State space models

Kjartan Halvorsen

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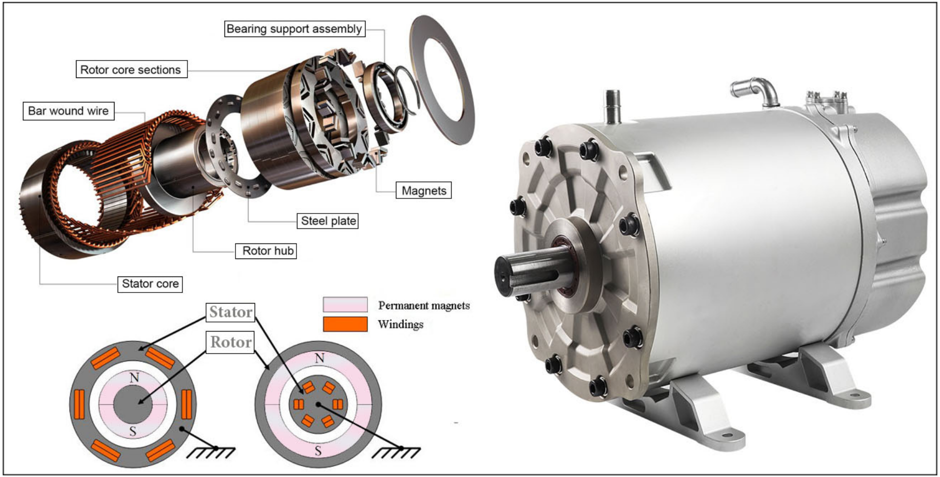
Obtain state-space model from discrete-time pulse-transfer function

The permanent magnet synchronous motor



Permanent Magnet Synchronous Motor Construction

The PMSM

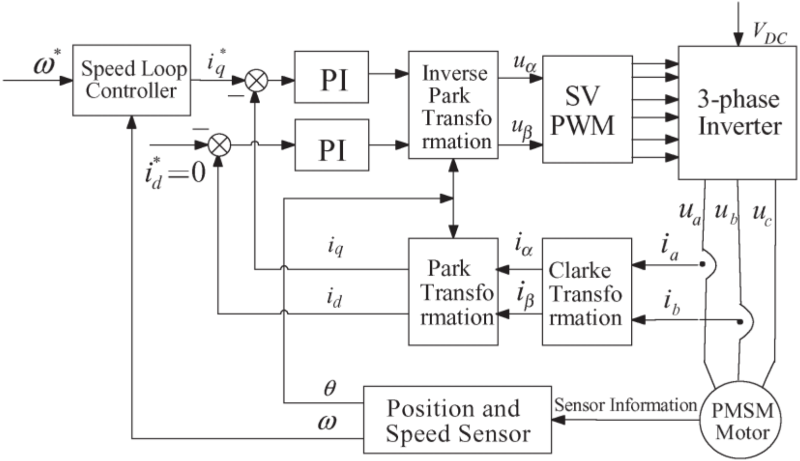


Fig. 1. Block diagram of the PMSM control system.

De Liu and Li "Speed control for PMSM servo system", IEEE Transactions on Industrial Electronics, 2012.

Identified model

Two poles, two zeros, one delay

$$u(k) \longrightarrow \boxed{\frac{1}{z}} \longrightarrow \boxed{\frac{b_0 z^2+b_1 z+b_2}{z^2+a_1 z+a_2}} \longrightarrow y(k)$$

$$\Updownarrow$$

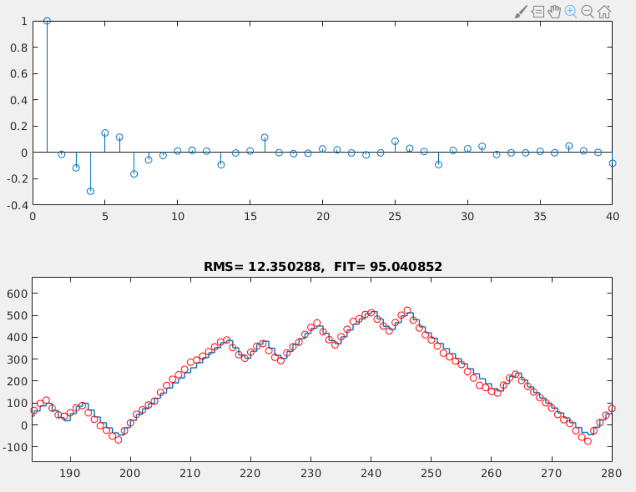
$$u(k) \longrightarrow \boxed{\frac{b_0 z^2+b_1 z+b_2}{z^2+a_1 z+a_2}} \longrightarrow \boxed{\frac{1}{z}} \longrightarrow y(k)$$

$$\Updownarrow$$

$$u(k) \longrightarrow \boxed{\frac{b_0 z^2+b_1 z+b_2}{z(z^2+a_1 z+a_2})} \longrightarrow y(k)$$

Identified model

$$H(z) = \frac{6.91z^2 + 16.48z - 17.87}{z(z^2 - 1.766z + 0.7665)} = \frac{6.91(z+3.19)(z-0.81)}{z(z-0.998)(z-0.768)}$$



From pulse-transfer function to state space model

$$u(k) \longrightarrow \boxed{H(z) = \frac{b_0 z^2 + b_1 z + b_2}{z(z^2 + a_1 z + a_2)}} \longrightarrow y(k)$$

$$\Updownarrow$$

$$u(k) \longrightarrow \boxed{\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= C x(k) \end{aligned}} \longrightarrow y(k)$$

Canonical forms

Given pulse-transfer function

$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

Find a representation in state space form

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned}$$

Canonical forms

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$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

Find a representation in state space form

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= C x(k) \end{aligned}$$

- Controlable canonical form
- Observable canonical form

Controlable canonical form

Given pulse-transfer function

$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

$$x(k+1) = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} x(k)$$

Observable canonical form

Given pulse-transfer function

$$H(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}.$$

$$x(k+1) = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k)$$

Canonical forms

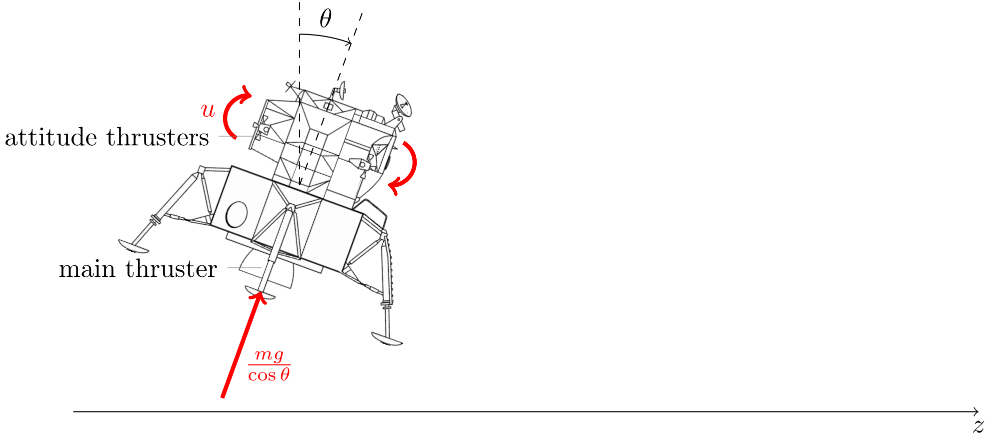
Activity Find the controlable abd observable canonical forms for the pulse-transfer function of the motor. Answer on Canvas (questions 1 and 2 on today's exercises).

$$H(z) = \frac{6.91z^2 + 16.48z - 17.87}{z(z^2 - 1.766z + 0.7665)} = \frac{6.91(z + 3.19)(z - 0.81)}{z(z - 0.998)(z - 0.768)}$$

Discrete-time state-space from continuous-time state space

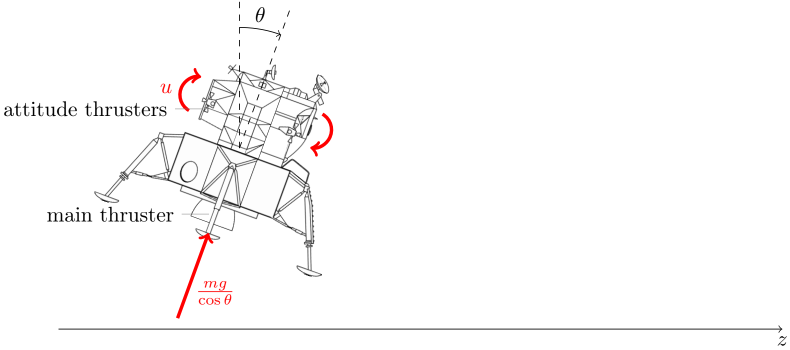
A.k.a. discretization

Example - the Apollo lunar module



$$u(t) \longrightarrow \boxed{\frac{k_1}{s}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{k_2} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

Example - the Apollo lunar module

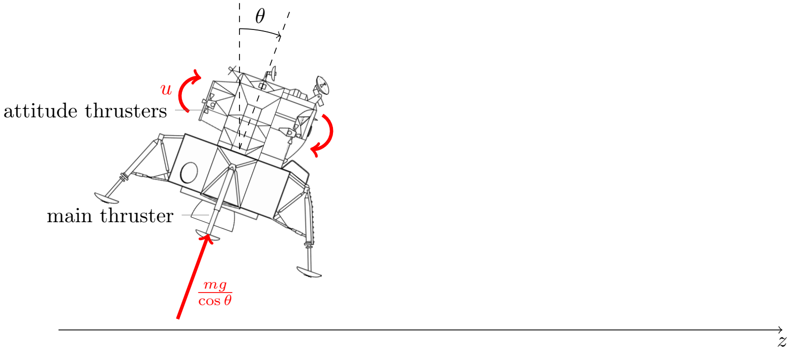


$$u(t) \longrightarrow \boxed{\frac{k_1}{s}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{k_2} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

Activity Which is the transfer function of the system?

$$1: \quad G(s) = \frac{k_1 k_2}{s^2} \qquad 2: \quad G(s) = \frac{k_1 k_2}{s(s^2+1)} \qquad 3: \quad G(s) = \frac{k_1 k_2}{s^3}$$

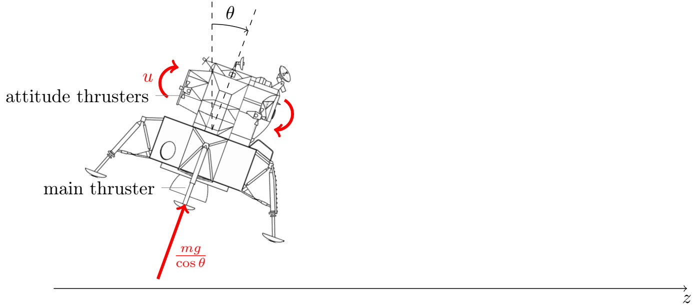
Example - the Apollo lunar module

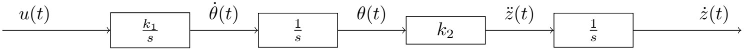


$$u(t) \longrightarrow \boxed{\frac{k_1}{s}} \xrightarrow{\dot{\theta}(t)} \boxed{\frac{1}{s}} \xrightarrow{\theta(t)} \boxed{k_2} \xrightarrow{\ddot{z}(t)} \boxed{\frac{1}{s}} \xrightarrow{\dot{z}(t)}$$

Activity What sensors are needed by the control system?

Example - the Apollo lunar module





State variables: $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} \dot{\theta} & \theta & \dot{z} \end{bmatrix}^T$. With the dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2\theta = k_2 x_2 \end{cases}$$

Example - the Apollo lunar module

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Activity Fill the matrix A and vector B.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}}_{B} u$$

Example - the Apollo lunar module

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State variables: $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} \dot{\theta} & \theta & \dot{z} \end{bmatrix}^T$. With dynamics

$$\begin{cases} \dot{x}_1 = \ddot{\theta} = k_1 u \\ \dot{x}_2 = \dot{\theta} = x_1 \\ \dot{x}_3 = \ddot{z} = k_2 \theta = k_2 x_2 \end{cases}$$

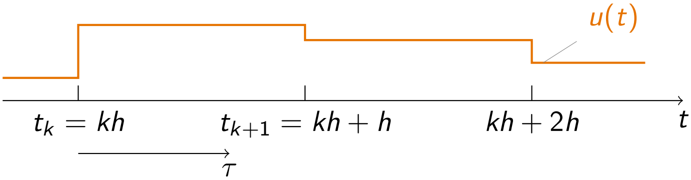
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

Discretizing a continuous-time state-space model

Discretización

The general solution to a linear, continuous-time state-space system

$$x(t_k+\tau) = \mathrm{e}^{A(\tau)}x(t_k) + \int_0^{\tau} \mathrm{e}^{As}Bu\big((t_k+\tau)-s\big)ds$$



$$\begin{aligned} x(kh+h) &= \mathrm{e}^{Ah}x(kh) + \int_0^{h} \mathrm{e}^{As}Bu(kh+h-s)ds \\ &= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^{h} \mathrm{e}^{As}Bds\right)}_{\Gamma(h)}u(kh) \end{aligned}$$

Discretization - The matrix exponential

Matriz A cuadrada. Variable t escalar.

$$e^{At} = 1 + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Transformada de Laplace:

$$\mathcal{L}\left\{e^{At}\right\} = (sI - A)^{-1}$$

Discretization - example

$$
\begin{aligned}
x(kh+h) &= \mathrm{e}^{Ah}x(kh) + \int_0^h \mathrm{e}^{As}Bu(kh+h-s)ds \\
&= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h \mathrm{e}^{As}Bds\right)}_{\Gamma(h)}u(kh)
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}, \quad A^3 = 0
$$

Entonces,

$$
\begin{aligned}
\Phi(h) &= \mathrm{e}^{Ah} = 1 + Ah + A^2h^2/2 + \cdots \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & k_2 & 0 \end{bmatrix} h + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \frac{h^2}{2} = \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2k_2}{2} & hk_2 & 1 \end{bmatrix}
\end{aligned}
$$

Discretization - example

$$x(kh+h) = \mathrm{e}^{Ah}x(kh) + \int_0^h \mathrm{e}^{As}Bu(kh+h-s)ds$$

$$= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)} x(kh) + \underbrace{\left(\int_0^h \mathrm{e}^{As}Bds\right)}_{\Gamma(h)} u(kh)$$

$$\mathrm{e}^{As}B = \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{s^2 k_2}{2} & sk_2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ s \\ \frac{k_2 s^2}{2} \end{bmatrix}$$

$$\Gamma(h) = \int_0^h \mathrm{e}^{As}Bds = k_1 \int_0^h \begin{bmatrix} 1 \\ s \\ \frac{k_2 s^2}{2} \end{bmatrix} ds = k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2 h^3}{6} \end{bmatrix}$$

Discretization - example

$$
\begin{aligned}
x(kh+h) &= \mathrm{e}^{Ah}x(kh) + \int_0^h \mathrm{e}^{As}Bu(kh+h-s)ds \\
&= \underbrace{\mathrm{e}^{Ah}}_{\Phi(h)}x(kh) + \underbrace{\left(\int_0^h \mathrm{e}^{As}Bds\right)}_{\Gamma(h)}u(kh) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2k_2}{2} & hk_2 & 1 \end{bmatrix} x(kh) + k_1 \begin{bmatrix} h \\ \frac{h^2}{2} \\ \frac{k_2h^3}{6} \end{bmatrix} u(kh)
\end{aligned}
$$

Discretization - exercise

Activity Discretize the system (question 3 on today's exercises on Canvas)

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$