Diff Eq Final Solutions

Monday, May 17, 2021 10:29 AM

Problem 1.

$$y'' - x^2 y' + xy = 0.$$

Guess $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}.$$

Plug in:

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - x^2 \sum_{n=1}^{\infty} c_n n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}}_{\text{let } k=n-2} - \underbrace{\sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1}}_{\text{let } k=n+1} = 0.$$

then at $n=2, k=0$

$n=k+2$

at $n=1, k=2$.

at $n=0$, $k=1$.

and $n=k-1$.

$$\underbrace{\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k}_{\text{pull out first two terms}} - \sum_{k=2}^{\infty} c_{k-1}(k-1)x^k + \underbrace{\sum_{k=1}^{\infty} c_{k-1} x^k}_{\text{pull out first term}} = 0.$$

$$c_2(2)(1)x^0 + c_3(3)(2)x^1 \quad + c_0 x^1$$

$$c_2(2)(1)x^0 + c_3(3)(2)x^1 + c_0 x^1$$

$$+ \sum_{k=2}^{\infty} \left[c_{k+2}(k+2)(k+1) - c_{k-1}(k-1) + c_{k-1} \right] x^k = 0.$$

Then $2c_2 = 0$

$$6c_3 + c_0 = 0.$$

and $c_{k+2}(k+2)(k+1) + (-k+2)c_{k-1} = 0.$

$$c_{k+2} = \frac{((k-2)\, c_{k-1}}{(k+2)(k+1)}.$$

<u>$k=1$:</u> $c_3 = \dfrac{c_0}{3 \cdot 2}$

<u>$k=2$:</u> $c_4 = \dfrac{0 \cdot c_1}{4 \cdot 3} = 0$

<u>$k=3$:</u> $c_5 = \dfrac{c_2}{5 \cdot 4} = 0$ since $2c_2 = 0.$

<u>$k=4$:</u> $c_6 = \dfrac{2c_3}{6 \cdot 5} = \dfrac{2}{6 \cdot 5} \cdot \dfrac{c_0}{3 \cdot 2}$

<u>$k=5$:</u> $c_7 = \dfrac{3c_4}{7 \cdot 6} = 0.$

<u>$k=6$:</u> $c_8 = \dfrac{4c_5}{8 \cdot 7} = 0.$

<u>$k=7$:</u> $c_9 = \dfrac{5c_6}{9 \cdot 8} = \dfrac{5}{9 \cdot 8} \cdot \dfrac{2}{6 \cdot 5} \cdot \dfrac{c_0}{3 \cdot 2}$

Plug into series:

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$= c_0 + c_1 x + c_3 x^3 + c_6 x^6 + c_9 x^9 + \cdots$$

$$y = c_0\left(1 + \frac{1}{2\cdot 3}x^3 + \underbrace{\frac{2}{6\cdot 5\cdot 3\cdot 2}}_{n=2}x^6 + \underbrace{\frac{\overset{=2\cdot 3-1}{5}\cdot 2}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}}_{n=3}x^9 + \cdots\right)$$

$$+ c_1\left(x + 0 + \cdots\right)$$

$$= c_0 \sum_{n=0}^{\infty} \frac{?}{\underbrace{(3n)(3n-1)\cdots 3\cdot 2}_{\text{done in class}}} x^{3n} + c_1 x.$$

← Not sure about numerator yet

k = 10:

$$c_{12} = \frac{8\cdot c_9}{12\cdot 11}$$

$$= \frac{8\cdot 5\cdot 2}{12\cdot 11\cdot 9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}c_0.$$

← at $n=4$

k = 13:

$$c_{15} = \frac{11\, c_{12}}{15\cdot 14}$$

$$= \frac{11\cdot 8\cdot 5\cdot 2\; c_0}{15\cdot 14\cdot 12\cdot 11\cdot 9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$$

← at $n=5$

looks like numerator is

$$\left(3(n-1) - 1\right)\cdots 2.$$

Check:

$$11 = 3(5-1) - 1$$
$$= 12 - 1 \checkmark$$
$$8 = 3(4-1) - 1$$

$$\left(\begin{aligned} 8 &= 3(4-1)-1 \\ &= 9-1 \checkmark \end{aligned} \right)$$

Finally

$$\boxed{y = c_0 \sum_{n=0}^{\infty} \frac{(3n-4)\cdots 2}{(3n)(3n-1)\cdots 3.2} x^{3n} + c_1 x}$$

Problem 2.

$$x^2 y'' + 8xy' + 6y = 0.$$

This is a Cauchy-Euler equation.

Guess $y = x^m$,

$$y' = m x^{m-1}$$

$$y'' = m(m-1) x^{m-2}$$

Plug in:

$$x^2 m(m-1) x^{m-2} + 8x\, m x^{m-1} + 6 x^m = 0$$

$$m(m-1) + 8m + 6 = 0$$

$$m^2 - m + 8m + 6 = 0$$

$$m^2 + 7m + 6 = 0.$$

$$(m+1)(m+6) = 0$$

$$m = -1, \quad m = -6$$

$$\boxed{y = c_1 x^{-1} + c_2 x^{-6}.}$$

Problem 3.

Logistic model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$$

$$\boxed{\frac{dP}{dt} = 3P\left(1 - \frac{P}{10{,}000}\right)}$$

Separable method.

$$\int \frac{1}{P(1 - P/k)}\, dP = \int 3\, dt$$

Partial fractions:

$$\frac{1}{P(1 - P/k)} = \frac{A}{P} + \frac{B}{1 - P/k}$$

$$1 = A(1 - P/k) + B \cdot P$$

At $P = 0$:

$$1 = A$$

At $P = k$:

$$1 = B \cdot k$$

$$B = \frac{1}{k}$$

Then

$$\frac{1}{P(1 - P/k)} = \frac{1}{P} + \frac{1}{k(1 - P/k)}$$

$P(1-P/K)$

$$= \frac{1}{P} + \frac{1}{K-P}.$$

Then we have

$$\int \left(\frac{1}{P} + \frac{1}{K-P} \right) dP = 3t + C$$

$$\ln|P| - \ln|K-P| = 3t + C$$

$$\ln\left|\frac{P}{K-P}\right| = 3t + C$$

$$\frac{P}{K-P} = e^{3t+C}$$

$$\boxed{\frac{P}{K-P} = Ae^{3t}} \quad *$$

at $t=0$, $P=400$.

$$\frac{400}{10{,}000-400} = Ae^0$$

$$\frac{4\cancel{00}}{96\cancel{00}} = A$$

$$A = \frac{2}{3}$$

$$P = Ae^{3t}(K-P)$$

$$P(1+Ae^{3t}) = KAe^{3t}$$

$$\boxed{P = \frac{KAe^{3t}}{1+Ae^{3t}}}$$

$$P(t) = \frac{10{,}000 \cdot \frac{2}{3}e^{3t}}{1 + \frac{2}{3}e^{3t}}$$

$$\boxed{P(t) = \frac{20{,}000e^{3t}}{3+2e^{3t}}}$$

When does $P(t) = 5{,}000$?

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Using ★,

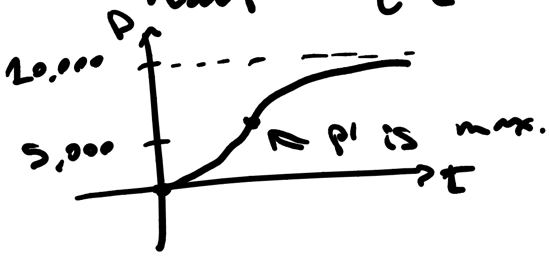
$$\frac{5{,}000}{10{,}000 - 5{,}000} = \frac{2}{3}e^{3t}$$

$$1 = \frac{2}{3}e^{3t}$$

$$\ln\left(\frac{3}{2}\right) = 3t$$

$$\boxed{\frac{1}{3}\ln\left(\frac{3}{2}\right) = t.}$$

The population grows fastest at half the carrying capacity.



Problem 4.

$$f(x) = \begin{cases} 0, & |x| > 1 \\ e^x \cos x, & |x| < 1. \end{cases}$$

Yes, this function has a Fourier transform, because it is piecewise continuous and integrable.

$$\mathcal{F}[f(x)] = \int_{-1}^{1} e^x \cos x \, e^{i\alpha x}\, dx$$

$$\mathcal{F}[f(x)] = \int_{-1} e \cos x \, e \, dx$$

$$= \int_{-1}^{1} e^{i\alpha x + x} \left(\frac{e^{ix} + e^{-ix}}{2} \right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(e^{x(i\alpha + i + 1)} + e^{x(i\alpha - i + 1)} \right) dx$$

$$= \frac{1}{2} \left(\frac{e^{x(i\alpha + i + 1)}}{i\alpha + i + 1} + \frac{e^{x(i\alpha - i + 1)}}{i\alpha - i + 1} \right) \Bigg|_{-1}^{1}$$

$$\boxed{\mathcal{F}[f(x)] = \frac{1}{2} \left[\frac{e^{i\alpha + i + 1}}{i\alpha + i + 1} - \frac{e^{-(i\alpha + i + 1)}}{i\alpha + i + 1} + \frac{e^{i\alpha - i + 1}}{i\alpha - i + 1} - \frac{e^{-(i\alpha - i + 1)}}{i\alpha - i + 1} \right]}$$