Computerized control - partial exam 3 (dummy) - modified from Fall 2015

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In the last lecture we looked at a state-space model of the harmonic oscillator with state variables corresponding to the position and velocity of the oscillating mass. If the frequency of the oscillations is $\omega = 1$ and we sample the system with zero-order-hold with a sampling period h such that $\omega h = 0.4$ we obtain the sampled system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.92 & 0.39 \\ -0.39 & 0.92 \end{bmatrix} x(k) + \begin{bmatrix} 0.079 \\ 0.39 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned} \tag{1}$$

Problem 1

Verify that the pulse-transfer function is

$$H(z) = \frac{0.079(z+1)}{z^2 - 1.84z + 1.0} = \frac{0.079(z+1)}{(z-0.92)^2 + 0.15}. \tag{2}$$

Plot the zeros and poles of the discrete-time system **and** of the continuous-time system in the complex plane in figure 1.

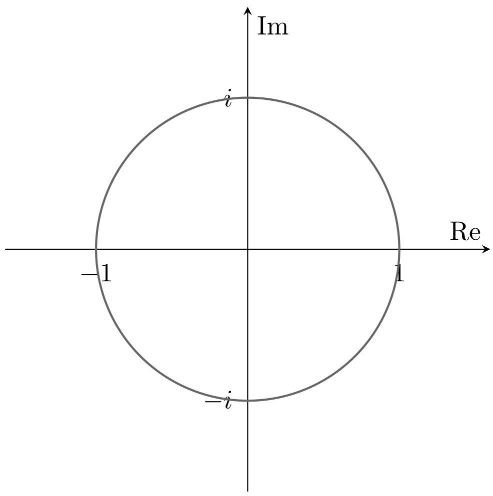


Figure 1: Problem 1: Plot the poles and zeros of the system (both discrete-time and continuous-time).

Problem 2

A PD-regulator for the continuous-time harmonic oscillator has been designed. It is sampled using Tustin's formula to give the discrete-time controller

$$F(z) = \frac{5z - 3.4}{z + 0.6}. \tag{3}$$

The system is controlled by error-feedback, according to figure 2. Calculate the pulse-transfer function for the closed-loop system from u_c to y.

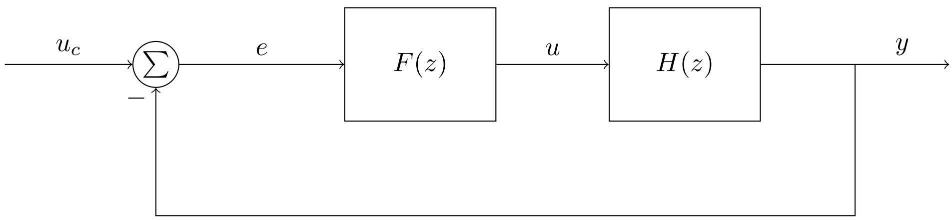


Figure 2: Problem 2: Error feedback with PD-control.

Problem 3

Assume that the requirements on the properties of the discrete-time closed-loop system is that it should have two complex-conjugated poles at

$$0.3 \pm 0.3i \tag{4}$$

In addition, there should be an observer pole in the origin (deadbeat observer). Hence, the desired closed-loop characteristic polynomial is

$$A_{cl} = A_c A_o = (z - 0.3 - 0.3i)(z - 0.3 + 0.3i)z = z^3 - 0.6z^2 + 0.18z, \tag{5}$$

with observer polynomial $A_o = z$. Design a RST-controller (see figure 3) that gives the desired closed-loop characteristic polynomial, and such that the pulse-transfer function from u_c to y has static gain equal to one ($G_c(1) = 1$). It is sufficient to set up the linear system of equations to solve for s_0, s_1 and

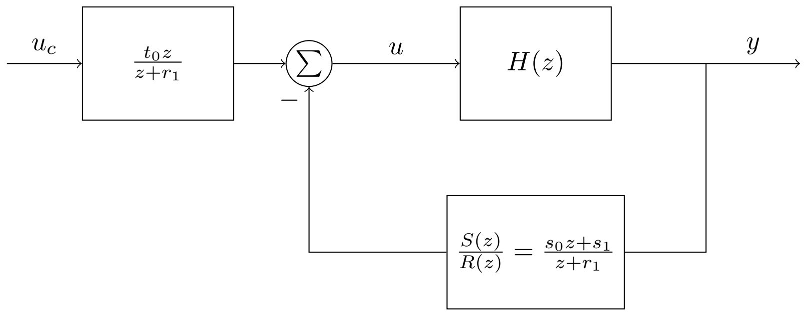


Figure 3: Problem 4: RST-controller.

r_1.

Problem 4

Assume that both the position and the velocity of the moving mass are measured. We thus have a measurement of the state $x(k)$ available, and we can use these two measurements to implement state feedback control according to the control law

$$u(k) = -l_1 x_1(k) - l_2 x_2(k) = -Lx(k). \tag{6}$$

Determine the state feedback gain L such that the closed-loop system has characteristic polynomial

$$z^2 - 0.6z + 0.18. \tag{7}$$

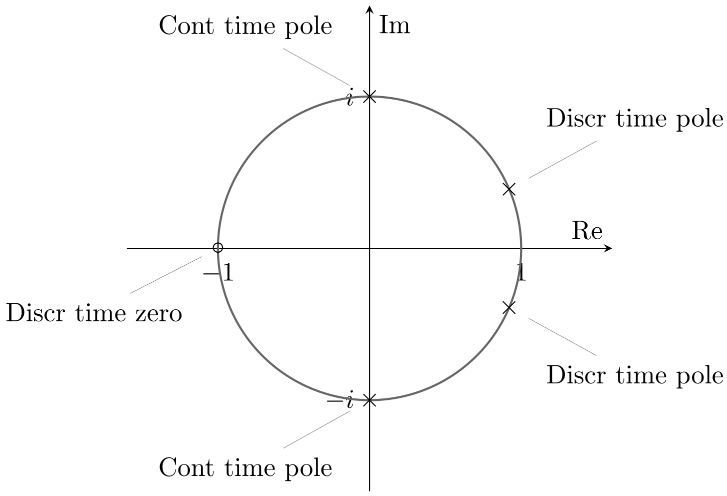
It is sufficient to set up the linear system of equations to solve for l_1 and l_2.

Solutions

Problem 1

The pulse-transfer function is given by

$$\begin{aligned}
H(z) &= C\,(zI-\Phi)^{-1}\,\Gamma \\
&= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} z-0.92 & -0.39 \\ 0.39 & z-0.92\end{bmatrix}^{-1}\begin{bmatrix}0.079 \\ 0.39\end{bmatrix} \\
&= \frac{1}{(z-0.92)^2+0.39^2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} z-0.92 & 0.39 \\ -0.39 & z-0.92\end{bmatrix}\begin{bmatrix}0.079 \\ 0.39\end{bmatrix} \\
&= \frac{0.079(z+1)}{(z-0.92)^2+0.15}.
\end{aligned}$$



Problem 2

Write the pulse-transfer function of the controller as

$$F(z) = \frac{B_f(z)}{A_f(z)},$$

and the plant as

$$H(z) = \frac{B(z)}{A(z)}.$$

The pulse transfer function of the closed-loop system is given by

$$\begin{aligned}
H_c(z) &= \frac{H(z)F(z)}{1+H(z)F(z)} = \frac{\frac{B(z)}{A(z)}\frac{B_f(z)}{A_f(z)}}{1+\frac{B(z)}{A(z)}\frac{B_f(z)}{A_f(z)}} \\
&= \frac{B(z)B_f(z)}{A(z)A_f(z)+B(z)B_f(z)} = \frac{0.079(z+1)(5z-3.4)}{(z^2-1.84z+1)(z+0.6)+0.079(z+1)(5z-3.4)} \\
&= \frac{0.395(z+1)(z-0.68)}{z^3-1.24z^2-0.104z+0.6+0.395(z^2+0.32z-0.68)} \\
&= \frac{0.395(z+1)(z-0.68)}{z^3-0.845z^2+0.0224z+0.3314}
\end{aligned}$$

Problem 3

With a first order controller

$$\frac{S(z)}{R(z)} = \frac{s_0 z + s_1}{z + r_z},$$

the diophantine equation $AR + BS = A_{cl}$ becomes

$$(z^2 - 1.84z + 1)(z + r_1) + 0.079(z + 1)(s_0 z + s_1) = z^3 - 0.6z^2 + 1.18z$$
$$z^3 + (-1.84 + r_1 + 0.079s_0)z^2 + (1 - 1.84r_1 + 0.079(s_0 + s_1))z + r_1 + 0.079s_1 = z^3 - 0.6z^2 + 1.18z$$

which leads to the following linear equation in the controller parameters s_0, s_1 and r_1:

$$\begin{bmatrix} 0.079 & 0 & 1 \\ 0.079 & 0.079 & -1.84 \\ 0 & 0.079 & 1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} -0.6 + 1.84 \\ 0.18 - 1 \\ 0 \end{bmatrix}$$

with solution

$$\begin{bmatrix} s_0 \\ s_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 8.90 \\ -6.80 \\ 0.54 \end{bmatrix}.$$

Problem 4

The desired closed-loop characteristic equation is

$$z^2 - 0.6z + 0.18.$$

With the state feedback

$$u(k) = -Lx(k) = -\begin{bmatrix} l_1 & l_2 \end{bmatrix} x(k)$$

the closed-loop characteristic equation becomes

$$\det\left(zI - \Phi + \Gamma L\right),$$

where

$$\Gamma L = \begin{bmatrix} 0.079 \\ 0.39 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} 0.079l_1 & 0.079l_2 \\ 0.39l_1 & 0.39l_2 \end{bmatrix}.$$

We get

$$\begin{aligned} \det\left(zI - \Phi + \Gamma L\right) &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0.92 & 0.39 \\ -0.39 & 0.92 \end{bmatrix} + \begin{bmatrix} 0.079l_1 & 0.079l_2 \\ 0.39l_1 & 0.39l_2 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} z - 0.92 + 0.079l_1 & -0.39 + 0.079l_2 \\ 0.39 + 0.39l_1 & z - 0.92 + 0.39l_2 \end{bmatrix} \\ &= (z - 0.92 + 0.079l_1)(z - 0.92 + 0.39l_2) - (-0.39 + 0.079l_2)(0.39 + 0.39l_1) \\ &= z^2 + (-1.84 + 0.079l_1 + 0.39l_2)z + 1 - 0.39l_2 + 0.079l_2. \end{aligned}$$

Comparing this to the desired characteristic polynomial gives the system of equations

$$\begin{bmatrix} 0.079 & 0.39 \\ 0.079 & -0.39 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -0.6 + 1.84 \\ 0.18 - 1 \end{bmatrix}$$

with solution

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 2.65 \\ 2.64 \end{bmatrix}.$$