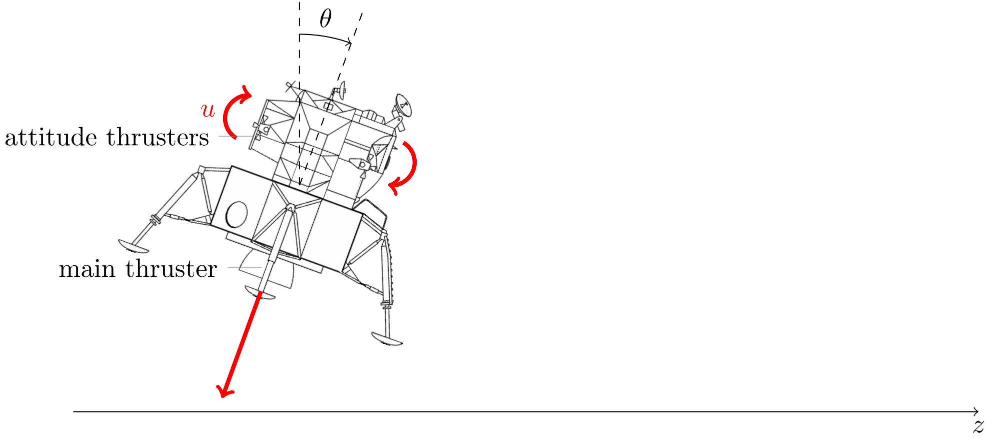
Control computarizado - Modelos en espacio de estados

Kjartan Halvorsen

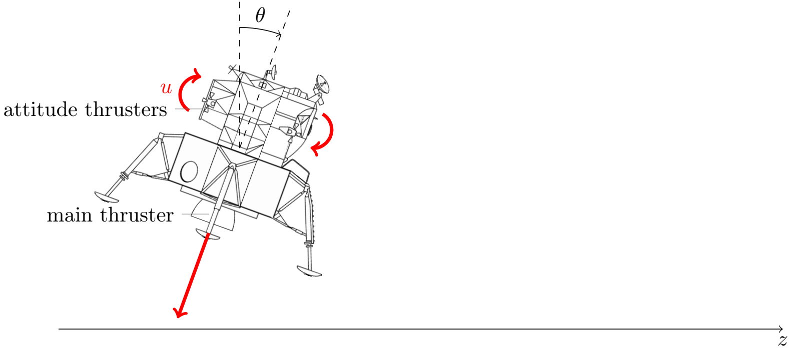
July 21, 2020

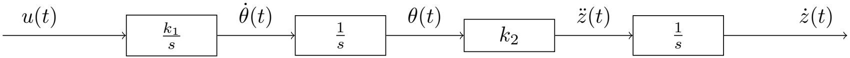
Ejemplo - El modulo lunar de Apollo



$$u(t) \longrightarrow \boxed{\frac{k_1}{s}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{k_2} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

Ejemplo - El modulo lunar de Apollo

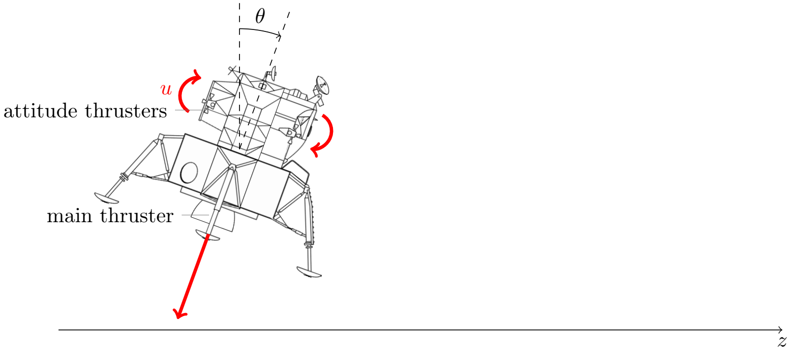




Actividad ¿Cuál es la función de transferencia del sistema?

$$1: \; G(s) = \frac{k_1 k_2}{s^2} \qquad 2: \; G(s) = \frac{k_1 k_2}{s(s^2+1)} \qquad 3: \; G(s) = \frac{k_1 k_2}{s^3}$$

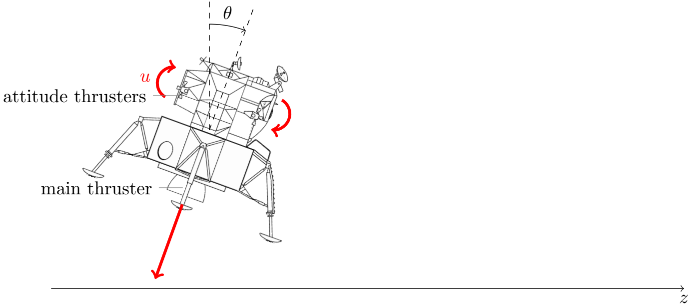
Ejemplo - El modulo lunar de Apollo



$$u(t) \longrightarrow \boxed{\frac{k_1}{s}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{k_2} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

Actividad ¿Que sensores relevantes se puede usar para el control?

Ejemplo - El modulo lunar de Apollo



$$u(t) \longrightarrow \boxed{\frac{k_1}{s}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{k_2} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

Variables del estado: $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} \dot{\theta} & \theta & \dot{z} \end{bmatrix}^T$. Con dinamica

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2 \theta = k_2 x_2 \end{cases}$$

Ejemplo - El modulo lunar de Apollo

Variables del estado: $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} \dot{\theta} & \theta & \dot{z} \end{bmatrix}^T$. Con dinamica

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2\theta = k_2 x_2 \end{cases}$$

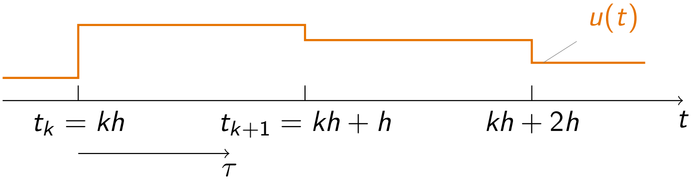
Actividad Llena los matrices y vectores en el modelo de espacio de estados

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}}_{B} u$$

Discretización

Solución general de un sistema lineal en espacio de estados

$$x(t_k + \tau) = \mathrm{e}^{A(\tau)} x(t_k) + \int_0^{\tau} \mathrm{e}^{As} B u((t_k + \tau) - s) ds$$



$$\begin{aligned} x(kh + h) &= \mathrm{e}^{Ah} x(kh) + \int_0^{h} \mathrm{e}^{As} B u(kh + h - s) ds \\ &= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)} x(kh) + \underbrace{\left(\int_0^{h} \mathrm{e}^{As} B ds \right)}_{\Gamma(h)} u(kh) \end{aligned}$$

Discretización - La exponencial de una matriz

Matriz A cuadrada. Variable t escalar.

$$\mathrm{e}^{At} = 1 + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Transformada de Laplace:

$$\mathcal{L}\left\{\mathrm{e}^{At}\right\} = (sI - A)^{-1}$$

Discretización - ejemplo

$$
\begin{aligned}
x(kh+h) &= \mathrm{e}^{Ah}x(kh) + \int_0^h \mathrm{e}^{As}Bu(kh+h-s)ds \\
&= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h \mathrm{e}^{As}Bds\right)}_{\Gamma(h)}u(kh)
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}, \quad A^3 = 0
$$

Entonces,

$$
\begin{aligned}
\Phi(h) &= \mathrm{e}^{Ah} = 1 + Ah + A^2h^2/2 + \cdots \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} h + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \frac{h^2}{2} = \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2k_2}{2} & hk_2 & 1 \end{bmatrix}
\end{aligned}
$$

Discretización - ejemplo

$$
\begin{aligned}
x(kh+h) &= \mathrm{e}^{Ah}x(kh) + \int_0^h \mathrm{e}^{As}Bu(kh+h-s)ds \\
&= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h \mathrm{e}^{As}Bds\right)}_{\Gamma(h)}u(kh)
\end{aligned}
$$

$$
\mathrm{e}^{As}B = \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{s^2k_2}{2} & sk_2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ s \\ \frac{k_2 s^2}{2} \end{bmatrix}
$$

$$
\Gamma(h) = \int_0^h \mathrm{e}^{As}Bds = k_1 \int_0^h \begin{bmatrix} 1 \\ s \\ \frac{k_2 s^2}{2} \end{bmatrix} ds = k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2 h^3}{6} \end{bmatrix}
$$

Discretización - ejemplo

$$\begin{aligned} x(kh+h) &= \mathrm{e}^{Ah}x(kh) + \int_0^h \mathrm{e}^{As}Bu(kh+h-s)ds \\ &= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)} x(kh) + \underbrace{\left(\int_0^h \mathrm{e}^{As}Bds\right)}_{\Gamma(h)} u(kh) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2 k_2}{2} & hk_2 & 1 \end{bmatrix} x(kh) + k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2 h^3}{6} \end{bmatrix} u(kh) \end{aligned}$$

Goal of today's lecture

- Understand state feedback design

Stability

A system

$$x(k+1) = \Phi x(k), \quad x(0) = x_0$$

is *asymptotically stable* if $\lim_{t \to \infty} x(kh) = 0$ for all $x_0 \in \mathbb{R}^n$.

A system is asymptotically stable if and only if all eigenvalues of Φ are inside the unit circle.

Reachability (controllability)

Reachability is the answer to the question "Can we by choosing a suitable input sequence $u(k),\ k = 0, 1, 2, \ldots, n-1$ reach any point in the state space?"

Consider

$$x(k+1) = \Phi x(k) + \Gamma u(k).$$

With initial state $x(0)$ given. The solution at time n where n is the order of the system is

$$\begin{aligned} x(n) &= \Phi^n x(0) + \Phi^{n-1}\Gamma u(0) + \cdots + \Gamma u(n-1) \\ &= \Phi^n x(0) + W_c U, \end{aligned} \tag{1}$$

where

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix}$$
$$U = \begin{bmatrix} u(n-1) & u(n-2) & \cdots & u(0) \end{bmatrix}^{\mathrm{T}}$$

Reachability (controllability), contd

To find the input sequence that takes the state to $x(n) = x_d$ we solve the equation

$$x_d = \Phi^n x(0) + W_c U$$

for U.

$$U = W_c^{-1} (x_d - \Phi^n x(0))$$

This requires the matrix W_x to be invertible. This gives Theorem 3.7 in Å&W:

THEOREM 3.7 REACHABILITY The state space system above is reachable if and only if the matrix W_c has rank n.

This is equivalent to

$$\det W_c \neq 0.$$

State feedback

Have state space model

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= Cx(k)
\end{aligned}
\tag{2}
$$

and measurements (or estimates) of the state vector $x(k)$.

Linear state feedback is the control law

$$
\begin{aligned}
u(k) = f\big((x(k), u_c(k)\big) &= -l_1 x_1(k) - l_2 x_2(k) - \cdots - l_n x_n(k) + m u_c(k) \\
&= -Lx(k) + m u_c(k),
\end{aligned}
$$

where

$$
L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}.
$$

Insert the control law into the state space model (3) to get

State feedback

Have state space model

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= Cx(k)
\end{aligned}
\tag{3}
$$

and measurements (or estimates) of the state vector $x(k)$.

Linear state feedback is the control law

$$
u(k) = -l_1 x_1(k) - l_2 x_2(k) - \cdots - l_n x_n(k) + m u_c(k) = -Lx(k) + m u_c(k),
$$

where

$$
L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}.
$$

Insert the control law into the state space model (3) to get

$$
\begin{aligned}
x(k+1) &= (\Phi - \Gamma L)\, x(k) + m\Gamma u_c(k) \\
y(k) &= Cx(k)
\end{aligned}
\tag{4}
$$

Pole placement by state feedback

Assume the desired performance of the control system is given as a set of desired closed loop poles $p_1, p_2, \ldots, p_n$, corresponding to the desired characteristic polynomial

$$a_c(z) = (z - p_1)(z - p_2)\cdots(z - p_n) = z^n + \alpha_1 z^{n-1} + \cdots \alpha_n. \tag{5}$$

With state feedback we get the the closed-loop system

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)\, x(k) + m\Gamma u_c(k) \\ y(k) &= Cx(k) \end{aligned} \tag{6}$$

with characteristic equation

$$\det\left(zI - (\Phi - \Gamma L)\right) = z^n + \beta_1(l_1, \ldots, l_n) z^{n-1} + \cdots \beta_n(l_1, \ldots, l_n). \tag{7}$$

Equate the coefficients in (5) and (7) to get the system of equations

$$\begin{aligned} \beta_1(l_1, \ldots, l_n) &= \alpha_1 \\ \beta_2(l_1, \ldots, l_n) &= \alpha_2 \\ &\vdots \\ \beta_n(l_1, \ldots, l_n) &= \alpha_n \end{aligned}$$

Pole placement by state feedback, contd.

The system of equations

$$\begin{aligned}
\beta_1(l_1, \ldots, l_n) &= \alpha_1 \\
\beta_2(l_1, \ldots, l_n) &= \alpha_2 \\
&\vdots \\
\beta_n(l_1, \ldots, l_n) &= \alpha_n
\end{aligned}$$

is always linear in the unknown controller parameters, so it can be written

$$AL^{\mathrm{T}} = \alpha,$$

Where $\alpha^{\mathrm{T}} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$.

Pole placement and reacability

It can be shown that the controllability matrix W_c is a factor of the matrix A

$$A = \bar{A} W_c.$$

Hence, in general the system of equations

$$\bar{A} W_c L^{\mathrm{T}} = \alpha \tag{8}$$

has a solution only if W_c is invertible, i.e. the system is *reachable*.

Note that equation (8) can still have a solution for unreachable systems if α is in the *column space* of A, i.e. α can be written

$$\alpha = b_1 A_{:,1} + b_2 A_{:,2} + \cdots + b_m A_{:,m}, \;\; m < n$$