Task 11. Модель Самуэльсона-Хикса

Условие:

$$x_t = \frac{11}{6}x_{t-1} - \frac{3}{2}x_{t-2} + 6 \quad (1)$$

Пусть $a_0 = 6, a_1 = \dfrac{11}{6}, a_2 = \dfrac{3}{2}$. Тогда

$$Z\{x_t\} = a_1 z^{-1}(x_{-1}z + \tilde{x}(z)) - a_2 z^{-2}(x_{-1}z + x_{-2}z^2 + \tilde{x}(z)) + Z\{a_0\} \quad (2)$$

$$\tilde{x}(z) = a_1 z^{-1} x_{-1} z + a_1 z^{-1}\tilde{x}(z) - a_2 z^{-2} x_{-1} z - a_2 x_{-2} - a_2 z^{-2}\tilde{x}(z) + a_0 \frac{z}{z-1} \quad (3)$$

$$\tilde{x}(z) = \frac{a_1 z^{-1} x_{-1} z - a_2 z^{-2} x_{-1} z - a_2 x_{-2} + a_0 \frac{z}{z-1}}{1 - a_1 z^{-1} + a_2 z^{-2}} =$$

$$= \frac{a_1 z^2 x_{-1} - a_2 z x_{-1} - a_2 x_{-2} z^2 + a_0 \frac{z^3}{z-1}}{z^2 - a_1 z + a_2} \quad (4)$$

Посчитаем дискриминант: $D = a_1^2 - 4a_2 < 0 \Rightarrow z_{1,2} = \frac{a_1 \pm i\sqrt{D'}}{2}; D' = -D$.

$$\tilde{x}(z) = \frac{z^2(a_1 x_{-1} - a_2 x_{-2}) - a_2 x_{-1} z + a_0 \frac{z^3}{z-1}}{z^2 - a_1 z + a_2} \quad (5)$$

$$x_t = (a_1 x_{-1} - a_2 x_{-2}) \sum_{j=1,2} Res_{z_j} \frac{z^{t+1}}{(z-z_1)(z-z_2)} -$$

$$- a_2 x_{-1} \sum_{j=1,2} Res_{z_j} \frac{z^t}{(z-z_1)(z-z_2)} + a_0 \sum_{j=1,2} Res_{z_j} \frac{z^{t+2}}{(z-1)(z-z_1)(z-z_2)} =$$

$$= (a_1 x_{-1} - a_2 x_{-2}) \frac{z_1^{t+1} - z_2^{t+1}}{z_1 - z_2} - a_2 x_{-1} \frac{z_1^t - z_2^t}{z_1 - z_2} + a_0 \frac{z_1^{t+2} - z_2^{t+2}}{(z_1 - 1)(z_1 - z_2)(z_2 - 1)} \quad (6)$$

Пусть $z_{1,2} = re^{\pm i\varphi} \Rightarrow z_1^{t+1} = r^{t+1}e^{i(t+1)\varphi}; r = \dfrac{\sqrt{a_1^2 + D'}}{2}$. Тогда

$$x_t = (a_1 x_{-1} - a_2 x_{-2}) r^t \frac{\sin{(t+1)\varphi}}{\sin\varphi} - a_2 x_{-1} r^{t-1} \frac{\sin t\varphi}{\sin\varphi} + \frac{a_0}{2} \frac{r^{t+2}\sin{(t+2)\varphi}}{\frac{r^2+1}{2} - \cos\varphi}$$