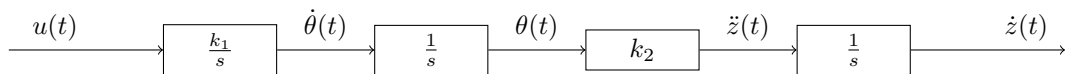
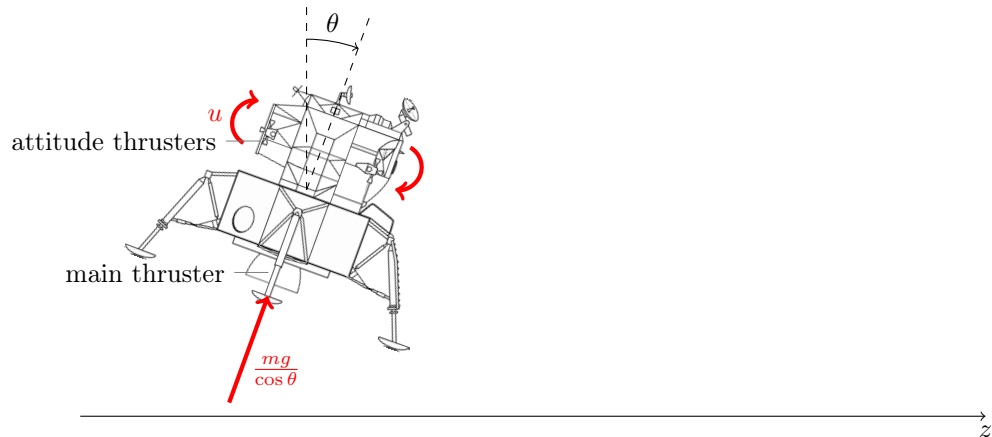


State feedback

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Example - The Apollo lunar module



Example - The Apollo lunar module

State variables: $x = [x_1 \ x_2 \ x_3]^T = [\dot{\theta} \ \theta \ \dot{z}]^T$. With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2 \theta = k_2 x_2 \end{cases}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}}_B u$$

Example - The Apollo lunar module

$$\begin{aligned}x(kh + h) &= e^{Ah}x(kh) + \int_0^h e^{As}Bu(kh + h - s)ds \\&= \underbrace{e^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h e^{As}Bds\right)}_{\Gamma(h)}u(kh) \\&= \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2 k_2}{2} & hk_2 & 1 \end{bmatrix} x(kh) + k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2 h^3}{6} \end{bmatrix} u(kh)\end{aligned}$$

Stability

Eigenvalues and eigenvectors

Definition The eigenvalues $\lambda_i \in \mathbb{R}$ and eigenvectors $v_i \in \mathbb{R}^n$ of a matrix $\Phi \in \mathbb{R}^{n \times n}$ are the n pairs $(\lambda_i, v_i \neq 0)$, $i = 1, 2, \dots, n$ that satisfy

$$\Phi v_i = \lambda_i v_i$$

Stability

The system

$$x(k+1) = \Phi x(k), \quad x(0) = x_0$$

is **stable** if $\lim_{t \rightarrow \infty} x(kh) = 0, \quad \forall x_0 \in \mathbb{R}^n$.

A necessary and sufficient requirement for stability is that **all the eigenvalues of Φ are inside the unit circle**.

The **eigenvalues** of Φ are the **poles** of the system.

Eigenvalues and eigenvectors - exercise

Activity Verify that the vector

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is an eigenvector of

$$\Phi = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

What is the corresponding eigenvalue?

Controllability

Controllability is the answer to the question *Can we drive the state of the system to any location in the state space by a suitable input sequence $u(k)$, $k = 0, 1, 2, \dots, n-1$?*

Consider

$$x(k+1) = \Phi x(k) + \Gamma u(k), \quad x(0) = x_0$$

with solution

$$\begin{aligned} x(n) &= \Phi^n x(0) + \Phi^{n-1} \Gamma u(0) + \Phi^{n-2} \Gamma u(1) + \dots + \Gamma u(n-1) \\ &= \Phi^n x(0) + W_c U, \end{aligned} \tag{1}$$

where

$$\begin{aligned} W_c &= [\Gamma \quad \Phi \Gamma \quad \dots \quad \Phi^{n-1} \Gamma] \\ U &= [u(n-1) \quad u(n-2) \quad \dots \quad u(0)]^T \end{aligned}$$

Controllability

To find the input sequence $u(k)$ that takes the state from $x(0) = x_0$ to $x(n) = x_d$ we may solve for U in the equation

$$x_d = \Phi^n x_0 + W_c U.$$

$$U = W_c^{-1} (x_d - \Phi^n x(0))$$

This is possible when the matrix W_c is **invertible**:

The state-space system above is controllable if and only if the *Controllability matrix* W_c has rank n , i.e.

$$\det W_c \neq 0.$$

Observability

Observability is the answer to the question "Can we determine the initial state $x(0)$ from measurements $y(k)$, $k = 0, 1, 2, \dots, n - 1$?"

The first n values of the output sequence (with $u = 0$) are given by

$$y(0) = Cx(0)$$

$$y(1) = Cx(1) = C\Phi x(0)$$

$$\vdots$$

$$y(n-1) = Cx(n-1) = C\Phi^{n-1}x(0).$$

This gives the equation

$$\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}$$

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$$\text{possible iff } W_o = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} \text{ has full rank.}$$

Observability, contd

The equation

$$\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y(1) - C\Gamma u(0) \\ \vdots \\ y(n-1) - CW_c U \end{bmatrix}$$

can be solved for $x(0)$ if and only if the matrix

$$W_o = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}$$

has full rank. If this is the case, the system is said to be **observable**.

State feedback control

State feedback control

Given

$$\begin{aligned}x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k)\end{aligned}\tag{2}$$

and measurements (or an estimate) of the state vector $x(k)$.

Linear state feedback is the control law

$$\begin{aligned}u(k) &= f(x(k), u_c(k)) = -l_1 x_1(k) - l_2 x_2(k) - \cdots - l_n x_n(k) + l_0 u_c(k) \\ &= -Lx(k) + l_0 u_c(k),\end{aligned}$$

where

$$L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}.$$

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Activity Substitute the control law $u = l_0 u_c(k) - Lx(k)$ into the state-space model (2)

$$\begin{aligned}x(k+1) &= (\Phi - \Gamma L)x(k) + l_0 \Gamma u_c(k) \\ y(k) &= Cx(k)\end{aligned}\tag{3}$$

Pole placement by state feedback

Given a desired placement of the closed-loop poles p_1, p_2, \dots, p_n , being roots of the desired characteristic polynomial

$$a_c(z) = (z - p_1)(z - p_2) \cdots (z - p_n) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n. \quad (4)$$

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Linear state feedback gives the system

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L) x(k) + l_0 \Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \quad (5)$$

with characteristic polynomial

$$\det(zI - (\Phi - \Gamma L)) = z^n + \beta_1(l_1, \dots, l_n) z^{n-1} + \cdots + \beta_n(l_1, \dots, l_n). \quad (6)$$

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Set the coefficients of the desired characteristic polynomial (8) equal to the coefficients of (6) to obtain the system of equations

$$\begin{aligned} \beta_1(l_1, \dots, l_n) &= \alpha_1 \\ \beta_2(l_1, \dots, l_n) &= \alpha_2 \\ &\vdots \\ \beta_n(l_1, \dots, l_n) &= \alpha_n \end{aligned}$$

Pole placement by state feedback

The system of equations

$$\beta_1(l_1, \dots, l_n) = \alpha_1$$

$$\beta_2(l_1, \dots, l_n) = \alpha_2$$

$$\vdots$$

$$\beta_n(l_1, \dots, l_n) = \alpha_n$$

is always linear in the parameters of the controller, henc

$$ML^T = \alpha,$$

where $\alpha^T = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]$.

Pole placement and controllability

It can be shown that the controllability matrix

$$W_c = [\Gamma \quad \Phi\Gamma \quad \dots \quad \Phi^{n-1}\Gamma]$$

is a factor of the matrix M

$$M = \bar{M}W_c.$$

Hence, in general, the equations

$$\bar{M}W_c L^T = \alpha \quad \Rightarrow \quad L^T = W_c^{-1} \bar{M}^{-1} \alpha \quad (7)$$

only has a solution if W_c is invertible, that is when the system is *controllable*.

Note that the equations (7) may also have solutions when the system is not controllable, if α is in the column space of M . That is

$$\alpha = b_1 M_{:,1} + b_2 M_{:,2} + \dots + b_m M_{:,m}, \quad m < n$$

Pole placement by state feedback

Given a desired placement of the closed-loop poles p_1, p_2, \dots, p_n , being roots of the desired characteristic polynomial

$$a_c(z) = (z - p_1)(z - p_2) \cdots (z - p_n) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n. \quad (8)$$

and closed-loop system

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)x(k) + l_0 \Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \quad (9)$$

The Matlab (*control systems toolbox*) has methods for computing the gain vector L

1. Ackerman's method

```
L = acker(Phi, Gamma, pd)
```

2. Numerically more stable method

```
L = place(Phi, Gamma, pd)
```


The reference input gain l_0

The closed-loop state space system

$$\begin{aligned}x(k+1) &= \underbrace{(\Phi - \Gamma L)}_{\Phi_c} x(k) + l_0 \Gamma u_c(k) \\y(k) &= Cx(k)\end{aligned}$$

has the steady-state solution ($x(k+1) = x(k)$) for constant reference signal $u_c(k) = u_{c,f}$

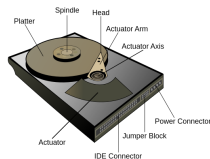
$$y_f = l_0 C(I - \Phi_c)^{-1} \Gamma u_{c,f}.$$

We want $y_f = u_{c,f}$,

$$\Rightarrow \quad l_0 = \frac{1}{C(I - \Phi_c)^{-1} \Gamma}$$

Exercise - The harddisk drive arm

The model of the arm of the harddisk drive



can, with suitable choice of sampling period, be written

$$x(k+1) = \Phi x(k) + \Gamma u(k) = \begin{bmatrix} 1 & 0.4 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.16 \\ 0.8 \end{bmatrix} u.$$

With linear state feedback $u(k) = -Lx(k) + l_0 u_c(k)$ the closed-loop system is

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L) x(k) + l_0 \Gamma u_c(k) \\ &= \begin{bmatrix} 1 - 0.16l_1 & 0.4 - 0.16l_2 \\ -0.8l_1 & 1 - 0.8l_2 \end{bmatrix} x(k) + l_0 \Gamma u_c(k). \end{aligned}$$

Determine the characteristic polynomial of the closed-loop system $\det(zI - (\Phi - \Gamma L))$