Matricula and name:

Computerized Control Final exam (28%)

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November 28, 2019

Time Thursday November 28 19:10 — 21.55

Place 5305

Permitted aids The single colored page with your own notes, table of Laplace transforms, calculator

All answers should be readable and well motivated (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam.

Good luck!

Problem 1 The linearized inverted-pendulum model is used in the control design for many types of systems, such as rockets and segways.

(a) Segway (b) Saturn rocket (c) Cart–pendulum system

Figure 2.5: Balance systems. (a) Segway Personal Transporter, (b) Saturn rocket and (c) inverted pendulum on a cart. Each of these examples uses forces at the bottom of the system to keep it upright.

From Åström & Murray "Feedback systems" Princeton University Press, 2008.

In the continuous-time domain the system has the transfer function

$$G(s) = \frac{\omega_0^2}{s^2 - \omega_0^2}. \tag{1}$$

It can also be described in discrete-time as the state-space model

$$\begin{aligned} x(k+1) &= \underbrace{\begin{bmatrix} 2\cosh(\omega_0 h) & 1 \\ -1 & 0 \end{bmatrix}}_{\Phi} x(k) + \underbrace{\begin{bmatrix} \cosh(\omega_0 h) - 1 \\ \cosh(\omega_0 h) - 1 \end{bmatrix}}_{\Gamma} u(k) \\ y(k) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(k) \end{aligned} \tag{2}$$

Note that $\cosh(\omega_0 h) = \frac{1}{2}(\mathrm{e}^{\omega_0 h} + \mathrm{e}^{-\omega_0 h})$.

(a) Assume that we want to control the inverted pendulum using linear state feedback so that the closed-loop system has a step-response as shown in figure 1. Determine a suitable sampling period h for the system, in terms of ω_0.

Answer and motivation for sampling period h:

(b) Determine the **poles** of the continuous-time system (1) and the discrete-time model (2),

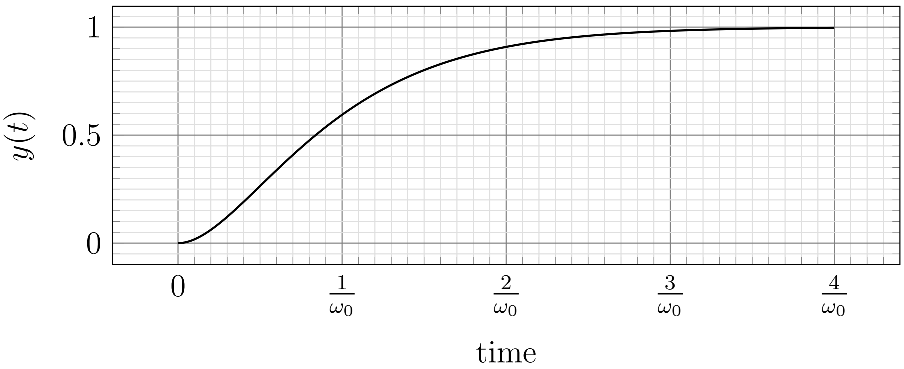


Figure 1: Step-response of desired closed-loop system (in continuous-time)

making use of the sampling period h you decided in (a). Also, **mark** the poles in figure 2.

Calculation of continuous- and discrete poles:

(c) Is the discrete-time state-space model (2) **observable**?

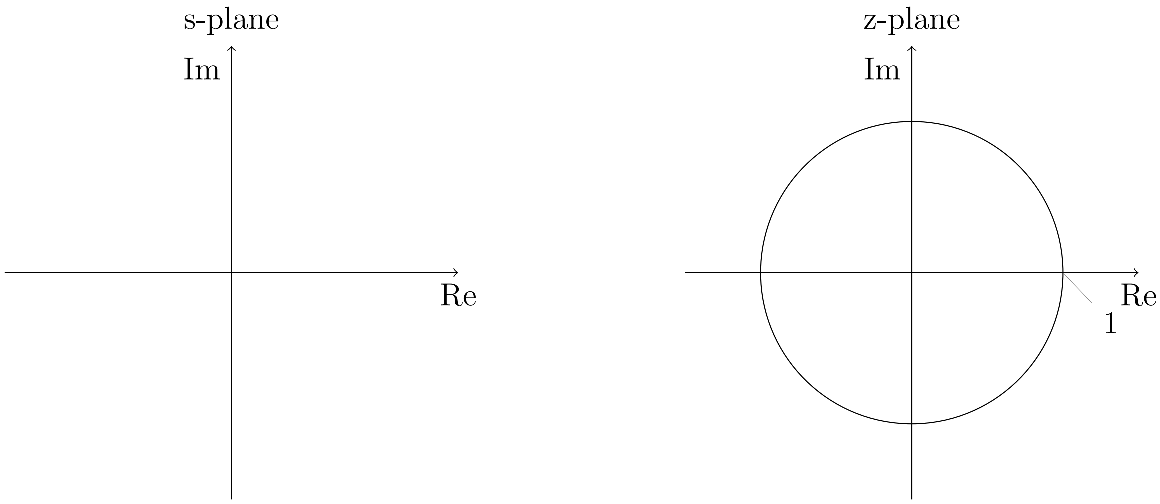


Figure 2: Mark the poles of the continuous-time system (1) and the discrete-time system (2).

Calculations:

(d) for a specific choice of sampling period h we obtain the following discrete-time state-space model for the linearized inverted pendulum

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2.04 & 1 \\ -1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned} \tag{3}$$

The following control law has been designed

$$u(k) = -Lx(k) + l_0 y_{ref}(k) = -19x_1(k) - 15x_2(k) + 2.7y_{ref}(k),$$

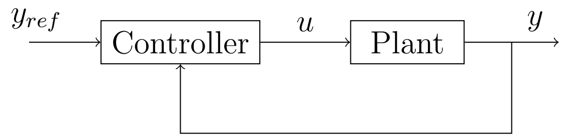
which gives a closed-loop system with step-response close to the desired response in figure 1. However, only measurements of the output $y(k)$ of the system are available, so it is necessary to use an observer. Which of the following state-space expressions is the correct one for the observer? No motivation required.

1. $\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k)$
2. $\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + K(y(k) - \hat{x}(k))$
3. $\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) - K\hat{x}(k)$
4. $\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + Ky(k) - KC\hat{x}(k)$

(e) With the state-space model (3), determine an observer gain vector such that the observer has all its poles in the origin (deadbeat observer).

Calculations:

(f) The output feedback controller when it is designed (gain vectors L and K determined) corresponds to a discrete-time dynamical system with two input signals and one output signal. The input signals are the reference signal $y_{ref}(k)$ and the feedback signal $y(k)$, and the output signal is the control signal $u(k)$.



On state space form this controller corresponds to the system

$$\begin{aligned} \hat{x}(k+1) &= (\Phi - \Gamma L - KC)\,\hat{x}(k) + l_0 \Gamma y_{ref}(k) + K y(k) \\ u(k) &= -L\hat{x}(k) + l_0 y_{ref}(k) \end{aligned} \tag{4}$$

This system can of course also be written on transfer-function form. For the particular system in this exercise, the controller becomes

$$u(k) = F_f(\mathrm{q}) y_{ref}(k) - F_b(\mathrm{q}) y(k) = \frac{2.7\,\mathrm{q}^2 + 1.8\,\mathrm{q} + 0.8}{\mathrm{q}^2 + 1.365\,\mathrm{q} + 0.68} y_{ref}(k) - \frac{24.2\,\mathrm{q} - 11.1}{\mathrm{q}^2 + 1.365\,\mathrm{q} + 0.68} y(k). \tag{5}$$

Write the controller as a **difference equation** with the newest value of u by it self on the left-hand side, just as you would write it in order to implement the controller in code (for instance on an arduino).

Calculations:

Problem 2 For each of the four pulse-transfer functions below, determine which of the step-responses in figure 3 it corresponds to.

$$G_1(z) = \frac{0.02(z+0.95)}{(z-0.8)(z-1.2)}$$
$$G_2(z) = \frac{0.08(z+0.95)}{(z-0.6)^2}$$
$$G_3(z) = \frac{0.02(z+0.95)}{(z-0.8+0.5i)(z-0.8-0.5j)}$$
$$G_4(z) = \frac{z+0.95}{(z-1)(z-0.8)}$$

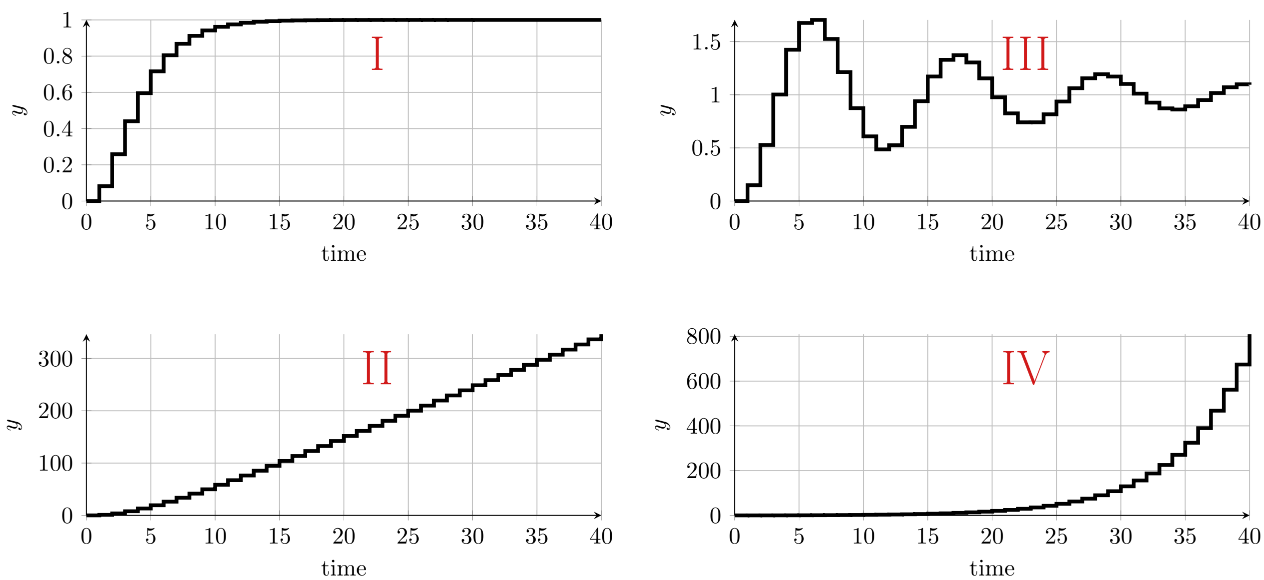


Figure 3: Exercise 2 Match the step-responses with the pulse-transfer functions.

Answer and motivation:

Solutions

Problem 1

(a) Choosing a suitable sampling period is based on the speed of either the open-loop system or the desired closed-loop system. Here we are given a desired step-response of the closed-loop system. One common rule-of-thumb is that we should have 4 to 10 samples in one rise-time. The rise-time is from 10% to 90% of the final value which gives (approximately) $t_r = \frac{2}{\omega_0} - \frac{0.3}{\omega_0} = \frac{1.7}{\omega_0}$. Since the open-loop system is unstable, it is good to be cautious and choose a short sampling period, for instance

$$h = \frac{0.2}{\omega_0}.$$

(b) For the continuous-time system the characteristic equation is $s^2 - \omega_0^2 = 0$ with solutions

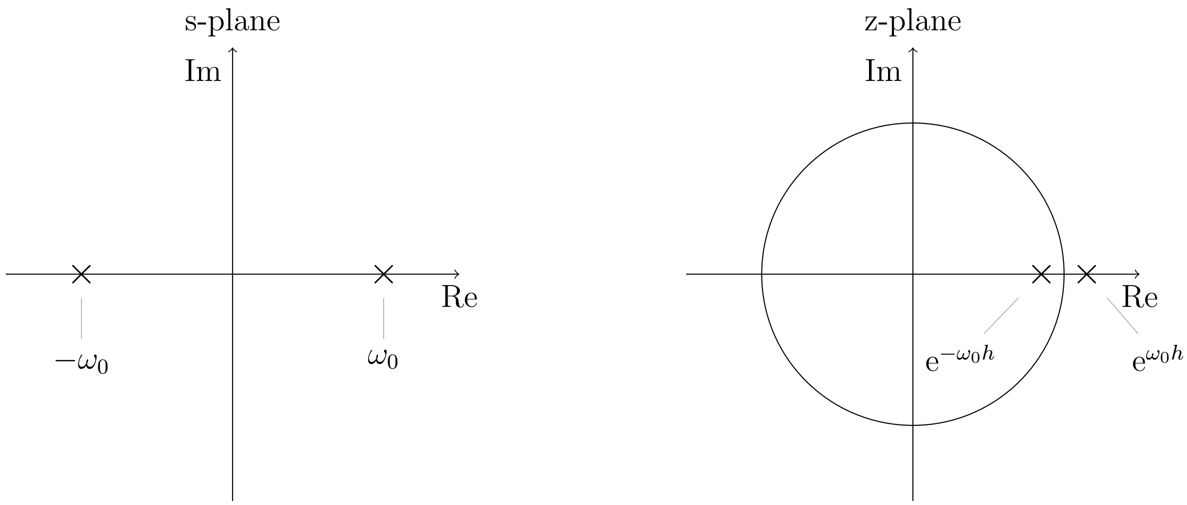
$$s = \pm\omega_0.$$

For the discrete-time state-space system the poles are the eigenvalues of the Φ-matrix. The characteristic equation becomes

$$\begin{aligned}
\det\left(zI - \Phi\right) &= 0 \\
\det \begin{bmatrix} z - 2\cosh(\omega_0 h) & -1 \\ 1 & z \end{bmatrix} &= 0 \\
(z - 2\cosh(\omega_0 h))z + 1 &= 0 \\
z^2 - 2\cosh(\omega_0 h)z + 1 &= 0 \\
z^2 - \left(\mathrm{e}^{\omega_0 h} + \mathrm{e}^{-\omega_0 h}\right)z + 1 &= 0 \\
\left(z - \mathrm{e}^{\omega_0 h 0}\right)\left(z - \mathrm{e}^{-\omega_0 h}\right) &= 0.
\end{aligned}$$

With solutions

$$z_1 = \mathrm{e}^{\omega_0 h} = \mathrm{e}^{0.2} \approx 1.22, \qquad z_1 = \mathrm{e}^{-\omega_0 h} = \mathrm{e}^{-0.2} \approx 0.82.$$



(c) To check for observability, form the observability matrix which for a second-order system is

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix},$$

and check that it is non-singular (that its determinant is not zero). We get

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2\cosh(\omega_0 h) & 1 \end{bmatrix},$$

with determinant

$$\det W_o = 1 \neq 0,$$

so the system is **observable**. In fact, the system is on observable canonical form, which means it must be observable.

(d) The correct expression for the observer is

$$hatx(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + Ky(k) - KC\hat{x}(k).$$

(e) The poles of the observer are given by the solutions to the characteristic equation

$$\det\big(zI - (\Phi - KC)\big) = 0.$$

Since $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^T$,

$$KC = \begin{bmatrix} k_1 & 0 \\ k_2 & 0 \end{bmatrix}$$

and

$$\Phi - KC = \begin{bmatrix} 2\cos(\omega_0 h) - k_1 & 1 \\ -1 - k_2 & 0 \end{bmatrix}$$

the characteristic polynomial becomes

$$\det\big(zI - (\Phi - KC)\big) = \det \begin{bmatrix} z - 2\cosh(\omega_0 h) + k_1 & -1 \\ 1 + k_2 & z \end{bmatrix} = z^2 + \big(k_1 - 2\cosh(\omega_0 h)\big)z + 1 + k_2.$$

Compare this with the desired characteristic polynomial for a deadbeat observer, which is z^2, to get the following equations for the observer gains

$$\begin{aligned} k_1 - 2\cosh(\omega_0 h) &= 0 \\ 1 + k_2 &= 0 \end{aligned}$$

with the obvious solution

$$k_1 = 2\cosh(\omega_0 h) = 2.04, \qquad k_2 = -1.$$

(f) The difference equation becomes

$$(\mathrm{q}^2 + 1.365\,\mathrm{q} + 0.68)u(k) = (2.7\,\mathrm{q}^2 + 1.8\,\mathrm{q} + 0.8)y_{ref}(k) - (24.2\,\mathrm{q} - 11.1)y(k)$$

$$u(k{+}2) + 1.365u(k{+}1) + 0.68u(k) = 2.7y_{ref}(k{+}2) + 1.8y_{ref}(k{+}1) + 0.8y_{ref}(k) - 24.2y(k{+}1) + 11.1y(k)$$

Shifting the difference equation back in time with two sampling period (multiplying both sides with q^{-2}) and rearranging gives

$$u(k) = -1.365u(k{-}1) - 0.68u(k{-}2) + 2.7y_{ref}(k) + 1.8y_{ref}(k{-}1) + 0.8y_{ref}(k{-}2) - 24.2y(k{-}1) + 11.1y(k{-}2).$$

This is what is implemented in the computer.

Problem 2 The idea is to figure out the poles of the four systems, and from this determine how the system should respond.

The system $G_1(z)$ poles in $z = 0.8$ and $z = 1.2$, the latter of which is outside the unit circle. This gives a diverging response. The system $G_2(z)$ is critically damped with two poles in $z = 0.6$. This has a stable, fast response with no overshoot. System $G_3(z)$ has complex-conjugated poles in $z = 0.8 \pm i0.5$ which are inside the unit circle, but with poor damping. This gives an oscillating, stable response. Finally, $G_4(z)$ has poles in $z = 1$ and $z = 0.8$, and so it contains an integration, and the response will grow with time. The analysis leads to the pairing: $G_1(z)$ — **IV**, $G_2(z)$ — **I**, $G_3(z)$ — **III**, $G_4(z)$ — **II**.