5

Pole-Placement Design: A Polynomial Approach

5.1 Introduction

In this chapter we will discuss the same design problems as in Chapter 4 but we will use polynomial calculations instead of matrix calculations. This gives new insights and new computational methods. In addition we will be able to investigate consequences of errors in the model used to design the controller. The idea of pole placement is to find a controller that gives a closed-loop system with a specified characteristic polynomial. It is natural to explore if this can be done directly by polynomial calculations.

We start by describing a process model and a controller as input-output systems characterized by rational transfer functions. The design problem is then solved in a simple setting in Sec. 5.2. The design problem is identical to the one posed in Secs. 4.2 and 4.5. A polynomial equation is a crucial part of the solution. This equation is investigated in Sec. 5.3, where we give conditions for solvability and algorithms. In Sec. 5.4 we solve more realistic design problems. We consider cancellation of poles and zeros, separation of command signal responses and disturbance responses, and improved responses to disturbances. In Sec. 5.5 we consider the problem of modeling errors, which is much more convenient to deal with in the input-output formulation than in the state-space formulation. In Sec. 5.6 we summarize results and obtain a general design procedure. Some practical aspects are also discussed in that section.

The chapter ends with several design examples that illustrate the procedure. Control of a double integrator is discussed in Sec. 5.7, an harmonic oscillator in Sec. 5.8, and a flexible robot arm in Sec. 5.9. Many other design procedures can be expressed in terms of pole placement. This gives insight and gives a unified view, as is illustrated in Sec. 5.10.

5.2 A Simple Design Problem

We will now discuss the same simple design problem that was solved by state-space methods in Sec. 4.6, namely, to find a two-degree-of-freedom controller for a linear system with output feedback. The design problem is stated, and the solution is given and illustrated by two examples. It turns out that an algebraic equation plays a key role in the solution. The properties of this equation will be explored in the next section, where we also will resolve some technicalities.

A general discussion of the design problem was given in Sec. 4.2. It is recommended to review that section before proceeding. In this case we will consider command signal following, attenuation of load disturbances, and effects of measurement noise.

It is assumed that the system has one control variable, u, and one measured output, y, which are related by the following input-output model:

$$A(q)y(k) = B(q)u(k) \tag{5.1}$$

where $A(q)$ and $B(q)$ are polynomials in the forward-shift operator q. It is assumed that the degree of $B(q)$ is less than the degree of $A(q)$, that the polynomials $A(q)$ and $B(q)$ do not have any common factors, and that the polynomial $A(q)$ is normalized so that the coefficient of the term with the highest power in q is one. Such a polynomial is called *monic.*

The dynamics of the process has the pulse-transfer function $B(z)/A(z)$, which includes a hold circuit, an actuator, a sensor, and antialiasing filter. Recall from Sec. 2.3 that the model of (5.1) may represent a discrete-time model of a continuous-time system with a rational transfer function and an arbitrary time delay.

As in Sec. 4.5 we will assume that the disturbances are widely spaced impulses. The response of the closed-loop system can thus be judged by how well it will respond to perturbations in initial conditions of the process.

In pole-placement design it is assumed that specifications are primarily given by the closed-loop characteristic polynomial. In addition it may be specified that the controller should have certain properties, for example, integral action. The design variables are the closed-loop characteristic polynomial and the sampling period. Notice that the sampling period appears implicitly in the model (5.1).

The controller has one output, u, and two inputs: the command signal, u_c, and the measured output, y. A general linear controller can be represented by where $R(q)$, $S(q)$, and $T(q)$ are polynomials in the forward-shift operator. The polynomial $R(q)$ can be chosen so that the coefficient of the term of highest power in q is unity.

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k) \tag{5.2}$$

The control law (5.2) represents a combination of a feedforward with the pulse-transfer function $H_{ff}(z) = T(z)/R(z)$ and a feedback with the pulse-transfer function $H_{fb}(z) = S(z)/R(z)$. To have a causal controller it must be required that the degree of $R(z)$ is larger than or equal to the degrees of $S(z)$ and $T(z)$.

Solving the Design Problem

The solution of the design problem is straightforward. We will simply determine the characteristic equation of the closed-loop system and explore the conditions that it has to satisfy.

Eliminating $u(k)$ between the process model (5.1) and the controller (5.2) gives

$$\Big(A(q)R(q) + B(q)S(q)\Big)y(k) = B(q)T(q)u_c(k) \tag{5.3}$$

The characteristic polynomial of the closed-loop system is

$$A_{cl}(z) = A(z)R(z) + B(z)S(z) \tag{5.4}$$

Pole-placement design thus reduces to the algebraic problem of finding polynomials $R(z)$ and $S(z)$ that satisfy Eq. (5.4) for given $A(z)$, $B(z)$, and $A_{cl}(z)$. Equation (5.4), which plays a central role in the polynomial approach, is called the *Diophantine equation*. A general discussion of this equation will be given later. Let it suffice for now that the problem always

can be solved if the polynomials $A(z)$ and $B\ (z)$ do not have common factors.

Additional insight is obtained by comparing with the state-space solution to the design problem in Sec. 4.5. There we found that the characteristic polynomial $A_{cl}(z)$ could be factored as

$$A_{cl}(z) = A_c(z)A_o(z) \tag{5.5}$$

where $A_c(z) = \det(zI - \Phi + \Gamma L)$ and $A_o(z) = \det(zI - \Phi + KC)$. This factorization corresponds to the separation of the controller into a state feedback and an observer. For this reason we call $A_c(z)$ the *controller polynomial* and $A_o(z)$ as the *observer polynomial*. Recall that it was found in Sec. 4.3 that the arbitrary eigenvalues could be assigned to $A_c(z)$ if the system is reachable and that arbitrary eigenvalues could be assigned to $A_o(z)$ if the system is observable.

To complete the design it remains to determine the polynomial $T(z)$. To do this we consider Eq. (5.3), which tells how the system reacts to command signals. The pulse-transfer function from command signal to output is given by

$$Y(z) = \frac{B(z)T(z)}{A_{cl}(z)}U_c(z) = \frac{B(z)T(z)}{A_c(z)A_o(z)}U_c(z) \tag{5.6}$$

This equation shows that the zeros of the open-loop system are also zeros of the closed-loop system, unless the polynomials $B(z)$ and $A_{cl}(z)$ have common factors. By referring to the solution of the design problem in Sec. 4.6 it is natural to choose the polynomial $T(z)$ so that it cancels the observer polynomial $A_o(z)$. This implies that command signals are introduced in such a way that they do not generate observer errors. Hence

$$T(z) = t_0A_o(z) \tag{5.7}$$

The response to command signals is then given by

$$Y(z) = \frac{t_0B(z)}{A_c(z)}U_c(z) \tag{5.8}$$

where the parameter t_0 is chosen to obtain the desired static gain of the system. For example, to have unit gain we have $t_0 = A_c(1)/B(1)$.

Summary

We have thus obtained the following design procedure.

ALGORITHM 5.1 SIMPLE POLE-PLACEMENT DESIGN

Data: A process model is specified by the pulse-transfer function $B(z)/A(z)$, where $A(z)$ and $B\ (z)$ do not have any common factors. Specifications are given in terms of a desired closed-loop characteristic polynomial $A_{cl}(z)$.

Step 1. Find polynomials $R(z)$ and $S(z)$, such that $\deg S\ (z) \le \deg R(z)$, which satisfy the equation

$$A(z)\ R\ (z) + B(z)\ S(z) = A_{cl}(z)$$

Step 2. Factor the closed-loop characteristic polynomial as $A_{cl}\ (z) = A_c\ (z)\ A_o(z)$, where $\deg A_o(z) \le \deg R(z)$, and choose

$$T\ (z) = t_0A_o(z)$$

where $t_0 = A_c\ (1)/B\ (1)$. The control law is

$$R\ (q)\ u(k) = T(q)\ u_c\ (k) - S(q)\ y(k)$$

and the response to command signals is given by

$$A_c\ (q)\ y(k) = t_0\ B\ (q)\ u_c\ (k)$$

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There are several details that have to be investigated. The most important is the solution of the Diophantine equation (5.4). Before doing this we will, however, consider an example.

Example 5.1 Control of a double integrator

For the double integrator we have

$$A(z) = (z-1)^2$$
$$B(z) = \frac{h^2}{2}(z+1)$$

and the Diophantine equation (5.4) becomes

$$(z^2 - 2z + 1)R(z) + \frac{h^2}{2}(z+1)S(z) = A_{cl}(z)$$

The closed-loop characteristic polynomial A_{cl} is a design parameter. Both its degree and its parameters will be selected to achieve the design goals. It is natural to look for as simple controllers as possible. This means that we will search for polynomials $R\ (z)$ and $S\ (z)$ of the lowest order that satisfies the Diophantine equation. The simplest case is $R\ (z) = 1$ and $S\ (z) = s_0$, that is, a proportional controller. This gives the equation

$$z^2 - 2z + 1 + \frac{s_0h^2}{2}(z+1) = A_{cl}(z)$$

which cannot be solved for an arbitrary $A_{cl}\ (z)$ of second order. With a first-order controller we have $R(z) = z + r_1$ and $S(z) = s_0z + s_1$, which gives

$$(z^2 - 2z + 1)(z + r_1) + \frac{h^2}{2}(z+1)(s_0z + s_1) = A_{cl}(z)$$

Hence

$$z^3 + \Big(r_1 + \frac{h^2}{2}s_0 - 2\Big)z^2 + \Big(1 - 2r_1 + \frac{h^2}{2}(s_0 + s_1)\Big)z + r_1 + s_1\frac{h^2}{2} = A_{cl}(z)$$

and we find that it is possible to select the controller coefficients r_1, s_0, and s_1 to obtain an arbitrary polynomial $A_{cl}(z)$ of third degree. Choosing

$$A_{cl}(z) = z^3 + p_1z^2 + p_2z + p_3$$

and identifying coefficients of powers of equal degree we find that

$$\begin{aligned} r_1 + \frac{h^2}{2}s_0 &= p_1 + 2 \\ -2r_1 + \frac{h^2}{2}(s_0 + s_1) &= p_2 - 1 \\ r_1 + s_1\frac{h^2}{2} &= p_3 \end{aligned}$$

This equation has the solution