Diff Eq Exam 1 Solutions

Monday, March 22, 2021 7:40 AM

Problem 1: $y'' + 3y' + 2y = x + e^{-x}$

First find y_c:

$$y'' + 3y' + 2y = 0$$

using $y = e^{mx}$, we get

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -2, \quad m = -1.$$

then $y_c = c_1 e^{-2x} + c_2 e^{-x}$.

Then using the superposition, we guess

$$y_p = Ax + B + Cxe^{-x},$$

since e^{-x} appears in our y_c.

Taking derivatives:

$$y_p' = A - Cxe^{-x} + Ce^{-x}$$

$$y_p'' = -C \cdot 1 \cdot e^{-x} + Cxe^{-x} - Ce^{-x}$$

$$= Cxe^{-x} - 2Ce^{-x}.$$

Plugging this in,

$$C\cancel{xe^{-x}} - 2Ce^{-x} + 3(A - \cancel{Cxe^{-x}} + Ce^{-x}) + 2(Ax + B + \cancel{Cxe^{-x}}) = x + e^{-x}$$

$$C\cancel{x}e^{-x} - 2Ce^{-x} + 3(A - C\cancel{x}e^{-x} + Ce^{-x}) + 2(Ax + B + C\cancel{x}e^{-x}) = x + e^{-x}$$

$$Ce^{-x} + 2Ax + 3A + B = x + e^{-x}$$

equating coefficients:

$$C = 1, \quad 2A = 1, \quad 3A + B = 0$$

$$A = 1/2 \rightarrow B = -3/2.$$

This gives us the particular solution

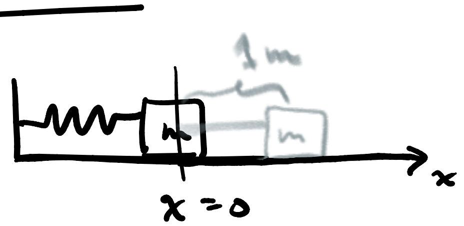
$$y_p = \frac{1}{2}x - \frac{3}{2} + xe^{-x}.$$

Adding y_p & y_c, the general solution is

$$y = y_p + y_c$$

$$\boxed{y = \frac{1}{2}x - \frac{3}{2} + xe^{-x} + c_1 e^{-2x} + c_2 e^{-x}.}$$

Problem 2.



Using Newton's 2nd law or eqn 10 in Section 5.1, we have

$$m x'' + \beta x' + kx = f(t),$$

where $f(t)$ is the external force.

In this problem $m = 1$, $k = 1$,

and $f(t) = \begin{cases} 0, & t \le \pi/4 \\ 1, & t > \pi/4 \end{cases}$

since it is to the right.

This gives:

$$\begin{cases} x'' + x = f(t) \\ x(0) = 1 \\ x'(0) = 0. \end{cases}$$

To solve this, we first solve the assoc. homog. eqn:

$$x'' + x = 0$$

Use $x = e^{mt}$,

$$m^2 + 1 = 0$$

$$m = \pm i.$$

Then

$$x_c = c_1 \cos(t) + c_2 \sin(t).$$

To find the particular solution, we must use Green's functions, since $f(t)$ is a piecewise fn.

First choose $x_1 = \cos t$, $x_2 = \sin t$. (You can also find c_1 & c_2 first - full credit either way.)

We also need to choose a new variable to integrate over - call it s.

To find $W(s)$, the Wronskian, we compute $x_1' = -\sin(s)$, $x_2' = \cos(s)$, then

$$W(s) = \det \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix}$$

$$= \cos^2(s) + \sin^2(s)$$

$$= \cos^2(s) + \sin^2(s)$$

$$W(s) = 1.$$

Then our Green's function

$$G(t,s) = \frac{x_1(s)x_2(t) - x_1(t)x_2(s)}{W(s)}$$

$$= \cos(s)\sin(t) - \cos(t)\sin(s).$$

The particular soln is

$$x_p(t) = \int_0^t G(t,s) f(s)\,ds.$$

Since f is piecewise, break up integral into

$$x_p(t) = \int_0^{\pi/4} G(t,s) \underbrace{f(s)}_{=0} ds + \int_{\pi/4}^{t} G(t,s) \underbrace{f(s)}_{=1} ds$$

$$= \int_{\pi/4}^{t} \left(\cos(s)\sin(t) - \cos(t)\sin(s)\right) ds$$

$$= \sin(s)\cdot\sin(t)\Big|_{s=\pi/4}^{s=t} - \cos(t)\cdot(-\cos(s))\Big|_{s=\pi/4}^{s=t}$$

$$x_p = \sin^2(t) - \sin(\tfrac{\pi}{4})\sin(t) + \cos^2(t) - \cos(\tfrac{\pi}{4})\cos(t)$$

$$\boxed{x_p = 1 - \frac{1}{\sqrt{2}}\left(\sin(t) + \cos(t)\right)}$$

Finally

$$x = x \;\; + \;\; x$$

Finally
$$x = x_p + x_c$$
$$= 1 - \frac{1}{\sqrt{2}}(\sin t + \cos t) + C_1 \cos t + C_2 \sin t.$$

Using $x(0) = 1$, $x'(0) = 1$:
$$1 = 1 - \frac{1}{\sqrt{2}} + C_1$$
$$C_1 = \frac{1}{\sqrt{2}}$$
$$x(t) = 1 + (C_2 - 1/\sqrt{2}) \sin t$$
$$x'(t) = \left(C_2 - 1/\sqrt{2} \right)\cos(t)$$
$$x'(0) = 1 \Rightarrow 1 = C_2 - 1/\sqrt{2}$$
$$\boxed{C_2 = 1 + 1/\sqrt{2}.}$$

then $\boxed{x(t) = 1 + \sin(t).}$

As $t \to \infty$, the mass continues to oscillate forever without changing amplitude, but its equilibrium position is shifted to the right by the external force.

Problem 3:
$$y'' + x(y')^2 = 0$$

This is a nonlinear eqn with

This is a nonlinear eqn with the independent variable missing.

Substitute $u = y'$,

$$\text{then } y'' = \frac{du}{dx}$$

$$= \frac{du}{dy} \cdot \frac{dy}{dx} \quad \text{by the Chain Rule.}$$

$$= u \cdot \frac{du}{dy}.$$

Plugging this in:

$$\cancel{u} \frac{du}{dy} + \frac{1}{y} u^2 = 0.$$

This is a separable equation:

$$\int \frac{1}{u}\, du = \int \frac{1}{y}\, dy$$

$$\ln|u| = \ln|y| + c$$

$$u = Cy$$

It's easiest to use our $y'(1) = 3$ condition here:

$$3 = C \cdot 1$$

$$c = 3.$$

Then

$$u = 3y,$$

we now have an equation in $y(x)$, since $u = y'$:

$$\frac{dy}{dx} = 3y$$

This is also separable:

$$\int \frac{1}{y}\, dy = \int 3\, dx$$

$$\ln(y) = 3x + c$$

$$y = Ae^{3x}$$

$$y(1) = 1, \text{ So}$$

$$1 = Ae^{3}$$

$$A = e^{-3}.$$

Then $\boxed{y = e^{3x-3}}$.

Problem 4. Using equation (10) from Section 3.3, we have

$$\frac{ds}{\;} = 2s - \frac{1}{\;} \quad (1) \qquad S(0) = 1000$$

$$\frac{ds}{dt} = 2s - \frac{1}{4}p \quad (1)$$

$S(0) = 1000$

$p(0) = 5000.$

$$\frac{dp}{dt} = 3p - \frac{1}{3}s. \quad (2)$$

Solve eqn (1) for p:

$$p = 4s' + 8s$$

differentiate: $p' = 4s'' + 8s'$

plug these into (2):

$$4s'' + 8s' = 3(4s' + 8s) - \frac{1}{3}s.$$

$$4s'' - 4s' + (1/3 - 24)s = 0$$

Using $s = e^{mt}$,

$$4m^2 - 4m + (1/3 - 24) = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(4)(1/3 - 24)}}{8}$$

$$m = \frac{1}{2} \pm \frac{1}{2}\sqrt{2/3 + 24}$$

Then $\boxed{S(t) = c_1 e^{1/2(1+\sqrt{2/3+24})t} + c_2 e^{1/2(1-\sqrt{2/3+24})t}}$

Next, use

$p = 4s' + 8s$. we get

$$p = 4s' + 8s, \text{ we get}$$

$$\boxed{p = 4c_1\left(\tfrac{1}{2}\left(1+\sqrt{2/3+24}\right)\right)e^{\frac{1}{2}\left(1+\sqrt{2/3+24}\right)t} + 4c_2\tfrac{1}{2}\left(1-\sqrt{2/3+24}\right)e^{\frac{1}{2}(1-\sqrt{\ })t} + 8c_1 e^{\frac{1}{2}(1+\sqrt{\ })t} + 8c_2 e^{\frac{1}{2}(1+\sqrt{\ })t}.}$$

Yikes! Finally, use our initial values:

$S(0) = 1000 \rightarrow c_1 + c_2 = 1000, \quad c_1 = 1000 - c_2.$

$p(0) = 5000 \rightarrow 2c_1(1+\sqrt{\ }) + 2c_2(1-\sqrt{\ }) + 8(c_1+c_2) = 5000$

plug in c_1:

$$2(1000-c_2)(1+\sqrt{\ }) + 2c_2(1-\sqrt{\ }) + 8\cdot 1000 = 5000$$

$$2000(1+\sqrt{\ }) + c_2\left(2(1-\sqrt{\ }) - 2(1+\sqrt{\ })\right) = -3000$$

$$c_2\left(-2\sqrt{2/3+24}\right) = -3000 - 2000\left(1+\sqrt{2/3+24}\right)$$

$$\boxed{c_2 = \frac{3000 + 2000\left(1+\sqrt{2/3+24}\right)}{2\sqrt{2/3+24}}} \approx 1500$$

then $\boxed{c_1 = 1000 - c_2,} \approx -500$ c_2 given above.

The solns to the initial value problem are

$$e^{\frac{1}{2}(1+\sqrt{2/3+24})t} \quad \frac{1}{2}(1-\sqrt{\ })$$

The sols to [illegible]

are

$$\boxed{\begin{aligned} S(t) &= c_1 e^{\frac{1}{2}\left(1+\sqrt{2/3+24}\right)t} + c_2 e^{\frac{1}{2}\left(1-\sqrt{2/3+24}\right)t} \\ p(t) &= c_2\left(2\left(1+\sqrt{2/3+24}\right)+8\right) e^{1/2\left(1+\sqrt{2/3+24}\right)t} \\ &\quad + c_2\left(2\left(1-\sqrt{2/3+24}\right)+8\right) e^{1/2\left(1-\sqrt{2/3+24}\right)t}, \end{aligned}}$$

where c_1 & c_2 are given above.

$$\lim_{t\to\infty}\left(\frac{S(t)}{p(t)}\right) = \lim_{t\to\infty}\left(\frac{c_1 e^{\sqrt{\ }t} + c_2 \cancel{e^{-\sqrt{\ }t}}^{\to 0}}{c_1(\underline{\quad})e^{\sqrt{\ }t} + c_2(\underline{\quad})\cancel{e^{-\sqrt{\ }t}}^{\to 0}}\right)$$

Notice $e^{1/2 t}$ cancels.

$$= \frac{c_1}{c_1\left(2\left(1+\sqrt{2/3+24}+8\right)\right)},$$

which is a constant. In other words, neither $S(t)$ nor $p(t)$ dies out, they both continue to exist, approaching this ratio.