Root locus

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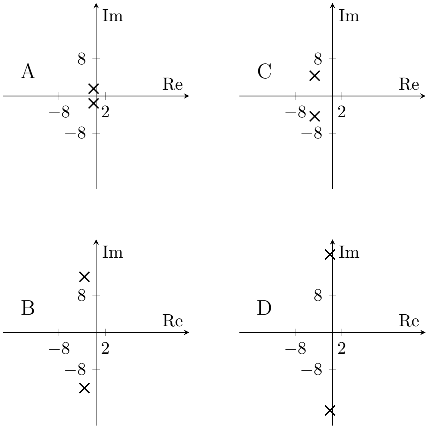
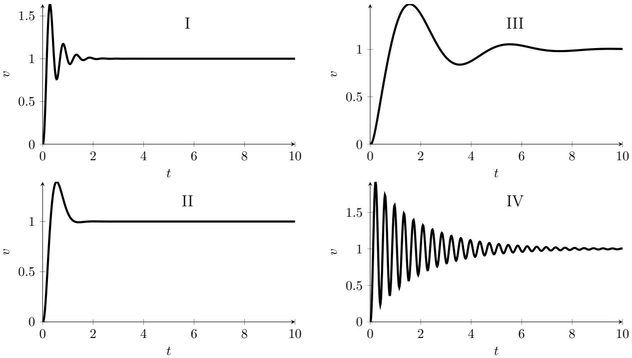
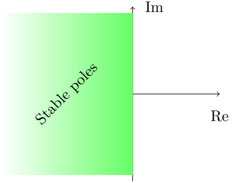
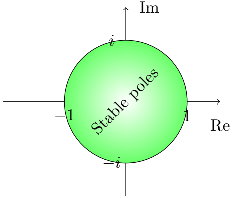
Pole-placement and time-response

Pair the pole-placement with the correct time-response (continuous time)!





Mapping of poles from continuous time to discrete time

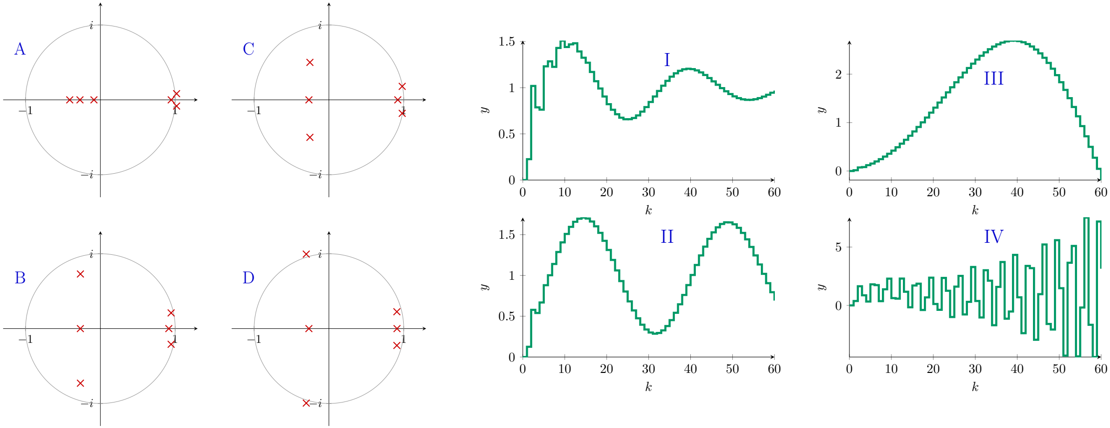
Continuous time	Discrete time
$Y(s) \triangleq \mathcal{L}\{y(t)\}$	$Y(z) \triangleq \mathcal{Z}\{y(kh)\}$
$Y(s) = G(s)U(s) = \frac{b}{s+a}U(s)$	$Y(z) = H(z)U(z) = \frac{\beta}{z+\alpha}U(z)$
Pole of the system: $s + a = 0 \;\Rightarrow\; s = -a$	Pole of the system: $z + \alpha = 0 \;\Rightarrow\; z = -\alpha$
	

The s-domain of continuous-time systems is related to the z-domain of discrete-time systems through

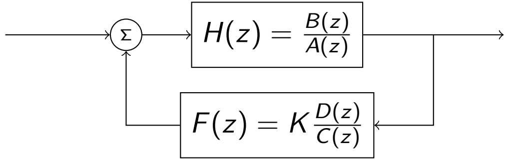
$$z = e^{sh}$$

Pole-placement and time-response

Pair the pole-placement with the correct time-response (discrete time)!



Root locus: A brief review



- The loop pulse-transfer function (loop gain) becomes

$$L(z) = H(z)F(z) = K\frac{\overbrace{B(z)D(z)}^{Q(z)}}{\underbrace{A(z)C(z)}_{P(z)}} = K\frac{Q(z)}{P(z)}.$$

- The roots of $Q(z)$ are called the open loop zeros.
- The roots of $P(z)$ are called the open loop poles.
- The characteristic equation for the closed-loop system is

$$1 + K\frac{Q(z)}{P(z)} = 0 \quad \Leftrightarrow \quad P(z) + KQ(z) = 0$$

Root locus: Definition

Let

$$\begin{cases} P(z) & = z^n + a_1 z^{n-1} + \cdots + a_n = (z - p_1)(z - p_2)\cdots(z - p_n) \\ Q(z) & = z^m + b_1 z^{m-1} + \cdots + b_m = (s - q_1)(z - q_2)\cdots(z - q_m) \end{cases}, \quad n \geq m$$

The root locus shows how the solution to the characteristic equation

$$P(z) + K \cdot Q(z) = 0, \quad 0 \leq K < \infty \tag{1}$$

depend on the parameter K. The root locus consists of the set of all points in the complex plane that are solutions to (1) for some non-negative value of K.

Root locus: Characteristics

Start points	The n roots of $P(z)$, marked by crosses
End points	The m roots of $Q(z)$, marked by circles
Asymptotes	Number equal to the *pole excess* $n - m$
Real axis	Some segments of the real axis belong to the root locus

Root locus: Direction of the asymptotes

The characteristic equation $P(z) + KQ(z) = 0$ can be written $\frac{P(z)}{Q(z)} = -K$ and for large z it can be approximated as

$$\frac{z^n}{z^m} = -K \quad \Leftrightarrow \quad z^{n-m} = -K.$$

Taking the argument of both sides of the equation gives
$(n-m)\arg z = \pi + k2\pi,\ \ k \in \mathbb{Z}$ So, the directions of the asymptotes are given by the expression

$$\theta_k = \arg z = \frac{(2k+1)\pi}{n-m},\ \ k \in \mathbb{Z}$$

Root locus: The asymptotes' intersection with the real axis

$$z_{ip} = \frac{\sum_{i=0}^{n} p_i - \sum_{i=0}^{m} q_i}{n - m},$$

where $\{p_i\}$ are the starting points (open-loop poles) and $\{q_i\}$ are the end points (open-loop zeros).

Root locus exerise: Pair the pulse-trf fcn and root locus

$$G_1(z) = K\frac{(z+2.9)(z+0.2)}{(z-1)^2(z-0.3)}$$

$$G_2(z) = K\frac{(z-0.5)(z+0.4)}{(z-1)(z-0.3)(z-0.1)}$$

$$G_3(z) = K\frac{(z-0.5)(z+0.8)}{(z-1)^2(z-0.3)}$$

$$G_4(z) = K\frac{z-0.6}{(z-1)(z-0.3)}$$