Polynomial pole placement

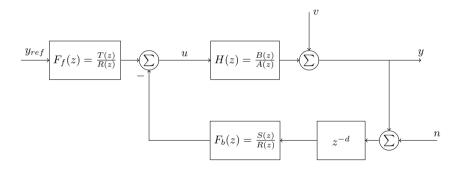
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Goal of today's lecture

Understand key concepts of the two-degree-of-freedom controller structure.

Two-degree-of-freedom controller

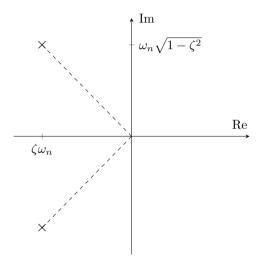


Three key concepts

- 1. Where to place the poles of the closed-loop system.
- 2. The sensitivity function and the complementary sensitivity function.
- 3. How to determine the order of the controller.

The closed-loop poles

Complex poles in the s-plane



The closed-loop poles

Given specifications on the velocity and damping of the closed-loop system

$$t_spprox rac{4}{\zeta\omega_n} < 1s \qquad \zetapprox rac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}, \quad OS < 10\%$$

gives

$$\zeta > 0.59, \qquad \zeta \omega_n > 4$$

$$0.1y_{fin}$$

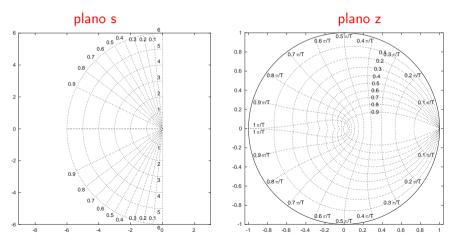
$$0.1y_{fin}$$

$$0 \downarrow t_p \qquad t_s$$

time

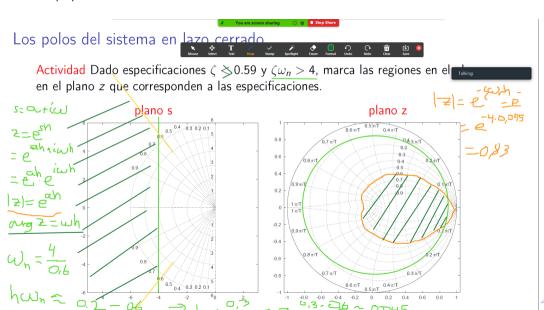
Closed-loop poles

Activity Given the specifications $\zeta > 0.59$ and $\zeta \omega_n > 4$, mark the regions in the s-plane and in the z-plane that corresponds to the specifications.



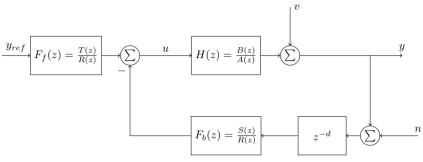
Closed-loop poles - Solution

Closed-loop poles - Solution



The sensitivity and the complementary sensitivity

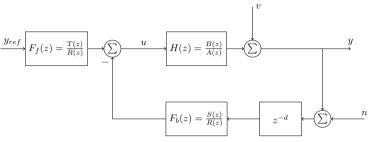
Two-degree-of-freedom controller



$$Y(z) = G_c(z)U_c(z) + S_s(z)V(z) - T_s(z)N(z)$$

$$= \frac{F_f(z)H(z)}{1 + F_b(z)z^{-d}H(z)}U_c(z) + \frac{1}{1 + F_b(z)z^{-d}H(z)}V(z) - \frac{z^{-d}F_b(z)H(z)}{1 + F_b(z)z^{-d}H(z)}N(z)$$

Two-degree-of-freedom controller



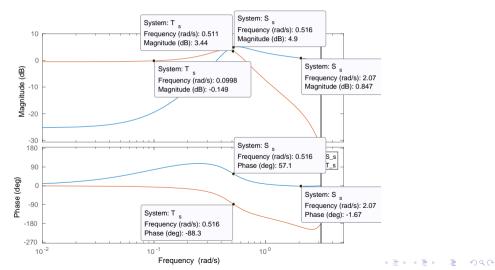
$$Y(z) = \frac{F_f(z)H(z)}{1 + z^{-d}F_b(z)H(z)}U_c(z) + \underbrace{\frac{S_s(z)}{1}}_{1 + z^{-d}F_b(z)H(z)}V(z) - \underbrace{\frac{z^{-d}F_b(z)H(z)}{1 + z^{-d}F_b(z)H(z)}}_{T_s(z)}N(z)$$

Evidently $S_s(z) + T_s(z) = 1$ Conclusion: One must find a balance between disturbance rejection and noise attenuation.



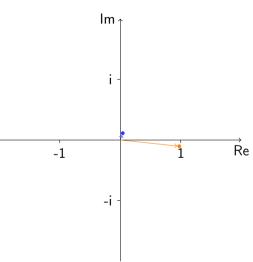
The sensitivity and the complementary sensitivity

Activity Mark in the complex plane the points indicated in the Bode plot of the two systems $S_s(z)$ and $T_s(z)$. Verify that the vector sum is equal to 1.



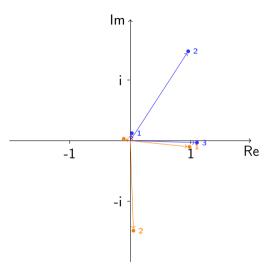
The sensitivity and the complementary sensitivity

Point 1:
$$\omega = 0.1$$
, $T_s(0.1) = 10^{-0.149/20} e^{-i6^{\circ}} = 0.98 e^{-i6^{\circ}} = 0.97 - i0.1$, $S_s(0.1) = 10^{-18/20} e^{i70^{\circ}} = 0.12 e^{i70^{\circ}} = 0.04 + i0.11$



The sensitivity and the complementary sensitivity - Solution

The sensitivity and the complementary sensitivity - Solution



Procedure

Given plant model $H(z) = \frac{B(z)}{A(z)}$ and specifications on the desired closed-loop poles $A_{cl}(z)$

1. Find polynomials R(z) and S(z) with $n_R \ge n_S$ such that

$$A(z)R(z)z^d + B(z)S(z) = A_{cl}(z)$$

2. Factor the closed-loop polynomials as $A_{cl}(z) = A_c(z)A_o(z)$, where $n_{A_o} \leq n_R$. Choose

$$T(z)=t_0A_o(z),$$

where $t_0 = \frac{A_c(1)}{B(1)}$.

The control law is then

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k).$$

The closed-loop response to the command signal is given by

$$A_c(q)y(k)=t_0B(q)u_c(k).$$



Determining the order of the controller

With Diophantine equation

$$A(z)R(z)z^d + B(z)S(z) = A_{cl}(z) \qquad (*)$$

and feedback controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \dots + s_n}{z^n + r_1 z^{n-1} + \dots + r_n}$$

How should we choose the order of the controller? Note:

- ▶ the controller has $n + n + 1 = 2 \deg R + 1$ unknown parameters
- ▶ the LHS of (*) has degree deg $(A(z)R(z)z^d + B(z)S(z)) = \deg A + \deg R + d$
- ► The diophantine gives as many (nontrivial) equations as the degree of the polynomials on each side when we set the coefficients equal.
 - \Rightarrow Choose deg R so that $2 \deg R + 1 = \deg A + \deg R + d$



Determining the order of the controller - Exercise

With the plant model

$$H(z) = \frac{B(z)}{A(z)} = \frac{b}{z+a}$$

and d = 0 (no delay), what is the appropriate degree of the controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \dots + s_n}{z^n + r_1 z^{n-1} + \dots + r_n}$$

so that all parameters can be determined from the diophantine equation

$$A(z)R(z) + B(z)S(z) = A_c(z)A_o(z)?$$

1.
$$n = 0$$
 2. $n = 1$

3.
$$n = 2$$
 4. $n = 3$

Determining the order of the controller - Exercise - Solution

With the plant model

$$H(z) = \frac{B(z)}{A(z)} = \frac{b}{z+a}$$

and d = 0 (no delay), what is the appropriate degree of the controller

$$F_b(z) = \frac{S(z)}{R(z)} = \frac{s_0 z^n + s_1 z^{n-1} + \dots + s_n}{z^n + r_1 z^{n-1} + \dots + r_n}$$

so that all parameters can be determined from the diophantine equation

$$A(z)R(z) + B(z)S(z) = A_c(z)A_o(z)$$
?
1. $n = 0$ 2.
3. 4.