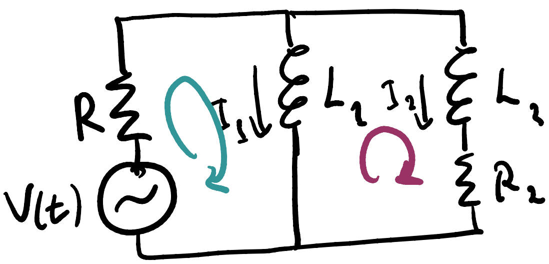
Diff Eq S21 Practice Exam 2

Tuesday, March 16, 2021 3:57 PM

Problem 1.



loop 1:

$$V(t) + (I_1+I_2)R + L_1\frac{dI_1}{dt} = 0$$

loop 2:

$$-L_1\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} + I_2R_2 = 0$$

putting in our values:

(1) $$\sin(2\pi t) + (I_1+I_2) + \frac{dI_1}{dt} = 0$$

(2) $$-\frac{dI_1}{dt} + 2\frac{dI_2}{dt} + 2I_2 = 0.$$

differentiate (1)

$$2\pi\cos(2\pi t) + I_1' + I_2' + I_1'' = 0 \quad (1*)$$

Solve (2) for I_1':

$$I_1' = 2I_2' + 2I_2 \quad (*)$$

differentiate: $$I_1'' = 2I_2'' + 2I_2' \quad (**)$$

plug $(*)$ & $(**)$ into $(1*)$:

$$2\pi\cos(2\pi t) + \underbrace{(2I_2' + 2I_2)}_{=I_1'} + I_2' + \underbrace{(2I_2'' + 2I_2')}_{=I_1''} = 0$$

$$2I_2'' + 5I_2' + 2I_2 = -2\pi\cos(2\pi t)$$

To solve this, first solve homog. eqn:

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$$2I_2'' + 5I_2' + 2I_2 = 0$$

use e^{mx} method

$$2m^2 + 5m + 2 = 0$$

$$m = \frac{-5 \pm \sqrt{25 - 4(2)(2)}}{2}$$

$$= -\frac{5}{2} \pm \frac{1}{2}\sqrt{\underbrace{25-16}_{=9}}$$

$$m = -\frac{5}{2} \pm \frac{3}{2} = -\frac{5}{2}+\frac{3}{2},\ -\frac{5}{2}-\frac{1}{2}$$

so then

$$= -1, -2$$

$$I_{2c} = c_1 e^{-t} + c_2 e^{-2t}$$

To find I_{2p}, use undetermined coefficients - superposition method.

Guess $I_{2p} = A\cos(2\pi t) + B\sin(2\pi t)$

$$I_{2p}' = -2\pi A\sin(2\pi t) + 2\pi B\cos(2\pi t)$$

$$I_{2p}'' = -4\pi^2 A\cos(2\pi t) - 4\pi^2 B\sin(2\pi t)$$

plug in:

$$2\left[-4\pi^2 A\cos(2\pi t) - 4\pi^2 B\sin(2\pi t)\right]$$

$$+5\left[-2\pi A\sin(2\pi t) + 2\pi B\cos(2\pi t)\right]$$

$$+2\left[A\cos(2\pi t) + B\sin(2\pi t)\right] = 2\pi\cos(2\pi t)$$

then equate coefficients:

then equate coefficients:

$$-8\pi^2 A + 10\pi B + 2A = 2\pi$$

$$-8\pi^2 B - 10\pi A + 2B = 0$$

Solve: $$A = \frac{(8\pi^2+2)B}{10\pi},$$

then

$$(2-8\pi^2)\cdot\left(\frac{8\pi^2+2}{10\pi}\right)B + 10\pi B = 2\pi$$

$$\boxed{B = \frac{20\pi^2}{100\pi^2 + (8\pi^2+2)(8\pi^2-2)}}$$

$$\boxed{A = \frac{(8\pi^2+2)2\pi}{100\pi^2+(8\pi^2+2)(8\pi^2-2)}}$$

so $I_{2p} = A\cos(2\pi t) + B\sin(2\pi t)$,

where A & B are given above.

Then $I_2 = I_{2c} + I_{2p}$

$$\boxed{I_2 = c_1 e^{-t} + c_2 e^{-2t} + A\cos(2\pi t) + B\sin(2\pi t),}$$

again A & B are given above.

Using our original equations (1) & (2),

$$I_1 = -\sin(2\pi t) - I_2 - I_1' \quad \text{(eqn 1)}$$

(use eqn 2)

$$I_1 = -\sin(2\pi t) - \underbrace{I_2 - (2I_2' + 2I_2)}_{=+I_2}$$

This allows us to find I_1 without having any extra constants.

$$I_2' = -c_1 e^{-t} - 2c_2 e^{-2t} - 2\pi A\sin(2\pi t) + 2\pi B\cos(2\pi t)$$

$$I_2' = -c_1 e^{-t} - 2c_2 e^{-2t} - 2\pi A \sin(2\pi t) + 2\pi B \cos(2\pi t)$$

then

$$I_1 = -\sin(2\pi t) + c_1 e^{-t} + c_2 e^{-2t} + A\cos(2\pi t) + B\sin(2\pi t)$$
$$-2\left(-c_1 e^{-t} - 2c_2 e^{-2t} - 2\pi A \sin(2\pi t) + 2\pi B \cos(2\pi t)\right)$$

$$\boxed{I_1 = 3c_1 e^{-t} + 5c_2 e^{-2t} + (A - 4\pi B)\cos(2\pi t) + (B + 4\pi A + 1)\sin(2\pi t)}$$

and again A & B are given above.

Problem 2.

$$x^2 y'' - xy' + y = f(x), \quad f(x) = \begin{cases} 0, & x<0 \\ 1, & x>0. \end{cases}$$

$y(1) = 1$, $y'(1) = 2$.

First solve associated homogeneous equation.

$$x^2 y'' - xy' + y = 0.$$

this is a Cauchy-Euler equation.

Guess $y = x^m$,

so $y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

plug in:

$$m(m-1)\cancel{x^m} - m\cancel{x^m} + \cancel{x^m} = 0 \qquad \downarrow x \neq 0$$

$$m(m-1) - m + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$(m-1)^2 = 0$$

$m = 1$, multiplicity 2.

Then using the result from reduction of order in Section 4.7, we have

$$\boxed{y_c = c_1 x + c_2 x \ln(x)}$$

To find y_p, we use Green's functions: identify $y_1 = x$, $y_2 = x\ln(x)$,

So that

$$W(t) = \det\begin{pmatrix} t & t\ln(t) \\ 1 & \ln(t)+1 \end{pmatrix}$$

$$= t(\ln(t)+1) + t\ln(t)$$

$$W(t) = 2t\ln(t) + 1.$$

Then

$$G(x,t) = \frac{y_1(t)\,y_2(x) - y_1(x)\,y_2(t)}{W(t)}$$

$$G(x,t) = \frac{t\cdot x\ln(x) - x\cdot t\ln(t)}{2t\ln(t)+1}$$

Now use

$$y_p = \int_1^x G(x,t)\, f(t)\, dt$$

$$f(t) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$$

over this interval $f(t) = 1$ - otherwise, break integral into

$$\boxed{y_p = \int_1^x \frac{tx\ln(x) - tx\ln(t)}{2t\ln(t)+1}\, dt}$$

otherwise, break integral into two pieces

Unfortunately, this integral is NOT doable. I warned you there may have been issues with the practice exam problems. I will make sure all the exam problems are doable, but this one we must leave as is.

Last,

$$y = y_c + y_p$$

$$y = c_1 x + c_2 x \ln x + \int_1^x \frac{tx\ln(x) - tx\ln t}{2t\ln t + 1}\,dt,$$

and to find c_1 & c_2, use

$$y(1) = 1$$

$$1 = c_1 + \overset{\ln(1)=0}{0} + \overset{\int_1^1 = 0}{0}$$

So $\boxed{c_1 = 1}$

and $y'(1) = 2$:

product rule & FTC for both terms of integral separately

$$y' = 1 + c_2(\ln(x) + 1) + \ln(x)\int_1^x \frac{t}{2t\ln(t)+1}\,dt - \int_1^x \frac{t\ln t}{2t\ln t + 1}\,dt$$
$$+ \int_1^x \frac{t}{2t\ln t + 1}\,dt - x\left(\frac{\ln(x)}{2x\ln x + 1}\right)$$
$$+ x\ln(x)\left(\frac{x}{2x\ln x + 1}\right)$$

$$2 = 1 + c_2(0 + 1) + 0$$ all these terms turn out to be zero!

$\boxed{c_2 = 1.}$

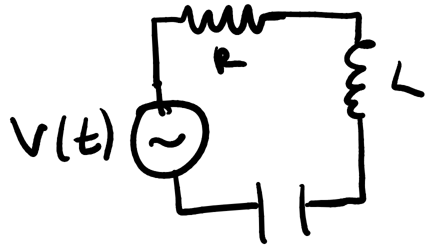
Finally,

$$y = x + x\ln(x) + \int^x \frac{tx\ln(x) - tx\ln t}{\cdots}\,dt$$

Finally,

$$\boxed{y = x + x\ln(x) + \int_1^x \frac{tx\ln(x) - tx\ln t}{2t\ln t + 1}\,dt.}$$

Problem 3:



$$V(t) + IR + L\frac{dI}{dt} + \frac{Q}{C} = 0$$

plug everything in & use $I = \frac{dQ}{dt}$

$$\sin(2\pi t) + \frac{dQ}{dt} + \frac{d^2Q}{dt^2} + Q = 0.$$

First solve homogeneous eqn:

$$Q'' + Q' + Q = 0$$, use e^{mx} method

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Then
$$Q_c = c_1 e^{(-1/2 + \sqrt{3}/2\, i)t} + c_2 e^{(-1/2 - \sqrt{3}/2\, i)t}$$

$$Q_c = e^{-t/2}\left(c_1\cos\left(\frac{\sqrt{3}}{2}t\right) + c_2\sin\left(\frac{\sqrt{3}}{2}t\right)\right),$$

as done in class.

Next to find Q_p, use superposition to guess

$$Q_p = A\cos(2\pi t) + B\sin(2\pi t)$$

$$Q_p' = 2\pi A\sin(2\pi t) - 2\pi B\cos(2\pi t)$$

$$Q_p' = 2\pi A\sin(2\pi t) - 2\pi B\cos(2\pi t)$$

$$Q_p'' = -4\pi^2 A\cos(2\pi t) - 4\pi^2 B\sin(2\pi t)$$

plug in:

$$-4\pi^2 A\cos(2\pi t) - 4\pi^2 B\sin(2\pi t) + 2\pi A\sin(2\pi t) - 2\pi B\cos(2\pi t) + A\cos(2\pi t) + B\sin(2\pi t) = \sin(2\pi t)$$

then equating coefficients we obtain the same result as in Problem 1,

then

$$Q = Q_c + Q_p$$

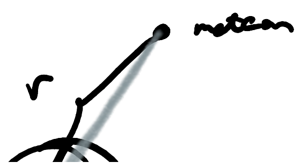
$$\boxed{Q = e^{-1/2 t}\left(c_1\cos\left(\tfrac{\sqrt{3}}{2}t\right) + c_2\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right) + A\cos(2\pi t) + B\sin(2\pi t),}$$

where A & B were found in Problem 1.

This function's maximum occurs at $t=0$, because of the decaying exponential & the fact that $\cos(t)$ is max at $t=0$ (assuming c_1 & $c_2 > 0$).

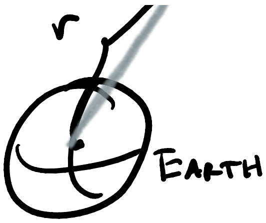
The max value is $\boxed{Q_{max} = c_1 + A,}$ where A is given above.

Problem 4.



Using $F = ma$,

we have



we have

$$\frac{GM\cancel{m}}{r^2} = \cancel{m}\,\frac{d^2r}{dt^2}$$

$$r'' = \frac{GM}{r^2}.$$

This is a second order, nonlinear equation, in which the independent variable is missing. We use the substitution $u = r'$.

Then
$$r'' = \frac{du}{dt}$$
$$= \frac{du}{dr}\cdot\frac{dr}{dt}$$
$$= \frac{du}{dr}\cdot u.$$

Now r is the dependent variable, & the equation is

$$\frac{du}{dr}\cdot u = \frac{GM}{r^2},$$

which is separable.

$$\int u\,du = \int \frac{GM}{r^2}\,dr$$

$$\frac{u^2}{2} = \frac{-GM}{r} + C_1$$

$$u = 2\sqrt{C_1 - \frac{GM}{r}}$$

Then
$$\frac{dr}{dt} = 2\sqrt{C_1 - \frac{GM}{r}},$$

another separable equation.

another separable equation.

$$\int \frac{1}{\sqrt{c_1 - \frac{GM}{r}}} dr = \int 2\,dt$$

let $u = \frac{1}{r}$, *Not the same u!

$du = \frac{-1}{r^2} dr$

$$-\int \frac{1}{\sqrt{c_1 - GM \cdot u} \cdot u^2} du = 2t + C_2$$

let $s = c_1 - GMu$ $\Rightarrow$ $u = \frac{c_1 - s}{GM}$

$ds = -GM\,du$ $\quad du = \frac{-1}{GM} ds$

$$= \frac{1}{(GM)^2} \int \frac{1}{\sqrt{s}\,(c_1 - s)^2} ds$$

let $x = \sqrt{s}$, then $dx = \frac{1}{2\sqrt{s}} ds$,

Notice

$y = \frac{\sqrt{c_1 - GM/r}}{c_1}$

$$= \frac{2}{(GM)^2} \int \frac{1}{(c_1 - x^2)^2} dx,$$

let's also factor out c_1,

$$= \frac{2}{(c_1 GM)^2} \int \frac{1}{(1 - (\frac{x}{c_1})^2)^2} dx$$

so let $y = \frac{x}{c_1}$, $dy = \frac{dx}{c_1}$.

$$= \frac{2}{c_1 (GM)^2} \int \frac{1}{(1 - y^2)^2} dy.$$

Next, use partial fractions:

$$\frac{1}{(1-y^2)^2} = \frac{1}{((y+1)(y-1))^2} = \frac{A}{y+1} + \frac{B}{(y+1)^2} + \frac{C}{y-1} + \frac{D}{(y-1)^2},$$

multiply everything by the common denominator:

$$1 = A(y+1)(y-1)^2 + B(y-1)^2 + C(y-1)(y+1)^2 + D(y+1)^2$$

At $u = 1$:

At $y = 1$:

$$1 = 4D$$
$$D = 1/4$$

At $y = -1$:

$$1 = 4B$$
$$B = 1/4$$

Then picking two more random values for y:

At $y = 0$:

$$1 = A + \frac{1}{4} - C + \frac{1}{4}$$
$$A - C = \frac{1}{2} \rightarrow A = \frac{1}{2} + C$$

At $y = 2$:

$$1 = 3A + \frac{1}{4} + 9C + \frac{9}{4}$$
$$\frac{-1}{2} = A + 3C$$
$$-\frac{1}{2} = \frac{1}{2} + C + 3C$$
$$-1 = 4C$$
$$\boxed{C = -1/4}$$

then $\boxed{A = 1/4}$.

Partial fractions result:

$$\frac{1}{(1-y^2)^2} = \frac{1/4}{y+1} + \frac{1/4}{(y+1)^2} - \frac{1/4}{y-1} + \frac{1/4}{(y-1)^2}$$

Plugging this back into the integral, we have

$$\frac{1}{2c_1(GM)^2}\int\left(\frac{1}{y+1}+\frac{1}{(y+1)^2}+\frac{1}{y-1}+\frac{1}{(y-1)^2}\right)dy = 2t+c_2$$

log rules   u sub

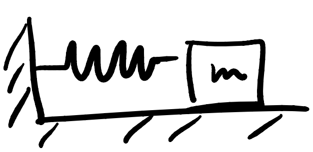
$$\frac{1}{2c_1(GM)^2}\left(\ln|(y+1)(y-1)| \underbrace{- \frac{1}{(y+1)} - \frac{1}{y-1}}_{= -1/(y^2-1)}\right) = 2t+c_2$$

Since this equation involves both $\ln y$ and y, we won't be able to solve for y explicitly (which would be clear on the real exam.) However, we can put our original variables back in:

$$\boxed{\ln\left|\left(\frac{\sqrt{c_1-\frac{GM}{r}}}{c_1}+1\right)\left(\frac{\sqrt{c_1-\frac{GM}{r}}}{c}-1\right)\right| - \frac{1}{\left(\frac{c_1-\frac{GM}{r}}{c_1^2}-1\right)} = 4c_1G^2M^2t+c_2}$$

We also will not be able to find c_1 & c_2, although you know we would use the initial conditions $r(0)=1$, $r'(0)=0$.

Problem 5.



$$F = ma$$

$$k_1x + k_2x^3 = mx''.$$

$$mx'' - k_1x - k_2x^3 = 0.$$

$$m x'' - k_1 x - k_2 x^3 = 0.$$

Again, this problem has the independent variable missing:

let $u = x'$

then $x'' = \frac{du}{dt}$

$= \frac{du}{dx} \cdot \frac{dx}{dt}$

$= \frac{du}{dx} \cdot u.$

Plugging this in, x is the indep. var:

$$m \cdot u \frac{du}{dx} = k_1 x + k_2 x^3$$

this equation is separable:

$$\int u \, du = \frac{1}{m} k_1 \int x \, dx + \frac{k_2}{m} \int x^3 dx$$

$$\frac{u^2}{2} = \frac{k_1}{m} \frac{x^2}{2} + \frac{k_2}{m} \frac{x^4}{4} + C_1$$

$$u = \sqrt{\frac{k_1}{m} x^2 + \frac{k_2}{2m} x^4 + C_1}$$

recall our original substitution

$$\frac{dx}{dt} = \sqrt{\frac{k_1}{m} x^2 + \frac{k_2}{2m} x^4 + C_1}$$

This is another separable eqn:

$$\boxed{\int \frac{1}{\sqrt{\frac{k_1}{m} x^2 + \frac{k_2}{2m} x^4 + C_1}} dx = t + C_2.}$$

$$\int \underbrace{\frac{1}{\sqrt{\frac{k_1}{m}x^2+\frac{k_3}{2m}x^4+c_1}}}\,dx = t + c_2.$$

This is another unsoluable integral – I promise to find soluable ones for the actual exam.

Problem 6:

$$y'' - xy' + 2y = 0$$

Guess $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=0}^{\infty} c_n n x^{n}$$

↓ $n=0$ term is zero

$$= \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$ ↓ same principle.

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} c_n n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

multiply x through

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

reindex this sum, letting $k = n-2$

rename $k = n$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum c_k k x^k + 2\sum c_k x^k$$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_k k x^k + 2\sum_{k=0}^{\infty} c_k x^k = 0$$

write out first terms to get index to match

$$c_2(2)(1) + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_k k x^k + 2c_0 + \sum_{k=1}^{\infty} 2c_k x^k = 0.$$

combine sums

$$2c_0 + 2c_2 + \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) + (2-k)c_k \right] x^k = 0.$$

Then $c_0 = -c_2$, and

$$c_{k+2}(k+2)(k+1) + (2-k)c_k = 0$$

$$c_{k+2} = \frac{(k-2)}{(k+2)(k+1)} c_k.$$

Let c_0 & c_1 be constants.

Then

$\underline{k=0:}$ $c_2 = \frac{-2}{2\cdot 1} c_0$

$\underline{k=1:}$ $c_3 = \frac{-1}{3\cdot 2} c_1$

$\underline{k=2:}$ $c_4 = 0$

$\underline{k=3:}$ $c_5 = \frac{1}{5\cdot 4} c_3 = \frac{1}{5\cdot 4\cdot 3\cdot 2} c_1$

$\underline{k=4:}$ $c_6 = \frac{4}{8\cdot 7} c_4 = 0$

even k will be zero

$\underline{k=5:}$ $c_7 = \frac{3}{7\cdot 6} c_5 = \frac{1}{7\cdot 6\cdot 5\cdot 4\cdot 2} c_1$

Then the solution is:

$y = c_1 \left(x - \frac{1}{[illegible]} x^3 + \frac{1}{[illegible]} x^5 + \dots \right)$

$$\boxed{y = c_1\left(x - \frac{1}{6}x^3 + \frac{1}{7\cdot 6\cdot 5\cdot 4\cdot 2}x^5 + \dots\right)}$$