# State feedback

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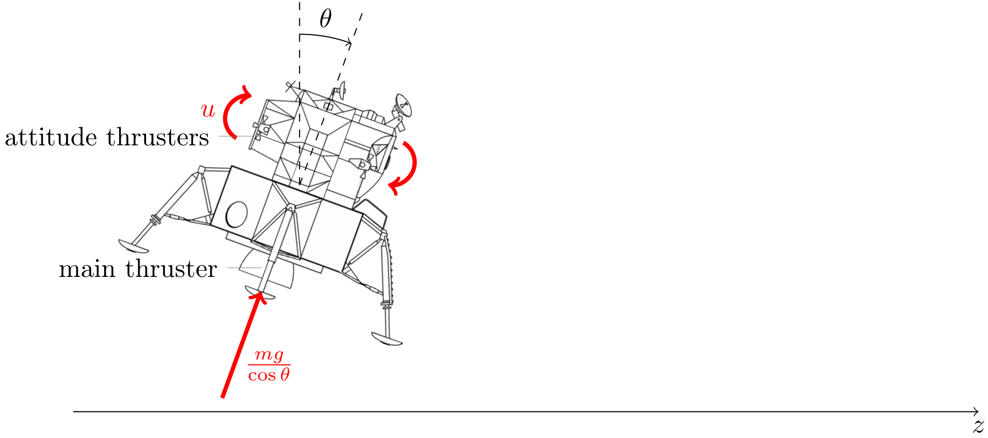
# Example - The Apollo lunar module




$$u(t) \longrightarrow \boxed{\frac{k_1}{s}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{k_2} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

# Example - The Apollo lunar module

State variables: $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} \dot{\theta} & \theta & \dot{z} \end{bmatrix}^T$. With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2 \theta = k_2 x_2 \end{cases}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

# Example - The Apollo lunar module

$$
\begin{aligned}
x(kh+h) &= \mathrm{e}^{Ah}x(kh) + \int_0^h \mathrm{e}^{As}Bu(kh+h-s)ds \\
&= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h \mathrm{e}^{As}Bds\right)}_{\Gamma(h)}u(kh) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2k_2}{2} & hk_2 & 1 \end{bmatrix} x(kh) + k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2h^3}{6} \end{bmatrix} u(kh)
\end{aligned}
$$

# Stability

# Eigenvalues and eigenvectors

**Definition** The eigenvalues $\lambda_i \in \mathbb{R}$ and eigenvectors $v_i \in \mathbb{R}^n$ of a matrix $\Phi \in \mathbb{R}^{n \times n}$ are the $n$ pairs $(\lambda_i, v_i \neq 0),\ i = 1, 2, \ldots, n$ that satisfy

$$\Phi v_i = \lambda_i v_i$$

# Stability

The system

$$x(k+1) = \Phi x(k), \quad x(0) = x_0$$

is stable if $\lim_{t \to \infty} x(kh) = 0, \quad \forall\, x_0 \in \mathbb{R}^n$.

A necessary and sufficient requirement for stability is that all the eigenvalues of $\Phi$ are inside the unit circle.

The eigenvalues of $\Phi$ are the poles of the system.

# Eigenvalues and eigenvectors - exercise

Activity Verify that the vector

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is an eigenvector of

$$\Phi = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

What is the corresponding eigenvalue?

# Controllability

Controllability is the answer to the question *Can we drive the state of the system to any location in the state space by a suitable input sequence* $u(k),\ k = 0, 1, 2, \ldots, n-1$*?*

Consider

$$x(k+1) = \Phi x(k) + \Gamma u(k), \quad x(0) = x_0$$

with solution

$$\begin{aligned} x(n) &= \Phi^n x(0) + \Phi^{n-1}\Gamma u(0) + \Phi^{n-2}\Gamma u(1) + \cdots + \Gamma u(n-1) \\ &= \Phi^n x(0) + W_c U, \end{aligned} \tag{1}$$

where

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}$$
$$U = \begin{bmatrix} u(n-1) & u(n-2) & \cdots & u(0) \end{bmatrix}^{\mathrm{T}}$$

# Controllability

To find the input sequence $u(k)$ that takes the state from $x(0) = x_0$ to $x(n) = x_d$ we may solve for $U$ in the equation

$$x_d = \Phi^n x_0 + W_c U.$$

$$U = W_c^{-1} \left( x_d - \Phi^n x(0) \right)$$

This is possible when the matrix $W_x$ is invertible:

The state-space system above is controllable if and only if the *Controllability matrix* $W_c$ has rank $n$, i.e.

$$\det W_c \neq 0.$$

# Observability

Observability is the answer to the question "Can we determine the initial state $x(0)$ if we only know $y(k),\ k = 0, 1, 2, \ldots, n-1$?"

The first $n$ values of the output sequence are given by

$$\begin{aligned} y(0) &= Cx(0) \\ y(1) &= Cx(1) = C\left(\Phi x(0) + \Gamma u(0)\right) \\ &\vdots \\ y(n-1) &= Cx(n-1) = C\left(\Phi^{n-1}x(0) + W_c U\right). \end{aligned}$$

This gives the equation

$$\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y(1) - C\Gamma u(0) \\ \vdots \\ y(n-1) - CW_c U \end{bmatrix}$$

which can be solved for $x(0)$ if and only if the matrix

$$W_o = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}$$

has full rank.

# Observability, contd

The equation

$$\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y(1) - C\Gamma u(0) \\ \vdots \\ y(n-1) - CW_c U \end{bmatrix}$$

can be solved for $x(0)$ if and only if the matrix

$$W_o = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}$$

has full rank. If this is the case, the system is said to be observable.

# State feedback control

# State feedback control

Given

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{2}$$

and measurements (or an estimate) of the state vector $x(k)$.

Linear state feedback is the control law

$$\begin{aligned} u(k) = f\big((x(k), u_c(k)\big) &= -l_1 x_1(k) - l_2 x_2(k) - \cdots - l_n x_n(k) + l_0 u_c(k) \\ &= -Lx(k) + l_0 u_c(k), \end{aligned}$$

where

$$L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}.$$

Substituting this in the state-space model (2) gives

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)\, x(k) + m\Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \tag{3}$$

# Pole placement by state feedback

Given a desired placement of the closed-loop poles $p_1, p_2, \ldots, p_n$, being roots of the desired characteristic polynomial

$$a_c(z) = (z - p_1)(z - p_2)\cdots(z - p_n) = z^n + \alpha_1 z^{n-1} + \cdots \alpha_n. \tag{4}$$

Linear state feedback gives the system

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)\, x(k) + l_0 \Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \tag{5}$$

with characteristic polynomial

$$\det\left(zI - (\Phi - \Gamma L)\right) = z^n + \beta_1(l_1, \ldots, l_n) z^{n-1} + \cdots \beta_n(l_1, \ldots, l_n). \tag{6}$$

Set the coefficients of the desired characteristic polynomial (8) equal to the coefficients of (6) to obtain the system of equations

$$\begin{aligned} \beta_1(l_1, \ldots, l_n) &= \alpha_1 \\ \beta_2(l_1, \ldots, l_n) &= \alpha_2 \\ &\vdots \\ \beta_n(l_1, \ldots, l_n) &= \alpha_n \end{aligned}$$

# Pole placement by state feedback

The system of equations

$$
\begin{aligned}
\beta_1(l_1, \ldots, l_n) &= \alpha_1 \\
\beta_2(l_1, \ldots, l_n) &= \alpha_2 \\
&\vdots \\
\beta_n(l_1, \ldots, l_n) &= \alpha_n
\end{aligned}
$$

is always linear in the parameters of the controller, henc

$$ML^{\mathrm{T}} = \alpha,$$

where $\alpha^{\mathrm{T}} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$.

# Pole placement and controllability

It can be shown that the controllability matrix

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}$$

is a factor of the matrix $M$

$$M = \bar{M} W_c.$$

Hence, in general, the equations

$$\bar{M} W_c L^{\mathrm{T}} = \alpha \qquad \Rightarrow \qquad L^{\mathrm{T}} = W_c^{-1} \bar{M}^{-1} \alpha \tag{7}$$

only has a solution if $W_c$ is invertible, that is when the system is *controllable*.

Note that the equations (7) may also have solutions when the system is not controllable, if $\alpha$ is in the column space of $M$. That is

$$\alpha = b_1 M_{:,1} + b_2 M_{:,2} + \cdots + b_{M:,m}, \ \ m < n$$

# Pole placement by state feedback

Given a desired placement of the closed-loop poles $p_1, p_2, \ldots, p_n$, being roots of the desired characteristic polynomial

$$a_c(z) = (z - p_1)(z - p_2)\cdots(z - p_n) = z^n + \alpha_1 z^{n-1} + \cdots \alpha_n. \tag{8}$$

and closed-loop system

$$\begin{aligned} x(k+1) &= \left(\Phi - \Gamma L\right) x(k) + l_0 \Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \tag{9}$$

The Matlab (*control systems toolbox*) has methods for computing the gain vector $L$

1. Ackerman's method

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L = acker(Phi, Gamma, pd)
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2. Numerically more stable method

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L = place(Phi, Gamma, pd)
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# The reference input gain $l_0$

The closed-loop state space system

$$x(k+1) = \underbrace{(\Phi - \Gamma L)}_{\Phi_c} x(k) + l_0 \Gamma u_c(k)$$

$$y(k) = Cx(k)$$

has the steady-state solution ($x(k+1) = x(k)$) for constant reference signal $u_c(k) = u_{c,f}$

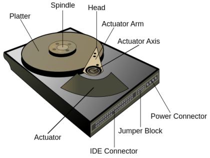
$$y_f = l_0 C(I - \Phi_c)^{-1} \Gamma u_{c,f}.$$

We want $y_f = u_{c,f}$,

$$\Rightarrow \qquad l_0 = \frac{1}{C(I - \Phi_c)^{-1}\Gamma}$$

# Exercise - The harddisk drive arm

The model of the arm of the harddisk drive




can, with suitable choice of sampling period, be written

$$x(k+1) = \Phi x(k) + \Gamma u(k) = \begin{bmatrix} 1 & 0.4 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.16 \\ 0.8 \end{bmatrix} u.$$

With linear state feedback $u(k) = -Lx(k) + l_0 u_c(k)$ the closed-loop system is

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)\, x(k) + l_0 \Gamma u_c(k) \\ &= \begin{bmatrix} 1 - 0.16 l_1 & 0.4 - 0.16 l_2 \\ -0.8 l_1 & 1 - 0.8 l_2 \end{bmatrix} x(k) + l_0 \Gamma u_c(k). \end{aligned}$$

**Determine** the characteristic polynomial of the closed-loop system $\det\Big(zI - (\Phi - \Gamma L)\Big)$