From analog to discrete-time systems

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The world according to the discrete-time controller

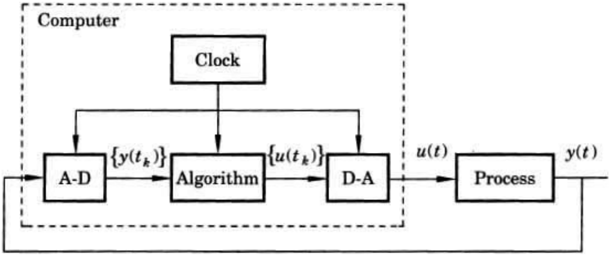

Figure 1.1 Schematic diagram of a computer-controlled system.

Source: Åström and Wittenmark "Computer-controlled systems".

Discrete-time Linear Shift-Invariant systems

$$\xrightarrow{\;u(k)\;} \boxed{\;g\;} \xrightarrow{\;y(k)\;}$$

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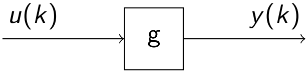

General case (non-causal)

$$
\begin{aligned}
y(k) = g * u &= \sum_{n=-\infty}^{\infty} g(n)u(k-n) \\
&= \cdots + g(-2)u(k+2) + g(-1)u(k+1) + g(0)u(k) + g(1)u(k-1) + \cdots
\end{aligned}
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Causal case

$$
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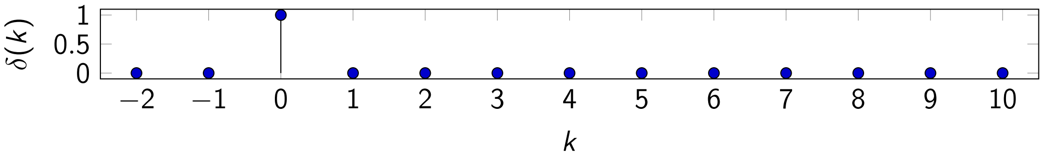
y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n) \\
= g(0)u(k) + g(1)u(k-1) + g(2)u(k-2) + g(3)u(k-3) + \cdots
\end{aligned}
$$

$g(k)$ is called the weighting sequence.

Discrete-time LSI systems

Impulse response

If the input signal is a unit discrete impulse



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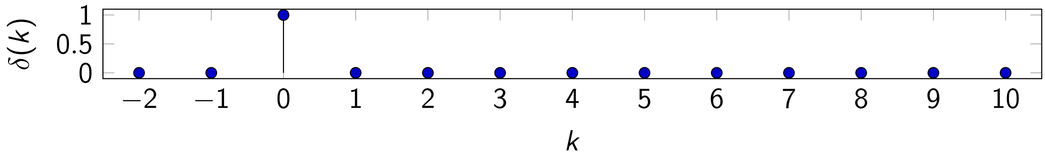


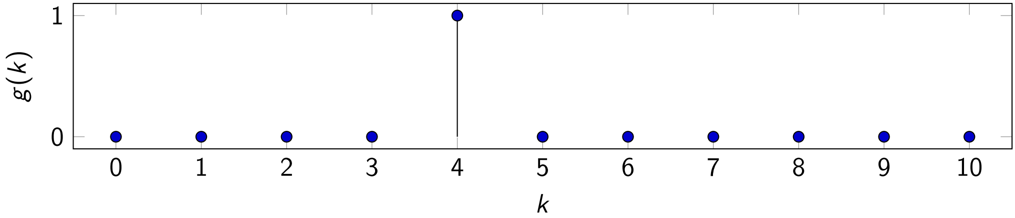
$$y(k) = \sum_{n=0}^{\infty} g(n)\delta(k-n) = g(k)$$

The weighting sequence is equal to the impulse response of the system.

The response of a discrete LSI system is a weighted sum of the current and previous values of the input

$$y(k) = g * u = \sum_{n=0}^{\infty} g(n)u(k-n) = g(0)u(k) + g(1)u(k-1) + g(2)u(k-2) + \cdots$$

Activity What is the response of a system to the input signal $u(k)$ if its impulse response (weighting sequence) is the one below?



$$y(k) =$$

The Laplace transform

Definition

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)\mathrm{e}^{-st}dt$$

Inverse transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\mathrm{e}^{st}\, ds$$

Note that in control engineering, the one-sided transform is used.

The z-transform

Definition

$$F(z) = \mathcal{Z}\left\{f(kh)\right\} = \sum_{k=0}^{\infty} f(kh) z^{-k}$$

Inverse transform

$$f(kh) = \frac{1}{2\pi i} \oint_r F(z) z^{k-1}\, dz$$

Note that in control engineering, the one-sided transform is used.

The Laplace transform of a sampled signal

Assume right-sided signal $f(t)$, meaning it is zero for negative times $t < 0$.

$$f_s(t) = f(t)m(t) = f(t)\sum_{k=0}^{\infty}\delta(t-kh) = \sum_{k=0}^{\infty} f(t)\delta(t-kh) = \sum_{k=0}^{\infty} f(kh)\delta(t-kh)$$

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$$\begin{aligned} F_s(s) = \mathcal{L}\left\{f_s(t)\right\} &= \int_0^{\infty}\left(\sum_{k=0}^{\infty} f(kh)\delta(t-kh)\right)\mathrm{e}^{-st}\,dt \\ &= \sum_{k=0}^{\infty}\int_0^{\infty} f(kh)\delta(t-kh)\mathrm{e}^{-st}\,dt = \sum_{k=0}^{\infty} f(kh)\mathrm{e}^{-skh} \\ &= \sum_{k=0}^{\infty} f(kh)\left(\mathrm{e}^{sh}\right)^{-k} \end{aligned}$$

The Laplace transform of a sampled signal

Note:

$$F_s(s) = \sum_{k=0}^{\infty} f(kh) \left(\mathrm{e}^{sh}\right)^{-k} \quad \text{Laplace transform}$$

$$F(z) = \sum_{k=0}^{\infty} f(kh) z^{-k} \quad \text{z-transform}$$

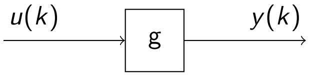
The z-transform of a sampled signal corresponds to its Laplace transform with the following relationship between the s-plane of the Laplace transform and the z-plane of the z-plane of the z-transform.

$$z = \mathrm{e}^{sh}$$

Exercise on the z-transform

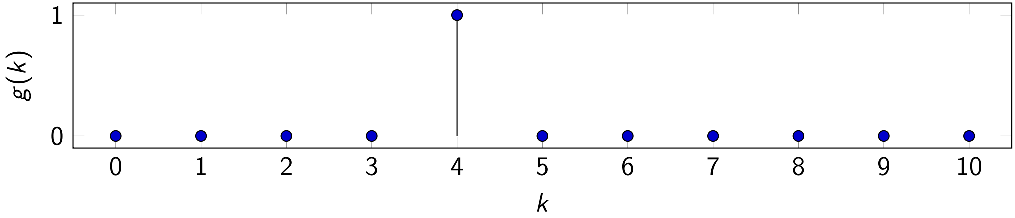
$$u(k) \longrightarrow \boxed{\text{g}} \longrightarrow y(k)$$

Exercise on the z-transform



$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k} \quad \text{z-transform}$$

Activity What is the z-transform of the delayed impulse signal $g(k) = \delta(k-4)$ below?



One of the most important transform pairs

$$f(kh) = \alpha^{kh}, \quad \alpha \in \mathbb{C}$$

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$$= \sum_{k=0}^{\infty} \left(\frac{\alpha^{h}}{z}\right)^{k} = \frac{1}{1 - \frac{\alpha^{h}}{z}} = \frac{z}{z - \alpha^{h}}, \quad \left|\frac{\alpha^{h}}{z}\right| < 1$$

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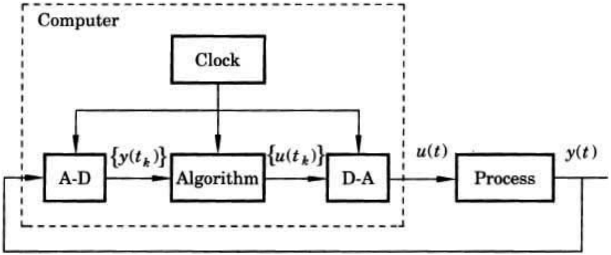
$$= \sum_{k=0}^{\infty} \left(\frac{\alpha^{h}}{z}\right)^{k} = \frac{1}{1 - \frac{\alpha^{h}}{z}} = \frac{z}{z - \alpha^{h}}, \quad |\frac{\alpha^{h}}{z}| < 1$$

$$\boxed{\alpha^{kh} \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z - \alpha^{h}}}$$

Step-invariant sampling (a.k.a ZOH sampling)



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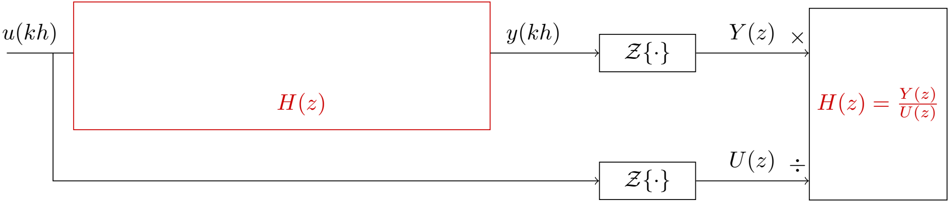

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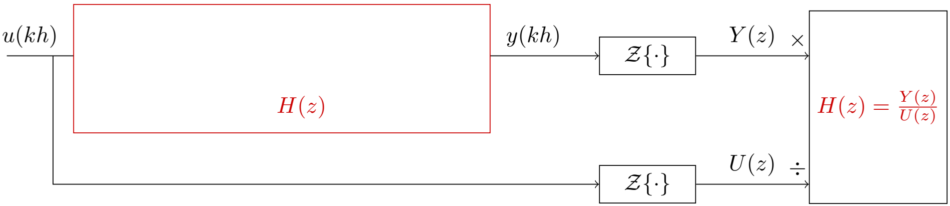
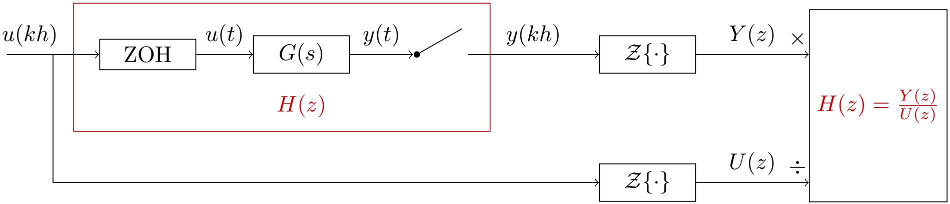


Step-invariant sampling (zero order hold): $u(kh) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

Step-invariant sampling (a.k.a ZOH sampling)

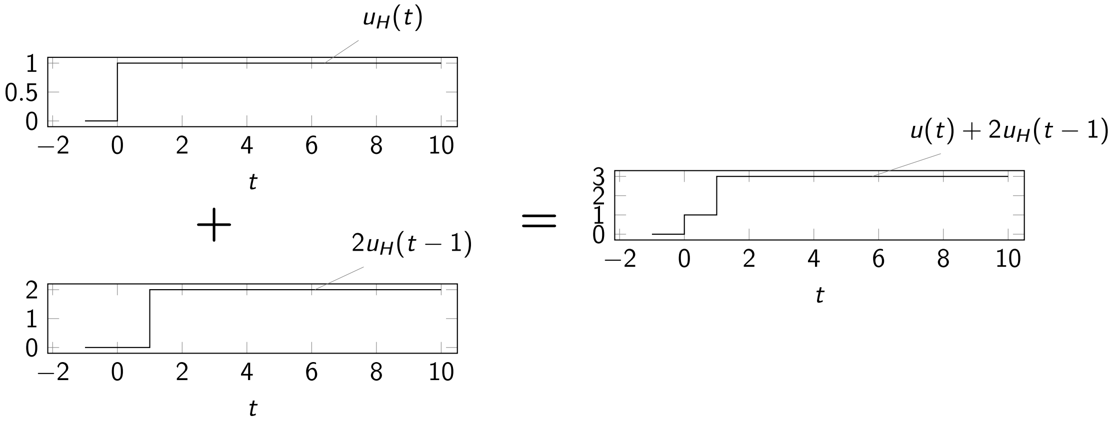
The idea is to sample the continuous-time system's response to a step input, in order to obtain a discrete approximation which is exact (at the sampling instants) for such an input signal.



Step-invariant sampling (zero order hold): $u(kh) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

Why is step-invariant sampling a good idea?

A piecewise constant (stair-case shaped) function can be written as a sum of delayed step-responses!



Step-invariant sampling, or zero-order-hold sampling

Let the input to the continuous-time system be a unit step $u(t) = u_H(t)$, which has Laplace transform $U(s) = \frac{1}{s}$. In the Laplace-domain we get

$$Y(s) = G(s)\frac{1}{s}$$

1. Obtain the time-response by inverse Laplace: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$
2. Sample the time-response to obtain the sequence $y(kh)$ and apply the z-transform to obtain $Y(z) = \mathcal{Z}\{y(kh)\}$
3. Calculate the pulse-transfer function by dividing with the z-transform of the input signal $U(z) = \frac{z}{z-1}$.

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z}Y(z)$$