Computerized Control Final Exam (31%)

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Time 2017-05-05 16:00 - 19:00

Place 5105

Permitted aids The single colored page with your own notes, table of transforms, calculator

All answers should be **readable and well motivated** (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam. Use the last page if more space is needed.

Good luck!

Matricula and name:

Problem 1 (70p)

Figure 1 shows a simple, one-dimensional magnetic suspension system. The current i in the windings generates a magnetic field which suspends the mass m. Ignoring friction, there are two forces acting on the mass: gravity and the magnetic force. The magnetic force is proportional to the square of the current i and inverse proportional to the square of the gap distance x. This gives the equation of motion

$$m\ddot{x} = -C\left(\frac{i}{x}\right)^2 + mg. \tag{1}$$

The system is non-linear, so in order to use linear control design, the system must be

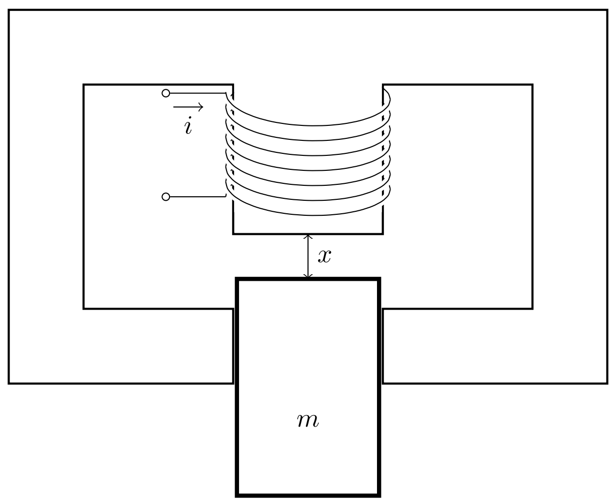


Figure 1: A magnetic suspension system. The force of gravity acts to pull the mass m downwards. The force due to the magnetic field generated by the current i keeps the mass from falling down.

linearized about an operating point. Introduce the variations y and u about the operating point

$$\begin{aligned} x &= x_0 + y \\ i &= i_0 - \tilde{i}. \end{aligned}$$

The input signal to the system (2) is the change $\tilde{i}$ in the current in the windings, and the output signal is the change y in the gap distance. The negative sign in the variation $i = i_0 - \tilde{i}$ is introduces so that a positive input signal leads to a positive change in the gap distance. With operating point

$$\frac{i_0}{x_0} = \sqrt{\frac{mg}{C}},$$

the linearized model becomes

$$\ddot{y} = \frac{2g}{x_0}y + \frac{2\sqrt{Cg}}{\sqrt{m}x_0}u \tag{2}$$

with transfer function

$$G(s) = \frac{\frac{2\sqrt{Cg}}{\sqrt{m}x_0}}{s^2 - \frac{2g}{x_0}}. \tag{3}$$

The system is unstable, with poles in $\pm\sqrt{\frac{2g}{x_0}}$. Normalizing the time (using the unit of time $T = \sqrt{\frac{x_0}{2g}}$) and setting $u(t) = \frac{2\sqrt{Cg}}{\sqrt{m}x_0}\tilde{i}(t)$ gives the plant model

$$Y(s) = G(s)U(s) = \frac{1}{s^2 - 1}U(s). \tag{4}$$

(a)

Show that discretizing the model (4) using zero-order-hold with $h = 0.1$ gives the discrete-time pulse transfer function

$$H(z) = 5.0 \cdot 10^{-3}\frac{z+1}{z^2 - 2.01z + 1}. \tag{5}$$

and plot the zeros and poles of the system in the z-plane.

Solution:

(b)

The magnetic suspension system is to be stabilized using the control law

$$U(z) = K\frac{z-0.9}{z}E(z) = K\frac{z-0.9}{z}\big(Y(z) - Y_{ref}(z)\big). \tag{6}$$

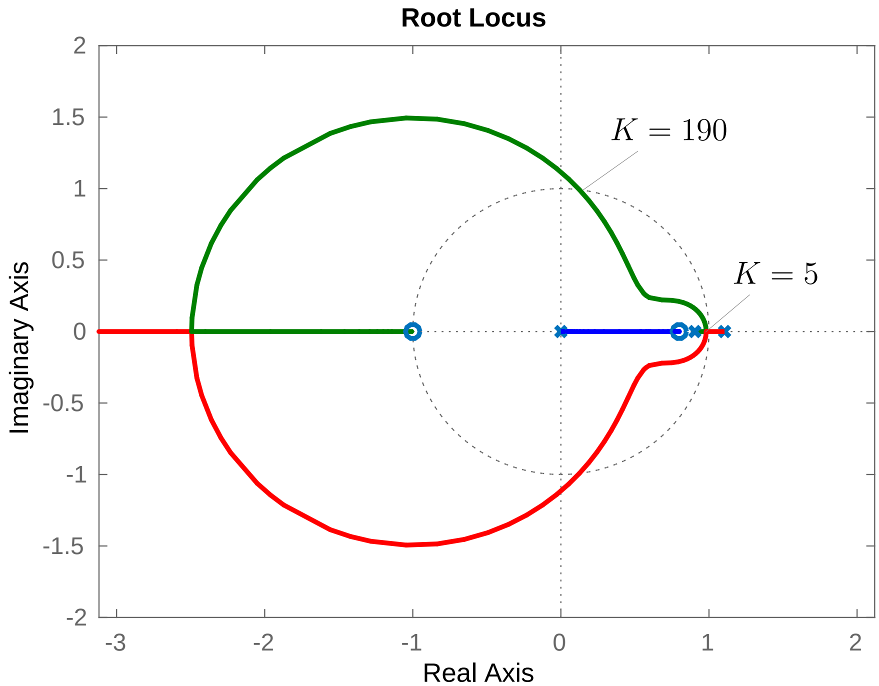
Write the control law as a difference equation. Explain also which values must be stored in the computer between sampling instants in order to calculate the control signal.

Solution:

$u(k+1) =$

(c)

With the control law (6), we obtain the below root locus for the closed-loop poles. Explain briefly (3-5 sentences) the closed-loop behaviour of the system for different values of the feedback gain K.



Answer:

(d)

Figure 2 shows four different step-responses, obtained with different values for the gain parameter K. Match the step-response to the gain. Motivate!

Answer and motivation:

Gain	$K = 6$	$K = 54$	$K = 180$	$K = 200$
Response				

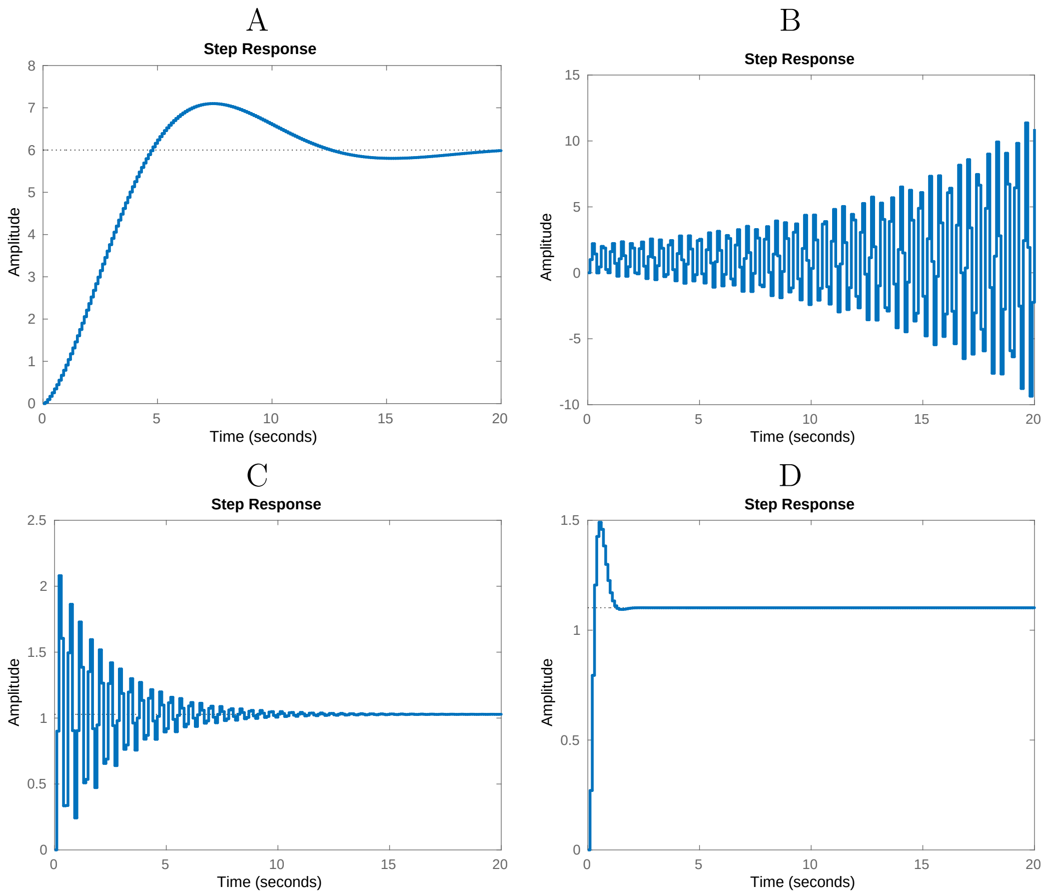


Figure 2: Step responses, problem 1 (d).

Problem 2 (30p)

The normalized magnetic suspension system can be represented on state-space form as

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 2.01 & -1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\
y(k) &= 5 \cdot 10^{-3} \begin{bmatrix} 1 & 1 \end{bmatrix} x(k)
\end{aligned}
\tag{7}
$$

(a)

Show that the system is reachable.

Solution:

(b)

Determine the state feedback gain

$$u(k) = u_c(k) - l_1 x_1(k) - l_2 x_2(k)$$

which gives a closed-loop system with poles in the origin. What is a controller with this choice of closed-loop poles called?

Controller design:

If necessary, you can continue your solutions on this sheet. Mark clearly which problem the solution corresponds to.

Solutions

Problem 1

(a)

The step-response of the system is

$$\begin{aligned}
Y(s) &= \frac{1}{(s+1)(s-1)s} = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1} - \frac{1}{s}\\
y(t) &= \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{t} - u_H(t)\\
y(kh) &= \frac{1}{2}\left(\mathrm{e}^{-h}\right)^k + \frac{1}{2}\left(\mathrm{e}^{h}\right)^k - u_H(k)
\end{aligned}$$

Taking the z-transform gives

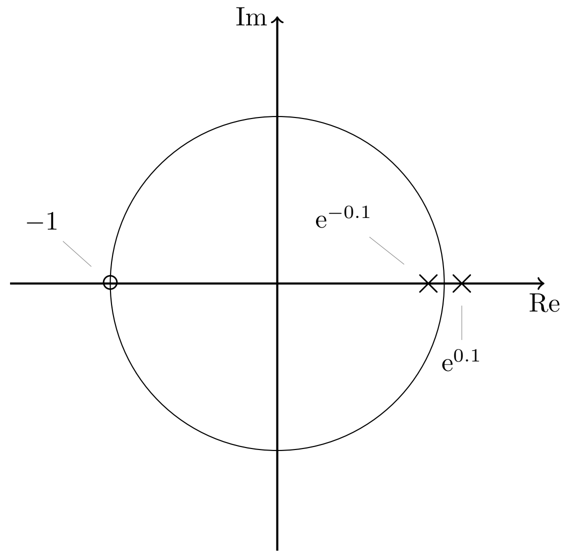
$$Y(z) = \frac{\frac{1}{2}z}{z-\mathrm{e}^{-h}} + \frac{\frac{1}{2}z}{z-\mathrm{e}^{h}} - \frac{z}{z-1}.$$

Dividing with the z-transform of the unit step $\frac{z}{z-1}$ gives

$$\begin{aligned}
H(z) = \frac{Y(z)}{U(z)} &= \frac{\frac{1}{2}(z-1)}{z-\mathrm{e}^{-h}} + \frac{\frac{1}{2}(z-1)}{z-\mathrm{e}^{h}} - 1\\
&= \frac{\frac{1}{2}(z-1)(z-\mathrm{e}^{h}) + \frac{1}{2}(z-1)(z-\mathrm{e}^{-h}) - (z-\mathrm{e}^{-h})(z-\mathrm{e}^{h})}{(z-\mathrm{e}^{-h})(z-\mathrm{e}^{h})}\\
&= \frac{\frac{1}{2}z^2 - \frac{1}{2}(1+\mathrm{e}^{h})z + \frac{1}{2}\mathrm{e}^{h} + \frac{1}{2}z^2 - \frac{1}{2}(1+\mathrm{e}^{-h})z + \frac{1}{2}\mathrm{e}^{-h} - z^2 + (\mathrm{e}^{-h}+\mathrm{e}^{h})z - 1}{(z-\mathrm{e}^{-h})(z-\mathrm{e}^{h})}\\
&= \frac{(-\frac{1}{2} - \frac{1}{2}\mathrm{e}^{h} - \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-h} + \mathrm{e}^{-h} + \mathrm{e}^{h})z + (\frac{1}{2}\mathrm{e}^{h} + \frac{1}{2}\mathrm{e}^{-h} - 1)}{z^2 - (\mathrm{e}^{-h}+\mathrm{e}^{h})z + 1}\\
&= \frac{(\frac{1}{2}(\mathrm{e}^{h}+\mathrm{e}^{-h}) - 1)(z+1)}{z^2 - (\mathrm{e}^{-h}+\mathrm{e}^{h})z + 1}.
\end{aligned}$$

Inserting $h = 0.1$ gives

$$H(z) = 5.0 \cdot 10^{-3}\frac{z+1}{z^2 - 2.01z + 1}.$$



(b)

Using the shift operator we have

$$\begin{aligned}
u(kh) &= K\frac{\mathrm{q}-0.9}{\mathrm{q}}(y(kh) - y_{ref}(kh)) \\
\mathrm{q}\,u(kh) &= K(\mathrm{q}-0.9)(y(kh) - y_{ref}(kh)) \\
u(kh+h) &= K\left(y(kh+h) - 0.9y(kh) - y_{ref}(kh+h) - 0.9y_{ref}(kh)\right).
\end{aligned} \tag{8}$$

In order to calculate the control signal at time $kh + h$ we need to store the previous output signal $y(kh)$ and the previous set-point $y_{ref}(kh)$.

(c)

For small gains $K < 5$ the system is unstable since one pole is outside the unit circle. When the gain increases from $K = 5$, the system is stable and at first dominated by the two poles close to the unit circle. When K increases further, these two poles break out into the imaginary plane, and the closed-loop system will have oscillations. The damping is quite good though, and the system response will be quite fast as the K increases. When $K > 190$ the poles break out of the unit circle and the system becomes unstable.

(d)

Base on the discussion in (c), we must have

Gain	$K = 6$	$K = 54$	$K = 180$	$K = 200$
Response	Slow, stable response, no oscillations: **A**	Fast, well damped response, since poles are far from unit circle:. **D**	Fast and oscillatory response, since poles are close to the unit circle: **C**	Unstable response: **B**

Problem 2

(a)

Reachability is tested by forming the matrix

$$W_r = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 1 & 2.01 \\ 0 & 1 \end{bmatrix}$$

and calculating the determinant

$$\det W_r = 1 \neq 0$$

so the system is reachable.

(b)

With the feedback

$$u(k) = u_c(k) - l_1 x_1(k) - l_2 x_2(k) = u_c(k) - Kx(k)$$

inserted into the state-space model we get the closed-loop state space model

$$x(k+1) = (\Phi - \Gamma L)\, x(k) + \Gamma u_c(k)$$

which have poles given by the characteristic equation of $\Phi - \Gamma L$

$$\det\left(zI - (\Phi - \Gamma L)\right) = 0.$$

We have

$$\Gamma L = \begin{bmatrix} l_1 & l_2 \\ 0 & 0 \end{bmatrix}$$

and

$$\Phi - \Gamma L = \begin{bmatrix} -2.01 - l_1 & 1 - l_2 \\ 1 & 0 \end{bmatrix}$$

which gives

$$\begin{aligned} \det\left(zI - (\Phi - \Gamma L)\right) &= \det \begin{bmatrix} z + 2.01 + l_1 & -1 + l_2 \\ -1 & z \end{bmatrix} \\ &= (z + 2.01 + l_1)z + (-1 + l_2) \\ &= z^2 + (2.01 + l_1)z + (-1 + l_2). \end{aligned}$$

We want this characteristic polynomial to equal z^2 which gives two poles in the origin. This gives the solution

$$\begin{aligned} l_1 &= -2.01 \\ l_2 &= 1. \end{aligned}$$