Diff Eq S21 Practice Final Solutions

Monday, May 10, 2021 10:53 AM

Problem 1.

$$y'' + xy' + 2y = 0, \qquad y(0) = 3, \quad y'(0) = -2.$$

First we substitute a guess

$$y = \sum_{n=0}^{\infty} c_n x^n$$

power rule

$$y' = \sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1}$$

first term is zero

$$y'' = \sum_{n=2}^{\infty} c_n \cdot n \cdot (n-1) \cdot x^{n-2}$$

(Same process)

Plug this in:

distributive property

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} c_n n x + \sum_{n=0}^{\infty} 2c_n x^n = 0.$$

We need the powers of x to be the same, and starting indices to be the same.

In 1st term, let $k = n-2$.

In 1^{st} term, let $k = n-2$.

Then at $n=2$, $k=0$,

& $n = k+2$.

In 2^{nd} & 3^{rd} term, let $k = n$.

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_k k x^k + \sum_{k=0}^{\infty} c_k x^k = 0.$$

we have to pull out the first terms so the starting indices are the same.

$$c_2(2)(1)x^0 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_k k x^k + c_0 + \sum_{k=1}^{\infty} c_k x^k = 0.$$

combining the sums & factoring out x^k:

$$2c_2 + c_0 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) + c_k k + c_k\right] x^k = 0.$$

Then $2c_2 + c_0 = 0, \Rightarrow c_2 = \frac{-c_0}{2}$

$$c_{k+2}(k+2)\cancel{(k+1)} + c_k\cancel{(k+1)} = 0$$

$$c_{k+2} = \frac{-c_k}{(k+2)}.$$

Plug in values of k, try to find pattern:

$\underline{k=0:}$ $\boxed{c_2 = -\frac{c_0}{2}}$

$k=0$: $\boxed{c_2 = \frac{-c_0}{2}}$

$k=1$: $\boxed{c_3 = \frac{-c_1}{3}}$

$k=2$: $c_4 = \frac{-c_2}{4} = \boxed{\frac{(-1)(-1)c_0}{4 \cdot 2}}$

$k=3$: $c_5 = \frac{-c_3}{5} = \boxed{\frac{(-1)(-1)c_1}{5 \cdot 3}}$

Guess a pattern:

even terms: $c_n = \frac{(-1)^n c_0}{2n(2n-2) \cdots 4 \cdot 2}$

odd terms: $c_n = \frac{(-1)^n c_1}{(2n+1)(2n-1) \cdots 5 \cdot 3 \cdot 1}$

Check the pattern:

$k=4$: $c_6 = \frac{-c_4}{6} = \frac{(-1)(-1)(-1)c_0}{6 \cdot 4 \cdot 2}$ ✓

$k=5$: $c_7 = \frac{-c_5}{7} = \frac{(-1)(-1)(-1)c_1}{7 \cdot 5 \cdot 3}$ ✓

Note that each of these is the third term in their series.

General Solution:

$$\boxed{y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n-2) \cdots 4 \cdot 2} + c_1 \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n+1)(2n-1) \cdots 3 \cdot 1}}$$

To solve the initial value problem, have to find c_0 & c_1: $y(0) = 3$, $y'(0) = -2$.

$c_0 \left(1 - \frac{x^2}{\ldots} \ldots\right) + c_1 \left(\ldots - \frac{x^3}{\ldots} \ldots\right)$

to find c_0 & c_1: $y(0) = 3$, $y'(0) = -2$.

$$y = c_0\left(1 - \frac{x^2}{2} + \cdots\right) + c_1\left(x - \frac{x^3}{3} + \cdots\right)$$

$$y(0) = 3 \Rightarrow \boxed{c_0 = 3.}$$

$$y'(x) = -\frac{2c_0 x}{2} + \cdots + c_1\left(0 - \frac{3x^2}{3} + \cdots\right)$$

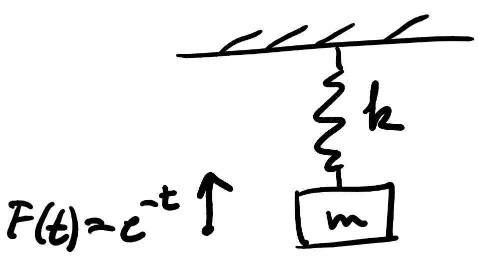
$$y'(0) = -2 \Rightarrow \boxed{c_1 = -2.}$$

Then the solution is:

$$\boxed{y(x) = 3\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n(2n-2)\cdots 2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n+1)(2n-1)\cdots 3\cdot 1}}$$

Problem 2:

$m = 1\,kg$, $k = 1\,\frac{N}{m}$



$$mx'' + kx = e^{-t},$$

using Newton's 2nd law, Hooke's law, & the external force.

Plugging in the numbers,

$$x'' + x = e^{-t}$$

First we solve the homogeneous equation, using $x = e^{mt}$ method:

$$m^2 + 1 = 0$$

$$m = \pm i,$$

Then the complementary solution is

$$x_c(t) = c_1 \cos(t) + c_2 \sin(t)$$

then the complementary solution is

$$x(t) = c_1 \cos(t) + c_2 \sin(t).$$

To find the particular solution, use superposition (or method of your choice) and guess $x_p = Ae^{-t}$,

$$x_p' = -Ae^{-t}$$

$$x_p'' = Ae^{-t},$$

plugging this in:

$$Ae^{-t} + Ae^{-t} = e^{-t},$$

$$2Ae^{-t} = e^{-t}$$

$$A = \frac{1}{2}.$$

Then the general solution is

$$x = x_c + x_p,$$

$$\boxed{x = c_1 \cos t + c_2 \sin t + \frac{1}{2}e^{-t}.}$$

Problem 3.

$$y'' + 16y = \delta(t - 2\pi), \quad y(0) = 0, \; y'(0) = 0.$$

We'll use Laplace transforms:

$$\mathcal{L}[y''] + 16\,\mathcal{L}[y] = \mathcal{L}[\delta(t-2\pi)]$$

let $\mathcal{L}[y] = Y(s)$,

$$s^2 Y(s) - sy(0) - y'(0) + 16\,Y(s) = e^{-2\pi s}$$

$$s^2 Y(s) - \underbrace{sy(0)}_{=0} - \underbrace{y'(0)}_{=0} + 16Y(s) = e^{-2\pi s}$$

$$Y(s)(s^2+16) = e^{-2\pi s}$$

$$Y(s) = \frac{e^{-2\pi s}}{s^2+16}$$

by the 2^{nd} translation theorem

Then $\boxed{y(t) = \frac{1}{4}\sin(4(t-2\pi))\,\mathcal{U}(t-2\pi)}$

Problem 4:

$$\vec{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\vec{X}, \quad \vec{X}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Use eigenvalue/eigenvector method:

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)(3-\lambda) - 2$$

$$= 6 - 2\lambda - 3\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 5\lambda + 4$$

$$= (\lambda-4)(\lambda-1) = 0,$$

$$\boxed{\lambda = 4, 1.}$$

$\lambda = 4$: $\quad A\vec{k}_1 = \lambda\vec{k}_1$

$$A\vec{k}_1 = \lambda \vec{k}_1$$

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 4\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$\cancel{2}k_1 + \cancel{2}k_2 = \cancel{4}k_1$$

$$k_1 + k_2 = 2k_1 \Rightarrow k_2 = k_1$$

$$k_1 + 3k_2 = 4k_2 \Rightarrow k_2 = k_1$$

let $k_1 = 1 \Rightarrow k_2 = 1.$

$$\boxed{\vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.}$$

$\underline{\lambda = 1:}$

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$2k_1 + 2k_2 = k_1 \Rightarrow 2k_2 = -k_1$$

$$k_1 + 3k_2 = k_2 \Rightarrow -2k_2 = k_1$$

let $k_2 = 1$, then $k_1 = -2$.

$$\boxed{\vec{K}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.}$$

Then the general solution is

$$\boxed{\vec{X}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{t}.}$$

With $\vec{X}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$,

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$2 = c_1 - 2c_2$$

$$4 = c_1 + c_2 \Rightarrow c_1 = 4 - c_2$$

$$\Rightarrow 2 = 4 - c_2 - 2c_2$$

$$-2 = -3c_2,$$

$$c_2 = 2/3$$

$$\Rightarrow c_1 = \frac{12}{3} - \frac{2}{3}$$

Then the IVP soln is:

$$c_1 = \frac{10}{3}.$$

$$\boxed{\vec{X}(t) = \frac{10}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \frac{2}{3}\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t}.}$$