

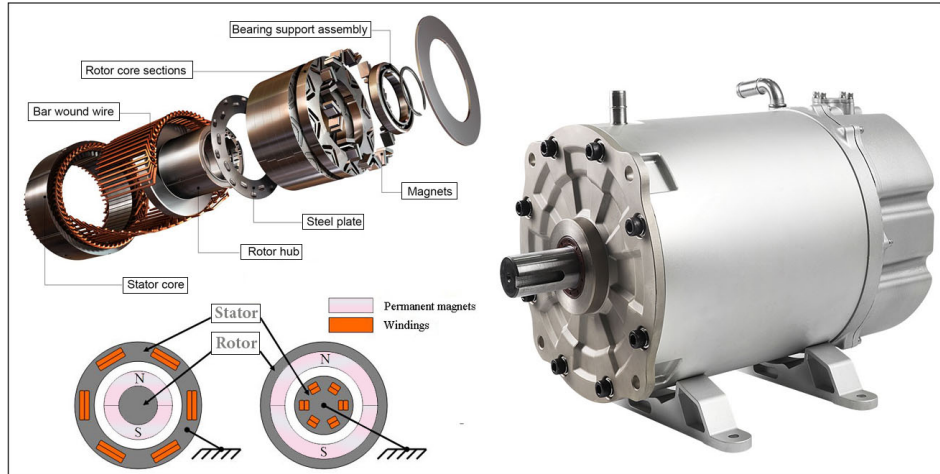
State space models

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Obtain state-space model from discrete-time pulse-transfer function

The permanent magnet synchronous motor



Permanent Magnet Synchronous Motor Construction

The PMSM

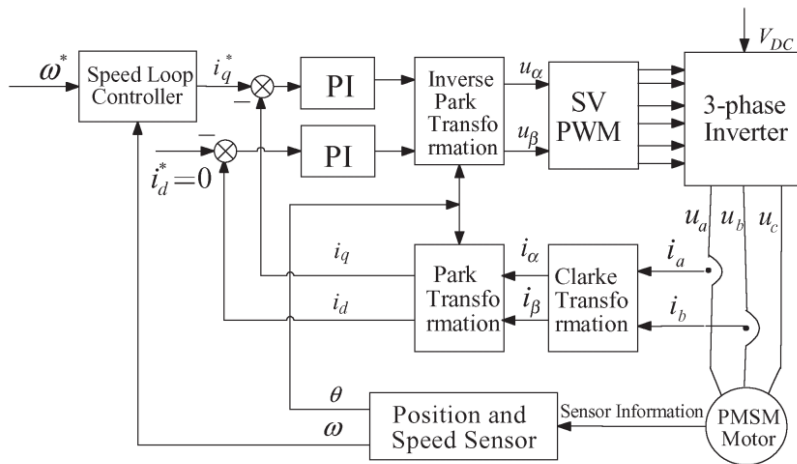
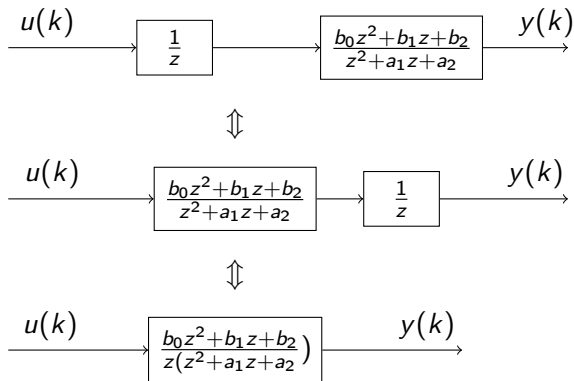


Fig. 1. Block diagram of the PMSM control system.

De Liu and Li "Speed control for PMSM servo system", IEEE Transactions on Industrial Electronics, 2012.

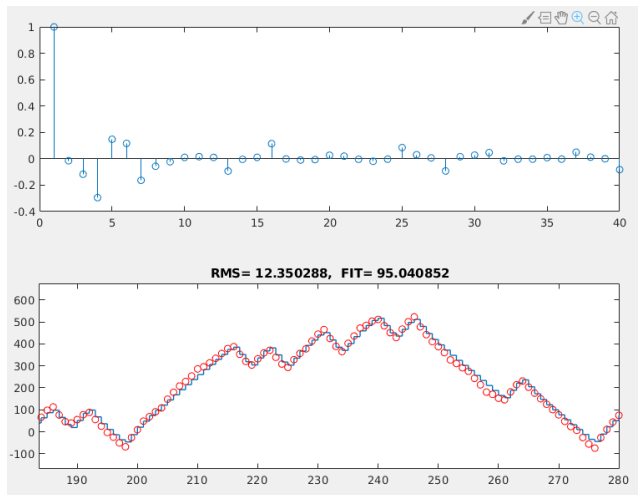
Identified model

Two poles, two zeros, one delay

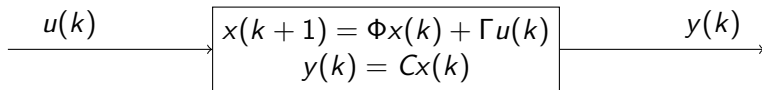
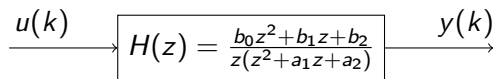


Identified model

$$H(z) = \frac{6.91z^2 + 16.48z - 17.87}{z(z^2 - 1.766z + 0.7665)} = \frac{6.91(z + 3.19)(z - 0.81)}{z(z - 0.998)(z - 0.768)}$$



From pulse-transfer function to state space model



Canonical forms

Given pulse-transfer function

$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

Find a representation in state space form

$$\begin{aligned}x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k)\end{aligned}$$

Canonical forms

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Find a representation in state space form

$$\begin{aligned}x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k)\end{aligned}$$

- ▶ Controllable canonical form
- ▶ Observable canonical form

Controlable canonical form

Given pulse-transfer function

$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

$$\begin{aligned} x(k+1) &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= [b_1 \quad b_2 \quad b_3] x(k) \end{aligned}$$

Observable canonical form

Given pulse-transfer function

$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

$$\begin{aligned}x(k+1) &= \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k)\end{aligned}$$

Canonical forms

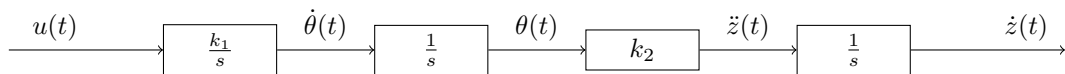
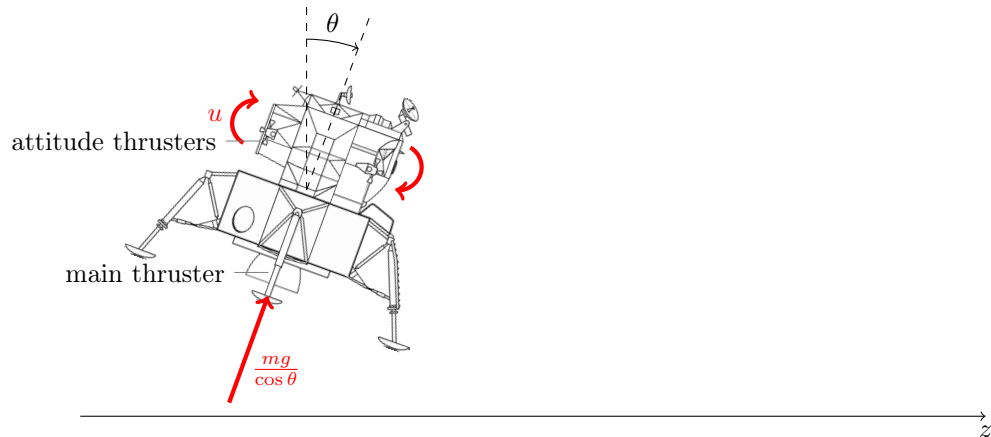
Activity Find the controllable and observable canonical forms for the pulse-transfer function of the motor. Answer on Canvas (questions 1 and 2 on today's exercises).

$$H(z) = \frac{6.91z^2 + 16.48z - 17.87}{z(z^2 - 1.766z + 0.7665)} = \frac{6.91(z + 3.19)(z - 0.81)}{z(z - 0.998)(z - 0.768)}$$

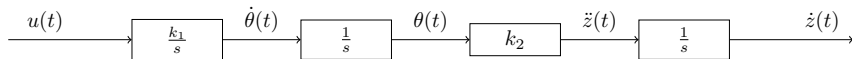
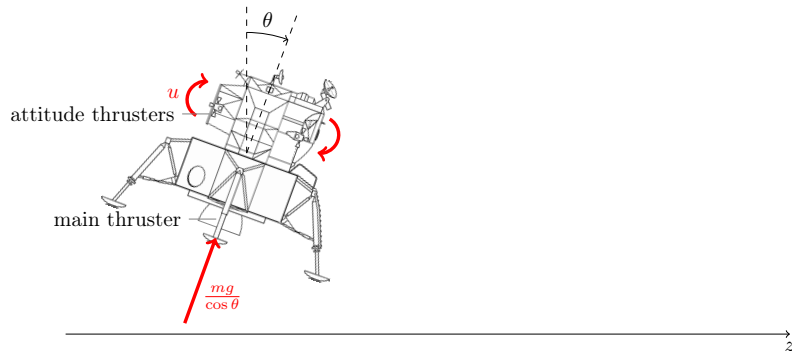
Discrete-time state-space from continuous-time state space

A.k.a. discretization

Example - the Apollo lunar module



Example - the Apollo lunar module



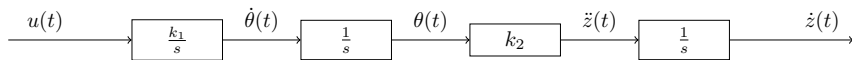
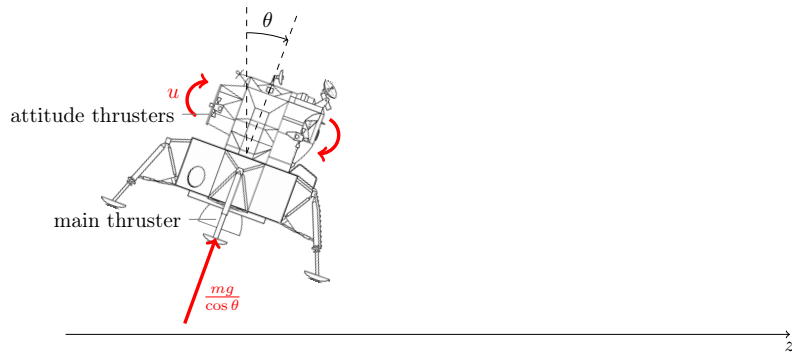
Activity Which is the transfer function of the system?

$$1: G(s) = \frac{k_1 k_2}{s^2}$$

$$2: G(s) = \frac{k_1 k_2}{s(s^2 + 1)}$$

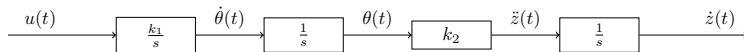
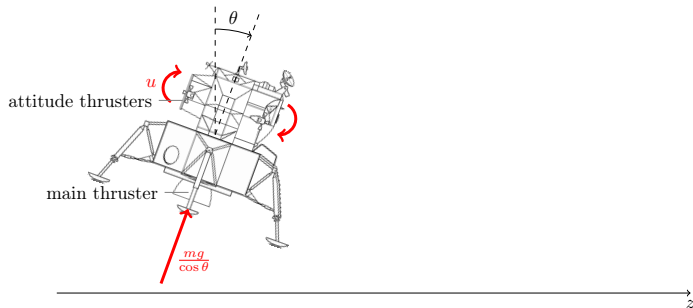
$$3: G(s) = \frac{k_1 k_2}{s^3}$$

Example - the Apollo lunar module



Activity What sensors are needed by the control system?

Example - the Apollo lunar module



State variables: $x = [x_1 \ x_2 \ x_3]^T = [\dot{\theta} \ \theta \ \dot{z}]^T$. With the dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2 \theta = k_2 x_2 \end{cases}$$

Example - the Apollo lunar module

State variables: $x = [x_1 \ x_2 \ x_3]^T = [\dot{\theta} \ \theta \ \dot{z}]^T$. With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2 \theta = k_2 x_2 \end{cases}$$

Activity Fill the matrix A and vector B .

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_B u$$

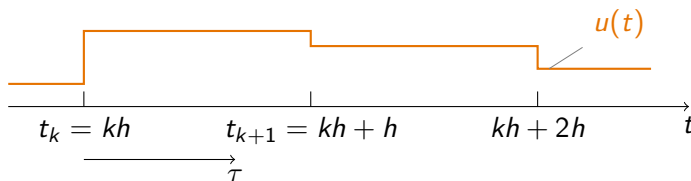
Example - the Apollo lunar module

Discretizing a continuous-time state-space model

Discretización

The general solution to a linear, continuous-time state-space system

$$x(t_k + \tau) = e^{A(\tau)}x(t_k) + \int_0^\tau e^{As}Bu((t_k + \tau) - s)ds$$



$$\begin{aligned} x(kh + h) &= e^{Ah}x(kh) + \int_0^h e^{As}Bu(kh + h - s)ds \\ &= \underbrace{e^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h e^{As}Bds\right)}_{\Gamma(h)}u(kh) \end{aligned}$$

Discretization - The matrix exponential

Square matrix A . Scalar variable t .

$$e^{At} = 1 + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots$$

Laplace transform

$$\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$$

Discretization - example

$$\begin{aligned}x(kh + h) &= e^{Ah}x(kh) + \int_0^h e^{As}Bu(kh + h - s)ds \\&= \underbrace{e^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h e^{As}Bds\right)}_{\Gamma(h)}u(kh)\end{aligned}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}, \quad A^3 = 0$$

So,

$$\Phi(h) = e^{Ah} = 1 + Ah + A^2h^2/2 + \dots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} h + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \frac{h^2}{2} = \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2k_2}{2} & hk_2 & 1 \end{bmatrix}$$

Discretization - example

$$\begin{aligned}x(kh + h) &= e^{Ah}x(kh) + \int_0^h e^{As}Bu(kh + h - s)ds \\&= \underbrace{e^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h e^{As}Bds\right)}_{\Gamma(h)}u(kh)\end{aligned}$$

$$e^{As}B = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ \frac{s^2 k_2}{2} & sk_2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ s \\ \frac{k_2 s^2}{2} \end{bmatrix}$$

$$\Gamma(h) = \int_0^h e^{As}Bds = k_1 \int_0^h \begin{bmatrix} 1 \\ s \\ \frac{k_2 s^2}{2} \end{bmatrix} ds = k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2 h^3}{6} \end{bmatrix}$$

Discretization - example

$$\begin{aligned}x(kh + h) &= e^{Ah}x(kh) + \int_0^h e^{As}Bu(kh + h - s)ds \\&= \underbrace{e^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h e^{As}Bds\right)}_{\Gamma(h)}u(kh) \\&= \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2 k_2}{2} & hk_2 & 1 \end{bmatrix} x(kh) + k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2 h^3}{6} \end{bmatrix} u(kh)\end{aligned}$$

Discretization - exercise

Activity Discretize the system (question 3 on today's exercises on Canvas)

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$