Computerized control - partial exam 1 (dummy)

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Problem 1

Consider the continuous-time system with the following transfer function

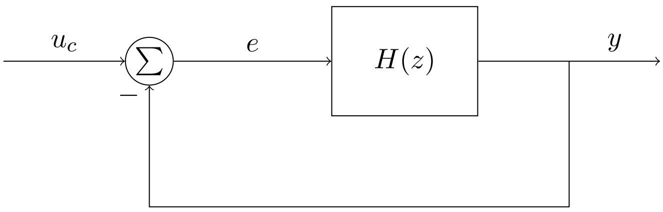
$$G(s) = \frac{s+1}{s(s+3)}.$$

The system is sampled with sampling interval h using zero-order hold. **Show that the pulse-transfer function for the sampled system is**

$$H(z) = \frac{(2z - 2 + 3h)\big(z - \mathrm{e}^{-3h}\big) - 2(z-1)^2}{9(z-1)\big(z - \mathrm{e}^{-3h}\big)}.$$

Problem 2

The sampled system in Problem 1 is controlled using proportional control with gain equal to 1.



1. Calculate the closed-loop pulse-transfer function
2. Let $h = \frac{\ln 2}{3} \approx 0.23$. Is the closed-loop system stable?

Solutions

Problem 1

First calculate the step-response of the continous-time system

$$G(s)\frac{1}{s} = \frac{s+1}{s^2(s+3)} = \frac{2}{9s} + \frac{1}{3s^2} - \frac{2}{9(s+3)}.$$

The inverse Laplace-transform gives

$$y(t) = \frac{2}{9} + \frac{1}{3}t - \frac{2}{9}\mathrm{e}^{-3t}.$$

Sampling this function gives

$$y(kh) = \frac{2}{9} + \frac{1}{3}kh - \frac{2}{9}\big(\mathrm{e}^{-3h}\big)^k,$$

which has the Z-transform

$$Y(z) = \frac{2z}{9(z-1)} + \frac{hz}{3(z-1)^2} - \frac{2z}{9\big(z-\mathrm{e}^{-3h}\big)}.$$

Dividing the z-transform of the system response to that of the input (the step) gives

$$\begin{aligned}
H(z) = \frac{Y(z)}{U(z)} &= \frac{z-1}{z}Y(z) = \frac{2}{9} + \frac{h}{3(z-1)} - \frac{2(z-1)}{9\big(z-\mathrm{e}^{-3h}\big)} \\
&= \frac{2(z-1)\big(z-\mathrm{e}^{-3h}\big) + 3h\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{9(z-1)\big(z-\mathrm{e}^{-3h}\big)} \\
&= \frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{9(z-1)\big(z-\mathrm{e}^{-3h}\big)}.
\end{aligned}$$

Problem 2

1. The closed loop system becomes

$$\begin{aligned}
H_c(z) &= \frac{H(z)}{1+H(z)} \\
&= \frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2 + 9(z-1)\big(z-\mathrm{e}^{-3h}\big)} \\
&= \frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{(2z-2+3h+9z-9)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2} \\
&= \frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{(11z+3h-11)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}
\end{aligned}$$

2. Stability of the closed-loop system. Substituting $h = \frac{\ln 2}{3}$ gives the characteristic equation

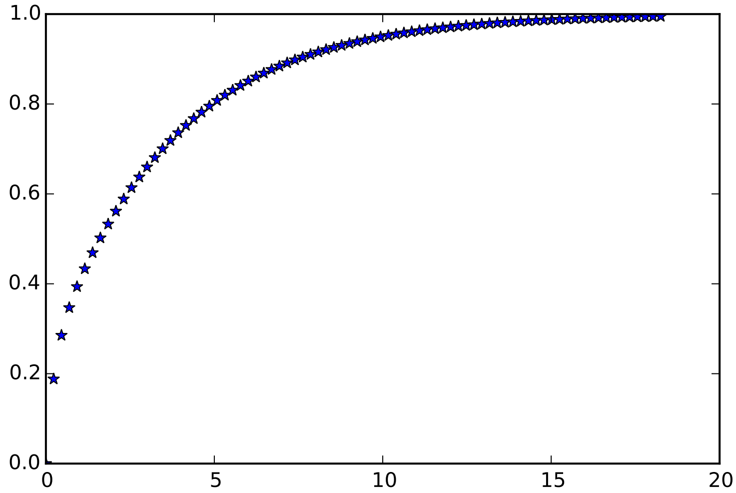
$$(11z+\ln 2-11)\big(z-\frac{1}{2}\big) - 2z^2 + 4z - 2 = 9z^2 - (12.5-\ln 2)z + 3.5 - 0.5\ln 2 = 0.$$

Apply Jury's criterion:

9	-11.81	3.15	
3.15	-11.81	9	$\alpha_2 = \frac{3.15}{9}$
$9 - \frac{3.15^2}{9} \approx 7.9$	$-11.81 + \frac{3.15}{9}11.81 \approx -7.68$		
-7.68	7.9		$\alpha_1 = -\frac{7.68}{7.9} \approx -0.97$
$7.9 - 0.96 \cdot 7.68 \approx 0.44$			

The first element in each odd row in the table are positive $\Rightarrow$ **The closed loop system is stable.**

Step response (not asked for):


1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20