

Problem 1: $y'' + 3y' + 2y = 2x + e^{-x}$

First find y_c :

$$y'' + 3y' + 2y = 0$$

using $y = e^{mx}$, we get

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -2, m = -1,$$

then $y_c = C_1 e^{-2x} + C_2 e^{-x}$.

Then using the superposition, we guess

$$y_p = Ax + B + Cxe^{-x},$$

since e^{-x} appears in our y_c .

Taking derivatives:

$$y_p' = A - Cxe^{-x} + Ce^{-x}$$

$$\begin{aligned} y_p'' &= -C \cdot 1 \cdot e^{-x} + Cx e^{-x} - Ce^{-x} \\ &= Cxe^{-x} - 2Ce^{-x}. \end{aligned}$$

Plugging this in,

$$\cancel{Cxe^{-x}} - 2\cancel{Ce^{-x}} + 3(A - \cancel{Cxe^{-x}} + \cancel{Ce^{-x}}) + 2(Ax + B + \cancel{Cxe^{-x}}) = 2x + e^{-x}$$

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$$Cx^2 - 2Cx + 5(A - Cx^2 + Ce^{-x}) + 2(Ax + B + Cxe^{-x}) = 2x + e^{-x}$$

$$Ce^{-x} + 2Ax + 3A + B = 2x + e^{-x}$$

Equating coefficients:

$$C = 1, \quad 2A = 2, \quad 3A + B = 0$$

$$A = 1 \rightarrow B = -3.$$

This gives us the particular solution

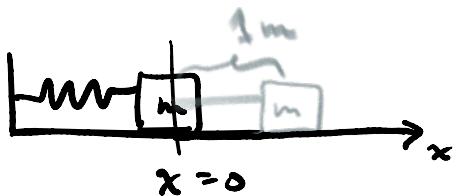
$$y_p = x - 3 + xe^{-x}.$$

Adding y_p & y_c , the general solution is

$$y = y_p + y_c$$

$$y = x - 3 + xe^{-x} + c_1 e^{-2x} + c_2 e^{-x}.$$

Problem 2.



Using Newton's 2nd law or
eqn 10 in Section S.1,
we have

$$m x'' + \beta x' + kx = f(t),$$

where $f(t)$ is the external force.

In this problem $m=1$, $k=1$,

$$\text{and } f(t) = \begin{cases} 0, & t \leq \pi/4 \\ 1, & t > \pi/4 \end{cases}$$

since it is to the right.

This gives:

$$\begin{cases} x'' + x = f(t) \\ x(0) = 1 \\ x'(0) = 0. \end{cases}$$

To solve this, we first solve the assoc. homog. eqn:

$$x'' + x = 0$$

$$\text{Use } x = e^{mt},$$

$$m^2 + 1 = 0$$

$$m = \pm i.$$

Then

$$x_c = C_1 \cos(t) + C_2 \sin(t).$$

To find the particular solution, we must use Green's functions, since $f(t)$ is a piecewise fn.

First choose $x_1 = \cos t$, $x_2 = \sin t$. (You can also

We also need to choose a new variable to integrate over - call it s.

find C_1 & C_2
first - full
credit either way.)

To find $W(s)$, the Wronskian,

we compute $x_1' = -\sin(s)$, $x_2' = \cos(s)$,

then

$$W(s) = \det \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix}$$

$$= \cos^2(s) + \sin^2(s)$$

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$$\omega(s) = 1.$$

Then our Green's function

$$G(t, s) = \frac{x_1(s)x_2(t) - x_1(t)x_2(s)}{\omega(s)}$$

$$= \cos(s)\sin(t) - \cos(t)\sin(s).$$

The particular soln is

$$x_p(t) = \int_0^t G(t, s) f(s) ds.$$

Since f is piecewise, break up integral into

$$x_p(t) = \int_0^{\pi/4} G(t, s) \underbrace{f(s)}_{=0} ds + \int_{\pi/4}^t G(t, s) \underbrace{f(s)}_{=1} ds$$

$$= \int_{\pi/4}^t (\cos(s)\sin(t) - \cos(t)\sin(s)) ds$$

$$= \left. \sin(s) \cdot \sin(t) \right|_{s=\pi/4}^{s=t} - \left. \cos(t) \cdot (-\cos(s)) \right|_{s=\pi/4}^{s=t}$$

$$x_p = \sin^2(t) - \sin(\frac{\pi}{4})\sin(t) + \cos^2(t) - \cos(\frac{\pi}{4})\cos(t)$$

$$x_p = 1 - \frac{1}{\sqrt{2}} (\sin(t) + \cos(t))$$

Finally

$$x = x_p + \dots$$

Finally

$$x = x_p + x_c$$

$$= 1 - \frac{1}{\sqrt{2}}(\sin t + \cos t) + C_1 \cos t + C_2 \sin t.$$

Using $x(0) = 1, x'(0) = 1$:

$$1 = 1 - \frac{1}{\sqrt{2}} + C_1$$

$$C_1 = \frac{1}{\sqrt{2}}$$

$$x(t) = 1 + (C_2 - \frac{1}{\sqrt{2}}) \sin t$$

$$x'(t) = (C_2 - \frac{1}{\sqrt{2}}) \cos t$$

$$x'(0) = 1 \Rightarrow 1 = C_2 - \frac{1}{\sqrt{2}}$$

$$\boxed{C_2 = 1 + \frac{1}{\sqrt{2}}}.$$

Then $\boxed{x(t) = 1 + \sin(t)}.$

As $t \rightarrow \infty$, the mass continues to oscillate forever without changing amplitude, but its equilibrium position is shifted to the right by the external force.

Problem 3:

$$y'' + 2y(y')^2 = 0$$

This is a nonlinear eqn with

This is a nonlinear eqn with
the independent variable missing.

Substitute $u = y'$,

$$\begin{aligned} \text{then } y'' &= \frac{du}{dx} \\ &= \frac{du}{dy} \cdot \frac{dy}{dx} \quad \text{by the Chain Rule.} \\ &= u \cdot \frac{du}{dy}. \end{aligned}$$

Plugging this in:

$$y \frac{du}{dy} + 2y u^2 = 0.$$

This is a separable equation:

$$\int \frac{1}{u} du = \int 2y dy$$

$$\ln|u| = y^2 + C$$

$$u = C e^{y^2}.$$

It's easiest to use our $y'(1) = \frac{1}{2}$
condition here:

$$\frac{1}{2} = C \cdot e$$

$$\text{Then } C = \frac{1}{2e}.$$

$$u = \frac{1}{2} e^{y^2-1}$$

we now have an equation
in $y(x)$, since $u = y^1$:

$$\frac{dy}{dx} = \frac{1}{2} e^{y^2-1}.$$

This is also separable:

$$\int e^{1-y^2} dy = \int \frac{1}{2} dx$$

The integral on the left is NOT
doable. When I did this problem initially,
I made a bad typo & it was solvable.

As promised in my email, I will give
full credit for getting to:

$$\boxed{\int e^{1-y^2} dy = \frac{1}{2}x + C}$$

You do not have
to find C .

I'll be generous with partial
credit as well.

Problem 4. Using equation (10) from
Section 3.3, we have

$$\frac{ds}{dt} = 2s - \frac{1}{4}p \quad (1) \quad s(0) = 1000$$

$$\frac{dp}{dt} = 3p - \frac{1}{3}s. \quad (2) \quad p(0) = 5000.$$

Solve eqn (1) for p :

$$p = 4s' + 8s$$

Differentiate: $p' = 4s'' + 8s'$

Plug this into (2):

$$4s'' + 8s' = 3(4s' + 8s) - \frac{1}{3}s.$$

$$4s'' - 4s' + (\frac{1}{3} - 24)s = 0$$

Using $s = e^{mt}$,

$$4m^2 - 4m + (\frac{1}{3} - 24) = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(4)(\frac{1}{3} - 24)}}{8}$$

$$m = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1}{3} + 24}$$

Then $\boxed{s(t) = c_1 e^{\frac{1}{2}(1+\sqrt{\frac{1}{3}+24})t} + c_2 e^{\frac{1}{2}(1-\sqrt{\frac{1}{3}+24})t}}$

Next, use

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$$P = 4S' + 8S, \text{ we get}$$

$$\boxed{P = 4C_1 \left(\frac{1}{2}(1 + \sqrt{\frac{2}{3} + 24}) \right) e^{\frac{1}{2}(1 + \sqrt{\frac{2}{3} + 24})t} + 4C_2 \frac{1}{2}(1 - \sqrt{\frac{2}{3} + 24}) e^{\frac{1}{2}(1 - \sqrt{\frac{2}{3} + 24})t} + 8C_1 e^{\frac{1}{2}(1 + \sqrt{1})t} + 8C_2 e^{\frac{1}{2}(1 - \sqrt{1})t}}.$$

Yikes! Finally, use our initial values:

$$S(0) = 1000 \rightarrow C_1 + C_2 = 1000, C_1 = 1000 - C_2.$$

$$P(0) = 5000 \rightarrow 2C_1(1 + \sqrt{1}) + 2C_2(1 - \sqrt{1}) + 8(C_1 + C_2) = 5000$$

Plug in C_1 :

$$2(1000 - C_2)(1 + \sqrt{1}) + 2C_2(1 - \sqrt{1}) + 8 \cdot 1000 = 5000$$

$$2000(1 + \sqrt{1}) + C_2(2(1 - \sqrt{1}) - 2(1 + \sqrt{1})) = -3000$$

$$C_2(-2\sqrt{\frac{2}{3} + 24}) = -3000 - 2000(1 + \sqrt{\frac{2}{3} + 24})$$

$$\boxed{C_2 = \frac{3000 + 2000(1 + \sqrt{\frac{2}{3} + 24})}{2\sqrt{\frac{2}{3} + 24}}} \approx 1500.$$

then $\boxed{C_1 = 1000 - C_2,} \approx -500$ given above.

The sols to the initial value problem

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are

$$S(t) = C_1 e^{\frac{1}{2}(1+\sqrt{3}+24)t} + C_2 e^{\frac{1}{2}(2-\sqrt{3}+24)t}$$

$$P(t) = C_2 \left(2(1+\sqrt{3}+24) + 8 \right) e^{\frac{1}{2}(1+\sqrt{3}+24)t}$$

$$+ C_2 \left(2(2-\sqrt{3}+24) + 8 \right) e^{\frac{1}{2}(2-\sqrt{3}+24)t},$$

where C_1 & C_2 are given above.

$$\lim_{t \rightarrow \infty} \left(\frac{S(t)}{P(t)} \right) = \lim_{t \rightarrow \infty} \left(\frac{C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t}}{C_1 (1+\sqrt{3}) e^{\sqrt{3}t} + C_2 (2-\sqrt{3}) e^{-\sqrt{3}t}} \right)$$

Notice
 $e^{\sqrt{3}t}$
cancels.

$$= \frac{C_1}{C_1 (2(1+\sqrt{3}+24) + 8)}$$

which is a constant. In other words,
neither $S(t)$ nor $P(t)$ dies out, they both
continue to exist, approaching this ratio.