Computerized Control partial exam 1 (18%)

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Time September 19 19:05-20.35

Place 5305

Permitted aids The single colored page with your own notes, table of Laplace transforms, calculator

All answers should be readable and well motivated (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam. Use the last page if more space is needed.

Good luck!

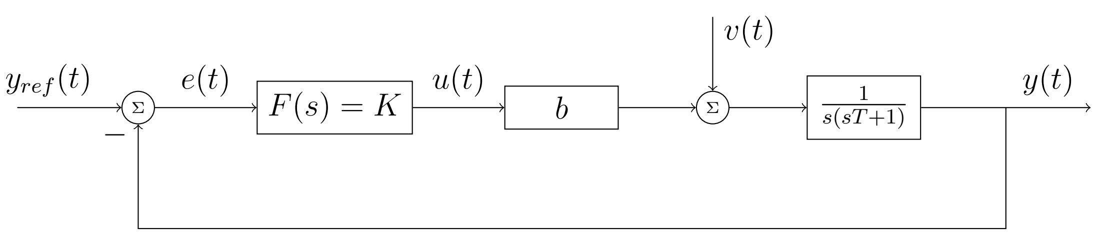
Matricula and name:

Position servo for a DC-motor

Consider a DC-motor connected to a load. The differential equation describing the dynamics of the system is

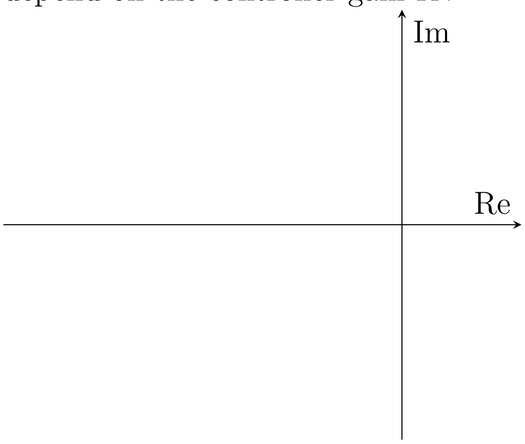
$$T\ddot{y}(t) + \dot{y}(t) = bu(t) + v(t), \tag{1}$$

where $y(t)$ is the angle of the shaft, $u(t)$ is the voltage applied to the circuit and $v(t)$ is a torque disturbance on the shaft. The time-constant of the system is T and b is a gain parameter. We want to control the angle of the shaft using a simple proportional controller.



Problem 1

(a) Sketch a root locus showing how the poles of the continous-time, closed-loop system depend on the controller gain K.



(b) Mark in the root locus above the location of a set of closed-loop poles (one on each branch) that gives the best performance possible with proportional control. Motivate your choice!

Motivation:

Problem 2 The controller we are actually going to use is implemented on a microcontroller. So to be able design a controller taking into account the discrete-time nature of the closed-loop system, we want to work with a discrete-time model of the plant.

(a) Show that zero-order-hold sampling of the DC-motor gives the pulse-transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{T\left(\frac{h}{T} - 1 + \mathrm{e}^{-\frac{h}{T}}\right)z + T\left(1 - \mathrm{e}^{-\frac{h}{T}} - \frac{h}{T}\mathrm{e}^{-\frac{h}{T}}\right)}{(z-1)\left(z - \mathrm{e}^{-\frac{h}{T}}\right)}.$$

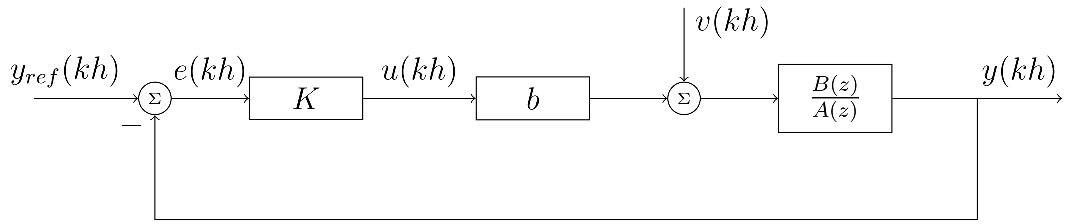
You may have use of the identity $\frac{1}{s^2(sT+1)} = \frac{T^2}{sT+1} - \frac{T}{s} + \frac{1}{s^2}$.

Calculations:

(b) State as a mathematical expression how the continuous-time poles (in the s-plane) and the corresponding discrete-time poles (in the z-plane) are related, and verify that the relationship holds in this particular case.

Answer:

Problem 3 Consider now doing proportional control in discrete time.



We simplify the pulse-transfer function of the DC-motor by writing

$$H(z) = \frac{B(z)}{A(z)} = \frac{T(\mathrm{e}^{-\frac{h}{T}} - 1 + \frac{h}{T})z + T(1 - \mathrm{e}^{-\frac{h}{T}} - \frac{h}{T}\mathrm{e}^{-\frac{h}{T}})}{(z-1)(z - \mathrm{e}^{-\frac{h}{T}})} = \frac{b_0 z + b_1}{(z-1)(z-p)}.$$

Using the approximation $\mathrm{e}^{-\frac{h}{T}} \approx 1 - \frac{h}{T} + 0.5(\frac{h}{T})^2$, the approximate location of the zero is

$$\begin{aligned} z = -\frac{b_1}{b_0} &= \frac{\mathrm{e}^{-\frac{h}{T}} - 1 + \frac{h}{T}\mathrm{e}^{-\frac{h}{T}}}{\mathrm{e}^{-\frac{h}{T}} - 1 + \frac{h}{T}} \approx \frac{1 - \frac{h}{T} + 0.5(\frac{h}{T})^2 - 1 + \frac{h}{T}(1 - \frac{h}{T} + 0.5(\frac{h}{T})^2)}{1 - \frac{h}{T} + 0.5(\frac{h}{T})^2 - 1 + \frac{h}{T}} \\ &= \frac{-0.5(\frac{h}{T})^2 + 0.5(\frac{h}{T})^3}{0.5(\frac{h}{T})^2} = -1 + 0.5\frac{h}{T}, \end{aligned}$$

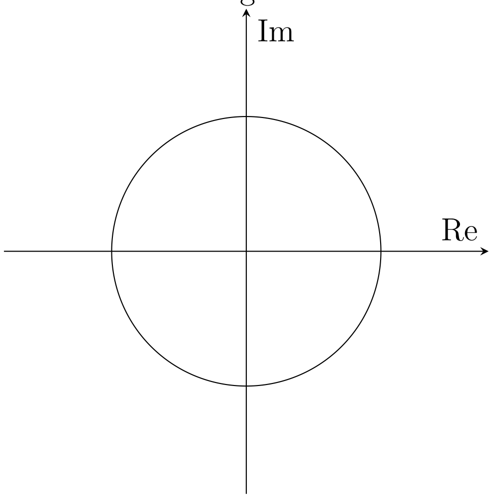
which for reasonable sampling periods h is inside the unit circle and close to -1.

(a) Show that the closed-loop pulse-transfer function from the disturbance $v(kh)$ to the output $y(kh)$ is

$$H_{cv}(z) = \frac{b_0 z + b_1}{(z-1)(z-p) + Kb(b_0 z + b_1)}.$$

Calculations:

(b) Sketch a root locus showing how the poles of the discrete-time closed-loop system depend on the controller gain K.



(c) Will the closed-loop system be stable for any value of the gain K?

Answer and motivation:

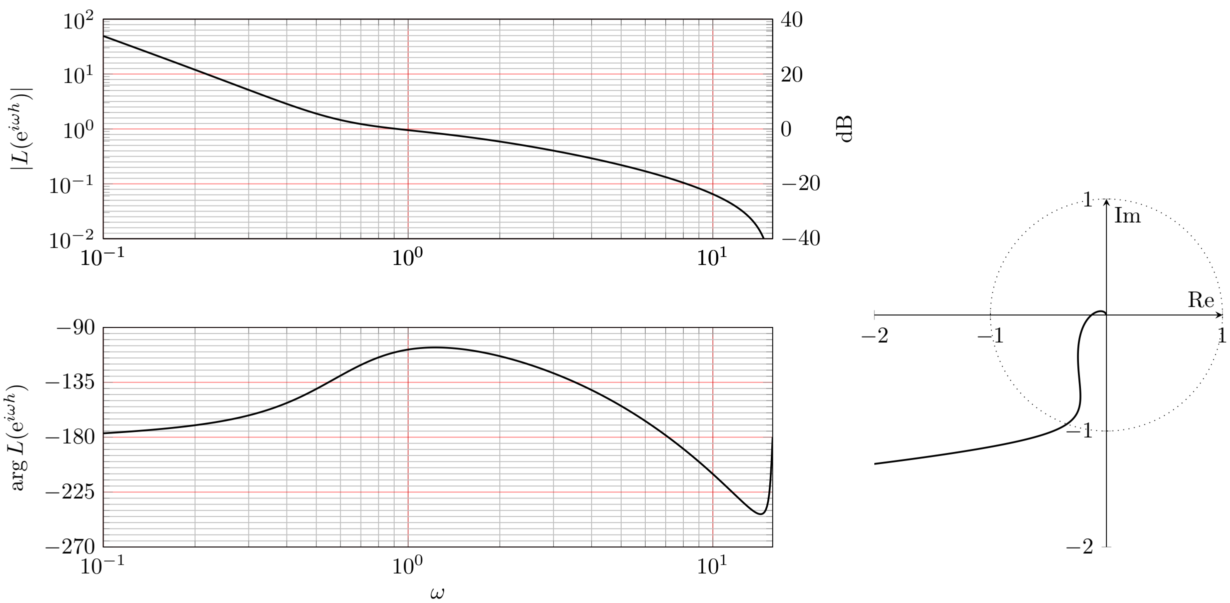
(d) Mark in your root locus for the discrete-time system the location of a set of closed-loop poles (one on each branch) that gives the best performance possible with proportional control. Motivate your choice!

Motivation:

Problem 4 For a particular DC-motor with time-constant $T = 1$, gain $b = 1$, and using the sampling period $h = 0.2$ s, the following discrete-time PID-controller is proposed

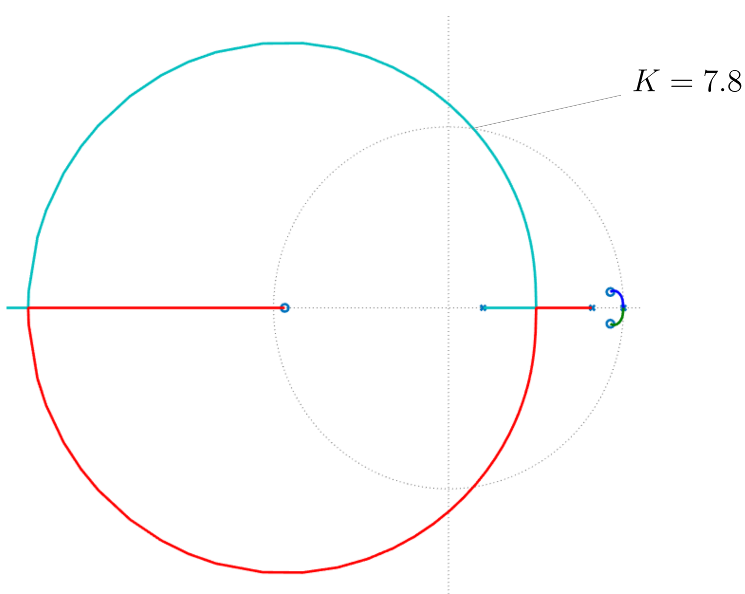
$$F(z) = K\frac{6z^2 - 11z + 5.2}{z^2 - 1.2z + 0.2}.$$

(a) With the suggested PID-controller and $K = 1$ we get the loop-gain $L(z) = H(z)F(z)$, whose frequency response is shown in the Bode diagram and Nyquist plot below. Determine (numerical values) and indicate (mark in the plots) the following: Crossover frequency (ω_c), phase margin (φ_m), phase-crossover frequency (ω_p) and amplitude margin (A_m)



ω_c=	φ_m=
ω_p=	A_m=

(b) The root locus of the closed-loop system w.r.t the gain K is given below



In figure 1, four different closed-loop responses to a step in the reference angle y_{ref} are shown. The different responses are for four different values of K. Identify (and circle) the corresponding step plot for each value of K in the table below. **Motivate your choice!**

K	Step plot			
0.5	I	II	III	IV
1.0	I	II	III	IV
2.0	I	II	III	IV
6.0	I	II	III	IV

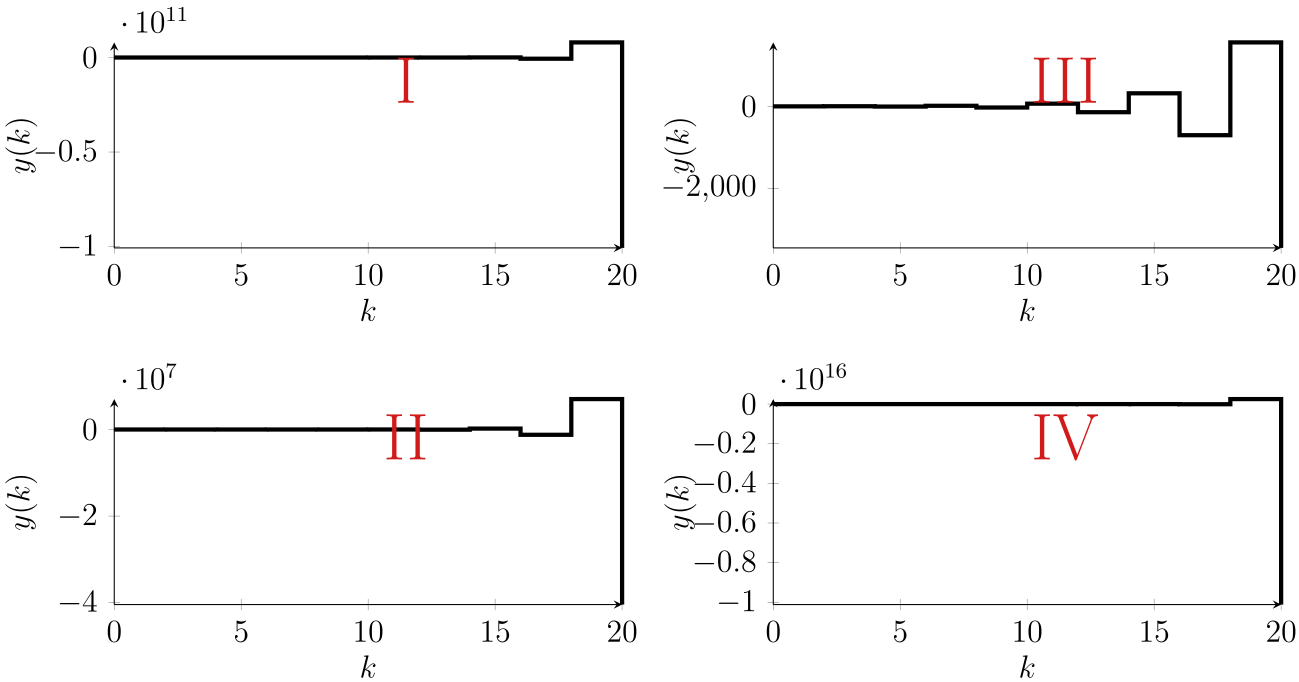


Figure 1: Step-responses of the closed-loop system.

Motivation:

(c) [Bonus problem worth 5p] If the gain is $K = 7.8$ the closed-loop system will have two complex-conjugated poles on the unit circle (see the root locus), and the response of the system

will have undamped oscillations. What will be the *period* of these oscillations (in seconds)?
Hint: For $K = 7.8$, the Nyquist curve passes through the point -1.

Calculation and answer:

Solutions

Problem 1

(a) We have loop gain

$$L(s) = KG(s) = K\frac{\overbrace{1}^{Q(s)}}{\underbrace{s(sT+1)}_{P(s)}},$$

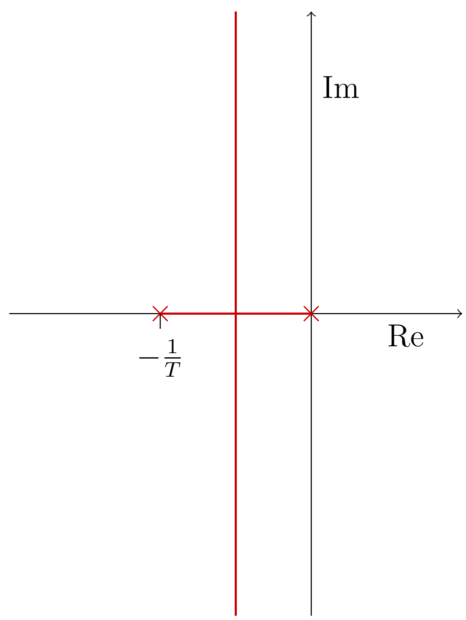
and the characteristic equation

$$s(sT+1) + K = 0$$

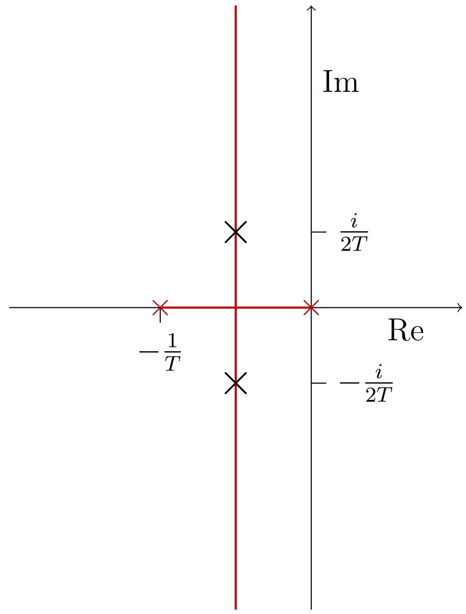
for the closed-loop system. We have $n = 2$ start points ($s = 0$, $s = -\frac{1}{T}$) and $m = 0$ end points. The $n - m = 2$ asymptotes have directions $\pm\frac{\pi}{2}$, and intersects the real axis in the point

$$i.p. = \frac{\sum p_i - \sum q_i}{n-m} = \frac{0 - \frac{1}{T_i}}{2} = -\frac{1}{2T_i}.$$

The root locus looks like below



(b) Good performance in general means fast and well-damped response. Assuming that some small overshoot in the step-response is acceptable, we may aim for the fastest possible poles with a damping ratio of $\zeta = 0.707$. This means poles which have imaginary part equal in magnitude to the real part. Note that for the pure continuous-case the system will stay stable (but be more and more oscillatory) no matter how large we choose the gain K.



Problem 2

(a) First we calculate the step-response of the DC-motor

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s(sT+1)}\frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(sT+1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{T^2}{sT+1} - \frac{T}{s} + \frac{1}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{T^2}{sT+1}\right\} - \mathcal{L}^{-1}\left\{\frac{T}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\
&= T\mathrm{e}^{-\frac{t}{T}} - T + t, \quad t \geq 0.
\end{aligned}
$$

Then we sample (set $t = kh$), and find the z-transform

$$
\begin{aligned}
Y(z) &= \mathcal{Z}\left\{T\mathrm{e}^{-\frac{h}{T}k}\right\} - T\mathcal{Z}\left\{u_H(k)\right\} + \mathcal{Z}\left\{kh\right\} \\
&= \frac{zT}{z - \mathrm{e}^{-\frac{h}{T}}} - \frac{zT}{z-1} + \frac{zh}{(z-1)^2}.
\end{aligned}
$$

Finally, form the pulse-transfer function

$$
\begin{aligned}
H(z) = \frac{Y(z)}{U(z)} &= \frac{z-1}{z}\left(\frac{zT}{z-\mathrm{e}^{-\frac{h}{T}}} - \frac{zT}{z-1} + \frac{zh}{(z-1)^2}\right) \\
&= \frac{T(z-1)}{z-\mathrm{e}^{-\frac{h}{T}}} - T + \frac{h}{(z-1)} \\
&= \frac{T(z-1)^2 - T(z-1)(z-\mathrm{e}^{-\frac{h}{T}}) + h(z-\mathrm{e}^{-\frac{h}{T}})}{(z-1)(z-\mathrm{e}^{-\frac{h}{T}})} \\
&= \frac{T(z^2-2z+1) - T(z^2-(1+\mathrm{e}^{-\frac{h}{T}})z+\mathrm{e}^{-\frac{h}{T}}) + hz - h\mathrm{e}^{-\frac{h}{T}}}{(z-1)(z-\mathrm{e}^{-\frac{h}{T}})} \\
&= \frac{T(\mathrm{e}^{-\frac{h}{T}} - 1 + \frac{h}{T})z + T(1-\mathrm{e}^{-\frac{h}{T}} - \frac{h}{T}\mathrm{e}^{-\frac{h}{T}})}{(z-1)(z-\mathrm{e}^{-\frac{h}{T}})},
\end{aligned}
$$

which correponds to the one given.

(b) When we sample a plant using ZOH, the poles are related through the expression $z = \mathrm{e}^{sh}$. The continuous-time poles of the DC-motor are $s_1 = 0$ and $s_2 = -\frac{1}{T}$, so using the expression we get

$$
z_1 = \mathrm{e}^{s_1 h} = \mathrm{e}^0 = 1, \qquad z_1 = \mathrm{e}^{s_2 h} = \mathrm{e}^{-\frac{h}{T}},
$$

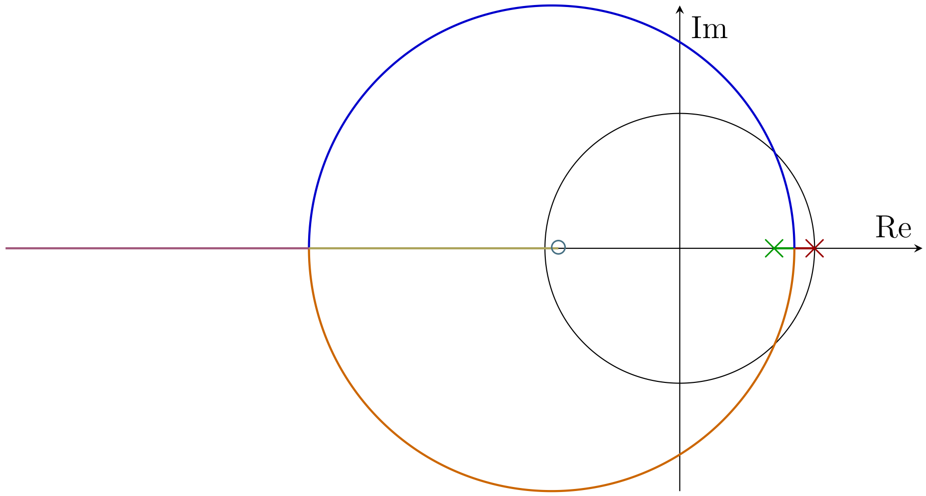
which are exactly the poles of the discrete-time pulse-transfer function.

Problem 3

(a) Using Mason's rule we have that the closed-loop pulse-transfer function is

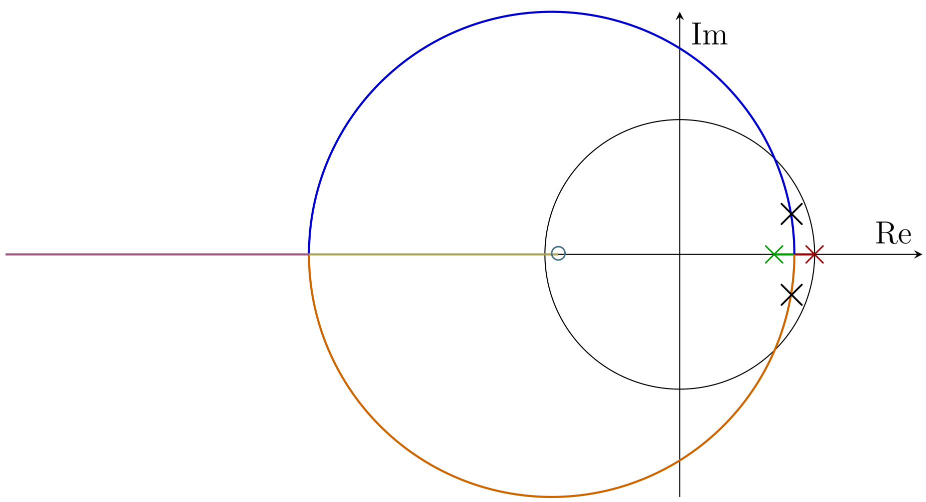
$$
H_{cv}(z) = \frac{\frac{B(z)}{A(z)}}{1 + Kb\frac{B(z)}{A(z)}} = \frac{B(z)}{A(z) + KbB(z)} = \frac{b_1 z + b_1}{(z-1)(z-p) + Kb(b_0 z + b_1)}.
$$

(b) There are $n = 2$ start points in $z = 1$ and $z = p = \mathrm{e}^{-\frac{h}{T}}$, and one end-point in the open-loop zero $z = -1 + 0.5\frac{h}{T}$. There is one asymptote with direction π. The part of the real axis between the two startpoints belongs to the root locus (the "odd rule") and also everything to the left of the end point. This means that the two branches must meet between the two start points, then branch out into the imaginary plane, loop to the left and meet again on the negative real axis outside the unit circle. After this, one branch goes toward infinity and one goes to the end point. It should look something like below



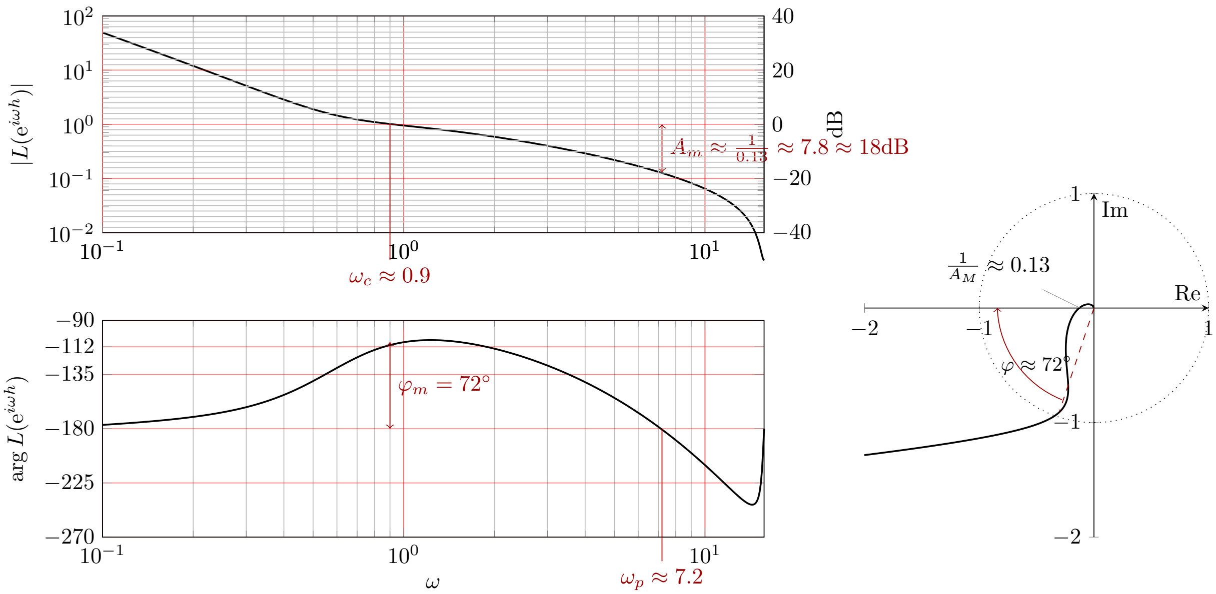
(c) No, clearly there is a limited range of gains K for which the system is stable. This is in contrast to the continuous-time case, where the system is stable for any gain K.

(d) With similar motivation as for the continous-time case, we want as fast system as possible with not too much oscillations. So complex-conjugated poles as far from $z = 1$ as possible, but with some distance to the unit circle is reasonable.



Problem 4

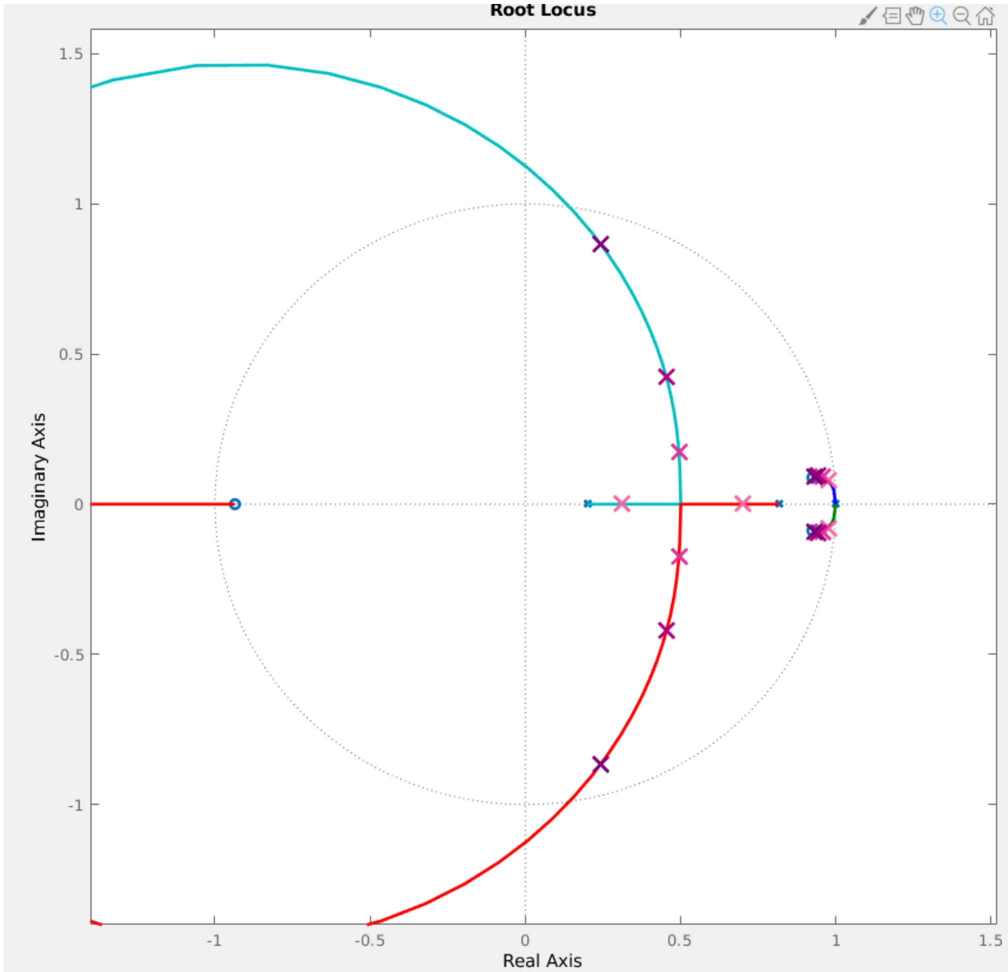
(a)



(b) From the root locus we see that we have four starting points. Two start points are in $z = 1$ (one from the pole of the DC-motor and one from the integrator of the PID controller), and two start points are well inside the unit circle. For small gains K, the behaviour will be dominated by the two slow poles close to $z = 1$. These two poles will never become unstable, but approach the two end points. The two poles starting inside the unit circle will meet and the for the gain $K = 7.8$ they will break out of the unit circle and the closed-loop system will be unstable. This means that for small gains the closed-loop system will be slow, and somewhat oscillatory (slow oscillations), with increasing gains, the response will be better damped initially, until the two poles (on the red and cyan branches) gets closer to the unit circle, which will give a response with fast and poorly damped oscillations. With this motivation, the answer becomes

K	Step plot			
0.5	I	II	III	IV
1.0	I	II	III	IV
2.0	I	II	III	IV
6.0	I	II	III	IV

In fact, the actual position of the poles for the four different values of the gain K is shown in the root locus below.



(c) Looking at the Bode diagram and Nyquist plot again, we see that the gain $K = 7.8$ is the same as the gain margin. So the Nyquist curve goes through the point -1 for $\omega = \omega_p \approx 7.2$ rad/s, and the closed-loop system will have undamped oscillations with this frequency. In discrete time, this corresponds to a closed-loop system with a pair of complex-conjugated poles on the unit circle at $z = \mathrm{e}^{\pm i\omega_p h} = \mathrm{e}^{\pm i 7.2 \cdot 0.2} = \mathrm{e}^{\pm i 1.44}$ which has argument 1.44 rad $= 82.5°$. This is exactly where the rootlocus intersects the unit circle. The period of the oscillations is $T = \frac{2\pi}{7.2} \approx 0.87$ s.