Task 10

Условие

$$u_s = \frac{31}{32}u_{s-1} + \frac{s+31}{s+32} \tag{1}$$

Метод итераций

Пусть дано u_0

$$u_1 = \frac{31}{32}u_0 + \frac{32}{33}$$
$$u_2 = \frac{31}{32}u_1 + \frac{33}{34} = \frac{31}{32}\left(\frac{31}{32}u_0 + \frac{32}{33}\right) + \frac{33}{34} = \left(\frac{31}{32}\right)^2 u_0 + \frac{31\cdot 32}{32\cdot 33} + \frac{33}{34}$$
$$u_3 = \frac{31}{32}u_2 + \frac{34}{35} = \frac{31}{32}\left(\left(\frac{31}{32}\right)^2 u_0 + \frac{31\cdot 32}{32\cdot 33} + \frac{33}{34}\right) + \frac{34}{35} = \left(\frac{31}{32}\right)^3 u_0 + \frac{31\cdot 31}{32\cdot 33} + \frac{31\cdot 33}{32\cdot 34} + \frac{34}{35} =$$
$$\left(\frac{31}{32}\right)^3 u_0 + \left(\frac{31}{32}\right)\left(\frac{33}{34} + \frac{34}{35}\right)$$
...
$$u_s = \left(\frac{31}{32}\right)^s u_0 + \sum_{i=0}^{s-1}\left(\frac{31}{32}\right)^i \frac{(s-i)+31}{(s-i)+32} \tag{2}$$

Z-преобразование

$$Z\{u_s\} = \frac{31}{32}z^{-1}(u_{-1}z + \tilde{u}(z)) + \tilde{x}(z)$$
$$\tilde{u}(z) - \frac{31}{32}z^{-1}\tilde{u}(z) = \frac{31}{32}z^{-1}u_{-1}z + \tilde{x}(z)$$
$$\tilde{u}(z) = \frac{\frac{31}{32}z^{-1}u_{-1}z + \tilde{x}(z)}{1 - \frac{31}{32}z^{-1}}$$
$$\tilde{u}(z) = \frac{\frac{31}{32}u_{-1}z + z\tilde{x}(z)}{z - \frac{31}{32}} \tag{3}$$

Так как получили простую дробь, то выполним сразу обратное Z-преобразование.

$$u_s = \sum_{\zeta_1=\frac{31}{32}} Res\frac{\left(\frac{31}{32}u_{-1} + \tilde{x}(z)\right)zz^{s-1}}{z - \frac{31}{32}} = \lim_{z\to\frac{31}{32}}(z - \frac{31}{32})\frac{z^s\frac{31}{32}u_{-1} + \tilde{x}(z)z^s}{z - \frac{31}{32}} = \left(\frac{31}{32}\right)^{s+1}u_{-1} + \tilde{x}\left(\frac{31}{32}\right)\left(\frac{31}{32}\right)^s$$

где $\tilde{x}\left(\frac{31}{32}\right) = \sum_{k=0}^{\infty} x_k\left(\frac{31}{32}\right)^{s-k} = \sum_{k=1}^{s} x_k\left(\frac{31}{32}\right)^{s-k} = \sum_{i=0}^{s-1} x_{s-i}\left(\frac{31}{32}\right)^i = \sum_{i=0}^{s-1}\frac{(s-i)+31}{(s-i)+32}\left(\frac{31}{32}\right)^i$.

Таким образом, получили следующее решение

$$u_s = \left(\frac{31}{32}\right)^{s+1} u_{-1} + \sum_{i=0}^{s-1} \frac{(s-i)+31}{(s-i)+32}\left(\frac{31}{32}\right)^i = \left(\frac{31}{32}\right)^s u_0 + \sum_{i=0}^{s-1} \frac{(s-i)+31}{(s-i)+32}\left(\frac{31}{32}\right)^i \quad (4)$$

Обратный оператор

$$\begin{aligned}
u_s &= \frac{31}{32}\mathcal{B}u_s + \frac{s+31}{s+32} \\
\left(1 - \frac{31}{32}\mathcal{B}\right)u_s &= \frac{s+31}{s+32} \\
u_s &= \left(1 - \frac{31}{32}\mathcal{B}\right)^{-1}\frac{s+31}{s+32} \\
u_s &= (1 + \alpha\mathcal{B} + \alpha^2\mathcal{B}^2 + \alpha^3\mathcal{B}^3 + ...)\frac{s+31}{s+32} \\
u_s &= \frac{s+31}{s+32} + \frac{31}{32}\frac{(s-1)+31}{(s-1)+32} + \left(\frac{31}{32}\right)^2\frac{(s-2)+31}{(s-2)+32} + ...
\end{aligned}$$