Computerized control partial exam 1 (15%)

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Time February 10 17:30

Place 5105

Permitted aids The single colored page with your own notes, table of Laplace transforms, calculator

All answers should be readable and well motivated (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam. Use the last page if more space is needed.

Good luck!

Matricula and name

The system

Consider the water tank illustrated in figure 1. After linearization and with certain values for the parameters A, a, and h_0, the system leads to the model shown in figure 2. Note that the variables are defined as deviations from some nominal values, i.e. deviations from an operating point. There is a valve controlling the flow of water into the tank, and $u(t)$ is the control signal to the valve. When the pressure in the water pipe is at the normal level, the valve reacts to the control signal with the (change in) flow $\bar{w}(t)$. Varying water pressure leads to variations in the actual flow, and this is modelled as an additive disturbance $v(t)$, so the actual change in flow is $w(t) = \bar{w}(t) + v(t)$.
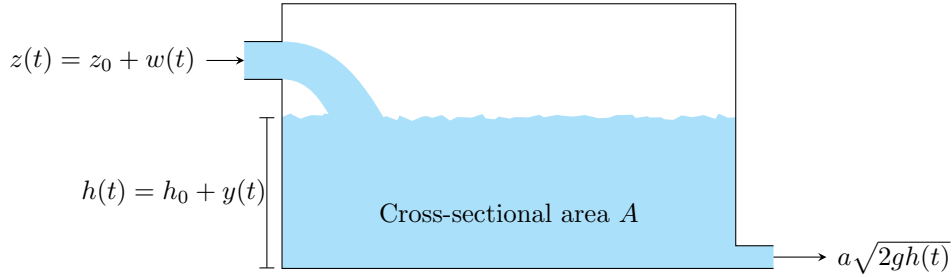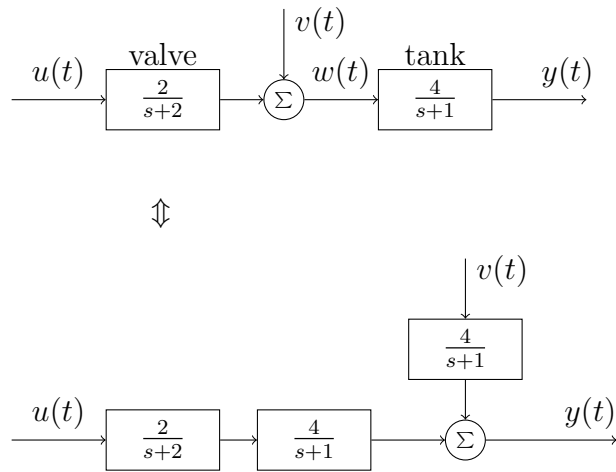
Figure 1: Tank with hole in the bottom.

Figure 2: Tank model with valve.

Problem 1 (50p)

Discretizing the system using step-invariant (zero-order-hold) sampling with sampling interval h gives the discrete-time system shown in figure 3. **Show that the two pulse-transfer functions $H_1(z)$ and $H_2(z)$ are given by**

$$H_1(z) = \frac{4(z - e^{-h})(z - e^{-2h}) - 8(z - 1)(z - e^{-2h}) + 4(z - 1)(z - e^{-h})}{(z - e^{-h})(z - e^{-2h})}$$

$$= \frac{\left(4 - 8e^{-h} + 4e^{-2h}\right)z + 4e^{-h}(e^{-2h} + 1) - 8e^{-2h}}{(z - e^{-h})(z - e^{-2h})}$$
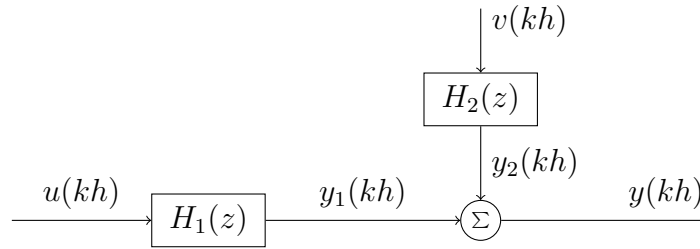
$$H_2(z) = \frac{4(1 - e^{-h})}{z - e^{-h}}.$$

Figure 3: Discrete-time model of tank with valve.

Derivation:

Problem 2 (20p)

Assume that the sampling period is $h = 0.2$. In figure 4 draw the poles (crosses) and zero (circle) for both the continuous-time plant model $G_1(s)$ and the discretized pulse-transfer function $H_1(z)$ you determined in Problem 1.
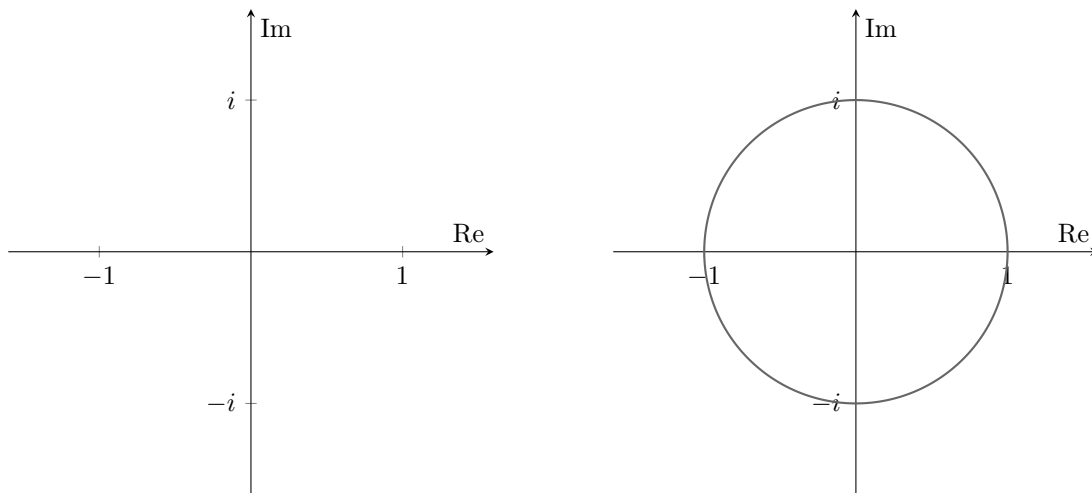

Figure 4: Problem 2: Plot the poles of the continuous-time system (on the left) and the poles and zero of the discrete-time system (on the right). Indicate (with arrows and/or colors) corresponding pairs of continuous-time and discrete-time poles.
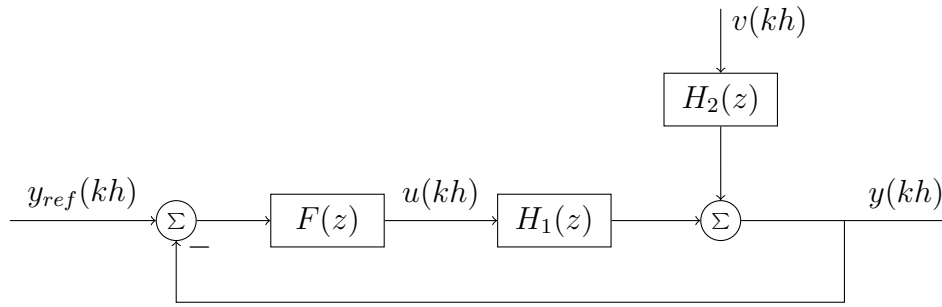
Calculations:

4

Figure 5: Feedback control from the error signal.

Problem 3 (30p)

The tank model is controlled using the feedback controller $F(z) = K\frac{1.1z-0.9}{z-1}$. See figure 5.

Figure 6 shows the root locus for the closed-loop poles with respect to the gain K. In figure 7, four different closed-loop responses to a step disturbance in the water pressure are shown. The different responses are for four different values of K. Identify (and circle) the corresponding step plot for each value of K in the table below. **Motivate your choice!**

K	Step plot			
0.1	A	B	C	D
0.8	A	B	C	D
2.0	A	B	C	D
4.0	A	B	C	D
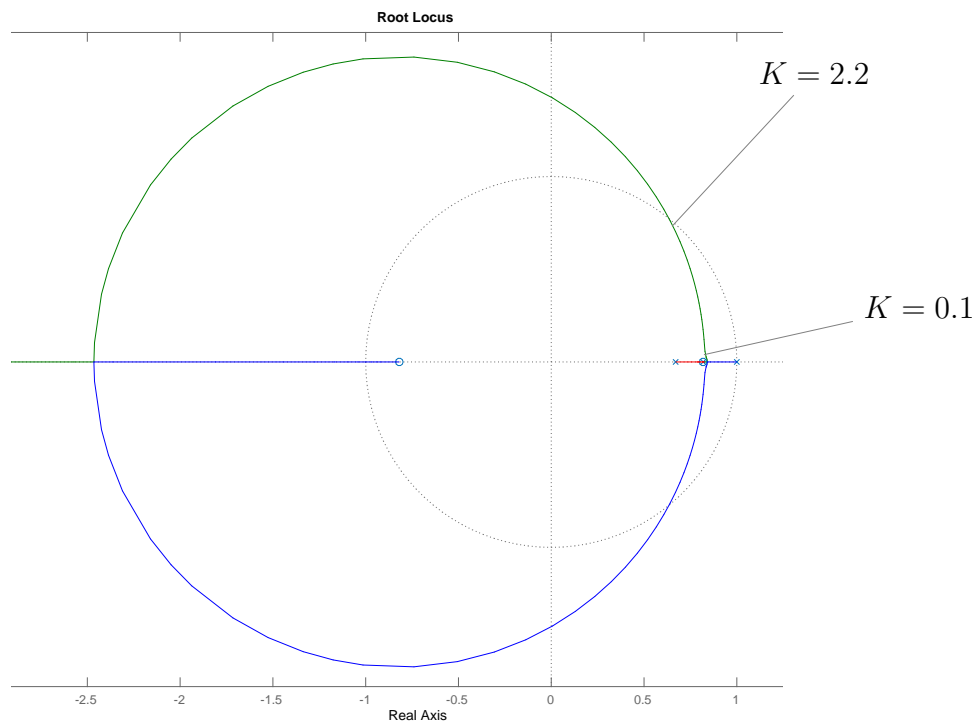
Figure 6: Root locus wrt the gain K.

A

Step Response

B

Step Response

C

Step Response

D

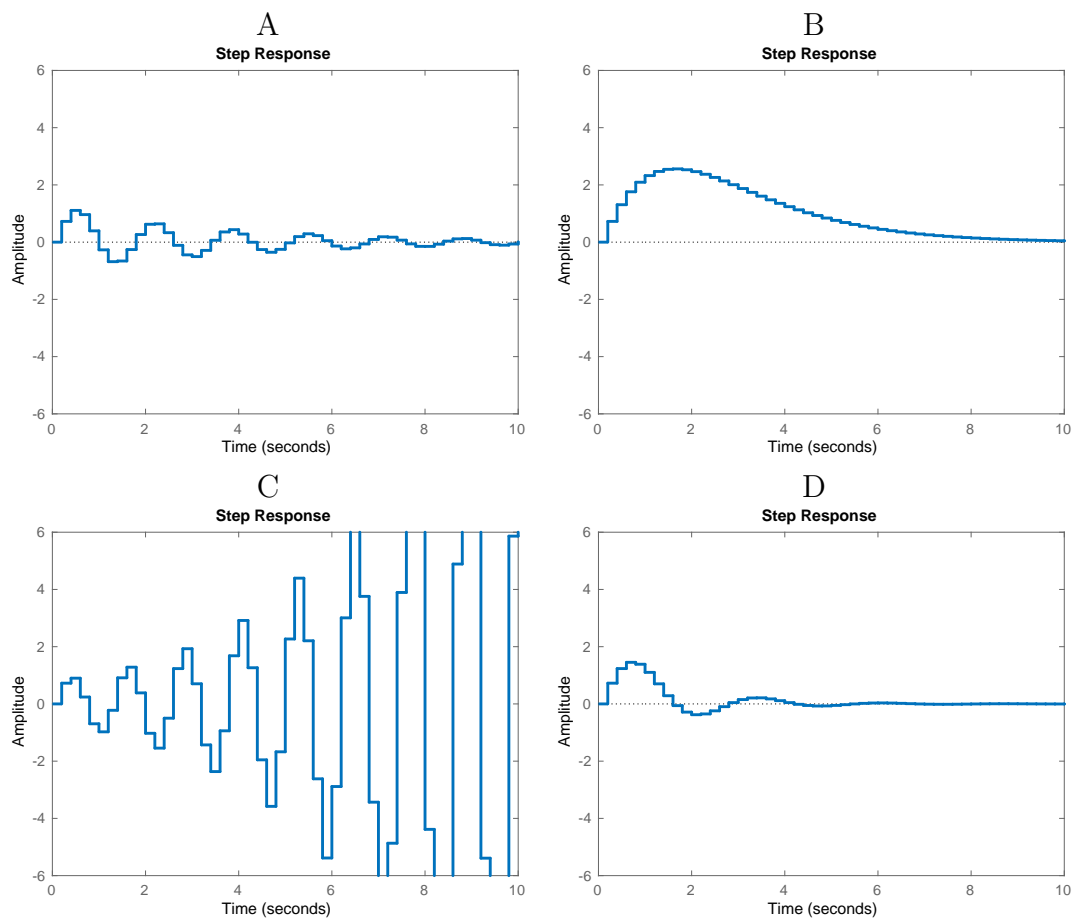**Step Response**

Figure 7: Step responses for different values of K.

Motivation:

If necessary, you can continue your solutions on this page. Mark clearly which problem the solution corresponds to.

Solutions

Problem 1

There are two continous-time systems to discretize $G_1(s) = \frac{8}{(s+1)(s+2)}$ and $G_2(s) = \frac{4}{s+1}$. First calculate the step-response of the system $G_1(s)$

$$Y_1(s) = G_1(s)\frac{1}{s} = \frac{8}{s(s+1)(s-1)} = \frac{4}{s} - \frac{8}{s+1} + \frac{4}{s+2}.$$

The inverse Laplace-transform gives

$$y_1(t) = \left(4 - 8e^{-t} + 4e^{-2t}\right)u_H(t),$$

where $u_H(t)$ is the Heaviside step function. Sampling $y(t)$ gives

$$y_1(kh) = \left(4 - 8e^{-kh} + 4e^{-2kh}\right)u_H(kh),$$

which has the Z-transform

$$Y_1(z) = \frac{4z}{z-1} - \frac{8z}{z-e^{-h}} + \frac{4z}{z-e^{-2h}}.$$

Dividing the z-transform of the system response to that of the input (the step) gives

$$\begin{aligned}
H_1(z) &= \frac{Y_1(z)}{U(z)} = \frac{z-1}{z}Y_1(z) = 4 - \frac{8(z-1)}{z-e^{-h}} + \frac{4(z-1)}{z-e^{-2h}} \\
&= \frac{4(z-e^{-h})(z-e^{-2h}) - 8(z-1)(z-e^{-2h}) + 4(z-1)(z-e^{-h})}{(z-e^{-h})(z-e^{-2h})} \\
&= \frac{4z^2 - 4(e^{-h}+e^{-2h})z + 4e^{-h}e^{-2h} - 8z^2 + 8(1+e^{-2h})z - 8e^{-2h} + 4z^2 - 4(1+e^{-h})z + 4e^{-h}}{(z-e^{-h})(z-e^{-2h})} \\
&= \frac{\left(-4e^{-h} - 4e^{-2h} + 8 + 8e^{-2h} - 4 - 4e^{-h}\right)z + 4e^{-h}e^{-2h} - 8e^{-2h} + 4e^{-h}}{(z-e^{-h})(z-e^{-2h})} \\
&= \frac{\left(4 - 8e^{-h} + 4e^{-2h}\right)z + 4e^{-h}(e^{-2h}+1) - 8e^{-2h}}{(z-e^{-h})(z-e^{-2h})}
\end{aligned}$$

The same procedure for the system $G_2(s)$ gives

$$Y_2(s) = G_2(s)\frac{1}{s} = \frac{4}{s(s+1)} = \frac{4}{s} - \frac{4}{s+1}.$$

Inverse Laplace gives

$$y_2(t) = (4 - 4e^{-t})u_H(t),$$

which sampled is

$$y_2(kh) = (4 - 4e^{-kh})u_H(kh).$$

The z-transform gives

$$Y_2(z) = \frac{4z}{z-1} - \frac{4z}{z-e^{-h}}.$$
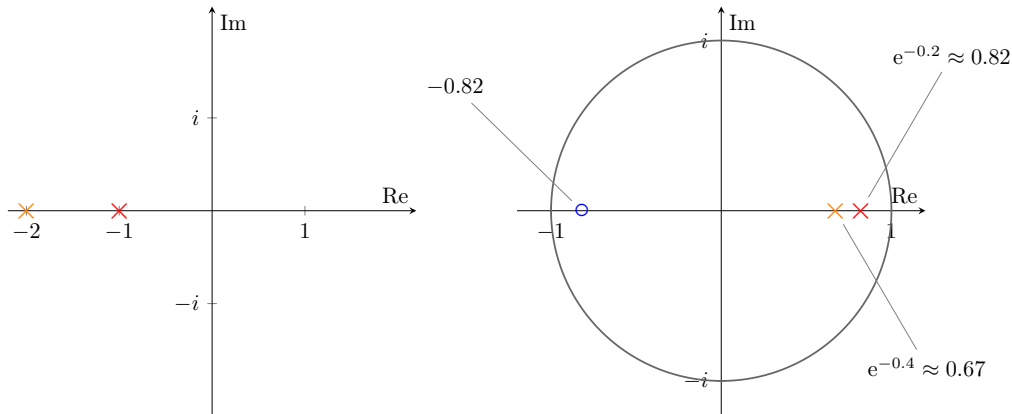
Dividing with the z-transform of the step input gives

$$\begin{aligned}
H_2(z) &= \frac{Y_2(z)}{U(z)} = \frac{z-1}{z}Y_2(z) = 4 - \frac{4(z-1)}{z-e^{-h}} \\
&= \frac{4(z-e^{-h}) - 4(z-1)}{z-e^{-h}} = \frac{4(1-e^{-h})}{z-e^{-h}}.
\end{aligned}$$

Problem 2

The discrete-time poles are in $z = \mathrm{e}^{-h} = \mathrm{e}^{-0.2}$ and $z = \mathrm{e}^{-2h} = \mathrm{e}^{-0.4}$. The zero is in

$$z = -\frac{4\mathrm{e}^{-h}(\mathrm{e}^{-2h}+1) - 8\mathrm{e}^{-2h}}{4 - 8\mathrm{e}^{-h} + 4\mathrm{e}^{-2h}} \approx -0.82$$

Problem 3

From the root locus we see that if the gain is $K > 2.2$, then the closed-loop system will have two complex-conjugated poles outside the unit circle, and the system will be unstable. So the gain $K = 4$ should give an unstable response. This is seen in the response C. As the gain increases from 0.1 we should see responses that are increasingly faster and with more pronounced and faster oscillations. Hence response B corresponds to the most damped and slowest response, response D is faster and with some damping, and finally response A has little damping and quite fast oscillations.

K	Step plot			
0.1	A	(B)	C	D
0.8	A	B	C	(D)
2.0	(A)	B	C	D
4.0	A	B	(C)	D

2