Computerized Control - analysis of discrete-time systems

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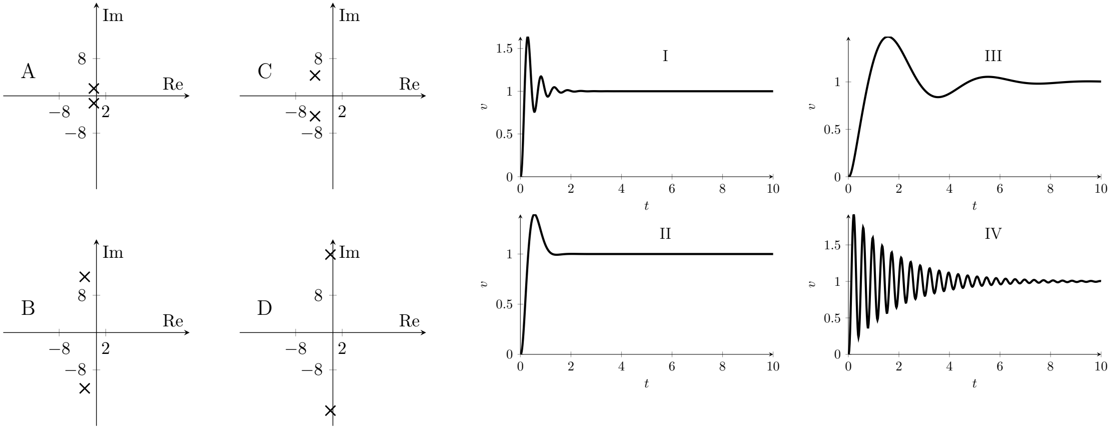
Result from quizz

- Very good results!
- Nyquist plot + criterion
- Robustness

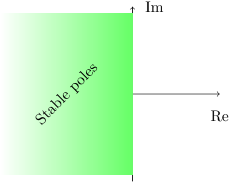
Pole-placement and time-response

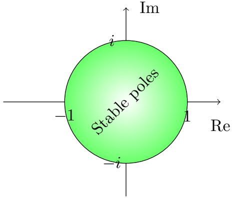
Pair the pole-placement with the correct time-response (from HW1)



Mapping of poles from continuous time to discrete time

Continuous time	Discrete time
$Y(s) \triangleq \mathcal{L}\{y(t)\}$	$Y(z) \triangleq \mathcal{Z}\{y(kh)\}$
$Y(s) = G(s)U(s) = \frac{b}{s+a}U(s)$	$Y(z) = H(z)U(z) = \frac{\beta}{z+\alpha}U(z)$
Pole of the system: $s + a = 0 \;\Rightarrow\; s = -a$	Pole of the system: $z + \alpha = 0 \;\Rightarrow\; z = -\alpha$





The s-domain of continuous-time systems is related to the z-domain of discrete-time systems as

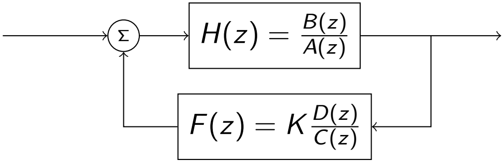
$$z = e^{sh}$$

Mapping of poles from continuous time to discrete time

Do excercise on paper!

Root locus: A brief review

LaTeX



- The loop pulse-transfer function (loop gain) becomes

$$L(z) = H(z)F(z) = K\frac{\overbrace{B(z)D(z)}^{Q(z)}}{\underbrace{A(z)C(z)}_{P(z)}} = K\frac{Q(z)}{P(z)}.$$

- The roots of $Q(z)$ are called the open loop zeros.
- The roots of $P(z)$ are called the open loop poles.
- The characteristic equation for the closed-loop system is

$$Q(z)$$

Root locus: Definition

Let

$$\begin{cases} P(z) & = z^n + a_1 z^{n-1} + \cdots + a_n = (z-p_1)(z-p_2)\cdots(z-p_n) \\ Q(z) & = z^m + b_1 z^{m-1} + \cdots + b_m = (s-q_1)(z-q_2)\cdots(z-q_m) \end{cases}, \quad n \geq m$$

The root locus shows how the solution to the characteristic equation

$$P(z) + K \cdot Q(z) = 0, \quad 0 \leq K < \infty \tag{1}$$

depend on the parameter K. The root locus consists of the set of all points in the complex plane that are solutions to (1) for some non-negative value of K.

Root locus: Characteristics

Start points	The n roots of $P(z)$, marked by crosses
End points	The m roots of $Q(z)$, marked by circles
Asymptotes	Number equal to the *pole excess* $n - m$
Real axis	Some segments of the real axis belong to the root locus

Root locus: Direction of the asymptotes

The characteristic equation $P(z) + KQ(z) = 0$ can be written $\frac{P(z)}{Q(z)} = -K$ and for large z it can be approximated as

$$\frac{z^n}{z^m} = -K \quad \Leftrightarrow \quad z^{n-m} = -K.$$

Taking the argument of both sides of the equation gives
$(n-m)\arg z = \pi + k2\pi,\ \ k \in \mathbb{Z}$ So, the directions of the asymptotes are given by the expression

$$\theta_k = \arg z = \frac{(2k+1)\pi}{n-m},\ \ k \in \mathbb{Z}$$

Root locus: The asymptotes' intersection with the real axis

$$z_{ip} = \frac{\sum_{i=0}^{n} p_i - \sum_{i=0}^{m} q_i}{n - m},$$

where $\{p_i\}$ are the starting points (open-loop poles) and $\{q_i\}$ are the end points (open-loop zeros).

Root locus exerise: Pair the pulse-trf fcn and root locus

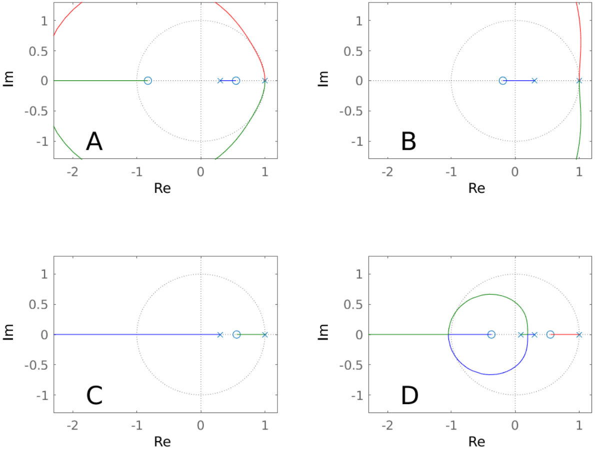
LaTeX

$$G_1(z) = K\frac{(z+2.9)(z+0.2)}{(z-1)^2(z-0.3)}$$

$$G_2(z) = K\frac{(z-0.5)(z+0.4)}{(z-1)(z-0.3)(z-0.1)}$$

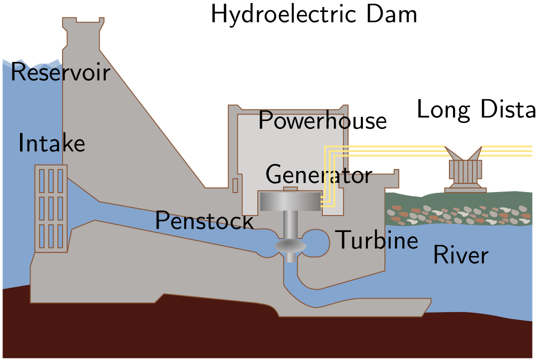
$$G_3(z) = K\frac{(z-0.5)(z+0.8)}{(z-1)^2(z-0.3)}$$

$$G_4(z) = K\frac{z-0.6}{(z-1)(z-0.3)}$$



Draw a root locus

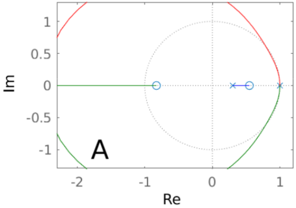
Level control in a hydro power plant dam



Discrete-time model: $y(k+1) - y(k) = \frac{h}{A}u(k) + \frac{h}{A}v(k)$, where $y(k)$ is the deviation in water level from a standard level, $u(k)$ is the (negative) deviation in flow through the dam ports and $v(k)$ is a deviation in other flows (disturbance).

What happens if the poles are on the unit circle?

Say, in $z = \mathrm{e}^{\pm i\omega_0}$



$$H_c(z) = \frac{kz}{(z - \mathrm{e}^{i\omega_0})(z - \mathrm{e}^{-i\omega_0})} \overbrace{+\cdots}^{\text{stable term}}$$

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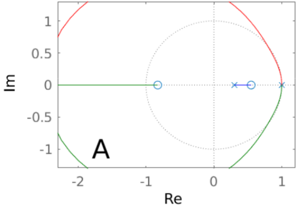
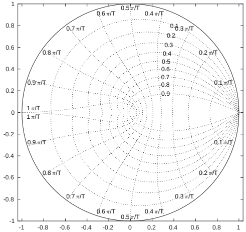
What happens if the poles are on the unit circle?

Say, in $z = \mathrm{e}^{\pm i\omega_0}$





$$H_c(z) = \frac{kz}{(z - \mathrm{e}^{i\omega_0})(z - \mathrm{e}^{-i\omega_0})} + \cdots$$
$$= \frac{kz}{z^2 - 2\cos\omega_0 z + 1} + \cdots$$

If $\omega_0 = \frac{\pi}{6}$ and the sampling period is $0.4\ \mathrm{s}$, what is the frequency (in $\mathrm{rad/s}$ and in Hz) of the oscillations in the pulse response?

Bode diagram and Nyquist plots

Sine in — sine out

LaTeX

$$u(kh) = \sin(\omega kh) \longrightarrow \boxed{H(z)} \longrightarrow y(kh) = |H(\mathrm{e}^{i\omega h})| \sin(\omega kh + \arg H(\mathrm{e}^{i\omega h}))$$

Sine in — sine out

LaTeX

$$u(kh) = \sin(\omega kh) \longrightarrow \boxed{H(z)} \longrightarrow y(kh) = |H(\mathrm{e}^{i\omega h})| \sin(\omega kh + \arg H(\mathrm{e}^{i\omega h}))$$

Prove it! Some hints:

- Write $\sin(\omega kh) = \mathrm{Im}\{\mathrm{e}^{i\omega kh}\}$.
- Use $H(z)U(z) \overset{\mathcal{Z}}{\leftrightarrow} h(k) * u(k)$, and write out the discrete-time convolution $h * u = \sum_{n=-\infty}^{\infty} h(n)u(k-n) = \sum_{n=0}^{\infty} h(n)u(k-n)$
- Try to rewrite to obtain as a factor $\sum_{n=0}^{\infty} h(n)\mathrm{e}^{-i\omega nh} = H(\mathrm{e}^{i\omega h})$.

The Nyquist plot

Example of a Nyquist plot or frequency curve. LaTeX

$H(z)$ evaluated at $z = \mathrm{e}^{i\omega h}$, $0 < \omega < \frac{\pi}{h}$

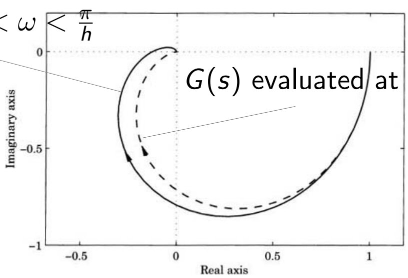


Figure 3.3 The frequency curve of (3.6) (dashed) and for (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

The system $G(s) = \frac{1}{s^2+1.4s+1}$ is sampled with ZOH-sampling ($h = 0.4$ s) to get $H(z) = \frac{0.066z+0.055}{z^2-1.45z+0.571}$

Simplified Nyquist criterion

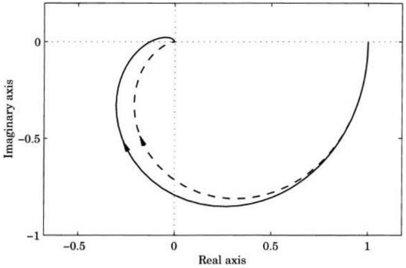
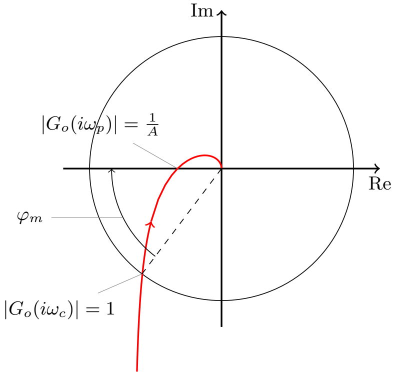


Figure 3.3 The frequency curve of (3.6) (dashed) and for (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

Consider the loop pulse-transfer function $L(z)$ of a closed-loop system. If $L(z)$ is stable (no poles outside the unit circle), then the closed-loop system with characteristic equation $1 + L(z) = 0$ will be stable iff $L(z)$ evaluated on the unit circle (i.e, the Nyquist plot of L) has the point -1 to the left.

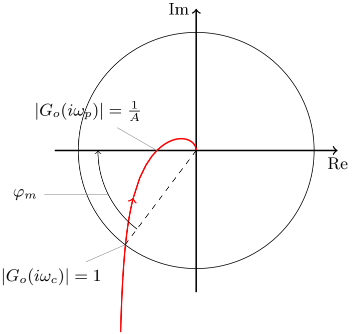
Stability margins - phase margin



- Cross-over frequency: The frequency ω_c for which $|L(\mathrm{e}^{i\omega h})| = 1$.
- Phase margin: The angle φ_m to the negative real axis for the point where the Nyquist curve intersects the unit circle.

$$\varphi_m = \arg L(\mathrm{e}^{i\omega_c h}) - (-180^\circ) = \arg L(\mathrm{e}^{i\omega_c h}) + 180^\circ$$

Stability margins - gain margin

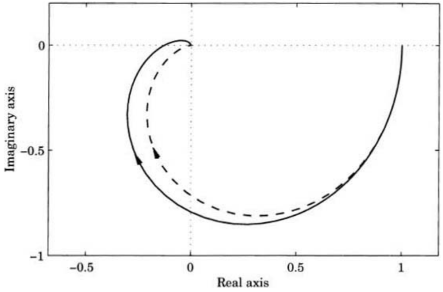


- phase-cross-over frequency: The frequency ω_p for which $\arg L(\mathrm{e}^{i\omega h}) = -180^\circ$.
- Gain margin: The gain $K = A$ that would make the Nyquist curve of $KL(\mathrm{e}^{i\omega h})$ go through the point $-1 + i0$. This means that

$$|L(\mathrm{e}^{i\omega_p h})| = \frac{1}{A}.$$

The relationship between Bode diagrams and frequency curves (Nyquist plots)

They are both showing the value of a pulse-transfer function $H(z)$ evaluated for $z = \mathrm{e}^{i\omega h}$, $0 < \omega \leq \frac{\pi}{h}$.



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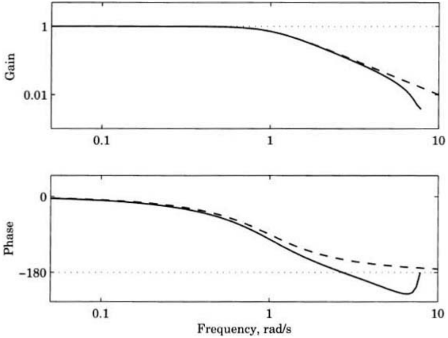

Figure 3.4 The Bode diagram of (3.6) (dashed) and of (3.6) sampled with zero-order hold when $h = 0.4$ (solid).