is called *integrated Brownian motion*. Obviously, $Z_0 = 0$.

Let $0 = s_0 < s_1 < \cdots < s_k < \cdots s_n = t$, with $s_k = \frac{kt}{n}$. Then Z_t can be written as a limit of Riemann sums

$$Z_t = \lim_{n\to\infty} \sum_{k=1}^n W_{s_k} \Delta s = t \lim_{n\to\infty} \frac{W_{s_1} + \cdots + W_{s_n}}{n},$$

where $\Delta s = s_{k+1} - s_k = \frac{t}{n}$. We are tempted to apply the Central Limit Theorem at this point, but W_{s_k} are not independent, so we first need to transform the sum into a sum of independent normally distributed random variables. A straightforward computation shows that

$$\begin{aligned} & W_{s_1} + \cdots + W_{s_n} \\ = \;& n(W_{s_1} - W_0) + (n-1)(W_{s_2} - W_{s_1}) + \cdots + (W_{s_n} - W_{s_{n-1}}) \\ = \;& X_1 + X_2 + \cdots + X_n. \end{aligned} \tag{2.3.1}$$

Since the increments of a Brownian motion are independent and normally distributed, we have

$$\begin{aligned} & X_1 \sim N\big(0, n^2 \Delta s\big) \\ & X_2 \sim N\big(0, (n-1)^2 \Delta s\big) \\ & X_3 \sim N\big(0, (n-2)^2 \Delta s\big) \\ & \vdots \\ & X_n \sim N\big(0, \Delta s\big). \end{aligned}$$

Recall now the following variant of the Central Limit Theorem:

Theorem 2.3.1 *If X_j are independent random variables normally distributed with mean μ_j and variance σ_j^2, then the sum $X_1+\cdots+X_n$ is also normally distributed with mean $\mu_1+\cdots+\mu_n$ and variance $\sigma_1^2+\cdots+\sigma_n^2$.*

Then

$$X_1+\cdots+X_n \sim N\Big(0,(1+2^2+3^2+\cdots+n^2)\Delta s\Big) = N\Big(0,\frac{n(n+1)(2n+1)}{6}\Delta s\Big),$$

with $\Delta s = \dfrac{t}{n}$. Using (2.3.1) yields

$$t\frac{W_{s_1}+\cdots+W_{s_n}}{n} \sim N\Big(0,\frac{(n+1)(2n+1)}{6n^2}t^3\Big).$$

"Taking the limit" we get

$$Z_t \sim N\Big(0,\frac{t^3}{3}\Big).$$

Proposition 2.3.2 *The integrated Brownian motion Z_t has a normal distribution with mean 0 and variance $t^3/3$.*

Remark 2.3.3 *The aforementioned limit was taken heuristically, without specifying the type of the convergence. In order to make this to work, the following result is usually used:*

If X_n is a sequence of normal random variables that converges in mean square to X, then the limit X is normal distributed, with $E[X_n]\to E[X]$ and $Var(X_n)\to Var(X)$, as $n\to\infty$.

The mean and the variance can also be computed in a direct way as follows. By Fubini's theorem we have

$$\begin{aligned} E[Z_t] &= E[\int_0^t W_s\,ds] = \int_{\mathbb{R}}\int_0^t W_s\,ds\,dP \\ &= \int_0^t\int_{\mathbb{R}} W_s\,dP\,ds = \int_0^t E[W_s]\,ds = 0, \end{aligned}$$

since $E[W_s]=0$. Then the variance is given by

$$\begin{aligned} Var[Z_t] &= E[Z_t^2]-E[Z_t]^2 = E[Z_t^2] \\ &= E[\int_0^t W_u\,du\cdot\int_0^t W_v\,dv] = E[\int_0^t\int_0^t W_uW_v\,dudv] \\ &= \int_0^t\int_0^t E[W_uW_v]\,dudv = \iint_{[0,t]\times[0,t]}\min\{u,v\}\,dudv \\ &= \iint_{D_1}\min\{u,v\}\,dudv + \iint_{D_2}\min\{u,v\}\,dudv, \end{aligned} \tag{2.3.2}$$

where

$$D_1=\{(u,v);u>v,0\le u\le t\},\qquad D_2=\{(u,v);u<v,0\le u\le t\}$$

The first integral can be evaluated using Fubini's theorem

$$\begin{aligned} \iint_{D_1}\min\{u,v\}\,dudv &= \iint_{D_1} v\,dudv \\ &= \int_0^t\Big(\int_0^u v\,dv\Big)du = \int_0^t\frac{u^2}{2}\,du = \frac{t^3}{6}. \end{aligned}$$

Similarly, the latter integral is equal to

$$\iint_{D_2} \min\{u,v\}\, dudv = \frac{t^3}{6}.$$

Substituting in (2.3.2) yields

$$Var[Z_t] = \frac{t^3}{6} + \frac{t^3}{6} = \frac{t^3}{3}.$$

Exercise 2.3.4 *(a) Prove that the moment generating function of Z_t is*

$$m(u) = e^{u^2 t^3/6}.$$

(b) Use the first part to find the mean and variance of Z_t.

Exercise 2.3.5 *Let $s < t$. Show that the covariance of the integrated Brownian motion is given by*

$$Cov\Big(Z_s, Z_t\Big) = s^2\Big(\frac{t}{2} - \frac{s}{6}\Big), \qquad s < t.$$

Exercise 2.3.6 *Show that*

(a) $Cov(Z_t, Z_t - Z_{t-h}) = \frac{1}{2}t^2 h + o(h)$, where $o(h)$ denotes a quantity such that $\lim_{h\to 0} o(h)/h = 0$;

(b) $Cov(Z_t, W_t) = \dfrac{t^2}{2}$.

Exercise 2.3.7 *Show that*

$$E[e^{W_s + W_u}] = e^{\frac{u+s}{2}} e^{\min\{s,u\}}.$$

Exercise 2.3.8 *Consider the process $X_t = \displaystyle\int_0^t e^{W_s}\, ds$.*
(a) Find the mean of X_t;
(b) Find the variance of X_t.

Exercise 2.3.9 *Consider the process $Z_t = \displaystyle\int_0^t W_u\, du$, $t > 0$.*
(a) Show that $E[Z_T|\mathcal{F}_t] = Z_t + W_t(T-t)$, for any $t < T$;
(b) Prove that the process $M_t = Z_t - tW_t$ is an $\mathcal{F}_t$-martingale.

2.4 Exponential Integrated Brownian Motion

If $Z_t = \int_0^t W_s\, ds$ denotes the integrated Brownian motion, the process

$$V_t = e^{Z_t}$$

is called *exponential integrated Brownian motion*. The process starts at $V_0 = e^0 = 1$. Since Z_t is normally distributed, then V_t is log-normally distributed. We compute the mean and the variance in a direct way. Using Exercises 2.2.5 and 2.3.4 we have

$$\begin{aligned}
E[V_t] &= E[e^{Z_t}] = m(1) = e^{\frac{t^3}{6}} \\
E[V_t^2] &= E[e^{2Z_t}] = m(2) = e^{\frac{4t^3}{6}} = e^{\frac{2t^3}{3}} \\
Var(V_t) &= E[V_t^2] - E[V_t]^2 = e^{\frac{2t^3}{3}} - e^{\frac{t^3}{3}} \\
Cov(V_s, V_t) &= e^{\frac{t+3s}{2}}.
\end{aligned}$$

Exercise 2.4.1 *Show that* $E[V_T|\mathcal{F}_t] = V_t e^{(T-t)W_t + \frac{(T-t)^3}{3}}$ *for* $t < T$.

2.5 Brownian Bridge

The process $X_t = W_t - tW_1$ is called the *Brownian bridge* fixed at both 0 and 1. Since we can also write

$$\begin{aligned}
X_t &= W_t - tW_t - tW_1 + tW_t \\
&= (1-t)(W_t - W_0) - t(W_1 - W_t),
\end{aligned}$$

using that the increments $W_t - W_0$ and $W_1 - W_t$ are independent and normally distributed, with

$$W_t - W_0 \sim N(0, t), \qquad W_1 - W_t \sim N(0, 1-t),$$

it follows that X_t is normally distributed with

$$\begin{aligned}
E[X_t] &= (1-t)E[(W_t - W_0)] - tE[(W_1 - W_t)] = 0 \\
Var[X_t] &= (1-t)^2 Var[(W_t - W_0)] + t^2 Var[(W_1 - W_t)] \\
&= (1-t)^2(t-0) + t^2(1-t) \\
&= t(1-t).
\end{aligned}$$

This can be also stated by saying that the Brownian bridge tied at 0 and 1 is a Gaussian process with mean 0 and variance $t(1-t)$, so $X_t \sim N\big(0, t(1-t)\big)$.

Exercise 2.5.1 *Let* $X_t = W_t - tW_1$, $0 \le t \le 1$ *be a Brownian bridge fixed at* 0 *and* 1. *Let* $Y_t = X_t^2$. *Show that* $Y_0 = Y_1 = 0$ *and find* $E[Y_t]$ *and* $Var(Y_t)$.

2.6 Brownian Motion with Drift

The process $Y_t = \mu t + W_t$, $t \ge 0$, is called *Brownian motion with drift.* The process Y_t tends to drift off at a rate μ. It starts at $Y_0 = 0$ and it is a Gaussian process with mean

$$E[Y_t] = \mu t + E[W_t] = \mu t$$

and variance

$$Var[Y_t] = Var[\mu t + W_t] = Var[W_t] = t.$$

Exercise 2.6.1 *Find the distribution and the density functions of the process* Y_t.

2.7 Bessel Process

This section deals with the process satisfied by the Euclidean distance from the origin to a particle following a Brownian motion in $\mathbb{R}^n$. More precisely, if $W_1(t), \cdots, W_n(t)$ are independent Brownian motions, let $W(t) = (W_1(t), \cdots, W_n(t))$ be a Brownian motion in $\mathbb{R}^n$, $n \geq 2$. The process

$$R_t = dist(O, W(t)) = \sqrt{W_1(t)^2 + \cdots + W_n(t)^2}$$

is called *n-dimensional Bessel process*.

The probability density of this process is given by the following result.

Proposition 2.7.1 *The probability density function of R_t, $t > 0$ is given by*

$$p_t(\rho) = \begin{cases} \frac{2}{(2t)^{n/2}\,\Gamma(n/2)}\,\rho^{n-1} e^{-\frac{\rho^2}{2t}}, & \rho \geq 0; \\ \\ 0, & \rho < 0 \end{cases}$$

with

$$\Gamma\Big(\frac{n}{2}\Big) = \begin{cases} (\frac{n}{2} - 1)! & \text{for } n \text{ even;} \\ \\ (\frac{n}{2} - 1)(\frac{n}{2} - 2) \cdots \frac{3}{2}\frac{1}{2}\sqrt{\pi}, & \text{for } n \text{ odd.} \end{cases}$$

Proof: Since the Brownian motions $W_1(t), \ldots, W_n(t)$ are independent, their joint density function is

$$\begin{aligned} f_{W_1 \cdots W_n}(t) &= f_{W_1}(t) \cdots f_{W_n}(t) \\ &= \frac{1}{(2\pi t)^{n/2}}\, e^{-(x_1^2 + \cdots + x_n^2)/(2t)}, \qquad t > 0. \end{aligned}$$

In the next computation we shall use the following formula of integration that follows from the use of polar coordinates

$$\int_{\{|x| \leq \rho\}} f(x)\, dx = \sigma(\mathbb{S}^{n-1}) \int_0^\rho r^{n-1} g(r)\, dr, \tag{2.7.3}$$

where $f(x) = g(|x|)$ is a function on $\mathbb{R}^n$ with spherical symmetry, and where

$$\sigma(\mathbb{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the area of the $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

Let $\rho \geq 0$. The distribution function of R_t is

$$\begin{aligned} F_R(\rho) &= P(R_t \leq \rho) = \int_{\{R_t \leq \rho\}} f_{W_1 \cdots W_n}(t)\, dx_1 \cdots dx_n \\ &= \int_{x_1^2 + \cdots + x_n^2 \leq \rho^2} \frac{1}{(2\pi t)^{n/2}}\, e^{-(x_1^2 + \cdots + x_n^2)/(2t)}\, dx_1 \cdots dx_n \\ &= \int_0^\rho r^{n-1} \left(\int_{\mathbb{S}(0,1)} \frac{1}{(2\pi t)^{n/2}}\, e^{-(x_1^2 + \cdots + x_n^2)/(2t)}\, d\sigma \right) dr \\ &= \frac{\sigma(\mathbb{S}^{n-1})}{(2\pi t)^{n/2}} \int_0^\rho r^{n-1}\, e^{-r^2/(2t)}\, dr. \end{aligned}$$

Differentiating yields

$$\begin{aligned} p_t(\rho) &= \frac{d}{d\rho}F_{R}(\rho) = \frac{\sigma(\mathbb{S}^{n-1})}{(2\pi t)^{n/2}}\,\rho^{n-1}e^{-\frac{\rho^2}{2t}} \\ &= \frac{2}{(2t)^{n/2}\Gamma(n/2)}\,\rho^{n-1}e^{-\frac{\rho^2}{2t}}, \qquad \rho > 0, t > 0. \end{aligned}$$

■

It is worth noting that in the 2-dimensional case the aforementioned density becomes a particular case of a Weibull distribution with parameters $m = 2$ and $\alpha = 2t$, called *Wald's distribution*

$$\boxed{p_t(x) = \frac{1}{t}xe^{-\frac{x^2}{2t}}, \qquad x > 0, t > 0.}$$

Exercise 2.7.2 *Let $P(R_t \leq t)$ be the probability of a 2-dimensional Brownian motion to be inside of the disk $D(0,\rho)$ at time $t > 0$. Show that*

$$\frac{\rho^2}{2t}\Big(1 - \frac{\rho^2}{4t}\Big) < P(R_t \leq t) < \frac{\rho^2}{2t}.$$

Exercise 2.7.3 *Let R_t be a 2-dimensional Bessel process. Show that*

(*a*) $E[R_t] = \sqrt{2\pi t}/2$;

(*b*) $Var(R_t) = 2t\big(1 - \frac{\pi}{4}\big)$.

Exercise 2.7.4 *Let $X_t = \dfrac{R_t}{t}$, $t > 0$, where R_t is a 2-dimensional Bessel process. Show that $X_t \to 0$ as $t \to \infty$ in mean square.*

2.8 The Poisson Process

A *Poisson process* describes the number of occurrences of a certain event before time t, such as: the number of electrons arriving at an anode until time t; the number of cars arriving at a gas station until time t; the number of phone calls received on a certain day until time t; the number of visitors entering a museum on a certain day until time t; the number of earthquakes that occurred in Chile during the time interval $[0, t]$; the number of shocks in the stock market from the beginning of the year until time t; the number of twisters that might hit Alabama from the beginning of the century until time t.

2.8.1 Definition and Properties

The definition of a Poisson process is stated more precisely in the following.

Definition 2.8.1 *A Poisson process is a stochastic process* N_t, $t \geq 0$, *which satisfies*

1. The process starts at the origin, $N_0 = 0$;

2. N_t *has stationary, independent increments;*

3. The process N_t *is right continuous in* t, *with left hand limits;*

4. The increments $N_t - N_s$, *with* $0 < s < t$, *have a Poisson distribution with parameter* $\lambda(t-s)$, *i.e.*

$$P(N_t - N_s = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}.$$

It can be shown that condition *4* in the previous definition can be replaced by the following two conditions:

$$P(N_t - N_s = 1) = \lambda(t-s) + o(t-s) \tag{2.8.4}$$

$$P(N_t - N_s \geq 2) = o(t-s), \tag{2.8.5}$$

where $o(h)$ denotes a quantity such that $\lim_{h\to 0} o(h)/h = 0$. Then the probability that a jump of size 1 occurs in the infinitesimal interval dt is equal to λdt, and the probability that at least 2 events occur in the same small interval is zero. This implies that the random variable dN_t may take only two values, 0 and 1, and hence satisfies

$$P(dN_t = 1) = \lambda\, dt \tag{2.8.6}$$

$$P(dN_t = 0) = 1 - \lambda\, dt. \tag{2.8.7}$$

Exercise 2.8.2 *Show that if condition* 4 *is satisfied, then conditions* $(2.8.4 - 2.8.5)$ *hold.*

Exercise 2.8.3 *Which of the following expressions are* $o(h)$*?*

(a) $f(h) = 3h^2 + h$;

(b) $f(h) = \sqrt{h} + 5$;

(c) $f(h) = h \ln |h|$;

(d) $f(h) = he^h$.

The fact that $N_t - N_s$ is stationary can be stated as

$$P(N_{t+s} - N_s \leq n) = P(N_t - N_0 \leq n) = P(N_t \leq n) = \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

From condition *4* we get the mean and variance of increments

$$E[N_t - N_s] = \lambda(t-s), \qquad Var[N_t - N_s] = \lambda(t-s).$$

In particular, the random variable N_t is Poisson distributed with $E[N_t] = \lambda t$ and $Var[N_t] = \lambda t$. The parameter λ is called the *rate* of the process. This means that the events occur at the constant rate λ.

Since the increments are independent, we have for $0 < s < t$

$$\begin{aligned} E[N_s N_t] &= E[(N_s - N_0)(N_t - N_s) + N_s^2] \\ &= E[N_s - N_0]E[N_t - N_s] + E[N_s^2] \\ &= \lambda s \cdot \lambda(t-s) + (Var[N_s] + E[N_s]^2) \\ &= \lambda^2 st + \lambda s. \end{aligned} \tag{2.8.8}$$

As a consequence we have the following result:

Proposition 2.8.4 *Let $0 \leq s \leq t$. Then*

1. $Cov(N_s, N_t) = \lambda s$;
2. $Corr(N_s, N_t) = \sqrt{\dfrac{s}{t}}$.

Proof: 1. Using (2.8.8) we have

$$\begin{aligned} Cov(N_s, N_t) &= E[N_s N_t] - E[N_s]E[N_t] \\ &= \lambda^2 st + \lambda s - \lambda s \lambda t \\ &= \lambda s. \end{aligned}$$

2. Using the formula for the correlation yields

$$Corr(N_s, N_t) = \frac{Cov(N_s, N_t)}{(Var[N_s]Var[N_t])^{1/2}} = \frac{\lambda s}{(\lambda s \lambda t)^{1/2}} = \sqrt{\frac{s}{t}}.$$

■

It worth noting the similarity with Proposition 2.1.8.

Proposition 2.8.5 *Let N_t be $\mathcal{F}_t$-adapted. Then the process $M_t = N_t - \lambda t$ is an $\mathcal{F}_t$-martingale.*

Proof: Let $s < t$ and write $N_t = N_s + (N_t - N_s)$. Then

$$\begin{aligned} E[N_t|\mathcal{F}_s] &= E[N_s + (N_t - N_s)|\mathcal{F}_s] \\ &= E[N_s|\mathcal{F}_s] + E[N_t - N_s|\mathcal{F}_s] \\ &= N_s + E[N_t - N_s] \\ &= N_s + \lambda(t - s), \end{aligned}$$

where we used that N_s is $\mathcal{F}_s$-predictable (and hence $E[N_s|\mathcal{F}_s] = N_s$) and that the increment $N_t - N_s$ is independent of previous values of N_s and the information set $\mathcal{F}_s$. Subtracting λt yields

$$E[N_t - \lambda t|\mathcal{F}_s] = N_s - \lambda s,$$

or $E[M_t|\mathcal{F}_s] = M_s$. Since it is obvious that M_t is integrable and $\mathcal{F}_t$-adapted, it follows that M_t is a martingale. ■

It is worth noting that the Poisson process N_t is not a martingale. The martingale process $M_t = N_t - \lambda t$ is called the *compensated Poisson process.*

Exercise 2.8.6 *Compute $E[N_t^2|\mathcal{F}_s]$ for $s < t$. Is the process N_t^2 an $\mathcal{F}_s$-martingale?*

Exercise 2.8.7 (*a*) *Show that the moment generating function of the random variable N_t is*

$$m_{N_t}(x) = e^{\lambda t(e^x - 1)}.$$

(*b*) *Deduct the expressions for the first few moments*

$$\begin{aligned} E[N_t] &= \lambda t \\ E[N_t^2] &= \lambda^2 t^2 + \lambda t \\ E[N_t^3] &= \lambda^3 t^3 + 3\lambda^2 t^2 + \lambda t \\ E[N_t^4] &= \lambda^4 t^4 + 6\lambda^3 t^3 + 7\lambda^2 t^2 + \lambda t. \end{aligned}$$

(*c*) *Show that the first few central moments are given by*

$$\begin{aligned} E[N_t - \lambda t] &= 0 \\ E[(N_t - \lambda t)^2] &= \lambda t \\ E[(N_t - \lambda t)^3] &= \lambda t \\ E[(N_t - \lambda t)^4] &= 3\lambda^2 t^2 + \lambda t. \end{aligned}$$

Exercise 2.8.8 *Find the mean and variance of the process* $X_t = e^{N_t}$.

Exercise 2.8.9 (*a*) *Show that the moment generating function of the random variable* M_t *is*

$$m_{M_t}(x) = e^{\lambda t(e^x - x - 1)}.$$

(*b*) *Let* $s < t$. *Verify that*

$$\begin{aligned} E[M_t - M_s] &= 0, \\ E[(M_t - M_s)^2] &= \lambda(t-s), \\ E[(M_t - M_s)^3] &= \lambda(t-s), \\ E[(M_t - M_s)^4] &= \lambda(t-s) + 3\lambda^2(t-s)^2. \end{aligned}$$

Exercise 2.8.10 *Let* $s < t$. *Show that*

$$Var[(M_t - M_s)^2] = \lambda(t-s) + 2\lambda^2(t-s)^2.$$

2.8.2 Interarrival times

For each state of the world, ω, the path $t \to N_t(\omega)$ is a step function that exhibits unit jumps. Each jump in the path corresponds to an occurrence of a new event. Let T_1 be the random variable which describes the time of the 1st jump. Let T_2 be the time between the 1st jump and the second one. In general, denote by T_n the time elapsed between the $(n-1)$th and nth jumps. The random variables T_n are called *interarrival times*.

Proposition 2.8.11 *The random variables* T_n *are independent and exponentially distributed with mean* $E[T_n] = 1/\lambda$.

Proof: We start by noticing that the events $\{T_1 > t\}$ and $\{N_t = 0\}$ are the same, since both describe the situation that no events occurred until after time t. Then

$$P(T_1 > t) = P(N_t = 0) = P(N_t - N_0 = 0) = e^{-\lambda t},$$

and hence the distribution function of T_1 is

$$F_{T_1}(t) = P(T_1 \le t) = 1 - P(T_1 > t) = 1 - e^{-\lambda t}.$$

Differentiating yields the density function

$$f_{T_1}(t) = \frac{d}{dt} F_{T_1}(t) = \lambda e^{-\lambda t}.$$

It follows that T_1 is has an exponential distribution, with $E[T_1] = 1/\lambda$.

In order to show that the random variables T_1 and T_2 are independent, if suffices to show that

$$P(T_2 \le t) = P(T_2 \le t | T_1 = s),$$

i.e. the distribution function of T_2 is independent of the values of T_1. We note first that from the independent increments property

$$\begin{aligned} & P\big(0 \text{ jumps in } (s, s+t], 1 \text{ jump in } (0, s]\big) = P(N_{s+t} - N_s = 0, N_s - N_0 = 1) \\ & = P(N_{s+t} - N_s = 0) P(N_s - N_0 = 1) = P\big(0 \text{ jumps in } (s, s+t]\big) P\big(1 \text{ jump in } (0, s]\big). \end{aligned}$$

Then the conditional distribution of T_2 is

$$\begin{aligned} F(t|s) &= P(T_2 \le t | T_1 = s) = 1 - P(T_2 > t | T_1 = s) \\ &= 1 - \frac{P(T_2 > t, T_1 = s)}{P(T_1 = s)} \\ &= 1 - \frac{P\big(0 \text{ jumps in } (s, s+t], 1 \text{ jump in } (0, s]\big)}{P(T_1 = s)} \\ &= 1 - \frac{P\big(0 \text{ jumps in } (s, s+t]\big) P\big(1 \text{ jump in } (0, s]\big)}{P\big(1 \text{ jump in } (0, s]\big)} = 1 - P\big(0 \text{ jumps in } (s, s+t]\big) \\ &= 1 - P(N_{s+t} - N_s = 0) = 1 - e^{-\lambda t}, \end{aligned}$$

which is independent of s. Then T_2 is independent of T_1 and exponentially distributed. A similar argument for any T_n leads to the desired result.

■

2.8.3 Waiting times

The random variable $S_n = T_1 + T_2 + \cdots + T_n$ is called the *waiting time until the nth jump*. The event $\{S_n \le t\}$ means that there are n jumps that occurred before or at time t, i.e. there are at least n events that happened up to time t; the event is equal to $\{N_t \ge n\}$. Hence the distribution function of S_n is given by

$$F_{S_n}(t) = P(S_n \le t) = P(N_t \ge n) = e^{-\lambda t} \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!}\cdot$$

Differentiating we obtain the density function of the waiting time S_n

$$f_{S_n}(t) = \frac{d}{dt} F_{S_n}(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}\cdot$$

Writing

$$f_{S_n}(t) = \frac{t^{n-1} e^{-\lambda t}}{(1/\lambda)^n \Gamma(n)},$$

it turns out that S_n has a gamma distribution with parameters $\alpha = n$ and $\beta = 1/\lambda$. It follows that

$$E[S_n] = \frac{n}{\lambda}, \quad Var[S_n] = \frac{n}{\lambda^2}\cdot$$

The relation $\lim\limits_{n \to \infty} E[S_n] = \infty$ states that the expectation of the waiting time is unbounded as $n \to \infty$.

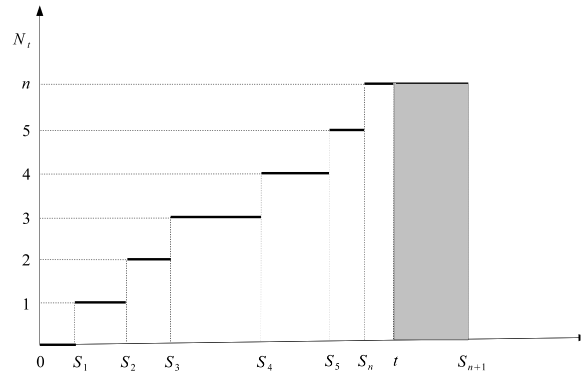


Figure 2.2: *The Poisson process N_t and the waiting times $S_1, S_2, \cdots S_n$. The shaded rectangle has area $n(S_{n+1} - t)$.*

Exercise 2.8.12 *Prove that* $\dfrac{d}{dt} F_{S_n}(t) = \dfrac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\cdot$

Exercise 2.8.13 *Using that the interarrival times $T_1, T_2, \cdots$ are independent and exponentially distributed, compute directly the mean $E[S_n]$ and variance $Var(S_n)$.*

2.8.4 The Integrated Poisson Process

The function $u \to N_u$ is continuous with the exception of a set of countable jumps of size 1. It is known that such functions are Riemann integrable, so it makes sense to define the process

$$U_t = \int_0^t N_u \, du,$$

called the *integrated Poisson process.* The next result provides a relation between the process U_t and the partial sum of the waiting times S_k.

Proposition 2.8.14 *The integrated Poisson process can be expressed as*

$$U_t = tN_t - \sum_{k=1}^{N_t} S_k.$$

Let $N_t = n$. Since N_t is equal to k between the waiting times S_k and S_{k+1}, the process U_t, which is equal to the area of the subgraph of N_u between 0 and t, can be expressed as

$$U_t = \int_0^t N_u \, du = 1 \cdot (S_2 - S_1) + 2 \cdot (S_3 - S_2) + \cdots + n(S_{n+1} - S_n) - n(S_{n+1} - t).$$

Since $S_n < t < S_{n+1}$, the difference of the last two terms represents the area of last the rectangle, which has the length $t - S_n$ and the height n. Using associativity, a computation yields

$$1 \cdot (S_2 - S_1) + 2 \cdot (S_3 - S_2) + \cdots + n(S_{n+1} - S_n) = nS_{n+1} - (S_1 + S_2 + \cdots + S_n).$$

Substituting in the aforementioned relation yields

$$\begin{aligned} U_t &= nS_{n+1} - (S_1 + S_2 + \cdots + S_n) - n(S_{n+1} - t) \\ &= nt - (S_1 + S_2 + \cdots + S_n) \\ &= tN_t - \sum_{k=1}^{N_t} S_k, \end{aligned}$$

where we replaced n by N_t.

The conditional distribution of the waiting times is provided by the following useful result.

Theorem 2.8.15 *Given that $N_t = n$, the waiting times $S_1, S_2, \cdots, S_n$ have the the joint density function given by*

$$f(s_1, s_2, \cdots, s_n) = \frac{n!}{t^n}, \qquad 0 < s_1 \leq s_2 \leq \cdots \leq s_n < t.$$

This is the same as the density of an ordered sample of size n from a uniform distribution on the interval $(0, t)$. A naive explanation of this result is as follows. If we know that there will be exactly n events during the time interval $(0, t)$, since the events can occur at any time, each of them can be considered uniformly distributed, with the density $f(s_k) = 1/t$. Since it makes sense to consider the events independent, taking into consideration all possible $n!$ permutaions, the joint density function becomes $f(s_1, \cdots, s_n) = n! f(s_1) \cdots f(s_n) = \dfrac{n!}{t^n}$.

Exercise 2.8.16 *Find the following means*

(a) $E[U_t]$.

(b) $E\big[\sum_{k=1}^{N_t} S_k\big]$.

Exercise 2.8.17 *Show that* $Var(U_t) = \dfrac{\lambda t^3}{3}$.

Exercise 2.8.18 *Can you apply a similar proof as in Proposition* 2.3.2 *to show that the integrated Poisson process U_t is also a Poisson process?*

Exercise 2.8.19 *Let $Y : \Omega \to \mathbb{N}$ be a discrete random variable. Then for any random variable X we have*

$$E[X] = \sum_{y \geq 0} E[X|Y = y] P(Y = y).$$

Exercise 2.8.20 *Use Exercise 2.8.19 to solve Exercise 2.8.16* (b).

Exercise 2.8.21 (a) *Let T_k be the kth interarrival time. Show that*

$$E[e^{-\sigma T_k}] = \frac{\lambda}{\lambda + \sigma}, \qquad \sigma > 0.$$

(b) *Let $n = N_t$. Show that*

$$U_t = nt - [nT_1 + (n-1)T_2 + \cdots + 2T_{n-1} + T_n].$$

(c) *Find the conditional expectation*

$$E\Big[e^{-\sigma U_t}\Big|N_t = n\Big].$$

(Hint: If know that there are exactly n jumps in the interval $[0, T]$, it makes sense to consider the arrival time of the jumps T_i independent and uniformly distributed on $[0, T]$).
(d) *Find the expectation*

$$E\Big[e^{-\sigma U_t}\Big].$$

2.9 Submartingales

A stochastic process X_t on the probability space $(\Omega, \mathcal{F}, P)$ is called *submartingale* with respect to the filtration $\mathcal{F}_t$ if:

(a) $\int_\Omega |X_t|\, dP < \infty$ (X_t integrable);

(b) X_t is known if $\mathcal{F}_t$ is given (X_t is adaptable to $\mathcal{F}_t$);

(c) $E[X_{t+s}|\mathcal{F}_t] \geq X_t, \forall t, s \geq 0$ (future predictions exceed the present value).

Example 2.9.1 *We shall prove that the process $X_t = \mu t + \sigma W_t$, with $\mu > 0$ is a submartingale.*

The integrability follows from the inequality $|X_t(\omega)| \leq \mu t + |W_t(\omega)|$ and integrability of W_t. The adaptability of X_t is obvious, and the last property follows from the computation:

$$\begin{aligned} E[X_{t+s}|\mathcal{F}_t] &= E[\mu t + \sigma W_{t+s}|\mathcal{F}_t] + \mu s > E[\mu t + \sigma W_{t+s}|\mathcal{F}_t] \\ &= \mu t + \sigma E[W_{t+s}|\mathcal{F}_t] = \mu t + \sigma W_t = X_t, \end{aligned}$$

where we used that W_t is a martingale.

Example 2.9.2 *We shall show that the square of the Brownian motion, W_t^2, is a submartingale.*

Using that $W_t^2 - t$ is a martingale, we have

$$\begin{aligned} E[W_{t+s}^2|\mathcal{F}_t] &= E[W_{t+s}^2 - (t+s)|\mathcal{F}_t] + t + s = W_t^2 - t + t + s \\ &= W_t^2 + s \geq W_t^2. \end{aligned}$$

The following result supplies examples of submatingales starting from martingales or submartingales.

Proposition 2.9.3 (a) *If X_t is a martingale and ϕ a convex function such that $\phi(X_t)$ is integrable, then the process $Y_t = \phi(X_t)$ is a submartingale.*

(b) *If X_t is a submartingale and ϕ an increasing convex function such that $\phi(X_t)$ is integrable, then the process $Y_t = \phi(X_t)$ is a submartingale.*

Proof: (a) Using Jensen's inequality for conditional probabilities, Exercise 1.12.6, we have

$$E[Y_{t+s}|\mathcal{F}_t] = E[\phi(X_{t+s})|\mathcal{F}_t] \geq \phi\Big(E[X_{t+s}|\mathcal{F}_t]\Big) = \phi(X_t) = Y_t.$$

(b) From the submatingale property and monotonicity of ϕ we have

$$\phi\Big(E[X_{t+s}|\mathcal{F}_t]\Big) \geq \phi(X_t).$$

Then apply a similar computation as in part (a). ∎

Corollary 2.9.4 *(a) Let X_t be a martingale. Then X_t^2, $|X_t|$, e^{X_t} are submartingales.*
(b) Let $\mu > 0$. Then $e^{\mu t + \sigma W_t}$ is a submartingale.

Proof: (a) Results from part (a) of Proposition 2.9.3.
(b) It follows from Example 2.9 and part (b) of Proposition 2.9.3. ∎

The following result provides important inequalities involving submartingales.

Proposition 2.9.5 (Doob's Submartingale Inequality) *(a) Let X_t be a non-negative submartingale. Then*

$$P(\sup_{s\leq t} X_s \geq x) \leq \frac{E[X_t]}{x}, \qquad \forall x > 0.$$

(b) If X_t is a right continuous submartingale, then for any $x > 0$

$$P(\sup_{s\leq t} X_t \geq x) \leq \frac{E[X_t^+]}{x},$$

where $X_t^+ = \max\{X_t, 0\}$.

Exercise 2.9.6 *Let $x > 0$. Show the inequalities:*
(a) $P(\sup_{s\leq t} W_s^2 \geq x) \leq \dfrac{t}{x}$.
(b) $P(\sup_{s\leq t} |W_s| \geq x) \leq \dfrac{\sqrt{2t/\pi}}{x}$.

Exercise 2.9.7 *Show that st-$\lim\limits_{t\to\infty} \dfrac{\sup_{s\leq t}|W_s|}{t} = 0$.*

Exercise 2.9.8 *Show that for any martingale X_t we have the inequality*

$$P(\sup_{s\leq t} X_t^2 > x) \leq \frac{E[X_t^2]}{x}, \qquad \forall x > 0.$$

It is worth noting that Doob's inequality implies Markov's inequality. Since $\sup\limits_{s\leq t} X_s \geq X_t$, then $P(X_t \geq x) \leq P(\sup\limits_{s\leq t} X_s \geq x)$. Then Doob's inequality

$$P(\sup_{s\leq t} X_s \geq x) \leq \frac{E[X_t]}{x}$$

implies Markov's inequality (see Theorem 1.12.8)

$$P(X_t \geq x) \leq \frac{E[X_t]}{x}.$$

Exercise 2.9.9 *Let N_t denote the Poisson process and consider the information set $\mathcal{F}_t = \sigma\{N_s; s \leq t\}$.*

(*a*) *Show that N_t is a submartingale;*

(*b*) *Is N_t^2 a submartingale?*

Exercise 2.9.10 *It can be shown that for any $0 < \sigma < \tau$ we have the inequality*

$$E\Big[\sum_{\sigma \leq t \leq \tau} \Big(\frac{N_t}{t} - \lambda\Big)^2\Big] \leq \frac{4\tau\lambda}{\sigma^2}.$$

Using this inequality prove that $\text{ms-}\lim\limits_{t\to\infty} \dfrac{N_t}{t} = \lambda.$

Chapter 3

Properties of Stochastic Processes

3.1 Stopping Times

Consider the probability space $(\Omega, \mathcal{F}, P)$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e.

$$\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}, \qquad \forall t < s.$$

Assume that the decision to stop playing a game before or at time t is determined by the information $\mathcal{F}_t$ available at time t. Then this decision can be modeled by a random variable $\tau : \Omega \to [0, \infty]$ which satisfies

$$\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

This means that given the information set $\mathcal{F}_t$, we know whether the event $\{\omega; \tau(\omega) \leq t\}$ had occurred or not. We note that the possibility $\tau = \infty$ is included, since the decision to continue the game for ever is also a possible event. However, we ask the condition $P(\tau < \infty) = 1$. A random variable τ with the previous properties is called a *stopping time*.
The next example illustrates a few cases when a decision is or is not a stopping time. In order to accomplish this, think of the situation that τ is the time when some random event related to a given stochastic process occurs first.

Example 3.1.1 *Let $\mathcal{F}_t$ be the information available until time t regarding the evolution of a stock. Assume the price of the stock at time $t = 0$ is \$50 per share. The following decisions are stopping times:*

(*a*) *Sell the stock when it reaches for the first time the price of* \$100 *per share;*

(*b*) *Buy the stock when it reaches for the first time the price of* \$10 *per share;*

(*c*) *Sell the stock at the end of the year;*

(*d*) *Sell the stock either when it reaches for the first time* \$80 *or at the end of the year.*

(*e*) *Keep the stock either until the initial investment doubles or until the end of the year;*

The following decisions are not stopping times:

(*f*) *Sell the stock when it reaches the maximum level it will ever be;*

(*g*) *Keep the stock until the initial investment at least doubles.*

Part (f) is not a stopping time because it requires information about the future that is not contained in $\mathcal{F}_t$. For part (g), since the initial stock price is $S_0 = \$50$, the general theory of stock prices state

$$P(S_t \geq 2S_0) = P(S_t \geq 100) < 1,$$

i.e. there is a positive probability that the stock never doubles its value. This contradicts the condition $P(\tau = \infty) = 0$. In part (e) there are two conditions; the latter one has the occurring probability equal to 1.

Exercise 3.1.2 *Show that any positive constant, $\tau = c$, is a stopping time with respect to any filtration.*

Exercise 3.1.3 *Let $\tau(\omega) = \inf\{t > 0; |W_t(\omega)| > K\}$, with $K > 0$ constant. Show that τ is a stopping time with respect to the filtration $\mathcal{F}_t = \sigma(W_s; s \leq t)$.*

The random variable τ is called the *first exit time* of the Brownian motion W_t from the interval $(-K, K)$. In a similar way one can define the *first exit time* of the process X_t from the interval (a, b):

$$\tau(\omega) = \inf\{t > 0; X_t(\omega) \notin (a, b)\} = \inf\{t > 0; X_t(\omega) > b \text{ or } X_t(\omega) < a)\}.$$

Let $X_0 < a$. The *first entry time* of X_t in the interval (a, b) is defined as

$$\tau(\omega) = \inf\{t > 0; X_t(\omega) \in (a, b)\} = \inf\{t > 0; X_t(\omega) > a \text{ or } X_t(\omega) < b)\}.$$

If let $b = \infty$, we obtain the *first hitting time* of the level a

$$\tau(\omega) = \inf\{t > 0; X_t(\omega) > a)\}.$$

We shall deal with hitting times in more detail in section 3.3.

Exercise 3.1.4 *Let X_t be a continuous stochastic process. Prove that the first exit time of X_t from the interval (a, b) is a stopping time.*

We shall present in the following some properties regarding operations with stopping times. Consider the notations $\tau_1 \vee \tau_2 = \max\{\tau_1, \tau_2\}$, $\tau_1 \wedge \tau_2 = \min\{\tau_1, \tau_2\}$, $\bar{\tau}_n = \sup_{n\geq 1} \tau_n$ and $\underline{\tau}_n = \inf_{n\geq 1} \tau_n$.

Proposition 3.1.5 *Let τ_1 and τ_2 be two stopping times with respect to the filtration $\mathcal{F}_t$. Then*

1. $\tau_1 \vee \tau_2$
2. $\tau_1 \wedge \tau_2$
3. $\tau_1 + \tau_2$

are stopping times.

Proof: 1. We have

$$\{\omega; \tau_1 \vee \tau_2 \leq t\} = \{\omega; \tau_1 \leq t\} \cap \{\omega; \tau_2 \leq t\} \in \mathcal{F}_t,$$

since $\{\omega; \tau_1 \leq t\} \in \mathcal{F}_t$ and $\{\omega; \tau_2 \leq t\} \in \mathcal{F}_t$. From the subadditivity of P we get

$$\begin{aligned} P\Big(\{\omega; \tau_1 \vee \tau_2 = \infty\}\Big) &= P\Big(\{\omega; \tau_1 = \infty\} \cup \{\omega; \tau_2 = \infty\}\Big) \\ &\leq P\Big(\{\omega; \tau_1 = \infty\}\Big) + P\Big(\{\omega; \tau_2 = \infty\}\Big). \end{aligned}$$

Since

$$P\Big(\{\omega; \tau_i = \infty\}\Big) = 1 - P\Big(\{\omega; \tau_i < \infty\}\Big) = 0, \qquad i = 1, 2,$$

it follows that $P\Big(\{\omega; \tau_1 \vee \tau_2 = \infty\}\Big) = 0$ and hence $P\Big(\{\omega; \tau_1 \vee \tau_2 < \infty\}\Big) = 1$. Then $\tau_1 \vee \tau_2$ is a stopping time.

2. The event $\{\omega; \tau_1 \wedge \tau_2 \le t\} \in \mathcal{F}_t$ if and only if $\{\omega; \tau_1 \wedge \tau_2 > t\} \in \mathcal{F}_t$.

$$\{\omega; \tau_1 \wedge \tau_2 > t\} = \{\omega; \tau_1 > t\} \cap \{\omega; \tau_2 > t\} \in \mathcal{F}_t,$$

since $\{\omega; \tau_1 > t\} \in \mathcal{F}_t$ and $\{\omega; \tau_2 > t\} \in \mathcal{F}_t$ (the σ-algebra $\mathcal{F}_t$ is closed to complements). The fact that $\tau_1 \wedge \tau_2 < \infty$ almost surely has a similar proof.

3. We note that $\tau_1 + \tau_2 \le t$ if there is a $c \in (0, t)$ such that

$$\tau_1 \le c, \qquad \tau_2 \le t - c.$$

Using that the rational numbers are dense in $\mathbb{R}$, we can write

$$\{\omega; \tau_1 + \tau_2 \le t\} = \bigcup_{0<c<t, c\in\mathbb{Q}} \Big(\{\omega; \tau_1 \le c\} \cap \{\omega; \tau_2 \le t - c\}\Big) \in \mathcal{F}_t,$$

since

$$\{\omega; \tau_1 \le c\} \in \mathcal{F}_c \subset \mathcal{F}_t, \qquad \{\omega; \tau_2 \le t - c\} \in \mathcal{F}_{t-c} \subset \mathcal{F}_t.$$

Writing

$$\{\omega; \tau_1 + \tau_2 = \infty\} = \{\omega; \tau_1 = \infty\} \cup \{\omega; \tau_2 = \infty\}$$

yields

$$P\Big(\{\omega; \tau_1 + \tau_2 = \infty\}\Big) \le P\Big(\{\omega; \tau_1 = \infty\}\Big) + P\Big(\{\omega; \tau_2 = \infty\}\Big) = 0,$$

Hence $P\Big(\{\omega; \tau_1 + \tau_2 < \infty\}\Big) = 1$. It follows that $\tau_1 + \tau_2$ is a stopping time. ■

A filtration $\mathcal{F}_t$ is called *right-continuous* if $\mathcal{F}_t = \bigcap_{n=1}^{\infty} \mathcal{F}_{t+\frac{1}{n}}$, for $t > 0$. This means that the information available at time t is a good approximation for any future infinitesimal information $\mathcal{F}_{t+\epsilon}$; or, equivalently, nothing more can be learned by peeking infinitesimally far into the future.

Exercise 3.1.6 *(a) Let $\mathcal{F}_t = \sigma\{W_s; s \le t\}$, where W_t is a Brownian motion. Show that $\mathcal{F}_t$ is right-continuous.*

(b) Let $\mathcal{N}_t = \sigma\{N_s; s \le t\}$, where N_t is a Poisson motion. Is $\mathcal{N}_t$ right-continuous?

Proposition 3.1.7 *Let $\mathcal{F}_t$ be right-continuous and $(\tau_n)_{n\ge1}$ be a sequence of bounded stopping times.*

(a) Then $\sup_n \tau$ and $\inf \tau_n$ are stopping times.

(b) If the sequence $(\tau_n)_{n\ge1}$ converges to τ, $\tau \ne 0$, then τ is a stopping time.

Proof: (a) The fact that $\bar{\tau}_n$ is a stopping time follows from

$$\{\omega; \bar{\tau}_n \le t\} \subset \bigcap_{n\ge 1} \{\omega; \tau_n \le t\} \in \mathcal{F}_t,$$

and from the boundedness, which implies $P(\tau_n < \infty) = 1$.

In order to show that $\underline{\tau}_n$ is a stopping time we shall proceed as in the following. Using that $\mathcal{F}_t$ is right-continuous and closed to complements, it suffices to show that $\{\omega; \underline{\tau}_n \ge t\} \in \mathcal{F}_t$. This follows from

$$\{\omega; \underline{\tau}_n \ge t\} = \bigcap_{n\ge 1} \{\omega; \tau_n > t\} \in \mathcal{F}_t.$$

(b) Let $\lim\limits_{n\to\infty} \tau_n = \tau$. Then there is an increasing (or decreasing) subsequence τ_{n_k} of stopping times that tends to τ, so $\sup \tau_{n_k} = \tau$ (or $\inf \tau_{n_k} = \tau$). Since τ_{n_k} are stopping times, by part (a), it follows that τ is a stopping time. ■

The condition that τ_n is bounded is significant, since if take $\tau_n = n$ as stopping times, then $\sup_n \tau_n = \infty$ with probability 1, which does not satisfy the stopping time definition.

Exercise 3.1.8 *Let τ be a stopping time.*

(a) *Let $c \ge 1$ be a constant. Show that $c\tau$ is a stopping time.*

(b) *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous, increasing function satisfying $f(t) \ge t$. Prove that $f(\tau)$ is a stopping time.*

(c) *Show that e^τ is a stopping time.*

Exercise 3.1.9 *Let τ be a stopping time and $c > 0$ a constant. Prove that $\tau + c$ is a stopping time.*

Exercise 3.1.10 *Let a be a constant and define $\tau = \inf\{t \ge 0; W_t = a\}$. Is τ a stopping time?*

Exercise 3.1.11 *Let τ be a stopping time. Consider the following sequence $\tau_n = (m+1)2^{-n}$ if $m2^{-n} \le \tau < (m+1)2^{-n}$ (stop at the first time of the form $k2^{-n}$ after τ). Prove that τ_n is a stopping time.*

3.2 Stopping Theorem for Martingales

The next result states that in a fair game, the expected final fortune of a gambler, who is using a stopping time to quit the game, is the same as the expected initial fortune. From the financially point of view, the theorem says that if you buy an asset at some initial time and adopt a strategy of deciding when to sell it, then the expected price at the selling time is the initial price; so one cannot make money by buying and selling an asset whose price is a martingale. Fortunately, the price of a stock is not a martingale, and people can still expect to make money buying and selling stocks.

If $(M_t)_{t\ge 0}$ is an $\mathcal{F}_t$-martingale, then taking the expectation in

$$E[M_t|\mathcal{F}_s] = M_s, \qquad \forall s < t$$

and using Example 1.11.2 yields

$$E[M_t] = E[M_s], \qquad \forall s < t.$$

In particular, $E[M_t] = E[M_0]$, for any $t > 0$. The next result states necessary conditions under which this identity holds if t is replaced by any stopping time τ.

Theorem 3.2.1 (Optional Stopping Theorem) *Let $(M_t)_{t\geq 0}$ be a right continuous $\mathcal{F}_t$-martingale and τ be a stopping time with respect to $\mathcal{F}_t$. If either one of the following conditions holds:*

1. *τ is bounded, i.e. $\exists N < \infty$ such that $\tau \leq N$;*
2. *$\exists c > 0$ such that $E[|M_t|] \leq c$, $\forall t > 0$,*

then $E[M_\tau] = E[M_0]$.

Proof: We shall sketch the proof for the case 1 only. Taking the expectation in relation

$$M_\tau = M_{\tau\wedge t} + (M_\tau - M_t)\mathbf{1}_{\{\tau>t\}},$$

see Exercise 3.2.3, yields

$$E[M_\tau] = E[M_{\tau\wedge t}] + E[M_\tau\mathbf{1}_{\{\tau>t\}}] - E[M_t\mathbf{1}_{\{\tau>t\}}].$$

Since $M_{\tau\wedge t}$ is a martingale, see Exercise 3.2.4 (b), then $E[M_{\tau\wedge t}] = E[M_0]$. The previous relation becomes

$$E[M_\tau] = E[M_0] + E[M_\tau\mathbf{1}_{\{\tau>t\}}] - E[M_t\mathbf{1}_{\{\tau>t\}}], \quad \forall t > 0.$$

Taking the limit yields

$$E[M_\tau] = E[M_0] + \lim_{t\to\infty} E[M_\tau\mathbf{1}_{\{\tau>t\}}] - \lim_{t\to\infty} E[M_t\mathbf{1}_{\{\tau>t\}}]. \tag{3.2.1}$$

We shall show that both limits are equal to zero.

Since $|M_\tau\mathbf{1}_{\{\tau>t\}}| \leq |M_\tau|$, $\forall t > 0$, and M_τ is integrable, see Exercise 3.2.4 (a), by the dominated convergence theorem we have

$$\lim_{t\to\infty} E[M_\tau\mathbf{1}_{\{\tau>t\}}] = \lim_{t\to\infty}\int_\Omega M_\tau\mathbf{1}_{\{\tau>t\}}\,dP = \int_\Omega \lim_{t\to\infty} M_\tau\mathbf{1}_{\{\tau>t\}}\,dP = 0.$$

For the second limit

$$\lim_{t\to\infty} E[M_t\mathbf{1}_{\{\tau>t\}}] = \lim_{t\to\infty}\int_\Omega M_t\mathbf{1}_{\{\tau>t\}}\,dP = 0,$$

since for $t > N$ the integrand vanishes. Hence relation (3.2.1) yields $E[M_\tau] = E[M_0]$. ■

It is worth noting that the previous theorem is a special case of the more general Optional Stopping Theorem of Doob:

Theorem 3.2.2 *Let M_t be a right continuous martingale and σ, τ be two bounded stopping times, with $\sigma \leq \tau$. Then M_σ, M_τ are integrable and*

$$E[M_\tau|\mathcal{F}_\sigma] = M_\sigma \quad a.s.$$

Exercise 3.2.3 *Show that*

$$M_\tau = M_{\tau\wedge t} + (M_\tau - M_t)\mathbf{1}_{\{\tau>t\}},$$

where

$$\mathbf{1}_{\{\tau>t\}}(\omega) = \begin{cases} 1, & \tau(\omega) > t; \\ 0, & \tau(\omega) \leq t \end{cases}$$

is the indicator function of the set $\{\tau > t\}$.

Exercise 3.2.4 *Let M_t be a right continuous martingale and τ be a stopping time. Show that*

(a) *M_τ is integrable;*

(b) *$M_{\tau\wedge t}$ is a martingale.*

Exercise 3.2.5 *Show that if let $\sigma = 0$ in Theorem 3.2.2 yields Theorem 3.2.1.*

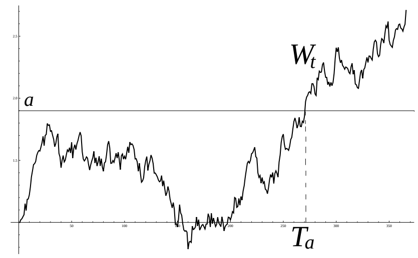


Figure 3.1: *The first hitting time T_a given by $W_{T_a} = a$.*

3.3 The First Passage of Time

The first passage of time is a particular type of hitting time, which is useful in finance when studying barrier options and lookback options. For instance, knock-in options enter into existence when the stock price hits for the first time a certain barrier before option maturity. A lookback option is priced using the maximum value of the stock until the present time. The stock price is not a Brownian motion, but it depends on one. Hence the need for studying the hitting time for the Brownian motion.

The first result deals with the first hitting time for a Brownian motion to reach the barrier $a \in \mathbb{R}$, see Fig.3.1.

Lemma 3.3.1 *Let T_a be the first time the Brownian motion W_t hits a. Then the distribution function of T_a is given by*

$$\boxed{P(T_a \le t) = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy.}$$

Proof: If A and B are two events, then

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap \overline{B}) \\ &= P(A|B)P(B) + P(A|\overline{B})P(\overline{B}). \end{aligned} \tag{3.3.2}$$

Let $a > 0$. Using formula (3.3.2) for $A = \{\omega; W_t(\omega) \ge a\}$ and $B = \{\omega; T_a(\omega) \le t\}$ yields

$$\begin{aligned} P(W_t \ge a) &= P(W_t \ge a | T_a \le t)P(T_a \le t) \\ &\quad + P(W_t \ge a | T_a > t)P(T_a > t) \end{aligned} \tag{3.3.3}$$

If $T_a > t$, the Brownian motion did not reach the barrier a yet, so we must have $W_t < a$. Therefore

$$P(W_t \ge a | T_a > t) = 0.$$

If $T_a \le t$, then $W_{T_a} = a$. Since the Brownian motion is a Markov process, it starts fresh at T_a. Due to symmetry of the density function of a normal variable, W_t has equal chances to go up or go down after the time interval $t - T_a$. It follows that

$$P(W_t \ge a | T_a \le t) = \frac{1}{2}.$$

Substituting into (3.3.3) yields

$$\begin{aligned} P(T_a \le t) &= 2P(W_t \ge a) \\ &= \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/(2t)}\, dx = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2}\, dy. \end{aligned}$$

If $a < 0$, symmetry implies that the distribution of T_a is the same as that of T_{-a}, so we get

$$P(T_a \le t) = P(T_{-a} \le t) = \frac{2}{\sqrt{2\pi}} \int_{-a/\sqrt{t}}^\infty e^{-y^2/2}\, dy.$$

■

Remark 3.3.2 *The previous proof is based on a more general principle called the Reflection Principle: If τ is a stopping time for the Brownian motion W_t, then the Brownian motion reflected at τ is also a Brownian motion.*

Theorem 3.3.3 *Let $a \in \mathbb{R}$ be fixed. Then the Brownian motion hits a (in a finite amount of time) with probability 1.*

Proof: The probability that W_t hits a (in a finite amount of time) is

$$\begin{aligned} P(T_a < \infty) &= \lim_{t\to\infty} P(T_a \le t) = \lim_{t\to\infty} \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^\infty e^{-y^2/2}\, dy \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2}\, dy = 1, \end{aligned}$$

where we used the well known integral

$$\int_0^\infty e^{-y^2/2}\, dy = \frac{1}{2} \int_{-\infty}^\infty e^{-y^2/2}\, dy = \frac{1}{2}\sqrt{2\pi}.$$

■

The previous result stated that the Brownian motion hits the barrier a almost surely. The next result shows that the expected time to hit the barrier is infinite.

Proposition 3.3.4 *The random variable T_a has a Pearson 5 distribution given by*

$$\boxed{p(t) = \frac{|a|}{\sqrt{2\pi}} e^{-\frac{a^2}{2t}} t^{-\frac{3}{2}}, \qquad t > 0.}$$

It has the mean $E[T_a] = \infty$ and the mode $\dfrac{a^2}{3}$.

Proof: Differentiating in the formula of distribution function[1]

$$F_{T_a}(t) = P(T_a \le t) = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2}\, dy$$

[1] One may use Leibnitz's formula $\dfrac{d}{dt}\displaystyle\int_{\varphi(t)}^{\psi(t)} f(u)\, du = f(\psi(t))\psi'(t) - f(\varphi(t))\varphi'(t)$.

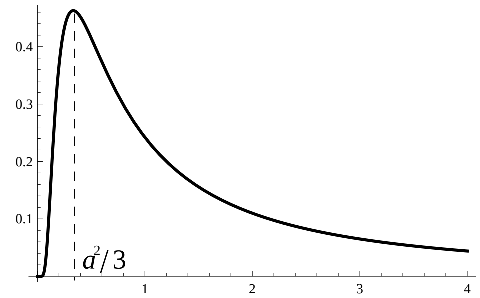


Figure 3.2: *The distribution of the first hitting time T_a.*

yields the following probability density function

$$p(t) = \frac{dF_{T_a}(t)}{dt} = \frac{a}{\sqrt{2\pi}} e^{-\frac{a^2}{2t}} t^{-\frac{3}{2}}, \qquad t > 0.$$

This is a Pearson 5 distribution with parameters $\alpha = 1/2$ and $\beta = a^2/2$. The expectation is

$$E[T_a] \;=\; \int_0^\infty tp(t)\,dt = \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{-\frac{a^2}{2t}}\,dt.$$

Using the inequality $e^{-\frac{a^2}{2t}} > 1 - \dfrac{a^2}{2t}$, $t > 0$, we have the estimation

$$E[T_a] \;>\; \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t}}\,dt - \frac{a^3}{2\sqrt{2\pi}} \int_0^\infty \frac{1}{t^{3/2}}\,dt = \infty,$$

since $\int_0^\infty \frac{1}{\sqrt{t}}\,dt$ is divergent and $\int_0^\infty \frac{1}{t^{3/2}}\,dt$ is convergent.

The mode of T_a is given by

$$\frac{\beta}{\alpha+1} = \frac{a^2}{2(\frac{1}{2}+1)} = \frac{a^2}{3}.$$

■

Remark 3.3.5 *The distribution has a peak at $a^2/3$. Then if we need to pick a small time interval $[t-dt, t+dt]$ in which the probability that the Brownian motion hits the barrier a is maximum, we need to choose $t = a^2/3$.*

Remark 3.3.6 *The expected waiting time for W_t to reach the barrier a is infinite. However, the expected waiting time for the Brownian motion W_t to hit either a or $-a$ is finite, see Exercise 3.3.9.*

Corollary 3.3.7 *A Brownian motion process returns to the origin in a finite amount time with probability 1.*

Proof: Choose $a = 0$ and apply Theorem 3.3.3. ■

Exercise 3.3.8 *Try to apply the proof of Lemma 3.3.1 for the following stochastic processes*

(a) $X_t = \mu t + \sigma W_t$, *with* $\mu, \sigma > 0$ *constants;*

(b) $X_t = \displaystyle\int_0^t W_s\,ds$.

Where is the difficulty?

Exercise 3.3.9 *Let* $a > 0$ *and consider the hitting time*

$$\tau_a = \inf\{t > 0; W_t = a \text{ or } W_t = -a\} = \inf\{t > 0; |W_t| = a\}.$$

Prove that $E[\tau_a] = a^2$.

Exercise 3.3.10 *(a) Show that the distribution function of the process*

$$X_t = \max_{s\in[0,t]} W_s$$

is given by

$$P(X_t \le a) = \frac{2}{\sqrt{2\pi}} \int_0^{a/\sqrt{t}} e^{-y^2/2}\,dy.$$

(b) Show that $E[X_t] = 0$ *and* $Var(X_t) = 2\sqrt{t}$.

Exercise 3.3.11 *(a) Find the distribution of* $Y_t = |W_t|$, $t \ge 0$;

(b) Show that $E\big[\max\limits_{0\le t\le T} |W_t|\big] = \sqrt{\dfrac{\pi T}{2}}$.

The fact that a Brownian motion returns to the origin or hits a barrier almost surely is a property characteristic to the first dimension only. The next result states that in larger dimensions this is no longer possible.

Theorem 3.3.12 *Let* $(a, b) \in \mathbb{R}^2$. *The 2-dimensional Brownian motion* $W(t) = \big(W_1(t), W_2(t)\big)$ *(with* $W_1(t)$ *and* $W_2(t)$ *independent) hits the point* (a, b) *with probability zero. The same result is valid for any* n*-dimensional Brownian motion, with* $n \ge 2$.

However, if the point (a, b) is replaced by the disk $D_\epsilon(\mathbf{x}_0) = \{x \in \mathbb{R}^2; |\mathbf{x} - \mathbf{x}_0| \le \epsilon\}$, there is a difference in the behavior of the Brownian motion from $n = 2$ to $n > 2$.

Theorem 3.3.13 *The 2-dimensional Brownian motion* $W(t) = \big(W_1(t), W_2(t)\big)$ *hits the disk* $D_\epsilon(\mathbf{x}_0)$ *with probability one.*

Theorem 3.3.14 *Let* $n > 2$. *The* n*-dimensional Brownian motion* $W(t) = \big(W_1(t), \cdots W_n(t)\big)$ *hits the ball* $D_\epsilon(\mathbf{x}_0)$ *with probability*

$$P = \Big(\frac{|\mathbf{x}_0|}{\epsilon}\Big)^{2-n} < 1.$$

The previous results can be stated by saying that that Brownian motion is transient in $\mathbb{R}^n$, for $n > 2$. If $n = 2$ the previous probability equals 1. We shall come back with proofs to the aforementioned results in a later chapter.

Remark 3.3.15 *If life spreads according to a Brownian motion, the aforementioned results explain why life is more extensive on earth rather than in space. The probability for a form of life to reach a planet of radius* R *situated at distance* d *is* $\frac{R}{d}$. *Since* d *is large the probability is very small, unlike in the plane, where the probability is always 1.*

Exercise 3.3.16 *Is the one-dimensional Brownian motion transient or recurrent in* $\mathbb{R}$*?*

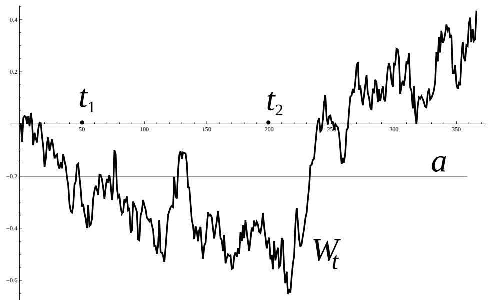


Figure 3.3: *The event $A(a; t_1, t_2)$ in the Law of Arc-sine.*

3.4 The Arc-sine Laws

In this section we present a few results which provide certain probabilities related with the behavior of a Brownian motion in terms the arc-sine of a quotient of two time instances. These results are generally known under the name of *law of arc-sines*.

The following result will be used in the proof of the first law of Arc-sine.

Proposition 3.4.1 *(a) If $X : \Omega \to \mathbb{N}$ is a discrete random variable, then for any subset $A \subset \Omega$, we have*

$$P(A) = \sum_{x \in \mathbb{N}} P(A|X = x)P(X = x).$$

(b) If $X : \Omega \to \mathbb{R}$ is a continuous random variable, then

$$P(A) = \int P(A|X = x)dP = \int P(A|X = x)f_X(x)\,dx.$$

Proof: (a) The sets $X^{-1}(x) = \{X = x\} = \{\omega; X(\omega) = x\}$ form a partition of the sample space Ω, i.e.:

(i) $\Omega = \bigcup_x X^{-1}(x)$;

(ii) $X^{-1}(x) \cap X^{-1}(y) = \emptyset$ for $x \neq y$.

Then $A = \bigcup_x \Big(A \cap X^{-1}(x)\Big) = \bigcup_x \Big(A \cap \{X = x\}\Big)$, and hence

$$\begin{aligned} P(A) &= \sum_x P\Big(A \cap \{X = x\}\Big) \\ &= \sum_x \frac{P(A \cap \{X = x\})}{P(\{X = x\})} P(\{X = x\}) \\ &= \sum_x P(A|X = x)P(X = x). \end{aligned}$$

(b) In the case when X is continuous, the sum is replaced by an integral and the probability $P(\{X = x\})$ by $f_X(x)dx$, where f_X is the density function of X. ∎

The *zero set* of a Brownian motion W_t is defined by $\{t \geq 0; W_t = 0\}$. Since W_t is continuous, the zero set is closed with no isolated points almost surely. The next result deals with the probability that the zero set does not intersect the interval (t_1, t_2).

Theorem 3.4.2 (The law of Arc-sine) *The probability that a Brownian motion W_t does not have any zeros in the interval (t_1, t_2) is equal to*

$$\boxed{P(W_t \neq 0, t_1 \leq t \leq t_2) = \frac{2}{\pi} \arcsin \sqrt{\frac{t_1}{t_2}}.}$$

Proof: Let $A(a; t_1, t_2)$ denote the event that the Brownian motion W_t takes on the value a between t_1 and t_2. In particular, $A(0; t_1, t_2)$ denotes the event that W_t has (at least) a zero between t_1 and t_2. Substituting $A = A(0; t_1, t_2)$ and $X = W_{t_1}$ into the formula provided by Proposition 3.4.1

$$P(A) = \int P(A|X = x) f_X(x)\, dx$$

yields

$$\begin{aligned} P\big(A(0; t_1, t_2)\big) &= \int P\big(A(0; t_1, t_2)|W_{t_1} = x\big) f_{W_{t_1}}(x)\, dx \\ &= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\infty} P\big(A(0; t_1, t_2)|W_{t_1} = x\big) e^{-\frac{x^2}{2t_1}}\, dx \end{aligned} \qquad (3.4.4)$$

Using the properties of W_t with respect to time translation and symmetry we have

$$\begin{aligned} P\big(A(0; t_1, t_2)|W_{t_1} = x\big) &= P\big(A(0; 0, t_2 - t_1)|W_0 = x\big) \\ &= P\big(A(-x; 0, t_2 - t_1)|W_0 = 0\big) \\ &= P\big(A(|x|; 0, t_2 - t_1)|W_0 = 0\big) \\ &= P\big(A(|x|; 0, t_2 - t_1)\big) \\ &= P\big(T_{|x|} \leq t_2 - t_1\big), \end{aligned}$$

the last identity stating that W_t hits $|x|$ before $t_2 - t_1$. Using Lemma 3.3.1 yields

$$P\big(A(0; t_1, t_2)|W_{t_1} = x\big) = \frac{2}{\sqrt{2\pi(t_2 - t_1)}} \int_{|x|}^{\infty} e^{-\frac{y^2}{2(t_2 - t_1)}}\, dy.$$

Substituting into (3.4.4) we obtain

$$\begin{aligned} P\big(A(0; t_1, t_2)\big) &= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\infty} \Big(\frac{2}{\sqrt{2\pi(t_2 - t_1)}} \int_{|x|}^{\infty} e^{-\frac{y^2}{2(t_2 - t_1)}}\, dy \Big) e^{-\frac{x^2}{2t_1}}\, dx \\ &= \frac{1}{\pi\sqrt{t_1(t_2 - t_1)}} \int_0^{\infty} \int_{|x|}^{\infty} e^{-\frac{y^2}{2(t_2 - t_1)} - \frac{x^2}{2t_1}}\, dy dx. \end{aligned}$$

The above integral can be evaluated to get (see Exercise 3.4.3)

$$P\big(A(0; t_1, t_2)\big) = 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{t_1}{t_2}}.$$

Using $P(W_t \neq 0, t_1 \leq t \leq t_2) = 1 - P\big(A(0; t_1, t_2)\big)$ we obtain the desired result. ■

Exercise 3.4.3 *Use polar coordinates to show*

$$\frac{1}{\pi\sqrt{t_1(t_2-t_1)}}\int_0^{\infty}\int_{|x|}^{\infty} e^{-\frac{y^2}{2(t_2-t_1)}-\frac{x^2}{2t_1}}\,dydx = 1-\frac{2}{\pi}\arcsin\sqrt{\frac{t_1}{t_2}}.$$

Exercise 3.4.4 *Find the probability that a 2-dimensional Brownian motion $W(t) = \big(W_1(t), W_2(t)\big)$ stays in the same quadrant for the time interval $t \in (t_1, t_2)$.*

Exercise 3.4.5 *Find the probability that a Brownian motion W_t does not take the value a in the interval (t_1, t_2).*

Exercise 3.4.6 *Let $a \neq b$. Find the probability that a Brownian motion W_t does not take any of the values $\{a, b\}$ in the interval (t_1, t_2). Formulate and prove a generalization.*

We provide below without proof a few similar results dealing with arc-sine probabilities. The first result deals with the amount of time spent by a Brownian motion on the positive half-axis.

Theorem 3.4.7 (Arc-sine Law of Lévy) *Let $L_t^+ = \int_0^t sgn^+ W_s\,ds$ be the amount of time a Brownian motion W_t is positive during the time interval $[0, t]$. Then*

$$P(L_t^+ \le \tau) = \frac{2}{\pi}\arcsin\sqrt{\frac{\tau}{t}}.$$

The next result deals with the Arc-sine law for the last exit time of a Brownian motion from 0.

Theorem 3.4.8 (Arc-sine Law of exit from 0) *Let $\gamma_t = \sup\{0 \le s \le t; W_s = 0\}$. Then*

$$P(\gamma_t \le \tau) = \frac{2}{\pi}\arcsin\sqrt{\frac{\tau}{t}}, \quad 0 \le \tau \le t.$$

The Arc-sine law for the time the Brownian motion attains its maximum on the interval $[0, t]$ is given by the next result.

Theorem 3.4.9 (Arc-sine Law of maximum) *Let $M_t = \max\limits_{0 \le s \le t} W_s$ and define*

$$\theta_t = \sup\{0 \le s \le t; W_s = M_t\}.$$

Then

$$P(\theta_t \le s) = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}}, \qquad 0 \le s \le t, t > 0.$$

3.5 More on Hitting Times

In this section we shall deal with results regarding hitting times of Brownian motion with drift. They will be useful in the sequel when pricing barrier options.

Theorem 3.5.1 *Let $X_t = \mu t + W_t$ denote a Brownian motion with nonzero drift rate μ, and consider $\alpha, \beta > 0$. Then*

$$\boxed{P(X_t \textit{ goes up to } \alpha \textit{ before down to } -\beta) = \frac{e^{2\mu\beta}-1}{e^{2\mu\beta}-e^{-2\mu\alpha}}.}$$

Proof: Let $T = \inf\{t > 0; X_t = \alpha \text{ or } X_t = -\beta\}$ be the first exit time of X_t from the interval $(-\beta, \alpha)$, which is a stopping time, see Exercise 3.1.4. The exponential process

$$M_t = e^{cW_t - \frac{c^2}{2}t}, \qquad t \geq 0$$

is a martingale, see Exercise 2.2.4(c). Then $E[M_t] = E[M_0] = 1$. By the Optional Stopping Theorem (see Theorem 3.2.1), we get $E[M_T] = 1$. This can be written as

$$1 \quad = \quad E[e^{cW_T - \frac{1}{2}c^2 T}] = E[e^{cX_T - (c\mu + \frac{1}{2}c^2)T}]. \tag{3.5.5}$$

Choosing $c = -2\mu$ yields $E[e^{-2\mu X_T}] = 1$. Since the random variable X_T takes only the values α and $-\beta$, if let $p_\alpha = P(X_T = \alpha)$, the previous relation becomes

$$e^{-2\mu\alpha} p_\alpha + e^{2\mu\beta}(1 - p_\alpha) = 1.$$

Solving for p_α yields

$$p_\alpha = \frac{e^{2\mu\beta} - 1}{e^{2\mu\beta} - e^{-2\mu\alpha}}. \tag{3.5.6}$$

Noting that

$$p_\alpha = P(X_t \text{ goes up to } \alpha \text{ before down to } -\beta)$$

leads to the desired answer. ■

It is worth noting how the previous formula changes in the case when the drift rate is zero, i.e. when $\mu = 0$, and $X_t = W_t$. The previous probability is computed by taking the limit $\mu \to 0$ and using L'Hospital's rule

$$\lim_{\mu \to 0} \frac{e^{2\mu\beta} - 1}{e^{2\mu\beta} - e^{-2\mu\alpha}} = \lim_{\mu \to 0} \frac{2\beta e^{2\mu\beta}}{2\beta e^{2\mu\beta} + 2\alpha e^{-2\mu\alpha}} = \frac{\beta}{\alpha + \beta}.$$

Hence

$$\boxed{P(W_t \text{ goes up to } \alpha \text{ before down to } -\beta) = \frac{\beta}{\alpha + \beta}.}$$

Taking the limit $\beta \to \infty$ we recover the following result

$$\boxed{P(W_t \text{ hits } \alpha) = 1.}$$

If $\alpha = \beta$ we obtain

$$\boxed{P(W_t \text{ goes up to } \alpha \text{ before down to } -\alpha) = \frac{1}{2},}$$

which shows that the Brownian motion is equally likely to go up or down an amount α in a given time interval.

If T_α and T_β denote the times when the process X_t reaches α and β, respectively, then the aforementioned probabilities can be written using inequalities. For instance the first identity becomes

$$P(T_\alpha \leq T_{-\beta}) = \frac{e^{2\mu\beta} - 1}{e^{2\mu\beta} - e^{-2\mu\alpha}}.$$

Exercise 3.5.2 *Let $X_t = \mu t + W_t$ denote a Brownian motion with nonzero drift rate μ, and consider $\alpha > 0$.*
(a) If $\mu > 0$ show that

$$P(X_t \text{ goes up to } \alpha) = 1.$$

(b) If $\mu < 0$ show that

$$P(X_t \text{ goes up to } \alpha) = e^{2\mu\alpha} < 1.$$

Formula (a) can be written equivalently as

$$P(\sup_{t\geq 0}(W_t + \mu t) \geq \alpha) = 1, \qquad \mu \geq 0,$$

while formula (b) becomes

$$P(\sup_{t\geq 0}(W_t + \mu t) \geq \alpha) = e^{2\mu\alpha}, \qquad \mu < 0,$$

or

$$P(\sup_{t\geq 0}(W_t - \gamma t) \geq \alpha) = e^{-2\gamma\alpha}, \qquad \gamma > 0,$$

which is known as one of the Doob's inequalities. This can be also described in terms of stopping times as follows. Define the stopping time $\tau_\alpha = \inf\{t > 0; W_t - \gamma t \geq \alpha\}$. Using

$$P(\tau_\alpha < \infty) = P\big(\sup_{t\geq 0}(W_t - \gamma t) \geq \alpha\big)$$

yields the identities

$$\begin{aligned} P(\tau_\alpha < \infty) &= e^{-2\alpha\gamma}, \qquad \gamma > 0, \\ P(\tau_\alpha < \infty) &= 1, \qquad \gamma \leq 0. \end{aligned}$$

Exercise 3.5.3 *Let $X_t = \mu t + W_t$ denote a Brownian motion with nonzero drift rate μ, and consider $\beta > 0$. Show that the probability that X_t never hits $-\beta$ is given by*

$$\begin{cases} 1 - e^{-2\mu\beta}, & \text{if } \mu > 0 \\ 0, & \text{if } \mu < 0. \end{cases}$$

Recall that T is the first time when the process X_t hits α or $-\beta$.

Exercise 3.5.4 (a) *Show that*

$$E[X_T] = \frac{\alpha e^{2\mu\beta} + \beta e^{-2\mu\alpha} - \alpha - \beta}{e^{2\mu\beta} - e^{-2\mu\alpha}}.$$

(b) Find $E[X_T^2]$;
(c) Compute $Var(X_T)$.

The next result deals with the time one has to wait (in expectation) for the process $X_t = \mu t + W_t$ to reach either α or $-\beta$.

Proposition 3.5.5 *The expected value of T is*

$$E[T] = \frac{\alpha e^{2\mu\beta} + \beta e^{-2\mu\alpha} - \alpha - \beta}{\mu(e^{2\mu\beta} - e^{-2\mu\alpha})}.$$

Proof: Using that W_t is a martingale, with $E[W_t] = E[W_0] = 0$, applying the Optional Stopping Theorem, Theorem 3.2.1, yields

$$0 = E[W_T] = E[X_T - \mu T] = E[X_T] - \mu E[T].$$

Then by Exercise 3.5.4(a) we get

$$E[T] = \frac{E[X_T]}{\mu} = \frac{\alpha e^{2\mu\beta} + b e^{-2\mu\alpha} - \alpha - \beta}{\mu(e^{2\mu\beta} - e^{-2\mu\alpha})}.$$

■

Exercise 3.5.6 *Take the limit $\mu \to 0$ in the formula provided by Proposition 3.5.5 to find the expected time for a Brownian motion to hit either α or $-\beta$.*

Exercise 3.5.7 *Find $E[T^2]$ and $Var(T)$.*

Exercise 3.5.8 (Wald's identities) *Let T be a finite stopping time for the Brownian motion W_t. Show that*
(a) $E[W_T] = 0$;
(b) $E[W_T^2] = E[T]$.

The previous techniques can be also applied to right continuous martingales. Let $a > 0$ and consider the hitting time of the Poisson process of the barrier a

$$\tau = \inf\{t > 0; N_t \geq a\}.$$

Proposition 3.5.9 *The expected waiting time for N_t to reach the barrier a is $E[\tau] = \frac{a}{\lambda}$.*

Proof: Since $M_t = N_t - \lambda t$ is a right continuous martingale, by the Optional Stopping Theorem $E[M_\tau] = E[M_0] = 0$. Then $E[N_\tau - \lambda\tau] = 0$ and hence $E[\tau] = \frac{1}{\lambda}E[N_\tau] = \frac{a}{\lambda}$. ■

3.6 The Inverse Laplace Transform Method

In this section we shall use the Optional Stopping Theorem in conjunction with the inverse Laplace transform to obtain the probability density for hitting times.

A. The case of standard Brownian motion Let $x > 0$. The first hitting time $\tau = T_x = \inf\{t > 0; W_t = x\}$ is a stopping time. Since $M_t = e^{cW_t - \frac{1}{2}c^2 t}$, $t \geq 0$, is a martingale, with $E[M_t] = E[M_0] = 1$, by the Optional Stopping Theorem, see Theorem 3.2.1, we have

$$E[M_\tau] = 1.$$

This can be written equivalently as $E[e^{cW_\tau} e^{-\frac{1}{2}c^2\tau}] = 1$. Using $W_\tau = x$, we get

$$E[e^{-\frac{1}{2}c^2\tau}] = e^{-cx}.$$

It is worth noting that $c > 0$. This is implied from the fact that $e^{-\frac{1}{2}c^2\tau} < 1$ and $\tau, x > 0$. Substituting $s = \frac{1}{2}c^2$, the previous relation becomes

$$E[e^{-s\tau}] = e^{-\sqrt{2s}x}. \tag{3.6.7}$$

This relation has a couple of useful applications.

Proposition 3.6.1 *The moments of the first hitting time are all infinite $E[\tau^n] = \infty$, $n \geq 1$.*

Proof: The nth moment of τ can be obtained by differentiating and taking $s = 0$

$$\frac{d^n}{ds^n} E[e^{-s\tau}]_{\big|_{s=0}} = E[(-\tau)^n e^{-s\tau}]_{\big|_{s=0}} = (-1)^n E[\tau^n].$$

Using (3.6.7) yields

$$E[\tau^n] = (-1)^n \frac{d^n}{ds^n} e^{-\sqrt{2s}x}_{\big|_{s=0}}.$$

Since by induction we have

$$\frac{d^n}{ds^n} e^{-\sqrt{2s}x} = (-1)^n e^{-\sqrt{2s}x} \sum_{k=0}^{n-1} \frac{M_k}{2^{r_k/2}} \frac{x^{n-k}}{s^{(n+k)/2}},$$

with M_k, r_k positive integers, it easily follows that $E[\tau^n] = \infty$.

For instance, in the case $n = 1$ we have

$$E[\tau] = -\frac{d}{ds} e^{-\sqrt{2s}x}_{\big|_{s=0}} = \lim_{s\to 0^+} e^{-\sqrt{2s}x} \frac{x}{2\sqrt{2sx}} = +\infty.$$

■

Another application involves the inverse Laplace transform to get the probability density. This way we can retrieve the result of Proposition 3.3.4.

Proposition 3.6.2 *The probability density of the hitting time τ is given by*

$$p(t) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}, \qquad t > 0. \tag{3.6.8}$$

Proof: Let $x > 0$. The expectation

$$E[e^{-s\tau}] = \int_0^\infty e^{-s\tau} p(\tau)\, d\tau = \mathcal{L}\{p(\tau)\}(s)$$

is the Laplace transform of $p(\tau)$. Applying the inverse Laplace transform yields

$$\begin{aligned} p(\tau) &= \mathcal{L}^{-1}\{E[e^{-s\tau}]\}(\tau) = \mathcal{L}^{-1}\{e^{-\sqrt{2s}x}\}(\tau) \\ &= \frac{x}{\sqrt{2\pi\tau^3}} e^{-\frac{x^2}{2\tau}}, \qquad \tau > 0. \end{aligned}$$

In the case $x < 0$ we obtain

$$p(\tau) = \frac{-x}{\sqrt{2\pi\tau^3}} e^{-\frac{x^2}{2\tau}}, \qquad \tau > 0,$$

which leads to (3.6.8). ■

Another application of formula (3.6.7) is the following inequality.

Exercise 3.6.3 *Find the following inverse Laplace transforms:*

(*a*) $\mathcal{L}^{-1}\{e^{-c\sqrt{s}}\}(\tau)$, *$c > 0$ constant;*

(*b*) $\mathcal{L}^{-1}\{e^{-\sqrt{2s}x}\}(\tau)$, *$x > 0$.*

Proposition 3.6.4 (Chernoff bound) *Let τ denote the first hitting time when the Brownian motion W_t hits the barrier x, $x>0$. Then*

$$P(\tau \le \lambda) \le e^{-\frac{x^2}{2\lambda}}, \qquad \forall \lambda > 0.$$

Proof: Let $s=-t$ in the part 2 of Theorem 1.12.10 and use (3.6.7) to get

$$P(\tau \le \lambda) \le \frac{E[e^{tX}]}{e^{\lambda t}} = \frac{E[e^{-sX}]}{e^{-\lambda s}} = e^{\lambda s - x\sqrt{2s}}, \qquad \forall s > 0.$$

Then $P(\tau \le \lambda) \le e^{\min_{s>0} f(s)}$, where $f(s) = \lambda s - x\sqrt{2s}$. Since $f'(s) = \lambda - \frac{x}{\sqrt{2s}}$, then $f(s)$ reaches its minimum at the critical point $s_0 = \frac{x^2}{2\lambda^2}$. The minimum value is

$$\min_{s>0} f(s) = f(s_0) = -\frac{x^2}{2\lambda}.$$

Substituting in the previous inequality leads to the required result. ∎

B. The case of Brownian motion with drift

Consider the Brownian motion with drift $X_t = \mu t + \sigma W_t$, with $\mu, \sigma > 0$. Let

$$\tau = \inf\{t > 0; X_t = x\}$$

denote the first hitting time of the barrier x, with $x > 0$. We shall compute the distribution of the random variable τ and its first two moments.

Applying the Optional Stopping Theorem (Theorem 3.2.1) to the martingale $M_t = e^{cW_t - \frac{1}{2}c^2 t}$ yields

$$E[M_\tau] = E[M_0] = 1.$$

Using that $W_\tau = \frac{1}{\sigma}(X_\tau - \mu\tau)$ and $X_\tau = x$, the previous relation becomes

$$E[e^{-(\frac{c\mu}{\sigma} + \frac{1}{2}c^2)\tau}] = e^{-\frac{c}{\sigma}x}. \tag{3.6.9}$$

Substituting $s = \dfrac{c\mu}{\sigma} + \dfrac{1}{2}c^2$ and completing to a square yields

$$2s + \frac{\mu^2}{\sigma^2} = \Big(c + \frac{\mu}{\sigma}\Big)^2.$$

Solving for c we get the solutions

$$c = -\frac{\mu}{\sigma} + \sqrt{2s + \frac{\mu^2}{\sigma^2}}, \qquad c = -\frac{\mu}{\sigma} - \sqrt{2s + \frac{\mu^2}{\sigma^2}}.$$

Assume $c < 0$. Then substituting the second solution into (3.6.9) yields

$$E[e^{-s\tau}] = e^{\frac{1}{\sigma^2}(\mu + \sqrt{2s\sigma^2 + \mu^2})x}.$$

This relation is contradictory since $e^{-s\tau} < 1$ while $e^{\frac{1}{\sigma^2}(\mu + \sqrt{2s\sigma + \mu^2})x} > 1$, where we used that $s, x, \tau > 0$. Hence it follows that $c > 0$. Substituting the first solution into (3.6.9) leads to

$$E[e^{-s\tau}] = e^{\frac{1}{\sigma^2}(\mu - \sqrt{2s\sigma^2 + \mu^2})x}.$$

We arrived at the following result:

Proposition 3.6.5 *Let τ be the time the process $X_t = \mu t + \sigma W_t$ hits x for the first time. Then for $s > 0, x > 0$ we have*

$$E[e^{-s\tau}] = e^{\frac{1}{\sigma^2}(\mu - \sqrt{2s\sigma^2+\mu^2})x}. \tag{3.6.10}$$

Proposition 3.6.6 *Let τ be the time the process $X_t = \mu t + \sigma W_t$ hits x, with $x > 0$ and $\mu > 0$.*
(a) *Then the density function of τ is given by*

$$p(\tau) = \frac{x}{\sigma\sqrt{2\pi}\tau^{3/2}} e^{-\frac{(x-\mu\tau)^2}{2\tau\sigma^2}}, \qquad \tau > 0. \tag{3.6.11}$$

(b) *The mean and variance of τ are*

$$E[\tau] = \frac{x}{\mu}, \qquad Var(\tau) = \frac{x\sigma^2}{\mu^3}.$$

Proof: (a) Let $p(\tau)$ be the density function of τ. Since

$$E[e^{-s\tau}] = \int_0^\infty e^{-s\tau} p(\tau)\, d\tau = \mathcal{L}\{p(\tau)\}(s)$$

is the Laplace transform of $p(\tau)$, applying the inverse Laplace transform yields

$$\begin{aligned} p(\tau) &= \mathcal{L}^{-1}\{E[e^{-s\tau}]\} = \mathcal{L}^{-1}\{e^{\frac{1}{\sigma^2}(\mu - \sqrt{2s\sigma^2+\mu^2})x}\} \\ &= \frac{x}{\sigma\sqrt{2\pi}\tau^{3/2}} e^{-\frac{(x-\mu\tau)^2}{2\tau\sigma^2}}, \qquad \tau > 0. \end{aligned}$$

(b) The moments are obtained by differentiating the moment generating function and taking the value at $s = 0$

$$\begin{aligned} E[\tau] &= -\frac{d}{ds}E[e^{-s\tau}]\Big|_{s=0} = -\frac{d}{ds}e^{\frac{1}{\sigma^2}(\mu - \sqrt{2s\sigma^2+\mu^2})x}\Big|_{s=0} \\ &= \frac{x}{\sqrt{\mu^2+2s\mu}} e^{\frac{1}{\sigma^2}(\mu - \sqrt{2s\sigma^2+\mu^2})x}\Big|_{s=0} \\ &= \frac{x}{\mu}. \end{aligned}$$

$$\begin{aligned} E[\tau^2] &= (-1)^2\frac{d^2}{ds^2}E[e^{-s\tau}]\Big|_{s=0} = \frac{d^2}{ds^2}e^{\frac{1}{\sigma^2}(\mu - \sqrt{2s\sigma^2+\mu^2})x}\Big|_{s=0} \\ &= \frac{x(\sigma^2 + x\sqrt{\mu^2+2s\sigma^2})}{(\mu^2+2s\sigma^2)^{3/2}} e^{\frac{1}{\sigma^2}(\mu - \sqrt{2s\sigma^2+\mu^2})x}\Big|_{s=0} \\ &= \frac{x\sigma^2}{\mu^3} + \frac{x^2}{\mu^2}. \end{aligned}$$

Hence

$$Var(\tau) = E[\tau^2] - E[\tau]^2 = \frac{x\sigma^2}{\mu^3}.$$

It is worth noting that we can arrive at the formula $E[\tau] = \frac{x}{\mu}$ in the following heuristic way. Taking the expectation in the equation $\mu\tau + \sigma W_\tau = x$ yields $\mu E[\tau] = x$, where we used that $E[W_\tau] = 0$ for any finite stopping time τ (see Exercise 3.5.8 (a)). Solving for $E[\tau]$ yields the aforementioned formula.

∎

Exercise 3.6.7 *Find the mode of distribution* (3.6.11).

Exercise 3.6.8 *Let* $X_t = 2t + 3W_t$ *and* $Y_t = 6t + W_t$.

(a) *Show that the expected times for* X_t *and* Y_t *to reach any barrier* $x > 0$ *are the same.*

(b) *If* X_t *and* Y_t *model the prices of two stocks, which one would you like to own?*

Exercise 3.6.9 *Does* $4t + 2W_t$ *hit* 9 *faster (in expectation) than* $5t + 3W_t$ *hits* 14*?*

Exercise 3.6.10 *Let* τ *be the first time the Brownian motion with drift* $X_t = \mu t + W_t$ *hits* x, *where* $\mu, x > 0$. *Prove the inequality*

$$P(\tau \le \lambda) \le e^{-\frac{x^2+\lambda^2\mu^2}{2\lambda}+\mu x}, \quad \forall \lambda > 0.$$

C. The double barrier case

In the following we shall consider the case of double barrier. Consider the Brownian motion with drift $X_t = \mu t + W_t$, $\mu > 0$. Let $\alpha, \beta > 0$ and define the stopping time

$$T = \inf\{t > 0; X_t = \alpha \text{ or } X_t = -\beta\}.$$

Relation (3.5.5) states

$$E[e^{cX_T} e^{-(c\mu+\frac{1}{2}c^2)T}] = 1.$$

Since the random variables T and X_T are independent (why?), we have

$$E[e^{cX_T}]E[e^{-(c\mu+\frac{1}{2}c^2)T}] = 1.$$

Using $E[e^{cX_T}] = e^{c\alpha}p_\alpha + e^{-c\beta}(1 - p_\alpha)$, with p_α given by (3.5.6), then

$$E[e^{-(c\mu+\frac{1}{2}c^2)T}] = \frac{1}{e^{c\alpha}p_\alpha + e^{-c\beta}(1 - p_\alpha)}.$$

If substitute $s = c\mu + \frac{1}{2}c^2$, then

$$E[e^{-sT}] = \frac{1}{e^{(-\mu+\sqrt{2s+\mu^2})\alpha}p_\alpha + e^{-(-\mu+\sqrt{2s+\mu^2})\beta}(1 - p_\alpha)}. \tag{3.6.12}$$

The probability density of the stopping time T is obtained by taking the inverse Laplace transform of the right side expression

$$p(T) = \mathcal{L}^{-1}\Big\{\frac{1}{e^{(-\mu+\sqrt{2s+\mu^2})\alpha}p_\alpha + e^{-(-\mu+\sqrt{2s+\mu^2})\beta}(1 - p_\alpha)}\Big\}(\tau),$$

an expression which is not feasible for having closed form solution. However, expression (3.6.12) would be useful for computing the price for double barrier derivatives.

Exercise 3.6.11 *Use formula* (3.6.12) *to find the expectation* $E[T]$.

Exercise 3.6.12 *Denote by $M_t = N_t - \lambda t$ the compensated Poisson process and let $c > 0$ be a constant.*
(a) Show that

$$X_t = e^{cM_t - \lambda t(e^c - c - 1)}$$

is an $\mathcal{F}_t$-martingale, with $\mathcal{F}_t = \sigma(N_u; u \le t)$.
(b) Let $a > 0$ and $T = \inf\{t > 0; M_t > a\}$ be the first hitting time of the level a. Use the Optional Stopping Theorem to show that

$$E[e^{-\lambda s T}] = e^{-\varphi(s)a}, \qquad s > 0,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is the inverse function of $f(x) = e^x - x - 1$.
(c) Show that $E[T] = \infty$.
(d) Can you use the inverse Laplace transform to find the probability density function of T?

3.7 Limits of Stochastic Processes

Let $(X_t)_{t\ge 0}$ be a stochastic process. We can make sense of the limit expression $X = \lim\limits_{t\to\infty} X_t$, in a similar way as we did in section 1.13 for sequences of random variables. We shall rewrite the definitions for the continuous case.

Almost Certain Limit

The process X_t converges *almost certainly* to X, if for all states of the world ω, except a set of probability zero, we have

$$\lim_{t\to\infty} X_t(\omega) = X(\omega).$$

We shall write $\text{ac-}\lim\limits_{t\to\infty} X_t = X$. It is also sometimes called *strong convergence*.

Mean Square Limit

We say that the process X_t converges to X in the *mean square* if

$$\lim_{t\to\infty} E[(X_t - X)^2] = 0.$$

In this case we write $\text{ms-}\lim\limits_{t\to\infty} X_t = X$.

Limit in Probability or Stochastic Limit

The stochastic process X_t converges in *stochastic limit* to X if

$$\lim_{t\to\infty} P\big(\omega; |X_t(\omega) - X(\omega)| > \epsilon\big) = 0.$$

This limit is abbreviated by $\text{st-}\lim\limits_{t\to\infty} X_t = X$.

It is worth noting that, like in the case of sequences of random variables, both almost certain convergence and convergence in mean square imply the stochastic convergence, which implies the limit in distribution.

Limit in Distribution

We say that X_t converges *in distribution* to X if for any continuous bounded function $\varphi(x)$ we have

$$\lim_{t\to\infty} \varphi(X_t) = \varphi(X).$$

It is worth noting that the stochastic convergence implies the convergence in distribution.

3.8 Convergence Theorems

The following property is a reformulation of Exercise 1.13.1 in the continuous setup.

Proposition 3.8.1 *Consider a stochastic process X_t such that $E[X_t] \to k$, a constant, and $Var(X_t) \to 0$ as $t \to \infty$. Then ms-$\lim\limits_{t\to\infty} X_t = k$.*

Next we shall provide a few applications.

Application 3.8.2 *If $\alpha > 1/2$, then*

$$ms\text{-}\lim_{t\to\infty} \frac{W_t}{t^\alpha} = 0.$$

Proof: Let $X_t = \dfrac{W_t}{t^\alpha}$. Then $E[X_t] = \dfrac{E[W_t]}{t^\alpha} = 0$, and $Var[X_t] = \dfrac{1}{t^{2\alpha}} Var[W_t] = \dfrac{t}{t^{2\alpha}} = \dfrac{1}{t^{2\alpha-1}}$, for any $t > 0$. Since $\dfrac{1}{t^{2\alpha-1}} \to 0$ as $t \to \infty$, applying Proposition 3.8.1 yields ms-$\lim\limits_{t\to\infty} \dfrac{W_t}{t^\alpha} = 0$. ■

Corollary 3.8.3 *ms*-$\lim\limits_{t\to\infty} \dfrac{W_t}{t} = 0$.

Application 3.8.4 *Let $Z_t = \int_0^t W_s\, ds$. If $\beta > 3/2$, then*

$$ms\text{-}\lim_{t\to\infty} \frac{Z_t}{t^\beta} = 0.$$

Proof: Let $X_t = \dfrac{Z_t}{t^\beta}$. Then $E[X_t] = \dfrac{E[Z_t]}{t^\beta} = 0$, and $Var[X_t] = \dfrac{1}{t^{2\beta}} Var[Z_t] = \dfrac{t^3}{3t^{2\beta}} = \dfrac{1}{3t^{2\beta-3}}$, for any $t > 0$. Since $\dfrac{1}{3t^{2\beta-3}} \to 0$ as $t \to \infty$, applying Proposition 3.8.1 leads to the desired result. ■

Application 3.8.5 *For any $p > 0$, $c \geq 1$ we have*

$$ms\text{-}\lim_{t\to\infty} \frac{e^{W_t - ct}}{t^p} = 0.$$

Proof: Consider the process $X_t = \dfrac{e^{W_t - ct}}{t^p} = \dfrac{e^{W_t}}{t^p e^{ct}}$. Since

$$\begin{aligned} E[X_t] &= \frac{E[e^{W_t}]}{t^p e^{ct}} = \frac{e^{t/2}}{t^p e^{ct}} = \frac{1}{e^{(c-\frac{1}{2})t}} \frac{1}{t^p} \to 0, \quad \text{as } t \to \infty \\ Var[X_t] &= \frac{Var[e^{W_t}]}{t^{2p} e^{2ct}} = \frac{e^{2t} - e^t}{t^{2p} e^{2ct}} = \frac{1}{t^{2p}} \Big(\frac{1}{e^{2t(c-1)}} - \frac{1}{e^{t(2c-1)}} \Big) \to 0, \end{aligned}$$

as $t \to \infty$, Proposition 3.8.1 leads to the desired result. ■

Application 3.8.6 *Show that*

$$ms\text{-}\lim_{t\to\infty} \frac{\max\limits_{0\le s\le t} W_s}{t} = 0.$$

Proof: Let $X_t = \dfrac{\max\limits_{0\le s\le t} W_s}{t}$. Since by Exercise 3.3.10

$$\begin{aligned} E[\max_{0\le s\le t} W_s] &= 0 \\ Var(\max_{0\le s\le t} W_s) &= 2\sqrt{t}, \end{aligned}$$

then

$$\begin{aligned} E[X_t] &= 0 \\ Var[X_t] &= \frac{2\sqrt{t}}{t^2} \to 0, \qquad t \to \infty. \end{aligned}$$

Apply Proposition 3.8.1 to get the desired result. ∎

Remark 3.8.7 *One of the strongest result regarding limits of Brownian motions is called the law of iterated logarithms and was first proved by Lamperti:*

$$\lim_{t\to\infty} \sup \frac{W_t}{\sqrt{2t\ln(\ln t)}} = 1,$$

almost certainly.

Exercise 3.8.8 *Use the law of iterated logarithms to show*

$$\lim_{t\to 0^+} \frac{W_t}{\sqrt{2t\ln\ln(1/t)}} = 1.$$

Application 3.8.9 *We shall show that* $ac\text{-}\lim\limits_{t\to\infty} \dfrac{W_t}{t} = 0$.

From the law of iterated logarithms $\dfrac{W_t}{\sqrt{2t\ln(\ln t)}} < 1$ for t large. Then

$$\frac{W_t}{t} = \frac{W_t}{\sqrt{2t\ln(\ln t)}} \frac{\sqrt{2\ln(\ln t)}}{\sqrt{t}} < \frac{\sqrt{2\ln(\ln t)}}{\sqrt{t}}.$$

Let $\epsilon_t = \dfrac{\sqrt{2\ln(\ln t)}}{\sqrt{t}}$. Then $\dfrac{W_t}{t} < \epsilon_t$ for t large. As an application of the l'Hospital rule, it is not hard to see that ϵ_t satisfies the following limits

$$\begin{aligned} \epsilon_t \to 0, &\qquad t \to \infty \\ \epsilon_t\sqrt{t} \to \infty, &\qquad t \to \infty. \end{aligned}$$

In order to show that ac-$\lim\limits_{t\to\infty} \frac{W_t}{t} = 0$, it suffices to prove

$$P\Big(\omega; \Big|\frac{W_t(\omega)}{t}\Big| < \epsilon_t\Big) \to 1, \qquad t \to \infty. \tag{3.8.13}$$

We have

$$\begin{aligned} P\Big(\omega; \Big|\frac{W_t(\omega)}{t}\Big| < \epsilon_t\Big) &= P\Big(\omega; -t\epsilon_t < W_t(\omega) < t\epsilon_t\Big) = \int_{-t\epsilon_t}^{t\epsilon_t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}}\, du \\ &= \int_{-\epsilon_t\sqrt{t}}^{\epsilon_t\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\, dv \to \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\, dv = 1, \qquad t \to \infty, \end{aligned}$$

where we used $\epsilon_t\sqrt{t} \to \infty$, as $t \to \infty$, which proves (3.8.13).

Proposition 3.8.10 *Let X_t be a stochastic process. Then*

$$ms\text{-}\lim_{t\to\infty} X_t = 0 \Longleftrightarrow ms\text{-}\lim_{t\to\infty} X_t^2 = 0.$$

Proof: Left as an exercise. ∎

Exercise 3.8.11 *Let X_t be a stochastic process. Show that*

$$ms\text{-}\lim_{t\to\infty} X_t = 0 \Longleftrightarrow ms\text{-}\lim_{t\to\infty} |X_t| = 0.$$

Another convergence result can be obtained if we consider the continuous analog of Example 1.13.6:

Proposition 3.8.12 *Let X_t be a stochastic process such that there is a $p > 0$ such that $E[|X_t|^p] \to 0$ as $t \to \infty$. Then st-$\lim\limits_{t\to\infty} X_t = 0$.*

Application 3.8.13 *We shall show that for any $\alpha > 1/2$*

$$st\text{-}\lim_{t\to\infty} \frac{W_t}{t^\alpha} = 0.$$

Proof: Consider the process $X_t = \frac{W_t}{t^\alpha}$. By Proposition 3.8.10 it suffices to show ms-$\lim\limits_{t\to\infty} X_t^2 = 0$. Since the mean square convergence implies the stochastic convergence, we get st-$\lim\limits_{t\to\infty} X_t^2 = 0$. Since

$$E[|X_t|^2] = E[X_t^2] = E\Big[\frac{W_t^2}{t^{2\alpha}}\Big] = \frac{E[W_t^2]}{t^{2\alpha}} = \frac{t}{t^{2\alpha}} = \frac{1}{t^{2\alpha-1}} \to 0, \quad t \to \infty,$$

then Proposition 3.8.12 yields st-$\lim\limits_{t\to\infty} X_t = 0$. ∎

The following result can be regarded as the L'Hospital's rule for sequences:

Lemma 3.8.14 (Cesaró-Stoltz) *Let x_n and y_n be two sequences of real numbers, $n \geq 1$. If the limit $\lim\limits_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$ exists and is equal to L, then the following limit exists*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

Proof: (Sketch) Assume there are differentiable functions f and g such that $f(n) = x_n$ and $g(n) = y_n$. (How do you construct these functions?) From Cauchy's theorem[2] there is a $c_n \in (n, n+1)$ such that

$$L = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{f(n+1) - f(n)}{g(n+1) - g(n)} = \lim_{n\to\infty} \frac{f'(c_n)}{g'(c_n)}.$$

Since $c_n \to \infty$ as $n \to \infty$, we can also write the aforementioned limit as

$$\lim_{t\to\infty} \frac{f'(t)}{g'(t)} = L.$$

(Here one may argue against this, but we recall the freedom of choice for the functions f and g such that c_n can be any number between n and $n+1$). By l'Hospital's rule we get

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = L.$$

Making $t = n$ yields $\lim\limits_{t\to\infty} \dfrac{x_n}{y_n} = L$. ■

The next application states that if a sequence is convergent, then the arithmetic average of its terms is also convergent, and the sequences have the same limit.

Example 3.8.1 *Let a_n be a convergent sequence with $\lim\limits_{n\to\infty} a_n = L$. Let*

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

be the arithmetic average of the first n terms. Then A_n is convergent and

$$\lim_{n\to\infty} A_n = L.$$

Proof: This is an application of Cesaró-Stoltz lemma. Consider the sequences $x_n = a_1 + a_2 + \cdots + a_n$ and $y_n = n$. Since

$$\frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \frac{(a_1 + \cdots + a_{n+1}) - (a_1 + \cdots + a_n)}{(n+1) - n} = \frac{a_{n+1}}{1},$$

then

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} a_{n+1} = L.$$

Applying the Cesaró-Stoltz lemma yields

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

■

[2] This says that if f and g are differentiable on (a, b) and continuous on $[a, b]$, then there is a $c \in (a, b)$ such that $\dfrac{f(a) - f(b)}{g(a) - g(b)} = \dfrac{f'(c)}{g'(c)}$.

Exercise 3.8.15 *Let b_n be a convergent sequence with $\lim_{n\to\infty} b_n = L$. Let*

$$G_n = (b_1 \cdot b_2 \cdot \dots \cdot b_n)^{1/n}$$

be the geometric average of the first n terms. Show that G_n is convergent and

$$\lim_{n\to\infty} G_n = L.$$

The following result extends the Cesaró-Stoltz lemma to sequences of random variables.

Proposition 3.8.16 *Let X_n be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, P)$, such that*

$$ac\text{-}\lim_{n\to\infty} \frac{X_{n+1} - X_n}{Y_{n+1} - Y_n} = L.$$

Then

$$ac\text{-}\lim_{n\to\infty} \frac{X_n}{Y_n} = L.$$

Proof: Consider the sets

$$\begin{aligned} A &= \{\omega \in \Omega; \lim_{n\to\infty} \frac{X_{n+1}(\omega) - X_n(\omega)}{Y_{n+1}(\omega) - Y_n(\omega)} = L\} \\ B &= \{\omega \in \Omega; \lim_{n\to\infty} \frac{X_n(\omega)}{Y_n(\omega)} = L\}. \end{aligned}$$

Since for any given state of the world ω, the sequences $x_n = X_n(\omega)$ and $y_n = Y_n(\omega)$ are numerical sequences, Lemma 3.8.14 yields the inclusion $A \subset B$. This implies $P(A) \le P(B)$. Since $P(A) = 1$, it follows that $P(B) = 1$, which leads to the desired conclusion. ∎

Example 3.8.17 *Let S_n denote the price of a stock on day n, and assume that*

$$ac\text{-}\lim_{n\to\infty} S_n = L.$$

Then

$$ac\text{-}\lim_{n\to\infty} \frac{S_1 + \dots + S_n}{n} = L \quad \text{and} \quad ac\text{-}\lim_{n\to\infty} (S_1 \cdot \dots \cdot S_n)^{1/n} = L.$$

This says that if almost all future simulations of the stock price approach the steady state limit L, the arithmetic and geometric averages converge to the same limit. The statement is a consequence of Proposition 3.8.16 and follows a similar proof as Example 3.8.1. Asian options have payoffs depending on these type of averages, as we shall see in Part II.

3.8.1 The Martingale Convergence Theorem

We state now, without proof, a result which is a powerful way of proving the almost certain convergence. We start with the discrete version:

Theorem 3.8.18 *Let X_n be a martingale with bounded means*

$$\exists M > 0 \ \text{such that} \ E[|X_n|] \le M, \qquad \forall n \ge 1. \tag{3.8.14}$$

Then there is $L < \infty$ such that $ac\text{-}\lim_{n\to\infty} X_n = L$, i.e.

$$P\big(\omega; \lim_{n\to\infty} X_n(\omega) = L\big) = 1.$$

Since $E[|X_n|]^2 \leq E[X_n^2]$, the condition (3.8.14) can be replaced by its stronger version

$$\exists M > 0 \text{ such that } E[X_n^2] \leq M, \qquad \forall n \geq 1.$$

The following result deals with the continuous version of the Martingale Convergence Theorem. Denote the infinite knowledge by $\mathcal{F}_\infty = \sigma\Big(\cup_t \mathcal{F}_t\Big)$.

Theorem 3.8.19 *Let X_t be an $\mathcal{F}_t$-martingale such that*

$$\exists M > 0 \text{ such that } E[|X_t|] < M, \qquad \forall t > 0.$$

Then there is an $\mathcal{F}_\infty$-measurable random variable X_∞ such that $X_t \to X_\infty$ a.c. as $t \to \infty$.

The next exrcise deals with a process that is a.c-convergent but is not ms-convergent.

Exercise 3.8.20 *It is known that $X_t = e^{W_t - t/2}$ is a martingale. Since*

$$E[|X_t|] = E[e^{W_t - t/2}] = e^{-t/2}E[e^{W_t}] = e^{-t/2}e^{t/2} = 1,$$

by the Martingale Convergence Theorem there is a number L such that $X_t \to L$ a.c. as $t \to \infty$.

(a) *What is the limit L? How did you make your guess?*

(b) *Show that*

$$E[|X_t - 1|^2] = Var(X_t) + \Big(E(X_t) - 1\Big)^2.$$

(c) *Show that X_t does not converge in the mean square to 1.*

(d) *Prove that the sequence X_t is a.c-convergent but it is not ms-convergent.*

3.8.2 The Squeeze Theorem

The following result is the analog of the Squeeze Theorem from usual Calculus.

Theorem 3.8.21 *Let X_n, Y_n, Z_n be sequences of random variables on the probability space $(\Omega, \mathcal{F}, P)$ such that*

$$X_n \leq Y_n \leq Z_n \ \ a.s. \ \forall n \geq 1.$$

If X_n and Z_n converge to L as $n \to \infty$ almost certainly (or in mean square, or stochastic or in distribution), then Y_n converges to L in a similar mode.

Proof: For any state of the world $\omega \in \Omega$ consider the sequences $x_n = X_n(\omega)$, $y_n = Y_n(\omega)$ and $z_n = Z_n(\omega)$ and apply the usual Squeeze Theorem to them. ∎

Remark 3.8.22 *The previous theorem remains valid if n is replaced by a continuous positive parameter t.*

Example 3.8.2 *Show that* $\text{ac-}\lim\limits_{t\to\infty} \dfrac{W_t \sin(W_t)}{t} = 0.$

Proof: Consider the sequences $X_t = 0$, $Y_t = \dfrac{W_t \sin(W_t)}{t}$ and $Z_t = \dfrac{W_t}{t}$. From Application 3.8.9 we have $\text{ac-}\lim\limits_{t\to\infty} Z_t = 0$. Applying the Squeeze Theorem we obtain the desired result. ∎

Exercise 3.8.23 *Use the Squeeze Theorem to find the following limits:*

(a) $ac\text{-}\lim\limits_{t\to\infty} \dfrac{\sin(W_t)}{t}$;

(b) $ac\text{-}\lim\limits_{t\to 0} t\cos W_t$;

(c) $ac\text{-}\lim\limits_{t\to -\infty} e^t(\sin W_t)^2$.

3.9 Quadratic Variation

For some stochastic processes the sum of squares of consecutive increments tends in mean square to a finite number, as the norm of the partition decreases to zero. We shall encounter in the following a few important examples that will be useful when dealing with stochastic integrals.

3.9.1 The Quadratic Variation of W_t

The next result states that the *quadratic variation* of the Brownian motion W_t on the interval $[0, T]$ is T. More precisely, we have:

Proposition 3.9.1 *Let $T > 0$ and consider the equidistant partition $0 = t_0 < t_1 < \cdots t_{n-1} < t_n = T$. Then*

$$\boxed{ms\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 = T.} \tag{3.9.15}$$

Proof: Consider the random variable

$$X_n = \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2.$$

Since the increments of a Brownian motion are independent, Proposition 4.0.7 yields

$$\begin{aligned} E[X_n] &= \sum_{i=0}^{n-1} E[(W_{t_{i+1}} - W_{t_i})^2] = \sum_{i=0}^{n-1}(t_{i+1} - t_i) \\ &= t_n - t_0 = T; \end{aligned}$$

$$\begin{aligned} Var(X_n) &= \sum_{i=0}^{n-1} Var[(W_{t_{i+1}} - W_{t_i})^2] = \sum_{i=0}^{n-1} 2(t_{i+1} - t_i)^2 \\ &= n \cdot 2\Big(\frac{T}{n}\Big)^2 = \frac{2T^2}{n}, \end{aligned}$$

where we used that the partition is equidistant. Since X_n satisfies the conditions

$$\begin{aligned} E[X_n] &= T, \quad \forall n \geq 1; \\ Var[X_n] &\to 0, \quad n \to \infty, \end{aligned}$$

by Proposition 3.8.1 we obtain $\text{ms-}\lim\limits_{n\to\infty} X_n = T$, or

$$\text{ms-}\lim_{n\to\infty} \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 = T. \tag{3.9.16}$$

■

Exercise 3.9.2 *Prove that the quadratic variation of the Brownian motion W_t on $[a, b]$ is equal to $b - a$.*

The Fundamental Relation $dW_t^2 = dt$

The relation discussed in this section can be regarded as the fundamental relation of Stochastic Calculus. We shall start by recalling relation (3.9.16)

$$\text{ms-}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 = T. \tag{3.9.17}$$

The right side can be regarded as a regular Riemann integral

$$T = \int_0^T dt,$$

while the left side can be regarded as a stochastic integral with respect to dW_t^2

$$\int_0^T (dW_t)^2 = \text{ms-}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2.$$

Substituting into (3.9.17) yields

$$\int_0^T (dW_t)^2 = \int_0^T dt, \quad \forall T > 0.$$

The differential form of this integral equation is

$$\boxed{dW_t^2 = dt.}$$

In fact, this expression also holds in the mean square sense, as it can be inferred from the next exercise.

Exercise 3.9.3 *Show that*

(*a*) $E[dW_t^2 - dt] = 0$;

(*b*) $Var(dW_t^2 - dt) = o(dt)$;

(*c*) $ms\text{-}\lim_{dt\to 0} (dW_t^2 - dt) = 0$.

Roughly speaking, the process dW_t^2, which is the square of infinitesimal increments of a Brownian motion, is totally predictable. This relation plays a central role in Stochastic Calculus and will be useful when dealing with Ito's lemma.

The following exercise states that $dtdW_t = 0$, which is another important stochastic relation useful in Ito's lemma.

Exercise 3.9.4 *Consider the equidistant partition $0 = t_0 < t_1 < \cdots t_{n-1} < t_n = T$. Then*

$$ms\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i) = 0. \tag{3.9.18}$$

3.9.2 The Quadratic Variation of $N_t - \lambda t$

The following result deals with the quadratic variation of the compensated Poisson process $M_t = N_t - \lambda t$.

Proposition 3.9.5 *Let $a < b$ and consider the partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Then*

$$\boxed{ms\text{–}\lim_{\|\Delta_n\|\to 0}\sum_{k=0}^{n-1}(M_{t_{k+1}} - M_{t_k})^2 = N_b - N_a,} \tag{3.9.19}$$

where $\|\Delta_n\| := \sup_{0\le k\le n-1}(t_{k+1} - t_k)$.

Proof: For the sake of simplicity we shall use the following notations:

$$\Delta t_k = t_{k+1} - t_k, \quad \Delta M_k = M_{t_{k+1}} - M_{t_k}, \quad \Delta N_k = N_{t_{k+1}} - N_{t_k}.$$

The relation we need to prove can also be written as

$$\text{ms-}\lim_{n\to\infty}\sum_{k=0}^{n-1}\Big[(\Delta M_k)^2 - \Delta N_k\Big] = 0.$$

Let

$$Y_k = (\Delta M_k)^2 - \Delta N_k = (\Delta M_k)^2 - \Delta M_k - \lambda\Delta t_k.$$

It suffices to show that

$$E\Big[\sum_{k=0}^{n-1} Y_k\Big] \quad = \quad 0, \tag{3.9.20}$$

$$\lim_{n\to\infty} Var\Big[\sum_{k=0}^{n-1} Y_k\Big] \quad = \quad 0. \tag{3.9.21}$$

The first identity follows from the properties of Poisson processes (see Exercise 2.8.9)

$$\begin{aligned}
E\Big[\sum_{k=0}^{n-1} Y_k\Big] &= \sum_{k=0}^{n-1} E[Y_k] = \sum_{k=0}^{n-1} E[(\Delta M_k)^2] - E[\Delta N_k] \\
&= \sum_{k=0}^{n-1}(\lambda\Delta t_k - \lambda\Delta t_k) = 0.
\end{aligned}$$

For the proof of the identity (3.9.21) we need to find first the variance of Y_k.

$$\begin{aligned}
Var[Y_k] &= Var[(\Delta M_k)^2 - (\Delta M_k + \lambda\Delta t_k)] = Var[(\Delta M_k)^2 - \Delta M_k] \\
&= Var[(\Delta M_k)^2] + Var[\Delta M_k] - 2Cov[\Delta M_k^2, \Delta M_k] \\
&= \lambda\Delta t_k + 2\lambda^2\Delta t_k^2 + \lambda\Delta t_k \\
&\quad -2\Big[E[(\Delta M_k)^3] - E[(\Delta M_k)^2]E[\Delta M_k]\Big] \\
&= 2\lambda^2(\Delta t_k)^2,
\end{aligned}$$

where we used Exercise 2.8.9 and the fact that $E[\Delta M_k] = 0$. Since M_t is a process with independent increments, then $Cov[Y_k, Y_j] = 0$ for $i \neq j$. Then

$$\begin{aligned} Var\Big[\sum_{k=0}^{n-1} Y_k\Big] &= \sum_{k=0}^{n-1} Var[Y_k] + 2\sum_{k\neq j} Cov[Y_k, Y_j] = \sum_{k=0}^{n-1} Var[Y_k] \\ &= 2\lambda^2 \sum_{k=0}^{n-1} (\Delta t_k)^2 \leq 2\lambda^2 \|\Delta_n\| \sum_{k=0}^{n-1} \Delta t_k = 2\lambda^2 (b-a)\|\Delta_n\|, \end{aligned}$$

and hence $Var\Big[\sum_{k=0}^{n-1} Y_n\Big] \to 0$ as $\|\Delta_n\| \to 0$. According to the Example 1.13.1, we obtain the desired limit in mean square. ■

The previous result states that the quadratic variation of the martingale M_t between a and b is equal to the jump of the Poisson process between a and b.

The Fundamental Relation $dM_t^2 = dN_t$

Recall relation (3.9.19)

$$\text{ms-}\lim_{n\to\infty} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = N_b - N_a. \tag{3.9.22}$$

The right side can be regarded as a Riemann-Stieltjes integral

$$N_b - N_a = \int_a^b dN_t,$$

while the left side can be regarded as a stochastic integral with respect to $(dM_t)^2$

$$\int_a^b (dM_t)^2 := \text{ms-}\lim_{n\to\infty} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2.$$

Substituting in (3.9.22) yields

$$\int_a^b (dM_t)^2 = \int_a^b dN_t,$$

for any $a < b$. The equivalent differential form is

$$\boxed{(dM_t)^2 = dN_t.} \tag{3.9.23}$$

The Relations $dt\, dM_t = 0$, $dW_t\, dM_t = 0$

In order to show that $dt\, dM_t = 0$ in the mean square sense, we need to prove the limit

$$\text{ms-}\lim_{n\to\infty} \sum_{k=0}^{n-1} (t_{k+1} - t_k)(M_{t_{k+1}} - M_{t_k}) = 0. \tag{3.9.24}$$

This can be thought as a vanishing integral of the increment process dM_t with respect to dt

$$\int_a^b dM_t\, dt = 0, \qquad \forall a, b \in \mathbb{R}.$$

Denote

$$X_n = \sum_{k=0}^{n-1} (t_{k+1} - t_k)(M_{t_{k+1}} - M_{t_k}) = \sum_{k=0}^{n-1} \Delta t_k \Delta M_k.$$

In order to show (3.9.24) it suffices to prove that

1. $E[X_n] = 0$;
2. $\lim\limits_{n\to\infty} Var[X_n] = 0$.

Using the additivity of the expectation and Exercise 2.8.9, (ii)

$$E[X_n] = E\Big[\sum_{k=0}^{n-1} \Delta t_k \Delta M_k\Big] = \sum_{k=0}^{n-1} \Delta t_k E[\Delta M_k] = 0.$$

Since the Poisson process N_t has independent increments, the same property holds for the compensated Poisson process M_t. Then $\Delta t_k \Delta M_k$ and $\Delta t_j \Delta M_j$ are independent for $k \neq j$, and using the properties of variance we have

$$Var[X_n] = Var\Big[\sum_{k=0}^{n-1} \Delta t_k \Delta M_k\Big] = \sum_{k=0}^{n-1} (\Delta t_k)^2 Var[\Delta M_k] = \lambda \sum_{k=0}^{n-1} (\Delta t_k)^3,$$

where we used

$$Var[\Delta M_k] = E[(\Delta M_k)^2] - (E[\Delta M_k])^2 = \lambda \Delta t_k,$$

see Exercise 2.8.9 (ii). If we let $\|\Delta_n\| = \max\limits_k \Delta t_k$, then

$$Var[X_n] = \lambda \sum_{k=0}^{n-1} (\Delta t_k)^3 \leq \lambda \|\Delta_n\|^2 \sum_{k=0}^{n-1} \Delta t_k = \lambda(b-a)\|\Delta_n\|^2 \to 0$$

as $n \to \infty$. Hence we proved the stochastic differential relation

$$\boxed{dt\, dM_t = 0.} \tag{3.9.25}$$

For showing the relation $dW_t\, dM_t = 0$, we need to prove

$$\text{ms-}\lim_{n\to\infty} Y_n = 0, \tag{3.9.26}$$

where we have denoted

$$Y_n = \sum_{k=0}^{n-1} (W_{k+1} - W_k)(M_{t_{k+1}} - M_{t_k}) = \sum_{k=0}^{n-1} \Delta W_k \Delta M_k.$$

Since the Brownian motion W_t and the process M_t have independent increments and ΔW_k is independent of ΔM_k, we have

$$E[Y_n] = \sum_{k=0}^{n-1} E[\Delta W_k \Delta M_k] = \sum_{k=0}^{n-1} E[\Delta W_k] E[\Delta M_k] = 0,$$

where we used $E[\Delta W_k] = E[\Delta M_k] = 0$. Using also $E[(\Delta W_k)^2] = \Delta t_k$, $E[(\Delta M_k)^2] = \lambda \Delta t_k$, and invoking the independence of ΔW_k and ΔM_k, we get

$$\begin{aligned} Var[\Delta W_k \Delta M_k] &= E[(\Delta W_k)^2 (\Delta M_k)^2] - (E[\Delta W_k \Delta M_k])^2 \\ &= E[(\Delta W_k)^2] E[(\Delta M_k)^2] - E[\Delta W_k]^2 E[\Delta M_k]^2 \\ &= \lambda (\Delta t_k)^2. \end{aligned}$$

Then using the independence of the terms in the sum, we get

$$\begin{aligned} Var[Y_n] &= \sum_{k=0}^{n-1} Var[\Delta W_k \Delta M_k] = \lambda \sum_{k=0}^{n-1} (\Delta t_k)^2 \\ &\le \lambda \|\Delta_n\| \sum_{k=0}^{n-1} \Delta t_k = \lambda (b-a) \|\Delta_n\| \to 0, \end{aligned}$$

as $n \to \infty$. Since Y_n is a random variable with mean zero and variance decreasing to zero, it follows that $Y_n \to 0$ in the mean square sense. Hence we proved that

$$\boxed{dW_t \, dM_t = 0.} \tag{3.9.27}$$

Exercise 3.9.6 *Show the following stochastic differential relations:*

(a) $dt\, dN_t = 0$; (b) $dW_t\, dN_t = 0$; (c) $dt\, dW_t = 0$;

(d) $(dN_t)^2 = dN_t$; (e) $(dM_t)^2 = dN_t$; (f) $(dM_t)^4 = dN_t$.

The relations proved in this section will be useful in the Part II when developing the stochastic model of a stock price that exhibits jumps modeled by a Poisson process.

Chapter 4

Stochastic Integration

This chapter deals with one of the most useful stochastic integrals, called the *Ito integral*. This type of integral was introduced in 1944 by the Japanese mathematician K. Ito, and was originally motivated by a construction of diffusion processes.

4.0.3 Nonanticipating Processes

Consider the Brownian motion W_t. A process F_t is called a *nonanticipating process* if F_t is independent of the increment $W_{t'} - W_t$ for any t and t' with $t < t'$. Consequently, the process F_t is independent of the behavior of the Brownian motion in the future, i.e. it cannot anticipate the future. For instance, W_t, e^{W_t}, $W_t^2 - W_t + t$ are examples of nonanticipating processes, while W_{t+1} or $\frac{1}{2}(W_{t+1} - W_t)^2$ are not.

Nonanticipating processes are important because the Ito integral concept applies only to them.

If $\mathcal{F}_t$ denotes the information known until time t, where this information is generated by the Brownian motion $\{W_s; s \leq t\}$, then any $\mathcal{F}_t$-adapted process F_t is nonanticipating.

4.0.4 Increments of Brownian Motions

In this section we shall discuss a few basic properties of the increments of a Brownian motion, which will be useful when computing stochastic integrals.

Proposition 4.0.7 *Let W_t be a Brownian motion. If $s < t$, we have*

1. $E[(W_t - W_s)^2] = t - s.$
2. $Var[(W_t - W_s)^2] = 2(t - s)^2.$

Proof: 1. Using that $W_t - W_s \sim N(0, t - s)$, we have

$$E[(W_t - W_s)^2] = E[(W_t - W_s)^2] - (E[W_t - W_s])^2 = Var(W_t - W_s) = t - s.$$

2. Dividing by the standard deviation yields the standard normal random variable $\dfrac{W_t - W_s}{\sqrt{t - s}} \sim N(0, 1)$. Its square, $\dfrac{(W_t - W_s)^2}{t - s}$ is χ^2-distributed with 1 degree of freedom.[1] Its mean is 1 and

[1] A χ^2-distributed random variable with n degrees of freedom has mean n and variance $2n$.

its variance is 2. This implies

$$\begin{aligned} E\Big[\frac{(W_t - W_s)^2}{t-s}\Big] &= 1 \Longrightarrow E[(W_t - W_s)^2] = t - s; \\ Var\Big[\frac{(W_t - W_s)^2}{t-s}\Big] &= 2 \Longrightarrow Var[(W_t - W_s)^2] = 2(t-s)^2. \end{aligned}$$

■

Remark 4.0.8 *The infinitesimal version of the previous result is obtained by replacing $t - s$ with dt*

1. $E[dW_t^2] = dt$;
2. $Var[dW_t^2] = 2dt^2$.

We shall see in an upcoming section that in fact dW_t^2 and dt are equal in a mean square sense.

Exercise 4.0.9 *Show that*

(*a*) $E[(W_t - W_s)^4] = 3(t-s)^2$;

(*b*) $E[(W_t - W_s)^6] = 15(t-s)^3$.

4.1 The Ito Integral

The Ito integral is defined in a way that is similar to the Riemann integral. The Ito integral is taken with respect to infinitesimal increments of a Brownian motion, dW_t, which are random variables, while the Riemann integral considers integration with respect to the predictable infinitesimal changes dt. It is worth noting that the Ito integral is a random variable, while the Riemann integral is just a real number. Despite this fact, there are several common properties and relations between these two types of integrals.

Consider $0 \le a < b$ and let $F_t = f(W_t, t)$ be a nonanticipating process with

$$E\Big[\int_a^b F_t^2\, dt\Big] < \infty. \tag{4.1.1}$$

Divide the interval $[a, b]$ into n subintervals using the partition points

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

and consider the partial sums

$$S_n = \sum_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}} - W_{t_i}).$$

We emphasize that the intermediate points are the left endpoints of each interval, and this is the way they should be always chosen. Since the process F_t is nonanticipative, the random variables F_{t_i} and $W_{t_{i+1}} - W_{t_i}$ are independent; this is an important feature in the definition of the Ito integral.

The *Ito integral* is the limit of the partial sums S_n

$$\text{ms-}\lim_{n\to\infty} S_n = \int_a^b F_t\, dW_t,$$

provided the limit exists. It can be shown that the choice of partition does not influence the value of the Ito integral. This is the reason why, for practical purposes, it suffices to assume the intervals equidistant, i.e.

$$t_{i+1} - t_i = a + \frac{(b-a)}{n}, \qquad i = 0, 1, \cdots, n-1.$$

The previous convergence is in the mean square sense, i.e.

$$\lim_{n\to\infty} E\Big[\Big(S_n - \int_a^b F_t\, dW_t\Big)^2\Big] = 0.$$

Existence of the Ito integral

It is known that the Ito stochastic integral $\int_a^b F_t\, dW_t$ exists if the process $F_t = f(W_t, t)$ satisfies the following properties:

1. The paths $t \to F_t(\omega)$ are continuous on $[a, b]$ for any state of the world $\omega \in \Omega$;
2. The process F_t is nonanticipating for $t \in [a, b]$;
3. $E\Big[\int_a^b F_t^2\, dt\Big] < \infty$.

For instance, the following stochastic integrals exist:

$$\int_0^T W_t^2\, dW_t, \quad \int_0^T \sin(W_t)\, dW_t, \quad \int_a^b \frac{\cos(W_t)}{t}\, dW_t.$$

4.2 Examples of Ito integrals

As in the case of the Riemann integral, using the definition is not an efficient way of computing integrals. The same philosophy applies to Ito integrals. We shall compute in the following two simple Ito integrals. In later sections we shall introduce more efficient methods for computing Ito integrals.

4.2.1 The case $F_t = c$, constant

In this case the partial sums can be computed explicitly

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}} - W_{t_i}) = \sum_{i=0}^{n-1} c(W_{t_{i+1}} - W_{t_i}) \\ &= c(W_b - W_a), \end{aligned}$$

and since the answer does not depend on n, we have

$$\int_a^b c\, dW_t = c(W_b - W_a).$$

In particular, taking $c = 1$, $a = 0$, and $b = T$, since the Brownian motion starts at 0, we have the following formula:

$$\boxed{\int_0^T dW_t = W_T.}$$

4.2.2 The case $F_t = W_t$

We shall integrate the process W_t between 0 and T. Considering an equidistant partition, we take $t_k = \dfrac{kT}{n}$, $k = 0, 1, \cdots, n-1$. The partial sums are given by

$$S_n = \sum_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}).$$

Since

$$xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2],$$

letting $x = W_{t_i}$ and $y = W_{t_{i+1}} - W_{t_i}$ yields

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{1}{2}W_{t_{i+1}}^2 - \frac{1}{2}W_{t_i}^2 - \frac{1}{2}(W_{t_{i+1}} - W_{t_i})^2.$$

Then after pair cancelations the sum becomes

$$\begin{aligned} S_n &= \frac{1}{2}\sum_{i=0}^{n-1} W_{t_{i+1}}^2 - \frac{1}{2}\sum_{i=0}^{n-1} W_{t_i}^2 - \frac{1}{2}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \\ &= \frac{1}{2}W_{t_n}^2 - \frac{1}{2}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2. \end{aligned}$$

Using $t_n = T$, we get

$$S_n = \frac{1}{2}W_T^2 - \frac{1}{2}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2.$$

Since the first term is independent of n, using Proposition 3.9.1, we have

$$\begin{aligned} \text{ms-}\lim_{n\to\infty} S_n &= \frac{1}{2}W_T^2 - \text{ms-}\lim_{n\to\infty}\frac{1}{2}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 && (4.2.2) \\ &= \frac{1}{2}W_T^2 - \frac{1}{2}T. && (4.2.3) \end{aligned}$$

We have now obtained the following explicit formula of a stochastic integral:

$$\boxed{\int_0^T W_t\, dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T.}$$

In a similar way one can obtain

$$\boxed{\int_a^b W_t\, dW_t = \frac{1}{2}(W_b^2 - W_a^2) - \frac{1}{2}(b-a).}$$

It is worth noting that the right side contains random variables depending on the limits of integration a and b.

Exercise 4.2.1 *Show the following identities:*

(a) $E[\int_0^T dW_t] = 0$;

(b) $E[\int_0^T W_t dW_t] = 0$;

(c) $Var[\int_0^T W_t dW_t] = \dfrac{T^2}{2}$.

4.3 Properties of the Ito Integral

We shall start with some properties which are similar with those of the Riemannian integral.

Proposition 4.3.1 *Let $f(W_t, t)$, $g(W_t, t)$ be nonanticipating processes and $c \in \mathbb{R}$. Then we have*

1. Additivity:

$$\int_0^T [f(W_t, t) + g(W_t, t)]\, dW_t = \int_0^T f(W_t, t)\, dW_t + \int_0^T g(W_t, t)\, dW_t.$$

2. Homogeneity:

$$\int_0^T c f(W_t, t)\, dW_t = c \int_0^T f(W_t, t)\, dW_t.$$

3. Partition property:

$$\int_0^T f(W_t, t)\, dW_t = \int_0^u f(W_t, t)\, dW_t + \int_u^T f(W_t, t)\, dW_t, \qquad \forall 0 < u < T.$$

Proof: 1. Consider the partial sum sequences

$$\begin{aligned} X_n &= \sum_{i=0}^{n-1} f(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}) \\ Y_n &= \sum_{i=0}^{n-1} g(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i}). \end{aligned}$$

Since $\text{ms-}\lim_{n\to\infty} X_n = \int_0^T f(W_t,t)\,dW_t$ and $\text{ms-}\lim_{n\to\infty} Y_n = \int_0^T g(W_t,t)\,dW_t$, using Proposition 1.14.2 yields

$$\begin{aligned}
&\int_0^T \Big(f(W_t,t)+g(W_t,t)\Big)\,dW_t \\
&= \text{ms-}\lim_{n\to\infty} \sum_{i=0}^{n-1} \Big(f(W_{t_i},t_i)+g(W_{t_i},t_i)\Big)(W_{t_{i+1}}-W_{t_i}) \\
&= \text{ms-}\lim_{n\to\infty} \Big[\sum_{i=0}^{n-1} \Big(f(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i}) + \sum_{i=0}^{n-1} g(W_{t_i},t_i)(W_{t_{i+1}}-W_{t_i})\Big] \\
&= \text{ms-}\lim_{n\to\infty} (X_n+Y_n) = \text{ms-}\lim_{n\to\infty} X_n + \text{ms-}\lim_{n\to\infty} Y_n \\
&= \int_0^T f(W_t,t)\,dW_t + \int_0^T g(W_t,t)\,dW_t.
\end{aligned}$$

The proofs of parts 2 and 3 are left as an exercise for the reader. ■

Some other properties, such as monotonicity, do not hold in general. It is possible to have a nonnegative random variable F_t for which the random variable $\int_0^T F_t\,dW_t$ has negative values.

Some of the random variable properties of the Ito integral are given by the following result:

Proposition 4.3.2 *We have*
1. Zero mean:

$$E\Big[\int_a^b f(W_t,t)\,dW_t\Big] = 0.$$

2. Isometry:

$$E\Big[\Big(\int_a^b f(W_t,t)\,dW_t\Big)^2\Big] = E\Big[\int_a^b f(W_t,t)^2\,dt\Big].$$

3. Covariance:

$$E\Big[\Big(\int_a^b f(W_t,t)\,dW_t\Big)\Big(\int_a^b g(W_t,t)\,dW_t\Big)\Big] = E\Big[\int_a^b f(W_t,t)g(W_t,t)\,dt\Big].$$

We shall discuss the previous properties giving rough reasons why they hold true. The detailed proofs are beyond the goal of this book.

1. The Ito integral is the mean square limit of the partial sums $S_n = \sum_{i=0}^{n-1} f_{t_i}(W_{t_{i+1}}-W_{t_i})$, where we denoted $f_{t_i} = f(W_{t_i},t_i)$. Since $f(W_t,t)$ is a nonanticipative process, then f_{t_i} is independent of the increments $W_{t_{i+1}}-W_{t_i}$, and hence we have

$$\begin{aligned}
E[S_n] &= E\Big[\sum_{i=0}^{n-1} f_{t_i}(W_{t_{i+1}}-W_{t_i})\Big] = \sum_{i=0}^{n-1} E[f_{t_i}(W_{t_{i+1}}-W_{t_i})] \\
&= \sum_{i=0}^{n-1} E[f_{t_i}]E[(W_{t_{i+1}}-W_{t_i})] = 0,
\end{aligned}$$

because the increments have mean zero. Since each partial sum has zero mean, their limit, which is the Ito Integral, will also have zero mean.

2. Since the square of the sum of partial sums can be written as

$$\begin{aligned} S_n^2 &= \Big(\sum_{i=0}^{n-1} f_{t_i}(W_{t_{i+1}} - W_{t_i})\Big)^2 \\ &= \sum_{i=0}^{n-1} f_{t_i}^2 (W_{t_{i+1}} - W_{t_i})^2 + 2\sum_{i\neq j} f_{t_i}(W_{t_{i+1}} - W_{t_i}) f_{t_j}(W_{t_{j+1}} - W_{t_j}), \end{aligned}$$

using the independence yields

$$\begin{aligned} E[S_n^2] &= \sum_{i=0}^{n-1} E[f_{t_i}^2] E[(W_{t_{i+1}} - W_{t_i})^2] \\ &\quad + 2\sum_{i\neq j} E[f_{t_i}] E[(W_{t_{i+1}} - W_{t_i})] E[f_{t_j}] E[(W_{t_{j+1}} - W_{t_j})] \\ &= \sum_{i=0}^{n-1} E[f_{t_i}^2](t_{i+1} - t_i), \end{aligned}$$

which are the Riemann sums of the integral $\int_a^b E[f_t^2]\, dt = E\Big[\int_a^b f_t^2\, dt\Big]$, where the last identity follows from Fubini's theorem. Hence $E[S_n^2]$ converges to the aforementioned integral. It is yet to be shown that the convergence holds also in mean square.

3. Consider the partial sums

$$S_n = \sum_{i=0}^{n-1} f_{t_i}(W_{t_{i+1}} - W_{t_i}), \qquad V_n = \sum_{j=0}^{n-1} g_{t_j}(W_{t_{j+1}} - W_{t_j}).$$

Their product is

$$\begin{aligned} S_n V_n &= \Big(\sum_{i=0}^{n-1} f_{t_i}(W_{t_{i+1}} - W_{t_i})\Big)\Big(\sum_{j=0}^{n-1} g_{t_j}(W_{t_{j+1}} - W_{t_j})\Big) \\ &= \sum_{i=0}^{n-1} f_{t_i} g_{t_i}(W_{t_{i+1}} - W_{t_i})^2 + \sum_{i\neq j}^{n-1} f_{t_i} g_{t_j}(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j}) \end{aligned}$$

Using that f_t and g_t are nonanticipative and that

$$\begin{aligned} E[(W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j})] &= E[W_{t_{i+1}} - W_{t_i}]E[W_{t_{j+1}} - W_{t_j}] = 0,\ i \neq j \\ E[(W_{t_{i+1}} - W_{t_i})^2] &= t_{i+1} - t_i, \end{aligned}$$

it follows that

$$\begin{aligned} E[S_n V_n] &= \sum_{i=0}^{n-1} E[f_{t_i} g_{t_i}] E[(W_{t_{i+1}} - W_{t_i})^2] \\ &= \sum_{i=0}^{n-1} E[f_{t_i} g_{t_i}](t_{i+1} - t_i), \end{aligned}$$

which is the Riemann sum for the integral $\int_a^b E[f_t g_t]\,dt$.

From 1 and 2 it follows that the random variable $\int_a^b f(W_t,t)\,dW_t$ has mean zero and variance

$$Var\Big[\int_a^b f(W_t,t)\,dW_t\Big] = E\Big[\int_a^b f(W_t,t)^2\,dt\Big].$$

From 1 and 3 it follows that

$$Cov\Big[\int_a^b f(W_t,t)\,dW_t, \int_a^b g(W_t,t)\,dW_t\Big] = \int_a^b E[f(W_t,t)g(W_t,t)]\,dt.$$

Corollary 4.3.3 (Cauchy's integral inequality) *Let $f(t) = f(W_t,t)$ and $g(t) = g(W_t,t)$. Then*

$$\Big(\int_a^b E[f_t g_t]\,dt\Big)^2 \le \Big(\int_a^b E[f_t^2]\,dt\Big)\Big(\int_a^b E[g_t^2]\,dt\Big).$$

Proof: It follows from the previous theorem and from the correlation formula $|Corr(X,Y)| = \dfrac{|Cov(X,Y)|}{[Var(X)Var(Y)]^{1/2}} \le 1$. ∎

Let $\mathcal{F}_t$ be the information set at time t. This implies that f_{t_i} and $W_{t_{i+1}} - W_{t_i}$ are known at time t, for any $t_{i+1} \le t$. It follows that the partial sum $S_n = \displaystyle\sum_{i=0}^{n-1} f_{t_i}(W_{t_{i+1}} - W_{t_i})$ is $\mathcal{F}_t$-predictable. The following result states that this is also valid in mean square:

Proposition 4.3.4 *The Ito integral $\int_0^t f_s\,dW_s$ is $\mathcal{F}_t$-predictable.*

The following two results state that if the upper limit of an Ito integral is replaced by the parameter t we obtain a continuous martingale.

Proposition 4.3.5 *For any $s < t$ we have*

$$E\Big[\int_0^t f(W_u,u)\,dW_u|\mathcal{F}_s\Big] = \int_0^s f(W_u,u)\,dW_u.$$

Proof: Using part 3 of Proposition 4.3.2 we get

$$\begin{aligned} & E\Big[\int_0^t f(W_u,u)\,dW_u|\mathcal{F}_s\Big] \\ = \;& E\Big[\int_0^s f(W_u,u)\,dW_u + \int_s^t f(W_u,u)\,dW_u|\mathcal{F}_s\Big] \\ = \;& E\Big[\int_0^s f(W_u,u)\,dW_u|\mathcal{F}_s\Big] + E\Big[\int_s^t f(W_u,u)\,dW_u|\mathcal{F}_s\Big]. \end{aligned} \tag{4.3.4}$$

Since $\int_0^s f(W_u,u)\,dW_u$ is $\mathcal{F}_s$-predictable (see Proposition 4.3.4), by part 2 of Proposition 1.11.4

$$E\Big[\int_0^s f(W_u,u)\,dW_u|\mathcal{F}_s\Big] = \int_0^s f(W_u,u)\,dW_u.$$

Since $\int_s^t f(W_u, u)\, dW_u$ contains only information between s and t, it is unpredictable given the information set $\mathcal{F}_s$, so

$$E\Big[\int_s^t f(W_u, u)\, dW_u | \mathcal{F}_s\Big] = 0.$$

Substituting into (4.3.4) yields the desired result. ■

Proposition 4.3.6 *Consider the process $X_t = \int_0^t f(W_s, s)\, dW_s$. Then X_t is continuous, i.e. for almost any state of the world $\omega \in \Omega$, the path $t \to X_t(\omega)$ is continuous.*

Proof: A rigorous proof is beyond the purpose of this book. We shall provide a rough sketch. Assume the process $f(W_t, t)$ satisfies $E[f(W_t, t)^2] < M$, for some $M > 0$. Let t_0 be fixed and consider $h > 0$. Consider the increment $Y_h = X_{t_0+h} - X_{t_0}$. Using the aforementioned properties of the Ito integral we have

$$\begin{aligned}
E[Y_h] &= E[X_{t_0+h} - X_{t_0}] = E\Big[\int_{t_0}^{t_0+h} f(W_t, t)\, dW_t\Big] = 0 \\
E[Y_h^2] &= E\Big[\Big(\int_{t_0}^{t_0+h} f(W_t, t)\, dW_t\Big)^2\Big] = \int_{t_0}^{t_0+h} E[f(W_t, t)^2]\, dt \\
&< M \int_{t_0}^{t_0+h} dt = Mh.
\end{aligned}$$

The process Y_h has zero mean for any $h > 0$ and its variance tends to 0 as $h \to 0$. Using a convergence theorem yields that Y_h tends to 0 in mean square, as $h \to 0$. This is equivalent with the continuity of X_t at t_0. ■

4.4 The Wiener Integral

The *Wiener integral* is a particular case of the Ito stochastic integral. It is obtained by replacing the nonanticipating stochastic process $f(W_t, t)$ by the deterministic function $f(t)$. The Wiener integral $\int_a^b f(t)\, dW_t$ is the mean square limit of the partial sums

$$S_n = \sum_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i}).$$

All properties of Ito integrals also hold for Wiener integrals. The Wiener integral is a random variable with zero mean

$$E\Big[\int_a^b f(t)\, dW_t\Big] = 0$$

and variance

$$E\Big[\Big(\int_a^b f(t)\, dW_t\Big)^2\Big] = \int_a^b f(t)^2\, dt.$$

However, in the case of Wiener integrals we can say something about their distribution.

Proposition 4.4.1 *The Wiener integral $I(f) = \int_a^b f(t)\, dW_t$ is a normal random variable with mean 0 and variance*

$$Var[I(f)] = \int_a^b f(t)^2\, dt := \|f\|^2_{L^2}.$$

Proof: Since increments $W_{t_{i+1}} - W_{t_i}$ are normally distributed with mean 0 and variance $t_{i+1} - t_i$, then

$$f(t_i)(W_{t_{i+1}} - W_{t_i}) \sim N(0, f(t_i)^2(t_{i+1} - t_i)).$$

Since these random variables are independent, by the Central Limit Theorem (see Theorem 2.3.1), their sum is also normally distributed, with

$$S_n = \sum_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i}) \sim N\Big(0, \sum_{i=0}^{n-1} f(t_i)^2(t_{i+1} - t_i)\Big).$$

Taking $n \to \infty$ and $\max_i \|t_{i+1} - t_i\| \to 0$, the normal distribution tends to

$$N\Big(0, \int_a^b f(t)^2\,dt\Big).$$

The previous convergence holds in distribution, and it still needs to be shown in the mean square. However, we shall omit this essential proof detail. ■

Exercise 4.4.2 *Show that the random variable* $X = \int_1^T \frac{1}{\sqrt{t}}\,dW_t$ *is normally distributed with mean* 0 *and variance* $\ln T$*.*

Exercise 4.4.3 *Let* $Y = \int_1^T \sqrt{t}\,dW_t$*. Show that* Y *is normally distributed with mean* 0 *and variance* $(T^2 - 1)/2$*.*

Exercise 4.4.4 *Find the distribution of the integral* $\int_0^t e^{t-s}\,dW_s$*.*

Exercise 4.4.5 *Show that* $X_t = \int_0^t (2t - u)\,dW_u$ *and* $Y_t = \int_0^t (3t - 4u)\,dW_u$ *are Gaussian processes with mean* 0 *and variance* $\frac{7}{3}t^3$*.*

Exercise 4.4.6 *Show that* $ms\text{-}\lim_{t\to 0} \dfrac{1}{t}\displaystyle\int_0^t u\,dW_u = 0$*.*

Exercise 4.4.7 *Find all constants* a, b *such that* $X_t = \int_0^t \Big(a + \frac{bu}{t}\Big)\,dW_u$ *is normally distributed with variance* t*.*

Exercise 4.4.8 *Let* n *be a positive integer. Prove that*

$$Cov\Big(W_t, \int_0^t u^n\,dW_u\Big) = \frac{t^{n+1}}{n+1}.$$

Formulate and prove a more general result.

4.5 Poisson Integration

In this section we deal with the integration with respect to the compensated Poisson process $M_t = N_t - \lambda t$, which is a martingale. Consider $0 \le a < b$ and let $F_t = F(t, M_t)$ be a nonanticipating process with

$$E\Big[\int_a^b F_t^2\,dt\Big] < \infty.$$

Consider the partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

of the interval $[a, b]$, and associate the partial sums

$$S_n = \sum_{i=0}^{n-1} F_{t_i-}(M_{t_{i+1}} - M_{t_i}),$$

where F_{t_i-} is the left-hand limit at t_i-. For predictability reasons, the intermediate points are the left-handed limit to the endpoints of each interval. Since the process F_t is nonanticipative, the random variables F_{t_i-} and $M_{t_{i+1}} - M_{t_i}$ are independent.

The integral of F_{t-} with respect to M_t is the mean square limit of the partial sum S_n

$$\text{ms-}\lim_{n\to\infty} S_n = \int_0^T F_{t-}\,dM_t,$$

provided the limit exists. More precisely, this convergence means that

$$\lim_{n\to\infty} E\Big[\Big(S_n - \int_a^b F_{t-}\,dM_t\Big)^2\Big] = 0.$$

4.5.1 A Work Out Example: the case $F_t = M_t$

We shall integrate the process M_{t-} between 0 and T with respect to M_t. Considering the partition points $t_k = \dfrac{kT}{n}$, $k = 0, 1, \cdots, n-1$. The partial sums are given by

$$S_n = \sum_{i=0}^{n-1} M_{t_i-}(M_{t_{i+1}} - M_{t_i}).$$

Using $xy = \dfrac{1}{2}[(x+y)^2 - x^2 - y^2]$, by letting $x = M_{t_i-}$ and $y = M_{t_{i+1}} - M_{t_i}$, we get (**Where does a minus go?**)

$$M_{t_i-}(M_{t_{i+1}} - M_{t_i}) = \frac{1}{2}M^2_{t_{i+1}} - \frac{1}{2}M^2_{t_i} - \frac{1}{2}(M_{t_{i+1}} - M_{t_i})^2.$$

After pair cancelations we have

$$\begin{aligned} S_n &= \frac{1}{2}\sum_{i=0}^{n-1} M^2_{t_{i+1}} - \frac{1}{2}\sum_{i=0}^{n-1} M^2_{t_i} - \frac{1}{2}\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 \\ &= \frac{1}{2}M^2_{t_n} - \frac{1}{2}\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 \end{aligned}$$

Since $t_n = T$, we get

$$S_n = \frac{1}{2}M_T^2 - \frac{1}{2}\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2.$$

The second term on the right is the quadratic variation of M_t. Using formula (3.9.19) yields that S_n converges in mean square towards $\frac{1}{2}M_T^2 - \frac{1}{2}N_T$, since $N_0 = 0$.

Hence we have arrived at the following formula

$$\boxed{\int_0^T M_{t-}\,dM_t = \frac{1}{2}M_T^2 - \frac{1}{2}N_T.}$$

Similarly, one can obtain

$$\boxed{\int_a^b M_{t-}\,dM_t = \frac{1}{2}(M_b^2 - M_a^2) - \frac{1}{2}(N_b - N_a).}$$

Exercise 4.5.1 (*a*) *Show that* $E\Big[\int_a^b M_{t-}\,dM_t\Big] = 0$,

(*b*) *Find* $Var\Big[\int_a^b M_t\,dM_t\Big]$.

Exercise 4.5.2 *Let ω be a fixed state of the world and assume the sample path $t \to N_t(\omega)$ has a jump in the interval (a,b). Show that the Riemann-Stieltjes integral*

$$\int_a^b N_t(\omega)\,dN_t$$

does not exist.

Exercise 4.5.3 *Let N_{t-} denote the left-hand limit of N_t. Show that N_{t-} is predictable, while N_t is not.*

The previous exercises provide the reason why in the following we shall work with M_{t-} instead of M_t: the integral $\int_a^b M_t\,dN_t$ might not exist, while $\int_a^b M_{t-}\,dN_t$ does exist.

Exercise 4.5.4 *Show that*

$$\int_0^T N_{t-}\,dM_t = \frac{1}{2}(N_t^2 - N_t) - \lambda\int_0^t N_t\,dt.$$

Exercise 4.5.5 *Find the variance of*

$$\int_0^T N_{t-}\,dM_t.$$

The following integrals with respect to a Poisson process N_t are considered in the Riemann-Stieltjes sense.

Proposition 4.5.6 *For any continuous function f we have*

(*a*) $E\Big[\int_0^t f(s)\,dN_s\Big] = \lambda\int_0^t f(s)\,ds$;

(*b*) $E\Big[\Big(\int_0^t f(s)\,dN_s\Big)^2\Big] = \lambda\int_0^t f(s)^2\,ds + \lambda^2\Big(\int_0^t f(s)\,ds\Big)^2$;

(*c*) $E\Big[e^{\int_0^t f(s)\,dN_s}\Big] = e^{\lambda\int_0^t (e^{f(s)}-1)\,ds}$.

Proof: (a) Consider the equidistant partition $0 = s_0 < s_1 < \cdots < s_n = t$, with $s_{k+1} - s_k = \Delta s$. Then

$$\begin{aligned}
E\Big[\int_0^t f(s)\,dN_s\Big] &= \lim_{n\to\infty} E\Big[\sum_{i=0}^{n-1} f(s_i)(N_{s_{i+1}} - N_{s_i})\Big] = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(s_i) E\Big[N_{s_{i+1}} - N_{s_i}\Big] \\
&= \lambda \lim_{n\to\infty} \sum_{i=0}^{n-1} f(s_i)(s_{i+1} - s_i) = \lambda \int_0^t f(s)\,ds.
\end{aligned}$$

(b) Using that N_t is stationary and has independent increments, we have respectively

$$\begin{aligned}
E[(N_{s_{i+1}} - N_{s_i})^2] &= E[N^2_{s_{i+1}-s_i}] = \lambda(s_{i+1} - s_i) + \lambda^2(s_{i+1} - s_i)^2 \\
&= \lambda\,\Delta s + \lambda^2(\Delta s)^2, \\
E[(N_{s_{i+1}} - N_{s_i})(N_{s_{j+1}} - N_{s_j})] &= E[(N_{s_{i+1}} - N_{s_i})]E[(N_{s_{j+1}} - N_{s_j})] \\
&= \lambda(s_{i+1} - s_i)\lambda(s_{j+1} - s_j) = \lambda^2(\Delta s)^2.
\end{aligned}$$

Applying the expectation to the formula

$$\begin{aligned}
\Big(\sum_{i=0}^{n-1} f(s_i)(N_{s_{i+1}} - N_{s_i})\Big)^2 &= \sum_{i=0}^{n-1} f(s_i)^2(N_{s_{i+1}} - N_{s_i})^2 \\
&\quad + 2\sum_{i\neq j} f(s_i)f(s_j)(N_{s_{i+1}} - N_{s_i})(N_{s_{j+1}} - N_{s_j})
\end{aligned}$$

yields

$$\begin{aligned}
E\Big[\Big(\sum_{i=0}^{n-1} f(s_i)(N_{s_{i+1}} - N_{s_i})\Big)^2\Big] &= \sum_{i=0}^{n-1} f(s_i)^2(\lambda\Delta s + \lambda^2(\Delta s)^2) + 2\sum_{i\neq j} f(s_i)f(s_j)\lambda^2(\Delta s)^2 \\
&= \lambda\sum_{i=0}^{n-1} f(s_i)^2\Delta s + \lambda^2\Big[\sum_{i=0}^{n-1} f(s_i)^2(\Delta s)^2 + 2\sum_{i\neq j} f(s_i)f(s_j)(\Delta s)^2\Big] \\
&= \lambda\sum_{i=0}^{n-1} f(s_i)^2\Delta s + \lambda^2\Big(\sum_{i=0}^{n-1} f(s_i)\,\Delta s\Big)^2 \\
&\to \lambda\int_0^t f(s)^2\,ds + \lambda^2\Big(\int_0^t f(s)\,ds\Big)^2, \text{ as } n\to\infty.
\end{aligned}$$

(c) Using that N_t is stationary with independent increments and has the moment generating function $E[e^{kN_t}] = e^{\lambda(e^k-1)t}$, we have

$$\begin{aligned}
E\Big[e^{\int_0^t f(s)\,dN_s}\Big] &= \lim_{n\to\infty} E\Big[e^{\sum_{i=0}^{n-1} f(s_i)(N_{s_{i+1}} - N_{s_i})}\Big] = \lim_{n\to\infty} E\Big[\prod_{i=0}^{n-1} e^{f(s_i)(N_{s_{i+1}} - N_{s_i})}\Big] \\
&= \lim_{n\to\infty} \prod_{i=0}^{n-1} E\Big[e^{f(s_i)(N_{s_{i+1}} - N_{s_i})}\Big] = \lim_{n\to\infty} \prod_{i=0}^{n-1} E\Big[e^{f(s_i)(N_{s_{i+1}-s_i})}\Big] \\
&= \lim_{n\to\infty} \prod_{i=0}^{n-1} e^{\lambda(e^{f(s_i)}-1)(s_{i+1}-s_i)} = \lim_{n\to\infty} e^{\lambda\sum_{i=0}^{n-1}(e^{f(s_i)}-1)(s_{i+1}-s_i)} \\
&= e^{\lambda\int_0^t (e^{f(s)}-1)\,ds}.
\end{aligned}$$

■

Since f is continuous, the Poisson integral $\int_0^t f(s)\,dN_s$ can be computed in terms of the waiting times S_k

$$\int_0^t f(s)\,dN_s = \sum_{k=1}^{N_t} f(S_k).$$

This formula can be used to give a proof for the previous result. For instance, taking the expectation and using conditions over $N_t = n$, yields

$$\begin{aligned}
E\Big[\int_0^t f(s)\,dN_s\Big] &= E\Big[\sum_{k=1}^{N_t} f(S_k)\Big] = \sum_{n\geq 0} E\Big[\sum_{k=1}^{n} f(S_k)|N_t = n\Big]P(N_t = n) \\
&= \sum_{n\geq 0} \frac{n}{t}\int_0^t f(x)\,dx\,\frac{(\lambda t)^n}{n!}e^{-\lambda t} = e^{-\lambda t}\int_0^t f(x)\,dx\,\frac{1}{t}\sum_{n\geq 0}\frac{(\lambda t)^n}{(n-1)!} \\
&= e^{-\lambda t}\int_0^t f(x)\,dx\,\lambda e^{\lambda t} = \lambda\int_0^t f(x)\,dx.
\end{aligned}$$

Exercise 4.5.7 *Solve parts (b) and (c) of Proposition 4.5.6 using a similar idea with the one presented above.*

Exercise 4.5.8 *Show that*

$$E\Big[\Big(\int_0^t f(s)\,dM_s\Big)^2\Big] = \lambda\int_0^t f(s)^2\,ds,$$

where $M_t = N_t - \lambda t$ is the compensated Poisson process.

Exercise 4.5.9 *Prove that*

$$Var\Big(\int_0^t f(s)\,dN_s\Big) = \lambda\int_0^t f(s)^2\,dN_s.$$

Exercise 4.5.10 *Find*

$$E\Big[e^{\int_0^t f(s)\,dM_s}\Big].$$

Proposition 4.5.11 *Let $\mathcal{F}_t = \sigma(N_s; 0 \leq s \leq t)$. Then for any constant c, the process*

$$M_t = e^{cN_t + \lambda(1-e^c)t}, \qquad t \geq 0$$

is an $\mathcal{F}_t$-martingale.

Proof: Let $s < t$. Since $N_t - N_s$ is independent of $\mathcal{F}_s$ and N_t is stationary, we have

$$\begin{aligned}
E[e^{c(N_t-N_s)}|\mathcal{F}_s] &= E[e^{c(N_t-N_s)}] = E[e^{cN_{t-s}}] \\
&= e^{\lambda(e^c-1)(t-s)}.
\end{aligned}$$

On the other side, taking out the predictable part yields

$$E[e^{c(N_t-N_s)}|\mathcal{F}_s] = e^{-cN_s}E[e^{cN_t}|\mathcal{F}_s].$$

Equating the last two relations we arrive at

$$E[e^{cN_t+(1-e^c)t}|\mathcal{F}_s] = e^{cN_s+\lambda(1-e^c)s},$$

which is equivalent with the martingale condition $E[M_t|\mathcal{F}_s] = M_s$. ∎

We shall present an application of the previous result. Consider the waiting time until the nth jump, $S_n = \inf\{t > 0; N_t = n\}$, which is a stopping time, and the filtration $\mathcal{F}_t = \sigma(N_s; 0 \leq s \leq t)$. Since

$$M_t = e^{cN_t+\lambda(1-e^c)t}$$

is an $\mathcal{F}_t$-martingale, by the Optional Stopping Theorem (Theorem 3.2.1) we have $E[M_{S_n}] = E[M_0] = 1$, which is equivalent with $E[e^{\lambda(1-e^c)S_n}] = e^{-cn}$. Substituting $s = -\lambda(1-e^c)$, then $c = \ln(1+\frac{s}{\lambda})$. Since $s, \lambda > 0$, then $c > 0$. The previous expression becomes

$$E[e^{-sS_n}] = e^{-n\ln(1+\frac{s}{\lambda})} = \Big(\frac{\lambda}{\lambda+s}\Big)^x.$$

Since the expectation on the left side is the Laplace transform of the probability density of S_n, then

$$\begin{aligned} p(S_n) &= \mathcal{L}^{-1}\{E[e^{-sS_n}]\} = \mathcal{L}^{-1}\Big\{\Big(\frac{\lambda}{\lambda+s}\Big)^x\Big\} \\ &= \frac{e^{-t\lambda}t^{n-1}\lambda^n}{\Gamma(n)}, \end{aligned}$$

which shows that S_n has a gamma distribution.

4.6 The distribution function of $X_T = \int_0^T g(t)\,dN_t$

In this section we consider the function $g(t)$ continuous. Let $S_1 < S_2 < \cdots < S_{N_t}$ denote the waiting times until time t. Since the increments dN_t are equal to 1 at S_k and 0 otherwise, the integral can be written as

$$X_T = \int_0^T g(t)\,dN_t = g(S_1) + \cdots + g(S_{N_t}).$$

The distribution function of the random variable $X_T = \int_0^T g(t)\,dN_t$ can be obtained conditioning over the N_t

$$\begin{aligned} P(X_T \leq u) &= \sum_{k\geq 0} P(X_T \leq u|N_T = k)\,P(N_T = k) \\ &= \sum_{k\geq 0} P(g(S_1) + \cdots + g(S_{N_t}) \leq u|N_T = k)\,P(N_T = k) \\ &= \sum_{k\geq 0} P(g(S_1) + \cdots + g(S_k) \leq u)\,P(N_T = k). \end{aligned} \tag{4.6.5}$$

Considering $S_1, S_2, \cdots, S_k$ independent and uniformly distributed over the interval $[0, T]$, we have

$$P\big(g(S_1)+\cdots+g(S_k)\leq u\big)=\int_{D_k}\frac{1}{T^k}\,dx_1\cdots dx_k=\frac{vol(D_k)}{T^k},$$

where

$$D_k=\{g(x_1)+g(x_2)+\cdots+g(x_k)\leq u\}\cap\{0\leq x_1,\cdots,x_k\leq T\}.$$

Substituting back in (4.6.5) yields

$$\begin{aligned}P(X_T\leq u)&=\sum_{k\geq 0}P(g(S_1)+\cdots+g(S_k)\leq u)\,P(N_T=k)\\&=\sum_{k\geq 0}\frac{vol(D_k)}{T^k}\frac{\lambda^kT^k}{k!}e^{-\lambda T}=e^{-\lambda T}\sum_{k\geq 0}\frac{\lambda^k vol(D_k)}{k!}.\end{aligned}\tag{4.6.6}$$

In general, the volume of the k-dimensional solid D_k is not obvious easy to obtain. However, there are simple cases when this can be computed explicitly.

A Particular Case. We shall do an explicit computation of the partition function of $X_T=\int_0^T s^2\,dN_s$. In this case the solid D_k is the intersection between the k-dimensional ball of radius $\sqrt{u}$ centered at the origin and the k-dimensional cube $[0,T]^k$. There are three possible shapes for D_k, which depend on the size of $\sqrt{u}$:

(a) if $0\leq\sqrt{u}<T$, then D_k is a $\frac{1}{2^k}$-part of a k-dimensional sphere;

(b) if $T\leq\sqrt{u}<T\sqrt{k}$, then D_k has a complicated shape;

(c) if $T\sqrt{k}\leq\sqrt{u}$, then D_k is the entire k-dimensional cube, and then $vol(D_k)=T^k$.

Since the volume of the k-dimensional ball of radius R is given by $\dfrac{\pi^{k/2}R^k}{\Gamma(\frac{k}{2}+1)}$, then the volume of D_k in case (**a**) becomes

$$vol(D_k)=\frac{\pi^{k/2}u^{k/2}}{2^k\Gamma(\frac{k}{2}+1)}.$$

Substituting in (4.6.6) yields

$$P(X_T\leq u)=e^{-\lambda T}\sum_{k\geq 0}\frac{(\lambda^2\pi u)^{k/2}}{k!\Gamma(\frac{k}{2}+1)},\qquad 0\leq\sqrt{u}<T.$$

It is worth noting that for $u\to\infty$, the inequality $T\sqrt{k}\leq\sqrt{u}$ is satisfied for all $k\geq 0$; hence relation (4.6.6) yields

$$\lim_{u\to\infty}P(X_T\leq u)=e^{-\lambda T}\sum_{k\geq 0}\frac{\lambda^kT^k}{k!}=e^{-kT}e^{kT}=1.$$

The computation in case (**b**) is more complicated and will be omitted.

Exercise 4.6.1 *Calculate the expectation* $E\Big[\int_0^T e^{ks}\,dN_s\Big]$ *and the variance* $Var\Big(\int_0^T e^{ks}\,dN_s\Big)$.

Exercise 4.6.2 *Compute the distribution function of* $X_t=\int_0^T s\,dN_s$.

Chapter 5

Stochastic Differentiation

5.1 Differentiation Rules

Most stochastic processes are not differentiable. For instance, the Brownian motion process W_t is a continuous process which is nowhere differentiable. Hence, derivatives like $\frac{dW_t}{dt}$ do not make sense in stochastic calculus. The only quantities allowed to be used are the infinitesimal changes of the process, in our case dW_t.

The infinitesimal change of a process
The change in the process X_t between instances t and $t + \Delta t$ is given by $\Delta X_t = X_{t+\Delta t} - X_t$. When Δt is infinitesimally small, we obtain the infinitesimal change of a process X_t

$$\boxed{dX_t = X_{t+dt} - X_t.}$$

Sometimes it is useful to use the equivalent formula $X_{t+dt} = X_t + dX_t$.

5.2 Basic Rules

The following rules are the analog of some familiar differentiation rules from elementary Calculus.

The constant multiple rule
If X_t is a stochastic process and c is a constant, then

$$\boxed{d(c\,X_t) = c\,dX_t.}$$

The verification follows from a straightforward application of the infinitesimal change formula

$$d(c\,X_t) \quad = \quad c\,X_{t+dt} - c\,X_t = c(X_{t+dt} - X_t) = c\,dX_t.$$

The sum rule
If X_t and Y_t are two stochastic processes, then

$$\boxed{d(X_t + Y_t) = dX_t + dY_t.}$$

The verification is as in the following:

$$\begin{aligned} d(X_t + Y_t) &= (X_{t+dt} + Y_{t+dt}) - (X_t + Y_t) \\ &= (X_{t+dt} - X_t) + (Y_{t+dt} - Y_t) \\ &= dX_t + dY_t. \end{aligned}$$

The difference rule
If X_t and Y_t are two stochastic processes, then

$$\boxed{d(X_t - Y_t) = dX_t - dY_t.}$$

The proof is similar to the one for the sum rule.

The product rule
If X_t and Y_t are two stochastic processes, then

$$\boxed{d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t.}$$

The proof is as follows:

$$\begin{aligned} d(X_t\, Y_t) &= X_{t+dt} Y_{t+dt} - X_t Y_t \\ &= X_t(Y_{t+dt} - Y_t) + Y_t(X_{t+dt} - X_t) + (X_{t+dt} - X_t)(Y_{t+dt} - Y_t) \\ &= X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t, \end{aligned}$$

where the second identity is verified by direct computation.

If the process X_t is replaced by the deterministic function $f(t)$, then the aforementioned formula becomes

$$\boxed{d(f(t)Y_t) = f(t)\, dY_t + Y_t\, df(t) + df(t)\, dY_t.}$$

Since in most practical cases the process Y_t is an Ito diffusion

$$dY_t = a(t, W_t)dt + b(t, W_t)dW_t,$$

using the relations $dt\, dW_t = dt^2 = 0$, the last term vanishes

$$df(t)\, dY_t = f'(t)dtdY_t = 0,$$

and hence

$$\boxed{d(f(t)Y_t) = f(t)\, dY_t + Y_t\, df(t).}$$

This relation looks like the usual product rule.

The quotient rule
If X_t and Y_t are two stochastic processes, then

$$\boxed{d\Big(\frac{X_t}{Y_t}\Big) = \frac{Y_t dX_t - X_t dY_t - dX_t dY_t}{Y_t^2} + \frac{X_t}{Y_t^3}(dY_t)^2.}$$

The proof follows from Ito's formula and will be addressed in section 5.3.3.

When the process Y_t is replaced by the deterministic function $f(t)$, and X_t is an Ito diffusion, then the previous formula becomes

$$\boxed{d\Big(\frac{X_t}{f(t)}\Big) = \frac{f(t)dX_t - X_t df(t)}{f(t)^2}.}$$

Example 5.2.1 *We shall show that*

$$\boxed{d(W_t^2) = 2W_t\, dW_t + dt.}$$

Applying the product rule and the fundamental relation $(dW_t)^2 = dt$, yields

$$d(W_t^2) = W_t\, dW_t + W_t\, dW_t + dW_t\, dW_t = 2W_t\, dW_t + dt.$$

Example 5.2.2 *Show that*

$$\boxed{d(W_t^3) = 3W_t^2\, dW_t + 3W_t dt.}$$

Applying the product rule and the previous exercise yields

$$\begin{aligned} d(W_t^3) &= d(W_t \cdot W_t^2) = W_t d(W_t^2) + W_t^2\, dW_t + d(W_t^2)\, dW_t \\ &= W_t(2W_t\, dW_t + dt) + W_t^2\, dW_t + dW_t(2W_t\, dW_t + dt) \\ &= 2W_t^2\, dW_t + W_t\, dt + W_t^2\, dW_t + 2W_t(dW_t)^2 + dt\, dW_t \\ &= 3W_t^2\, dW_t + 3W_t\, dt, \end{aligned}$$

where we used $(dW_t)^2 = dt$ and $dt\, dW_t = 0$.

Example 5.2.3 *Show that* $d(tW_t) = W_t\, dt + t\, dW_t$.

Using the product rule and $dt\, dW_t = 0$, we get

$$\begin{aligned} d(tW_t) &= W_t\, dt + t\, dW_t + dt\, dW_t \\ &= W_t\, dt + t\, dW_t. \end{aligned}$$

Example 5.2.4 *Let* $Z_t = \int_0^t W_u\, du$ *be the integrated Brownian motion. Show that*

$$\boxed{dZ_t = W_t\, dt.}$$

The infinitesimal change of Z_t is

$$dZ_t = Z_{t+dt} - Z_t = \int_t^{t+dt} W_s\, ds = W_t\, dt,$$

since W_s is a continuous function in s.

Example 5.2.5 *Let* $A_t = \frac{1}{t} Z_t = \frac{1}{t}\int_0^t W_u\, du$ *be the average of the Brownian motion on the time interval* $[0, t]$. *Show that*

$$\boxed{dA_t = \frac{1}{t}\Big(W_t - \frac{1}{t} Z_t\Big)\, dt.}$$

We have

$$\begin{aligned} dA_t &= d\Big(\frac{1}{t}\Big)\, Z_t + \frac{1}{t}\, dZ_t + d\Big(\frac{1}{t}\Big)\, dZ_t \\ &= \frac{-1}{t^2} Z_t\, dt + \frac{1}{t} W_t\, dt + \frac{-1}{t^2} W_t \underbrace{dt^2}_{=0} \\ &= \frac{1}{t}\Big(W_t - \frac{1}{t} Z_t\Big)\, dt. \end{aligned}$$

Exercise 5.2.1 *Let* $G_t = \frac{1}{t}\int_0^t e^{W_u}\, du$ *be the average of the geometric Brownian motion on* $[0, t]$. *Find* dG_t.

5.3 Ito's Formula

Ito's formula is the analog of the chain rule from elementary Calculus. We shall start by reviewing a few concepts regarding function approximations.

Let f be a differentiable function of a real variable x. Let x_0 be fixed and consider the changes $\Delta x = x - x_0$ and $\Delta f(x) = f(x) - f(x_0)$. It is known from Calculus that the following second order Taylor approximation holds

$$\Delta f(x) = f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + O(\Delta x)^3.$$

When x is infinitesimally close to x_0, we replace Δx by the differential dx and obtain

$$df(x) = f'(x)dx + \frac{1}{2}f''(x)(dx)^2 + O(dx)^3. \tag{5.3.1}$$

In the elementary Calculus, all terms involving terms of equal or higher order to dx^2 are neglected; then the aforementioned formula becomes

$$df(x) = f'(x)dx.$$

Now, if we consider $x = x(t)$ to be a differentiable function of t, substituting into the previous formula we obtain the differential form of the well known chain rule

$$df\big(x(t)\big) = f'\big(x(t)\big)dx(t) = f'\big(x(t)\big)x'(t)\,dt.$$

We shall present a similar formula for the stochastic environment. In this case the deterministic function $x(t)$ is replaced by a stochastic process X_t. The composition between the differentiable function f and the process X_t is denoted by $F_t = f(X_t)$. Since the increments involving powers of dt^2 or higher are neglected, we may assume that the same holds true for the increment dX_t. Then the expression (5.3.1) becomes

$$\boxed{dF_t = f'\big(X_t\big)dX_t + \frac{1}{2}f''\big(X_t\big)\big(dX_t\big)^2.} \tag{5.3.2}$$

In the computation of dX_t we may take into the account stochastic relations such as $dW_t^2 = dt$, or $dt\,dW_t = 0$.

5.3.1 Ito's formula for diffusions

The previous formula is a general case of *Ito's formula*. However, in most cases the increments dX_t are given by some particular relations. An important case is when the increment is given by

$$dX_t = a(W_t, t)dt + b(W_t, t)dW_t.$$

A process X_t satisfying this relation is called an *Ito diffusion*.

Theorem 5.3.1 (Ito's formula for diffusions) *If X_t is an Ito diffusion, and $F_t = f(X_t)$, then*

$$\boxed{dF_t = \Big[a(W_t, t)f'(X_t) + \frac{b(W_t, t)^2}{2}f''(X_t)\Big]dt + b(W_t, t)f'(X_t)\,dW_t.} \tag{5.3.3}$$

Proof: We shall provide a formal proof. Using relations $dW_t^2 = dt$ and $dt^2 = dW_t\, dt = 0$, we have

$$\begin{aligned}(dX_t)^2 &= \Big(a(W_t,t)dt + b(W_t,t)dW_t\Big)^2 \\ &= a(W_t,t)^2 dt^2 + 2a(W_t,t)b(W_t,t)dW_t dt + b(W_t,t)^2 dW_t^2 \\ &= b(W_t,t)^2 dt.\end{aligned}$$

Substituting into (5.3.2) yields

$$\begin{aligned}dF_t &= f'\big(X_t\big)dX_t + \frac{1}{2}f''\big(X_t\big)\big(dX_t\big)^2 \\ &= f'\big(X_t\big)\Big(a(W_t,t)dt + b(W_t,t)dW_t\Big) + \frac{1}{2}f''\big(X_t\big)b(W_t,t)^2 dt \\ &= \Big[a(W_t,t)f'(X_t) + \frac{b(W_t,t)^2}{2}f''(X_t)\Big]dt + b(W_t,t)f'(X_t)\, dW_t.\end{aligned}$$

■

In the case $X_t = W_t$ we obtain the following consequence:

Corollary 5.3.2 *Let $F_t = f(W_t)$. Then*

$$\boxed{dF_t = \frac{1}{2}f''(W_t)dt + f'(W_t)\, dW_t.} \tag{5.3.4}$$

Particular cases

1. If $f(x) = x^\alpha$, with α constant, then $f'(x) = \alpha x^{\alpha-1}$ and $f''(x) = \alpha(\alpha-1)x^{\alpha-2}$. Then (5.3.4) becomes the following useful formula

$$\boxed{d(W_t^\alpha) = \frac{1}{2}\alpha(\alpha-1)W_t^{\alpha-2}dt + \alpha W_t^{\alpha-1}\, dW_t.}$$

A couple of useful cases easily follow:

$$\begin{aligned}d(W_t^2) &= 2W_t\, dW_t + dt \\ d(W_t^3) &= 3W_t^2\, dW_t + 3W_t dt.\end{aligned}$$

2. If $f(x) = e^{kx}$, with k constant, $f'(x) = ke^{kx}$, $f''(x) = k^2 e^{kx}$. Therefore

$$\boxed{d(e^{kW_t}) = ke^{kW_t} dW_t + \frac{1}{2}k^2 e^{kW_t}\, dt.}$$

In particular, for $k = 1$, we obtain the increments of a geometric Brownian motion

$$d(e^{W_t}) = e^{W_t} dW_t + \frac{1}{2}e^{W_t}\, dt.$$

3. If $f(x) = \sin x$, then

$$\boxed{d(\sin W_t) = \cos W_t\, dW_t - \frac{1}{2}\sin W_t\, dt.}$$

Exercise 5.3.3 *Use the previous rules to find the following increments*

(a) $d(W_t e^{W_t})$

(b) $d(3W_t^2 + 2e^{5W_t})$

(c) $d(e^{t+W_t^2})$

(d) $d\big((t+W_t)^n\big)$.

(e) $d\Big(\frac{1}{t}\int_0^t W_u\,du\Big)$

(f) $d\Big(\frac{1}{t^\alpha}\int_0^t e^{W_u}\,du\Big)$, *where α is a constant.*

In the case when the function $f = f(t,x)$ is also time dependent, the analog of (5.3.1) is given by

$$df(t,x) = \partial_t f(t,x)dt + \partial_x f(t,x)dx + \frac{1}{2}\partial_x^2 f(t,x)(dx)^2 + O(dx)^3 + O(dt)^2. \qquad (5.3.5)$$

Substituting $x = X_t$ yields

$$df(t,X_t) = \partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t + \frac{1}{2}\partial_x^2 f(t,X_t)(dX_t)^2. \qquad (5.3.6)$$

If X_t is an Ito diffusion we obtain an extra-term in formula (5.3.3)

$$\begin{aligned} dF_t &= \Big[\partial_t f(t,X_t) + a(W_t,t)\partial_x f(t,X_t) + \frac{b(W_t,t)^2}{2}\partial_x^2 f(t,X_t)\Big]dt \\ &\quad + b(W_t,t)\partial_x f(t,X_t)\,dW_t. \end{aligned} \qquad (5.3.7)$$

Exercise 5.3.4 *Show that*

$$d(tW_t^2) = (t + W_t^2)dt + 2tW_t\,dW_t.$$

Exercise 5.3.5 *Find the following increments*

(a) $d(tW_t)$ (c) $d(t^2\cos W_t)$

(b) $d(e^t W_t)$ (d) $d(\sin t\, W_t^2)$.

5.3.2 Ito's formula for Poisson processes

Consider the process $F_t = F(M_t)$, where $M_t = N_t - \lambda t$ is the compensated Poisson process. Ito's formula for the process F_t takes the following integral form. For a proof the reader can consult Kuo [**?**].

Proposition 5.3.6 *Let F be a twice differentiable function. Then for any $a < t$ we have*

$$F_t = F_a + \int_a^t F'(M_{s-})\,dM_s + \sum_{a<s\le t}\Big(\Delta F(M_s) - F'(M_{s-})\Delta M_s\Big),$$

where $\Delta M_s = M_s - M_{s-}$ and $\Delta F(M_s) = F(M_s) - F(M_{s-})$.

We shall apply the aforementioned result for the case $F_t = F(M_t) = M_t^2$. We have

$$M_t^2 = M_a^2 + 2\int_a^t M_{s-}\,dM_s + \sum_{a<s\le t}\Big(M_s^2 - M_{s-}^2 - 2M_{s-}(M_s - M_{s-})\Big). \tag{5.3.8}$$

Since the jumps in N_s are of size 1, we have $(\Delta N_s)^2 = \Delta N_s$. Since the difference of the processes M_s and N_s is continuous, then $\Delta M_s = \Delta N_s$. Using these formulas we have

$$\begin{aligned}
\Big(M_s^2 - M_{s-}^2 - 2M_{s-}(M_s - M_{s-})\Big) &= (M_s - M_{s-})\Big(M_s + M_{s-} - 2M_{s-}\Big) \\
&= (M_s - M_{s-})^2 = (\Delta M_s)^2 = (\Delta N_s)^2 \\
&= \Delta N_s = N_s - N_{s-}.
\end{aligned}$$

Since the sum of the jumps between s and t is $\sum_{a<s\le t}\Delta N_s = N_t - N_a$, formula (5.3.8) becomes

$$M_t^2 = M_a^2 + 2\int_a^t M_{s-}\,dM_s + N_t - N_a. \tag{5.3.9}$$

The differential form is

$$d(M_t^2) = 2M_{t-}\,dM_t + dN_t,$$

which is equivalent with

$$d(M_t^2) = (1 + 2M_{t-})\,dM_t + \lambda dt,$$

since $dN_t = dM_t + \lambda dt$.

Exercise 5.3.7 *Show that*

$$\int_0^T M_{t-}\,dM_t = \frac{1}{2}(M_T^2 - N_T).$$

Exercise 5.3.8 *Use Ito's formula for the Poison process to find the conditional expectation $E[M_t^2|\mathcal{F}_s]$ for $s < t$.*

5.3.3 Ito's multidimensional formula

If the process F_t depends on several Ito diffusions, say $F_t = f(t, X_t, Y_t)$, then a similar formula to (5.3.7) leads to

$$\begin{aligned}
dF_t &= \frac{\partial f}{\partial t}(t,X_t,Y_t)dt + \frac{\partial f}{\partial x}(t,X_t,Y_t)dX_t + \frac{\partial f}{\partial y}(t,X_t,Y_t)dY_t \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t,Y_t)(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t,X_t,Y_t)(dY_t)^2 \\
&\quad + \frac{\partial^2 f}{\partial x \partial y}(t,X_t,Y_t)dX_t\,dY_t.
\end{aligned}$$

Particular cases

In the case when $F_t = f(X_t, Y_t)$, with $X_t = W_t^1$, $Y_t = W_t^2$ independent Brownian motions, we have

$$\begin{aligned}
dF_t &= \frac{\partial f}{\partial x}dW_t^1 + \frac{\partial f}{\partial y}dW_t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dW_t^1)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dW_t^2)^2 \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}dW_t^1\,dW_t^2 \\
&= \frac{\partial f}{\partial x}dW_t^1 + \frac{\partial f}{\partial y}dW_t^2 + \frac{1}{2}\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big)dt
\end{aligned}$$

The expression

$$\Delta f = \frac{1}{2}\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big)$$

is called the *Laplacian* of f. We can rewrite the previous formula as

$$dF_t \quad = \quad \frac{\partial f}{\partial x} dW_t^1 + \frac{\partial f}{\partial y} dW_t^2 + \Delta f \, dt$$

A function f with $\Delta f = 0$ is called *harmonic*. The aforementioned formula in the case of harmonic functions takes the simple form

$$\boxed{dF_t = \frac{\partial f}{\partial x} dW_t^1 + \frac{\partial f}{\partial y} dW_t^2.} \qquad (5.3.10)$$

Exercise 5.3.9 *Let W_t^1, W_t^2 be two independent Brownian motions. If the function f is harmonic, show that $F_t = f(W_t^1, W_t^2)$ is a martingale. Is the converse true?*

Exercise 5.3.10 *Use the previous formulas to find dF_t in the following cases*

(a) $\quad F_t = (W_t^1)^2 + (W_t^2)^2$

(b) $\quad F_t = \ln[(W_t^1)^2 + (W_t^2)^2]$.

Exercise 5.3.11 *Consider the Bessel process $R_t = \sqrt{(W_t^1)^2 + (W_t^2)^2}$, where W_t^1 and W_t^2 are two independent Brownian motions. Prove that*

$$dR_t = \frac{1}{2R_t} dt + \frac{W_t^1}{R_t} dW_t^1 + \frac{W_t^2}{R_t} dW_t^2.$$

Example 5.3.1 (The product rule) *Let X_t and Y_t be two processes. Show that*

$$d(X_t Y_t) = Y_t dX_t + X_t dY_t + dX_t dY_t.$$

Consider the function $f(x, y) = xy$. Since $\partial_x f = y$, $\partial_y f = x$, $\partial_x^2 f = \partial_y^2 f = 0$, $\partial_x \partial_y = 1$, then Ito's multidimensional formula yields

$$\begin{aligned} d(X_t Y_t) &= d\big(f(X, Y_t)\big) = \partial_x f \, dX_t + \partial_y f \, dY_t \\ &\quad + \frac{1}{2}\partial_x^2 f (dX_t)^2 + \frac{1}{2}\partial_y^2 f (dY_t)^2 + \partial_x \partial_y f \, dX_t dY_t \\ &= Y_t dX_t + X_t dY_t + dX_t dY_t. \end{aligned}$$

Example 5.3.2 (The quotient rule) *Let X_t and Y_t be two processes. Show that*

$$d\Big(\frac{X_t}{Y_t}\Big) = \frac{Y_t dX_t - X_t dY_t - dX_t dY_t}{Y_t^2} + \frac{X_t}{Y_t^2}(dY_t)^2.$$

Consider the function $f(x, y) = \frac{x}{y}$. Since $\partial_x f = \frac{1}{y}$, $\partial_y f = -\frac{x}{y^2}$, $\partial_x^2 f = 0$, $\partial_y^2 f = -\frac{x}{y^2}$, $\partial_x \partial_y = \frac{1}{y^2}$, then applying Ito's multidimensional formula yields

$$\begin{aligned} d\Big(\frac{X_t}{Y_t}\Big) &= d\big(f(X, Y_t)\big) = \partial_x f \, dX_t + \partial_y f \, dY_t \\ &\quad + \frac{1}{2}\partial_x^2 f (dX_t)^2 + \frac{1}{2}\partial_y^2 f (dY_t)^2 + \partial_x \partial_y f \, dX_t dY_t \\ &= \frac{1}{Y_t} dX_t - \frac{X_t}{Y_t^2} dY_t - \frac{1}{Y_t^2} dX_t dY_t \\ &= \frac{Y_t dX_t - X_t dY_t - dX_t dY_t}{Y_t^2} + \frac{X_t}{Y_t^2}(dY_t)^2. \end{aligned}$$

Chapter 6

Stochastic Integration Techniques

Computing a stochastic integral starting from the definition of the Ito integral is a quite inefficient method. Like in the elementary Calculus, several methods can be developed to compute stochastic integrals. In order to keep the analogy with the elementary Calculus, we have called them *Fundamental Theorem of Stochastic Calculus* and *Integration by Parts*. The integration by substitution is more complicated in the stochastic environment and we have considered only a particular case of it, which we called *The method of heat equation*.

6.1 Fundamental Theorem of Stochastic Calculus

Consider a process X_t whose increments satisfy the equation $dX_t = f(t, W_t)dW_t$. Integrating formally between a and t yields

$$\int_a^t dX_s = \int_a^t f(s, W_s)dW_s. \tag{6.1.1}$$

The integral on the left side can be computed as in the following. If we consider the partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$, then

$$\int_a^t dX_s = \text{ms-}\lim_{n\to\infty} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j}) = X_t - X_a,$$

since we canceled the terms in pairs. Substituting into formula (6.1.1) yields $X_t = X_a + \int_a^t f(s, W_s)dW_s$, and hence $dX_t = d\Big(\int_a^t f(s, W_s)dW_s\Big)$, since X_a is a constant.

Theorem 6.1.1 (The Fundamental Theorem of Stochastic Calculus)
(i) For any $a < t$, we have

$$\boxed{d\Big(\int_a^t f(s, W_s)dW_s\Big) = f(t, W_t)dW_t.}$$

(ii) If Y_t is a stochastic process, such that $Y_t dW_t = dF_t$, then

$$\boxed{\int_a^b Y_t\, dW_t = F_b - F_a.}$$

We shall provide a few applications of the aforementioned theorem.

Example 6.1.1 *Verify the stochastic formula*

$$\int_0^t W_s\,dW_s = \frac{W_t^2}{2} - \frac{t}{2}.$$

Let $X_t = \int_0^t W_s\,dW_s$ and $Y_t = \dfrac{W_t^2}{2} - \dfrac{t}{2}$. From Ito's formula

$$dY_t = d\Big(\frac{W_t^2}{2}\Big) - d\Big(\frac{t}{2}\Big) = \frac{1}{2}(2W_t\,dW_t + dt) - \frac{1}{2}dt = W_t\,dW_t,$$

and from the Fundamental Theorem of Stochastic Calculus

$$dX_t = d\Big(\int_0^t W_s\,dW_s\Big) = W_t\,dW_t.$$

Hence $dX_t = dY_t$, or $d(X_t - Y_t) = 0$. Since the process $X_t - Y_t$ has zero increments, then $X_t - Y_t = c$, constant. Taking $t = 0$, yields

$$c = X_0 - Y_0 = \int_0^0 W_s\,dW_s - \Big(\frac{W_0^2}{2} - \frac{0}{2}\Big) = 0,$$

and hence $c = 0$. It follows that $X_t = Y_t$, which verifies the desired relation.

Example 6.1.2 *Verify the formula*

$$\int_0^t sW_s\,dW_s = \frac{t}{2}\Big(W_t^2 - \frac{t}{2}\Big) - \frac{1}{2}\int_0^t W_s^2\,ds.$$

Consider the stochastic processes $X_t = \int_0^t sW_s\,dW_s$, $Y_t = \dfrac{t}{2}\Big(W_t^2 - 1\Big)$, and $Z_t = \frac{1}{2}\int_0^t W_s^2\,ds$. The Fundamental Theorem yields

$$\begin{aligned} dX_t &= tW_t\,dW_t \\ dZ_t &= \frac{1}{2}W_t^2 dt. \end{aligned}$$

Applying Ito's formula, we get

$$\begin{aligned} dY_t &= d\Big(\frac{t}{2}\Big(W_t^2 - \frac{t}{2}\Big)\Big) = \frac{1}{2}d(tW_t^2) - d\Big(\frac{t^2}{4}\Big) \\ &= \frac{1}{2}\Big[(t + W_t^2)dt + 2tW_t\,dW_t\Big] - \frac{1}{2}tdt \\ &= \frac{1}{2}W_t^2 dt + tW_t\,dW_t. \end{aligned}$$

We can easily see that

$$dX_t = dY_t - dZ_t.$$

This implies $d(X_t - Y_t + Z_t) = 0$, i.e. $X_t - Y_t + Z_t = c$, constant. Since $X_0 = Y_0 = Z_0 = 0$, it follows that $c = 0$. This proves the desired relation.

Example 6.1.3 *Show that*

$$\int_0^t (W_s^2 - s)\, dW_s = \frac{1}{3}W_t^3 - tW_t.$$

Consider the function $f(t,x) = \frac{1}{3}x^3 - tx$, and let $F_t = f(t, W_t)$. Since $\partial_t f = -x$, $\partial_x f = x^2 - t$, and $\partial_x^2 f = 2x$, then Ito's formula provides

$$\begin{aligned} dF_t &= \partial_t f\, dt + \partial_x f\, dW_t + \frac{1}{2}\partial_x^2 f\, (dW_t)^2 \\ &= -W_t dt + (W_t^2 - t)\, dW_t + \frac{1}{2}2W_t\, dt \\ &= (W_t^2 - t) dW_t. \end{aligned}$$

From the Fundamental Theorem we get

$$\int_0^t (W_s^2 - s)\, dW_s = \int_0^t dF_s = F_t - F_0 = F_t = \frac{1}{3}W_t^3 - tW_t.$$

6.2 Stochastic Integration by Parts

Consider the process $F_t = f(t)g(W_t)$, with f and g differentiable. Using the product rule yields

$$\begin{aligned} dF_t &= df(t)\, g(W_t) + f(t)\, dg(W_t) \\ &= f'(t)g(W_t)dt + f(t)\big(g'(W_t)dW_t + \frac{1}{2}g''(W_t)dt\big) \\ &= f'(t)g(W_t)dt + \frac{1}{2}f(t)g''(W_t)dt + f(t)g'(W_t)dW_t. \end{aligned}$$

Writing the relation in the integral form, we obtain the first integration by parts formula:

$$\boxed{\int_a^b f(t)g'(W_t)\, dW_t = f(t)g(W_t)\Big|_a^b - \int_a^b f'(t)g(W_t)\, dt - \frac{1}{2}\int_a^b f(t)g''(W_t)\, dt.}$$

This formula is to be used when integrating a product between a function of t and a function of the Brownian motion W_t, for which an antiderivative is known. The following two particular cases are important and useful in applications.

1. If $g(W_t) = W_t$, the aforementioned formula takes the simple form

$$\boxed{\int_a^b f(t)\, dW_t = f(t)W_t\Big|_{t=a}^{t=b} - \int_a^b f'(t)W_t\, dt.} \tag{6.2.2}$$

It is worth noting that the left side is a Wiener integral.

2. If $f(t) = 1$, then the formula becomes

$$\boxed{\int_a^b g'(W_t)\, dW_t = g(W_t)\Big|_{t=a}^{t=b} - \frac{1}{2}\int_a^b g''(W_t)\, dt.} \tag{6.2.3}$$

Application 1 Consider the Wiener integral $I_T = \int_0^T t\,dW_t$. From the general theory, see Proposition 4.4.1, it is known that I is a random variable normally distributed with mean 0 and variance

$$Var[I_T] = \int_0^T t^2\,dt = \frac{T^3}{3}.$$

Recall the definition of integrated Brownian motion

$$Z_t = \int_0^t W_u\,du.$$

Formula (6.2.2) yields a relationship between I and the integrated Brownian motion

$$I_T = \int_0^T t\,dW_t = TW_T - \int_0^T W_t\,dt = TW_T - Z_T,$$

and hence $I_T + Z_T = TW_T$. This relation can be used to compute the covariance between I_T and Z_T.

$$\begin{aligned}
Cov(I_T + Z_T, I_T + Z_T) &= Var[TW_T] \Longleftrightarrow \\
Var[I_T] + Var[Z_T] + 2Cov(I_T, Z_T) &= T^2 Var[W_T] \Longleftrightarrow \\
T^3/3 + T^3/3 + 2Cov(I_T, Z_T) &= T^3 \Longleftrightarrow \\
Cov(I_T, Z_T) &= T^3/6,
\end{aligned}$$

where we used that $Var[Z_T] = T^3/3$. The processes I_t and Z_t are not independent. Their correlation coefficient is 0.5 as the following calculation shows

$$\begin{aligned}
Corr(I_T, Z_T) &= \frac{Cov(I_T, Z_T)}{\Big(Var[I_T]Var[Z_T]\Big)^{1/2}} = \frac{T^3/6}{T^3/3} \\
&= 1/2.
\end{aligned}$$

Application 2 If we let $g(x) = \frac{x^2}{2}$ in formula (6.2.3), we get

$$\boxed{\int_a^b W_t\,dW_t = \frac{W_b^2 - W_a^2}{2} - \frac{1}{2}(b-a).}$$

It is worth noting that letting $a = 0$ and $b = T$, we retrieve a formula that was proved by direct methods in chapter 2

$$\boxed{\int_0^T W_t\,dW_t = \frac{W_T^2}{2} - \frac{T}{2}.}$$

Similarly, if we let $g(x) = \frac{x^3}{3}$ in (6.2.3) yields

$$\boxed{\int_a^b W_t^2\,dW_t = \frac{W_t^3}{3}\Big|_a^b - \int_a^b W_t\,dt.}$$

Application 3
Choosing $f(t) = e^{\alpha t}$ and $g(x) = \cos x$, we shall compute the stochastic integral $\int_0^T e^{\alpha t} \cos W_t \, dW_t$ using the formula of integration by parts

$$\begin{aligned}
\int_0^T e^{\alpha t} \cos W_t \, dW_t &= \int_0^T e^{\alpha t} (\sin W_t)' \, dW_t \\
&= e^{\alpha t} \sin W_t \Big|_0^T - \int_0^T (e^{\alpha t})' \sin W_t \, dt - \frac{1}{2} \int_0^T e^{\alpha t} (\cos W_t)'' \, dt \\
&= e^{\alpha T} \sin W_T - \alpha \int_0^T e^{\alpha t} \sin W_t \, dt + \frac{1}{2} \int_0^T e^{\alpha t} \sin W_t \, dt \\
&= e^{\alpha T} \sin W_T - \Big(\alpha - \frac{1}{2}\Big) \int_0^T e^{\alpha t} \sin W_t \, dt.
\end{aligned}$$

The particular case $\alpha = \frac{1}{2}$ leads to the following exact formula of a stochastic integral

$$\boxed{\int_0^T e^{\frac{t}{2}} \cos W_t \, dW_t = e^{\frac{T}{2}} \sin W_T.} \tag{6.2.4}$$

In a similar way, we can obtain an exact formula for the stochastic integral $\int_0^T e^{\beta t} \sin W_t \, dW_t$ as follows

$$\begin{aligned}
\int_0^T e^{\beta t} \sin W_t \, dW_t &= -\int_0^T e^{\beta t} (\cos W_t)' \, dW_t \\
&= -e^{\beta t} \cos W_t \Big|_0^T + \beta \int_0^T e^{\beta t} \cos W_t \, dt - \frac{1}{2} \int_0^T e^{\beta t} \cos W_t \, dt.
\end{aligned}$$

Taking $\beta = \frac{1}{2}$ yields the closed form formula

$$\boxed{\int_0^T e^{\frac{t}{2}} \sin W_t \, dW_t = 1 - e^{\frac{T}{2}} \cos W_T.} \tag{6.2.5}$$

A consequence of the last two formulas and of Euler's formula

$$e^{iW_t} = \cos W_t + i \sin W_t,$$

is

$$\boxed{\int_0^T e^{\frac{t}{2} + iW_t} \, dW_t = i(1 - e^{\frac{T}{2} + iW_T}).}$$

The proof details are left to the reader.

A general form of the integration by parts formula
In general, if X_t and Y_t are two Ito diffusions, from the product formula

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \, dY_t.$$

Integrating between the limits a and b

$$\int_a^b d(X_t Y_t) = \int_a^b X_t dY_t + \int_a^b Y_t dX_t + \int_a^b dX_t \, dY_t.$$

From the Fundamental Theorem

$$\int_a^b d(X_tY_t) = X_bY_b - X_aY_a,$$

so the previous formula takes the following form of integration by parts

$$\boxed{\int_a^b X_t dY_t = X_bY_b - X_aY_a - \int_a^b Y_t dX_t - \int_a^b dX_t\, dY_t.}$$

This formula is of theoretical value. In practice, the term $dX_t\, dY_t$ needs to be computed using the rules $W_t^2 = dt$, and $dt\, dW_t = 0$.

Exercise 6.2.1 (*a*) *Use integration by parts to get*

$$\int_0^T \frac{1}{1+W_t^2} dW_t = \tan^{-1}(W_T) + \int_0^T \frac{W_t}{(1+W_t^2)^2}\, dt, \quad T > 0.$$

(*b*) *Show that*

$$E[\tan^{-1}(W_T)] = -\int_0^T E\Big[\frac{W_t}{(1+W_t^2)^2}\Big]\, dt.$$

(*c*) *Prove the double inequality*

$$-\frac{3\sqrt{3}}{16} \leq \frac{x}{(1+x^2)^2} \leq \frac{3\sqrt{3}}{16}, \qquad \forall x \in \mathbb{R}.$$

(*d*) *Use part* (*c*) *to obtain*

$$-\frac{3\sqrt{3}}{16} T \leq \int_0^T \frac{W_t}{(1+W_t^2)^2}\, dt \leq \frac{3\sqrt{3}}{16} T.$$

(*e*) *Use part* (*d*) *to get*

$$-\frac{3\sqrt{3}}{16} T \leq E[\tan^{-1}(W_T)] \leq \frac{3\sqrt{3}}{16} T.$$

(*f*) *Does part* (*e*) *contradict the inequality*

$$-\frac{\pi}{2} < \tan^{-1}(W_T) < \frac{\pi}{2}?$$

Exercise 6.2.2 (*a*) *Show the relation*

$$\int_0^T e^{W_t}\, dW_t = e^{W_T} - 1 - \frac{1}{2}\int_0^T e^{W_t}\, dt.$$

(*b*) *Use part* (*a*) *to find* $E[e^{W_t}]$.

Exercise 6.2.3 (*a*) *Use integration by parts to show*

$$\int_0^T W_t e^{W_t}\, dW_t = 1 + W_T e^{W_T} - e^{W_T} - \frac{1}{2}\int_0^T e^{W_t}(1+W_t)\, dt;$$

(*b*) *Use part* (*a*) *to find* $E[W_t e^{W_t}]$;
(*c*) *Show that* $Cov(W_t, e^{W_t}) = te^{t/2}$;
(*d*) *Prove that* $Corr(W_t, e^{W_t}) = \sqrt{\dfrac{t}{e^t - 1}}$, *and compute the limits as* $t \to 0$ *and* $t \to \infty$.

Exercise 6.2.4 *(a) Let $T > 0$. Show the following relation using integration by parts*

$$\int_0^T \frac{2W_t}{1+W_t^2}\,dW_t = \ln(1+W_T^2) - \int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}\,dt.$$

(b) Show that for any real number x the following double inequality holds

$$-\frac{1}{8} \le \frac{1-x^2}{(1+x^2)^2} \le 1.$$

(c) Use part (b) to show that

$$-\frac{1}{8}T \le \int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}\,dt \le T.$$

(d) Use parts (a) and (c) to get

$$-\frac{T}{8} \le E[\ln(1+W_T^2)] \le T.$$

(e) Use Jensen's inequality to get

$$E[\ln(1+W_T^2)] \le \ln(1+T).$$

Does this contradict the upper bound provided in (d)?

6.3 The Heat Equation Method

In elementary Calculus, integration by substitution is the inverse application of the chain rule. In the stochastic environment, this will be the inverse application of Ito's formula. This is difficult to apply in general, but there is a particular case of great importance.

Let $\varphi(t,x)$ be a solution of the equation

$$\partial_t\varphi + \frac{1}{2}\partial_x^2\varphi = 0. \tag{6.3.6}$$

This is called the *heat equation without sources.* The non-homogeneous equation

$$\partial_t\varphi + \frac{1}{2}\partial_x^2\varphi = G(t,x) \tag{6.3.7}$$

is called *heat equation with sources.* The function $G(t,x)$ represents the density of heat sources, while the function $\varphi(t,x)$ is the temperature at point x at time t in a one-dimensional wire. If the heat source is time independent, then $G = G(x)$, i.e. G is a function of x only.

Example 6.3.1 *Find all solutions of the equation (6.3.6) of type $\varphi(t,x) = a(t) + b(x)$.*

Substituting into equation (6.3.6) yields

$$\frac{1}{2}b''(x) = -a'(t).$$

Since the left side is a function of x only, while the right side is a function of variable t, the only where the previous equation is satisfied is when both sides are equal to the same constant C. This is called a separation constant. Therefore $a(t)$ and $b(x)$ satisfy the equations

$$a'(t) = -C, \qquad \frac{1}{2}b''(x) = C.$$

Integrating yields $a(t) = -Ct + C_0$ and $b(x) = Cx^2 + C_1x + C_2$. It follows that

$$\varphi(t,x) = C(x^2 - t) + C_1x + C_3,$$

with C_0, C_1, C_2, C_3 arbitrary constants.

Example 6.3.2 *Find all solutions of the equation (6.3.6) of the type $\varphi(t,x) = a(t)b(x)$.*

Substituting into the equation and dividing by $a(t)b(x)$ yields

$$\frac{a'(t)}{a(t)} + \frac{1}{2}\frac{b''(x)}{b(x)} = 0.$$

There is a separation constant C such that $\dfrac{a'(t)}{a(t)} = -C$ and $\dfrac{b''(x)}{b(x)} = 2C$. There are three distinct cases to discuss:

1. $C = 0$. In this case $a(t) = a_0$ and $b(x) = b_1x + b_0$, with a_0, a_1, b_0, b_1 real constants. Then

$$\varphi(t,x) = a(t)b(x) = c_1x + c_0, \qquad c_0, c_1 \in \mathbb{R}$$

is just a linear function in x.

2. $C > 0$. Let $\lambda > 0$ such that $2C = \lambda^2$. Then $a'(t) = -\frac{\lambda^2}{2}a(t)$ and $b''(x) = \lambda^2 b(x)$, with solutions

$$\begin{aligned} a(t) &= a_0e^{-\lambda^2t/2} \\ b(x) &= c_1e^{\lambda x} + c_2e^{-\lambda x}. \end{aligned}$$

The general solution of (6.3.6) is

$$\varphi(t,x) = e^{-\lambda^2t/2}(c_1e^{\lambda x} + c_2e^{-\lambda x}), \quad c_1, c_2 \in \mathbb{R}.$$

3. $C < 0$. Let $\lambda > 0$ such that $2C = -\lambda^2$. Then $a'(t) = \frac{\lambda^2}{2}a(t)$ and $b''(x) = -\lambda^2 b(x)$. Solving yields

$$\begin{aligned} a(t) &= a_0e^{\lambda^2t/2} \\ b(x) &= c_1\sin(\lambda x) + c_2\cos(\lambda x). \end{aligned}$$

The general solution of (6.3.6) in this case is

$$\varphi(t,x) = e^{\lambda^2t/2}\big(c_1\sin(\lambda x) + c_2\cos(\lambda x)\big), \quad c_1, c_2 \in \mathbb{R}.$$

In particular, the functions x, $x^2 - t$, $e^{x-t/2}$, $e^{-x-t/2}$, $e^{t/2}\sin x$ and $e^{t/2}\cos x$, or any linear combination of them are solutions of the heat equation (6.3.6). However, there are other solutions which are not of the previous type.

Exercise 6.3.1 *Prove that* $\varphi(t,x) = \frac{1}{3}x^3 - tx$ *is a solution of the heat equation (6.3.6).*

Exercise 6.3.2 *Show that* $\varphi(t,x) = t^{-1/2}e^{-x^2/(2t)}$ *is a solution of the heat equation (6.3.6) for* $t > 0$.

Exercise 6.3.3 *Let* $\varphi = u(\lambda)$, *with* $\lambda = \dfrac{x}{2\sqrt{t}}$, $t > 0$. *Show that* φ *satisfies the heat equation (6.3.6) if and only if* $u'' + 2\lambda u' = 0$.

Exercise 6.3.4 *Let* $erfc(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_x^\infty e^{-r^2}\,dr$. *Show that* $\varphi = erfc(x/(2\sqrt{t}))$ *is a solution of the equation (6.3.6).*

Exercise 6.3.5 (the fundamental solution) *Show that* $\varphi(t,x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$, $t > 0$ *satisfies the equation (6.3.6).*

Sometimes it is useful to generate new solutions for the heat equation from other solutions. Below we present a few ways to accomplish this:

(i) by linear combination: if φ_1 and φ_2 are solutions, then $a_1\varphi_1 + a_1\varphi_2$ is a solution, where a_1, a_2 constants.

(ii) by translation: if $\varphi(t,x)$ is a solution, then $\varphi(t-\tau, x-\xi)$ is a solution, where (τ,ξ) is a translation vector.

(iii) by affine transforms: if $\varphi(t,x)$ is a solution, then $\varphi(\lambda\tau, \lambda^2 x)$ is a solution, for any constant λ.

(iv) by differentiation: if $\varphi(t,x)$ is a solution, then $\dfrac{\partial^{n+m}}{\partial^n x\partial^m t}\varphi(t,x)$ is a solution.

(v) by convolution: if $\varphi(t,x)$ is a solution, then so are

$$\int_a^b \varphi(t, x-\xi)f(\xi)\,d\xi$$
$$\int_a^b \varphi(t-\tau, x)g(\tau)\,dt.$$

For more detail on the subject the reader can consult Widder [**?**] and Cannon [**?**].

Theorem 6.3.6 *Let* $\varphi(t,x)$ *be a solution of the heat equation (6.3.6) and denote* $f(t,x) = \partial_x\varphi(t,x)$. *Then*

$$\boxed{\int_a^b f(t, W_t)\,dW_t = \varphi(b, W_b) - \varphi(a, W_a).}$$

Proof: Let $F_t = \varphi(t, W_t)$. Applying Ito's formula we get

$$dF_t = \partial_x\varphi(t,W_t)\,dW_t + \Big(\partial_t\varphi + \frac{1}{2}\partial_x^2\varphi\Big)dt.$$

Since $\partial_t\varphi + \frac{1}{2}\partial_x^2\varphi = 0$ and $\partial_x\varphi(t,W_t) = f(t,W_t)$, we have

$$dF_t = f(t,W_t)\,dW_t.$$

Applying the Fundamental Theorem yields

$$\int_a^b f(t, W_t)\, dW_t = \int_a^b dF_t = F_b - F_a = \varphi(b, W_b) - \varphi(a, W_a).$$

■

Application 6.3.7 *Show that*

$$\int_0^T W_t\, dW_t = \frac{1}{2} W_T^2 - \frac{1}{2} T.$$

Choose the solution of the heat equation (6.3.6) given by $\varphi(t, x) = x^2 - t$. Then $f(t, x) = \partial_x \varphi(t, x) = 2x$. Theorem 6.3.6 yields

$$\int_0^T 2W_t\, dW_t = \int_0^T f(t, W_t)\, dW_t = \varphi(t, x)\Big|_0^T = W_T^2 - T.$$

Dividing by 2 leads to the desired result.

Application 6.3.8 *Show that*

$$\int_0^T (W_t^2 - t)\, dW_t = \frac{1}{3} W_T^3 - T W_T.$$

Consider the function $\varphi(t, x) = \frac{1}{3}x^3 - tx$, which is a solution of the heat equation (6.3.6), see Exercise 6.3.1. Then $f(t, x) = \partial_x \varphi(t, x) = x^2 - t$. Applying Theorem 6.3.6 yields

$$\int_0^T (W_t^2 - t)\, dW_t = \int_0^T f(t, W_t)\, dW_t = \varphi(t, W_t)\Big|_0^T = \frac{1}{3} W_T^3 - T W_T.$$

Application 6.3.9 *Let* $\lambda > 0$. *Prove the identities*

$$\boxed{\int_0^T e^{-\frac{\lambda^2 t}{2} \pm \lambda W_t}\, dW_t = \frac{1}{\pm\lambda}\Big(e^{-\frac{\lambda^2 T}{2} \pm \lambda W_T} - 1\Big).}$$

Consider the function $\varphi(t, x) = e^{-\frac{\lambda^2 t}{2} \pm \lambda x}$, which is a solution of the homogeneous heat equation (6.3.6), see Example 6.3.2. Then $f(t, x) = \partial_x \varphi(t, x) = \pm\lambda e^{-\frac{\lambda^2 t}{2} \pm \lambda x}$. Apply Theorem 6.3.6 to get

$$\int_0^T \pm\lambda e^{-\frac{\lambda^2 t}{2} \pm \lambda x}\, dW_t = \int_0^T f(t, W_t)\, dW_t = \varphi(t, W_t)\Big|_0^T = e^{-\frac{\lambda^2 T}{2} \pm \lambda W_T} - 1.$$

Dividing by the constant $\pm\lambda$ ends the proof.

In particular, for $\lambda = 1$ the aforementioned formula becomes

$$\boxed{\int_0^T e^{-\frac{t}{2} + W_t}\, dW_t = e^{-\frac{T}{2} + W_T} - 1.} \tag{6.3.8}$$

Application 6.3.10 *Let $\lambda > 0$. Prove the identity*

$$\boxed{\int_0^T e^{\frac{\lambda^2 t}{2}} \cos(\lambda W_t)\, dW_t = \frac{1}{\lambda} e^{\frac{\lambda^2 T}{2}} \sin(\lambda W_T).}$$

From the Example 6.3.2 we know that $\varphi(t,x) = e^{\frac{\lambda^2 t}{2}} \sin(\lambda x)$ is a solution of the heat equation. Applying Theorem 6.3.6 to the function $f(t,x) = \partial_x \varphi(t,x) = \lambda e^{\frac{\lambda^2 t}{2}} \cos(\lambda x)$, yields

$$\begin{aligned}
\int_0^T \lambda e^{\frac{\lambda^2 t}{2}} \cos(\lambda W_t)\, dW_t &= \int_0^T f(t,W_t)\, dW_t = \varphi(t,W_t)\Big|_0^T \\
&= e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t)\Big|_0^T = e^{\frac{\lambda^2 T}{2}} \sin(\lambda W_T).
\end{aligned}$$

Divide by λ to end the proof.

If we choose $\lambda = 1$ we recover a result already familiar to the reader from section 6.2

$$\boxed{\int_0^T e^{\frac{t}{2}} \cos(W_t)\, dW_t = e^{\frac{T}{2}} \sin W_T.} \tag{6.3.9}$$

Application 6.3.11 *Let $\lambda > 0$. Show that*

$$\boxed{\int_0^T e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t)\, dW_t = \frac{1}{\lambda}\Big(1 - e^{\frac{\lambda^2 T}{2}} \cos(\lambda W_T)\Big).}$$

Choose $\varphi(t,x) = e^{\frac{\lambda^2 t}{2}} \cos(\lambda x)$ to be a solution of the heat equation. Apply Theorem 6.3.6 for the function $f(t,x) = \partial_x \varphi(t,x) = -\lambda e^{\frac{\lambda^2 t}{2}} \sin(\lambda x)$ to get

$$\begin{aligned}
\int_0^T (-\lambda) e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t)\, dW_t &= \varphi(t,W_t)\Big|_0^T \\
&= e^{\frac{\lambda^2 T}{2}} \cos(\lambda W_t)\Big|_0^T = e^{\frac{\lambda^2 T}{2}} \cos(\lambda W_T) - 1,
\end{aligned}$$

and then divide by $-\lambda$.

Application 6.3.12 *Let $0 < a < b$. Show that*

$$\boxed{\int_a^b t^{-\frac{3}{2}} W_t e^{-\frac{W_t^2}{2t}}\, dW_t = a^{-\frac{1}{2}} e^{-\frac{W_a^2}{2a}} - b^{-\frac{1}{2}} e^{-\frac{W_b^2}{2b}}.} \tag{6.3.10}$$

From Exercise 6.3.2 we have that $\varphi(t,x) = t^{-1/2} e^{-x^2/(2t)}$ is a solution of the homogeneous heat equation. Since $f(t,x) = \partial_x \varphi(t,x) = -t^{-3/2} x e^{-x^2/(2t)}$, applying Theorem 6.3.6 yields to the desired result. The reader can easily fill in the details.

Integration techniques will be used when solving stochastic differential equations in the next chapter.

Exercise 6.3.13 *Find the value of the following stochastic integrals*

(a) $\displaystyle\int_0^1 e^t \cos(\sqrt{2}W_t)\, dW_t$

(b) $\displaystyle\int_0^3 e^{2t} \cos(2W_t)\, dW_t$

(c) $\displaystyle\int_0^4 e^{-t+\sqrt{2}W_t}\, dW_t.$

Exercise 6.3.14 *Let $\varphi(t,x)$ be a solution of the following non-homogeneous heat equation with time-dependent and uniform heat source $G(t)$*

$$\partial_t \varphi + \frac{1}{2}\partial_x^2 \varphi = G(t).$$

Denote $f(t,x) = \partial_x \varphi(t,x)$. Show that

$$\boxed{\int_a^b f(t, W_t)\, dW_t = \varphi(b, W_b) - \varphi(a, W_a) - \int_a^b G(t)\, dt.}$$

How does the formula change if the heat source G is constant?

6.4 Table of Usual Stochastic Integrals

Since there are so many formulas in this chapter that we need to use in next chapters, we shall present them in the following in a single user-friendly table. Let $a < b$ and $0 < T$. Then we have:

1. $\displaystyle\int_a^b dW_t = W_b - W_a;$

2. $\displaystyle\int_0^T W_t\, dW_t = \frac{W_T^2}{2} - \frac{T}{2};$

3. $\displaystyle\int_0^T (W_t^2 - t)\, dW_t = \frac{W_T^2}{3} - TW_T;$

4. $\displaystyle\int_0^T t\, dW_t = TW_T - \int_0^T W_t\, dt, \quad 0 < T;$

5. $\displaystyle\int_0^T W_t^2\, dW_t = \frac{W_T^3}{3} - \int_0^T W_t\, dt;$

6. $\displaystyle\int_0^T e^{\frac{t}{2}} \cos W_t\, dW_t = e^{\frac{T}{2}} \sin W_T;$

7. $\displaystyle\int_0^T e^{\frac{t}{2}} \sin W_t\, dW_t = 1 - e^{\frac{T}{2}} \cos W_T;$

8. $\displaystyle\int_0^T e^{-\frac{t}{2}+W_t} = e^{-\frac{T}{2}+W_T} - 1;$

9. $\displaystyle\int_0^T e^{\frac{\lambda^2 t}{2}} \cos(\lambda W_t)\, dW_t = \frac{1}{\lambda} e^{\frac{\lambda^2 T}{2}} \sin(\lambda W_T);$

10. $\displaystyle\int_0^T e^{\frac{\lambda^2 t}{2}} \sin(\lambda W_t)\, dW_t = \frac{1}{\lambda}\Big(1 - e^{\frac{\lambda^2 T}{2}} \cos(\lambda W_T)\Big);$

11. $\displaystyle\int_0^T e^{-\frac{\lambda^2 t}{2} \pm \lambda W_t}\, dW_t = \frac{1}{\pm\lambda}\Big(e^{-\frac{\lambda^2 T}{2} \pm \lambda W_T} - 1\Big);$

12. $\displaystyle\int_a^b t^{-\frac{3}{2}} W_t e^{-\frac{W_t^2}{2t}}\, dW_t = a^{-\frac{1}{2}} e^{-\frac{W_a^2}{2a}} - b^{-\frac{1}{2}} e^{-\frac{W_b^2}{2b}};$

13. $\displaystyle d\Big(\int_a^t f(s, W_s)\, dW_s\Big) = f(t, W_t)\, dW_t;$

14. $\displaystyle\int_a^b Y_t\, dW_t = F_b - F_a$, if $Y_t dW_t = dF_t;$

15. $\displaystyle\int_a^b f(t)\, dW_t = f(t)W_t\big|_a^b - \int_a^b f'(t) W_t\, dt;$

16. $\displaystyle\int_a^b g'(W_t)\, dW_t = g(W_t)\Big|_a^b - \frac{1}{2}\int_a^b g''(W_t)\, dt.$

Chapter 7

Stochastic Differential Equations

7.1 Definitions and Examples

Let X_t be a continuous stochastic process. If small changes in the process X_t can be written as a linear combination of small changes in t and small increments of the Brownian motion W_t, we may write

$$\boxed{dX_t = a(t, W_t, X_t)dt + b(t, W_t, X_t)\, dW_t} \tag{7.1.1}$$

and call it a *stochastic differential equation*. In fact, this differential relation has the following integral meaning:

$$\boxed{X_t = X_0 + \int_0^t a(s, W_s, X_s)\, ds + \int_0^t b(s, W_s, X_s)\, dW_s,} \tag{7.1.2}$$

where the last integral is taken in the Ito sense. Relation (7.1.2) is taken as the definition for the stochastic differential equation (7.1.1). However, since it is convenient to use stochastic differentials informally, we shall approach stochastic differential equations by analogy with the ordinary differential equations, and try to present the same methods of solving equations in the new stochastic environment.

The functions $a(t, W_t, X_t)$ and $b(t, W_t, X_t)$ are called *drift rate* and *volatility*, respectively. A process X_t is called a (strong) *solution* for the stochastic equation (7.1.1) if it satisfies the equation. We shall start with an example.

Example 7.1.1 (The Brownian bridge) *Let $a, b \in \mathbb{R}$. Show that the process*

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\, dW_s, \quad 0 \le t < 1$$

is a solution of the stochastic differential equation

$$dX_t = \frac{b - X_t}{1-t}dt + dW_t, \qquad 0 \le t < 1, X_0 = a.$$

We shall perform a routine verification to show that X_t is a solution. First we compute the quotient $\dfrac{b - X_t}{1-t}$:

$$
\begin{aligned}
b - X_t &= b - a(1-t) - bt - (1-t)\int_0^t \frac{1}{1-s}\,dW_s \\
&= (b-a)(1-t) - (1-t)\int_0^t \frac{1}{1-s}\,dW_s,
\end{aligned}
$$

and dividing by $1-t$ yields

$$
\frac{b - X_t}{1-t} = b - a - \int_0^t \frac{1}{1-s}\,dW_s. \tag{7.1.3}
$$

Using

$$
d\Big(\int_0^t \frac{1}{1-s}\,dW_s\Big) = \frac{1}{1-t}\,dW_t,
$$

the product rule yields

$$
\begin{aligned}
dX_t &= a\,d(1-t) + bdt + d(1-t)\int_0^t \frac{1}{1-s}\,dW_s + (1-t)d\Big(\int_0^t \frac{1}{1-s}\,dW_s\Big) \\
&= \Big(b - a - \int_0^t \frac{1}{1-s}\,dW_s\Big)dt + dW_t \\
&= \frac{b - X_t}{1-t}dt + dW_t,
\end{aligned}
$$

where the last identity comes from (7.1.3). We just verified that the process X_t is a solution of the given stochastic equation. The question of *how this solution was obtained in the first place*, is the subject of study for the next few sections.

7.2 Finding Mean and Variance from the Equation

For most practical purposes, the most important information one needs to know about a process is its mean and variance. These can be found directly from the stochastic equation in some particular cases without solving explicitly the equation. We shall deal in the present section with this problem.

Taking the expectation in (7.1.2) and using the property of the Ito integral as a zero mean random variable yields

$$
E[X_t] = X_0 + \int_0^t E[a(s, W_s, X_s)]\,ds. \tag{7.2.4}
$$

Applying the Fundamental Theorem of Calculus we obtain

$$
\frac{d}{dt}E[X_t] = E[a(t, W_t, X_t)].
$$

We note that X_t is not differentiable, but its expectation $E[X_t]$ is. This equation can be solved exactly in a few particular cases.

1. If $a(t, W_t, X_t) = a(t)$, then $\frac{d}{dt}E[X_t] = a(t)$ with the exact solution $E[X_t] = X_0 + \int_0^t a(s)\,ds$.

2. If $a(t, W_t, X_t) = \alpha(t)X_t + \beta(t)$, with $\alpha(t)$ and $\beta(t)$ continuous deterministic function, then

$$\frac{d}{dt}E[X_t] = \alpha(t)E[X_t] + \beta(t),$$

which is a linear differential equation in $E[X_t]$. Its solution is given by

$$E[X_t] = e^{A(t)}\Big(X_0 + \int_0^t e^{-A(s)}\beta(s)\,ds\Big), \tag{7.2.5}$$

where $A(t) = \int_0^t \alpha(s)\,ds$. It is worth noting that the expectation $E[X_t]$ does not depend on the volatility term $b(t, W_t, X_t)$.

Exercise 7.2.1 *If $dX_t = (2X_t + e^{2t})dt + b(t, W_t, X_t)dW_t$, then*

$$E[X_t] = e^{2t}(X_0 + t).$$

Proposition 7.2.2 *Let X_t be a process satisfying the stochastic equation*

$$dX_t = \alpha(t)X_t dt + b(t)dW_t.$$

Then the mean and variance of X_t are given by

$$\begin{aligned} E[X_t] &= e^{A(t)}X_0 \\ Var[X_t] &= e^{2A(t)}\int_0^t e^{-A(s)}b^2(s)\,ds, \end{aligned}$$

where $A(t) = \int_0^t \alpha(s)\,ds$.

Proof: The expression of $E[X_t]$ follows directly from formula (7.2.5) with $\beta = 0$. In order to compute the second moment we first compute

$$\begin{aligned} (dX_t)^2 &= b^2(t)\,dt; \\ d(X_t^2) &= 2X_t dX_t + (dX_t)^2 \\ &= 2X_t\big(\alpha(t)X_t dt + b(t)dW_t\big) + b^2(t)dt \\ &= \big(2\alpha(t)X_t^2 + b^2(t)\big)dt + 2b(t)X_t dW_t, \end{aligned}$$

where we used Ito's formula. If we let $Y_t = X_t^2$, the previous equation becomes

$$dY_t = \big(2\alpha(t)Y_t + b^2(t)\big)dt + 2b(t)\sqrt{Y_t}\,dW_t.$$

Applying formula (7.2.5) with $\alpha(t)$ replaced by $2\alpha(t)$ and $\beta(t)$ by $b^2(t)$, yields

$$E[Y_t] = e^{2A(t)}\Big(Y_0 + \int_0^t e^{-2A(s)}b^2(s)\,ds\Big),$$

which is equivalent with

$$E[X_t^2] = e^{2A(t)}\Big(X_0^2 + \int_0^t e^{-2A(s)}b^2(s)\,ds\Big).$$

It follows that the variance is

$$Var[X_t] = E[X_t^2] - (E[X_t])^2 = e^{2A(t)}\int_0^t e^{-2A(s)}b^2(s)\,ds.$$

■

Remark 7.2.3 *We note that the previous equation is of linear type. This shall be solved explicitly in a future section.*

The mean and variance for a given stochastic process can be computed by working out the associated stochastic equation. We shall provide next a few examples.

Example 7.2.1 *Find the mean and variance of e^{kW_t}, with k constant.*

From Ito's formula

$$d(e^{kW_t}) = ke^{kW_t}dW_t + \frac{1}{2}k^2e^{kW_t}dt,$$

and integrating yields

$$e^{kW_t} = 1 + k\int_0^t e^{kW_s}\,dW_s + \frac{1}{2}k^2\int_0^t e^{kW_s}\,ds.$$

Taking the expectations we have

$$E[e^{kW_t}] = 1 + \frac{1}{2}k^2\int_0^t E[e^{kW_s}]\,ds.$$

If we let $f(t) = E[e^{kW_t}]$, then differentiating the previous relations yields the differential equation

$$f'(t) = \frac{1}{2}k^2f(t)$$

with the initial condition $f(0) = E[e^{kW_0}] = 1$. The solution is $f(t) = e^{k^2t/2}$, and hence

$$\boxed{E[e^{kW_t}] = e^{k^2t/2}.}$$

The variance is

$$\begin{aligned} Var(e^{kW_t}) &= E[e^{2kW_t}] - (E[e^{kW_t}])^2 = e^{4k^2t/2} - e^{k^2t} \\ &= e^{k^2t}(e^{k^2t} - 1). \end{aligned}$$

Example 7.2.2 *Find the mean of the process $W_te^{W_t}$.*

We shall set up a stochastic differential equation for $W_te^{W_t}$. Using the product formula and Ito's formula yields

$$\begin{aligned} d(W_te^{W_t}) &= e^{W_t}dW_t + W_td(e^{W_t}) + dW_t\,d(e^{W_t}) \\ &= e^{W_t}dW_t + (W_t + dW_t)(e^{W_t}dW_t + \frac{1}{2}e^{W_t}dt) \\ &= (\frac{1}{2}W_te^{W_t} + e^{W_t})dt + (e^{W_t} + W_te^{W_t})dW_t. \end{aligned}$$

Integrating and using that $W_0e^{W_0} = 0$ yields

$$W_te^{W_t} = \int_0^t (\frac{1}{2}W_se^{W_s} + e^{W_s})\,ds + \int_0^t (e^{W_s} + W_se^{W_s})\,dW_s.$$

Since the expectation of an Ito integral is zero, we have

$$E[W_t e^{W_t}] = \int_0^t \big(\frac{1}{2}E[W_s e^{W_s}] + E[e^{W_s}]\big)\, ds.$$

Let $f(t) = E[W_t e^{W_t}]$. Using $E[e^{W_s}] = e^{s/2}$, the previous integral equation becomes

$$f(t) = \int_0^t (\frac{1}{2}f(s) + e^{s/2})\, ds,$$

Differentiating yields the following linear differential equation

$$f'(t) = \frac{1}{2}f(t) + e^{t/2}$$

with the initial condition $f(0) = 0$. Multiplying by $e^{-t/2}$ yields the following exact equation

$$(e^{-t/2}f(t))' = 1.$$

The solution is $f(t) = te^{t/2}$. Hence we obtained that

$$E[W_t e^{W_t}] = te^{t/2}.$$

Exercise 7.2.4 *Find a* $E[W_t^2 e^{W_t}]$; *(b)* $E[W_t e^{kW_t}]$.

Example 7.2.3 *Show that for any integer* $k \geq 0$ *we have*

$$\boxed{E[W_t^{2k}] = \frac{(2k)!}{2^k k!}t^k, \qquad E[W_t^{2k+1}] = 0.}$$

In particular, $E[W_t^4] = 3t^2$, $E[W_t^6] = 15t^3$.

From Ito's formula we have

$$d(W_t^n) = nW_t^{n-1}dW_t + \frac{n(n-1)}{2}W_t^{n-2}dt.$$

Integrate and get

$$W_t^n = n\int_0^t W_s^{n-1}\, dW_s + \frac{n(n-1)}{2}\int_0^t W_s^{n-2}\, ds.$$

Since the expectation of the first integral on the right side is zero, taking the expectation yields the following recursive relation

$$E[W_t^n] = \frac{n(n-1)}{2}\int_0^t E[W_s^{n-2}]\, ds.$$

Using the initial values $E[W_t] = 0$ and $E[W_t^2] = t$, the method of mathematical induction implies that $E[W_t^{2k+1}] = 0$ and $E[W_t^{2k}] = \frac{(2k)!}{2^k k!}t^k$.

Exercise 7.2.5 *(a) Is* $W_t^4 - 3t^2$ *an* $\mathcal{F}_t$*-martingale?*
(b) What about W_t^3*?*

Example 7.2.4 *Find* $E[\sin W_t]$.

From Ito's formula

$$d(\sin W_t) = \cos W_t \, dW_t - \frac{1}{2}\sin W_t \, dt,$$

then integrating yields

$$\sin W_t = \int_0^t \cos W_s \, dW_s - \frac{1}{2}\int_0^t \sin W_s \, ds.$$

Taking expectations we arrive at the integral equation

$$E[\sin W_t] = -\frac{1}{2}\int_0^t E[\sin W_s] \, ds.$$

Let $f(t) = E[\sin W_t]$. Differentiating yields the equation $f'(t) = -\frac{1}{2}f(t)$ with $f(0) = E[\sin W_0] = 0$. The unique solution is $f(t) = 0$. Hence

$$E[\sin W_t] = 0.$$

Exercise 7.2.6 *Let* σ *be a constant. Show that*

(*a*) $E[\sin(\sigma W_t)] = 0$;

(*b*) $E[\cos(\sigma W_t)] = e^{-\sigma^2 t/2}$;

(*c*) $E[\sin(t + \sigma W_t)] = e^{-\sigma^2 t/2}\sin t$;

(*d*) $E[\cos(t + \sigma W_t)] = e^{-\sigma^2 t/2}\cos t$;

Exercise 7.2.7 *Use the previous exercise and the definition of expectation to show that*

(*a*) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\cos x \, dx = \frac{\pi^{1/2}}{e^{1/4}}$;

(*b*) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\cos x \, dx = \sqrt{\frac{2\pi}{e}}$.

Exercise 7.2.8 *Using expectations show that*

(*a*) $\displaystyle\int_{-\infty}^{\infty} xe^{-ax^2+bx} \, dx = \sqrt{\frac{\pi}{a}}\Big(\frac{b}{2a}\Big)e^{b^2/(4a)}$;

(*b*) $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-ax^2+bx} \, dx = \sqrt{\frac{\pi}{a}}\frac{1}{2a}\Big(1 + \frac{b^2}{2a}\Big)e^{b^2/(4a)}$;

(*c*) *Can you apply a similar method to find a close form expression for the integral*

$$\int_{-\infty}^{\infty} x^n e^{-ax^2+bx} \, dx?$$

Exercise 7.2.9 *Using the result given by Example 7.2.3 show that*

(*a*) $E[\cos(tW_t)] = e^{-t^3/2}$;

(*b*) $E[\sin(tW_t)] = 0$;

(*c*) $E[e^{tW_t}] = 0$.

For general drift rates we cannot find the mean, but in the case of concave drift rates we can find an upper bound for the expectation $E[X_t]$. The following result will be useful.

Lemma 7.2.10 (Gronwall's inequality) *Let $f(t)$ be a non-negative function satisfying the inequality*

$$f(t) \leq C + M \int_0^t f(s)\,ds$$

for $0 \leq t \leq T$, with C, M constants. Then

$$f(t) \leq Ce^{Mt}, \qquad 0 \leq t \leq T.$$

Proposition 7.2.11 *Let X_t be a continuous stochastic process such that*

$$dX_t = a(X_t)dt + b(t, W_t, X_t)\,dW_t,$$

with the function $a(\cdot)$ satisfying the following conditions

1. *$a(x) \geq 0$, for $0 \leq x \leq T$;*
2. *$a''(x) < 0$, for $0 \leq x \leq T$;*
3. *$a'(0) = M$.*

Then $E[X_t] \leq X_0 e^{Mt}$, for $0 \leq X_t \leq T$.

Proof: From the mean value theorem there is $\xi \in (0, x)$ such that

$$a(x) = a(x) - a(0) = (x - 0)a'(\xi) \leq xa'(0) = Mx, \tag{7.2.6}$$

where we used that $a'(x)$ is a decreasing function. Applying Jensen's inequality for concave functions yields

$$E[a(X_t)] \leq a(E[X_t]).$$

Combining with (7.2.6) we obtain $E[a(X_t)] \leq ME[X_t]$. Substituting in the identity (7.2.4) implies

$$E[X_t] \leq X_0 + M \int_0^t E[X_s]\,ds.$$

Applying Gronwall's inequality we obtain $E[X_t] \leq X_0 e^{Mt}$. ■

Exercise 7.2.12 *State the previous result in the particular case when $a(x) = \sin x$, with $0 \leq x \leq \pi$.*

Not in all cases can the mean and the variance be obtained directly from the stochastic equation. In these cases we need more powerful methods that produce closed form solutions. In the next sections we shall discuss several methods of solving stochastic differential equation.

7.3 The Integration Technique

We shall start with the simple case when both the drift and the volatility are just functions of time t.

Proposition 7.3.1 *The solution X_t of the stochastic differential equation*

$$dX_t = a(t)dt + b(t)dW_t$$

is Gaussian distributed with mean $X_0 + \int_0^t a(s)\,ds$ and variance $\int_0^t b^2(s)\,ds$.

Proof: Integrating in the equation yields

$$X_t - X_0 = \int_0^t dX_s = \int_0^t a(s)\,ds + \int_0^t b(s)\,dW_s.$$

Using the property of Wiener integrals, $\int_0^t b(s)\,dW_s$ is Gaussian distributed with mean 0 and variance $\int_0^t b^2(s)\,ds$. Then X_t is Gaussian (as a sum between a predictable function and a Gaussian), with

$$\begin{aligned}
E[X_t] &= E[X_0 + \int_0^t a(s)\,ds + \int_0^t b(s)\,dW_s] \\
&= X_0 + \int_0^t a(s)\,ds + E[\int_0^t b(s)\,dW_s] \\
&= X_0 + \int_0^t a(s)\,ds, \\
Var[X_t] &= Var[X_0 + \int_0^t a(s)\,ds + \int_0^t b(s)\,dW_s] \\
&= Var\Big[\int_0^t b(s)\,dW_s\Big] \\
&= \int_0^t b^2(s)\,ds,
\end{aligned}$$

which ends the proof. ■

Exercise 7.3.2 *Solve the following stochastic differential equations for $t \geq 0$ and determine the mean and the variance of the solution*

(a) $dX_t = \cos t\,dt - \sin t\,dW_t,\ X_0 = 1.$

(b) $dX_t = e^t\,dt + \sqrt{t}\,dW_t,\ X_0 = 0.$

(c) $dX_t = \frac{t}{1+t^2}\,dt + t^{3/2}\,dW_t,\ X_0 = 1.$

If the drift and the volatility depend on both variables t and W_t, the stochastic differential equation

$$dX_t = a(t, W_t)dt + b(t, W_t)dW_t, \qquad t \geq 0$$

defines an *Ito diffusion*. Integrating yields the solution

$$X_t = X_0 + \int_0^t a(s, W_s)\,ds + \int_0^t b(s, W_s)\,dW_s.$$

There are several cases when both integrals can be computed explicitly.

Example 7.3.1 *Find the solution of the stochastic differential equation*

$$dX_t = dt + W_t\, dW_t, \qquad X_0 = 1.$$

Integrate between 0 and t and get

$$\begin{aligned} X_t &= 1 + \int_0^t ds + \int_0^t W_s\, dW_s = t + \frac{W_t^2}{2} - \frac{t}{2} \\ &= \frac{1}{2}(W_t^2 + t). \end{aligned}$$

Example 7.3.2 *Solve the stochastic differential equation*

$$dX_t = (W_t - 1)dt + W_t^2\, dW_t, \qquad X_0 = 0.$$

Let $Z_t = \int_0^t W_s\, ds$ denote the integrated Brownian motion process. Integrating the equation between 0 and t yields

$$\begin{aligned} X_t &= \int_0^s dX_s = \int_0^t (W_s - 1)ds + \int_0^t W_s^2\, dW_s \\ &= Z_t - t + \frac{1}{3}W_t^3 - \frac{1}{2}W_t^2 - \frac{t}{2} \\ &= Z_t + \frac{1}{3}W_t^3 - \frac{1}{2}W_t^2 - \frac{t}{2}. \end{aligned}$$

Example 7.3.3 *Solve the stochastic differential equation*

$$dX_t = t^2 dt + e^{t/2} \cos W_t\, dW_t, \qquad X_0 = 0,$$

and find $E[X_t]$ and $Var(X_t)$.

Integrating yields

$$\begin{aligned} X_t &= \int_0^t s^2\, ds + \int_0^t e^{s/2} \cos W_s\, dW_s \\ &= \frac{t^3}{3} + e^{t/2} \sin W_t, \end{aligned} \tag{7.3.7}$$

where we used (6.3.9). Even if the process X_t is not Gaussian, we can still compute its mean and variance. By Ito's formula we have

$$d(\sin W_t) = \cos W_t\, dW_t - \frac{1}{2} \sin W_t\, dt$$

Integrating between 0 and t yields

$$\sin W_t = \int_0^t \cos W_s\, dW_s - \frac{1}{2} \int_0^t \sin W_s\, ds,$$

where we used that $\sin W_0 = \sin 0 = 0$. Taking the expectation in the previous relation yields

$$E[\sin W_t] = E\Big[\int_0^t \cos W_s\, dW_s\Big] - \frac{1}{2} \int_0^t E[\sin W_s]\, ds.$$

From the properties of the Ito integral, the first expectation on the right side is zero. Denoting $\mu(t) = E[\sin W_t]$, we obtain the integral equation

$$\mu(t) = -\frac{1}{2}\int_0^t \mu(s)\,ds.$$

Differentiating yields the differential equation

$$\mu'(t) = -\frac{1}{2}\mu(t)$$

with the solution $\mu(t) = ke^{-t/2}$. Since $k = \mu(0) = E[\sin W_0] = 0$, it follows that $\mu(t) = 0$. Hence

$$E[\sin W_t] = 0.$$

Taking expectation in (7.3.7) leads to

$$E[X_t] = E\Big[\frac{t^3}{3}\Big] + e^{t/2}E[\sin W_t] = \frac{t^3}{3}.$$

Since the variance of predictable functions is zero,

$$\begin{aligned} Var[X_t] &= Var\Big[\frac{t^3}{3} + e^{t/2}\sin W_t\Big] = (e^{t/2})^2 Var[\sin W_t] \\ &= e^t E[\sin^2 W_t] = \frac{e^t}{2}(1 - E[\cos 2W_t]). \end{aligned} \tag{7.3.8}$$

In order to compute the last expectation we use Ito's formula

$$d(\cos 2W_t) = -2\sin 2W_t\,dW_t - 2\cos 2W_t\,dt$$

and integrate to get

$$\cos 2W_t = \cos 2W_0 - 2\int_0^t \sin 2W_s\,dW_s - 2\int_0^t \cos 2W_s\,ds$$

Taking the expectation and using that Ito integrals have zero expectation, yields

$$E[\cos 2W_t] = 1 - 2\int_0^t E[\cos 2W_s]\,ds.$$

If we denote $m(t) = E[\cos 2W_t]$, the previous relation becomes an integral equation

$$m(t) = 1 - 2\int_0^t m(s)\,ds.$$

Differentiate and get

$$m'(t) = -2m(t),$$

with the solution $m(t) = ke^{-2t}$. Since $k = m(0) = E[\cos 2W_0] = 1$, we have $m(t) = e^{-2t}$. Substituting into (7.3.8) yields

$$Var[X_t] = \frac{e^t}{2}(1 - e^{-2t}) = \frac{e^t - e^{-t}}{2} = \sinh t.$$

In conclusion, the solution X_t has the mean and the variance given by

$$E[X_t] = \frac{t^3}{3}, \qquad Var[X_t] = \sinh t.$$

Example 7.3.4 *Solve the following stochastic differential equation*

$$e^{t/2} dX_t = dt + e^{W_t}\, dW_t, \qquad X_0 = 0,$$

and find the distribution of the solution X_t and its mean and variance.

Dividing by $e^{t/2}$, integrating between 0 and t, and using formula (6.3.8) yields

$$\begin{aligned} X_t &= \int_0^t e^{-s/2}\, ds + \int_0^t e^{-s/2+W_s}\, dW_s \\ &= 2(1-e^{-t/2}) + e^{-t/2}e^{W_t} - 1 \\ &= 1 + e^{-t/2}(e^{W_t} - 2). \end{aligned}$$

Since e^{W_t} is a geometric Brownian motion, using Proposition 2.2.2 yields

$$\begin{aligned} E[X_t] &= E[1 + e^{-t/2}(e^{W_t} - 2)] = 1 - 2e^{-t/2} + e^{-t/2}E[e_t^W] \\ &= 2 - 2e^{-t/2}. \\ Var(X_t) &= Var[1 + e^{-t/2}(e^{W_t} - 2)] = Var[e^{-t/2}e^{W_t}] = e^{-t}Var[e^{W_t}] \\ &= e^{-t}(e^{2t} - e^t) = e^t - 1. \end{aligned}$$

The process X_t has the following distribution:

$$\begin{aligned} F(y) &= P(X_t \le y) = P\big(1 + e^{-t/2}(e^{W_t} - 2) \le y\big) \\ &= P\Big(W_t \le \ln\big(2 + e^{t/2}(y-1)\big)\Big) = P\Big(\frac{W_t}{\sqrt{t}} \le \frac{1}{\sqrt{t}} \ln\big(2 + e^{t/2}(y-1)\big)\Big) \\ &= N\Big(\frac{1}{\sqrt{t}} \ln\big(2 + e^{t/2}(y-1)\big)\Big), \end{aligned}$$

where $N(u) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^u e^{-s^2/2}\, ds$ is the distribution function of a standard normal distributed random variable.

Example 7.3.5 *Solve the stochastic differential equation*

$$dX_t = dt + t^{-3/2}W_t e^{-W_t^2/(2t)}\, dW_t, \qquad X_1 = 1.$$

Integrating between 1 and t and applying formula (6.3.10) yields

$$\begin{aligned} X_t &= X_1 + \int_1^t ds + \int_1^t s^{-3/2}W_s e^{-W_s^2/(2s)}\, dW_s \\ &= 1 + t - 1 - e^{-W_1^2/2} - \frac{1}{t^{1/2}}e^{-W_t^2/(2t)} \\ &= t - e^{-W_1^2/2} - \frac{1}{t^{1/2}}e^{-W_t^2/(2t)}, \qquad \forall t \ge 1. \end{aligned}$$

7.4 Exact Stochastic Equations

The stochastic differential equation

$$dX_t = a(t, W_t)dt + b(t, W_t)dW_t \tag{7.4.9}$$

is called *exact* if there is a differentiable function $f(t, x)$ such that

$$a(t, x) = \partial_t f(t, x) + \frac{1}{2}\partial_x^2 f(t, x) \tag{7.4.10}$$

$$b(t, x) = \partial_x f(t, x). \tag{7.4.11}$$

Assume the equation is exact. Then substituting in (7.4.9) yields

$$dX_t = \Big(\partial_t f(t, W_t) + \frac{1}{2}\partial_x^2 f(t, W_t)\Big)dt + \partial_x f(t, W_t)dW_t.$$

Applying Ito's formula, the previous equations becomes

$$dX_t = d\big(f(t, W_t)\big),$$

which implies $X_t = f(t, W_t) + c$, with c constant.

Solving the partial differential equations system (7.4.10)–(7.4.10) requires the following steps:

1. Integrating partially with respect to x in the second equation to obtain $f(t, x)$ up to an additive function $T(t)$;
2. Substitute into the first equation and determine the function $T(t)$;
3. The solution is $X_t = f(t, W_t) + c$, with c determined from the initial condition on X_t.

Example 7.4.1 *Solve the stochastic differential equation*

$$dX_t = e^t(1 + W_t^2)dt + (1 + 2e^t W_t)dW_t, \quad X_0 = 0.$$

In this case $a(t, x) = e^t(1 + x^2)$ and $b(t, x) = 1 + 2e^t x$. The associated system is

$$e^t(1 + x^2) = \partial_t f(t, x) + \frac{1}{2}\partial_x^2 f(t, x)$$

$$1 + 2e^t x = \partial_x f(t, x).$$

Integrate partially in x in the second equation yields

$$f(t, x) = \int (1 + 2e^t x)\, dx = x + e^t x^2 + T(t).$$

Then $\partial_t f = e^t x^2 + T'(t)$ and $\partial_x^2 f = 2e^t$. Substituting in the first equation yields

$$e^t(1 + x^2) = e^t x^2 + T'(t) + e^t.$$

This implies $T'(t) = 0$, or $T = c$ constant. Hence $f(t, x) = x + e^t x^2 + c$, and $X_t = f(t, W_t) = W_t + e^t W_t^2 + c$. Since $X_0 = 0$, it follows that $c = 0$. The solution is $X_t = W_t + e^t W_t^2$.

Example 7.4.2 *Find the solution of*

$$dX_t = \big(2tW_t^3 + 3t^2(1 + W_t)\big)dt + (3t^2W_t^2 + 1)dW_t, \quad X_0 = 0.$$

The coefficient functions are $a(t, x) = 2tx^3 + 3t^2(1 + x)$ and $b(t, x) = 3t^2x^2 + 1$. The associated system is given by

$$\begin{aligned} 2tx^3 + 3t^2(1 + x) &= \partial_t f(t, x) + \frac{1}{2}\partial_x^2 f(t, x) \\ 3t^2x^2 + 1 &= \partial_x f(t, x). \end{aligned}$$

Integrating partially in the second equation yields

$$f(t, x) = \int (3t^2x^2 + 1)\, dx = t^2x^3 + x + T(t).$$

Then $\partial_t f = 2tx^3 + T'(t)$ and $\partial_x^2 f = 6t^2x$, and substituting into the first equation we get

$$2tx^3 + 3t^2(1 + x) = 2tx^3 + T'(t) + \frac{1}{2}6t^2x.$$

After cancellations we get $T'(t) = 3t^2$, so $T(t) = t^3 + c$. Then

$$f(t, x) = t^2x^3 + x + 3t^2 = t^2(x^3 + 1) + x + c.$$

The solution process is given by $X_t = f(t, W_t) = t^2(W_t^3 + 1) + W_t + c$. Using $X_0 = 0$ we get $c = 0$. Hence the solution is $X_t = t^2(W_t^3 + 1) + W_t$.

The next result deals with a condition regarding the closeness of the stochastic differential equation.

Theorem 7.4.1 *If the stochastic differential equation (7.4.9) is exact, then the coefficient functions $a(t, x)$ and $b(t, x)$ satisfy the condition*

$$\boxed{\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b.} \tag{7.4.12}$$

Proof: If the stochastic equation is exact, there is a function $f(t, x)$ satisfying the system (7.4.10)–(7.4.10). Differentiating the first equation of the system with respect to x yields

$$\partial_x a = \partial_t \partial_x f + \frac{1}{2}\partial_x^2 \partial_x f.$$

Substituting $b = \partial_x f$ yields the desired relation. ■

Remark 7.4.2 *The equation (7.4.12) has the meaning of a heat equation. The function $b(t, x)$ represents the temperature measured at x at the instance t, while $\partial_x a$ is the density of heat sources. The function $a(t, x)$ can be regarded as the potential from which the density of heat sources is derived by taking the gradient in x.*

It is worth noting that equation (7.4.12) is a just necessary condition for exactness. This means that if this condition is not satisfied, then the equation is not exact. In that case we need to try a different method to solve the equation.

Example 7.4.3 *Is the stochastic differential equation*

$$dX_t = (1 + W_t^2)dt + (t^4 + W_t^2)dW_t$$

exact?

Collecting the coefficients, we have $a(t, x) = 1 + x^2$, $b(t, x) = t^4 + x^2$. Since $\partial_x a = 2x$, $\partial_t b = 4t^3$, and $\partial_x^2 b = 2$, the condition (7.4.12) is not satisfied, and hence the equation is not exact.

7.5 Integration by Inspection

When solving a stochastic differential equation by inspection we look for opportunities to apply the product or the quotient formulas:

$$\boxed{d(f(t)Y_t) = f(t)\,dY_t + Y_t\,df(t).}$$

$$\boxed{d\Big(\frac{X_t}{f(t)}\Big) = \frac{f(t)dX_t - X_t df(t)}{f(t)^2}.}$$

For instance, if a stochastic differential equation can be written as

$$dX_t = f'(t)W_t dt + f(t)dW_t,$$

the product rule brings the equation into the exact form

$$dX_t = d\Big(f(t)W_t\Big),$$

which after integration leads to the solution

$$X_t = X_0 + f(t)W_t.$$

Example 7.5.1 *Solve*

$$dX_t = (t + W_t^2)dt + 2tW_t dW_t, \qquad X_0 = a.$$

We can write the equation as

$$dX_t = W_t^2 dt + t(2W_t dW_t + dt),$$

which can be contracted to

$$dX_t = W_t^2 dt + td(W_t^2).$$

Using the product rule we can bring it to the exact form

$$dX_t = d(tW_t^2),$$

with the solution $X_t = tW_t^2 + a$.

Example 7.5.2 *Solve the stochastic differential equation*

$$dX_t = (W_t + 3t^2)dt + tdW_t.$$

If we rewrite the equation as

$$dX_t = 3t^2dt + (W_tdt + tdW_t),$$

we note the exact expression formed by the last two terms $W_tdt + tdW_t = d(tW_t)$. Then

$$dX_t = d(t^3) + d(tW_t),$$

which is equivalent with $d(X_t) = d(t^3 + tW_t)$. Hence $X_t = t^3 + tW_t + c$, $c \in \mathbb{R}$.

Example 7.5.3 *Solve the stochastic differential equation*

$$e^{-2t}dX_t = (1 + 2W_t^2)dt + 2W_tdW_t.$$

Multiply by e^{2t} to get

$$dX_t = e^{2t}(1 + 2W_t^2)dt + 2e^{2t}W_tdW_t.$$

After regrouping, this becomes

$$dX_t = (2e^{2t}dt)W_t^2 + e^{2t}(2W_tdW_t + dt).$$

Since $d(e^{2t}) = 2e^{2t}dt$ and $d(W_t^2) = 2W_tdW_t + dt$, the previous relation becomes

$$dX_t = d(e^{2t})W_t^2 + e^{2t}d(W_t^2).$$

By the product rule, the right side becomes exact

$$dX_t = d(e^{2t}W_t^2),$$

and hence the solution is $X_t = e^{2t}W_t^2 + c$, $c \in \mathbb{R}$.

Example 7.5.4 *Solve the equation*

$$t^3dX_t = (3t^2X_t + t)dt + t^6dW_t, \qquad X_1 = 0.$$

The equation can be written as

$$t^3dX_t - 3X_tt^2dt = tdt + t^6dW_t.$$

Divide by t^6:

$$\frac{t^3dX_t - X_td(t^3)}{(t^3)^2} = t^{-5}dt + dW_t.$$

Applying the quotient rule yields

$$d\Big(\frac{X_t}{t^3}\Big) = -d\Big(\frac{t^{-4}}{4}\Big) + dW_t.$$

Integrating between 1 and t, yields

$$\frac{X_t}{t^3} = -\frac{t^{-4}}{4} + W_t - W_1 + c$$

so

$$X_t = ct^3 - \frac{1}{4t} + t^3(W_t - W_1), \quad c \in \mathbb{R}.$$

Using $X_1 = 0$ yields $c = 1/4$ and hence the solution is

$$X_t = \frac{1}{4}\Big(t^3 - \frac{1}{t}\Big) + t^3(W_t - W_1), \quad c \in \mathbb{R}.$$

Exercise 7.5.1 *Solve the following stochastic differential equations by the inspection method*

(a) $dX_t = (1 + W_t)dt + (t + 2W_t)dW_t, \qquad X_0 = 0;$

(b) $t^2 dX_t = (2t^3 - W_t)dt + tdW_t, \qquad X_1 = 0;$

(c) $e^{-t/2}dX_t = \frac{1}{2}W_t dt + dW_t, \qquad X_0 = 0;$

(d) $dX_t = 2tW_t dW_t + W_t^2 dt, \qquad X_0 = 0;$

(e) $dX_t = \Big(1 + \frac{1}{2\sqrt{t}}W_t\Big)dt + \sqrt{t}\,dW_t, \qquad X_1 = 0.$

7.6 Linear Stochastic Differential Equations

Consider the stochastic differential equation with drift term linear in X_t

$$dX_t = \big(\alpha(t)X_t + \beta(t)\big)dt + b(t, W_t)dW_t, \qquad t \geq 0.$$

This also can be written as

$$dX_t - \alpha(t)X_t dt = \beta(t)dt + b(t, W_t)dW_t.$$

Let $A(t) = \int_0^t \alpha(s)\,ds$. Multiplying by the integrating factor $e^{-A(t)}$, the left side of the previous equation becomes an exact expression

$$\begin{aligned} e^{-A(t)}\Big(dX_t - \alpha(t)X_t dt\Big) &= e^{-A(t)}\beta(t)dt + e^{-A(t)}b(t, W_t)dW_t \\ d\Big(e^{-A(t)}X_t\Big) &= e^{-A(t)}\beta(t)dt + e^{-A(t)}b(t, W_t)dW_t. \end{aligned}$$

Integrating yields

$$\begin{aligned} e^{-A(t)}X_t &= X_0 + \int_0^t e^{-A(s)}\beta(s)\,ds + \int_0^t e^{-A(s)}b(s, W_s)\,dW_s \\ X_t &= X_0 e^{A(t)} + e^{A(t)}\Big(\int_0^t e^{-A(s)}\beta(s)\,ds + \int_0^t e^{-A(s)}b(s, W_s)\,dW_s\Big). \end{aligned}$$

The first integral within the previous parentheses is a Riemann integral, and the latter one is an Ito stochastic integral. Sometimes, in practical applications these integrals can be computed explicitly.

When $b(t, W_t) = b(t)$, the latter integral becomes a Wiener integral. In this case the solution X_t is Gaussian with mean and variance given by

$$\begin{aligned} E[X_t] &= X_0 e^{A(t)} + e^{A(t)}\int_0^t e^{-A(s)}\beta(s)\,ds \\ Var[X_t] &= e^{2A(t)}\int_0^t e^{-2A(s)}b(s)^2\,ds. \end{aligned}$$

Another important particular case is when $\alpha(t) = \alpha \neq 0$, $\beta(t) = \beta$ are constants and $b(t, W_t) = b(t)$. The equation in this case is

$$dX_t = \big(\alpha X_t + \beta\big)dt + b(t)dW_t, \qquad t \geq 0,$$

and the solution takes the form

$$X_t = X_0 e^{\alpha t} + \frac{\beta}{\alpha}(e^{\alpha t} - 1) + \int_0^t e^{\alpha(t-s)}b(s)\,dW_s.$$

Example 7.6.1 *Solve the linear stochastic differential equation*

$$dX_t = (2X_t + 1)dt + e^{2t}dW_t.$$

Write the equation as

$$dX_t - 2X_t dt = dt + e^{2t}dW_t$$

and multiply by the integrating factor e^{-2t} to get

$$d\Big(e^{-2t}X_t\Big) = e^{-2t}dt + dW_t.$$

Integrate between 0 and t and multiply by e^{2t}, and obtain

$$\begin{aligned} X_t &= X_0 e^{2t} + e^{2t}\int_0^t e^{-2s}ds + e^{2t}\int_0^t dW_s \\ &= X_0 e^{2t} + \frac{1}{2}(e^{2t} - 1) + e^{2t}W_t. \end{aligned}$$

Example 7.6.2 *Solve the linear stochastic differential equation*

$$dX_t = (2 - X_t)dt + e^{-t}W_t dW_t.$$

Multiplying by the integrating factor e^t yields

$$e^t(dX_t + X_t dt) = 2e^t dt + W_t\, dW_t.$$

Since $e^t(dX_t + X_t dt) = d(e^t X_t)$, integrating between 0 and t we get

$$e^t X_t = X_0 + \int_0^t 2e^t\, dt + \int_0^t W_s\, dW_s.$$

Dividing by e^t and performing the integration yields

$$X_t = X_0 e^{-t} + 2(1 - e^{-t}) + \frac{1}{2}e^{-t}(W_t^2 - t).$$

Example 7.6.3 *Solve the linear stochastic differential equation*

$$dX_t = (\frac{1}{2}X_t + 1)dt + e^t \cos W_t\, dW_t.$$

Write the equation as

$$dX_t - \frac{1}{2}X_t dt = dt + e^t \cos W_t\, dW_t$$

and multiply by the integrating factor $e^{-t/2}$ to get

$$d(e^{-t/2}X_t) = e^{-t/2}dt + e^{t/2}\cos W_t\, dW_t.$$

Integrating yields

$$e^{-t/2}X_t = X_0 + \int_0^t e^{-s/2}\, ds + \int_0^t e^{s/2}\cos W_s\, dW_s$$

Multiply by $e^{t/2}$ and use formula (6.3.9) to obtain the solution

$$X_t = X_0 e^{t/2} + 2(e^{t/2} - 1) + e^t \sin W_t.$$

Exercise 7.6.1 *Solve the following linear stochastic differential equations*

(a) $dX_t = (4X_t - 1)dt + 2dW_t$;

(b) $dX_t = (3X_t - 2)dt + e^{3t}dW_t$;

(c) $dX_t = (1 + X_t)dt + e^t W_t dW_t$;

(d) $dX_t = (4X_t + t)dt + e^{4t}dW_t$;

(e) $dX_t = \Big(t + \frac{1}{2}X_t\Big)dt + e^t \sin W_t\, dW_t$;

(f) $dX_t = -X_t dt + e^{-t}\, dW_t$.

In the following we present an important example of stochastic differential equations, which can be solved by the method presented in this section.

Proposition 7.6.2 (The mean-reverting Ornstein-Uhlenbeck process) *Let m and α be two constants. Then the solution X_t of the stochastic equation*

$$\boxed{dX_t = (m - X_t)dt + \alpha dW_t} \tag{7.6.13}$$

is given by

$$\boxed{X_t = m + (X_0 - m)e^{-t} + \alpha \int_0^t e^{s-t}\, dW_s.} \tag{7.6.14}$$

X_t is Gaussian with mean and variance given by

$$\begin{aligned} E[X_t] &= m + (X_0 - m)e^{-t} \\ Var(X_t) &= \frac{\alpha^2}{2}(1 - e^{-2t}). \end{aligned}$$

Proof: Adding $X_t dt$ to both sides and multiplying by the integrating factor e^t we get

$$d(e^t X_t) = me^t dt + \alpha e^t dW_t,$$

which after integration yields

$$e^t X_t = X_0 + m(e^t - 1) + \alpha \int_0^t e^s\, dW_s.$$

Hence

$$\begin{aligned} X_t &= X_0 e^{-t} + m - e^{-t} + \alpha e^{-t} \int_0^t e^s\, dW_s \\ &= m + (X_0 - m)e^{-t} + \alpha \int_0^t e^{s-t}\, dW_s. \end{aligned}$$

Since X_t is the sum between a predictable function and a Wiener integral, then we can use Proposition 4.4.1 and it follows that X_t is Gaussian, with

$$\begin{aligned} E[X_t] &= m + (X_0 - m)e^{-t} + E\Big[\alpha \int_0^t e^{s-t}\, dW_s\Big] = m + (X_0 - m)e^{-t} \\ Var(X_t) &= Var\Big[\alpha \int_0^t e^{s-t}\, dW_s\Big] = \alpha^2 e^{-2t} \int_0^t e^{2s}\, ds \\ &= \alpha^2 e^{-2t} \frac{e^{2t} - 1}{2} = \frac{1}{2}\alpha^2(1 - e^{-2t}). \end{aligned}$$

■

The name *mean-reverting* comes from the fact that

$$\lim_{t\to\infty} E[X_t] = m.$$

The variance also tends to zero exponentially, $\lim\limits_{t\to\infty} Var[X_t] = 0$. According to Proposition 3.8.1, the process X_t tends to m in the mean square sense.

Proposition 7.6.3 (The Brownian bridge) *For $a, b \in \mathbb{R}$ fixed, the stochastic differential equation*

$$dX_t = \frac{b - X_t}{1-t}dt + dW_t, \quad 0 \le t < 1, X_0 = a$$

has the solution

$$\boxed{X_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}dW_s,\ 0 \le t < 1.} \tag{7.6.15}$$

The solution has the property $\lim_{t\to 1} X_t = b$, almost certainly.

Proof: If we let $Y_t = b - X_t$ the equation becomes linear in Y_t

$$dY_t + \frac{1}{1-t}Y_t dt = -dW_t.$$

Multiplying by the integrating factor $\rho(t) = \frac{1}{1-t}$ yields

$$d\Big(\frac{Y_t}{1-t}\Big) = -\frac{1}{1-t}dW_t,$$

which leads by integration to

$$\frac{Y_t}{1-t} = c - \int_0^t \frac{1}{1-s}\,dW_s.$$

Making $t = 0$ yields $c = a - b$, so

$$\frac{b - X_t}{1-t} = a - b - \int_0^t \frac{1}{1-s}\,dW_s.$$

Solving for X_t yields

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}dW_s,\ 0 \le t < 1.$$

Let $U_t = (1-t)\int_0^t \frac{1}{1-s}\,dW_s$. First we notice that

$$\begin{aligned}
E[U_t] &= (1-t)E\Big[\int_0^t \frac{1}{1-s}\,dW_s\Big] = 0,\\
Var(U_t) &= (1-t)^2 Var\Big[\int_0^t \frac{1}{1-s}\,dW_s\Big] = (1-t)^2\int_0^t \frac{1}{(1-s)^2}\,ds\\
&= (1-t)^2\Big(\frac{1}{1-t} - 1\Big) = t(1-t).
\end{aligned}$$

In order to show ac-$\lim_{t\to 1} X_t = b$, we need to prove

$$P\big(\omega; \lim_{t\to 1} X_t(\omega) = b\big) = 1.$$

Since $X_t = a(1-t) + bt + U_t$, it suffices to show that

$$P\big(\omega; \lim_{t\to 1} U_t(\omega) = 0\big) = 1. \tag{7.6.16}$$

We evaluate the probability of the complementary event

$$P\big(\omega; \lim_{t\to 1} U_t(\omega) \neq 0\big) = P\big(\omega; |U_t(\omega)| > \epsilon, \forall t\big),$$

for some $\epsilon > 0$. Since by Markov's inequality

$$P\big(\omega; |U_t(\omega)| > \epsilon\big) < \frac{Var(U_t)}{\epsilon^2} = \frac{t(1-t)}{\epsilon^2}$$

holds for any $0 \le t < 1$, choosing $t \to 1$ implies that

$$P\big(\omega; |U_t(\omega)| > \epsilon, \forall t\big) = 0,$$

which implies (7.6.16). ■

The process (7.6.15) is called the *Brownian bridge* because it joins $X_0 = a$ with $X_1 = b$. Since X_t is the sum between a deterministic linear function in t and a Wiener integral, it follows that it is a Gaussian process, with mean and variance

$$\begin{aligned} E[X_t] &= a(1-t) + bt \\ Var(X_t) &= Var(U_t) = t(1-t). \end{aligned}$$

It is worth noting that the variance is maximum at the midpoint $t = (b-a)/2$ and zero at the end points a and b.

Example 7.6.4 *Find $Cov(X_s, X_t)$, $0 < s < t$ for the following cases:*

(a) X_t is a mean reverting Orstein-Uhlenbeck process;

(b) X_t is a Brownian bridge process.

Stochastic equations with respect to a Poisson process

Similar techniques can be applied in the case when the Brownian motion process W_t is replaced by a Poisson process N_t with constant rate λ. For instance, the stochastic differential equation

$$\begin{aligned} dX_t &= 3X_t dt + e^{3t} dN_t \\ X_0 &= 1 \end{aligned}$$

can be solved multiplying by the integrating factor e^{-3t} to obtain

$$d(e^{-3t} X_t) = dN_t.$$

Integrating yields $e^{-3t} X_t = N_t + C$. Making $t = 0$ yields $C = 1$, so the solution is given by $X_t = e^{3t}(1 + N_t)$.

The following equation

$$dX_t = (m - X_t)dt + \alpha dN_t$$

is similar with the equation defining the mean-reverting Ornstein-Uhlenbeck process. As we shall see, in this case, the process is no more mean-reverting. It reverts though to a certain constant. A similar method yields the solution

$$X_t = m + (X_0 - m)e^{-t} + \alpha e^{-t} \int_0^t e^s \, dN_s.$$

Since from Proposition 4.5.6 and Exercise 4.5.9 we have

$$\begin{aligned} E\Big[\int_0^t e^s \, dN_s\Big] &= \lambda \int_0^t e^s \, ds = \lambda(e^t - 1) \\ Var\Big(\int_0^t e^s \, dN_s\Big) &= \lambda \int_0^t e^{2s} \, ds = \frac{\lambda}{2}(e^{2t} - 1), \end{aligned}$$

it follows that

$$\begin{aligned} E[X_t] &= m + (X_0 - m)e^{-t} + \alpha\lambda(1 - e^{-t}) \\ Var(X_t) &= \frac{\lambda\alpha^2}{2}(1 - e^{-2t}). \end{aligned}$$

It is worth noting that in this case the process X_t is no more Gaussian.

7.7 The Method of Variation of Parameters

Let's start by considering the following stochastic equation

$$dX_t = \alpha X_t dW_t, \tag{7.7.17}$$

with α constant. This is the equation which, in physics is known to model the *linear noise*. Dividing by X_t yields

$$\frac{dX_t}{X_t} = \alpha \, dW_t.$$

Switch to the integral form

$$\int \frac{dX_t}{X_t} = \int \alpha \, dW_t,$$

and integrate "blindly" to get $\ln X_t = \alpha W_t + c$, with c an integration constant. This leads to the "pseudo-solution"

$$X_t = e^{\alpha W_t + c}.$$

The nomination "pseudo" stands for the fact that X_t does not satisfy the initial equation. We shall find a correct solution by letting the parameter c be a function of t. In other words, we are looking for a solution of the following type:

$$X_t = e^{\alpha W_t + c(t)}, \tag{7.7.18}$$

where the function $c(t)$ is subject to be determined. Using Ito's formula we get

$$\begin{aligned} dX_t &= d(e^{W_t + c(t)}) = e^{\alpha W_t + c(t)}(c'(t) + \alpha^2/2)dt + \alpha e^{\alpha W_t + c(t)} dW_t \\ &= X_t(c'(t) + \alpha^2/2)dt + \alpha X_t \, dW_t. \end{aligned}$$

Substituting the last term from the initial equation (7.7.17) yields

$$dX_t = X_t(c'(t) + \alpha^2/2)dt + dX_t,$$

which leads to the equation

$$c'(t) + \alpha^2/2 = 0$$

with the solution $c(t) = -\frac{\alpha^2}{2}t + k$. Substituting into (7.7.18) yields

$$X_t = e^{\alpha W_t - \frac{\alpha^2}{2}t + k}.$$

The value of the constant k is determined by taking $t = 0$. This leads to $X_0 = e^k$. Hence we have obtained the solution of the equation (7.7.17)

$$X_t = X_0 e^{\alpha W_t - \frac{\alpha^2}{2}t}.$$

Example 7.7.1 *Use the method of variation of parameters to solve the equation*

$$dX_t = X_t W_t dW_t.$$

Dividing by X_t and converting the differential equation into the equivalent integral form, we get

$$\int \frac{1}{X_t}\, dX_t = \int W_t\, dW_t.$$

The right side is a well-known stochastic integral given by

$$\int W_t\, dW_t = \frac{W_t^2}{2} - \frac{t}{2} + C.$$

The left side will be integrated "blindly" according to the rules of elementary Calculus

$$\int \frac{1}{X_t}\, dX_t = \ln X_t + C.$$

Equating the last two relations and solving for X_t we obtain the "pseudo-solution"

$$X_t = e^{\frac{W_t^2}{2} - \frac{t}{2} + c},$$

with c constant. In order to get a correct solution, we let c to depend on t and W_t. We shall assume that $c(t, W_t) = a(t) + b(W_t)$, so we are looking for a solution of the form

$$X_t = e^{\frac{W_t^2}{2} - \frac{t}{2} + a(t) + b(W_t)}.$$

Applying Ito's formula, we have

$$dX_t \quad = \quad X_t\big[-\frac{1}{2} + a'(t) + \frac{1}{2}(1 + b''(W_t))\big]dt + X_t\big(W_t + b'(W_t)\big)dW_t.$$

Subtracting the initial equation $dX_t = X_t W_t dW_t$ yields

$$0 \quad = \quad X_t\big(a'(t) + \frac{1}{2}b''(W_t)\big)dt + X_t b'(W_t)dW_t.$$

This equation is satisfied if we are able to choose the functions $a(t)$ and $b(W_t)$ such that the coefficients of dt and dW_t vanish

$$b'(W_t) = 0, \qquad a'(t) + \frac{1}{2}b''(W_t) = 0.$$

From the first equation b must be a constant. Substituting into the second equation it follows that a is also a constant. It turns out that the aforementioned "pseudo-solution" is in fact a solution. The constant $c = a + b$ is obtained letting $t = 0$. Hence the solution is given by

$$X_t = X_0 e^{\frac{W_t^2}{2} - \frac{t}{2}}.$$

Example 7.7.2 *Use the method of variation of parameters to solve the stochastic differential equation*

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

with μ and σ constants.

After dividing by X_t we bring the equation into the equivalent integral form

$$\int \frac{dX_t}{X_t} = \int \mu\, dt + \int \sigma\, dW_t.$$

Integrate on the left "blindly" and get

$$\ln X_t = \mu t + \sigma W_t + c,$$

where c is an integration constant. We arrive at the following "pseudo-solution"

$$X_t = e^{\mu t + \sigma W_t + c}.$$

Assume the constant c is replaced by a function $c(t)$, so we are looking for a solution of the form

$$X_t = e^{\mu t + \sigma W_t + c(t)}. \tag{7.7.19}$$

Apply Ito's formula and get

$$dX_t \quad = \quad X_t\big(\mu + c'(t) + \frac{\sigma^2}{2}\big)dt + \sigma X_t dW_t.$$

Subtracting the initial equation yields

$$\big(c'(t) + \frac{\sigma^2}{2}\big)dt = 0,$$

which is satisfied for $c'(t) = -\frac{\sigma^2}{2}$, with the solution $c(t) = -\frac{\sigma^2}{2}t + k$, $k \in \mathbb{R}$. Substituting into (7.7.19) yields the solution

$$X_t = e^{\mu t + \sigma W_t - \frac{\sigma^2}{2}t + k} = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t + k} = X_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}.$$

7.8 Integrating Factors

The method of integrating factors can be applied to a class of stochastic differential equations of the type

$$dX_t = f(t, X_t)dt + g(t)X_t dW_t, \tag{7.8.20}$$

where f and g are continuous deterministic functions. The integrating factor is given by

$$\rho_t = e^{-\int_0^t g(s)\, dW_s + \frac{1}{2}\int_0^t g^2(s)\, ds}.$$

The equation can be brought into the following exact form

$$d(\rho_t X_t) = \rho_t f(t, X_t)dt.$$

Substituting $Y_t = \rho_t X_t$, we obtain that Y_t satisfies the deterministic differential equation

$$dY_t = \rho_t f(t, Y_t/\rho_t)dt,$$

which can be solved by either integration or as an exact equation. We shall exemplify the method of integrating factors with a few examples.

Example 7.8.1 *Solve the stochastic differential equation*

$$dX_t = rdt + \alpha X_t dW_t, \tag{7.8.21}$$

with r and α constants.

The integrating factor is given by $\rho_t = e^{\frac{1}{2}\alpha^2 t - \alpha W_t}$. Using Ito's formula, we can easily check that

$$d\rho_t = \rho_t(\alpha^2 dt - \alpha dW_t).$$

Using $dt^2 = dt\, dW_t = 0$, $(dW_t)^2 = dt$ we obtain

$$dX_t d\rho_t = -\alpha^2 \rho_t X_t dt.$$

Multiplying by ρ_t, the initial equation becomes

$$\rho_t dX_t - \alpha \rho_t X_t dW_t = r\rho_t dt,$$

and adding and subtracting $\alpha^2 \rho_t X_t dt$ from the left side yields

$$\rho_t dX_t - \alpha \rho_t X_t dW_t + \alpha^2 \rho_t X_t dt - \alpha^2 \rho_t X_t dt = r\rho_t dt.$$

This can be written as

$$\rho_t dX_t + X_t d\rho_t + d\rho_t dX_t = r\rho_t dt,$$

which, with the virtue of the product rule, becomes

$$d(\rho_t X_t) = r\rho_t dt.$$

Integrating yields

$$\rho_t X_t = \rho_0 X_0 + r\int_0^t \rho_s\, ds$$

and hence the solution is

$$\begin{aligned} X_t &= \frac{1}{\rho_t} X_0 + \frac{r}{\rho_t}\int_0^t \rho_s\, ds \\ &= X_0 e^{\alpha W_t - \frac{1}{2}\alpha^2 t} + r\int_0^t e^{-\frac{1}{2}\alpha^2(t-s) + \alpha(W_t - W_s)}\, ds. \end{aligned}$$

Exercise 7.8.1 *Let α be a constant. Solve the following stochastic differential equations by the method of integrating factors*

(a) $dX_t = \alpha X_t dW_t$;

(b) $dX_t = X_t dt + \alpha X_t dW_t$;

(c) $dX_t = \dfrac{1}{X_t}dt + \alpha X_t dW_t,\ X_0 > 0.$

Exercise 7.8.2 *Let X_t be the solution of the stochastic equation $dX_t = \sigma X_t\, dW_t$, with σ constant. Let $A_t = \frac{1}{t}\int_0^t X_s\, dW_s$ be the stochastic average of X_t. Find the stochastic equation satisfied by A_t, the mean and variance of A_t.*

7.9 Existence and Uniqueness

An exploding solution
Consider the non-linear stochastic differential equation

$$dX_t = X_t^3 dt + X_t^2\, dW_t, \qquad X_0 = 1/a. \tag{7.9.22}$$

We shall look for a solution of the type $X_t = f(W_t)$. Ito's formula yields

$$dX_t \ = \ f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

Equating the coefficients of dt and dW_t in the last two equations yields

$$f'(W_t) \ = \ X_t^2 \Longrightarrow f'(W_t) = f(W_t)^2 \tag{7.9.23}$$
$$\frac{1}{2}f''(W_t) \ = \ X_t^3 \Longrightarrow f''(W_t) = 2f(W_t)^3 \tag{7.9.24}$$

We note that equation (7.9.23) implies (7.9.24) by differentiation. So it suffices to solve only the ordinary differential equation

$$f'(x) = f(x)^2, \qquad f(0) = 1/a.$$

Separating and integrating we have

$$\int \frac{df}{f(x)^2} = \int ds \Longrightarrow f(x) = \frac{1}{a-x}.$$

Hence a solution of equation (7.9.22) is

$$X_t = \frac{1}{a - W_t}.$$

Let T_a be the first time the Brownian motion W_t hits a. Then the process X_t is defined only for $0 \le t < T_a$. T_a is a random variable with $P(T_a < \infty) = 1$ and $E[T_a] = \infty$, see section 3.3.

The following theorem is the analog of Picard's uniqueness result from ordinary differential equations:

Theorem 7.9.1 (Existence and Uniqueness) *Consider the stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = c$$

where c is a constant and b and σ are continuous functions on $[0,T] \times \mathbb{R}$ satisfying

1. $|b(t,x)| + |\sigma(t,x)| \le C(1 + |x|); \qquad x \in R, t \in [0,T]$
2. $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x - y|, \qquad x, y \in R, t \in [0,T]$

with C, K positive constants. There there is a unique solution process X_t that is continuous and satisfies

$$E\Big[\int_0^T X_t^2\,dt\Big] < \infty.$$

The first condition says that the drift and volatility increase no faster than a linear function in x. The second conditions states that the functions are Lipschitz in the second argument.

Example 7.9.1 *Consider the stochastic differential equation*

$$dX_t = \Big(\sqrt{1 + X_t^2} + \frac{1}{2}X_t\Big)dt + \sqrt{1 + X_t^2}\,dW_t, \quad X_0 = x_0.$$

(a) *Solve the equation;*

(b) *Show that there a unique solution.*

Chapter 8

Applications of Brownian Motion

8.1 The Generator of an Ito Diffusion

Let $(X_t)_{t\geq 0}$ be a stochastic process with $X_0 = x_0$. In this section we shall deal with the operator associated with X_t. This is an operator which describes infinitesimally the rate of change of a function which depends smoothly on X_t.

More precisely, the *generator* of the stochastic process X_t is the second order partial differential operator A defined by

$$Af(x) = \lim_{t\searrow 0} \frac{E[f(X_t)] - f(x)}{t},$$

for any smooth function (at least of class C^2) with compact support $f : \mathbb{R}^n \to \mathbb{R}$. Here E stands for the expectation operator taken at $t = 0$, i.e.,

$$E[f(X_t)] = \int_{\mathbb{R}^n} f(y)p_t(x,y)\,dy,$$

where $p_t(x,y)$ is the transition density of X_t.

In the following we shall find the generator associated with the Ito diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW(t), \qquad t \geq 0, X_0 = x, \tag{8.1.1}$$

where $W(t) = \big(W_1(t), \dots, W_m(t)\big)$ is an m-dimensional Brownian motion, $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable functions.

The main tool used in deriving the formula for the generator A is Ito's formula in several variables. If let $F_t = f(X_t)$, then using Ito's formula we have

$$dF_t = \sum_i \frac{\partial f}{\partial x_i}(X_t)\,dX_t^i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t)\,dX_t^i\,dX_t^j, \tag{8.1.2}$$

where $X_t = (X_t^1, \cdots, X_t^n)$ satisfies the Ito diffusion (8.1.1) on components, i.e.,

$$\begin{aligned} dX_t^i &= b_i(X_t)dt + [\sigma(X_t)\,dW(t)]_i \\ &= b_i(X_t)dt + \sum_k \sigma_{ik}\,dW_k(t). \end{aligned} \tag{8.1.3}$$

Using the stochastic relations $dt^2 = dt\, dW_k(t) = 0$ and $dW_k(t)\, dW_r(t) = \delta_{kr} dt$, a computation provides

$$
\begin{aligned}
dX_t^i \, dX_t^j &= \Big(b_i dt + \sum_k \sigma_{ik} dW_k(t)\Big)\Big(b_j dt + \sum_k \sigma_{jk} dW_k(t)\Big) \\
&= \Big(\sum_k \sigma_{ik} dW_k(t)\Big)\Big(\sum_r \sigma_{jr} dW_r(t)\Big) \\
&= \sum_{k,r} \sigma_{ik}\sigma_{jr}\, dW_k(t) dW_r(t) = \sum_k \sigma_{ik}\sigma_{jk}\, dt \\
&= (\sigma\sigma^T)_{ij}\, dt.
\end{aligned}
$$

Therefore

$$dX_t^i \, dX_t^j = (\sigma\sigma^T)_{ij}\, dt. \tag{8.1.4}$$

Substitute (8.1.3) and (8.1.4) into (8.1.2) yields

$$
\begin{aligned}
dF_t &= \Big[\frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t)(\sigma\sigma^T)_{ij} + \sum_i b_i(X_t)\frac{\partial f}{\partial x_i}(X_t)\Big] dt \\
&\quad + \sum_{i,k} \frac{\partial f}{\partial x_i}(X_t)\sigma_{ik}(X_t)\, dW_k(t).
\end{aligned}
$$

Integrate and obtain

$$
\begin{aligned}
F_t &= F_0 + \int_0^t \Big[\frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\sigma\sigma^T)_{ij} + \sum_i b_i \frac{\partial f}{\partial x_i}\Big](X_s)\, ds \\
&\quad + \sum_k \int_0^t \sum_i \sigma_{ik}\frac{\partial f}{\partial x_i}(X_s)\, dW_k(s).
\end{aligned}
$$

Since $F_0 = f(X_0) = f(x)$ and $E(f(x)) = f(x)$, applying the expectation operator in the previous relation we obtain

$$E(F_t) = f(x) + E\Big[\int_0^t \Big(\frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial f}{\partial x_i}\Big)(X_s)\, ds\Big]. \tag{8.1.5}$$

Using the commutation between the operator E and the integral $\int_0^t$ yields

$$\lim_{t\searrow 0} \frac{E(F_t) - f(x)}{t} = \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_k b_k \frac{\partial f(x)}{\partial x_k}.$$

We conclude the previous computations with the following result.

Theorem 8.1.1 *The generator of the Ito diffusion (8.1.1) is given by*

$$A = \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_k b_k \frac{\partial}{\partial x_k}. \tag{8.1.6}$$

The matrix σ is called *dispersion* and the product $\sigma\sigma^T$ is called *diffusion* matrix. These names are related with their physical significance Substituting (8.1.6) in (8.1.5) we obtain the following formula

$$E[f(X_t)] = f(x) + E\Big[\int_0^t Af(X_s)\,ds\Big], \tag{8.1.7}$$

for any $f \in C_0^2(\mathbb{R}^n)$.

Exercise 8.1.2 *Find the generator operator associated with the n-dimensional Brownian motion.*

Exercise 8.1.3 *Find the Ito diffusion corresponding to the generator $Af(x) = f''(x) + f'(x)$.*

Exercise 8.1.4 *Let $\Delta_G = \frac{1}{2}(\partial_{x_1}^2 + x_1^2\partial_{x_2}^2)$ be the Grushin's operator.*

(*a*) *Find the diffusion process associated with the generator Δ_G.*

(*b*) *Find the diffusion and dispersion matrices and show that they are degenerate.*

8.2 Dynkin's Formula

Formula (8.1.7) holds under more general conditions, when t is a stopping time. First we need the following result, which deals with a continuity-type property in the upper limit of a Ito integral.

Lemma 8.2.1 *Let g be a bounded measurable function and τ be a stopping time for X_t with $E[\tau] < \infty$. Then*

$$\lim_{k\to\infty} E\Big[\int_0^{\tau\wedge k} g(X_s)\,dW_s\Big] = E\Big[\int_0^{\tau} g(X_s)\,dW_s\Big]; \tag{8.2.8}$$

$$\lim_{k\to\infty} E\Big[\int_0^{\tau\wedge k} g(X_s)\,ds\Big] = E\Big[\int_0^{\tau} g(X_s)\,ds\Big]. \tag{8.2.9}$$

Proof: Let $|g| < K$. Using the properties of Ito integrals, we have

$$\begin{aligned} & E\Big[\Big(\int_0^{\tau} g(X_s)\,dW_s - \int_0^{\tau\wedge k} g(X_t)\,dW_s\Big)^2\Big] = E\Big[\Big(\int_{\tau\wedge k}^{\tau} g(X_s)\,dW_s\Big)^2\Big] \\ = \; & E\Big[\int_{\tau\wedge k}^{\tau} g^2(X_s)\,ds\Big] \le K^2 E[\tau - \tau\wedge k] \to 0, \qquad k \to \infty. \end{aligned}$$

Since $E[X^2] \le E[X]^2$, it follows that

$$E\Big[\int_0^{\tau} g(X_s)\,dW_s - \int_0^{\tau\wedge k} g(X_t)\,dW_s\Big] \to 0, \qquad k \to \infty,$$

which is equivalent with relation (8.2.8).

The second relation can be proved similar and is left as an exercise for the reader. ∎

Exercise 8.2.2 *Assume the hypothesis of the previous lemma. Let* $\mathbf{1}_{\{s<\tau\}}$ *be the characteristic function of the interval* $(-\infty, \tau)$

$$\mathbf{1}_{\{s<\tau\}}(u) = \begin{cases} 1, & \text{if } u < \tau \\ 0, & \text{otherwise.} \end{cases}$$

Show that

(a) $\displaystyle\int_0^{\tau\wedge k} g(X_s)\,dW_s = \int_0^k \mathbf{1}_{\{s<\tau\}}\,g(X_s)\,dW_s,$

(b) $\displaystyle\int_0^{\tau\wedge k} g(X_s)\,ds = \int_0^k \mathbf{1}_{\{s<\tau\}}\,g(X_s)\,ds.$

Theorem 8.2.3 (Dynkin's formula) *Let* $f \in C_0^2(\mathbb{R}^n)$*, and* X_t *be an Ito diffusion starting at* x*. If* τ *is a stopping time with* $E[\tau] < \infty$*, then*

$$E[f(X_\tau)] = f(x) + E\Big[\int_0^\tau Af(X_s)\,ds\Big], \tag{8.2.10}$$

where A *is the infinitesimal generator of* X_t*.*

Proof: Replace t by k and f by $\mathbf{1}_{\{s<\tau\}}f$ in (8.1.7) and obtain

$$E[1_{s<\tau}f(X_k)] = \mathbf{1}_{\{s<\tau\}}f(x) + E\Big[\int_0^k A(\mathbf{1}_{\{s<\tau\}}f)(X_s)\,ds\Big],$$

which can be written as

$$\begin{aligned} E[f(X_{k\wedge\tau})] &= \mathbf{1}_{\{s<\tau\}}f(x) + E\Big[\int_0^k \mathbf{1}_{\{s<\tau\}}(s)A(f)(X_s)\,ds\Big] \\ &= \mathbf{1}_{\{s<\tau\}}f(x) + E\Big[\int_0^{k\wedge\tau} A(f)(X_s)\,ds\Big]. \end{aligned} \tag{8.2.11}$$

Since by Lemma 8.2.1 (b)

$$E[f(X_{k\wedge\tau})] \to E[f(X_\tau)], \qquad k \to \infty$$

$$E\Big[\int_0^{k\wedge\tau} A(f)(X_s)\,ds\Big] \to E\Big[\int_0^\tau A(f)(X_s)\,ds\Big], \qquad k \to \infty,$$

using Exercise (8.2.2) relation (8.2.11) yields (8.2.10). ∎

8.3 Applications

In this section we shall present a few important results of stochastic calculus which can be obtained as a consequence of Dynkin's formula.

8.3.1 Kolmogorov's Backward Equation

For any function $f \in C_0^2(\mathbb{R}^n)$ let $v(t,x) = E[f(X_t)]$, given that $X_0 = x$. As usual, E denotes the expectation at time $t = 0$. Then $v(0,x) = f(x)$, and differentiating in Dynkin's formula (8.1.7)

$$v(t,x) = f(x) + \int_0^t E[Af(X_s)]\,ds$$

yields

$$\frac{\partial v}{\partial t} = E[Af(X_t)] = AE[f(X_t)] = Av(t,x).$$

We arrived at the following result.

Theorem 8.3.1 (Kolmogorov's backward equation) *For any $f \in C_0^2(\mathbb{R}^n)$ the function $v(t,x) = E[f(X_t)]$ satisfies the following Cauchy's problem*

$$\begin{aligned} \frac{\partial v}{\partial t} &= Av, \qquad t > 0 \\ v(0,x) &= f(x), \end{aligned}$$

where A denotes the generator of the Ito's diffusion (8.1.1).

8.3.2 Exit Time from an Interval

Let $X_t = x_0 + W_t$ e a one-dimensional Brownian motion starting at x_0. Consider the exit time of the process X_t from the strip (a,b)

$$\tau = \inf\{t > 0; X_t \notin (a,b)\}.$$

Assuming $E[\tau] < 0$, applying Dynkin's formula yields

$$E\Big[f(X_\tau)\Big] = f(x_0) + E\Big[\int_0^\tau \frac{1}{2}\frac{d^2}{dx^2} f(X_s)\,ds\Big]. \tag{8.3.12}$$

Choosing $f(x) = x$ in (8.3.12) we obtain

$$E[X_\tau] = x_0 \tag{8.3.13}$$

Exercise 8.3.2 *Prove relation (8.3.13) using the Optional Stopping Theorem for the martingale X_t.*

Let $p_a = P(X_\tau = a)$ and $p_b = P(X_\tau = b)$ be the exit probabilities from the interval (a,b). Obviously, $p_a + p_b = 1$, since the probability that the Brownian motion will stay for ever inside the bounded interval is zero. Using the expectation definition, relation (8.3.13) yields

$$ap_a + b(1 - p_a) = x_0.$$

Solving for p_a and p_b we get the following exit probabilities

$$p_a = \frac{b - x_0}{b - a} \tag{8.3.14}$$

$$p_b = 1 - p_a = \frac{x_0 - a}{b - a}. \tag{8.3.15}$$

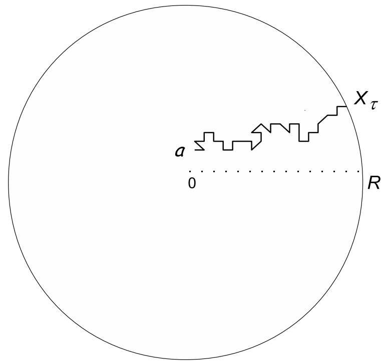


Figure 8.1: *The Brownian motion X_t in the ball $B(0, R)$.*

It is worth noting that if $b \to \infty$ then $p_a \to 1$ and if $a \to -\infty$ then $p_b \to 1$. This can be stated by saying that a Brownian motion starting at x_0 reaches any level (below or above x_0) with probability 1.

Next we shall compute the mean of the exit time, $E[\tau]$. Choosing $f(x) = x^2$ in (8.3.12) yields

$$E[(X_\tau)^2] = x_0^2 + E[\tau].$$

From the definition of the mean and formulas (8.3.14)-(8.3.15) we obtain

$$\begin{aligned} E[\tau] &= a^2 p_a + b^2 p_b - x_0^2 = a^2 \frac{b - x_0}{b - a} + b^2 \frac{x_0 - a}{b - a} - x_0 \\ &= \frac{1}{b - a}\Big[ba^2 - ab^2 + x_0(b - a)(b + a)\Big] - x_0^2 \\ &= -ab + x_0(b + a) - x_0^2 \\ &= (b - x_0)(x_0 - a). \end{aligned} \tag{8.3.16}$$

Exercise 8.3.3 (a) *Show that the equation $x^2 - (b - a)x + E[\tau] = 0$ cannot have complex roots;*

(b) *Prove that $E[\tau] \leq \dfrac{(b - a)^2}{4}$;*

(c) *Find the point $x_0 \in (a, b)$ such that the expectation of the exit time, $E[\tau]$, is maximum.*

8.3.3 Transience and Recurrence of Brownian Motion

We shall consider first the expectation of the exit time from a ball. Then we shall extend it to an annulus and compute the transience probabilities.

1. Consider the process $X_t = a + W(t)$, where $W(t) = \big(W_1(t), \ldots, W_n(t)\big)$ is an n-dimensional Brownian motion, and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is a fixed vector, see Fig. 8.1. Let $R > 0$ such

that $R > |a|$. Consider the exit time of the process X_t from the ball $B(0, R)$

$$\tau = \inf\{t > 0; |X_t| > R\}. \tag{8.3.17}$$

Assuming $E[\tau] < \infty$ and letting $f(x) = |x|^2 = x_1^2 + \cdots + x_n^2$ in Dynkin's formula

$$E\big[f(X_\tau)\big] = f(x) + E\Big[\int_0^\tau \frac{1}{2}\Delta f(X_s)\, ds\Big]$$

yields

$$R^2 = |a|^2 + E\Big[\int_0^\tau n\, ds\Big],$$

and hence

$$E[\tau] = \frac{R^2 - |a|^2}{n}. \tag{8.3.18}$$

In particular, if the Brownian motion starts from the center, i.e. $a = 0$, the expectation of the exit time is

$$E[\tau] = \frac{R^2}{n}.$$

(i) Since $R^2/2 > R^2/3$, the previous relation implies that it takes longer for a Brownian motion to exit a disk of radius R rather than a ball of the same radius.

(ii) The probability that a Brownian motion leaves the interval $(-R, R)$ is twice the probability that a 2-dimensional Brownian motion exits the disk $B(0, R)$.

Exercise 8.3.4 *Prove that $E[\tau] < \infty$, where τ is given by (8.3.17).*

Exercise 8.3.5 *Apply the Optional Stopping Theorem for the martingale $W_t = W_t^2 - t$ to show that $E[\tau] = R^2$, where*

$$\tau = \inf\{t > 0; |W_t| > R\}$$

is the first exit time of the Brownian motion from $(-R, R)$.

2. Let $b \in \mathbb{R}^n$ such that $b \notin B(0, R)$, i.e. $|b| > R$, and consider the annulus

$$A_k = \{x; R < |x| < kR\}$$

where $k > 0$ such that $b \in A_k$. Consider the process $X_t = b + W(t)$ and let

$$\tau_k = \inf\{t > 0; X_t \notin A_k\}$$

be the first exit time of X_t from the annulus A_k. Let $f : A_k \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -\ln|x|, & \text{if } n = 2 \\ \frac{1}{|x|^{n-2}}, & \text{if } n > 2. \end{cases}$$

A straightforward computation shows that $\Delta f = 0$. Substituting into Dynkin's formula

$$E\big[f(X_{\tau_k})\big] = f(b) + E\Big[\int_0^{\tau_k} \Big(\frac{1}{2}\Delta f\Big)(X_s)\, ds\Big]$$

yields

$$E\big[f(X_{\tau_k})\big] = f(b). \tag{8.3.19}$$

This can be stated by saying that the value of f at a point b in the annulus is equal to the expected value of f at the first exit time of a Brownian motion starting at b.

Since $|X_{\tau_k}|$ is a random variable with two outcomes, we have

$$E\big[f(X_{\tau_k})\big] = p_k f(R) + q_k f(kR),$$

where $p_k = P(|X_{\tau_k}| = R)$, $q_k = P(|X_{X_{\tau_k}}|) = kR$ and $p_k + q_k = 1$. Substituting in (8.3.19) yields

$$p_k f(R) + q_k f(kR) = f(b). \tag{8.3.20}$$

There are the following two distinguished cases:

(i) If $n = 2$ we obtain

$$-p_k \ln R - q_k(\ln k + \ln R) = -\ln b.$$

Using $p_k = 1 - q_k$, solving for p_k yields

$$p_k = 1 - \frac{\ln(\frac{b}{R})}{\ln k}.$$

Hence

$$P(\tau < \infty) = \lim_{k\to\infty} p_k = 1,$$

where $\tau = \inf\{t > 0; |X_t| < R\}$ is the first time X_t hits the ball $B(0, R)$. Hence in $\mathbb{R}^2$ a Brownian motion hits with probability 1 any ball. This is stated equivalently by saying that the Brownian motion is *recurrent* in $\mathbb{R}^2$.

(ii) If $n > 2$ the equation (8.3.20) becomes

$$\frac{p_k}{R^{n-2}} + \frac{q_k}{k^{n-2}R^{n-2}} = \frac{1}{b^{n-2}}.$$

Taking the limit $k \to \infty$ yields

$$\lim_{k\to\infty} p_k = \Big(\frac{R}{b}\Big)^{n-2} < 1.$$

Then in $\mathbb{R}^n$, $n > 2$, a Brownian motion starting outside of a ball hits it with a probability less than 1. This is usually stated by saying that the Brownian motion is *transient*.

3. We shall recover the previous results using the n-dimensional Bessel process

$$\mathcal{R}_t = dist\big(0, W(t)\big) = \sqrt{W_1(t)^2 + \cdots + W_n(t)^2}.$$

Consider the process $Y_t = \alpha + \mathcal{R}_t$, with $0 \le \alpha < R$. The generator of Y_t is the Bessel operator of order n

$$A = \frac{1}{2}\frac{d^2}{dx^2} + \frac{n-1}{2x}\frac{d}{dx},$$

see section 2.7. Consider the exit time

$$\tau = \{t > 0; Y_t > R\}.$$

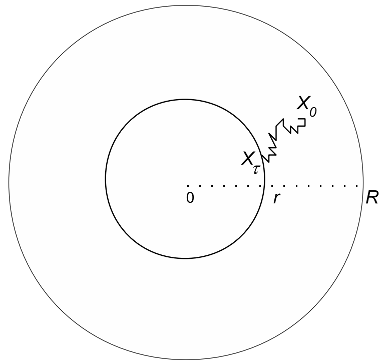


Figure 8.2: *The Brownian motion X_t in the annulus $A_{r,R}$.*

Applying Dynkin's formula

$$E\big[f(Y_\tau)\big] = f(Y_0) + E\Big[\int_0^\tau (Af)(Y_s)\,ds\Big]$$

for $f(x) = x^2$ yields $R^2 = \alpha^2 + E\big[\int_0^\tau n ds\big]$. This leads to

$$E[\tau] = \frac{R^2 - \alpha^2}{n}.$$

which recovers (8.3.18) with $\alpha = |a|$.

In the following assume $n \geq 3$ and consider the annulus

$$A_{r,R} = \{x \in \mathbb{R}^n; r < |x| < R\}.$$

Consider the stopping time $\tau = \inf\{t > 0; X_t \notin A_{r,R}\} = \inf\{t > 0; Y_t \notin (r, R)\}$, where $|Y_0| = \alpha \in (r, R)$. Applying Dynkin's formula for $f(x) = x^{2-n}$ yields $E\big[f(Y_\tau\big] = f(\alpha)$. This can be written as

$$p_r r^{2-n} + p_R R^{2-n} = \alpha^{2-n},$$

where

$$p_r = P(|X_t| = r), \qquad p_R = P(|X_t| = R), \qquad p_r + p_R = 1.$$

Solving for p_r and p_R yields

$$p_r \;=\; \frac{\left(\frac{R}{\alpha}\right)^{n-2} - 1}{\left(\frac{R}{r}\right)^{n-2} - 1}, \qquad p_R = \frac{\left(\frac{r}{\alpha}\right)^{n-2} - 1}{\left(\frac{r}{R}\right)^{n-2} - 1}.$$

The transience probability is obtained by taking the limit to infinity

$$p_r \;=\; \lim_{R\to\infty} p_{r,R} = \lim_{R\to\infty} \frac{\alpha^{2-n} R^{n-2} - 1}{r^{2-n} R^{n-2} - 1} = \left(\frac{r}{\alpha}\right)^{n-2},$$

where p_r is the probability that a Brownian motion starting outside the ball of radius r will hit the ball, see Fig. 8.2.

Exercise 8.3.6 *Solve the equation* $\frac{1}{2}f''(x)+\frac{n-1}{2x}f'(x)=0$ *by looking for a solution of monomial type* $f(x)=x^k$.

8.4 Application to Solving Parabolic Equations

This section deals with solving first and second order parabolic equations using the integral of the cost function along a certain *characteristic* solution. The first order equations are related to predictable characteristic curves, while the second order equations depend on stochastic characteristic curves.

1. Predictable characteristics Let $\varphi(s)$ be the solution of the following one-dimensional ODE

$$\begin{aligned}\frac{dX(s)}{ds} &= a\big(s, X(s)\big), \qquad t\le s\le T\\ X(t) &= x,\end{aligned}$$

and define the cumulative cost between t and T along the solution φ

$$u(t,x)=\int_t^T c\big(s,\varphi(s)\big)\,ds, \tag{8.4.21}$$

where c denotes a continuous cost function. Differentiate both sides with respect to t

$$\begin{aligned}\frac{\partial}{\partial t}u\big(t,\varphi(t)\big) &= \frac{\partial}{\partial t}\int_t^T c\big(s,\varphi(s)\big)\,ds\\ \partial_t u+\partial_x u\,\varphi'(t) &= -c\big(t,\varphi(t)\big).\end{aligned}$$

Hence (8.4.21) is a solution of the following final value problem

$$\begin{aligned}\partial_t u(t,x)+a(t,x)\partial_x u(t,x) &= -c(t,x)\\ u(T,x) &= 0.\end{aligned}$$

It is worth mentioning that this is a variant of the method of characteristics.[1] The curve given by the solution $\varphi(s)$ is called a characteristic curve.

Exercise 8.4.1 *Using the previous method solve the following final boundary problems:*
(a)

$$\begin{aligned}\partial_t u+x\partial_x u &= -x\\ u(T,x) &= 0.\end{aligned}$$

(b)

$$\begin{aligned}\partial_t u+tx\partial_x u &= \ln x, \qquad x>0\\ u(T,x) &= 0.\end{aligned}$$

[1] This is a well known method of solving linear partial differential equations.

2. Stochastic characteristics Consider the diffusion

$$\begin{aligned} dX_s &= a(s, X_s)ds + b(s, X_s)dW_s, \qquad t \le s \le T \\ X_t &= x, \end{aligned}$$

and define the stochastic cumulative cost function

$$u(t, X_t) = \int_t^T c(s, X_s)\, ds, \tag{8.4.22}$$

with the conditional expectation

$$\begin{aligned} u(t, x) &= E\Big[u(t, X_t)|X_t = x\Big] \\ &= E\Big[\int_t^T c(s, X_s)\, ds|X_t = x\Big]. \end{aligned}$$

Taking increments in both sides of (8.4.22) yields

$$du(t, X_t) = d\int_t^T c(s, X_s)\, ds.$$

Applying Ito's formula on one side and the Fundamental Theorem of Calculus on the other, we obtain

$$\partial_t u(t, x)dt + \partial_x u(t, X_t)dX_t + \frac{1}{2}\partial_x^2 u(t, t, X_t)dX_t^2 = -c(t, X_t)dt.$$

Taking the expectation $E[\,\cdot\,|X_t = x]$ on both sides yields

$$\partial_t u(t, x)dt + \partial_x u(t, x)a(t, x)dt + \frac{1}{2}\partial_x^2 u(t, x)b^2(t, x)dt = -c(t, x)dt.$$

Hence, the expected cost

$$u(t, x) = E\Big[\int_t^T c(s, X_s)\, ds|X_t = x\Big]$$

is a solution of the following second order parabolic equation

$$\begin{aligned} \partial_t u + a(t, x)\partial_x u + \frac{1}{2}b^2(t, x)\partial_x^2 u(t, x) &= -c(t, x) \\ u(T, x) &= 0. \end{aligned}$$

This represents the probabilistic interpretation of the solution of a parabolic equation.

Exercise 8.4.2 *Solve the following final boundary problems:*
(*a*)

$$\begin{aligned} \partial_t u + \partial_x u + \frac{1}{2}\partial_x^2 u &= -x \\ u(T, x) &= 0. \end{aligned}$$

(b)

$$\begin{aligned}
\partial_t u + \partial_x u + \frac{1}{2}\partial_x^2 u &= e^x, \\
u(T,x) &= 0.
\end{aligned}$$

(c)

$$\begin{aligned}
\partial_t u + \mu x \partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u &= -x, \\
u(T,x) &= 0.
\end{aligned}$$

Chapter 9

Martingales

9.1 Examples of Martingales

In this section we shall use the knowledge acquired in previous chapters to present a few important examples of martingales. These will be useful in the proof of Girsanov's theorem in the next section.

We start by recalling that a process X_t, $0 \leq t \leq T$, is an $\mathcal{F}_t$-*martingale* if

1. $E[|X_t|] < \infty$ (X_t integrable for each t);
2. X_t is $\mathcal{F}_t$-predictable;
3. the forecast of future values is the last observation: $E[X_t|\mathcal{F}_s] = X_s$, $\forall s < t$.

We shall present three important examples of martingales and some of their particular cases.

Example 9.1.1 *If $v(s)$ is a continuous function on $[0, T]$, then*

$$\boxed{X_t = \int_0^t v(s)\, dW_s}$$

is an $\mathcal{F}_t$-martingale.

Taking out the predictable part

$$\begin{aligned} E[X_t|\mathcal{F}_s] &= E\Big[\int_0^s v(\tau)\, dW_\tau + \int_s^t v(\tau)\, dW_\tau \Big| \mathcal{F}_s\Big] \\ &= X_s + E\Big[\int_s^t v(\tau)\, dW_\tau \Big| \mathcal{F}_s\Big] = X_s, \end{aligned}$$

where we used that $\int_s^t v(\tau)\, dW_\tau$ is independent of $\mathcal{F}_s$ and the conditional expectation equals the usual expectation

$$E\Big[\int_s^t v(\tau)\, dW_\tau \Big| \mathcal{F}_s\Big] = E\Big[\int_s^t v(\tau)\, dW_\tau\Big] = 0.$$

Example 9.1.2 *Let* $X_t = \int_0^t v(s)\,dW_s$ *be a process as in Example 9.1.1. Then*

$$\boxed{M_t = X_t^2 - \int_0^t v^2(s)\,ds}$$

is an $\mathcal{F}_t$*-martingale.*

The process X_t satisfies the stochastic equation $dX_t = v(t)dW_t$. By Ito's formula

$$d(X_t^2) = 2X_t dX_t + (dX_t)^2 = 2v(t)X_t dW_t + v^2(t)dt. \tag{9.1.1}$$

Integrating between s and t yields

$$X_t^2 - X_s^2 = 2\int_s^t X_\tau v(\tau)\,dW_\tau + \int_s^t v^2(\tau)\,d\tau.$$

Then separating the predictable from the unpredictable we have

$$\begin{aligned} E[M_t|\mathcal{F}_s] &= E\Big[X_t^2 - \int_0^t v^2(\tau)\,d\tau\Big|\mathcal{F}_s\Big] \\ &= E\Big[X_t^2 - X_s^2 - \int_s^t v^2(\tau)\,d\tau + X_s^2 - \int_0^s v^2(\tau)\,d\tau\Big|\mathcal{F}_s\Big] \\ &= X_s^2 - \int_0^s v^2(\tau)\,d\tau + E\Big[X_t^2 - X_s^2 - \int_s^t v^2(\tau)\,d\tau\Big|\mathcal{F}_s\Big] \\ &= M_s + 2E\Big[\int_s^t X_\tau v(\tau)\,dW_\tau\Big|\mathcal{F}_s\Big] = M_s, \end{aligned}$$

where we used relation (9.1.1) and that $\int_s^t X_\tau v(\tau)\,dW_\tau$ is totally unpredictable given the information set $\mathcal{F}_s$. In the following we shall mention a few particular cases.

1. If $v(s) = 1$, then $X_t = W_t$. In this case $M_t = W_t^2 - t$ is an $\mathcal{F}_t$-martingale.
2. If $v(s) = s$, then $X_t = \int_0^t s\,dW_s$, and hence

$$M_t = \Big(\int_0^t s\,dW_s\Big)^2 - \frac{t^3}{3}$$

is an $\mathcal{F}_t$-martingale.

Example 9.1.3 *Let* $u : [0,T] \to \mathbb{R}$ *be a continuous function. Then*

$$\boxed{M_t = e^{\int_0^t u(s)\,dW_s - \frac{1}{2}\int_0^t u^2(s)\,ds}}$$

is an $\mathcal{F}_t$*-martingale for* $0 \le t \le T$.

Consider the process $U_t = \int_0^t u(s)\,dW_s - \frac{1}{2}\int_0^t u^2(s)\,ds$. Then

$$\begin{aligned}
dU_t &= u(t)dW_t - \frac{1}{2}u^2(t)dt \\
(dU_t)^2 &= u(t)dt.
\end{aligned}$$

Then Ito's formula yields

$$\begin{aligned}
dM_t &= d(e^{U_t}) = e^{U_t}dU_t + \frac{1}{2}e^{U_t}(dU_t)^2 \\
&= e^{U_t}\Big(u(t)dW_t - \frac{1}{2}u^2(t)dt + \frac{1}{2}u^2(t)dt\Big) \\
&= u(t)M_t dW_t.
\end{aligned}$$

Integrating between s and t yields

$$M_t = M_s + \int_s^t u(\tau)M_\tau \, dW_\tau$$

Since $\displaystyle\int_s^t u(\tau)M_\tau \, dW_\tau$ is independent of $\mathcal{F}_s$, then

$$E\Big[\int_s^t u(\tau)M_\tau \, dW_\tau|\mathcal{F}_s\Big] = E\big[\int_s^t u(\tau)M_\tau \, dW_\tau\big] = 0,$$

and hence

$$E[M_t|\mathcal{F}_s] = E[M_s + \int_s^t u(\tau)M_\tau \, dW_\tau|\mathcal{F}_s] = M_s.$$

Remark 9.1.4 *The condition that $u(s)$ is continuous on $[0,T]$ can be relaxed by asking only*

$$u \in L^2[0,T] = \{u : [0,T] \to \mathbb{R}; \textit{measurable and } \int_0^t |u(s)|^2 \, ds < \infty\}.$$

It is worth noting that the conclusion still holds if the function $u(s)$ is replaced by a stochastic process $u(t,\omega)$ satisfying Novikov's condition

$$E\big[e^{\frac{1}{2}\int_0^T u^2(s,\omega)\,ds}\big] < \infty.$$

The previous process has a distinguished importance in the theory of martingales and it will be useful when proving the Girsanov theorem.

Definition 9.1.5 *Let $u \in L^2[0,T]$ be a deterministic function. Then the stochastic process*

$$M_t = e^{\int_0^t u(s)\,dW_s - \frac{1}{2}\int_0^t u^2(s)\,ds}$$

is called the exponential process induced by u.

Particular cases of exponential processes.

1. Let $u(s) = \sigma$, constant, then $M_t = e^{\sigma W_t - \frac{\sigma^2}{2}t}$ is an $\mathcal{F}_t$-martingale.

2. Let $u(s) = s$. Integrating in $d(tW_t) = tdW_t - W_t dt$ yields

$$\int_0^t s\,dW_s = tW_t - \int_0^t W_s\,ds.$$

Let $Z_t = \int_0^t W_s\,ds$ be the integrated Brownian motion. Then

$$\begin{aligned} M_t &= e^{\int_0^t d\,dW_s - \frac{1}{2}\int_0^t s^2\,ds} \\ &= e^{tW_t - \frac{t^3}{6} - Z_t} \end{aligned}$$

is an $\mathcal{F}_t$-martingale.

Example 9.1.6 *Let X_t be a solution of $dX_t = u(t)dt + dW_t$, with $u(s)$ a bounded function. Consider the exponential process*

$$M_t = e^{-\int_0^t u(s)\,dW_s - \frac{1}{2}\int_0^t u^2(s)\,ds}. \tag{9.1.2}$$

Then $Y_t = M_t X_t$ is an $\mathcal{F}_t$-martingale.

In Example 9.1.3 we obtained $dM_t = -u(t)M_t dW_t$. Then

$$dM_t\,dX_t = -u(t)M_t dt.$$

The product rule yields

$$\begin{aligned} dY_t &= M_t dX_t + X_t dM_t + dM_t\,dX_t \\ &= M_t\big(u(t)dt + dW_t\big) - X_t u(t) M_t dW_t - u(t)M_t dt \\ &= M_t\big(1 - u(t)X_t\big)dW_t. \end{aligned}$$

Integrating between s and t yields

$$Y_t = Y_s + \int_s^t M_\tau\big(1 - u(\tau)X_\tau\big)dW_\tau.$$

Since $\int_s^t M_\tau\big(1 - u(\tau)X_\tau\big)dW_\tau$ is independent of $\mathcal{F}_s$,

$$E\Big[\int_s^t M_\tau\big(1 - u(\tau)X_\tau\big)dW_\tau|\mathcal{F}_s\Big] = E\Big[\int_s^t M_\tau\big(1 - u(\tau)X_\tau\big)dW_\tau\Big] = 0,$$

and hence

$$E[Y_t|\mathcal{F}_s] = Y_s.$$

Exercise 9.1.7 *Prove that $(W_t + t)e^{-W_t - \frac{1}{2}t}$ is an $\mathcal{F}_t$-martingale.*

Exercise 9.1.8 *Let h be a continuous function. Using the properties of the Wiener integral and log-normal random variables, show that*

$$E\Big[e^{\int_0^t h(s)\,dW_s}\Big] = e^{\frac{1}{2}\int_0^t h(s)^2\,ds}.$$

Exercise 9.1.9 *Let M_t be the exponential process (9.1.2). Use the previous exercise to show that for any $t > 0$*

$$(a)\ E[M_t] = 1 \qquad\qquad (b)\ E[M_t^2] = e^{\int_0^t u(s)^2\, ds}.$$

Exercise 9.1.10 *Let $\mathcal{F}_t = \sigma\{W_u; u \le t\}$. Show that the following processes are $\mathcal{F}_t$-martingales:*

(a) $e^{t/2}\cos W_t$;

(b) $e^{t/2}\sin W_t$.

Recall that the Laplacian of a twice differentiable function f is defined by $\Delta f(x) = \sum_{j=1}^n \partial^2_{x_j} f$.

Example 9.1.11 *Consider the smooth function $f : \mathbb{R}^n \to R$, such that*

(i) $\Delta f = 0$;

(ii) $E\big[|f(W_t)|\big] < \infty$, $\forall t > 0$ *and* $x \in R$.

Then the process $X_t = f(W_t)$ is an $\mathcal{F}_t$-martingale.

Proof: It follows from the more general Example 9.1.13.

Exercise 9.1.12 *Let $W_1(t)$ and $W_2(t)$ be two independent Brownian motions. Show that $X_t = e^{W_1(t)}\cos W_2(t)$ is a martingale.*

Example 9.1.13 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that*

(i) $E\big[|f(W_t)|\big] < \infty$;

(ii) $E\Big[\int_0^t |\Delta f(W_s)|\, ds\Big] < \infty$.

Then the process $X_t = f(W_t) - \frac{1}{2}\int_0^t \Delta f(W_s)\, ds$ is a martingale.

Proof: For $0 \le s < t$ we have

$$\begin{aligned} E\big[X_t|\mathcal{F}_s\big] &= E\big[f(W_t)|\mathcal{F}_s\big] - E\Big[\frac{1}{2}\int_0^t \Delta f(W_u)\, du|\mathcal{F}_s\Big] \\ &= E\big[f(W_t)|\mathcal{F}_s\big] - \frac{1}{2}\int_0^s \Delta f(W_u)\, du - \int_s^t E\Big[\frac{1}{2}\Delta f(W_u)|\mathcal{F}_s\Big]\, du \qquad (9.1.3) \end{aligned}$$

Let $p(t, y, x)$ be the probability density function of W_t. Integrating by parts and using that p satisfies the Kolmogorov's backward equation, we have

$$\begin{aligned} E\Big[\frac{1}{2}\Delta f(W_u)|\mathcal{F}_s\Big] &= \frac{1}{2}\int p(u-s, W_s, x)\, \Delta f(x)\, dx = \frac{1}{2}\int \Delta_x p(u-s, W_s, x) f(x)\, dx \\ &= \int \frac{\partial}{\partial u} p(u-s, W_s, x) f(x)\, dx. \end{aligned}$$

Then, using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} \int_s^t E\Big[\frac{1}{2}\Delta f(W_u)|\mathcal{F}_s\Big]\, du &= \int_s^t \Big(\frac{\partial}{\partial u}\int p(u-s, W_s, x) f(x)\, dx\Big)\, du \\ &= \int p(t-s, W_s, x) f(x)\, dx - \lim_{\epsilon \searrow 0}\int p(\epsilon, W_s, x) f(x)\, dx \\ &= E\big[f(W_t)|\mathcal{F}_s\big] - \int \delta(x = W_s) f(x)\, dx \\ &= E\big[f(W_t)|\mathcal{F}_s\big] - f(W_s). \end{aligned}$$

Substituting in (9.1.3) yields

$$E\big[X_t|\mathcal{F}_s\big] = E\big[f(W_t)|\mathcal{F}_s\big] - \frac{1}{2}\int_0^s \Delta f(W_u)\,du - E\big[f(W_t)|\mathcal{F}_s\big] + f(W_s) \qquad (9.1.4)$$

$$= f(W_s) - \frac{1}{2}\int_0^s \Delta f(W_u)\,du \qquad (9.1.5)$$

$$= X_s. \qquad (9.1.6)$$

Hence X_t is an $\mathcal{F}_t$-martingale. ■

Exercise 9.1.14 *Use the Example 9.1.13 to show that the following processes are martingales:*

(*a*) $X_t = W_t^2 - t$;

(*b*) $X_t = W_t^3 - 3\int_0^t W_s\,ds$;

(*c*) $X_t = \frac{1}{n(n-1)}W_t^n - \frac{1}{2}\int_0^t W_s^{n-2}\,ds$;

(*d*) $X_t = e^{cW_t} - \frac{1}{2}c^2\int_0^t e^{W_s}\,ds$, *with* c *constant;*

(*e*) $X_t = \sin(cW_t) + \frac{1}{2}c^2\int_0^t \sin(cW_s)\,ds$, *with* c *constant.*

Exercise 9.1.15 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function such that*

(*i*) $E\big[|f(W_t)|\big] < \infty$;

(*ii*) $\Delta f = \lambda f$, λ *constant.*

Show that the process $X_t = f(W_t) - \frac{\lambda}{2}\int_0^t f(W_s)\,ds$ *is a martingale.*

9.2 Girsanov's Theorem

In this section we shall present and prove a particular version of Girsanov's theorem, which will suffice for the purpose of later applications. The application of Girsanov's theorem to finance is to show that in the study of security markets the differences between the mean rates of return can be removed.

We shall recall first a few basic notions. Let $(\Omega, \mathcal{F}, P)$ be a probability space. When dealing with an $\mathcal{F}_t$-martingale on the aforementioned probability space, the filtration $\mathcal{F}_t$ is considered to be the sigma-algebra generated by the given Brownian motion W_t, i.e. $\mathcal{F}_t = \sigma\{W_u; 0 \le u \le s\}$. By default, a martingale is considered with respect to the probability measure P, in the sense that the expectations involve an integration with respect to P

$$E^P[X] = \int_\Omega X(\omega)dP(\omega).$$

We have not used the upper script until now since there was no doubt which probability measure was used. In this section we shall use also another probability measure given by

$$dQ = M_T dP,$$

where M_T is an exponential process. This means that $Q : \mathcal{F} \to \mathbb{R}$ is given by

$$Q(A) = \int_A dQ = \int_A M_T dP, \qquad \forall A \in \mathcal{F}.$$

Since $M_T > 0$, $M_0 = 1$, using the martingale property of M_t yields

$$\begin{aligned} Q(A) &> 0, \qquad A \neq \emptyset; \\ Q(\Omega) &= \int_\Omega M_T dP = E^P[M_T] = E^P[M_T|\mathcal{F}_0] = M_0 = 1, \end{aligned}$$

which shows that Q is a probability on $\mathcal{F}$, and hence $(\Omega, \mathcal{F}, Q)$ becomes a probability space. The following transformation of expectation from the probability measure Q to P will be useful in part II. If X is a random variable

$$\begin{aligned} E^Q[X] &= \int_\Omega X(\omega)\, dQ(\omega) = \int_\Omega X(\omega) M_T(\omega)\, dP(\omega) \\ &= E^P[X M_T]. \end{aligned}$$

The following result will play a central role in proving Girsanov's theorem:

Lemma 9.2.1 *Let X_t be the Ito process*

$$dX_t = u(t)dt + dW_t, \qquad X_0 = 0,\ 0 \leq t \leq T,$$

with $u(s)$ a bounded function. Consider the exponential process

$$M_t = e^{-\int_0^t u(s)\, dW_s - \frac{1}{2}\int_0^t u^2(s)\, ds}.$$

Then X_t is an $\mathcal{F}_t$-martingale with respect to the measure

$$dQ(\omega) = M_T(\omega) dP(\omega).$$

Proof: We need to prove that X_t is an $\mathcal{F}_t$-martingale with respect to Q, so it suffices to show the following three properties:

1. Integrability of X_t. This part usually follows from standard manipulations of norms estimations. We shall do it here in detail, but we shall omit it in other proofs. Integrating in the equation of X_t between 0 and t yields

$$X_t = \int_0^t u(s)\, ds + W_t. \tag{9.2.7}$$

We start with an estimation of the expectation with respect to P

$$\begin{aligned} E^P[X_t^2] &= E^P\Big[\Big(\int_0^t u(s)\, ds\Big)^2 + 2\int_0^t u(s)\, ds\, W_t + W_t^2\Big] \\ &= \Big(\int_0^t u(s)\, ds\Big)^2 + 2\int_0^t u(s)\, ds\, E^P[W_t] + E^P[W_t^2] \\ &= \Big(\int_0^t u(s)\, ds\Big)^2 + t < \infty, \qquad \forall 0 \leq t \leq T \end{aligned}$$

where the last inequality follows from the norm estimation

$$\begin{aligned} \int_0^t u(s)\, ds &\leq \int_0^t |u(s)|\, ds \leq \Big[t\int_0^t |u(s)|^2\, ds\Big]^{1/2} \\ &\leq \Big[t\int_0^T |u(s)|^2\, ds\Big]^{1/2} = T^{1/2}\|u\|_{L^2[0,T]}. \end{aligned}$$

Next we obtain an estimation with respect to Q

$$\begin{aligned} E^Q[|X_t|]^2 &= \Big(\int_\Omega |X_t| M_T\, dP\Big)^2 \le \int_\Omega |X_t|^2\, dP \int_\Omega M_T^2\, dP \\ &= E^P[X_t^2]\, E^P[M_T^2] < \infty, \end{aligned}$$

since $E^P[X_t^2] < \infty$ and $E^P[M_T^2] = e^{\int_0^T u(s)^2\, ds} = e^{\|u\|^2_{L^2[0,T]}}$, see Exercise 9.1.9.

2. $\mathcal{F}_t$-predictability of X_t. This follows from equation (9.2.7) and the fact that W_t is $\mathcal{F}_t$-predictable.

3. Conditional expectation of X_t. From Examples 9.1.3 and 9.1.6 recall that for any $0 \le t \le T$

M_t is an $\mathcal{F}_t$-martingale with respect to probability measure P;
$X_t M_t$ is an $\mathcal{F}_t$-martingale with respect to probability measure P.

We need to verify that

$$E^Q[X_t|\mathcal{F}_s] = X_s, \qquad \forall s \le t.$$

which can be written as

$$\int_A X_t\, dQ = \int_A X_s\, dQ, \qquad \forall A \in \mathcal{F}_s.$$

Since $dQ = M_T dP$, the previous relation becomes

$$\int_A X_t M_T\, dP = \int_A X_s M_T\, dP, \qquad \forall A \in \mathcal{F}_s.$$

This can be written in terms of conditional expectation as

$$E^P[X_t M_T|\mathcal{F}_s] = E^P[X_s M_T|\mathcal{F}_s]. \tag{9.2.8}$$

We shall prove this identity by showing that both terms are equal to $X_s M_s$. Since X_s is $\mathcal{F}_s$-predictable and M_t is a martingale, the right side term becomes

$$E^P[X_s M_T|\mathcal{F}_s] = X_s E^P[M_T|\mathcal{F}_s] = X_s M_s, \qquad \forall s \le T.$$

Let $s < t$. Using the tower property (see Proposition 1.11.4, part 3), the left side term becomes

$$\begin{aligned} E^P[X_t M_T|\mathcal{F}_s] &= E^P\big[E^P[X_t M_T|\mathcal{F}_t]|\mathcal{F}_s\big] = E^P\big[X_t E^P[M_T|\mathcal{F}_t]|\mathcal{F}_s\big] \\ &= E^P\big[X_t M_t|\mathcal{F}_s\big] = X_s M_s, \end{aligned}$$

where we used that M_t and $X_t M_t$ are martingales and X_t is $\mathcal{F}_t$-predictable. Hence (9.2.8) holds and X_t is an $\mathcal{F}_t$-martingale with respect to the probability measure Q. ■

Lemma 9.2.2 *Consider the process*

$$X_t = \int_0^t u(s)\, ds + W_t, \qquad 0 \le t \le T,$$

with $u \in L^2[0,T]$ a deterministic function, and let $dQ = M_T dP$. Then

$$E^Q[X_t^2] = t.$$

Proof: Denote $U(t) = \int_0^t u(s)\,ds$. Then

$$\begin{aligned} E^Q[X_t^2] &= E^P[X_t^2 M_T] = E^P[U^2(t)M_T + 2U(t)W_t M_T + W_t^2 M_T] \\ &= U^2(t)E^P[M_T] + 2U(t)E^P[W_t M_T] + E^P[W_t^2 M_T]. \end{aligned} \tag{9.2.9}$$

From Exercise 9.1.9 (a) we have $E^P[M_T] = 1$. In order to compute $E^P[W_t M_T]$ we use the tower property and the martingale property of M_t

$$\begin{aligned} E^P[W_t M_T] &= E[E^P[W_t M_T|\mathcal{F}_t]] = E[W_t E^P[M_T|\mathcal{F}_t]] \\ &= E[W_t M_t]. \end{aligned} \tag{9.2.10}$$

Using the product rule

$$\begin{aligned} d(W_t M_t) &= M_t dW_t + W_t dM_t + dW_t dM_t \\ &= \big(M_t - u(t)M_t W_t\big)dW_t - u(t)M_t dt, \end{aligned}$$

where we used $dM_t = -u(t)M_t dW_t$. Integrating between 0 and t yields

$$W_t M_t = \int_0^t \big(M_s - u(s)M_s W_s\big)dW_s - \int_0^t u(s)M_s\,ds.$$

Taking the expectation and using the property of Ito integrals we have

$$E[W_t M_t] = -\int_0^t u(s)E[M_s]\,ds = -\int_0^t u(s)\,ds = -U(t). \tag{9.2.11}$$

Substituting into (9.2.10) yields

$$E^P[W_t M_T] = -U(t). \tag{9.2.12}$$

For computing $E^P[W_t^2 M_T]$ we proceed in a similar way

$$\begin{aligned} E^P[W_t^2 M_T] &= E[E^P[W_t^2 M_T|\mathcal{F}_t]] = E[W_t^2 E^P[M_T|\mathcal{F}_t]] \\ &= E[W_t^2 M_t]. \end{aligned} \tag{9.2.13}$$

Using the product rule yields

$$\begin{aligned} d(W_t^2 M_t) &= M_t d(W_t^2) + W_t^2 dM_t + d(W_t^2)dM_t \\ &= M_t(2W_t dW_t + dt) - W_t^2\big(u(t)M_t dW_t\big) \\ &\qquad -(2W_t dW_t + dt)\big(u(t)M_t dW_t\big) \\ &= M_t W_t\big(2 - u(t)W_t\big)dW_t + \big(M_t - 2u(t)W_t M_t\big)dt. \end{aligned}$$

Integrate between 0 and t

$$W_t^2 M_t = \int_0^t [M_s W_s\big(2 - u(s)W_s\big)]\,dW_s + \int_0^t \big(M_s - 2u(s)W_s M_s\big)\,ds,$$

and take the expected value to get

$$\begin{aligned} E[W_t^2 M_t] &= \int_0^t \big(E[M_s] - 2u(s)E[W_s M_s]\big)\,ds \\ &= \int_0^t \big(1 + 2u(s)U(s)\big)\,ds \\ &= t + U^2(t), \end{aligned}$$

where we used (9.2.11). Substituting into (9.2.13) yields

$$E^P[W_t^2 M_T] = t + U^2(t). \tag{9.2.14}$$

Substituting (9.2.12) and (9.2.14) into relation (9.2.9) yields

$$E^Q[X_t^2] \quad = \quad U^2(t) - 2U(t)^2 + t + U^2(t) = t, \tag{9.2.15}$$

which ends the proof of the lemma. ■

Now we are prepared to prove one of the most important results of stochastic calculus.

Theorem 9.2.3 (Girsanov Theorem) *Let $u \in L^2[0,T]$ be a deterministic function. Then the process*

$$X_t = \int_0^t u(s)\,ds + W_t, \qquad 0 \le t \le T$$

is a Brownian motion with respect to the probability measure Q given by

$$dQ = e^{-\int_0^T u(s)\,dW_s - \frac{1}{2}\int_0^T u(s)^2\,ds} dP.$$

Proof: In order to prove that X_t is a Brownian motion on the probability space $(\Omega, \mathcal{F}, Q)$ we shall apply Lévy's characterization theorem, see Theorem 2.1.5. Hence it suffices to show that X_t is a Wiener process. Lemma 9.2.1 implies that the process X_t satisfies the following properties:

1. $X_0 = 0$;
2. X_t is continuous in t;
3. X_t is a square integrable $\mathcal{F}_t$-martingale on the space $(\Omega, \mathcal{F}, Q)$.

The only property we still need to show is

4. $E^Q[(X_t - X_s)^2] = t - s, \qquad s < t.$

Using Lemma 9.2.2, the martingale property of W_t, the additivity and the tower property of expectations, we have

$$\begin{aligned}
E^Q[(X_t - X_s)^2] &= E^Q[X_t^2] - 2E^Q[X_t X_s] + E^Q[X_s^2] \\
&= t - 2E^Q[X_t X_s] + s \\
&= t - 2E^Q[E^Q[X_t X_s|\mathcal{F}_s]] + s \\
&= t - 2E^Q[X_s E^Q[X_t|\mathcal{F}_s]] + s \\
&= t - 2E^Q[X_s^2] + s \\
&= t - 2s + s = t - s,
\end{aligned}$$

which proves relation 4. ■

Choosing $u(s) = \lambda$, constant, we obtain the following consequence that will be useful later in finance applications.

Corollary 9.2.4 *Let W_t be a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Then the process*

$$X_t = \lambda t + W_t, \qquad 0 \le t \le T$$

is a Brownian motion on the probability space $(\Omega, \mathcal{F}, Q)$, where

$$dQ = e^{-\frac{1}{2}\lambda^2 T - \lambda W_T}\,dP.$$

Chapter 1

1.6.1 Let $X \sim N(\mu, \sigma^2)$. Then the distribution function of Y is

$$\begin{aligned} F_Y(y) &= P(Y < y) = P(\alpha X + \beta < y) = P\Big(X < \frac{y-\beta}{\alpha}\Big) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\frac{y-\beta}{\alpha}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi}\alpha\sigma} \int_{-\infty}^{y} e^{-\frac{\left(z-(\alpha\mu+\beta)\right)^2}{2\alpha^2\sigma^2}}\, dz \\ &= \frac{1}{\sqrt{2\pi}\sigma'} \int_{-\infty}^{y} e^{-\frac{(z-\mu')^2}{2(\sigma')^2}}\, dz, \end{aligned}$$

with $\mu' = \alpha\mu + \beta$, $\sigma' = \alpha\sigma$.

1.6.2 (a) Making $t = n$ yields $E[Y^n] = E[e^{nX}] = e^{\mu n + n^2\sigma^2/2}$.
(b) Let $n = 1$ and $n = 2$ in (a) to get the first two moments and then use the formula of variance.

1.11.5 The tower property

$$E\big[E[X|\mathcal{G}]|\mathcal{H}\big] = E[X|\mathcal{H}], \qquad \mathcal{H} \subset \mathcal{G}$$

is equivalent with

$$\int_A E[X|\mathcal{G}]\, dP = \int X\, dP, \qquad \forall A \in \mathcal{H}.$$

Since $A \in \mathcal{G}$, the previous relation holds by the definition on $E[X|\mathcal{G}]$.

1.11.6 (a) Direct application of the definition.
(b) $P(A) = \int_A dP = \int_\Omega \chi_A(\omega)\, dP(\omega) = E[\chi_A]$.
(d) $E[\chi_A X] = E[\chi_A]E[X] = P(A)E[X]$.
(e) We have the sequence of equivalencies

$$\begin{aligned} E[X|\mathcal{G}] &= E[X] \Leftrightarrow \int_A E[X]\, dP = \int_A X\, dP, \forall A \in \mathcal{G} \Leftrightarrow \\ E[X]P(A) &= \int_A X\, dP \Leftrightarrow E[X]P(A) = E[\chi_A], \end{aligned}$$

which follows from (d).

1.12.4 If $\mu = E[X]$ then

$$\begin{aligned} E[(X - E[X])^2] &= E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - E[X]^2 = Var[X]. \end{aligned}$$

1.12.5 From **1.12.4** we have $Var(X) = 0 \Leftrightarrow X = E[X]$, i.e. X is a constant.

1.12.6 The same proof as the one in Jensen's inequality.

1.12.7 It follows from Jensen's inequality or using properties of integrals.

1.12.12 (a) $m(t) = E[e^{tX}] = \sum_k e^{tk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{\lambda(e^t - 1)}$;
(b) It follows from the first Chernoff bound.

1.12.16 Choose $f(x) = x^{2k+1}$ and $g(x) = x^{2n+1}$.

1.13.2 By direct computation we have

$$\begin{aligned} E[(X-Y)^2] &= E[X^2] + E[Y^2] - 2E[XY] \\ &= Var(X) + E[X]^2 + Var[Y] + E[Y]^2 - 2E[X]E[Y] \\ &\quad +2E[X]E[Y] - 2E[XY] \\ &= Var(X) + Var[Y] + (E[X] - E[Y])^2 - 2Cov(X,Y). \end{aligned}$$

1.13.3 (a) Since

$$E[(X - X_n)^2] \geq E[X - X_n]^2 = (E[X_n] - E[X])^2 \geq 0,$$

the Squeeze theorem yields $\lim_{n\to\infty}(E[X_n] - E[X]) = 0$.
(b) Writing

$$X_n^2 - X^2 = (X_n - X)^2 - 2X(X - X_n),$$

and taking the expectation we get

$$E[X_n^2] - E[X^2] = E[(X_n - X)^2] - 2E[X(X - X_n)].$$

The right side tends to zero since

$$\begin{aligned} E[(X_n - X)^2] &\to 0 \\ |E[X(X - X_n)]| &\leq \int_\Omega |X(X - X_n)|\, dP \\ &\leq \Big(\int_\Omega X^2\, dP\Big)^{1/2}\Big(\int_\Omega (X - X_n)^2\, dP\Big)^{1/2} \\ &= \sqrt{E[X^2]E[(X - X_n)^2]} \to 0. \end{aligned}$$

(c) It follows from part (b).

(d) Apply Exercise **1.13.2**.

1.13.4 Using Jensen's inequality we have

$$\begin{aligned} E\big[\big(E[X_n|\mathcal{H}] - E[X|\mathcal{H}]\big)^2\big] &= E\big[\big(E[X_n - X|\mathcal{H}]\big)^2\big] \\ &\leq E\big[E[(X_n - X)^2|\mathcal{H}]\big] \\ &= E[(X_n - X)^2] \to 0, \end{aligned}$$

as $n \to 0$.

1.15.7 The integrability of X_t follows from

$$E[|X_t|] = E\big[|E[X|\mathcal{F}_t]|\big] \leq E\big[E[|X|\,|\mathcal{F}_t]\big] = E[|X|] < \infty.$$

X_t is $\mathcal{F}_t$-predictable by the definition of the conditional expectation. Using the tower property yields

$$E[X_t|\mathcal{F}_s] = E\big[E[X|\mathcal{F}_t]|\mathcal{F}_s\big] = E[X|\mathcal{F}_s] = X_s, \qquad s < t.$$

1.15.8 Since

$$E[|Z_t|] = E[|aX_t + bY_t + c|] \le |a|E[|X_t|] + |b|E[|Y_t|] + |c| < \infty$$

then Z_t is integrable. For $s < t$, using the martingale property of X_t and Y_t we have

$$E[Z_t|\mathcal{F}_s] = aE[X_t|\mathcal{F}_s] + bE[Y_t|\mathcal{F}_s] + c = aX_s + bY_s + c = Z_s.$$

1.15.9 In general the answer is no. For instance, if $X_t = Y_t$ the process X_t^2 is not a martingale, since the Jensen's inequality

$$E[X_t^2|\mathcal{F}_s] \ge \Big(E[X_t|\mathcal{F}_s]\Big)^2 = X_s^2$$

is not necessarily an identity. For instance B_t^2 is not a martingale, with B_t the Brownian motion process.

1.15.10 It follows from the identity

$$E[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = E[X_tY_t - X_sY_s|\mathcal{F}_s].$$

1.15.11 (a) Let $Y_n = S_n - E[S_n]$. We have

$$\begin{aligned} Y_{n+k} &= S_{n+k} - E[S_{n+k}] \\ &= Y_n + \sum_{j=1}^{k} X_{n+j} - \sum_{j=1}^{k} E[X_{n+j}]. \end{aligned}$$

Using the properties of expectation we have

$$\begin{aligned} E[Y_{n+k}|\mathcal{F}_n] &= Y_n + \sum_{j=1}^{k} E[X_{n+j}|\mathcal{F}_n] - \sum_{j=1}^{k} E[E[X_{n+j}]|\mathcal{F}_n] \\ &= Y_n + \sum_{j=1}^{k} E[X_{n+j}] - \sum_{j=1}^{k} E[X_{n+j}] \\ &= Y_n. \end{aligned}$$

(b) Let $Z_n = S_n^2 - Var(S_n)$. The process Z_n is an $\mathcal{F}_n$-martingale iff

$$E[Z_{n+k} - Z_n|\mathcal{F}_n] = 0.$$

Let $U = S_{n+k} - S_n$. Using the independence we have

$$\begin{aligned} Z_{n+k} - Z_n &= (S_{n+k}^2 - S_n^2) - \big(Var(S_{n+k} - Var(S_n)\big) \\ &= (S_n + U)^2 - S_n^2 - \big(Var(S_{n+k} - Var(S_n)\big) \\ &= U^2 + 2US_n - Var(U), \end{aligned}$$

so

$$\begin{aligned} E[Z_{n+k} - Z_n|\mathcal{F}_n] &= E[U^2] + 2S_nE[U] - Var(U) \\ &= E[U^2] - (E[U^2] - E[U]^2) \\ &= 0, \end{aligned}$$

since $E[U]=0$.

1.15.12 Let $\mathcal{F}_n=\sigma(X_k; k\leq n)$. Using the independence

$$E[|P_n|]=E[|X_0|]\cdots E[|X_n|]<\infty,$$

so $|P_n|$ integrable. Taking out the predictable part we have

$$\begin{aligned}E[P_{n+k}|\mathcal{F}_n] &= E[P_nX_{n+1}\cdots X_{n+k}|\mathcal{F}_n]=P_nE[X_{n+1}\cdots X_{n+k}|\mathcal{F}_n]\\ &= P_nE[X_{n+1}]\cdots E[X_{n+k}]=P_n.\end{aligned}$$

1.15.13 (a) Since the random variable $Y=\theta X$ is normally distributed with mean $\theta\mu$ and variance $\theta^2\sigma^2$, then

$$E[e^{\theta X}]=e^{\theta\mu+\frac{1}{2}\theta^2\sigma^2}.$$

Hence $E[e^{\theta X}]=1$ iff $\theta\mu+\frac{1}{2}\theta^2\sigma^2=0$ which has the nonzero solution $\theta=-2\mu/\sigma^2$.

(b) Since $e^{\theta X_i}$ are independent, integrable and satisfy $E[e^{\theta X_i}]=1$, by Exercise **1.15.12** we get that the product $Z_n=e^{\theta S_n}=e^{\theta X_1}\cdots e^{\theta X_n}$ is a martingale.

Chapter 2

2.1.4 B_t starts at 0 and is continuous in t. By Proposition 2.1.2 B_t is a martingale with $E[B_t^2]=t<\infty$. Since $B_t-B_s\sim N(0,|t-s|)$, then $E[(B_t-B_s)^2]=|t-s|$.

2.1.10 It is obvious that $X_t=W_{t+t_0}-W_{t_0}$ satisfies $X_0=0$ and that X_t is continuous in t. The increments are normal distributed $X_t-X_s=W_{t+t_0}-W_{s+t_0}\sim N(0,|t-s|)$. If $0<t_1<\cdots<t_n$, then $0<t_0<t_1+t_0<\cdots<t_n+t_0$. The increments $X_{t_{k+1}}-X_{t_k}=W_{t_{k+1}+t_0}-W_{t_k+t_0}$ are obviously independent and stationary.

2.1.11 For any $\lambda>0$, show that the process $X_t=\frac{1}{\sqrt{\lambda}}W_{\lambda t}$ is a Brownian motion. This says that the Brownian motion is invariant by scaling. Let $s<t$. Then $X_t-X_s=\frac{1}{\sqrt{\lambda}}(W_{\lambda t}-W_{\lambda s})\sim\frac{1}{\sqrt{\lambda}}N\big(0,\lambda(t-s)\big)=N(0,t-s)$. The other properties are obvious.

2.1.12 Apply Property 2.1.8.

2.1.13 Using the moment generating function, we get $E[W_t^3]=0$, $E[W_t^4]=3t^2$.

2.1.14 (a) Let $s<t$. Then

$$\begin{aligned}E[(W_t^2-t)(W_s^2-s)] &= E\Big[E[(W_t^2-t)(W_s^2-s)]|\mathcal{F}_s\Big]=E\Big[(W_s^2-s)E[(W_t^2-t)]|\mathcal{F}_s\Big]\\ &= E\Big[(W_s^2-s)^2\Big]=E\Big[W_s^4-2sW_s^2+s^2\Big]\\ &= E[W_s^4]-2sE[W_s^2]+s^2=3s^2-2s^2+s^2=2s^2.\end{aligned}$$

(b) Using part (a) we have

$$\begin{aligned}2s^2=E[(W_t^2-t)(W_s^2-s)] &= E[W_s^2W_t^2]-sE[W_t^2]-tE[W_s^2]+ts\\ &= E[W_s^2W_t^2]-st.\end{aligned}$$

Therefore $E[W_s^2W_t^2]=ts+2s^2$.

(c) $Cov(W_t^2,W_s^2)=E[W_s^2W_t^2]-E[W_s^2]E[W_t^2]=ts+2s^2-ts=2s^2$.

(d) $Corr(W_t^2, W_s^2) = \frac{2s^2}{2ts} = \frac{s}{t}$, where we used

$$Var(W_t^2) = E[W_t^4] - E[W_t^2]^2 = 3t^2 - t^2 = 2t^2.$$

2.1.15 (a) The distribution function of Y_t is given by

$$\begin{aligned} F(x) &= P(Y_t \le x) = P(tW_{1/t} \le x) = P(W_{1/t} \le x/t) \\ &= \int_0^{x/t} \phi_{1/t}(y)\, dy = \int_0^{x/t} \sqrt{t/(2\pi)} e^{ty^2/2}\, dy \\ &= \int_0^{x/\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du. \end{aligned}$$

(b) The probability density of Y_t is obtained by differentiating $F(x)$

$$p(x) = F'(x) = \frac{d}{dx}\int_0^{x/\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)},$$

and hence $Y_t \sim N(0,t)$.

(c) Using that Y_t has independent increments we have

$$\begin{aligned} Cov(Y_s, Y_t) &= E[Y_sY_t] - E[Y_s]E[Y_t] = E[Y_sY_t] \\ &= E\Big[Y_s(Y_t - Y_s) + Y_s^2\Big] = E[Y_s]E[Y_t - Y_s] + E[Y_s^2] \\ &= 0 + s = s. \end{aligned}$$

(d) Since

$$Y_t - Y_s = (t-s)(W_{1/t} - W_0) - s(W_{1/s} - W_{1/t})$$
$$E[Y_t - Y_s] = (t-s)E[W_{1/t}] - sE[W_{1/s} - W_{1/t}] = 0,$$

and

$$\begin{aligned} Var(Y_t - Y_s) &= E[(Y_t - Y_s)^2] = (t-s)^2\frac{1}{t} + s^2(\frac{1}{s} - \frac{1}{t}) \\ &= \frac{(t-s)^2 + s(t-s)}{t} = t - s. \end{aligned}$$

2.1.16 (a) Applying the definition of expectation we have

$$\begin{aligned} E[|W_t|] &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx = \int_0^{\infty} 2x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_0^{\infty} e^{-\frac{y}{2t}}\, dy = \sqrt{2t/\pi}. \end{aligned}$$

(b) Since $E[|W_t|^2] = E[W_t^2] = t$, we have

$$Var(|W_t|) = E[|W_t|^2] - E[|W_t|]^2 = t - \frac{2t}{\pi} = t(1 - \frac{2}{\pi}).$$

2.1.17 By the martingale property of $W_t^2 - t$ we have

$$E[W_t^2|\mathcal{F}_s] = E[W_t^2 - t|\mathcal{F}_s] + t = W_s^2 + t - s.$$

2.1.18 (a) Expanding

$$(W_t - W_s)^3 = W_t^3 - 3W_t^2 W_s + 3W_t W_s^2 - W_s^3$$

and taking the expectation

$$\begin{aligned} E[(W_t - W_s)^3|\mathcal{F}_s] &= E[W_t^3|\mathcal{F}_s] - 3W_s E[W_t^2] + 3W_s^2 E[W_t|\mathcal{F}_s] - W_s^3 \\ &= E[W_t^3|\mathcal{F}_s] - 3(t-s)W_s - W_s^3, \end{aligned}$$

so

$$E[W_t^3|\mathcal{F}_s] = 3(t-s)W_s + W_s^3,$$

since

$$E[(W_t - W_s)^3|\mathcal{F}_s] = E[(W_t - W_s)^3] = E[W_{t-s}^3] = 0.$$

(b) Hint: Start from the expansion of $(W_t - W_s)^4$.

2.2.3 Using that $e^{W_t - W_s}$ is stationary, we have

$$E[e^{W_t - W_s}] = E[e^{W_{t-s}}] = e^{\frac{1}{2}(t-s)}.$$

2.2.4 (a)

$$\begin{aligned} E[X_t|\mathcal{F}_s] &= E[e^{W_t}|\mathcal{F}_s] = E[e^{W_t - W_s} e^{W_s}|\mathcal{F}_s] \\ &= e^{W_s} E[e^{W_t - W_s}|\mathcal{F}_s] = e^{W_s} E[e^{W_t - W_s}] \\ &= e^{W_s} e^{t/2} e^{-s/2}. \end{aligned}$$

(b) This can be written also as

$$E[e^{-t/2} e^{W_t}|\mathcal{F}_s] = e^{-s/2} e^{W_s},$$

which shows that $e^{-t/2} e^{W_t}$ is a martingale.
(c) From the stationarity we have

$$E[e^{cW_t - cW_s}] = E[e^{c(W_t - W_s)}] = E[e^{cW_{t-s}}] = e^{\frac{1}{2}c^2(t-s)}.$$

Then for any $s < t$ we have

$$\begin{aligned} E[e^{cW_t}|\mathcal{F}_s] &= E[e^{c(W_t - W_s)} e^{cW_s}|\mathcal{F}_s] = e^{cW_s} E[e^{c(W_t - W_s)}|\mathcal{F}_s] \\ &= e^{cW_s} E[e^{c(W_t - W_s)}] = e^{cW_s} e^{\frac{1}{2}c^2(t-s)} = Y_s e^{\frac{1}{2}c^2 t}. \end{aligned}$$

Multiplying by $e^{-\frac{1}{2}c^2 t}$ yields the desired result.

2.2.5 (a) Using Exercise **2.2.3** we have

$$\begin{aligned} Cov(X_s, X_t) &= E[X_s X_t] - E[X_s]E[X_t] = E[X_s X_t] - e^{t/2} e^{s/2} \\ &= E[e^{W_s + W_t}] - e^{t/2} e^{s/2} = E[e^{W_t - W_s} e^{2(W_s - W_0)}] - e^{t/2} e^{s/2} \\ &= E[e^{W_t - W_s}] E[e^{2(W_s - W_0)}] - e^{t/2} e^{s/2} = e^{\frac{t-s}{2}} e^{2s} - e^{t/2} e^{s/2} \\ &= e^{\frac{t+3s}{2}} - e^{(t+s)/2}. \end{aligned}$$

(b) Using Exercise **2.2.4** (b), we have

$$\begin{aligned} E[X_sX_t] &= E\Big[E[X_sX_t|\mathcal{F}_s]\Big] = E\Big[X_sE[X_t|\mathcal{F}_s]\Big] \\ &= e^{t/2}E\Big[X_sE[e^{-t/2}X_t|\mathcal{F}_s]\Big] = e^{t/2}E\Big[X_s\,e^{-s/2}X_s]\Big] \\ &= e^{(t-s)/2}E[X_s^2] = e^{(t-s)/2}E[e^{2W_s}] \\ &= e^{(t-s)/2}e^{2s} = e^{\frac{t+3s}{2}}, \end{aligned}$$

and continue like in part (a).

2.2.6 Using the definition of the expectation we have

$$\begin{aligned} E[e^{2W_t^2}] &= \int e^{2x^2}\phi_t(x)\,dx = \frac{1}{\sqrt{2\pi t}}\int e^{2x^2}e^{-\frac{x^2}{2t}}\,dx \\ &= \frac{1}{\sqrt{2\pi t}}\int e^{2x^2}e^{-\frac{1-4t}{2t}x^2}\,dx = \frac{1}{\sqrt{1-4t}}, \end{aligned}$$

if $1-4t>0$. Otherwise, the integral is infinite. We used the standard integral $\displaystyle\int e^{-ax^2} = \sqrt{\pi/a}$, $a>0$.

2.3.4 It follows from the fact that Z_t is normally distributed.

2.3.5 Using the definition of covariance we have

$$\begin{aligned} Cov\Big(Z_s, Z_t\Big) &= E[Z_sZ_t] - E[Z_s]E[Z_t] = E[Z_sZ_t] \\ &= E\Big[\int_0^s W_u\,du\cdot\int_0^t W_v\,dv\Big] = E\Big[\int_0^s\int_0^t W_uW_v\,dudv\Big] \\ &= \int_0^s\int_0^t E[W_uW_v]\,dudv = \int_0^s\int_0^t \min\{u,v\}\,dudv \\ &= s^2\Big(\frac{t}{2}-\frac{s}{6}\Big), \qquad s<t. \end{aligned}$$

2.3.6 (a) Using Exercise **2.3.5**

$$\begin{aligned} Cov(Z_t, Z_t - Z_{t-h}) &= Cov(Z_t, Z_t) - Cov(Z_t, Z_{t-h}) \\ &= \frac{t^3}{3} - (t-h)^2\Big(\frac{t}{2}-\frac{t-h}{6}\Big) \\ &= \frac{1}{2}t^2h + o(h). \end{aligned}$$

(b) Using $Z_t - Z_{t-h} = \int_{t-h}^t W_u\,du = hW_t + o(h)$,

$$\begin{aligned} Cov(Z_t, W_t) &= \frac{1}{h}Cov(Z_t, Z_t - Z_{t-h}) \\ &= \frac{1}{h}\Big(\frac{1}{2}t^2h + o(h)\Big) = \frac{1}{2}t^2. \end{aligned}$$

2.3.7 Let $s < u$. Since W_t has independent increments, taking the expectation in

$$e^{W_s+W_t} = e^{W_t-W_s}e^{2(W_s-W_0)}$$

we obtain

$$\begin{aligned} E[e^{W_s+W_t}] &= E[e^{W_t-W_s}]E[e^{2(W_s-W_0)}] = e^{\frac{u-s}{2}}e^{2s} \\ &= e^{\frac{u+s}{2}}e^{s} = e^{\frac{u+s}{2}}e^{\min\{s,t\}}. \end{aligned}$$

2.3.8 (a) $E[X_t] = \int_0^t E[e^{W_s}]\,ds = \int_0^t E[e^{s/2}]\,ds = 2(e^{t/2}-1)$
(b) Since $Var(X_t) = E[X_t^2] - E[X_t]^2$, it suffices to compute $E[X_t^2]$. Using Exercise **2.3.7** we have

$$\begin{aligned} E[X_t^2] &= E\Big[\int_0^t e^{W_t}\,ds \cdot \int_0^t e^{W_u}\,du\Big] = E\Big[\int_0^t\int_0^t e^{W_s}e^{W_u}\,dsdu\Big] \\ &= \int_0^t\int_0^t E[e^{W_s+W_u}]\,dsdu = \int_0^t\int_0^t e^{\frac{u+s}{2}}e^{\min\{s,t\}}\,dsdu \\ &= \iint_{D_1} e^{\frac{u+s}{2}}e^{s}duds + \iint_{D_2} e^{\frac{u+s}{2}}e^{u}duds \\ &= 2\iint_{D_2} e^{\frac{u+s}{2}}e^{u}duds = \frac{4}{3}\Big(\frac{1}{2}e^{2t} - 2e^{t/2} + \frac{3}{2}\Big), \end{aligned}$$

where $D_1\{0 \le s < u \le t\}$ and $D_2\{0 \le u < s \le t\}$. In the last identity we applied Fubini's theorem. For the variance use the formula $Var(X_t) = E[X_t^2] - E[X_t]^2$.

2.3.9 (a) Splitting the integral at t and taking out the predictable part, we have

$$\begin{aligned} E[Z_T|\mathcal{F}_t] &= E[\int_0^T W_u\,du|\mathcal{F}_t] = E[\int_0^t W_u\,du|\mathcal{F}_t] + E[\int_t^T W_u\,du|\mathcal{F}_t] \\ &= Z_t + E[\int_t^T W_u\,du|\mathcal{F}_t] \\ &= Z_t + E[\int_t^T (W_u - W_t + W_t)\,du|\mathcal{F}_t] \\ &= Z_t + E[\int_t^T (W_u - W_t)\,du|\mathcal{F}_t] + W_t(T-t) \\ &= Z_t + E[\int_t^T (W_u - W_t)\,du] + W_t(T-t) \\ &= Z_t + \int_t^T E[W_u - W_t]\,du + W_t(T-t) \\ &= Z_t + W_t(T-t), \end{aligned}$$

since $E[W_u - W_t] = 0$.
(b) Let $0 < t < T$. Using (a) we have

$$\begin{aligned} E[Z_T - TW_T|\mathcal{F}_t] &= E[Z_T|\mathcal{F}_t] - TE[W_T|\mathcal{F}_t] \\ &= Z_t + W_t(T-t) - TW_t \\ &= Z_t - tW_t. \end{aligned}$$

2.4.1

$$\begin{aligned}
E[V_T|\mathcal{F}_t] &= E\Big[e^{\int_0^t W_u\,du+\int_t^T W_u\,du}|\mathcal{F}_t\Big]\\
&= e^{\int_0^t W_u\,du}E\Big[e^{\int_t^T W_u\,du}|\mathcal{F}_t\Big]\\
&= e^{\int_0^t W_u\,du}E\Big[e^{\int_t^T (W_u-W_t)\,du+(T-t)W_t}|\mathcal{F}_t\Big]\\
&= V_t e^{(T-t)W_t}E\Big[e^{\int_t^T (W_u-W_t)\,du}|\mathcal{F}_t\Big]\\
&= V_t e^{(T-t)W_t}E\Big[e^{\int_t^T (W_u-W_t)\,du}\Big]\\
&= V_t e^{(T-t)W_t}E\Big[e^{\int_0^{T-t} W_\tau\,d\tau}\Big]\\
&= V_t e^{(T-t)W_t}e^{\frac{1}{2}\frac{(T-t)^3}{3}}.
\end{aligned}$$

2.6.1

$$\begin{aligned}
F(x) &= P(Y_t\le x)=P(\mu t+W_t\le x)=P(W_t\le x-\mu t)\\
&= \int_0^{x-\mu t}\frac{1}{\sqrt{2\pi t}}e^{-\frac{u^2}{2t}}\,du;\\
f(x) &= F'(x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-\mu t)^2}{2t}}.
\end{aligned}$$

2.7.2 Since

$$P(R_t\le\rho)=\int_0^\rho \frac{1}{t}xe^{-\frac{x^2}{2t}}\,dx,$$

use the inequality

$$1-\frac{x^2}{2t}<e^{-\frac{x^2}{2t}}<1$$

to get the desired result.

2.7.3

$$\begin{aligned}
E[R_t] &= \int_0^\infty xp_t(x)\,dx=\int_0^\infty \frac{1}{t}x^2e^{-\frac{x^2}{2t}}\,dx\\
&= \frac{1}{2t}\int_0^\infty y^{1/2}e^{-\frac{y}{2t}}\,dy=\sqrt{2t}\int_0^\infty z^{\frac{3}{2}-1}e^{-z}\,dz\\
&= \sqrt{2t}\,\Gamma\big(\frac{3}{2}\big)=\frac{\sqrt{2\pi t}}{2}.
\end{aligned}$$

Since $E[R_t^2]=E[W_1(t)^2+W_2(t)^2]=2t$, then

$$Var(R_t) = 2t-\frac{2\pi t}{4}=2t\big(1-\frac{\pi}{4}\big).$$

2.7.4

$$\begin{aligned}
E[X_t] &= \frac{E[X_t]}{t}=\frac{\sqrt{2\pi t}}{2t}=\sqrt{\frac{\pi}{2t}}\to 0, \qquad t\to\infty;\\
Var(X_t) &= \frac{1}{t^2}Var(R_t)=\frac{2}{t}(1-\frac{\pi}{4})\to 0, \qquad t\to\infty.
\end{aligned}$$

By Example 1.13.1 we get $X_t \to 0$, $t \to \infty$ in mean square.

2.8.2

$$\begin{aligned} P(N_t - N_s = 1) &= \lambda(t-s)e^{-\lambda(t-s)} \\ &= \lambda(t-s)\big(1 - \lambda(t-s) + o(t-s)\big) \\ &= \lambda(t-s) + o(t-s). \end{aligned}$$

$$\begin{aligned} P(N_t - N_s \geq 1) &= 1 - P(N_t - N_s = 0) + P(N_t - N_s = 1) \\ &= 1 - e^{-\lambda(t-s)} - \lambda(t-s)e^{-\lambda(t-s)} \\ &= 1 - \Big(1 - \lambda(t-s) + o(t-s)\Big) \\ & \quad -\lambda(t-s)\Big(1 - \lambda(t-s) + o(t-s)\Big) \\ &= \lambda^2(t-s)^2 = o(t-s). \end{aligned}$$

2.8.6 Write first as

$$\begin{aligned} N_t^2 &= N_t(N_t - N_s) + N_t N_s \\ &= (N_t - N_s)^2 + N_s(N_t - N_s) + (N_t - \lambda t)N_s + \lambda t N_s, \end{aligned}$$

then

$$\begin{aligned} E[N_t^2|\mathcal{F}_s] &= E[(N_t - N_s)^2|\mathcal{F}_s] + N_s E[N_t - N_s|\mathcal{F}_s] + E[N_t - \lambda t|\mathcal{F}_s]N_s + \lambda t N_s \\ &= E[(N_t - N_s)^2] + N_s E[N_t - N_s] + E[N_t - \lambda t]N_s + \lambda t N_s \\ &= \lambda(t-s) + \lambda^2(t-s)^2 + \lambda t N_s + N_s^2 - \lambda s N_s + \lambda t N_s \\ &= \lambda(t-s) + \lambda^2(t-s)^2 + 2\lambda(t-s)N_s + N_s^2 \\ &= \lambda(t-s) + [N_s + \lambda(t-s)]^2. \end{aligned}$$

Hence $E[N_t^2|\mathcal{F}_s] \neq N_s^2$ and hence the process N_t^2 is not an $\mathcal{F}_s$-martingale.

2.8.7 (a)

$$\begin{aligned} m_{N_t}(x) &= E[e^{xN_t}] = \sum_{k\geq 0} e^{xh} P(N_t = k) \\ &= \sum_{k\geq 0} e^{xk} e^{-\lambda t} \frac{\lambda^k t^k}{k!} \\ &= e^{-\lambda t} e^{\lambda t e^x} = e^{\lambda t(e^x - 1)}. \end{aligned}$$

(b) $E[N_t^2] = m''_{N_t}(0) = \lambda^2 t^2 + \lambda t$. Similarly for the other relations.

2.8.8 $E[X_t] = E[e^{N_t}] = m_{N_t}(1) = e^{\lambda t(e-1)}$.

2.8.9 (a) Since $e^{xM_t} = e^{x(N_t - \lambda t)} = e^{-\lambda t x} e^{xN_t}$, the moment generating function is

$$\begin{aligned} m_{M_t}(x) &= E[e^{xM_t}] = e^{-\lambda t x} E[e^{xN_t}] \\ &= e^{-\lambda t x} e^{\lambda t(e^x - 1)} = e^{\lambda t(e^x - x - 1)}. \end{aligned}$$

(b) For instance

$$E[M_t^3] = m'''_{M_t}(0) = \lambda t.$$

Since M_t is a stationary process, $E[(M_t - M_s)^3] = \lambda(t-s)$.

2.8.10

$$\begin{aligned} Var[(M_t - M_s)^2] &= E[(M_t - M_s)^4] - E[(M_t - M_s)^2]^2 \\ &= \lambda(t-s) + 3\lambda^2(t-s)^2 - \lambda^2(t-s)^2 \\ &= \lambda(t-s) + 2\lambda^2(t-s)^2. \end{aligned}$$

2.8.16 (a) $\quad E[U_t] = E\Big[\int_0^t N_u\,du\Big] = \int_0^t E[N_u]\,du = \int_0^t \lambda u\,du = \frac{\lambda t^2}{2}.$

$(b)\quad E\big[\sum_{k=1}^{N_t} S_k\big] = E[tN_t - U_t] = t\lambda t - \dfrac{\lambda t^2}{2} = \dfrac{\lambda t^2}{2}.$

2.8.18 The proof cannot go through because a product between a constant and a Poisson process is not a Poisson process.

2.8.19 Let $p_X(x)$ be the probability density of X. If $p(x,y)$ is the joint probability density of X and Y, then $p_X(x) = \sum_y p(x,y)$. We have

$$\begin{aligned} \sum_{y\geq 0} E[X|Y=y]P(Y=y) &= \sum_{y\geq 0}\int x p_{X|Y=y}(x|y)P(Y=y)\,dx \\ &= \sum_{y\geq 0}\int \frac{xp(x,y)}{P(Y=y)}P(Y=y)\,dx = \int x\sum_{y\geq 0} p(x,y)\,dx \\ &= \int x p_X(x)\,dx = E[X]. \end{aligned}$$

2.8.21 (a) Since T_k has an exponential distribution with parameter λ

$$E[e^{-\sigma T_k}] = \int_0^\infty e^{-\sigma x}\lambda e^{-\lambda x}\,dx = \frac{\lambda}{\lambda+\sigma}.$$

(b) We have

$$\begin{aligned} U_t &= T_2 + 2T_3 + 3T_4 + \cdots + (n-2)T_{n-1} + (n-1)T_n + (t - S_n)n \\ &= T_2 + 2T_3 + 3T_4 + \cdots + (n-2)T_{n-1} + (n-1)T_n + nt - n(T_1 + T_2 + \cdots + T_n) \\ &= nt - [nT_1 + (n-1)T_2 + \cdots + T_n]. \end{aligned}$$

(c) Using that the arrival times S_k, $k = 1, 2, \dots n$, have the same distribution as the order statistics $U_{(k)}$ corresponding to n independent random variables uniformly distributed on the interval $(0, t)$, we get

$$\begin{aligned} E\Big[e^{-\sigma U_t}\Big|N_t = n\Big] &= E[e^{-\sigma(tN_t - \sum_{k=1}^{N_t} S_k)}|N_t = n] \\ &= e^{-n\sigma t}E[e^{\sigma\sum_{i=1}^n U_{(i)}}] = e^{-n\sigma t}E[e^{\sigma\sum_{i=1}^n U_i}] \\ &= e^{-n\sigma t}E[e^{\sigma U_1}]\cdots E[e^{\sigma U_n}] \\ &= e^{-n\sigma t}\frac{1}{t}\int_0^t e^{\sigma x_1}\,dx_1 \cdots \frac{1}{t}\int_0^t e^{\sigma x_n}\,dx_n \\ &= \frac{(1-e^{-\sigma t})^n}{\sigma^n t^n}. \end{aligned}$$

(d) Using Exercise **2.8.19** we have

$$\begin{aligned} E\Big[e^{-\sigma U_t}\Big] &= \sum_{n\geq 0} P(N_t = n) E\Big[e^{-\sigma U_t}\Big| N_t = n\Big] \\ &= \sum_{n\geq 0} \frac{e^{-\lambda} t^n \lambda^n}{n!} \frac{(1-e^{-\sigma t})^n}{\sigma^n t^n} \\ &= e^{\lambda(1-e^{-\sigma t})/\sigma - \lambda}. \end{aligned}$$

3.9.6 (a) $dt\,dN_t = dt(dM_t + \lambda dt) = dt\,dM_t + \lambda dt^2 = 0$

(b) $dW_t\,dN_t = dW_t(dM_t + \lambda dt) = dW_t dM_t + \lambda dW_t dt = 0$

2.9.6 Use Doob's inequality for the submartingales W_t^2 and $|W_t|$, and use that $E[W_t^2] = t$ and $E[|W_t|] = \sqrt{2t/\pi}$, see Exercise 2.1.16 (a).

2.9.7 Divide by t in the inequality from Exercise 2.9.6 part (b).

2.9.10 Let $\sigma = n$ and $\tau = n + 1$. Then

$$E\Big[\sup_{n\leq t\leq n=1} \Big(\frac{N_t}{t} - \lambda\Big)^2\Big] \leq \frac{4\lambda(n+1)}{n^2}.$$

The result follows by taking $n \to \infty$ in the sequence of inequalities

$$0 \leq E\Big[\Big(\frac{N_t}{t} - \lambda\Big)^2\Big] \leq E\Big[\sup_{n\leq t\leq n=1} \Big(\frac{N_t}{t} - \lambda\Big)^2\Big] \leq \frac{4\lambda(n+1)}{n^2}.$$

Chapter 3

3.1.2 We have

$$\{\omega; \tau(\omega) \leq t\} = \begin{cases} \Omega, & \text{if } c \leq t \\ \emptyset, & \text{if } c > t \end{cases}$$

and use that $\emptyset, \Omega \in \mathcal{F}_t$.

3.1.3 First we note that

$$\{\omega; \tau(\omega) < t\} = \bigcup_{0<s<t} \{\omega; |W_s(\omega)| > K\}. \qquad (9.2.16)$$

This can be shown by double inclusion. Let $A_s = \{\omega; |W_s(\omega)| > K\}$.
" $\subset$ " Let $\omega \in \{\omega; \tau(\omega) < t\}$, so $\inf\{s > 0; |W_s(\omega)| > K\} < t$. Then exists $\tau < u < t$ such that $|W_u(\omega)| > K$, and hence $\omega \in A_u$.
" $\supset$ " Let $\omega \in \bigcup_{0<s<t}\{\omega; |W_s(\omega)| > K\}$. Then there is $0 < s < t$ such that $|W_s(\omega)| > K$. This implies $\tau(\omega) < s$ and since $s < t$ it follows that $\tau(\omega) < t$.
Since W_t is continuous, then (9.2.16) can be also written as

$$\{\omega; \tau(\omega) < t\} = \bigcup_{0<r<t, r\in\mathbb{Q}} \{\omega; |W_r(\omega)| > K\},$$

which implies $\{\omega; \tau(\omega) < t\} \in \mathcal{F}_t$ since $\{\omega; |W_r(\omega)| > K\} \in \mathcal{F}_t$, for $0 < r < t$. Next we shall show that $P(\tau < \infty) = 1$.

$$\begin{aligned} P(\{\omega; \tau(\omega) < \infty\}) &= P\Big(\bigcup_{0<s}\{\omega; |W_s(\omega)| > K\}\Big) > P(\{\omega; |W_s(\omega)| > K\}) \\ &= 1 - \int_{|x|<K} \frac{1}{\sqrt{2\pi s}} e^{-\frac{y^2}{2s}}\, dy > 1 - \frac{2K}{\sqrt{2\pi s}} \to 1, \quad s \to \infty. \end{aligned}$$

Hence τ is a stopping time.

3.1.4 Let $K_m = [a + \frac{1}{m}, b - \frac{1}{m}]$. We can write

$$\{\omega; \tau \leq t\} = \bigcap_{m\geq 1} \bigcup_{r<t, r\in\mathbb{Q}} \{\omega; X_r \notin K_m\} \in \mathcal{F}_t,$$

since $\{\omega; X_r \notin K_m\} = \{\omega; X_r \in \overline{K}_m\} \in \mathcal{F}_r \subset \mathcal{F}_t$.

3.1.8 (a) We have $\{\omega; c\tau \leq t\} = \{\omega; \tau \leq t/c\} \in \mathcal{F}_{t/c} \subset \mathcal{F}_t$. And $P(c\tau < \infty) = P(\tau < \infty) = 1$.
(b) $\{\omega; f(\tau) \leq t\} = \{\omega; \tau \leq f^{-1}(t)\} = \mathcal{F}_{f^{-1}(t)} \subset \mathcal{F}_t$, since $f^{-1}(t) \leq t$. If f is bounded, then it is obvious that $P(f(\tau) < \infty) = 1$. If $\lim_{t\to\infty} f(t) = \infty$, then $P(f(\tau) < \infty) = P\big(\tau < f^{-1}(\infty)\big) = P(\tau < \infty) = 1$.
(c) Apply (b) with $f(x) = e^x$.

3.1.10 If let $G(n) = \{x; |x-a| < \frac{1}{n}\}$, then $\{a\} = \bigcap_{n\geq 1} G(n)$. Then $\tau_n = \inf\{t \geq 0; W_t \in G(n)\}$ are stopping times. Since $\sup_n \tau_n = \tau$, then τ is a stopping time.

3.2.3 The relation is proved by verifying two cases:
(i) If $\omega \in \{\omega; \tau > t\}$ then $(\tau \wedge t)(\omega) = t$ and the relation becomes

$$M_\tau(\omega) = M_t(\omega) + M_\tau(\omega) - M_t(\omega).$$

(ii) If $\omega \in \{\omega; \tau \leq t\}$ then $(\tau \wedge t)(\omega) = \tau(\omega)$ and the relation is equivalent with the obvious relation

$$M_\tau = M_\tau.$$

3.2.5 Taking the expectation in $E[M_\tau|\mathcal{F}_\sigma] = M_\sigma$ yields $E[M_\tau] = E[M_\sigma]$, and then make $\sigma = 0$.

3.3.9 Since $M_t = W_T^2 - t$ is a martingale with $E[M_t] = 0$, by the Optional Stopping Theorem we get $E[M_{\tau_a}] = E[M_0] = 0$, so $E[W_{\tau_a}^2 - \tau_a] = 0$, from where $E[\tau_a] = E[W_{\tau_a}^2] = a^2$, since $W_{\tau_a} = a$.

3.3.10 (a)

$$\begin{aligned} F(a) = P(X_t \leq a) &= 1 - P(X_t > a) = 1 - P(\max_{0\leq s\leq t} W_s > a) \\ &= 1 - P(T_a \leq t) = 1 - \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2/2}\, dy - \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{|a|/\sqrt{t}} e^{-y^2/2}\, dy. \end{aligned}$$

(b) The density function is $p(a) = F'(a) = \frac{2}{\sqrt{2\pi t}} e^{-a^2/(2t)}$, $a > 0$. Then

$$E[X_t] \quad = \quad \int_0^\infty x p(x)\, dx = \int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)}\, dy = \frac{1}{2}.$$

3.3.16 It is recurrent since $P(\exists t > 0 : a < W_t < b) = 1$.

3.4.4 Since

$$P(W_t > 0; t_1 \le t \le t_2) = \frac{1}{2} P(W_t \neq 0; t_1 \le t \le t_2) = \frac{1}{\pi} \arcsin \sqrt{\frac{t_1}{t_2}},$$

using the independence

$$P(W_t^1 > 0, W_t^2) = P(W_t^1 > 0) P(W_t^2 > 0) = \frac{1}{\pi^2} \Big(\arcsin \sqrt{\frac{t_1}{t_2}} \Big)^2.$$

The probability for $W_t = (W_t^1, W_t^2)$ to be in one of the quadrants is $\dfrac{4}{\pi^2} \Big(\arcsin \sqrt{\dfrac{t_1}{t_2}} \Big)^2$.

3.5.2 (a) We have

$$P(X_t \text{ goes up to } \alpha) = P(X_t \text{ goes up to } \alpha \text{ before down to } -\infty) = \lim_{\beta \to \infty} \frac{e^{2\mu\beta} - 1}{e^{2\mu\beta} - e^{-2\mu\alpha}} = 1.$$

3.5.3

$$P(X_t \text{ never hits } -\beta) = P(X_t \text{ goes up to } \infty \text{ before down to } -\beta) = \lim_{\alpha \to \infty} \frac{e^{2\mu\beta} - 1}{e^{2\mu\beta} - e^{-2\mu\alpha}}.$$

3.5.4 (a) Use that $E[X_T] = \alpha p_\alpha - \beta(1 - p_\alpha)$; (b) $E[X_T^2] = \alpha^2 p_\alpha + \beta^2 (1 - p_\alpha)$, with $p_\alpha = \frac{e^{2\mu\beta} - 1}{e^{2\mu\beta} - e^{-2\mu\alpha}}$; (c) Use $Var(T) = E[T^2] - E[T]^2$.

3.5.7 Since $M_t = W_t^2 - t$ is a martingale, with $E[M_t] = 0$, by the Optional Stopping Theorem we get $E[W_T^2 - T] = 0$. Using $W_T = X_T - \mu T$ yields

$$E[X_T^2 - 2\mu T X_T + \mu^2 T^2] = E[T].$$

Then

$$E[T^2] = \frac{E[T](1 + 2\mu E[X_T]) - E[X_T^2]}{\mu^2}.$$

Substitute $E[X_T]$ and $E[X_T^2]$ from Exercise **3.5.4** and $E[T]$ from Proposition 3.5.5.

3.6.10 See the proof of Proposition 3.6.4.

3.6.12 (b) Applying the Optional Stopping Theorem

$$\begin{aligned} E[e^{cM_T - \lambda T(e^c - c - 1)}] = E[X_0] &= 1 \\ E[e^{ca - \lambda T f(c)}] &= 1 \\ E[e^{-\lambda T f(c)}] &= e^{-ac}. \end{aligned}$$

Let $s = f(c)$, so $c = \varphi(s)$. Then $E[e^{-\lambda s T}] = e^{-a\varphi(s)}$.

(c) Differentiating and taking $s = 0$ yields

$$\begin{aligned} -\lambda E[T] &= -ae^{-a\varphi(0)}\varphi'(0) \\ &= -a\frac{1}{f'(0)} = -\infty, \end{aligned}$$

so $E[T] = \infty$.

(d) The inverse Laplace transform $\mathcal{L}^{-1}\Big(e^{-a\varphi(s)}\Big)$ cannot be represented by elementary functions.

3.8.8 Use that if W_t is a Brownian motion then also $tW_{1/t}$ is a Brownian motion.

3.8.11 Use $E[(|X_t| - 0)^2] = E[|X_t|^2] = E[(X_t - 0)^2]$.

3.8.15 Let $a_n = \ln b_n$. Then $G_n = e^{\ln G_n} = e^{\frac{a_1+\cdots+a_n}{n}} \to e^{\ln L} = L$.

3.8.20 (a) $L = 1$. (b) A computation shows

$$\begin{aligned} E[(X_t - 1)^2] &= E[X_t^2 - 2X_t + 1] = E[X_t^2] - 2E[X_t] + 1 \\ &= Var(X_t) + (E[X_t] - 1)^2. \end{aligned}$$

(c) Since $E[X_t] = 1$, we have $E[(X_t - 1)^2] = Var(X_t)$. Since

$$Var(X_t) = e^{-t}E[e^{W_t}] = e^{-t}(e^{2t} - e^t) = e^t - 1,$$

then $E[(X_t - 1)^2]$ does not tend to 0 as $t \to \infty$.

Chapter 4

4.0.9 (a) Use either the definition or the moment generation function to show that $E[W_t^4] = 3t^2$. Using stationarity, $E[(W_t - W_s)^4] = E[W_{t-s}^4] = 3(t-s)^2$.

4.2.1 (a) $E[\int_0^T dW_t] = E[W_T] = 0$.

(b) $E[\int_0^T W_t\, dW_t] = E[\frac{1}{2}W_T^2 - \frac{1}{2}T] = 0$.

(c) $Var(\int_0^T W_t\, dW_t) = E[(\int_0^T W_t\, dW_t)^2] = E[\frac{1}{4}W_T^2 + \frac{1}{4}T^2 - \frac{1}{2}TW_T^2] = \frac{T^2}{2}$.

3.9.3 (a) $E[(dW_t)^2 - dt^2] = E[(dW_t)^2] - dt^2 = 0$.

$$(b) \qquad \begin{aligned} Var((dW_t)^2 - dt) &= E[(dW_t^2 - dt)^2] = E[(dW_t)^4 - 2dtdW_t + dt^2] \\ &= 3st^2 - 2dt \cdot 0 + dt^2 = 4dt^2. \end{aligned}$$

4.4.2 $X \sim N(0, \int_1^T \frac{1}{t}dt) = N(0, \ln T)$.

4.4.3 $Y \sim N(0, \int_1^T t dt) = N\Big(0, \frac{1}{2}(T^2 - 1)\Big)$

4.4.4 Normally distributed with zero mean and variance $\displaystyle\int_0^t e^{2(t-s)}\, ds = \frac{1}{2}(e^{2t} - 1)$.

4.4.5 Using the property of Wiener integrals, both integrals have zero mean and variance $\dfrac{7t^3}{3}$.

4.4.6 The mean is zero and the variance is $t/3 \to 0$ as $t \to 0$.

4.4.7 Since it is a Wiener integral, X_t is normally distributed with zero mean and variance

$$\int_0^t \Big(a + \frac{bu}{t}\Big)^2 du = (a^2 + \frac{b^2}{3} + ab)t.$$

Hence $a^2 + \frac{b^2}{3} + ab = 1$.

4.4.8 Since both W_t and $\int_0^t f(s)\,dW_s$ have the mean equal to zero,

$$\begin{aligned} Cov\Big(W_t, \int_0^t f(s)\,dW_s\Big) &= E[W_t, \int_0^t f(s)\,dW_s] = E[\int_0^t dW_s \int_0^t f(s)\,dW_s] \\ &= E[\int_0^t f(u)\,ds] = \int_0^t f(u)\,ds. \end{aligned}$$

The general result is

$$Cov\Big(W_t, \int_0^t f(s)\,dW_s\Big) = \int_0^t f(s)\,ds.$$

Choosing $f(u) = u^n$ yields the desired identity.

4.5.7 Apply the expectation to

$$\Big(\sum_{k=1}^{N_t} f(S_k)\Big)^2 = \sum_{k=1}^{N_t} f^2(S_k) + 2\sum_{k\neq j}^{N_t} f(S_k)f(S_j).$$

4.6.1

$$\begin{aligned} E\Big[\int_0^T e^{ks}\,dN_s\Big] &= \frac{\lambda}{k}(e^{kT} - 1) \\ Var\Big(\int_0^T e^{ks}\,dN_s\Big) &= \frac{\lambda}{2k}(e^{2kT} - 1). \end{aligned}$$

Chapter 5

5.2.1 Let $X_t = \int_0^t e^{W_u}\,du$. Then

$$dG_t = d(\frac{X_t}{t}) = \frac{tdX_t - X_t dt}{t^2} = \frac{te^{W_t}dt - X_t dt}{t^2} = \frac{1}{t}\Big(e^{W_t} - G_t\Big)dt.$$

5.3.3

(a) $e^{W_t}(1 + \frac{1}{2}W_t)dt + e^{W_t}(1 + W_t)dW_t$;

(b) $(6W_t + 10e^{5W_t})dW_t + (3 + 25e^{5W_t})dt$;

(c) $2e^{t+W_t^2}(1 + W_t^2)dt + 2e^{t+W_t^2}W_t dW_t$;

(d) $n(t + W_t)^{n-2}\Big((t + W_t + \frac{n-1}{2})dt + (t + W_t)dW_t\Big)$;

(e) $\dfrac{1}{t}\left(W_t - \dfrac{1}{t}\displaystyle\int_0^t W_u\,du\right)dt;$

(f) $\dfrac{1}{t^\alpha}\left(e^{W_t} - \dfrac{\alpha}{t}\displaystyle\int_0^t e^{W_u}\,du\right)dt.$

5.3.4

$$d(tW_t^2) = td(W_t^2) + W_t^2 dt = t(2W_t dW_t + dt) + W_t^2 dt = (t + W_t^2)dt + 2tW_t dW_t.$$

5.3.5 (a) $tdW_t + W_t dt;$
(b) $e^t(W_t dt + dW_t);$
(c) $(2 - t/2)t\cos W_t\,dt - t^2\sin W_t\,dW_t;$
(d) $(\sin t + W_t^2\cos t)dt + 2\sin t\,W_t\,dW_t;$

5.3.7 It follows from (5.3.9).

5.3.8 Take the conditional expectation in

$$M_t^2 = M_s^2 + 2\int_s^t M_{u-}\,dM_u + N_t - N_s$$

and obtain

$$\begin{aligned} E[M_t^2|\mathcal{F}_s] &= M_s^2 + 2E[\int_s^t M_{u-}\,dM_u|\mathcal{F}_s] + E[N_t|\mathcal{F}_s] - N_s \\ &= M_s^2 + E[M_t + \lambda t|\mathcal{F}_s] - N_s \\ &= M_s^2 + M_s + \lambda t - N_s \\ &= M_s^2 + \lambda(t-s). \end{aligned}$$

5.3.9 Integrating in (5.3.10) yields

$$F_t = F_s + \int_s^t \frac{\partial f}{\partial x}\,dW_t^1 + \int_s^t \frac{\partial f}{\partial y}\,dW_t^2.$$

One can check that $E[F_t|\mathcal{F}_s] = F_s$.

5.3.10 (a) $\quad dF_t = 2W_t^1 dW_t^1 + 2W_t^2 dW_t^2 + 2dt;$ (b) $\quad dF_t = \dfrac{2W_t^1 dW_t^1 + 2W_t^2 dW_t^2}{(W_t^1)^2 + (W_t^2)^2}.$

5.3.11 Consider the function $f(x,y) = \sqrt{x^2+y^2}$. Since $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$, $\Delta f = \dfrac{1}{2\sqrt{x^2+y^2}}$, we get

$$dR_t = \frac{\partial f}{\partial x}dW_t^1 + \frac{\partial f}{\partial y}dW_t^2 + \Delta f dt = \frac{W_t^1}{R_t}dW_t^1 + \frac{W_t^2}{R_t}dW_t^2 + \frac{1}{2R_t}dt.$$

Chapter 6

6.2.1 (a) Use integration formula with $g(x) = \tan^{-1}(x)$.

$$\int_0^T \frac{1}{1+W_t^2}\,dW_t = \int_0^T (\tan^{-1})'(W_t)\,dW_t = \tan^{-1} W_T + \frac{1}{2}\int_0^T \frac{2W_t}{(1+W_t)^2}\,dt.$$

(b) Use $E\Big[\int_0^T \frac{1}{1+W_t^2}\,dW_t\Big] = 0$.

(c) Use Calculus to find minima and maxima of the function $\varphi(x) = \frac{x}{(1+x^2)^2}$.

6.2.2 (a) Use integration by parts with $g(x) = e^x$ and get

$$\int_0^T e^{W_t}\,dW_t = e^{W_T} - 1 - \frac{1}{2}\int_0^T e^{W_t}\,dt.$$

(b) Applying the expectation we obtain

$$E[e^{W_T}] = 1 + \frac{1}{2}\int_0^T E[e^{W_t}]\,dt.$$

If let $\phi(T) = E[e^{W_T}]$, then ϕ satisfies the integral equation

$$\phi(T) = 1 + \frac{1}{2}\int_0^T \phi(t)\,dt.$$

Differentiating yields the ODE $\phi'(T) = \frac{1}{2}\phi(T)$, with $\phi(0) = 1$. Solving yields $\phi(T) = e^{T/2}$.

6.2.3 (a) Apply integration by parts with $g(x) = (x-1)e^x$ to get

$$\int_0^T W_t e^{W_t}\,dW_t = \int_0^T g'(W_t)\,dW_t = g(W_T) - g(0) - \frac{1}{2}\int_0^T g''(W_t)\,dt.$$

(b) Applying the expectation yields

$$\begin{aligned} E[W_T e^{W_T}] &= E[e^{W_T}] - 1 + \frac{1}{2}\int_0^T \Big[E[e^{W_t}] + E[W_t e^{W_t}]\Big]\,dt \\ &= e^{T/2} - 1 + \frac{1}{2}\int_0^T \Big(e^{t/2} + E[W_t e^{W_t}]\Big)\,dt. \end{aligned}$$

Then $\phi(T) = E[W_T e^{W_T}]$ satisfies the ODE $\phi'(T) - \phi(T) = e^{T/2}$ with $\phi(0) = 0$.

6.2.4 (a) Use integration by parts with $g(x) = \ln(1+x^2)$.
(e) Since $\ln(1+T) \le T$, the upper bound obtained in (e) is better than the one in (d), without contradicting it.

6.3.1 By straightforward computation.

6.3.2 By computation.

6.3.13 (a) $\frac{e}{\sqrt{2}}\sin(\sqrt{2}W_1)$; (b) $\frac{1}{2}e^{2T}\sin(2W_3)$; (c) $\frac{1}{\sqrt{2}}(e^{\sqrt{2}W_4-4} - 1)$.

6.3.14 Apply Ito's formula to get

$$\begin{aligned} d\varphi(t, W_t) &= (\partial_t \varphi(t, W_t) + \frac{1}{2}\partial_x^2 \varphi(t, W_t))dt + \partial_x \varphi(t, W_t) dW_t \\ &= G(t)dt + f(t, W_t)\, dW_t. \end{aligned}$$

Integrating between a and b yields

$$\varphi(t, W_t)|_a^b = \int_a^b G(t)\, dt + \int_a^b f(t, W_t)\, dW_t.$$

Chapter 7

7.2.1 Integrating yields $X_t = X_0 + \int_0^t (2X_s + e^{2s})\, ds + \int_0^t b\, dW_s$. Taking the expectation we get

$$E[X_t] = X_0 + \int_0^t (2E[X_s] + e^{2s})\, ds.$$

Differentiating we obtain $f'(t) = 2f(t) + e^{2t}$, where $f(t) = E[X_t]$, with $f(0) = X_0$. Multiplying by the integrating factor e^{-2t} yields $(e^{-2t} f(t))' = 1$. Integrating yields $f(t) = e^{2t}(t + X_0)$.

7.2.4 (a) Using product rule and Ito's formula, we get

$$d(W_t^2 e^{W_t}) = e^{W_t}(1 + 2W_t + \frac{1}{2}W_t^2)dt + e^{W_t}(2W_t + W_t^2)dW_t.$$

Integrating and taking expectations yields

$$E[W_t^2 e^{W_t}] = \int_0^t \Big(E[e^{W_s}] + 2E[W_s e^{W_s}] + \frac{1}{2}E[W_s^2 e^{W_s}] \Big)\, ds.$$

Since $E[e^{W_s}] = e^{t/2}$, $E[W_s e^{W_s}] = te^{t/2}$, if let $f(t) = E[W_t^2 e^{W_t}]$, we get by differentiation

$$f'(t) = e^{t/2} + 2te^{t/2} + \frac{1}{2}f(t), \qquad f(0) = 0.$$

Multiplying by the integrating factor $e^{-t/2}$ yields $(f(t)e^{-t/2})' = 1 + 2t$. Integrating yields the solution $f(t) = t(1+t)e^{t/2}$. (b) Similar method.

7.2.5 (a) Using Exercise 2.1.18

$$\begin{aligned} E[W_t^4 - 3t^2 | \mathcal{F}_t] &= E[W_t^4|\mathcal{F}_t] - 3t^2 \\ &= 3(t-s)^2 + 6(t-s)W_s^2 + W_s^4 - 3t^2 \\ &= (W_s^4 - 3s^2) + 6s^2 - 6ts + 6(t-s)W_s^2 \neq W_s^4 - 3s^2 \end{aligned}$$

Hence $W_t^4 - 3t^2$ is not a martingale. (b)

$$E[W_t^3|\mathcal{F}_s] \quad = \quad W_s^3 + 3(t-s)W_s \neq W_s^3,$$

and hence W_t^3 is not a martingale.

7.2.6 (a) Similar method as in Example 7.2.4.
(b) Applying Ito's formula

$$d(\cos(\sigma W_t)) = -\sigma \sin(\sigma W_t) dW_t - \frac{1}{2}\sigma^2 \cos(\sigma W_t)dt.$$

Let $f(t) = E[\cos(\sigma W_t)]$. Then $f'(t) = -\frac{\sigma^2}{2} f(t)$, $f(0) = 1$. The solution is $f(t) = e^{-\frac{\sigma^2}{2}t}$.
(c) Since $\sin(t + \sigma W_t) = \sin t \cos(\sigma W_t) + \cos t \sin(\sigma W_t)$, taking the expectation and using (a) and (b) yields

$$E[\sin(t + \sigma W_t)] = \sin t E[\cos(\sigma W_t)] = e^{-\frac{\sigma^2}{2}t} \sin t.$$

(d) Similarly starting from $\cos(t + \sigma W_t) = \cos t \cos(\sigma W_t) - \sin t \sin(\sigma W_t)$.

7.2.7 From Exercise 7.2.6 (b) we have $E[\cos(W_t)] = e^{-t/2}$. From the definition of expectation

$$E[\cos(W_t)] = \int_{-\infty}^{\infty} \cos x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx.$$

Then choose $t = 1/2$ and $t = 1$ to get (a) and (b), respectively.

7.2.8 (a) Using a standard method involving Ito's formula we can get $E(W_t e^{bW_t}) = bte^{b^2 t/2}$. Let $a = 1/(2t)$. We can write

$$\begin{aligned}
\int xe^{-ax^2+bx}\, dx &= \sqrt{2\pi t} \int xe^{bx} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx \\
&= \sqrt{2\pi t} E(W_t e^{bW_t}) = \sqrt{2\pi t} bte^{b^2 t/2} = \sqrt{\frac{\pi}{a}} \Big(\frac{b}{2a}\Big) e^{b^2/(4a)}.
\end{aligned}$$

The same method for (b) and (c).

7.2.9 (a) We have

$$\begin{aligned}
E[\cos(tW_t)] &= E\big[\sum_{n\ge 0} (-1)^n \frac{W_t^{2n} t^{2n}}{(2n)!}\big] = \sum_{n\ge 0} (-1)^n \frac{E[W_t^{2n}] t^{2n}}{(2n)!} \\
&= \sum_{n\ge 0} (-1)^n \frac{t^{2n}}{(2n)!} \frac{(2n)! t^n}{2^n n!} = \sum_{n\ge 0} (-1)^n \frac{t^{3n}}{2^n n!} \\
&= e^{-t^3/2}.
\end{aligned}$$

(b) Similar computation using $E[W_t^{2n+1}] = 0$.

7.3.2 (a) $X_t = 1 + \sin t - \int_0^t \sin s\, dW_s$, $E[X_t] = 1 + \sin t$, $Var[X_t] = \int_0^t (\sin s)^2\, ds = \frac{t}{2} - \frac{1}{4}\sin(2t)$;
(b) $X_t = e^t - 1 + \int_0^t \sqrt{s}\, dW_s$, $E[X_t] = e^t - 1$, $Var[X_t] = \frac{t^2}{2}$;
(c) $X_t = 1 + \frac{1}{2}\ln(1 + t^2) + \int_0^t s^{3/2}\, dW_s$, $E[X_t] = 1 + \frac{1}{2}\ln(1 + t^2)$, $Var(X_t) = \frac{t^4}{4}$.

7.3.4

$$\begin{aligned}
X_t &= 2(1 - e^{-t/2}) + \int_0^t e^{-\frac{s}{2} + W_s}\, dW_s = 1 - 2e^{-t/2} + e^{-t/2 + W_t} \\
&= 1 + e^{t/2}(e^{W_t} - 2);
\end{aligned}$$

Its distribution function is given by

$$\begin{aligned}
F(y) &= P(X_t \le y) = P(1 + e^{t/2}(e^{W_t} - 2) \le y) = P\big(W_t \le \ln(2 + (y-1)e^{t/2})\big) \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\ln(2+(y-1)e^{t/2})} e^{-\frac{x^2}{2t}}\, dx.
\end{aligned}$$

$E[X_t] = 2(1 - e^{-t/2})$, $Var(X_t) = Var(e^{-t/2}e^{W_t}) = e^{-t}Var(e^{W_t}) = e^t - 1$.

7.5.1 (*a*)

$$\begin{aligned} dX_t &= (2W_t dW_t + dt) + W_t dt + t dW_t \\ &= d(W_t^2) + d(tW_t) = d(tW_t + W_t^2) \end{aligned}$$

so $X_t = tW_t + W_t^2$. (*b*) We have

$$\begin{aligned} dX_t &= (2t - \frac{1}{t^2}W_t)dt + \frac{1}{t}dW_t \\ &= 2tdt + d(\frac{1}{t}W_t) = d(t^2 + \frac{1}{t}W_t), \end{aligned}$$

so $X_t = t^2 + \frac{1}{t}W_t - 1 - W_1$.

(*c*) $dX_t = \dfrac{1}{2}e^{t/2}W_t dt + e^{t/2}dW_t = d(e^{t/2}W_t)$, so $X_t = e^{t/2}W_t$. (*d*) We have

$$\begin{aligned} dX_t &= t(2W_t dW_t + dt) - tdt + W_t^2 dt \\ &= td(W_t^2) + W_t^2 dt - \frac{1}{2}d(t^2) \\ &= d(tW_t^2 - \frac{t^2}{2}), \end{aligned}$$

so $X_t = tW_t - \frac{t^2}{2}$. (*e*) $dX_t = dt + d(\sqrt{t}W_t) = d(t + \sqrt{t}W_t)$, so $X_t = t + \sqrt{t}W_t - W_1$.

7.6.1 (*a*) $X_t = X_0 e^{4t} + \dfrac{1}{4}(1 - e^{4t}) + 2\displaystyle\int_0^t e^{4(t-s)}\, dW_s$;

(*b*) $X_t = X_0 e^{3t} + \dfrac{2}{3}(1 - e^{3t}) + e^{3t}W_t$;

(*c*) $X_t = e^t(X_0 + 1 + \dfrac{1}{2}W_t^2 - \dfrac{t}{2}) - 1$;

(*d*) $X_t = X_0 e^{4t} - \dfrac{t}{4} - \dfrac{1}{16}(1 - e^{4t}) + e^{4t}W_t$;

(*e*) $X_t = X_0 e^{t/2} - 2t - 4 + 5e^{t/2} - e^t \cos W_t$;

(*f*) $X_t = X_0 e^{-t} + e^{-t}W_t$.

7.8.1 (*a*) The integrating factor is $\rho_t = e^{-\int_0^t \alpha\, dW_s + \frac{1}{2}\int_0^t \alpha^2 ds} = e^{-\alpha W_t + \frac{\alpha^2}{2}t}$, which transforms the equation in the exact form $d(\rho_t X_t) = 0$. Then $\rho_t X_t = X_0$ and hence $X_t = X_0 e^{\alpha W_t - \frac{\alpha^2}{2}t}$.

(*b*) $\rho_t = e^{-\alpha W_t + \frac{\alpha^2}{2}t}$, $d(\rho_t X_t) = \rho_t X_t dt$, $dY_t = Y_t dt$, $Y_t = Y_0 e^t$, $\rho_t X_t = X_0 e^t$, $X_t = X_0 e^{(1-\frac{\alpha^2}{2})t + \alpha W_t}$.

7.8.2 $\sigma t\, dA_t = dX_t - \sigma A_t\, dt$, $E[A_t] = 0$, $Var(A_t) = E[A_t^2] = \frac{1}{t^2}\int_0^t E[X_s]^2\, ds = \frac{X_0^2}{t}$.

Chapter 8

8.1.2 $A = \frac{1}{2}\Delta = \frac{1}{2}\sum_{k=1}^n \partial_k^2$.

8.1.4

$$\begin{aligned} X_1(t) &= x_1^0 + W_1(t) \\ X_2(t) &= x_2^0 + \int_0^t X_1(s)\, dW_2(s) \\ &= x_2^0 + x_1^0 W_2(t) + \int_0^t W_1(s)\, dW_2(s). \end{aligned}$$

8.3.3 $E[\tau]$ is maximum if $b - x_0 = x_0 - a$, i.e. when $x_0 = (a+b)/2$. The maximum value is $(b-a)^2/4$.

8.3.4 Let $\tau_k = \min(k, \tau) \nearrow \tau$ and $k \to \infty$. Apply Dynkin's formula for τ_k to show that

$$E[\tau_k] \leq \frac{1}{n}(R^2 - |a|^2),$$

and take $k \to \infty$.

8.3.6 x^0 and x^{2-n}.

8.4.1 (a) We have $a(t,x) = x$, $c(t,x) = x$, $\varphi(s) = xe^{s-t}$ and $u(t,x) = \int_t^T xe^{s-t}\,ds = x(e^{T-t}-1)$. (b) $a(t,x) = tx$, $c(t,x) = -\ln x$, $\varphi(s) = xe^{(s^2-t^2)/2}$ and $u(t,x) = -\int_t^T \ln\big(xe^{(s^2-t^2)/2}\big)\,ds = -(T-t)[\ln x + \frac{T}{6}(T+t) - \frac{t^2}{3}]$.

8.4.2 (a) $u(t,x) = x(T-t) + \frac{1}{2}(T-t)^2$. (b) $u(t,x) = \frac{2}{3}e^x\big(e^{\frac{3}{2}(T-t)} - 1\big)$. (c) We have $a(t,x) = \mu x$, $b(t,x) = \sigma x$, $c(t,x) = x$. The associated diffusion is $dX_s = \mu X_s ds + \sigma X_s dW_s$, $X_t = x$, which is the geometric Brownian motion

$$X_s = xe^{(\mu - \frac{1}{2}\sigma^2)(s-t) + \sigma(W_s - W_t)}, \qquad s \geq t.$$

The solution is

$$\begin{aligned}
u(t,x) &= E\Big[\int_t^T xe^{(\mu-\frac{1}{2}\sigma^2)(s-t)+\sigma(W_s-W_t)}\,ds\Big] \\
&= x\int_t^T e^{(\mu-\frac{1}{2}(s-t))(s-t)}E[e^{\sigma(s-t)}]\,ds \\
&= x\int_t^T e^{(\mu-\frac{1}{2}(s-t))(s-t)}e^{\sigma^2(s-t)/2}\,ds \\
&= x\int_t^T e^{\mu(s-t)}\,ds = \frac{x}{\mu}\Big(e^{\mu(T-t)} - 1\Big).
\end{aligned}$$

Chapter 9

9.1.7 Apply Example 9.1.6 with $u = 1$.

9.1.8 $X_t = \int_0^t h(s)\,dW_s \sim N(0, \int_0^t h^2(s)\,ds)$. Then e^{X_t} is log-normal with $E[e^{X_t}] = e^{\frac{1}{2}Var(X_t)} = e^{\frac{1}{2}\int_0^t h(s)^2\,ds}$.

9.1.9 (a) Using Exercise 9.1.8 we have

$$\begin{aligned}
E[M_t] &= E[e^{-\int_0^t u(s)\,dW_s}e^{-\frac{1}{2}\int_0^t u(s)^2\,ds}] \\
&= e^{-\frac{1}{2}\int_0^t u(s)^2\,ds}E[e^{-\int_0^t u(s)\,dW_s}] = e^{-\frac{1}{2}\int_0^t u(s)^2\,ds}e^{\frac{1}{2}\int_0^t u(s)^2\,ds} = 1.
\end{aligned}$$

(b) Similar computation with (a).

9.1.10 (a) Applying the product and Ito's formulas we get

$$d(e^{t/2}\cos W_t) = -e^{-t/2}\sin W_t\,dW_t.$$

Integrating yields

$$e^{t/2}\cos W_t = 1 - \int_0^t e^{-s/2}\sin W_s\, dW_s,$$

which is an Ito integral, and hence a martingale; (*b*) Similarly.

9.1.12 Use that the function $f(x_1, x_2) = e^{x_1}\cos x_2$ satisfies $\Delta f = 0$.

9.1.14 (*a*) $f(x) = x^2$; (*b*) $f(x) = x^3$; (*c*) $f(x) = x^n/(n(n-1))$; (*d*) $f(x) = e^{cx}$; (*e*) $f(x) = \sin(cx)$.