# Polynomial pole placement - part 2

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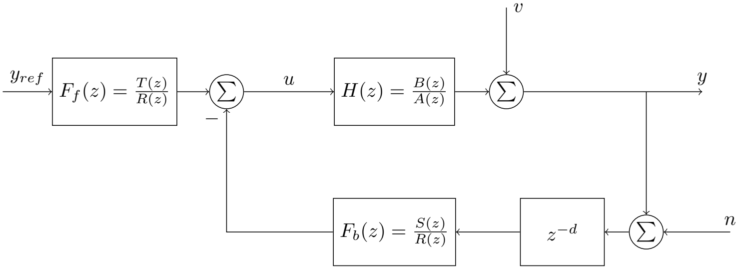
# Goal of today's lecture

Understand the design procedure of polynomial pole placement

# Two-degree-of-freedom controller




$$Y(z) = \frac{F_f(z)H(z)}{1 + z^{-d}F_b(z)H(z)} Y_{ref}(z) + \overbrace{\frac{1}{1 + z^{-d}F_b(z)H(z)}}^{S_s(z)} V(z) - \overbrace{\frac{z^{-d}F_b(z)H(z)}{1 + z^{-d}F_b(z)H(z)}}^{T_s(z)} N(z)$$

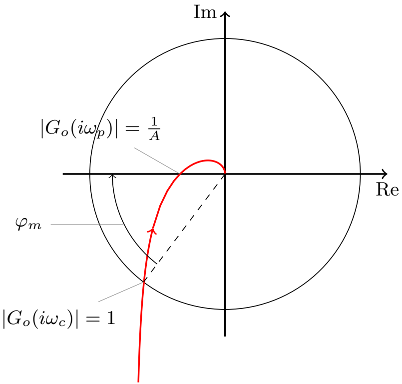
Evidently $S_s(z) + T_s(z) = 1$ Conclusion: One must find a balance between disturbance rejection and noise attenuation.

# The sensitivity function

$$S_s(z) = \frac{1}{1 + z^{-d}F_b(z)H(z)} = \frac{1}{1 + G_o(z)} = \frac{1}{G_o(z) - (-1)}$$

$$|S_s(\mathrm{e}^{i\omega h})| = |S_s(i\omega)| = \frac{1}{|G_o(i\omega) - (-1)|}$$

The magnitude of the sensitivity function is inverse proportional to the distance of the Nyquist curve to the critical point -1



# The design procedure

# The design procedure

Given plant model $H(z) = \frac{B(z)}{A(z)}$ and specifications on the desired closed-loop poles $A_{cl}(z)$

1. Find polynomials $R(z)$ and $S(z)$ with $n_R \geq n_S$ such that

$$A(z)R(z)z^d + B(z)S(z) = A_{cl}(z)$$

2. Factor the closed-loop polynomial as $A_{cl}(z) = A_c(z)A_o(z)$, where $n_{A_o} \leq n_R$. Choose

$$T(z) = t_0 A_o(z),$$

   where $t_0 = \frac{A_c(1)}{B(1)}$.

The control law is then

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k).$$

The closed-loop response to the command signal is given by

$$A_c(q)y(k) = t_0 B(q)u_c(k).$$

# Determining the order of the controller

With Diophantine equation

$$A(z)R(z)z^d + B(z)S(z) = A_{cl}(z) \qquad (*)$$

and feedback controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \cdots + s_n}{z^n + r_1 z^{n-1} + \cdots + r_n}$$

How should we choose the order of the controller? Note:

- the controller has $n + n + 1 = 2 \deg R + 1$ unknown parameters
- the LHS of $(*)$ has degree $\deg \big(A(z)R(z)z^d + B(z)S(z)\big) = \deg A + \deg R + d$
- The diophantine gives as many (nontrivial) equations as the degree of the polynomials on each side when we set the coefficients equal.

$\Rightarrow$ Choose $\deg R$ so that $2 \deg R + 1 = \deg A + \deg R + d$

# Determining the order of the controller - Exercise

With the plant model

$$H(z) = \frac{B(z)}{A(z)} = \frac{b}{z+a}$$

and $d = 0$ (no delay), what is the appropriate degree of the controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \cdots + s_n}{z^n + r_1 z^{n-1} + \cdots + r_n}$$

so that all parameters can be determined from the diophantine equation

$$A(z)R(z) + B(z)S(z) = A_c(z)A_o(z)?$$

1. $n = 0$
2. $n = 1$
3. $n = 2$
4. $n = 3$

# Determining the order of the controller - Exercise - Solution

With the plant model

$$H(z) = \frac{B(z)}{A(z)} = \frac{b}{z+a}$$

and $d = 0$ (no delay), what is the appropriate degree of the controller

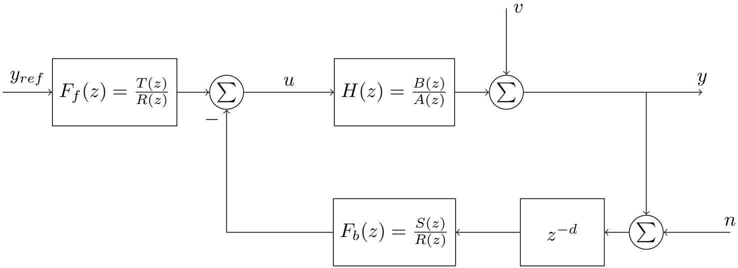
$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \cdots + s_n}{z^n + r_1 z^{n-1} + \cdots + r_n}$$

so that all parameters can be determined from the diophantine equation

$$A(z)R(z) + B(z)S(z) = A_c(z)A_o(z)?$$

| | |
|---|---|
| 1. $n = 0$ | 2. |
| 3. | 4. |

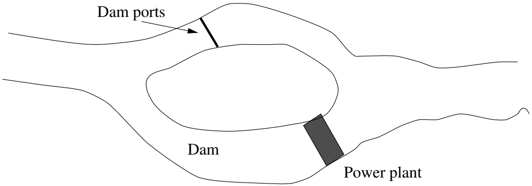
# Two-degree-of-freedom controller, the importance of the observer poles




$$Y(z) = \frac{t_0 B(z) z^d}{A_c(z)} Y_{ref}(z) + \frac{A(z)R(z)z^d}{A_c(z)A_o(z)} V(z) - \frac{S(z)B(z)}{A_c(z)A_o(z)} N(z)$$

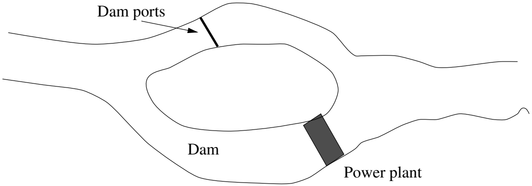
Conclusiones 1) There is a partial separation between designing for reference tracking and designing for perturbance rejection. 2) The observer poles (the roots of $A_o(z)$) can be used to determine a balance between disturbance rejection and noise attenuation.

# Example - Level control of a dam



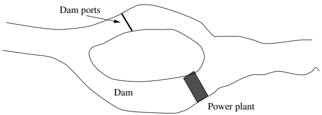

Objective Design a control system to maintain the water level under influence of disturbances.

# Example - Level control of a dam




$$
\begin{aligned}
\text{Amount of water in dam:} \quad & x(t) = x_0 + y(t) \\
\text{Flow through dam ports:} \quad & f(t) = f_0 - u(t) \\
\text{Uncontrolled flows out:} \quad & d(t) = f_p(t) - f_i(t) = d_0 + v(t) \\
\text{Dynamics:} \quad & \frac{d}{dt}x(t) = \frac{d}{dt}y(t) \\
& \quad = -f(t) - d(t) = \underbrace{-f_0 - d_0}_{=0} + u(t) - v(t)
\end{aligned}
$$

# Example - Level control of a dam



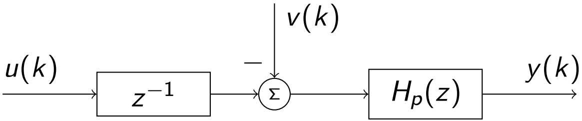

## Discrete-time process dynamics

Deviation in the level of water

Deviation in the controlled flow

$$y(k) = y(k-1) - v(k-1) + u(k-2)$$

Deviation in the non-controlled flows



# Example - Level control of a dam

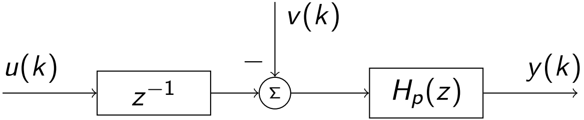
The process dynamics

Deviation in the level of water

Deviation in the controlled flow

$$y(k) = y(k-1) - v(k-1) + u(k-2)$$

Deviation in the non-controlled flows




Activity What is the transfer function from $u(k)$ to $y(k)$?

$$1:\ H(z) = \frac{z}{z-1} \qquad 2:\ H(z) = \frac{1}{z-1} \qquad 3:\ H(z) = \frac{1}{z(z-1)}$$

# Example - Level control of a dam

Given process $H(z) = \frac{B(z)}{A(z)} = \frac{1}{z(z-1)}$ and desired poles in $z = 0.9$.

1. The Diophantine equation $A(z)R(z)z^d + B(z)S(z) = A_{cl}(z)$

$$z(z-1)R(z) + S(z) = A_{cl}(z)$$

   The order of the controller is

$$\deg R = \deg A + d - 1 = 2 - 1 = 1, \quad \Rightarrow \quad F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z + s_1}{z + r_1}$$

2. Resulting Diophantine equation

$$z(z-1)(z + r_1) + s_0 z + s_1 = A_{cl}(z)$$

   The degree of $A_{cl}(z)$ is 3. Choose $A_o(z) = z$, ( $\deg A_o = \deg R$)

$$A_{cl}(z) = A_o(z)A_c(z) = z(z-0.9)^2$$

# Example - Level control of a dam

3. From the Diophantine equation

$$z(z-1)(z+r_1)+s_0 z+s_1 = z(z-0.9)^2$$

$$z^3+(r_1-1)z^2-r_1 z+s_0 z+s_1 = z^3-1.8z^2+0.81z$$

we obtain the equations

$$\begin{cases} z^2 & : & r_1-1=-1.8 \\ z^1 & : & -r_1+s_0=0.81 \\ z^0 & : & s_1=0 \end{cases} \Rightarrow \begin{cases} r_1 & =-0.8 \\ s_0 & =0.01 \\ s_1 & =0 \end{cases}$$

$$F_b(z)=\frac{0.01z}{z-0.8}$$

# Example - Level control of a dam

4. We have $A_o(z) = z$, so

$$T(z) = t_0 A_o(z) = t_0 z$$

$$G_c(z) = \frac{T(z)B(z)}{A_o(z)A_c(z)} = \frac{t_0 B(z)}{A_c(z)}, \quad \text{queremos } G_c(1) = 1$$

$$t_0 = \frac{A_c(1)}{B(1)} = \frac{(1-0.9)^2}{1} = 0.01$$

## Control law

$$R(\mathrm{q})u(kh) = T(\mathrm{q})u_c(kh) - S(\mathrm{q})y(kh)$$

$$(\mathrm{q} - 0.8)u(kh) = 0.01\,\mathrm{q}\,u_c(kh) - 0.01\,\mathrm{q}\,y(kh)$$

$$u(kh+h) = 0.8u(kh) + 0.01u_c(kh+h) - 0.01y(kh+h)$$