Computerized Control - Sampling and aliasing

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Repetition

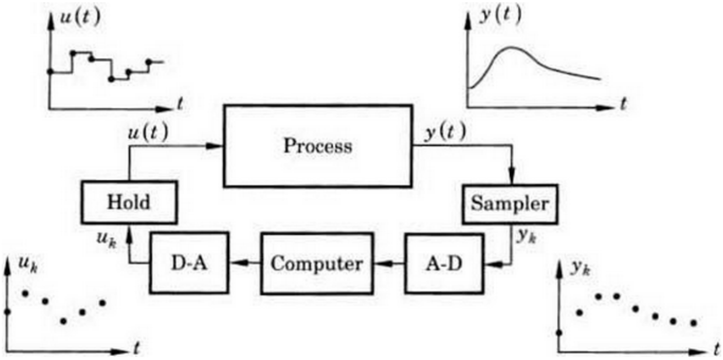
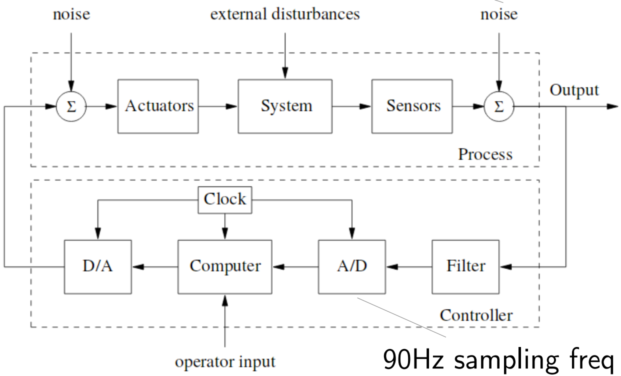


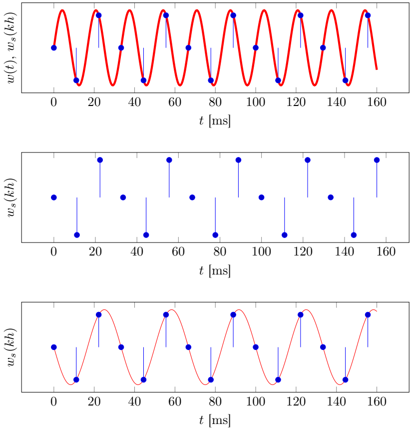
Figure 7.2 Relationships among the measured signal, control signal, and their representations in the computer.

Challenges with computerized control - aliasing

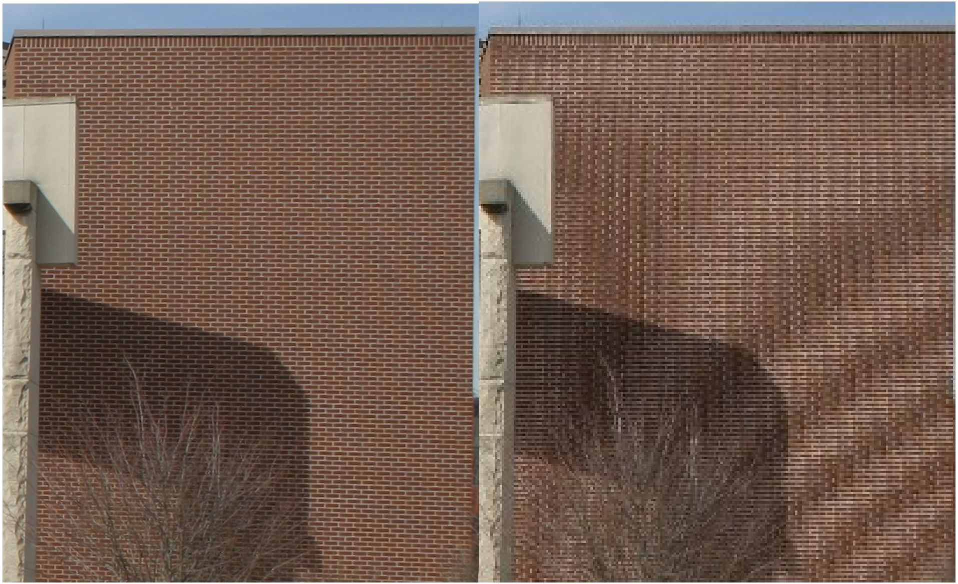
60Hz mains hum



90Hz sampling freq



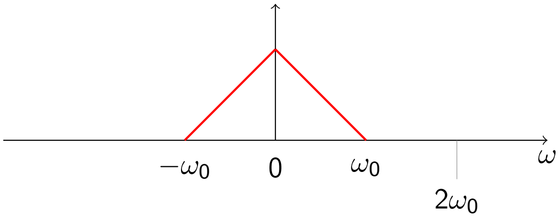
Spatial aliasing



The sampling theorem

Shannon and Nyquist:

A continuous-time signal with Fourier transform that is zero outside the interval $(-\omega_0, \omega_0)$ can be completely reconstructed from equidistant samples of the signal, as long as the sampling frequency is at least $2\omega_0$.

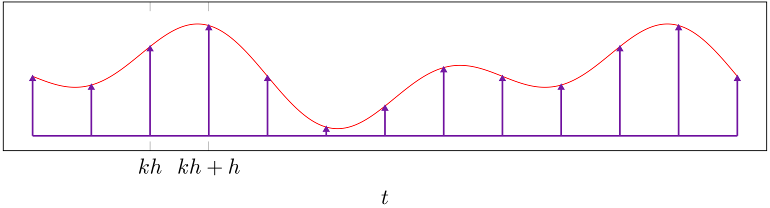


The impulse modulation model

The impulse train, a.k.a the Dirac comb:

$$m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh)$$

$$f_s(t) = f(t)m(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kh) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - kh) = \sum_{k=-\infty}^{\infty} f(kh)\delta(t - kh)$$



Fourier transform of the sampled signal

The Fourier transform of f_s and the Fourier transform of f are related as

$$F_s(\omega) = \frac{1}{h} \sum_n F(\omega + n\omega_s).$$

Because the Fourier transform of the sampled signal equals the Fourier transform of the continuous-time signal repeated at every multiple of the sampling frequency and added, we get *frequency-folding* or *aliasing*.

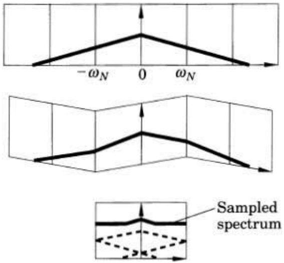
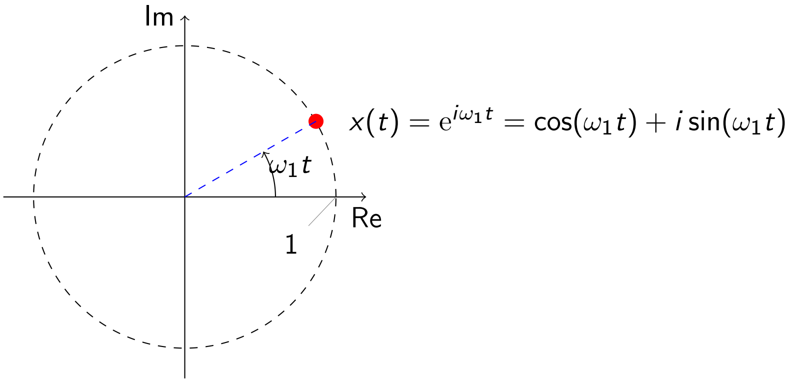


Figure 7.11 Frequency folding.

Fourier transform of a complex exponential

The function $x(t) = \mathrm{e}^{i\omega_1 t}$

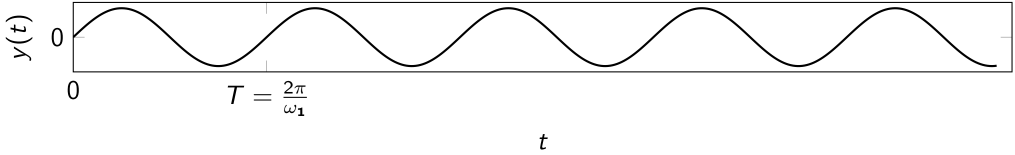


has Fourier transform

$$X(i\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-i\omega t}dt = \int_{-\infty}^{\infty} \mathrm{e}^{i(\omega_1-\omega)t}dt = \delta(\omega_1 - \omega)$$

Fourier transform of a sinusoid

A sinusoidal signal $y(t) = \sin(\omega_1 t)$ has all its power concentrated at one single frequency, $\omega = \omega_1 \text{ rad/s}$.



Since

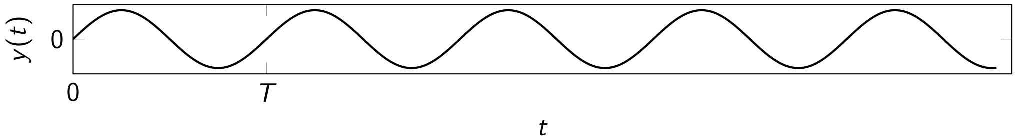
$$y(t) = \sin(\omega_1 t) = \frac{1}{2i}\left(e^{i\omega_1 t} - e^{i\omega_1 t}\right)$$

the Fourier transform of a sinusoid becomes

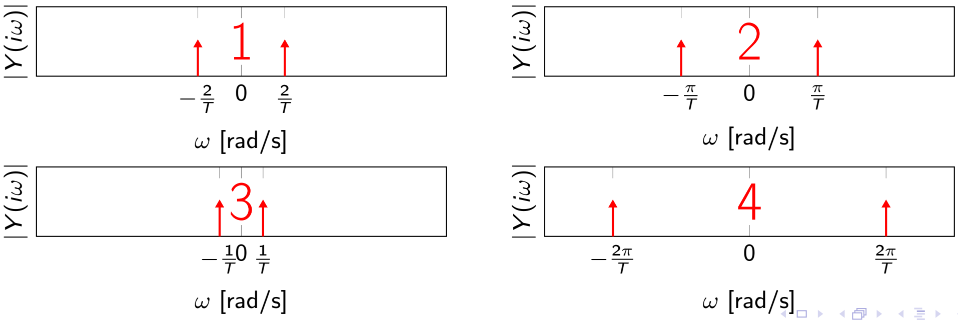
$$Y(i\omega) = \frac{1}{2i}\left(\delta(\omega_1 - \omega) - \delta(\omega_1 + \omega)\right)$$

Exercise 1: Fourier transform of a sinusoid

Consider the signal below

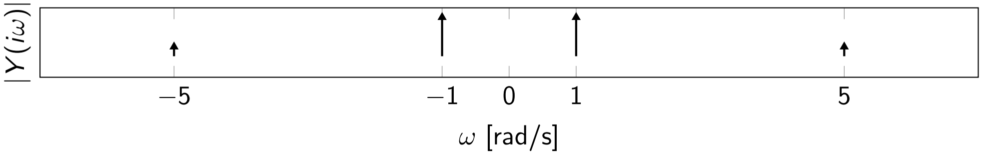


Which of the below is the correct Fourier transform (magnitude plot shown)?

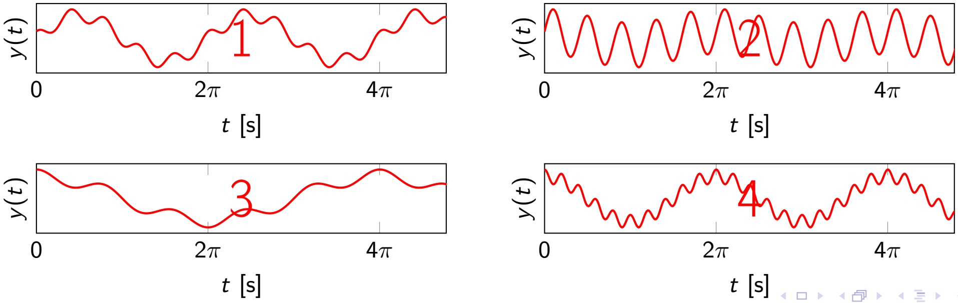


Exercise 2: Two sinusoids

Consider a signal with Fourier transform

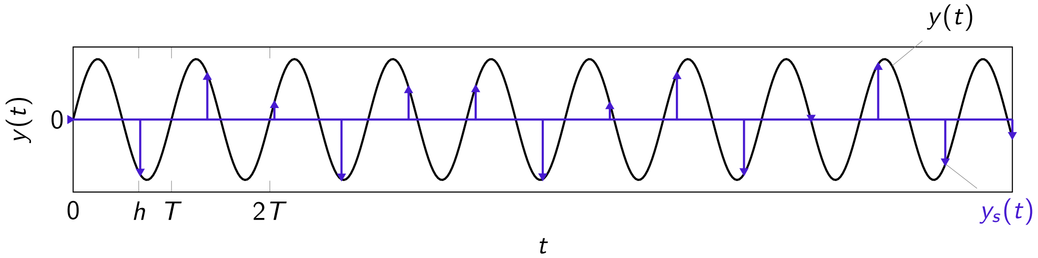


Which of the below time series could this Fourier transform correspond to?



Exercise 3: Fourier transform of a sampled sinusoid

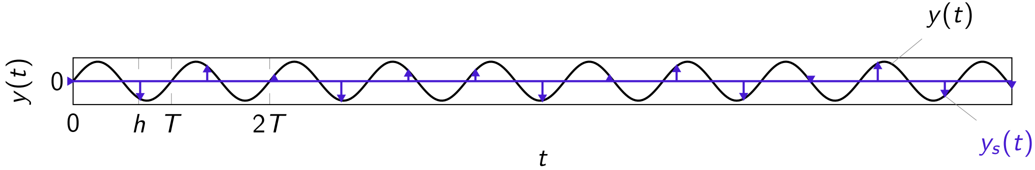
Consider the continuous and sampled signals with sampling period $h = \frac{2}{3}T$



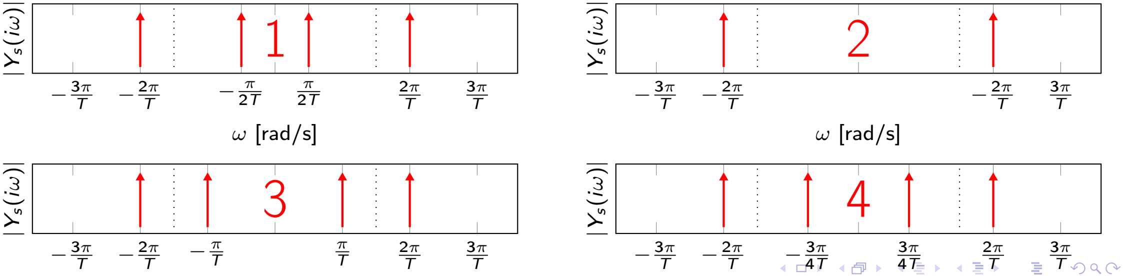
What is the frequency of the sinusoid? What is the sampling frequency ω_s and the Nyquist frequency ω_N?

Exercise 3: Fourier transform of a sampled sinusoid

Consider the continuous and sampled signals with sampling period $h = \frac{2}{3}T$



Which of the below corresponds to the Fourier transform of the sampled signal?



Alias frequency

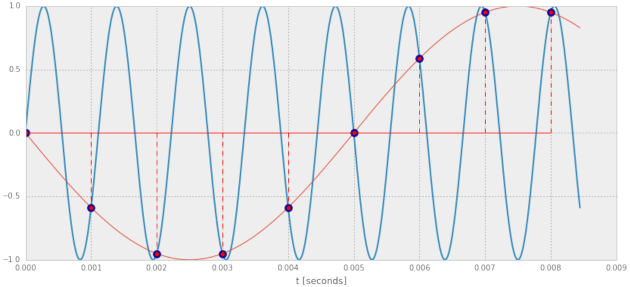
To find the low frequency alias $\omega_a < \omega_N$ of a high freqency sinusoid ω_1, The expression

$$\omega_a = \left| \left((\omega_1 + \omega_N) \bmod \omega_s \right) - \omega_N \right|$$

can be used.

Aliasing example

If a continuous-time signal with frequency content (bandwidth) ω_B is sampled at too low sampling rate ($\omega_s < 2\omega_B$), then the energy at higher frequencies is folded onto lower frequencies.



A high-frequency sinusoid ($\omega_1 = 1800\pi$ rad/s) masquerading as a lower frequency sinusoid (200π rad/s) due to aliasing when sampled with $h = 10^{-3}$ s.

Draw the spectrum (lines) of the two sinusoids. Mark the Nyquist frequency and verify that the alias frequency is obtained by folding about the Nyquist frequency.

Group exercise: Alias frequency

A $f_1 = 60$Hz noise signal is sampled at $f_s = 90$Hz.

1. Determine the alias frequency using the expression

$$f_a = \left| \left((f_1 + f_N) \bmod f_s \right) - f_N \right|$$

2. Verify in the plot that your calculation is correct
3. Draw the spectrum lines of the two sinusoids. Mark the Nyquist frequency and verify that the alias frequency is obtained by folding about the Nyquist frequency