Discretizing continuous-time controllers

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Context

- Controller $F(s)$ obtained from a design in continuous time.
- Need discrete approxmation in order to implement on a computer

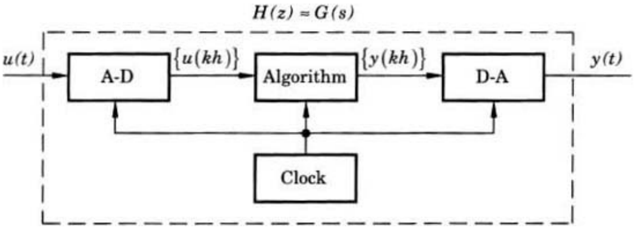
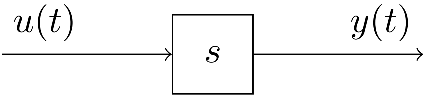


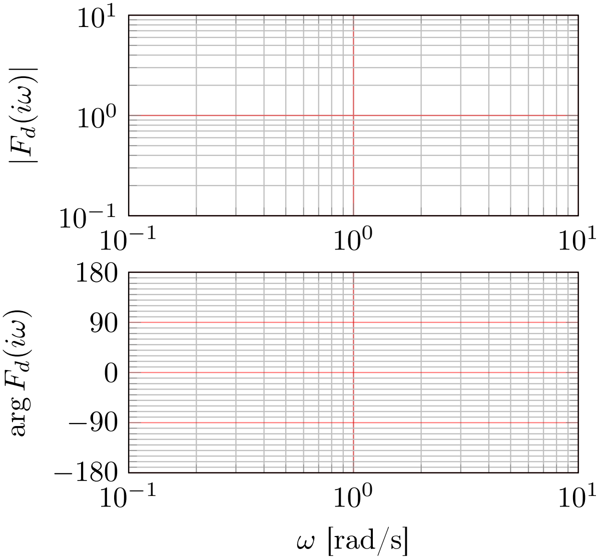
Figure 8.1 Approximating a continuous-time transfer function, $G(s)$, using a computer.

Source: Åström & Wittenmark

Warm-up exercise



Draw the Bode diagram for the transfer function



Discrete-time differentiation

$$u(kh) \longrightarrow \boxed{z} \longrightarrow y(kh) \quad \Leftrightarrow \quad u(kh) \longrightarrow \boxed{\mathrm{q}} \longrightarrow y(kh)$$

$$u(kh) \longrightarrow \boxed{\frac{\mathrm{q}-1}{h}} \longrightarrow y(kh)$$

$$\begin{aligned} y(kh) &= \frac{\mathrm{q}-1}{h} u(kh) = \frac{\mathrm{q}\, u(kh) - u(kh)}{h} \\ &= \frac{u(kh+h) - u(kh)}{h} \end{aligned}$$

$$u(kh) \longrightarrow \boxed{\frac{1-\mathrm{q}^{-1}}{h} = \frac{\mathrm{q}-1}{\mathrm{q}\,h}} \longrightarrow y(kh)$$

Discretization methods

1. Forward difference. Substitute
$$s = \frac{z-1}{h}$$
in $F(s)$ to get
$$F_d(z) = F(s')\big|_{s'=\frac{z-1}{h}}.$$

2. Backward difference. Substitute
$$s = \frac{z-1}{zh}$$
in $F(s)$ to get
$$F_d(z) = F(s')\big|_{s'=\frac{z-1}{zh}}.$$

Discretization methods, contd.

3. Tustin's method (also known as the bilinear transform). Substitute

$$s = \frac{2}{h}\frac{z-1}{z+1}$$

in $F(s)$ to get

$$F_d(z) = F(s')\Big|_{s'=\frac{2}{h}\cdot\frac{z-1}{z+1}}.$$

4. Ramp invariance. This is similar to ZoH, which is step-invariant approximation. Since a unit ramp has z-transform $\frac{zh}{(z-1)^2}$ and Laplace-transform $1/s^2$, the discretization becomes

$$F_d(z) = \frac{(z-1)^2}{zh}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\}\right\}.$$

Frequency warping using Tustin's

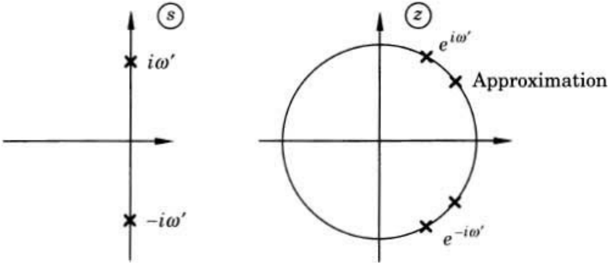
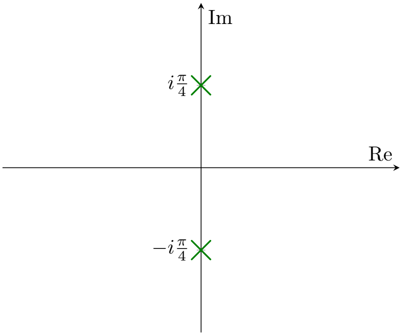


Figure 8.3 Frequency distortion (warping) obtained with approximation.

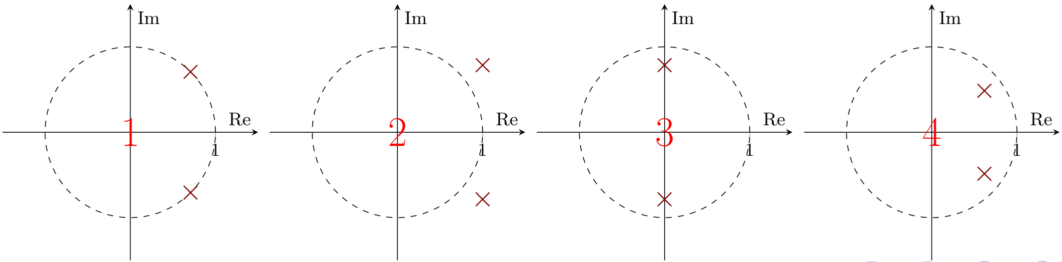
The infinite positive imaginary axis in the s-plane is mapped to the finite-length upper half of the unit circle in the z-plane.

Forward difference exercise

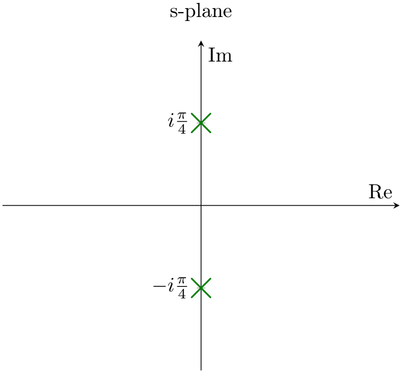
s-plane



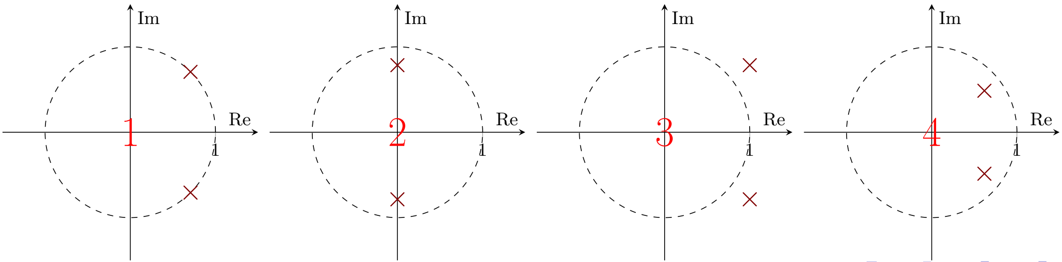
Which of the below figures shows the correct mapping of the continuous-time poles using the **forward difference** $z = 1 + sh$ with $h = 1$?



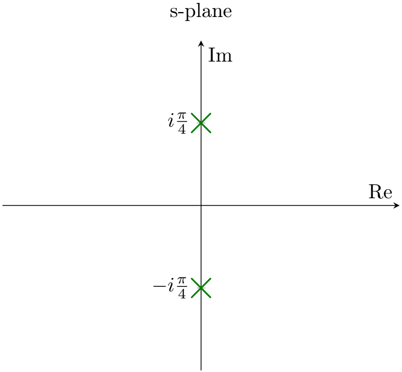
Backward difference exercise



Which of the below figures shows the correct mapping of the continuous-time poles using the **backward difference** $z = \frac{1}{1-sh}$ with $h = 1$?



Bilinear transformation exercise



Which of the below figures shows the correct mapping of the continuous-time poles using the **Tustin's approximation** $z = \frac{1+\frac{sh}{2}}{1-\frac{sh}{2}}$ with $h = 1$? (Hint: Warping of the frequency axis)