Control Computarizado - La transformada z

Kjartan Halvorsen

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El mundo según el controlador discreto

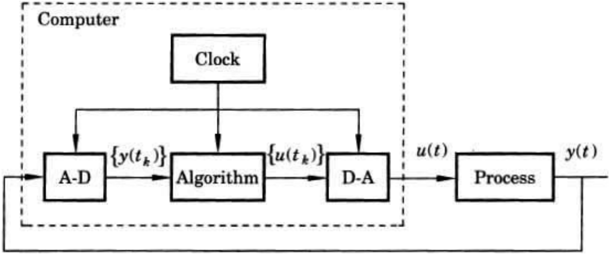
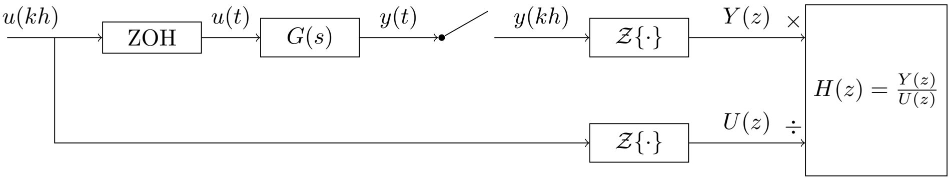


Figure 1.1 Schematic diagram of a computer-controlled system.

Discretizacion invariante al paso (*step-invariant* o *zero-order-hold sampling*)

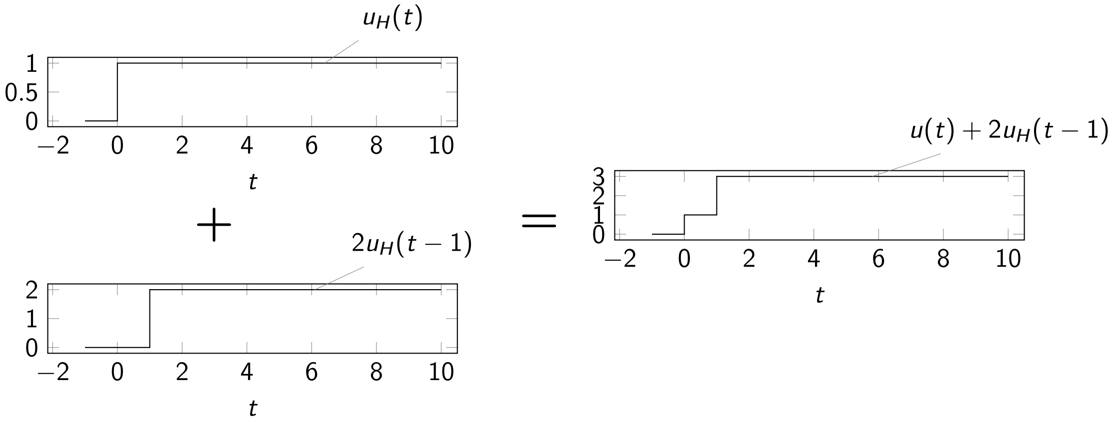
El idea es muestrear la respuesta de paso del sistema continuo para obtener un modelo discreto que es exacto (en los instantes de muestreo) para señales de entrada que son combinaciones de pasos (funciones constantes por partes)



$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Porqué discretizacion invariante al paso?

A piecewise constant (stair-case shaped) function can be written as a sum of delayed step-responses! LaTeX



Why is step-invariant sampling a good idea? (contd)

Due to the system being LTI (linear time-invariant), the output to a sum of delayed step functions, is the same sum of delayed step-responses.

LaTeX

$$u_H(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y_H(t)$$

Hence, $u(t) = \sum_i \alpha_i u_H(t - \tau_i)$ has the response $y(t) =$.

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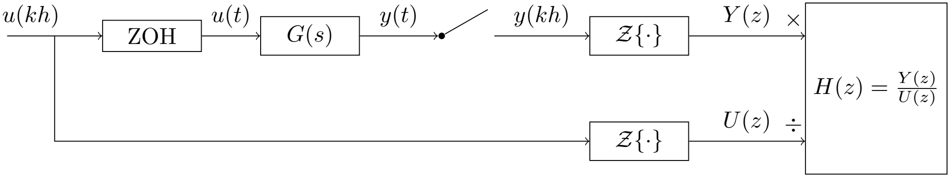
If the sampling method is exact for step input signals, it will also be exact for piecwise-constant step input signals, and this is exactly what the ZOH-block produces!

Impulse- step- and ramp-invariant sampling



- Impulse-invariant sampling: $u(t) = \delta(t)$
- Step-invariant sampling (zero order hold): $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
- Ramp-invariant sampling: $u(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Step-invariant sampling, or zero-order-hold sampling

Let the input to the continuous-time system be a step $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$, which has Laplace transform $U(s) = \frac{1}{s}$. In the Laplace-domain we get

$$Y(s) = G(s)\frac{1}{s}$$

1. Obtain the time-response by inverse Laplace: $y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$
2. Sample the time-response to obtain the sequence $y(kh)$ and apply the z-transform to obtain $Y(z) = \mathcal{Z}\left\{y(kh)\right\}$
3. Calculate the pulse-transfer function by dividing with the z-transform of the input signal $U(z) = \frac{z}{z-1}$.

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z}Y(z)$$

Example: First-order system

Let's apply step-invariant sampling to the system

$$G(s) = \frac{1}{s+a}.$$

Do on your own: The double integrator

$$G(s) = \frac{1}{s^2}$$

Another important property of the z-transform

The z-transform and the solution to difference equations

Taking the z-transform of a difference equation

$$(\mathrm{q}^2 + a_1 \mathrm{q} + a_2) y_k = \left(b_0 \mathrm{q}^2 + b_1 \mathrm{q} + b_2\right) u_k$$

gives

$$\begin{aligned} z^2 Y - z^2 y(0) - zy(1) + a_1 zY - a_1 zy(0) + a_2 Y = \\ b_0 z^2 U - b_0 z^2 u(0) - b_0 zu(1) + b_1 zU - b_1 zu(0) + b_2 U \end{aligned}$$

$$Y(z) = \underbrace{\frac{\big(y(0) - b_0 u(0)\big) z^2 + \big(y(1) + a_1 y(0) - b_0 u(1) - b_1 u(0)\big) z}{z^2 + a_1 z + a_2}}_{\text{transient response}}$$

$$+ \underbrace{\underbrace{\frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}}_{\text{pulse-transfer function}} U(z)}_{\text{response to input}}$$

The z-transform and the solution to difference equations

In general, the output of the discrete-time LTI

$$\left(\mathrm{q}^n + a_1 \mathrm{q}^{n-1} + \cdots + a_n\right) y(k) = \left(b_0 \mathrm{q}^m + b_1 \mathrm{q}^{m-1} + \cdots + b_m\right) u(k)$$

is

$$Y(z) = \frac{\beta(z)}{A(z)} + \frac{B(z)}{A(z)} U(z)$$

For systems that are intially at rest

$$Y(z) = \frac{B(z)}{A(z)} U(z) = G(z) U(z)$$

Convolution in the time-domain is multiplication in the z-domain

$$\mathcal{Z}\left\{g * u)\right\} = \mathcal{Z}\left\{g(kh)\right\}\mathcal{Z}\left\{u(kh)\right\} = \left(\sum_{k=0}^{\infty} g(kh)z^{-k}\right)\left(\sum_{k=0}^{\infty} u(kh)z^{-k}\right)$$

LaTeX

$$\xrightarrow{u(kh)} \boxed{G(z)} \xrightarrow{y(kh)}$$

$$y(kh) = g(kh) * u(kh)$$

$$\mathcal{Z}\left\{y(kh)\right\} = \mathcal{Z}\left\{g(kh) * u(kh)\right\}$$

$$Y(z) = G(z)U(z).$$

The z-transform plays the same role for discrete-time control systems as the Laplace transform for continuous-time ontrol systems!