Step-invariant sampling - example

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The controller sees the world as being discrete

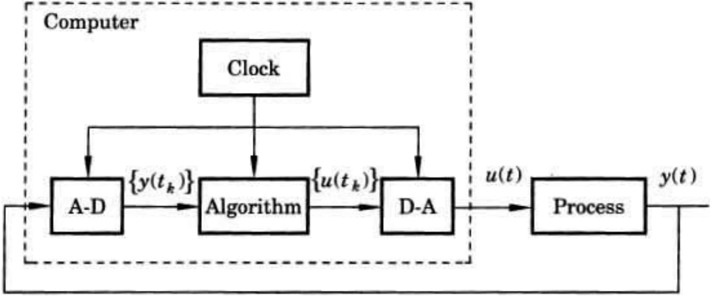


Figure 1.1 Schematic diagram of a computer-controlled system.

Åström & Wittenmark *Computer-controlled systems*

Sampling a continuous-time system

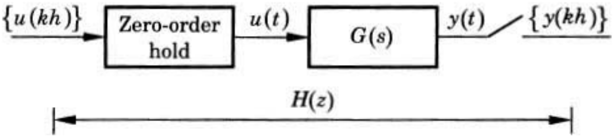
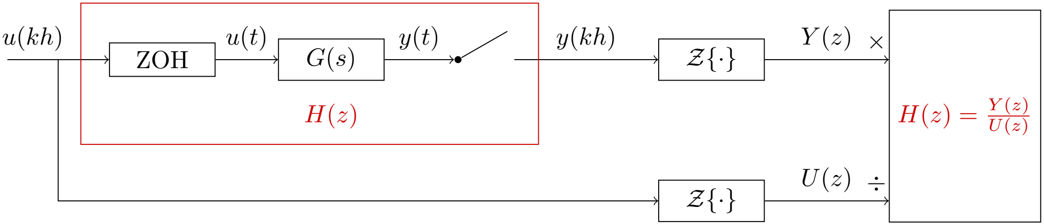


Figure 2.4 Sampling a continuous-time system.

Åström & Wittenmark *Computer-controlled systems*

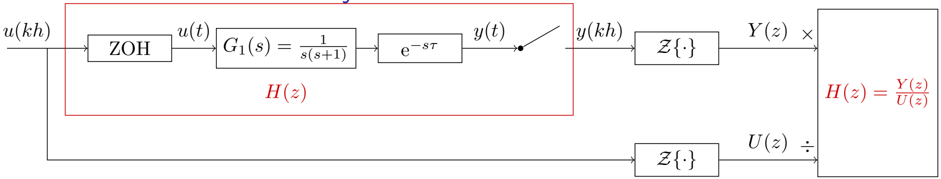
Step-invariant sampling



$$u(kh) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$H(z) = \frac{z-1}{z} \mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{G(s)}{s} \right\} \right\}$$

Example: DC motor with delay



$$G(s) = G_1(s)e^{-s\tau} = \frac{e^{-s\tau}}{s(s+1)}$$

1. Step-response without delay

$$\frac{G(s)}{s} = \frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

$$y_1(t) = \mathcal{L}^{-1}\{-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\} = u_H(t)\left(t - 1 + e^{-t}\right).$$

2. Step-reponse with delay

$$y(t) = y_1(t-\tau) = -u_H(t-\tau) + u_H(t-\tau)(t-\tau) + u_H(t-\tau)e^{-(t-\tau)}$$

Example: DC motor with delay

Assuming $\tau = nh$

$$\mathcal{Z}\left\{f(kh - nh)\right\} = z^{-n}\mathcal{Z}\left\{f(kh)\right\}.$$

3. Z-transform of the sampled response w/o delay Usando las transformadas

$$\begin{aligned} u_H(kh) \quad &\overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z-1} \\ u_H(kh)kh \quad &\overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{zh}{(z-1)^2} \\ u_H(kh)\mathrm{e}^{-a(kh)} \quad &\overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z-\mathrm{e}^{-ah}} \end{aligned}$$

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u_H(kh)\mathrm{e}^{-a(kh)} &\quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z-\mathrm{e}^{-ah}}
\end{aligned}$$

$$Y_1(z) = -\frac{z}{z-1} + \frac{zh}{(z-1)^2} + \frac{z}{z-\mathrm{e}^{-h}}$$

Example: DC motor with delay

4. Z-transform of the delayed response

$$Y(z) = z^{-n}\left(-\frac{z}{z-1} + \frac{zh}{(z-1)^2} + \frac{z}{z-\mathrm{e}^{-h}}\right)$$

5. Dividing with the z-transform of the step

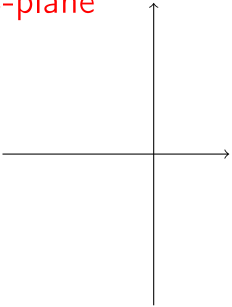
$$\begin{aligned} H(z) &= \frac{Y(z)}{U(z)} = \frac{z-1}{z} z^{-n}\left(-\frac{z}{z-1} + \frac{zh}{(z-1)^2} + \frac{z}{z-\mathrm{e}^{-h}}\right) \\ &= z^{-n}\left(-1 + \frac{h}{z-1} + \frac{z-1}{z-\mathrm{e}^{-h}}\right) \\ &= \frac{z\big(h-1+\mathrm{e}^{-h}\big) - \big(\mathrm{e}^{-h}(1+h)-1\big)}{z^n(z-1)(z-\mathrm{e}^{-h})} \end{aligned}$$

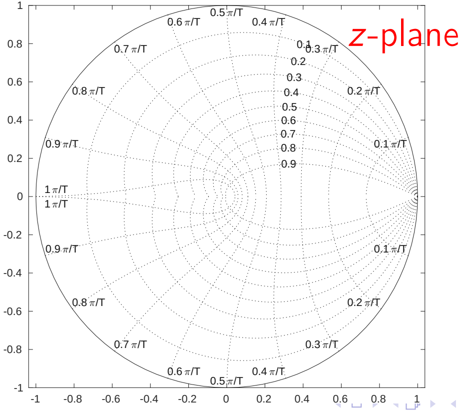
Example: DC motor with delay

$$G(s) = \frac{\mathrm{e}^{-s(nh)}}{s(s+1)} \quad \longrightarrow \quad H(z) = \frac{z\left(h - 1 + \mathrm{e}^{-h}\right) - \left(\mathrm{e}^{-h}(1+h) - 1\right)}{z^n(z-1)(z-\mathrm{e}^{-h})}$$

Activity Determine the zero and the poles for the case $n = 1$ y $h = 0.2$, and mark them in the corresponding diagrams (mark zero with $\bigcirc$ and poles with $\times$).

s-plane





Mapping from the s-plane to the z-plane

$$z = \mathrm{e}^{sh} \qquad \Leftrightarrow \qquad s = \frac{1}{h} \ln z$$

Important example The left half-plane of the s-plane :
$s = a + i\omega,\ a < 0,\ -\infty < \omega < \infty$

$$z = \mathrm{e}^{sh} = \mathrm{e}^{(a+i\omega)h} = \mathrm{e}^{ah}\mathrm{e}^{i\omega h}, \quad |z| = |\mathrm{e}^{ah}|\,|\mathrm{e}^{i\omega h}| = |\mathrm{e}^{ah}| < 1,\ a < 0$$