Relative stability

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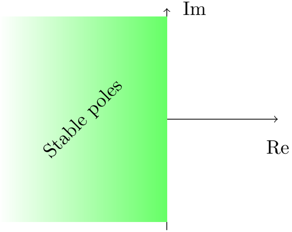
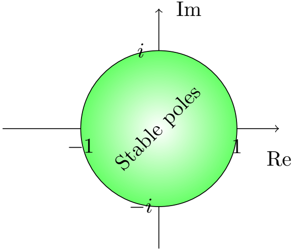
Root locus

The rules for drawing the root locus is the same in continous as in discrete time. But the interpretation differs.

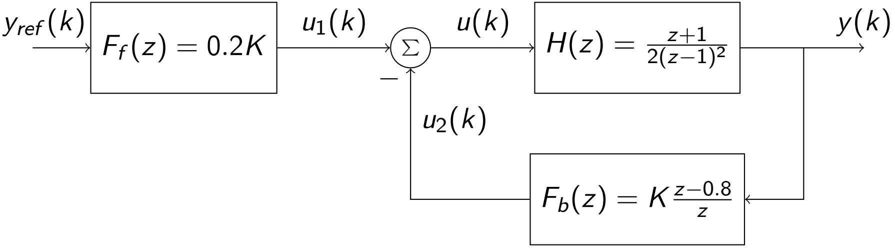
Continuous time



Discrete time



Example: Position servo for the hard disk drive arm



Characteristic equation

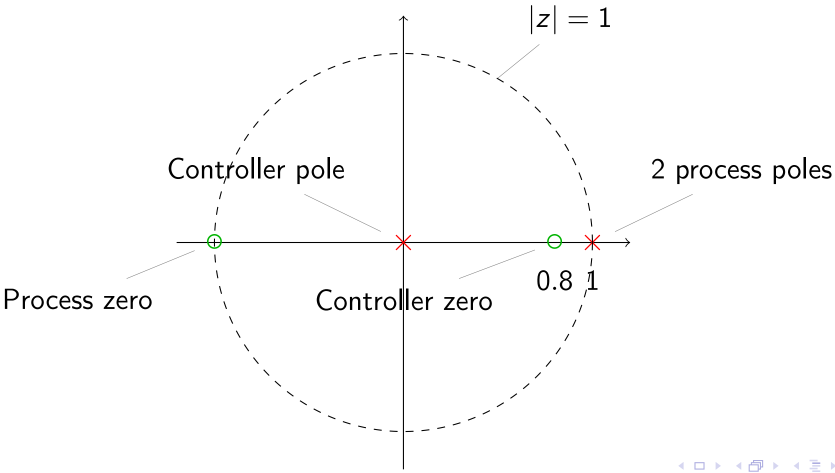
$$
\begin{aligned}
1 + H(z)F_b(z) &= 0 \\
1 + \frac{z+1}{2(z-1)^2} K \frac{z-0.8}{z} &= 0 \\
(z-1)^2 z + \frac{K}{2}(z+1)(z-0.8) &= 0
\end{aligned}
$$

For which values of K will the system be stable?

Example: Position servo for the hard disk drive arm

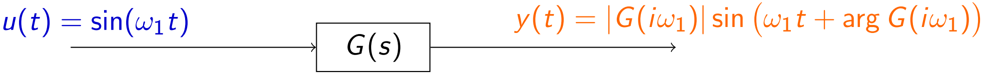
Activity Complete the root locus!

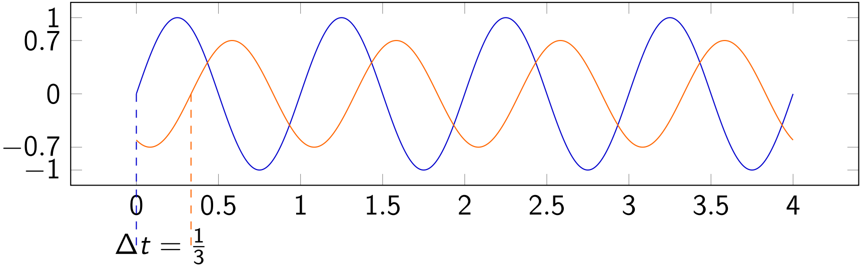
$$(z-1)^2 z + \frac{K}{2}(z+1)(z-0.8) = 0$$



Relative stability

Sinusoid in - sinusoid out





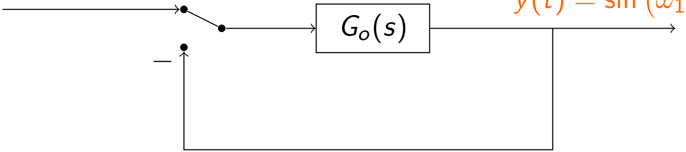
$$\omega_1 = \frac{2\pi}{T} = 2\pi,\ |G(i\omega_1)| = 0.7,\ \arg G(i\omega_1) = -\omega_1 \Delta t = -2\pi\frac{1}{3} = -\frac{2\pi}{3}$$

If the phase shift is π

$$G_o(i\omega_1) = -1,\ |G_o(i\omega_1)| = 1,\ \arg G_o(i\omega_1) = -\pi$$

$u(t) = \sin(\omega_1 t)$

$y(t) = \sin\left(\omega_1 t - \pi\right) = -\sin(\omega_1 t)$



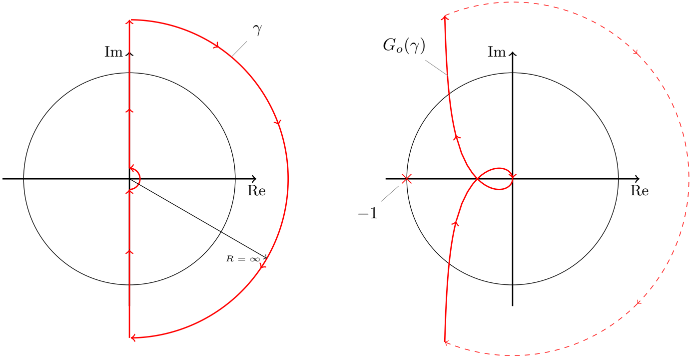
Closed-loop transfer function: $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$

We want

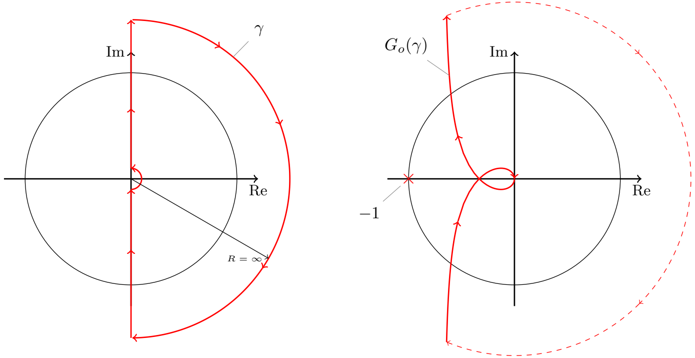
$$1 + G_o(i\omega) \neq 0, \quad \forall\omega$$

If not, then the closed-loop system will have poles on the imaginary axis (in the s-domain).

The simplified Nyquist criterion in the s-plane

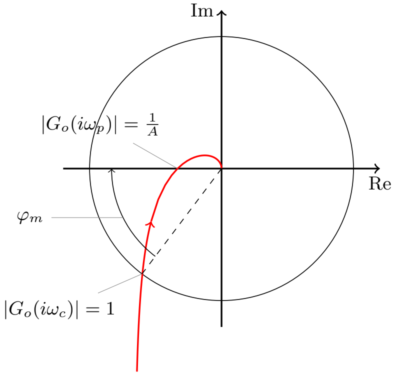


The simplified Nyquist criterion in the s-plane



If the open-loop system (the loop gain) is not unstable, i.e. $G_o(s)$ has no poles in the right-half plane, then the closed-loop system will be stable if the Nyquist curve **do not encircle the point** $s = -1$. The point $s = -1$ should stay on the left side of the Nyquist curve when we go along the curve from low to high frequencies.

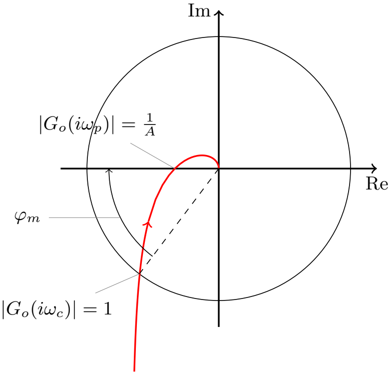
Stability margins



- Cross-over frequency: The frequency ω_c for which $|G_o(i\omega)| = 1$.
- Phase margin: The angle φ_m to the negative real axis for the point where the Nyquist curve intersects the unit circle.

$$\varphi_m = \arg G_o(i\omega_c) - (-180^\circ) = \arg G_o(i\omega_c) + 180^\circ$$

Stability margins



- phase-cross-over frequency: The frequency ω_p for which $\arg G_o(i\omega) = -180°$.
- Gain margin: The gain $K = A$ that would make the Nyquist curve of $KG_o(i\omega h)$ go through the point $-1 + i0$. This means that
$$|G_o(i\omega_p h| = \frac{1}{A}.$$

The effect of sampling on the stability margins

$$G(s) = \frac{1}{s^2 + 1.4s + 1} \quad \xrightarrow{h=0.4} \quad H(z) = \frac{0.066z + 0.055}{z^2 - 1.450z + 0.571}$$

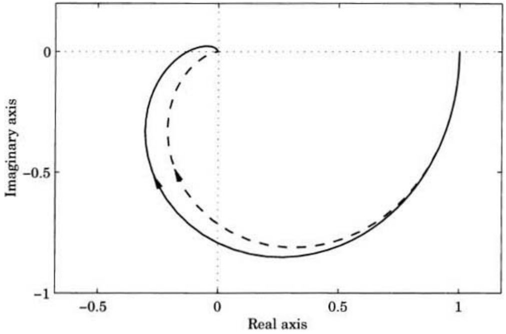


Figure 3.3 The frequency curve of (3.6) (dashed) and for (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

Source: Åström & Wittenmark

The effect of sampling on the stability margins

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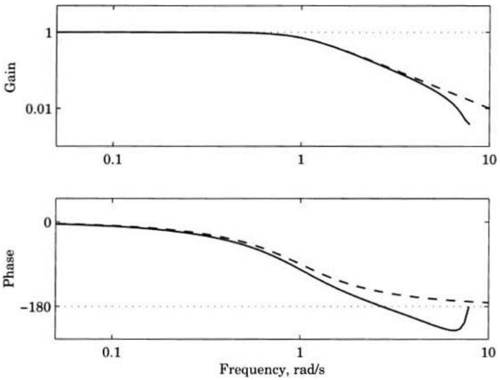

Figure 3.4 The Bode diagram of (3.6) (dashed) and of (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

Source: Åström & Wittenmark

Selecting the sampling period

One can use the phase margin to determine a suitable sampling period. Given a desired cross-over frequency ω_c and a maximum acceptable negative change in the phase margin $\Delta\varphi \approx 5^\circ \ - \ 15^\circ \approx 0.09\text{rad} \ - \ 0.26\text{rad}$ (a rule-of-thumb).

$$u_s(t) \longrightarrow \underset{\text{ZOH}}{\boxed{G_{ZOH}(s) = \frac{1-\mathrm{e}^{-sh}}{s} \approx \mathrm{e}^{-s\frac{h}{2}}}} \longrightarrow u(t)$$

$$\arg G_{ZOH}(i\omega_c) \approx \arg \mathrm{e}^{-i\omega_c \frac{h}{2}} = -\omega_c \frac{h}{2} \approx -0.09\text{rad} \ - \ -0.26\text{rad}$$

Activity Use the rule-of-thumb above to calculate a sampling period for the case $\omega_c = 20 \text{ rad/s}$ and $\Delta\varphi = 0.2 \text{ rad}$.