Computerized control partial exam 1 – Dummy

Kjartan Halvorsen

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Time Whenever suits you best. Each problem should not take more than 30 min to solve. **The actual exam will have only three problems.**

Place Somewhere quiet

Permitted aids For the exam: The single colored page with your own notes, table of Laplace transforms, calculator

All answers should be readable and well motivated (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam. Use the last page if more space is needed.

Good luck!

Matricula and name

Problem 1

Consider the continuous-time system with the following transfer function

$$G(s) = \frac{s+1}{s(s+3)}.$$

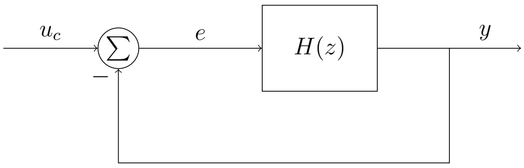
The system is sampled with sampling interval h using step-invariant (zero-order hold) sampling. **Show that the pulse-transfer function for the sampled system is**

$$H(z) = \frac{(2z - 2 + 3h)\left(z - \mathrm{e}^{-3h}\right) - 2(z-1)^2}{9(z-1)\left(z - \mathrm{e}^{-3h}\right)}.$$

Derivation:

Problem 2

The sampled system in Problem 1 is controlled using proportional control with gain equal to 1.



Calculate the closed-loop pulse-transfer function

Solutions:

Problem 3

What is the steady-state value of the control error $e(kh) = y_{ref}(kh) - y(kh)$? when $y_{ref}(kh)$ is a step?

Solution:

Problem 4

Instead of designing a discrete-time controller, a continuous-time controller was designed, given by

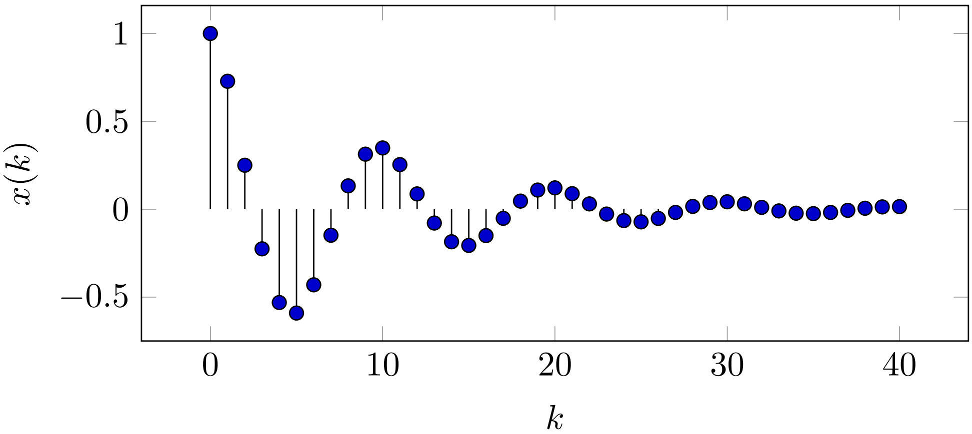
$$2\frac{d}{dt}u + u = 2\frac{d}{dt}e + 5e$$

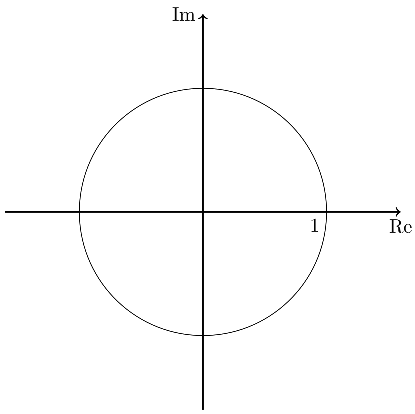
Discretize the controller using Tustin's approximation. Determine the poles and zeroes of the discrete-time controller.

Solution:

Problem 5

Below is a plot of a discrete-time signal $x(k) = \mathrm{Re}\left\{a^k\right\}$. Mark out a in the complex plane below. Motivate your answer.





Motivation:

If necessary, you can continue your solutions on this page. Mark clearly which problem the solution corresponds to.

Solutions

Problem 1

First calculate the step-response of the continous-time system

$$G(s)\frac{1}{s} = \frac{s+1}{s^2(s+3)} = \frac{2}{9s} + \frac{1}{3s^2} - \frac{2}{9(s+3)}.$$

The inverse Laplace-transform gives

$$y(t) = \frac{2}{9} + \frac{1}{3}t - \frac{2}{9}\mathrm{e}^{-3t}.$$

Sampling this function gives

$$y(kh) = \frac{2}{9} + \frac{1}{3}kh - \frac{2}{9}\big(\mathrm{e}^{-3h}\big)^k,$$

which has the Z-transform

$$Y(z) = \frac{2z}{9(z-1)} + \frac{hz}{3(z-1)^2} - \frac{2z}{9\big(z-\mathrm{e}^{-3h}\big)}.$$

Dividing the z-transform of the system response to that of the input (the step) gives

$$\begin{aligned} H(z) = \frac{Y(z)}{U(z)} &= \frac{z-1}{z}Y(z) = \frac{2}{9} + \frac{h}{3(z-1)} - \frac{2(z-1)}{9\big(z-\mathrm{e}^{-3h}\big)} \\ &= \frac{2(z-1)\big(z-\mathrm{e}^{-3h}\big) + 3h\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{9(z-1)\big(z-\mathrm{e}^{-3h}\big)} \\ &= \frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{9(z-1)\big(z-\mathrm{e}^{-3h}\big)}. \end{aligned}$$

Problem 2

Write the open-loop pulse-transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{9(z-1)\big(z-\mathrm{e}^{-3h}\big)}.$$

The closed-loop pulse transfer function from the reference signal to the output becomes

$$\begin{aligned} H_c(z) &= \frac{H(z)}{1+H(z)} = \frac{B(z)}{A(z)+B(z)} \\ &= \frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2}{9(z-1)\big(z-\mathrm{e}^{-3h}\big) + (2z-2+3h)\big(z-\mathrm{e}^{-3h}\big) - 2(z-1)^2} \end{aligned}$$

Problem 3

The steady-state control error becomes

$$\lim_{k\to\infty}\big(y_{ref}(kh)-y(kh)\big)=\lim_{k\to\infty}y_{ref}(kh)-\lim_{k\to\infty}y(kh).$$

The first limit is simply the steady-state value of the unit step input signal which is 1. The second limit can be computed using the final value theorem

$$\begin{aligned}\lim_{k\to\infty}y(kh)&=\lim_{z\to1}(z-1)Y(z)=\lim_{z\to1}(z-1)\frac{H(z)}{1+H(z)}Y_{ref}(z)\\&=\lim_{z\to1}(z-1)\frac{B(z)}{A(z)+B(z)}\frac{z}{z-1}=\lim_{z\to1}\frac{zB(z)}{A(z)+B(z)}\\&=\lim_{z\to1}z\frac{(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big)-2(z-1)^2}{9(z-1)\big(z-\mathrm{e}^{-3h}\big)+(2z-2+3h)\big(z-\mathrm{e}^{-3h}\big)-2(z-1)^2}=\frac{3h(1-\mathrm{e}^{-3h})}{3h(1-\mathrm{e}^{-3h})}=1.\end{aligned}$$

So the steady-state error is zero.

Problem 4

The controller has transfer function

$$F(s)=\frac{2s+5}{2s+1}.$$

Inserting for the Tustin's approximation gives

$$\begin{aligned}F_d(z)&=F(s)\big|_{s=\frac{2}{h}\frac{z-1}{z+1}}\\&=\frac{2\frac{2}{h}\frac{z-1}{z+1}+5}{2\frac{2}{h}\frac{z-1}{z+1}+1}\\&=\frac{4(z-1)+5h(z+1)}{4(z-1)+h(z+1)}=\frac{(4+5h)z-(4-5h)}{(4+h)z-(4-h)}\end{aligned}$$

The pole is in $z=\frac{4-h}{4+h}$ and the zero in $z=\frac{4-5h}{4+5h}$.

Problem 5

$\mathrm{Re}\{a^k\}$ is the operation of taking the real part of the expression a^k, where a is a complex number. Seen in the complex plane, we project the point a^k onto the real line in order to find the real part.

In polar form we have

$$x(k)=\mathrm{Re}\left\{\left(r\mathrm{e}^{i\theta}\right)^k\right\}.$$

The discrete-time signal in the graph is a decaying discrete cosine. The period is clearly 10 samples, so we must have

$$x(k)=\mathrm{Re}\left\{\left(r\mathrm{e}^{i\frac{\pi}{5}}\right)^k\right\}.$$

The signal is decaying at a rate such that the amplitude is approximately 0.1 after 20 samples.

$$r^{20} \approx 0.1$$

Hence

$$r \approx 0.1^{1/20} = 0.89$$

(In fact, $r = 0.9$)