Task 11. Модель Самуэльсона-Хикса

Условие:

$$x_t = \frac{11}{6}x_{t-1} - \frac{3}{2}x_{t-2} + 6 \tag{1}$$

Пусть $a_0 = 6, a_1 = \frac{11}{6}, a_2 = \frac{3}{2}$. Тогда

1 способ

$$\check{x}_t = a_1 x_{t-1} + a_2 x_{t-2} \tag{2}$$

$$x_t = Ch^t \tag{3}$$

$$H(h) = h^2 - a_1 h - a_2 = 0 \tag{4}$$

$D = a_1 - 4a_2 = \frac{11}{6} - 6 < 0; D' = D$. Тогда $h_{1,2} = v \pm w = \frac{a_1 \pm i\sqrt{D'}}{2} \Rightarrow v = \frac{a_1}{2}; w = \frac{\sqrt{D'}}{2} \Rightarrow h_{1,2} = re^{\pm i\varphi} = r(\cos\varphi \pm i\sin\varphi); r = \sqrt{v^2 + w^2}; \cos\varphi = \frac{v}{r}; \sin\varphi = \frac{w}{r}$.

$$\check{x}_t = r^t[(C_1 + C_2)\cos t\varphi + (C_1 - C_2)\sin t\varphi] \tag{5}$$

$$\hat{x}_t = (1 - a_1 B - a_2 B^2)^{-1} a_0 = (1 - (a_1 B + a_2 B^2))^{-1} a_0 = a_0 \sum_{k=0}^{\infty}(a_1 + a_2)^k \tag{6}$$

$$x_t = r^t[(C_1 + C_2)\cos t\varphi + (C_1 - C_2)\sin t\varphi] + a_0 \sum_{k=0}^{\infty}(a_1 + a_2)^k \tag{7}$$

2 способ. Z-transform

$$Z\{x_t\} = a_1 z^{-1}(x_{-1}z + \tilde{x}(z)) - a_2 z^{-2}(x_{-1}z + x_{-2}z^2 + \tilde{x}(z)) + Z\{a_0\} \tag{8}$$

$$\tilde{x}(z) = a_1 z^{-1} x_{-1} z + a_1 z^{-1}\tilde{x}(z) - a_2 z^{-2} x_{-1} z - a_2 x_{-2} - a_2 z^{-2}\tilde{x}(z) + a_0 \frac{z}{z-1} \tag{9}$$

$$\begin{aligned} \tilde{x}(z) &= \frac{a_1 z^{-1} x_{-1} z - a_2 z^{-2} x_{-1} z - a_2 x_{-2} + a_0 \frac{z}{z-1}}{1 - a_1 z^{-1} + a_2 z^{-2}} = \\ &= \frac{a_1 z^2 x_{-1} - a_2 z x_{-1} - a_2 x_{-2} z^2 + a_0 \frac{z^3}{z-1}}{z^2 - a_1 z + a_2} \end{aligned} \tag{10}$$

Посчитаем дискриминант: $D = a_1^2 - 4a_2 < 0 \Rightarrow z_{1,2} = \frac{a_1 \pm i\sqrt{D'}}{2}; D' = -D$.

$$\tilde{x}(z) = \frac{z^2(a_1x_{-1} - a_2x_{-2}) - a_2x_{-1}z + a_0\frac{z^3}{z-1}}{z^2 - a_1z + a_2} \tag{11}$$

$$\begin{aligned} x_t &= (a_1x_{-1} - a_2x_{-2})\sum_{j=1,2} Res_{z_j}\frac{z^{t+1}}{(z-z_1)(z-z_2)} - \\ &- a_2x_{-1}\sum_{j=1,2} Res_{z_j}\frac{z^t}{(z-z_1)(z-z_2)} + a_0\sum_{j=1,2} Res_{z_j}\frac{z^{t+2}}{(z-1)(z-z_1)(z-z_2)} = \\ &= (a_1x_{-1} - a_2x_{-2})\frac{z_1^{t+1} - z_2^{t+1}}{z_1 - z_2} - a_2x_{-1}\frac{z_1^t - z_2^t}{z_1 - z_2} + a_0\frac{z_1^{t+2} - z_2^{t+2}}{(z_1-1)(z_1-z_2)(z_2-1)} \end{aligned} \tag{12}$$

Пусть $z_{1,2} = re^{\pm i\varphi} \Rightarrow z_1^{t+1} = r^{t+1}e^{i(t+1)\varphi}; r = \dfrac{\sqrt{a_1^2 + D'}}{2}$. Тогда

$$x_t = (a_1x_{-1} - a_2x_{-2})r^t\frac{\sin{(t+1)\varphi}}{\sin\varphi} - a_2x_{-1}r^{t-1}\frac{\sin t\varphi}{\sin\varphi} + \frac{a_0}{2}\frac{r^{t+2}\sin{(t+2)\varphi}}{\frac{r^2+1}{2} - \cos\varphi} \tag{13}$$

$$x_t = (\frac{11}{6}x_{-1} - \frac{3}{2}x_{-2})r^t\frac{\sin{(t+1)\varphi}}{\sin\varphi} - \frac{3}{2}x_{-1}r^{t-1}\frac{\sin t\varphi}{\sin\varphi} + 3\frac{r^{t+2}\sin{(t+2)\varphi}}{\frac{r^2+1}{2} - \cos\varphi} \tag{14}$$

Task 12. Паутинная модель рынка

Условие:

$$p_t = \left(-\frac{a}{b} - \frac{m}{b} - \frac{n}{b}p_{t-1}\right) \tag{15}$$

$$a = 6; b = \frac{4}{3}; m = 3; n = \frac{2}{3}$$

Пусть $a_0 = -\dfrac{a}{b} - \dfrac{m}{b}; a_1 = -\dfrac{n}{b}$

$$p_t = a_0 + a_1p_{t-1} \tag{16}$$

Рассмотрим однородное уравнение $p_t = a_1p_{t-1}$.

$$H(h) = h - a_1 = 0 \tag{17}$$

$$\check{p}_t = Ca_1^t \tag{18}$$

$$A(B)p_t = (1 - a_1B)p_t = a_0 \tag{19}$$

$$\hat{p}_t = (1 - a_1B)^{-1}a_0 = (1 + a_1B + a_1^2B^2 + ...)a_0 \tag{20}$$

$$\hat{p}_t = a_0\sum_{k=0}^{\infty} a_1^k \tag{21}$$

$$\Rightarrow p_t = Ca_1^t + a_0\sum_{k=0}^{\infty} a_1^k = -C\left(\frac{1}{2}\right)^t - \frac{27}{4}\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k \tag{22}$$