(Informal) Introduction to Stochastic Calculus

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Disclaimer

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Main References

- D. Brigo and F. Mercurio
  Interest Rate Models – Theory and Practice. With Smile, Inflation and Credit
  Springer (2006)
  Appendix C
- S. Shreve
  Stochastic Calculus for Finance II
  Springer (2004)
  Chapters 1-6

Outline

Preamble

[...] In **continuous-time finance**, *we work within the framework of a* **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$. *We normally have a fixed final time* T, *and then have a filtration, which is a collection of* σ*-algebras* $\{\mathcal{F}(t); 0 \leq t \leq T\}$, *indexed by the time variable* t. *We interpret* $\mathcal{F}(t)$ *as the* **information** *available at time* t. *For* $0 \leq s \leq t \leq T$, *every set in* $\mathcal{F}(s)$ *is also in* $\mathcal{F}(t)$. *In other words, information increases over time. Within this context, an* **adapted stochastic process** *is a collection of random variables* $\{X(t); 0 \leq t \leq T\}$, *also indexed by time, such that for every t, X(t) is* $\mathcal{F}(t)$*-measurable; the information at time t is sufficient to evaluate the random variable* $X(t)$. *We think of* $X(t)$ *as the* **price of some asset** *at time* t *and* $\mathcal{F}(t)$ *as the information obtained by watching all the prices in the* **market** *up to time* t.

Two important classes of adapted stochastic processes are **martingales** *and* **Markov processes**.

`Shreve, Chapter 2`

Probability Space: Definition

A probability space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbb{P})$ can be interpreted as an **experiment**, where:

- $\omega \in \Omega$ represents a **generic result** of the experiment
- $\boldsymbol{\Omega}$ is the set of **all possible outcomes** of the random experiment
- a subset $\mathbf{A} \subset \Omega$ represents an **event**
- $\mathcal{F}$ is a **collection of subsets** of Ω which forms a σ-**algebra** (σ-field)
- $\mathbb{P}$ is a **probability measure**

(See Shreve, Chapter 1)

Information

In order to price a derivative security in the no-arbitrage framework, we need to model mathematically the **information** on which our future decisions (**contingency plans**) are based.

- Given a non empty set Ω and a positive number T, assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}_{\mathbf{t}}$.
  $\mathcal{F}_{\mathbf{t}}$ represents the **information up to time** t.
- If $t \leq u$, every set in $\mathcal{F}_t$ is also in $\mathcal{F}_u$, i.e. $\mathcal{F}_t \subseteq \mathcal{F}_u \subseteq \mathcal{F}$.
  "The information increases in time", never exceeding the whole set of events
- The family of σ-fields $(\mathcal{F}_{\mathbf{t}})_{\mathbf{t} \geq \mathbf{0}}$ is called **filtration**.
  A filtration tells us the information that we will have at future times, i.e. when we get to time t we will know for each set in $\mathcal{F}_t$ whether the true ω lies in that set.

(See Shreve, Chapter 2)

Random Variables and Stochastic Processes: Definitions

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t)_t$, $0 \leq t \leq T$:

- A **random variable** X is defined as a **measurable function** from the set of possible outcomes Ω to $\mathbb{R}$, i.e. $X : \Omega \to \mathbb{R}$

  (+ some technical conditions – See Shreve, Chapter 1)

- A **stochastic process** $(X_t)_t$ is defined as a **collection of random variables, indexed** by $t \in [0, T]$.

  For each experiment result ω, **the map** $\mathbf{t} \mapsto \mathbf{X_t}(\omega)$ is called the **path** of X associated to ω.

- A stochastic process is said to be **adapted** if, for each t, the random variable X_t is $\mathcal{F}_t$-measurable.

Expectations: Definitions

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- The **expectation** (or expected value) of X is defined as

$$\boxed{\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)}$$

  provided that X is integrable i.e. $\int_\Omega |X(\omega)|\, d\mathbb{P}(\omega) < \infty$

- Let $\mathcal{G} \subset \mathcal{F}$ be a sub-algebra of $\mathcal{F}$.
  The **conditional expectation** of X given $\mathcal{G}$ is any **random variable** which satisfies:
  1. **Measurability:** $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$- measurable
  2. **Partial averaging:**

$$\boxed{\int_A \mathbb{E}[X|\mathcal{G}](\omega)\, d\mathbb{P}(\omega) = \int_A X(\omega)\, d\mathbb{P}(\omega) \quad \forall A \in \mathcal{G}}$$

(See Shreve, Chapter 1,2)

Conditional Expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ and X, Y be (integrable) random variables.

- **Linearity of conditional expectations**

$$\mathbb{E}[a\,X + b\,Y|\mathcal{G}] = a\,\mathbb{E}[X|\mathcal{G}] + b\,\mathbb{E}[Y|\mathcal{G}]$$

- **Taking out what is known**
  If Y and XY are integral r.v and X is $\mathcal{G}$-measurable then:

$$\mathbb{E}[XY|\mathcal{G}] = X\,\mathbb{E}[Y|\mathcal{G}]$$

- **Independence**
  If X is integrable and independent of $\mathcal{G}$

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$$

- **Iterated conditioning (tower rule)**
  If $\mathcal{H} \subset \mathcal{G}$ and X is an integrable r.v., then:

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$$

(See Shreve, Chapter 2)

Martingales I

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_t$, where $0 \leq t \leq T$. Consider a process $(X_t)_t$ satisfying the following conditions:

- **Measurability:**
  $\mathcal{F}_t$ includes all the information on X_t up to time t, i.e. $(X_t)_t$ is **adapted** to $(\mathcal{F}_t)_t$.
- **Integrability:**
  The relevant expected values exist.

If:

$$\mathbb{E}[X_T|\mathcal{F}_t] = X_t \qquad \text{for each } 0 \leq t \leq T \tag{1}$$

we say the process is a **martingale**. It has no tendency to rise or fall.

Martingales II

In other words...

- if t is the present time, the expected value at a future time T, given the current information, is equal to the **current value**
- a martingale represents a picture of a **fair game**, where it is not possible to lose or gain on average
- the martingale property is suited to model the **absence of arbitrage**, i.e. there is no safe way to make money from nothing (**no free lunch**)

Go to No-Arbitrage Pricing

Submartingales, Supermartingales and Semimartingales

- A **submartingale** is a similar process $(X_t)_t$ satisfying:

$$\mathbb{E}[X_T|\mathcal{F}_t] \geq X_t \qquad \text{for each } t \leq T$$

i.e. the expected value of the process grows in time.

- A **supermartingale** satisfies:

$$\mathbb{E}[X_T|\mathcal{F}_t] \leq X_t \qquad \text{for each } t \leq T$$

i.e. the expected value of the process decreases in time.

- A process $(X_t)_t$ that is either a submartingale or a supermartingale is termed a **semimartingale**.

Go to Martingales: Exercises

Quadratic Variation: Definition

Given a stochastic process Y_t with continuous paths, its **quadratic variation** is defined as:

$$\boxed{\langle Y \rangle_T = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} \left[Y_{t_i}(\omega) - Y_{t_{i-1}}(\omega) \right]^2}$$

where $0 = t_0 < t_1 < \ldots < t_n = T$ and $\Pi = \{t_0, t_1, \ldots, t_n\}$ is a **partition** of the interval $[0, T]$. $\|\Pi\|$ represents the maximum step size of the partition.

In form of a **second order integral**:

$$\langle Y \rangle_T = \text{``} \int_0^T \left[dY_s(\omega) \right]^2 \text{''}$$

or in **differential form**:

$$\text{``} d\langle Y \rangle_t = dY_t(\omega)\, dY_t(\omega) \text{''}$$

Quadratic Variation: Deterministic Process

A process whose paths are **differentiable** for almost all ω satisfies $\langle Y \rangle_t = 0$.

If Y is the **deterministic process** $Y : t \mapsto t$, then $dY_t = 0$ and

$$\boxed{dt\,dt = 0}$$

Quadratic Covariation: Definition

The **quadratic covariation** of two stochastic processes Y_t and Z_t, with continuous paths, is defined as follows:

$$\boxed{\langle Y, Z \rangle_T = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} \left[Y_{t_i}(\omega) - Y_{t_{i-1}}(\omega) \right] \left[Z_{t_i}(\omega) - Z_{t_{i-1}}(\omega) \right]}$$

or in form of a **second order integral**:

$$\langle Y, Z \rangle_T = \text{“} \int_0^T dY_s(\omega) dZ_s(\omega) \text{”}$$

or in **differential form**:

$$\text{“} d\langle Y, Z \rangle_t = dY_t(\omega)\, dZ_t(\omega) \text{”}$$

Deterministic Differential Equations (DDE)

EXAMPLE: Population Growth Model

- Let $x(t) = x_t \in \mathbb{R}, x_t \geq 0$, denote the population at time t, and assume for simplicity a constant (proportional) population growth rate, so that the change in the population at t is given by the **deterministic differential equation**:

$$dx_t = K\, x_t\, dt, \quad x_0$$

  where K is a real constant.

- Assume now that $\mathbf{x_0}$ **is a random variable** $\mathbf{X_0}(\omega)$ and that the population growth is modeled by the following differential equation:

$$dX_t(\omega) = K\, X_t(\omega) dt, \quad X_0(\omega)$$

From Deterministic to Stochastic Differential Equations

- The solution to this equation is:

$$X_t(\omega) = X_0(\omega)\, e^{Kt}$$

  where all the randomness comes from the initial condition $X_0(\omega)$.

- As a further step, suppose that even K is not known for certain, but that also our knowledge of K is perturbed by some randomness, which we model as the increment of a stochastic process $\{W_t(\omega)\}$, $t \geq 0$, so that

$$dX_t(\omega) = (K\, dt + dW_t(\omega))\, X_t(\omega), \quad X_0(\omega), \quad K \geq 0 \tag{2}$$

  where $\mathbf{dW_t}(\omega)$ represents a **noise process** that adds randomness to K.

  Eq. (2) represents an example of **stochastic differential equation (SDE)**.

Stochastic Differential Equations (SDE)

- More generally, a **SDE** is written as

$$\boxed{dX_t(\omega) = f_t(X_t(\omega))\,dt + \sigma_t(X_t(\omega))\,dW_t(\omega), \quad X_0(\omega)} \tag{3}$$

  The function **f** corresponds to the deterministic part of the SDE and is called the **drift**. The function $\sigma_\mathbf{t}$ is called the **diffusion coefficient**.

  The randomness enters the SDE from two sources:

  - the **noise term** $\mathbf{dW_t}(\omega)$
  - the **initial condition** $\mathbf{X_0}(\omega)$

- The solution **X** of the SDE is also called a **diffusion process**. In general the corresponding paths $t \mapsto X_t(\omega)$ are continuous.

Noise Term

Which kind of **process** is suitable to describe the **noise term $\mathbf{dW_t}(\omega)$?**

Brownian Motion

- The process whose increments $\mathbf{dW_t}(\omega)$ are candidates to represent the noise process in the SDE given by Eq. (3) is the **Brownian motion**.
- The **most important properties** of Brownian motion are that:
  - it is a **martingale**
  - it accumulates **quadratic variation at rate one per unit time**.
    This makes **stochastic calculus** different from **ordinary calculus**.

Brownian Motion: Definition

Definition

Given a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, for each $\omega \in \Omega$ there is a **continuous function** W_t, $t \geq 0$ such that it depends on ω and $\mathbf{W_0 = 0}$. Then, $\mathbf{W_t}$ is a **Brownian motion** if and only if for any $0 < s < t < u$ and any $h > 0$ it has:

- **Independent increments**: $W_u(\omega) - W_t(\omega)$ independent of $W_t(\omega) - W_s(\omega)$
- **Stationary increments**: $W_{t+h}(\omega) - W_{s+h}(\omega) \sim W_t(\omega) - W_s(\omega)$
- **Gaussian increments**: $W_t(\omega) - W_s(\omega) \sim \mathcal{N}(0, t-s)$

Although the paths are continuous, they are (almost surely) **nowhere differentiable**, i.e.

$$\dot{W}_t(\omega) = \frac{d}{dt} W_t(\omega)$$

does not exist.

Go to Brownian Motion: Exercises

Property 1: Martingality

Brownian motion is a **martingale**.

PROOF
Let $0 \leq s \leq t$. Then:

$$\begin{aligned}
\mathbb{E}[W_t|\mathcal{F}_s] &= \mathbb{E}[W_t - W_s + W_s|\mathcal{F}_s] \\
&= \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] \\
&= \mathbb{E}[W_t - W_s] + W_s \\
&= W_s
\end{aligned}$$

□

Property 2: Quadratic Variation

The **quadratic variation** of a Brownian motion W is given *almost surely* by:

$$\langle W \rangle_T = T \qquad \text{for each } T$$

or, equivalently:

$$\boxed{dW_t(\omega)\, dW_t(\omega) = dt}$$

Brownian motion **accumulates quadratic variation at rate one per unit time**

This comes from the fact that the Brownian motion moves so quickly that **second order effects cannot be neglected**. *Instead, a process with differentiable trajectories cannot move so quickly and therefore second order effects do not contribute (derivatives are continuous).*

See Shreve, Chapter 3 for the proof.

Property 2.bis: Quadratic Covariation

If W is a **Brownian motion** and Z a **deterministic** process $t \mapsto t$ it follows:

$$\langle W, t \rangle_T = 0 \qquad \text{for each } \; T$$

or, equivalently:

$$\boxed{dW_t(\omega)\, dt = 0}$$

Integral Form of an SDE

The integral form of the general SDE, given by Eq. (3), i.e.

$$\boxed{X_t(\omega) = X_0(\omega) + \int_0^t f_s(X_s(\omega))\, ds + \int_0^t \sigma_s(X_s(\omega))\, dW_s(\omega)} \tag{4}$$

contains two types of integrals:

- $\int_0^t f_s(X_s(\omega))\, ds$ is a **Riemann-Stieltjes integral**.
- $\int_0^t \sigma_s(X_s(\omega))\, dW_s(\omega)$ is a **stochastic generalization of the Riemann-Stieltjes integral**, such that the result depends on the chosen points of the sub-partitions used in the limit that defines the integral.

Stochastic Integrals

A **stochastic integral** is an integral of the type:

$$\int_0^T \phi_t(\omega) dW_t(\omega)$$

where ϕ_t is an adapted process, and W_t a Brownian motion.

The problem we face when trying to **assign a meaning** *to the above integral, is that the Brownian motion paths cannot be differentiated w.r.t. time. If $g(t)$ is a differentiable function, then we can define:*

$$\int_0^T \phi_t(\omega) dg(t) = \int_0^T \phi_t(\omega) g'(t) dt$$

where the right end side is an ordinary integral w.r.t time. This will not work with Brownian motion.

Stochastic Integrals: Partitions

Take the interval $[0, T]$ and consider the following partitions of this interval:

$$T_i^n = \min\left(T, \frac{i}{2^n}\right) \qquad i = 0, 1, \ldots, \infty$$

where n is an integer.

- For all $i > 2^n T$ all terms collapse to T, i.e. $T_i^n = T$.
- For each n we have such a partition, and when n increases the partition contains more elements, giving a better discrete approximation of the continuous interval $[0, T]$.

Ito and Stratonovich Integrals

Then define the integral as:

$$\int_0^T \phi_s(\omega) dW_s(\omega) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \phi_{t_i^n}(\omega) \left[W_{T_{i+1}^n}(\omega) - W_{T_i^n}(\omega) \right]$$

where t_i^n is any point in the interval $\left[T_i^n, T_{i+1}^n\right)$. By choosing:

- $\mathbf{t_i^n} := \mathbf{T_i^n}$ (initial point), we have the **Ito integral**;
- $\mathbf{t_i^n} := \frac{\mathbf{T_i^n} + \mathbf{T_{i+1}^n}}{\mathbf{2}}$ (middle point), we have the **Stratonovich integral**.

Properties of Ito Integral

Let $I(t) = \int_0^t \phi(u) dW(u)$ be an **Ito integral**. $I(t)$ has the following properties:

1. **Continuity:** as a function of the upper limit t, the paths of $I(t)$ are continuous
2. **Adaptivity:** for each t, $I(t)$ is $\mathcal{F}_t$-measurable
3. **Linearity:** for $J(t) = \int_0^t \gamma(u)\, dW(u)$ then

$$I(t) \pm J(t) = \int_0^t [\phi(u) \pm \gamma(u)]\, dW(u)$$

   and $cI(t) = \int_0^t c\,\phi(u)\, dW(u)$

4. **Martingality:** $I(t)$ is a martingale
5. **Ito isometry:** $\mathbb{E}[I^2(t)] = \mathbb{E}[\int_0^t \phi^2(u)\, du]$
6. **Quadratic variation** $\langle I \rangle_t = \int_0^t \phi^2(u)\, du$

Ito Integral *vs* Stratonovich Integral

$$\text{Ito} \Longrightarrow \int_0^t W_s(\omega) dW_s(\omega) = \frac{W_t(\omega)^2}{2} - \frac{1}{2}\, t$$

$$\text{Stratonovich} \Longrightarrow \int_0^t W_s(\omega) dW_s(\omega) = \frac{W_t(\omega)^2}{2}$$

Ito

- Martingale property
- No standard chain rule

Stratonovich

- No martingale property
- Standard chain rule

Solution to a General SDE

Consider the general SDE, given by Eq. (3):

$$dX_t(\omega) = f_t(t, X_t(\omega))\, dt + \sigma_t(t, X_t(\omega))\, dW_t(\omega), \quad X_0(\omega)$$

Existence and uniqueness of the solution are guaranteed by:

- **Lipschitz continuity**:

$$|f(t,x) - f(t,y)| \leq C|x-y| \text{ and } |\sigma(t,x) - \sigma(t,y)| \leq C|x-y| \quad x, y \in \mathbb{R}^d$$

- **Linear growth bound**:

$$|f(t,x)| \leq D(1+|x|) \text{ and } |\sigma(t,x)| \leq D(1+|x|) \quad x \in \mathbb{R}^d$$

(See Øksendal (1992) for the details.)

Interpretation of the Coefficients: DDE Case

For a **deterministic differential equation**

$$dx_t = f(x_t)dt$$

with f a smooth function, we have:

$$\lim_{h \to 0} \frac{x_{t+h} - x_t}{h} \bigg|_{x_t = y} = f(y)$$

$$\lim_{h \to 0} \frac{(x_{t+h} - x_t)^2}{h} \bigg|_{x_t = y} = 0$$

Interpretation of the Coefficients: SDE Case

For a **stochastic differential equation**

$$dX_t(\omega) = f(X_t(\omega))dt + \sigma(X_t(\omega))dW_t(\omega)$$

functions f and σ can be interpreted as:

$$\lim_{h \to 0} \mathbb{E}\left\{ \frac{X_{t+h}(\omega) - X_t(\omega)}{h} \,\middle|\, X_t = y \right\} = f(y)$$

$$\lim_{h \to 0} \mathbb{E}\left\{ \frac{[X_{t+h}(\omega) - X_t(\omega)]^2}{h} \,\middle|\, X_t = y \right\} = \sigma^2(y)$$

The second limit is non-zero because of the *infinite velocity* of the Brownian motion. Moreover, if the **drift f is zero**, the solution is a **martingale**.

Outline

Deterministic Case

For a deterministic differential equation such as

$$dx_t = f(x_t)dt$$

given a smooth transformation $\phi(t, x)$, we can write the evolution of $\phi(t, x_t)$ via the **standard chain rule**:

$$d\phi(t, x_t) = \frac{\partial \phi}{\partial t}(t, x_t)dt + \frac{\partial \phi}{\partial x}(t, x_t)dx_t \tag{5}$$

Stochastic Case: Ito's Formula I

Let $\phi(t,x)$ be a smooth function and $X_t(\omega)$ the unique solution to the SDE (3).

The **chain rule – Ito's formula** reads as:

$$\begin{aligned} d\phi(t,X_t(\omega)) = & \frac{\partial \phi}{\partial t}(t,X_t(\omega))dt + \frac{\partial \phi}{\partial x}(t,X_t(\omega))dX_t(\omega) \\ & + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(t,X_t(\omega))dX_t(\omega)dX_t(\omega) \end{aligned} \tag{6}$$

or, in compact notation:

$$\boxed{d\phi(t,X_t) = \frac{\partial \phi}{\partial t}(t,X_t)dt + \frac{\partial \phi}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(t,X_t)d\langle X\rangle_t}$$

Stochastic Case: Ito's Formula II

The term $dX_t(\omega)dX_t(\omega)$ can be developed by recalling the rules on quadratic variation and covariation:

$$dW_t(\omega)dW_t(\omega) = dt, \qquad dW_t(\omega)dt = 0, \qquad dtdt = 0$$

thus giving:

$$\begin{aligned} d\phi(t, X_t(\omega)) = & \Big[\frac{\partial \phi}{\partial t}(t, X_t(\omega)) + \frac{\partial \phi}{\partial x}(t, X_t(\omega))f(X_t(\omega)) \\ & + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(t, X_t(\omega))\sigma^2(X_t(\omega))\Big] dt \\ & + \frac{\partial \phi}{\partial x}(t, X_t(\omega))\sigma(X_t(\omega))dW_t(\omega) \end{aligned}$$

▸ Go to Ito's Formula: Exercises

Leibniz Rule

It applies to **differentiation of a product of functions**.

- **Deterministic Leibniz rule**

  For deterministic and differentiable functions x and y:

$$d(x_t\, y_t) = x_t\, dy_t + y_t\, dx_t$$

- **Stochastic Leibniz rule**

  For two diffusion processes $X_t(\omega)$ and $Y_t(\omega)$:

$$d(X_t(\omega)\, Y_t(\omega)) = X_t(\omega)\, dY_t(\omega) + Y_t(\omega)\, dX_t(\omega) + dX_t(\omega) dY_t(\omega)$$

  or, in compact notation:

$$\boxed{d(X_t\, Y_t) = X_t\, dY_t + Y_t\, dX_t + d\langle X, Y\rangle_t}$$

Outline

Linear SDE with Deterministic Diffusion Coefficient I

A SDE is **linear** if both its drift and diffusion coefficients are first order polynomials in the state variable.

Consider the **particular case**:

$$dX_t(\omega) = (\alpha_t + \beta_t X_t(\omega))\, dt + v_t dW_t(\omega), \qquad X_0(\omega) = x_0 \tag{7}$$

where α, β, v are deterministic functions of time, regular enough to ensure existence and uniqueness of the solution.

The **solution** is:

$$\begin{aligned} X_t(\omega) &= e^{\int_0^t \beta_s ds} \left[x_0 + \int_0^t e^{-\int_0^s \beta_u du}\, \alpha_s\, ds + \int_0^t e^{-\int_0^s \beta_u du}\, v_s\, dW_s(\omega) \right] \\ &= x_0 e^{\int_0^t \beta_s ds} + \int_0^t e^{\int_s^t \beta_u du}\, \alpha_s\, ds + \int_0^t e^{\int_s^t \beta_u du}\, v_s\, dW_s(\omega) \end{aligned} \tag{8}$$

Linear SDE with Deterministic Diffusion Coefficient II

The **distribution** of the solution $X_t(\omega)$ is **normal** at each time t:

$$X_t \sim \mathcal{N}\left(x_0 e^{\int_0^t \beta_s ds} + \int_0^t e^{\int_s^t \beta_u du}\, \alpha_s\, ds, \int_0^t e^{2\int_s^t \beta_u du}\, v_s^2\, ds\right)$$

Major examples: Vasicek SDE (1978) and Hull and White SDE (1990).

Go to Vasicek Model

Lognormal Linear SDE

The **lognormal** SDE can be obtained as an exponential of a linear equation with deterministic diffusion coefficient.

Let us take $Y_t = \exp(X_t)$, where X_t evolves according to (7), i.e.:

$$d \ln Y_t(\omega) = (\alpha_t + \beta_t \ln Y_t(\omega))\, dt + v_t dW_t(\omega), \qquad Y_0(\omega) = \exp(x_0)$$

Equivalently, by Ito's formula we can write:

$$\begin{aligned} dY_t(\omega) &= de^{X_t(\omega)} = e^{X_t(\omega)} dX_t(\omega) + \frac{1}{2} e^{X_t(\omega)} dX_t(\omega) dX_t(\omega) \\ &= \left[\alpha_t + \beta_t \ln Y_t(\omega) + \frac{1}{2} v_t^2\right] Y_t dt + v_t Y_t(\omega) dW_t(\omega) \end{aligned}$$

The process Y has a **lognormal marginal density**. Major examples: Black Karasinski model (1991) and **Geometric Brownian Motion**.

Geometric Brownian Motion I

The GBM is a particular case of **lognormal linear** process.

Its **evolution** is defined by:

$$dX_t(\omega) = \mu\, X_t(\omega)dt + \sigma\, X_t(\omega)\, dW_t(\omega), \qquad X_0(\omega) = X_0$$

where μ and σ are positive constants.

By Ito's formula, one can solve the SDE, by computing $d \ln X_t$:

$$\boxed{X_t(\omega) = X_0\, \exp\left\{\left(\mu - \frac{1}{2}\,\sigma^2\right) t + \sigma\, W_t(\omega)\right\}}$$

From the work of **Black and Scholes** (1973) on, processes of this type are frequently used in **option pricing theory** to model the asset price dynamics.

Geometric Brownian Motion II

The GBM process is a **submartingale**:

$$\mathbb{E}[X_T|\mathcal{F}_t] = e^{\mu(T-t)} X_t \geq X_t$$

The process $Y_t(\omega) = e^{-\mu t} X_t(\omega)$ is a **martingale**, since we obtain:

$$dY_t(\omega) = \sigma \, Y_t(\omega) \, dW_t(\omega)$$

i.e. the drift of the process is zero.

Go to Geometric Brownian Motion: Excercise

Square Root Process

It is characterized by a **non-linear** SDE:

$$dX_t(\omega) = (\alpha_t + \beta_t X_t(\omega))\, dt + v_t \sqrt{X_t(\omega)}\, dW_t(\omega), \qquad X_0(\omega) = X_0$$

Square root processes are naturally linked to **non-central ξ-square distributions**.

Major examples: the **Cox Ingersoll and Ross (CIR)** model (1985) and a particular case of the **constant-elasticity variance (CEV)** model for stock prices:

$$dX_t(\omega) = \mu\, X_t(\omega) dt + \sigma \sqrt{X_t(\omega)}\, dW_t(\omega), \qquad X_0(\omega) = X_0$$

Go to Cox Ingersoll Ross Model

Go to SDE: Excercise

Outline

Change of Measure

The way a **change** in the underlying **probability measure** affects a SDE is defined by the **Girsanov theorem**

The theorem is based on the following facts:

- the SDE drift depends on the particular probability measure $\mathbb{P}$
- if we change the probability measure in a "regular" way, the **drift of the equation changes while the diffusion coefficient remains the same**.

The Girsanov theorem can be useful when we want to **modify the drift** coefficient of a SDE.

Radon-Nikodym Derivative

Two measures $\mathbb{P}^*$ and $\mathbb{P}$ on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$ are said to be **equivalent**, i.e. $\mathbb{P}^* \sim \mathbb{P}$, if they share the same sets of null probability.

When two measures are equivalent, it is possible to express the first in terms of the second through the **Radon-Nikodym derivative**.

There exists a martingale ρ_t on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that

$$\mathbb{P}^* = \int_A \rho_t(\omega)\, d\mathbb{P}(\omega), \quad A \in \mathcal{F}_t$$

which can be written as:

$$\boxed{\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \rho_t}$$

The process ρ_t is called the Radon-Nikodym derivative of $\mathbb{P}^*$ with respect to $\mathbb{P}$ restricted to $\mathcal{F}_t$.

Expected Values

When in need of computing the expected value of an integrable random variable X, it may be useful to switch from one measure to another equivalent one.

- **Expectations**

$$\mathbb{E}^*[X] = \int_\Omega X(\omega)\, d\mathbb{P}^*(\omega) = \int_\Omega X(\omega)\, \frac{d\mathbb{P}^*}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}\left[X\, \frac{d\mathbb{P}^*}{d\mathbb{P}}\right]$$

- **Conditional expectations**

$$\mathbb{E}^*[X \mid \mathcal{F}_t] = \frac{\mathbb{E}\left[X\, \frac{d\mathbb{P}^*}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]}{\rho_t}$$

Girsanov Theorem

Consider SDE, with Lipschitz coefficients, under $d\mathbb{P}$:

$$\boxed{dX_t(\omega) = f(X_t(\omega))dt + \sigma(X_t(\omega))dW_t(\omega), \qquad x_0}$$

Let be given a new drift $f^*(x)$ and assume $(f^*(x) - f(x))/\sigma(x)$ to be bounded. Define the measure $\mathbb{P}^*$ through the Radon-Nikodym derivative:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}(\omega)\bigg|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{f^*(X_s(\omega)) - f(X_s(\omega))}{\sigma(X_s(\omega))}\right)^2 ds + \int_0^t \frac{f^*(X_s(\omega)) - f(X_s(\omega))}{\sigma(X_s(\omega))}\, dW_s(\omega)\right\}$$

Then $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ and the process W^* defined by:

$$dW_t^*(\omega) = -\left[\frac{f^*(X_t(\omega)) - f(X_t(\omega))}{\sigma(X_t(\omega))}\right] dt + dW_t(\omega)$$

is a Brownian motion under $\mathbb{P}^*$ and

$$\boxed{dX_t(\omega) = f^*(X_t(\omega))dt + \sigma(X_t(\omega))dW_t^*(\omega), \qquad x_0}$$

Example: from $\mathbb{P}$ to $\mathbb{Q}$ I

A classical example involves moving from the **real world asset price dynamics** $\mathbb{P}$ to the **risk neutral** one, $\mathbb{Q}$, i.e. from

$$dS_t = \boxed{\mu} S_t dt + \sigma S_t \boxed{dW_t^{\mathbb{P}}} \qquad \text{under } \mathbb{P} \tag{9}$$

to

$$dS_t = \boxed{r} S_t dt + \sigma S_t \boxed{dW_t^{\mathbb{Q}}} \qquad \text{under } \mathbb{Q} \tag{10}$$

The **risk neutral measure** $\mathbb{Q}$ *is used in* **pricing problems** *while the* **real-world (or historical) measure** $\mathbb{P}$ *is used in* **risk management**.

Example: from $\mathbb{P}$ to $\mathbb{Q}$ II

- Start from the asset dynamics under the **real-world measure** $\mathbb{P}$, eq. (9):

$$dS_t = \mu\, S_t dt + \sigma S_t dW_t^{\mathbb{P}}$$

- Consider the **discounted asset price process** $\tilde{S}_t = S_t\, e^{-rt}$.
  This process satisfies the following SDE:

$$d\tilde{S}_t = (\mu - r)\, \tilde{S}_t\, dt + \sigma\, \tilde{S}_t\, dW_t^{\mathbb{P}} \tag{11}$$

- The goal is to find a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the discounted asset price process is a **martingale** under the new measure, i.e.

$$d\tilde{S}_t = \sigma\, \tilde{S}_t\, dW_t^{\mathbb{Q}} \tag{12}$$

Example: from $\mathbb{P}$ to $\mathbb{Q}$ III

- To this purpose, rewrite eq. (11) as follows:

$$d\tilde{S}_t = \left[(\mu - r)\, dt + \sigma\, dW_t^{\mathbb{P}}\right] \tilde{S}_t = \left[\frac{\mu - r}{\sigma}\, dt + \, dW_t^{\mathbb{P}}\right] \sigma\, \tilde{S}_t$$

and define, according to **Girsanov theorem**, a new Brownian process:

$$dW_t^* \equiv \frac{\mu - r}{\sigma}\, dt + \, dW_t^{\mathbb{P}} \, .$$

Therefore,

$$d\tilde{S}_t = \sigma\, \tilde{S}_t\, dW_t^* \tag{13}$$

i.e. $\tilde{S}_t$ is a martingale under the equivalent measure $\mathbb{P}^*$.

- Comparing (12) with (13), we obtain $\mathbb{P}^* \equiv \mathbb{Q}$.
- Going back to the **asset price process** $S_t = \tilde{S}_t\, e^{rt}$, we finally get eq. (10):

$$dS_t = r\, S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

Outline

No-Arbitrage Pricing

We refer to Brigo and Mercurio, Chapter 2.

As already mentioned, **absence of arbitrage** is equivalent to the impossibility to invest zero today and receive tomorrow a non-negative amount that is positive with positive probability. In other words, two portfolios having the same payoff at a given future date must have the same price today.

Historically, **Black and Scholes (1973)** showed that, by constructing a suitable portfolio having the same instantaneous return as that of a risk-less investment, the portfolio instantaneous return was indeed equal to the instantaneous risk-free rate, which led to their celebrated **option-pricing formula**.

Harrison and Pliska Result (1983)

A financial market is **arbitrage free and complete** if and only if there exists a **unique equivalent (risk-neutral or risk-adjusted) martingale measure**.

Stylized characterization of **no-arbitrage theory via martingales**:

- The market is **free of arbitrage** if (and only if) there exists a martingale measure
- The market is **complete** if and only if the martingale measure is unique
- In an arbitrage-free market, not necessarily complete, the **price** π_t of any attainable claim is uniquely given, either by the value of the associated replicating strategy, or by the **risk neutral expectation of the discounted claim payoff under any of the equivalent (risk-neutral) martingale measures**:

$$\boxed{\pi_t = \mathbb{E}[D(t,T)\Pi_T|\mathcal{F}_t]}$$

Go Back

Outline

Brownian Motion: Exercises

1. **Time reversal**

   Prove that the continuous time stochastic process defined by:

   $$B_t = W_T - W_{T-t}, \qquad t \in [0, T]$$

   is a standard Brownian motion.

2. **Brownian scaling**

   Let W_t be a standard Brownian motion. Given a constant $c > 0$, show that the stochastic process X_t defined by:

   $$X_t = \frac{1}{\sqrt{c}} W_{ct}, \qquad t > 0$$

   is a standard Brownian motion.

Go Back

Martingales: Exercises

1. Let $X_1, X_2, \ldots$ be a sequence of independent random variables in $\mathcal{L}^1$ such that $\mathbb{E}[X_n] = 0$ for all n. If we set:

$$S_0 = 0, \quad S_n = X_1 + X_2 + \ldots + X_n, \quad \text{for } n \geq 1$$
$$\mathcal{F}_0 = \{\varnothing, \Omega\} \text{ and } \mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n), \quad \text{for } n \geq 1$$

prove that the process $(S_n)_n$ is an $(\mathcal{F}_n)_n$-martingale.

2. Consider a filtration $(\mathcal{F}_n)_n$ and an $\mathcal{F}_n$-adapted stochastic process $(X_n)_n$ such that $X_0 = 0$ and $\mathbb{E}[|X_n|] \leq \infty$ for all $n \geq 0$. Also, let $(c_n)_n$ be a sequence of constants. Define $M_0 = 0$ and

$$M_n = c_n X_n - \sum_{j=1}^{n} c_j \, \mathbb{E}[X_j - X_{j-1} | \mathcal{F}_{j-1}] - \sum_{j=1}^{n} (c_j - c_{j-1}) X_{j-1}, \quad \text{for } n \geq 1$$

Prove that $(M_n)_n$ is an $\mathcal{F}_n$-martingale.

Go Back

Ito's formula: Exercises

1. Consider a standard one-dimensional Brownian motion W_t. Use Ito's formula to calculate:

$$W_t^2 = t + 2\int_0^t W_s dW_s$$

and

$$W_t^{27} = 351\int_0^t W_s^{25} ds + 27\int_0^t W_s^{26} dW_s$$

2. Consider a standard one-dimensional Brownian motion W_t. Given $k \geq 2$ and $t \geq 0$, use Ito's formula to prove that:

$$\mathbb{E}[W_t^k] = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[W_u^{k-2}]du$$

Use this expression to calculate $\mathbb{E}[W_t^4]$ and $\mathbb{E}[W_t^6]$.

Hint: stochastic integrals are martingales, so their expectation is zero.

Go Back

Vasicek Model

In the Vasicek model (1977) for interest rates, the dynamics of the short rate process r_t is given by the following SDE:

$$dr_t = k(\theta - r_t)\, dt + \sigma dW_t$$

with k, θ and σ strictly positive constants, and initial condition r_0.

1. Show that the solution to the above SDE is given by:

$$r_t = \theta + (r_0 - \theta)\, e^{-kt} + \sigma \int_0^t e^{k(s-t)} dW_s$$

   Hint: Consider the Ito processes X_t and Y_t defined by $X_t = e^{kt}$ and $Y_t = r_t$ and integrate by parts.

2. Calculate the mean $\mathbb{E}[r_t]$ and the variance $\mathrm{var}(r_t)$ of the random variable r_t.
   Hint: Use Ito's isometry and assume that all stochastic integrals are martingales, so they have zero expectation.

Go Back

Geometric Brownian Motion: Exercise

Consider the Geometric Brownian motion SDE:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

with deterministic time-dependent coefficients, μ_t and σ_t, and initial condition S_0.

Prove that its solution is given by:

$$S_t = S_0 \exp\left\{ \int_0^t \left(\mu_u - \frac{\sigma_u^2}{2} \right) du + \int_0^t \sigma_u \, dW_u \right\}$$

Go Back

Cox Ingersoll Ross Model

In the Cox Ingersoll Ross model (1985) for interest rates, the dynamics of the short rate process r_t is given by the following SDE:

$$dr_t = k(\theta - r_t)\, dt + \sigma \sqrt{r_t} dW_t$$

with k, θ and σ strictly positive constants, and initial condition r_0.

1. Show that the solution to the above SDE is given by:

$$r_t = \theta + (r_0 - \theta)\, e^{-kt} + \sigma \int_0^t e^{k(s-t)} \sqrt{r_s} dW_s$$

   Hint: Consider the Ito processes X_t and Y_t defined by $X_t = e^{kt}$ and $Y_t = r_t$ and integrate by parts.

2. Calculate the mean $\mathbb{E}[r_t]$ and the variance $\mathrm{var}(r_t)$ of the random variable r_t.
   Hint: Use Ito's isometry and assume that all stochastic integrals are martingales, so they have zero expectation.

Go Back

SDE: Exercise

Consider the following integral SDE:

$$Z_t = -\int_0^t Z_u du + \int_0^t e^{-u} dW_u$$

Prove that:

$$Z_t = e^{-t} W_t$$

Go Back