# Sampling and aliasing

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# Computer-controlled system

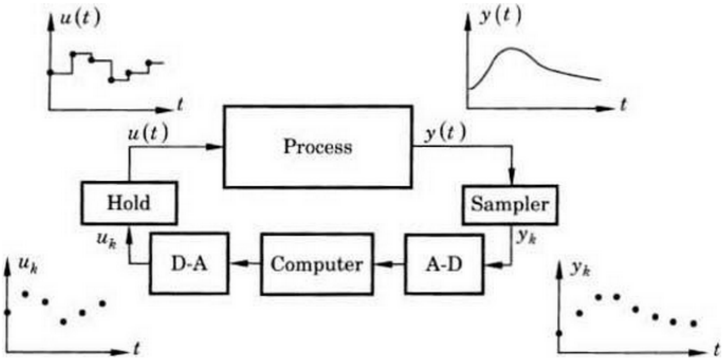
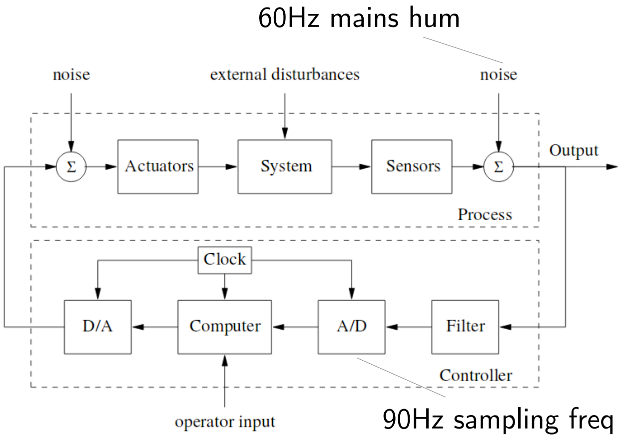
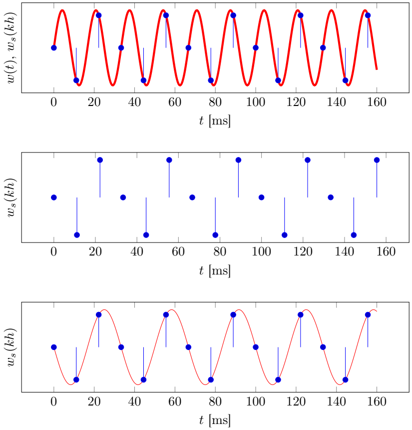



**Figure 7.2** Relationships among the measured signal, control signal, and their representations in the computer.

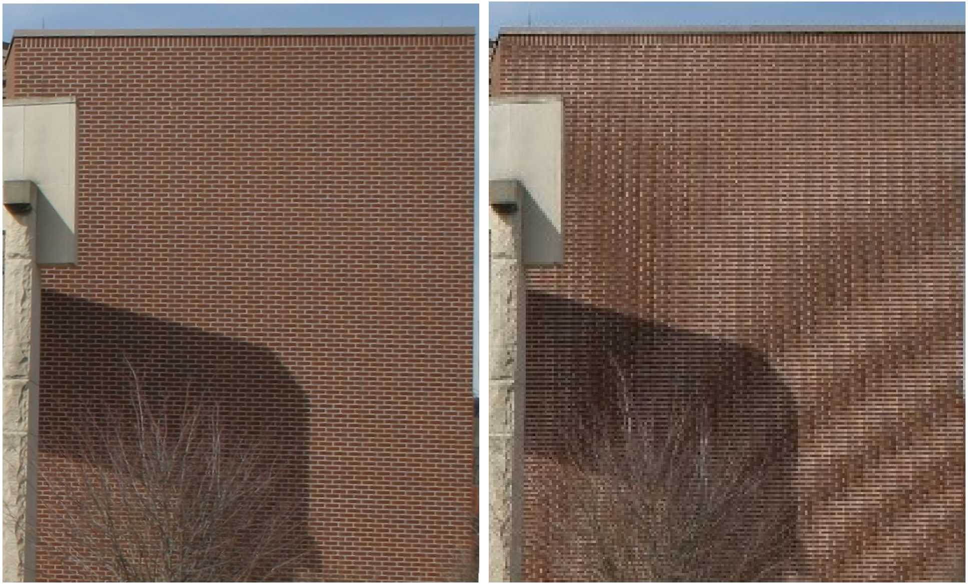
Source: Åström & Wittenmark

# Challenges with computerized control - aliasing



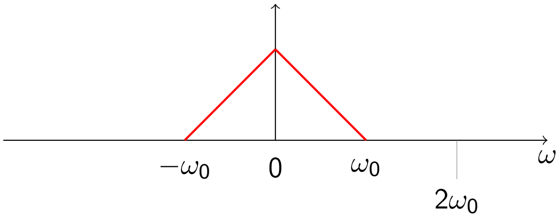
# Spatial aliasing



# The sampling theorem

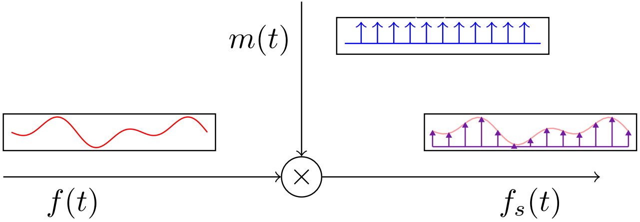
Shannon and Nyquist:

A continuous-time signal with Fourier transform that is zero outside the interval $(-\omega_0, \omega_0)$ can be completely reconstructed from equidistant samples of the signal, as long as the sampling frequency is at least $2\omega_0$.



# The impulse modulation model

The impulse train, a.k.a the Dirac comb:




$m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh)$

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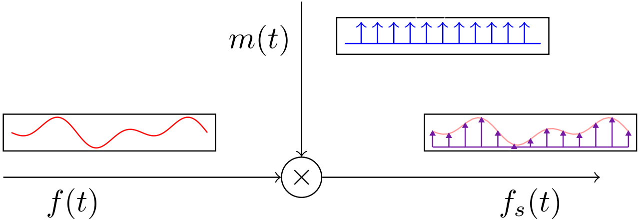


$$m(t) = \sum_{k=-\infty}^{\infty} \delta(t - kh)$$

$$f_s(t) = f(t)m(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kh) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad F_s(\omega) = F(\omega) * M(\omega)$$

# The impulse modulation model

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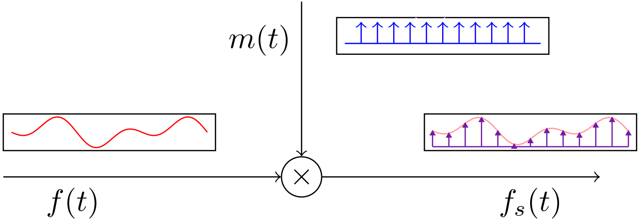


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## Resources

Dr. Trefor Bazett on the Dirac delta function, Prof Iain Collings explains convolution with the delta, Steven Fenton on the dirac comb and sampling

# Fourier transform of the sampled signal

The Fourier transform of $f_s$ and the Fourier transform of $f$ are related as

$$F_s(\omega) = \frac{1}{h} \sum_n F(\omega + n\omega_s).$$

Because the Fourier transform of the sampled signal equals the Fourier transform of the continuous-time signal repeated at every multiple of the sampling frequency and added, we get *frequency-folding* or *aliasing*.

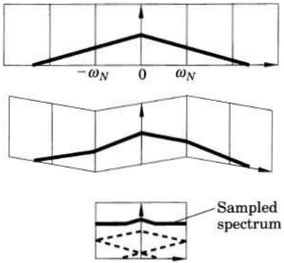
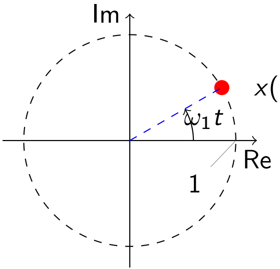



**Figure 7.11** Frequency folding.

# Fourier transform of a complex exponential

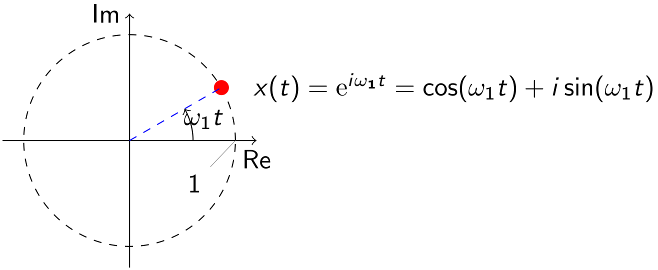
The function $x(t) = \mathrm{e}^{i\omega_{\mathbf{1}} t}$




$x(t) = \mathrm{e}^{i\omega_{\mathbf{1}} t} = \cos(\omega_1 t) + i\sin(\omega_1 t)$

# Fourier transform of a complex exponential

The function $x(t) = \mathrm{e}^{i\omega_1 t}$




has Fourier transform

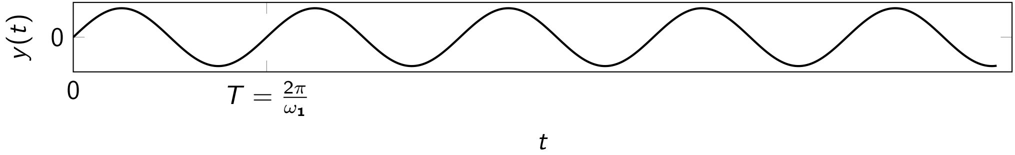
$$X(i\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-i\omega t}dt = \int_{-\infty}^{\infty} \mathrm{e}^{i(\omega_1 - \omega)t}dt = 2\pi\delta(\omega_1 - \omega)$$

## Resources

Prof Iain Collings on Complex numbers and real signals

# Fourier transform of a sinusoid

A sinusoidal signal $y(t) = \sin(\omega_1 t)$ has all its power concentrated at one single frequency, $\omega = \omega_1$ rad/s.




Since

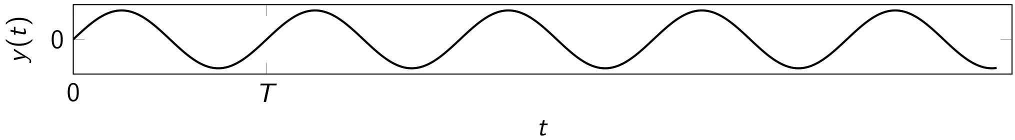
$$y(t) = \sin(\omega_1 t) = \frac{1}{2i}\left(e^{i\omega_1 t} - e^{-i\omega_1 t}\right)$$

the Fourier transform of a sinusoid becomes

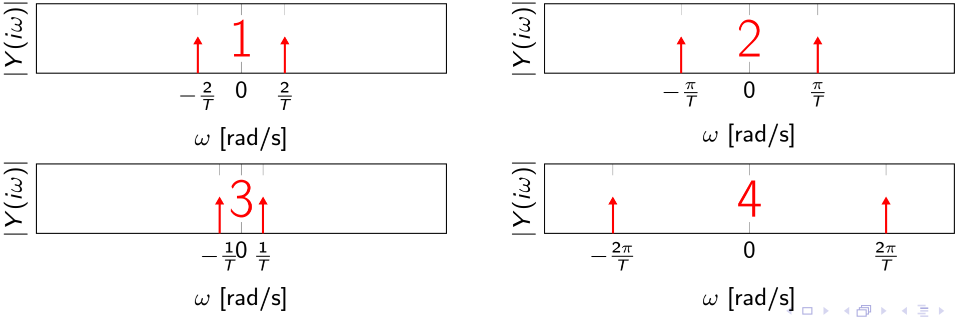
$$Y(i\omega) = \frac{1}{2i}\left(2\pi\delta(\omega_1 - \omega) - 2\pi\delta(\omega_1 + \omega)\right)$$

# Exercise 1: Fourier transform of a sinusoid

Consider the signal below

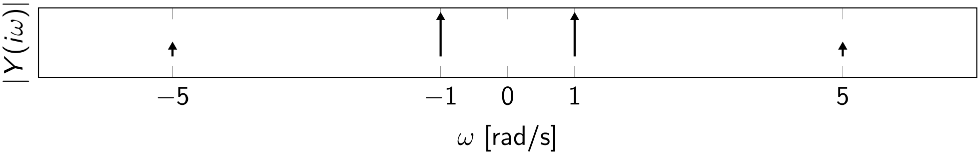



Which of the below is the correct Fourier transform (magnitude plot shown)?

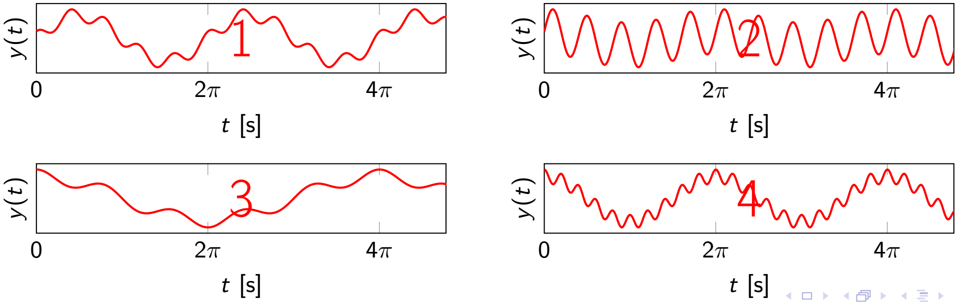


# Exercise 2: Two sinusoids

Consider a signal with Fourier transform

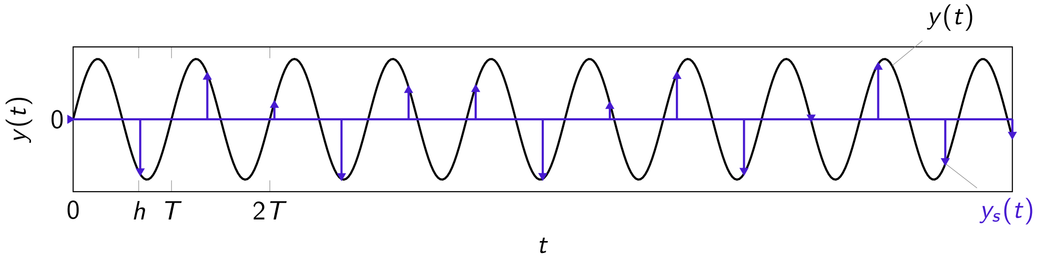



Which of the below time series could this Fourier transform correspond to?



# Exercise 3: Fourier transform of a sampled sinusoid

Consider the continuous and sampled signals with sampling period $h = \frac{2}{3}T$




What is the frequency of the sinusoid? What is the sampling frequency $\omega_s$ and the Nyquist frequency $\omega_N$ in terms of the period $T$ of the sinusoid?

# Exercise 3: Fourier transform of a sampled sinusoid

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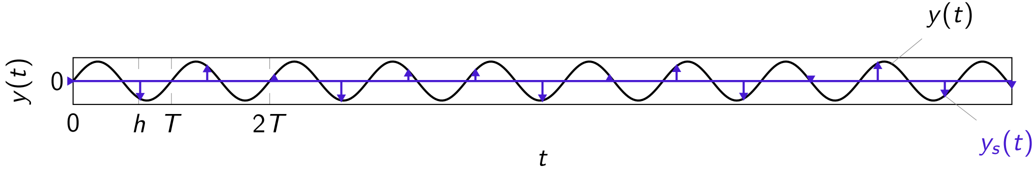
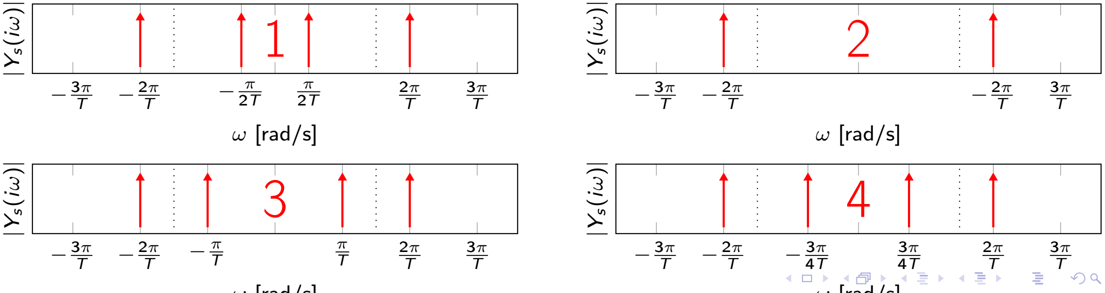



Which of the below corresponds to the Fourier transform of the sampled signal?



# Alias frequency

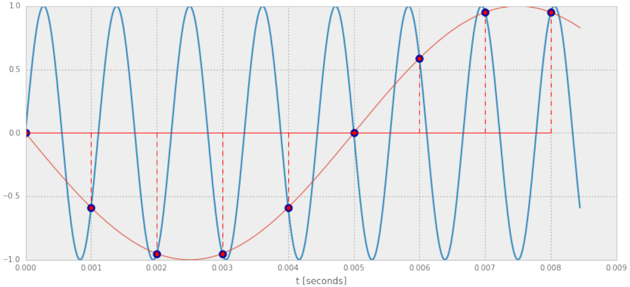
To find the low frequency alias $\omega_a < \omega_N$ of a high freqency sinusoid $\omega_1$, The expression

$$\omega_a = \left|\left((\omega_1 + \omega_N) \bmod \omega_s\right) - \omega_N\right|$$

can be used.

# Aliasing example

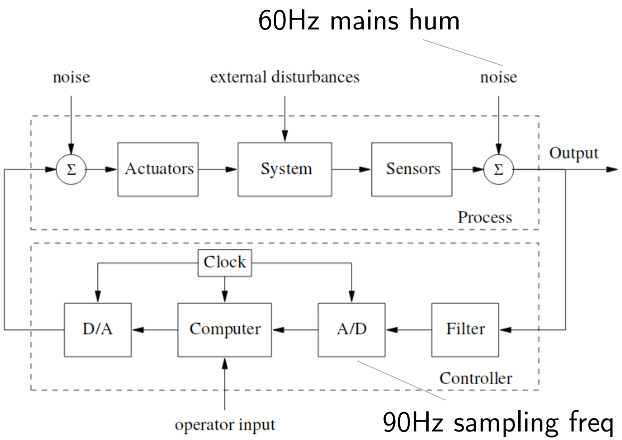
If a continuous-time signal with frequency content (bandwidth) $\omega_B$ is sampled at too low sampling rate ( $\omega_s < 2\omega_B$ ), then the energy at higher frequencies is folded onto lower frequencies.

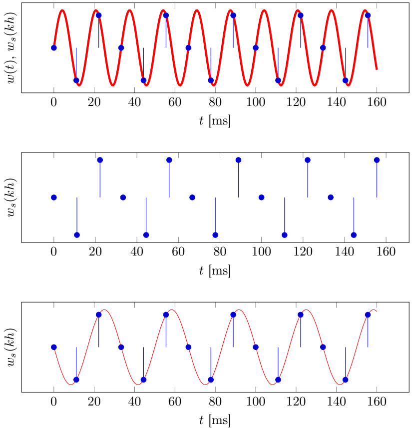



A high-frequency sinusoid ( $\omega_1 = 1800\pi$ rad/s ) masquerading as a lower frequency sinusoid ( $200\pi$ rad/s ) due to aliasing when sampled with $h = 10^{-3}$ s.

Draw the spectrum (lines) of the two sinusoids. Mark the Nyquist frequency and verify that the alias frequency is obtained by folding about the Nyquist frequency.

# Noisy measurements



# Antialiasing filter

The Bessel filter is often used. From wikipedia:

> *In electronics and signal processing, a Bessel filter is a type of analog linear filter with a maximally flat group/phase delay (maximally linear phase response), which preserves the wave shape of filtered signals in the passband. Bessel filters are often used in audio systems.*

Why use a Bessel filter as antialiasing filter?

# Antialiasing filter

The Bessel filter is often used. From wikipedia:

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Why use as antialiasing filter?

- Preserves wave shapes $\Rightarrow$ very little distortion of signals in the passband
- Maximally linear phase response $\Rightarrow \arg H \approx -T\omega$, Can be modelled as a pure delay