Computerized Control - difference equations, LSI, impulse response

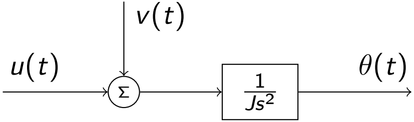
Kjartan Halvorsen

2020-06-30

Position control of a diskdrive arm

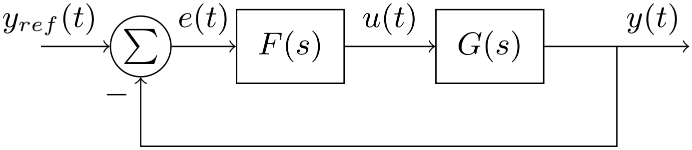
"Laptop-hard-drive-exposed" by Evan-Amos - Own work. Licensed under CC BY-SA 3.0 via Commons

$$J\ddot{\theta}(t) = u(t) + v(t)$$



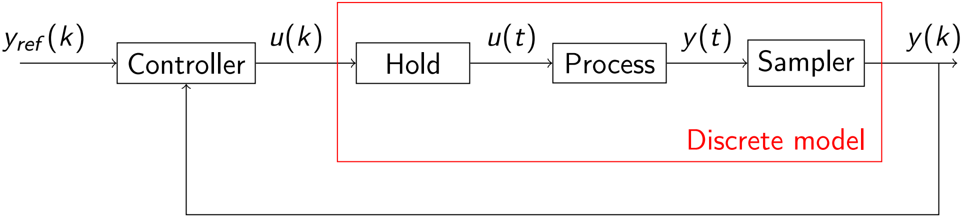
Two approaches to designing a discrete-time controller

1. Do design the controller in the continuous-time domain (methods from control engineering class). Then discretize the continuous-time controller.



$$F_d(z) \approx F(s)$$

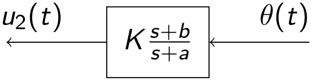
2. Determine discrete-time model of the plant. Do design in discrete-time domain.



Difference equations

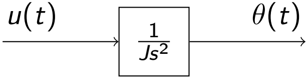
We saw two difference equations in the introductory lecture

1. Discretized version of continuous-time lead-compensator $F(s) = K\frac{s+b}{s+a}$



$$u_2(kh+h) - (1-ah)u_2(kh) = K\theta(kh+h) - K(1-bh)\theta(kh)$$

2. Discretized version of the double integrator $G(s) = \frac{1}{Js^2}$

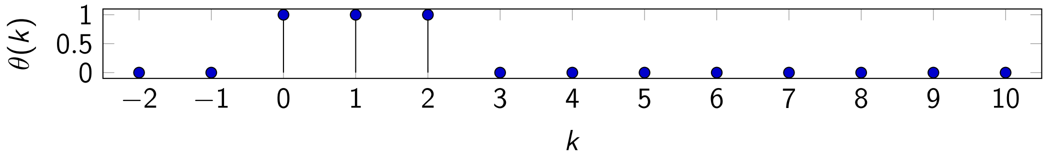


$$\theta(k+2) - 2\theta(k+1) + \theta(k) = \frac{h^2}{J}u(k)$$

Direct solution by iteration

The difference equation can be used as a recipe to calculate the solution numerically

Example Calculate the response of the lead-compensator
$u_2(kh+h) - (1-ah)u_2(kh) = K\theta(kh+h) - K(1-bh)\theta(kh)$ with $a = 8,\ b = 1,$
$h = 0.1,\ K = 1$ to the signal below



$$u_2(k+1) = 0.2u_2(k) + \theta(k+1) - 0.9\theta(k)$$

$$u_2(0) = 0.2u_2(-1) + \theta(0) - 0.9\theta(-1) = 0 + 1 - 0 = 1$$

$$u_2(1) = 0.2u_2(0) + \theta(1) - 0.9\theta(0) = 0.2 + 1 - 0.9 = 0.3$$

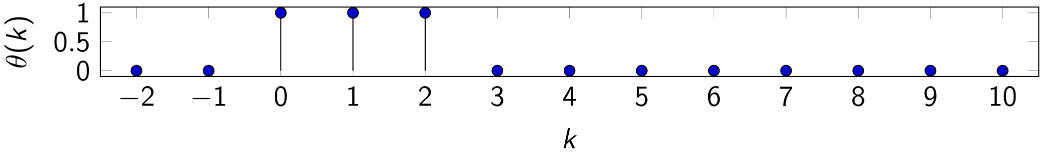
$$u_2(2) = 0.2u_2(1) + \theta(2) - 0.9\theta(1) = 0.06 + 1 - 0.9 = 0.16$$

$$u_2(3) = 0.2u_2(2) + \theta(3) - 0.9\theta(2) = 0.032 + 0 - 0.9 = -0.868$$

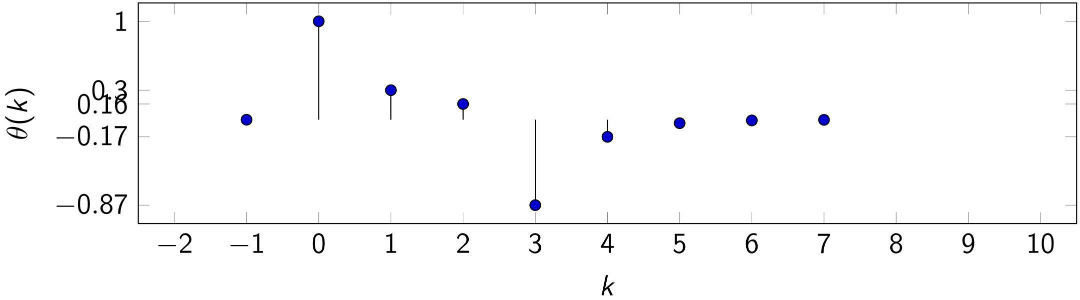
$$u_2(4) = 0.2u_2(3) + \theta(4) - 0.9\theta(3) = -0.1736 + 0 - 0 = -0.1736$$

Direct solution by iteration

Input



Output



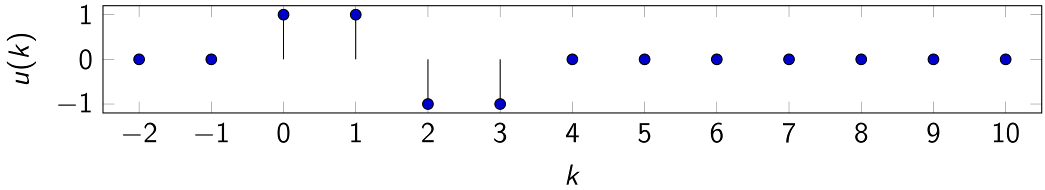
Do on your own: Solution to the discrete double-integrator

Activity Calculate the response of the double-integrator

$$\theta(k+2) - 2\theta(k+1) + \theta(k) = \frac{h^2}{J} u(k)$$

with $h = 1,\ J = 1$

and initial values zero: $\theta(k) = 0,\ k \le 0$ to the signal below



Sketch your solution as a graph on paper, photograph and send as PM on Remind.

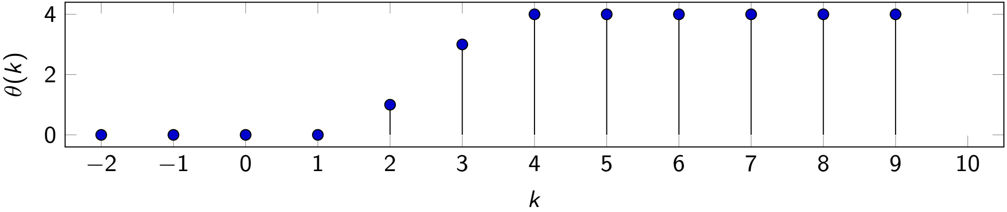
Do on your own: Solution to the discrete double-integrator

Solution

$$\theta(k+2) = 2\theta(k+1) - \theta(k) + u(k)$$

and initial values zero: $\theta(k) = 0, \; k \leq 0$

$$\begin{aligned}
\theta(0) &= 2\theta(-1) - \theta(-2) + u(-2) = 0 \\
\theta(1) &= 2\theta(0) - \theta(-1) + u(-1) = 0 \\
\theta(2) &= 2\theta(1) - \theta(0) + u(0) = 0 - 0 + 1 = 1 \\
\theta(3) &= 2\theta(2) - \theta(1) + u(1) = 2 - 0 + 1 = 3 \\
\theta(4) &= 2\theta(3) - \theta(2) + u(2) = 6 - 1 - 1 = 4 \\
\theta(5) &= 2\theta(4) - \theta(3) + u(3) = 8 - 3 - 1 = 4 \\
\theta(6) &= 2\theta(5) - \theta(4) + u(4) = 8 - 4 + 0 = 4 \\
\theta(7) &= 2\theta(6) - \theta(5) + u(5) = 8 - 4 + 0 = 4
\end{aligned}$$



A short intermezzo - the shift operator

Recall the definition of the shift operator q:

$$\mathrm{q}\, f(k) = f(k+1), \quad \text{for double-infinite sequence } f(k)$$

Activity Show that the discretized lead-compensator

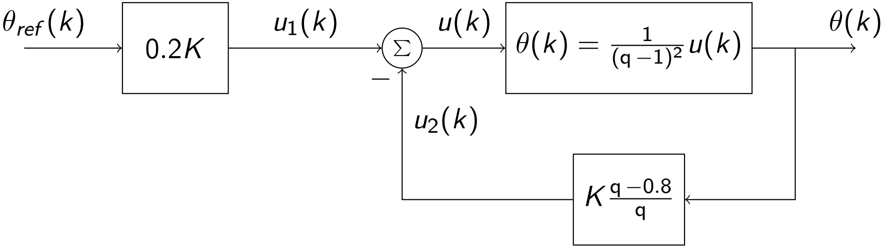
$$u_2(k+1) = 0.2u_2(k) + \theta(k+1) - 0.9\theta(k)$$

can be written using the shift operator as

$$u_2(k) = \frac{\mathrm{q} - 0.9}{\mathrm{q} - 0.2}\theta(k).$$

Simple discrete-time controller for the disk drive arm

$J = 1$ and $h = 1$ for simplicity.



Combine

$$\theta(k+2) - 2\theta(k+1) + \theta(k) = u(k)$$

with the control-law

$$\begin{aligned} u(k) &= u_1(k) - u_2(k) = 0.2K\theta_{ref}(k) - K(1 - 0.8\,\mathrm{q}^{-1})\theta(k) \\ &= 0.2K\theta_{ref}(k) - K\big(\theta(k) - 0.8\theta(k-1)\big) \end{aligned}$$

to get

$$\theta(k+2) - 2\theta(k+1) + \theta(k) = 0.2K\theta_{ref}(k) - K\big(\theta(k) - 0.8\theta(k-1)\big)$$

Simple discrete-time controller for the disk drive arm, contd

$$\theta(k+2) - 2\theta(k+1) + \theta(k) = 0.2K\theta_{ref}(k) - K\big(\theta(k) - 0.8\theta(k-1)\big)$$
$$\theta(k+2) - 2\theta(k+1) + (1+K)\theta(k) - 0.8K\theta(k-1) = 0.2K\theta_{ref}(k)$$

The solution consists of the sum of the homogenous solution and the particular solution

$$\theta(k) = \theta_H(k) + \theta_P(k)$$

Homogenous solution by ansatz: $\theta(k) = \alpha^k$

$$\theta(k+2) - 2\theta(k+1) + (1+K)\theta(k) - 0.8K\theta(k-1) = 0$$
$$\alpha^{k+2} - 2\alpha^{k+1} + (1+K)\alpha^k - 0.8K\alpha^{k-1} = 0$$
$$(\alpha^3 - 2\alpha^2 + (1+K)\alpha - 0.8K)\alpha^{k-1} = 0$$

A non-trivial solution $\alpha \neq 0$ requires that the paranthesis is zero:

Characteristic equation: $\alpha^3 - 2\alpha^2 + (1+K)\alpha - 0.8K = 0$

Simple discrete-time controller for the disk drive arm, contd

The characteristic equation

$$\alpha^3 - 2\alpha^2 + (1 + K)\alpha - 0.8K = 0$$

is of third order and will therefore have three solutions α_1, α_2 and α_3. These are the POLES of the closed-loop system!

The homogenous solution will consist of the linear combination

$$\theta_H(k) = c_1\alpha_1^k + c_2\alpha_2^k + c_3\alpha_3^k.$$

The solution will be stable if and only if

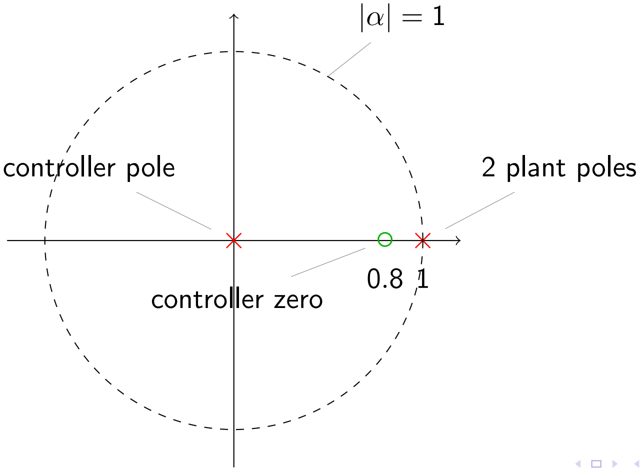
$$|\alpha_j| < 1, \quad j = 1, 2, 3$$

Simple discrete-time controller for the disk drive arm, contd

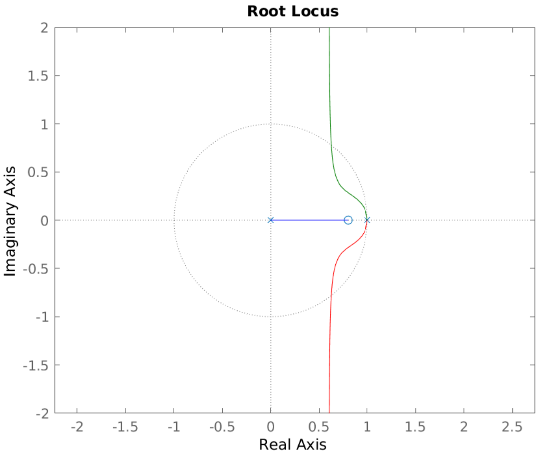
Studying the characteristic equation using a root locus

$$\alpha(\alpha - 1)^2 + K(\alpha - 0.8) = 0$$

Group activity Complete the root locus and choose a set of reasonable poles for the closed-loop system!



Simple discrete-time controller for the disk drive arm, solution



Sampled systems are not time invariant

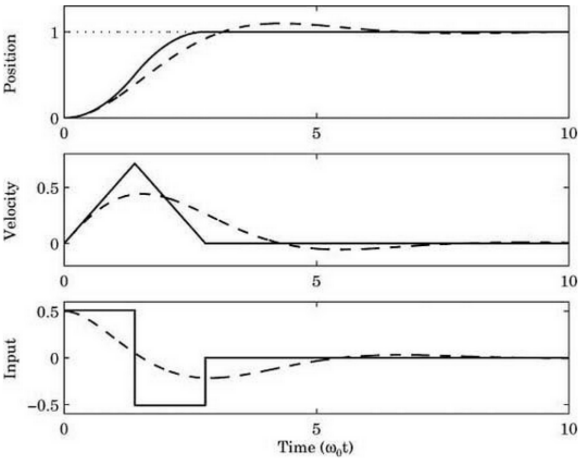


Figure 1.9 Simulation of the disk arm servo with deadbeat control (solid). The sampling period is $h = 1.4/\omega_0$. The analog controller from Example 1.2 is also shown (dashed).

The discrete-time causal linear shift-invariant system

$$u(k) \longrightarrow \boxed{g} \longrightarrow y(k)$$

$g(k)$ is called the weighting sequence.

General (non-causal) LSI

$$y(k) = g * u = \sum_{n=-\infty}^{\infty} g(n)u(k-n)$$

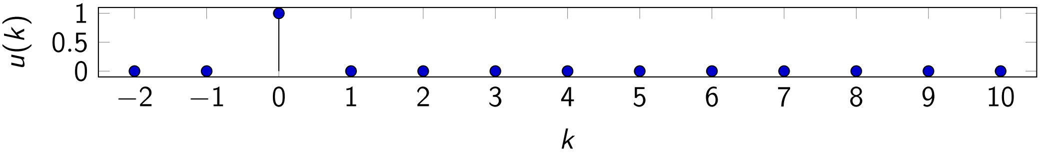
Causal LSI

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

The discrete-time causal linear shift-invariant system

Impulse response

If input signal is a unit pulse



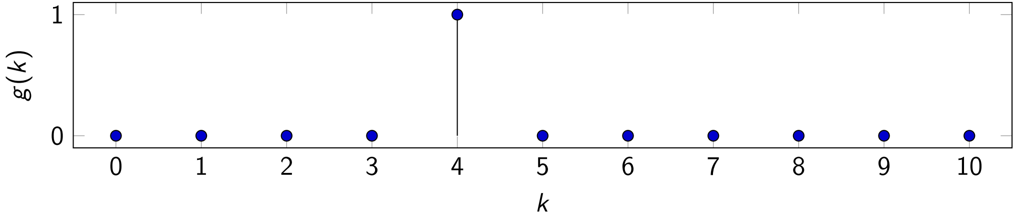
$$y(k) = \sum_{n=0}^{\infty} g(n)\delta(k-n) = g(k)$$

The output of a causal, linear discrete-time system is a weighted sum of previous input

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n)$$

The weighting sequence $g(k)$ is the impulse response of the system.

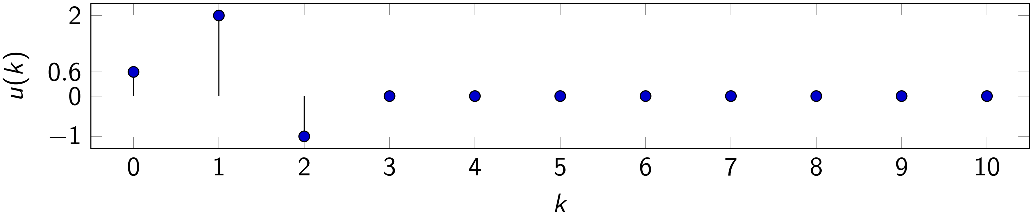
What if the weighting sequence looks like this



$$y(k) =$$

Linearity, shift invariance and the impulse response

The input signal



Can be written

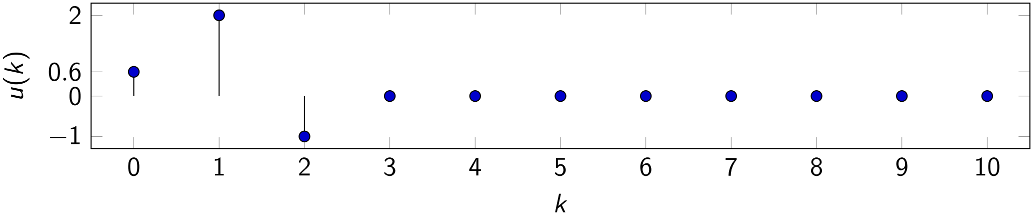
$$u(k) = 0.6\delta(k) + 2\delta(k-1) - \delta(k-2)$$

Since the system's response to a pulse is given by $g(k)$, the output signal is

$$y(k) = ?$$

Linearity, shift invariance and the impulse response

The input signal



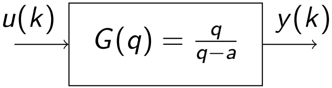
Can be written

$$u(k) = 0.6\delta(k) + 2\delta(k-1) - \delta(k-2)$$

Since the system's response to a pulse is given by $g(k)$, the output signal is

$$y(k) = 0.6g(k) + 2g(k-1) - g(k-2)$$

First order system



The system with pulse-transfer operator $G(q) = \frac{q}{q-a}$ corresponds to the difference equation

$$y(k) = G(q)u(k) \Leftrightarrow y(k) = \frac{q}{q-a}u(k)$$

$y(k+1) = ay(k) + u(k+1)$. If $a = 1$, the system is a discrete-time integrator

Impulse-response of a first order system

$$y(k+1) = ay(k) + u(k+1), \quad y(k) = 0, \; k < 0$$

$$u(k) = \delta(k) = \begin{cases} 1, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution

$$\begin{aligned} y(0) &= ay(-1) + u(0) = 1 \\ y(1) &= ay(0) + u(1) = a + 0 = a \\ y(2) &= ay(1) + u(2) = a * a + 0 = a^2 \\ &\vdots \\ y(n) &= a^n \end{aligned}$$

Impulse response of a first order system

$$y(k+1) = ay(k) + u(k+1)$$

Activity Pair the impulse response to each of the values of a

I) $a = 1$ II) $a = 2$ III) $a = 0.5$ IV) $a = -0.9$