Polynomial pole placement

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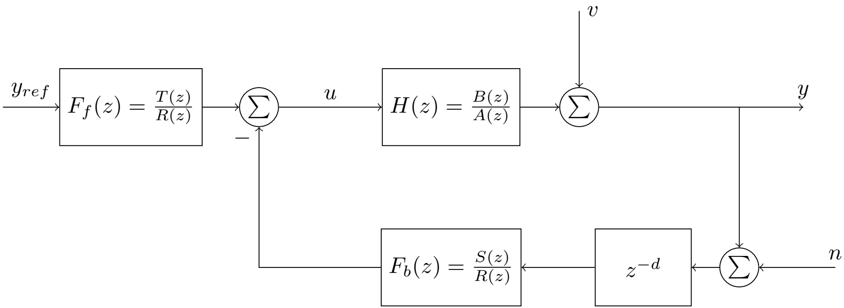
Goal of today's lecture

Understand key concepts of the two-degree-of-freedom controller structure.

Two-degree-of-freedom controller

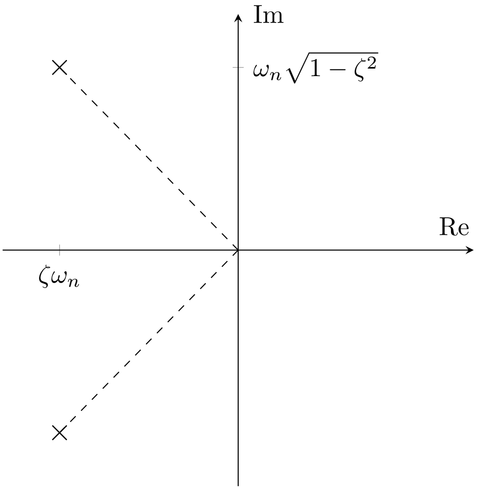


Three key concepts

1. Where to place the poles of the closed-loop system.
2. The *sensitivity function* and the *complementary sensitivity function*.
3. How to determine the order of the controller.

The closed-loop poles

Complex poles in the s-plane



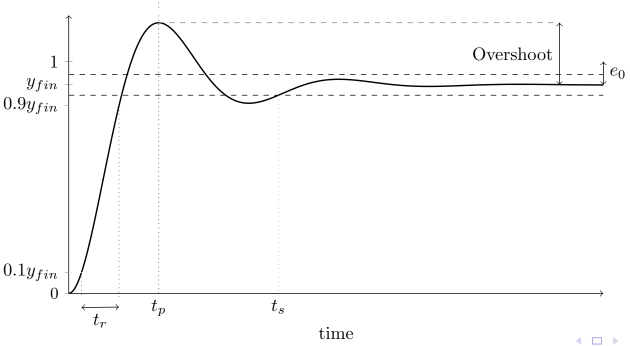
The closed-loop poles

Given specifications on the velocity and damping of the closed-loop system

$$t_s \approx \frac{4}{\zeta\omega_n} < 1s \qquad \zeta \approx \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}, \quad OS < 10\%$$

gives

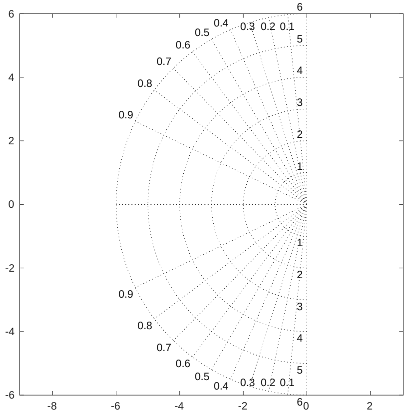
$$\zeta > 0.59, \qquad \zeta\omega_n > 4$$



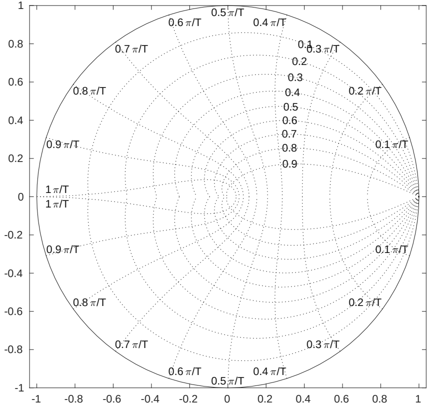
Closed-loop poles

Activity Given specifications $\zeta > 0.59$ and $\zeta\omega_n > 4$, mark the regions in the s-plane and in the z-plane that corresponds to the specifications. Use sampling period $h = 0.2$.

s-plane

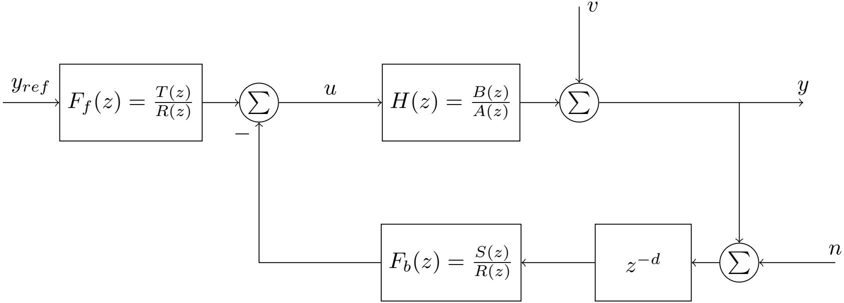


z-plane



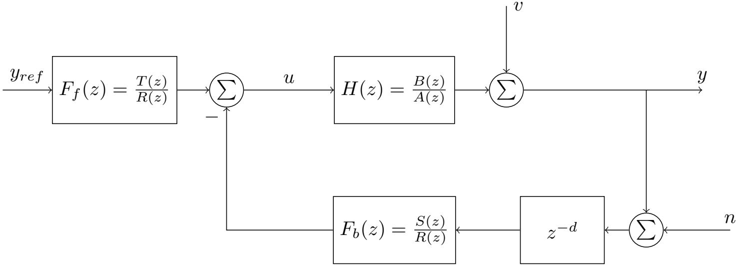
The sensitivity and the complementary sensitivity

Two-degree-of-freedom controller



$$
\begin{aligned}
Y(z) &= G_c(z)Y_{ref}(z) + \overbrace{S_s(z)}^{\text{sens}} V(z) - \overbrace{T_s(z)}^{\text{compl sens}} N(z) \\
&= \frac{F_f(z)H(z)}{1 + F_b(z)z^{-d}H(z)} U_c(z) + \frac{1}{1 + F_b(z)z^{-d}H(z)} V(z) - \frac{z^{-d}F_b(z)H(z)}{1 + F_b(z)z^{-d}H(z)} N(z)
\end{aligned}
$$

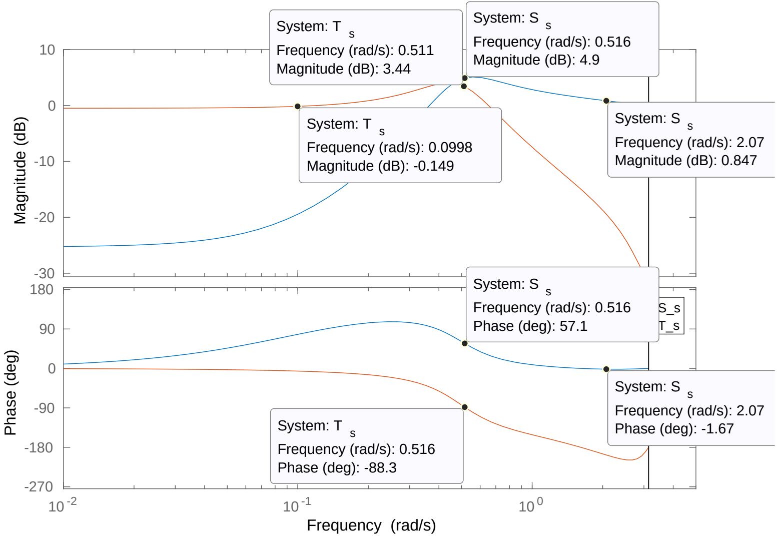
Two-degree-of-freedom controller



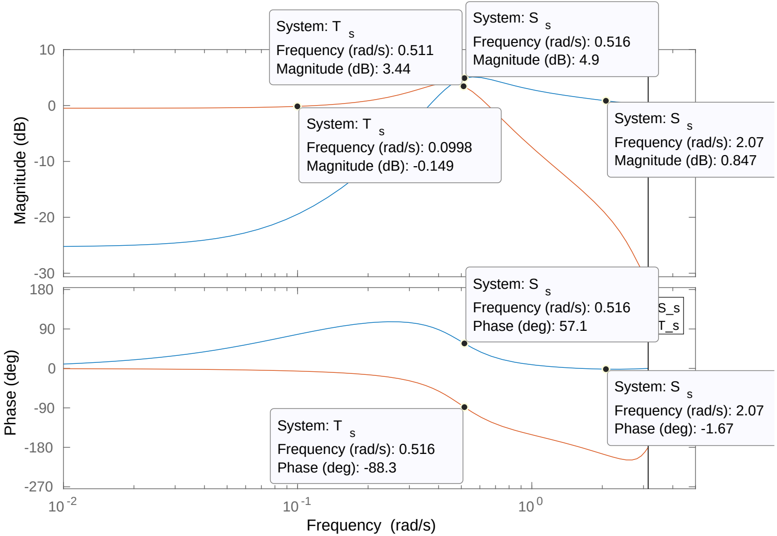
$$Y(z) = \frac{F_f(z)H(z)}{1 + z^{-d}F_b(z)H(z)} Y_{ref}(z) + \overbrace{\frac{1}{1 + z^{-d}F_b(z)H(z)}}^{S_s(z)} V(z) - \overbrace{\frac{z^{-d}F_b(z)H(z)}{1 + z^{-d}F_b(z)H(z)}}^{T_s(z)} N(z)$$

Evidently $S_s(z) + T_s(z) = 1$ Conclusion: One must find a balance between disturbance rejection and noise attenuation.

The sensitivity and the complementary sensitivity



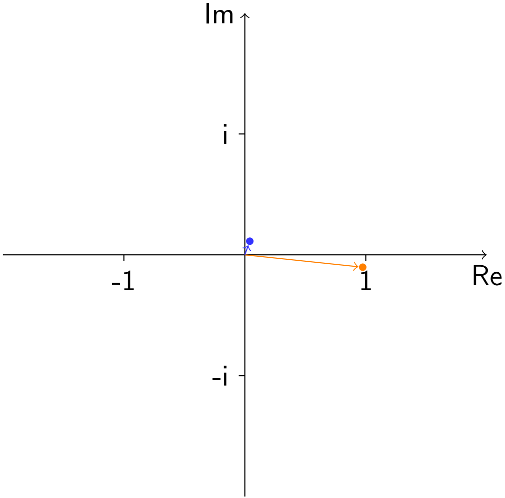
The sensitivity and the complementary sensitivity



Activity Mark in the complex plane the values of $S_s(z)$ and $T_s(z)$ for the frequencies 0.511, 0.516 and 2.07 rad/s. Verify that the vector sum is equal to 1.

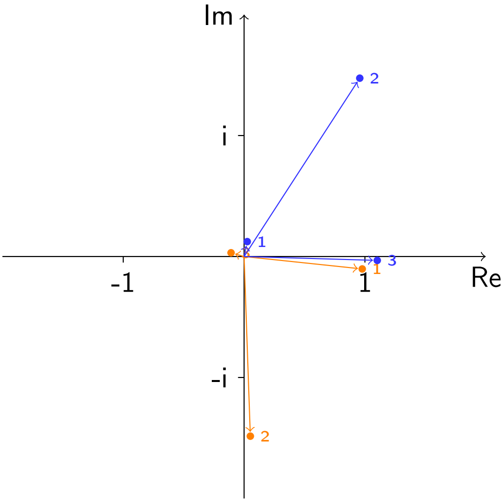
The sensitivity and the complementary sensitivity

Point 1: $\omega = 0.1,\ T_s(0.1) = 10^{-0.149/20}\mathrm{e}^{-i6^o} = 0.98\mathrm{e}^{-i6^o} = 0.97 - i0.1,$
$S_s(0.1) = 10^{-18/20}\mathrm{e}^{i70^o} = 0.12\mathrm{e}^{i70^o} = 0.04 + i0.11$



The sensitivity and the complementary sensitivity - Solution

The sensitivity and the complementary sensitivity - Solution



Procedure

Given plant model $H(z) = \frac{B(z)}{A(z)}$ and specifications on the desired closed-loop poles $A_{cl}(z)$

1. Find polynomials $R(z)$ and $S(z)$ with $n_R \geq n_S$ such that

$$A(z)R(z)z^d + B(z)S(z) = A_{cl}(z)$$

2. Factor the closed-loop polynomials as $A_{cl}(z) = A_c(z)A_o(z)$, where $n_{A_o} \leq n_R$. Choose

$$T(z) = t_0 A_o(z),$$

   where $t_0 = \frac{A_c(1)}{B(1)}$.

The control law is then

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k).$$

The closed-loop response to the command signal is given by

$$A_c(q)y(k) = t_0 B(q)u_c(k).$$

Determining the order of the controller

With Diophantine equation

$$A(z)R(z)z^d + B(z)S(z) = A_{cl}(z) \qquad (*)$$

and feedback controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \cdots + s_n}{z^n + r_1 z^{n-1} + \cdots + r_n}$$

How should we choose the order of the controller? Note:

- the controller has $n + n + 1 = 2\deg R + 1$ unknown parameters
- the LHS of $(*)$ has degree $\deg\big(A(z)R(z)z^d + B(z)S(z)\big) = \deg A + \deg R + d$
- The diophantine gives as many (nontrivial) equations as the degree of the polynomials on each side when we set the coefficients equal.

$\Rightarrow$ Choose $\deg R$ so that $2\deg R + 1 = \deg A + \deg R + d$

Determining the order of the controller - Exercise

With the plant model

$$H(z) = \frac{B(z)}{A(z)} = \frac{b}{z+a}$$

and $d = 0$ (no delay), what is the appropriate degree of the controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \cdots + s_n}{z^n + r_1 z^{n-1} + \cdots + r_n}$$

so that all parameters can be determined from the diophantine equation

$$A(z)R(z) + B(z)S(z) = A_c(z)A_o(z)?$$

1. $n = 0$
2. $n = 1$
3. $n = 2$
4. $n = 3$

Determining the order of the controller - Exercise - Solution

With the plant model

$$H(z) = \frac{B(z)}{A(z)} = \frac{b}{z+a}$$

and $d = 0$ (no delay), what is the appropriate degree of the controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \cdots + s_n}{z^n + r_1 z^{n-1} + \cdots + r_n}$$

so that all parameters can be determined from the diophantine equation

$$A(z)R(z) + B(z)S(z) = A_c(z)A_o(z)?$$

1. $n = 0$	2.
3.	4.