Computerized control partial exam 1 from fall semester 2016, modified

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Time September 13 17:30

Place 4101

Permitted aids The single colored page with your own notes, table of Laplace transforms, calculator

All answers should be readable and well motivated (if nothing else is written). Solutions/motivations should be written on the provided spaces in this exam. Use the last page if more space is needed.

Good luck!

Matricula and name

The system

The dynamic model of a ship with input u being the rudder angle and the output y being the heading (see figure 1) can be described as a continuous-time second order system with a pole in the origin

$$G(s) = \frac{K}{s(s+a)}.$$

For fully loaded, large tankers this dynamics is often unstable, meaning that $a < 0$.

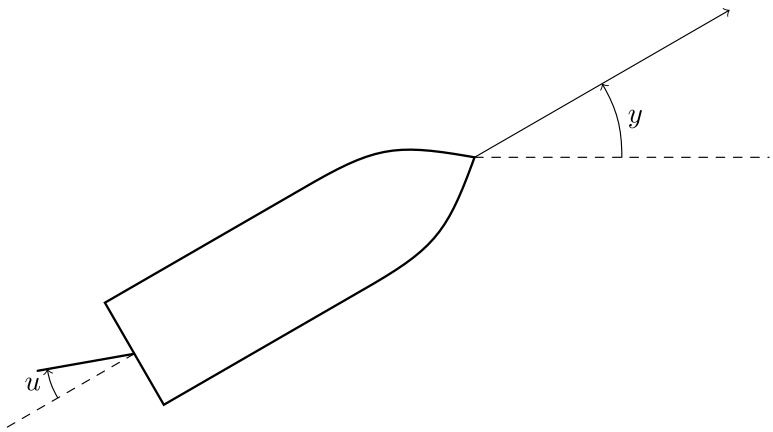


Figure 1: Heading of a ship controlled by rudder input.

Consider for this exam the normalized continuous-time model of the tanker

$$G(s) = \frac{1}{s(s-1)}.$$

Problem 1 (50p)

The system is sampled with sampling interval h using step-invariant (zero-order hold) sampling. **Circle the correct pulse-transfer function below, and show your calculations**

1. $H(z) = \frac{(1-e^h-h)z-\left((1-h)e^h-1\right)}{(z-1)(z-e^h)}$
2. $H(z) = \frac{(-1+e^h-h)z-\left((1-h)e^h-1\right)}{(z-1)(z-e^{2h})}$
3. $H(z) = \frac{(-1+e^h-h)z-\left((1-h)e^h-1\right)}{(z-1)(z-e^h)}$

Derivation:

Problem 2 (20p)

Assume that the sampling period is $h = 0.2$. In figure 2 draw the poles (crosses) and zero (circle) for both the continuous-time transfer function $G(s)$ and the discretized pulse-transfer function $H(z)$ you determined in Problem 1.

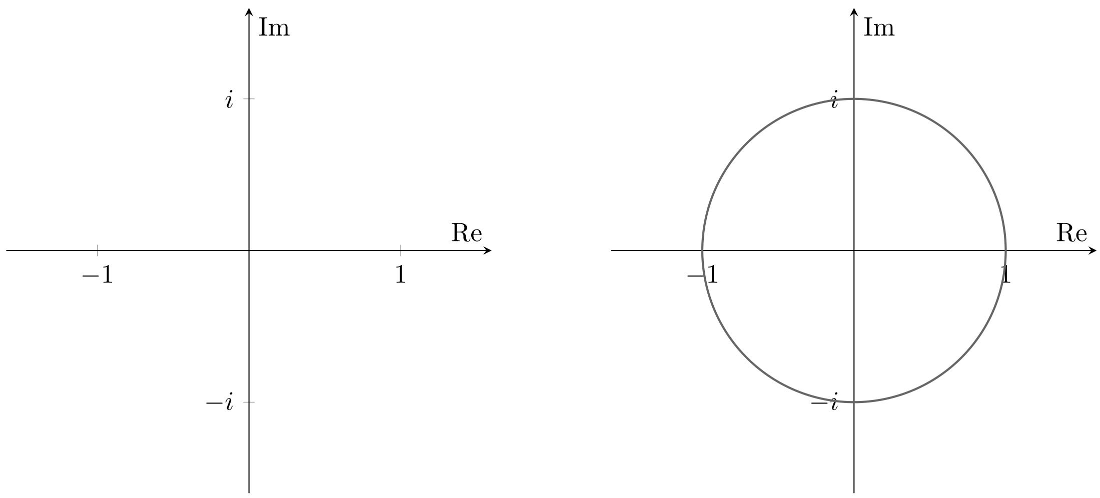


Figure 2: Problem 2: Plot the poles of the continuous-time system (on the left) and the poles and zero of the discrete-time system (on the right). Indicate (with arrows and/or colors) corresponding pairs of continuous-time and discrete-time poles.

Calculations:

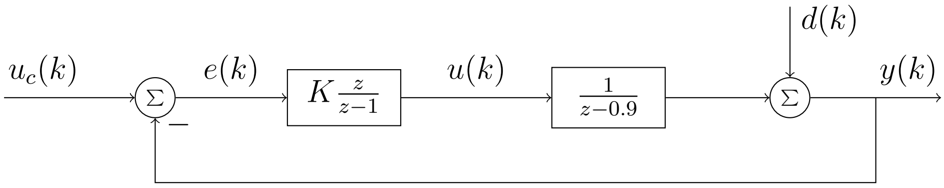


Figure 3: Feedback control from the error signal.

Problem 3 (40p)

Assume, now, that the plant $H(z) = \frac{1}{z-0.9}$ is controlled by feedback from the control error, as illustrated in figure 3, using the controller

$$F(z) = K\frac{z}{z-1}.$$

(a) 20p

Figure 4 shows the root locus for the closed-loop poles with respect to the gain K. In figure 5, four different step plots are shown for four different values of K. Identify (and circle) the corresponding step plot for each value of K in the table below.

K	Step plot			
0.002	A	B	C	D
1.0	A	B	C	D
3.0	A	B	C	D
4.0	A	B	C	D

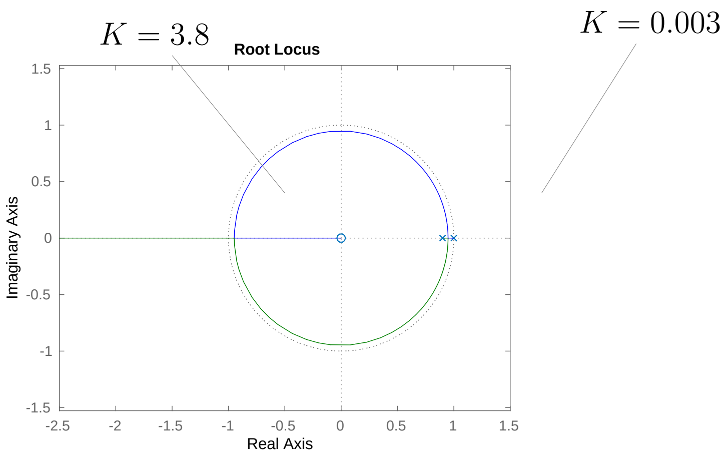


Figure 4: Root locus wrt the gain K.

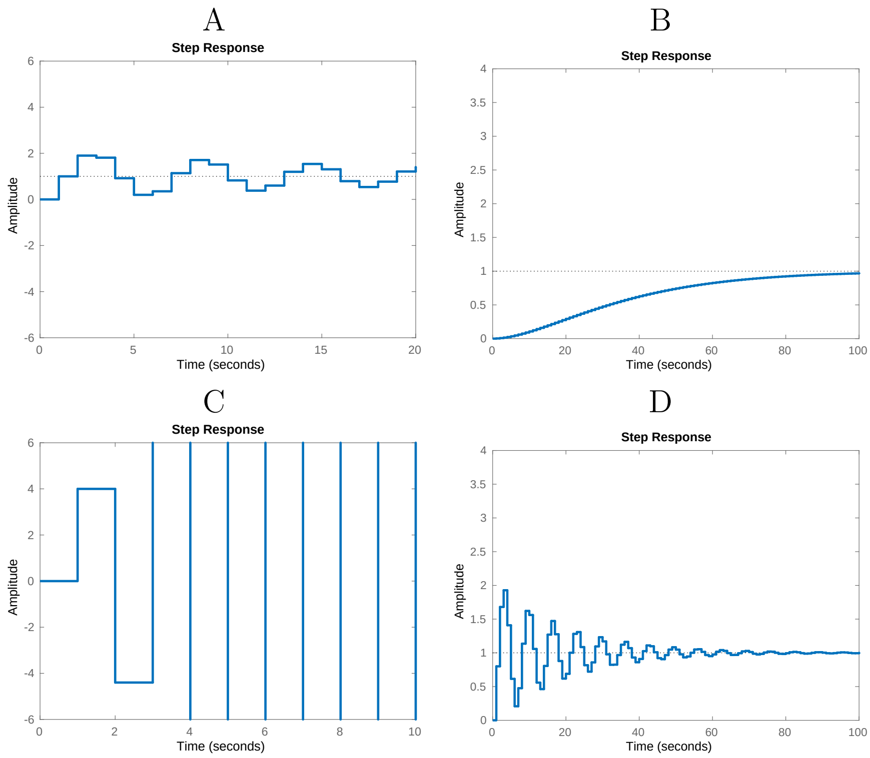


Figure 5: Step responses for different values of K.

(b) 20p

Determine the gain K so that the closed loop system has poles with realpart equal to 0.7.

Solution:

If necessary, you can continue your solutions on this page. Mark clearly which problem the solution corresponds to.

Solutions

Problem 1

First calculate the step-response of the continous-time system

$$Y(s) = G(s)\frac{1}{s} = \frac{1}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}.$$

The inverse Laplace-transform gives

$$y(t) = \mathrm{e}^t - 1 - t$$

Sampling this function gives

$$y(kh) = \mathrm{e}^{kh} - 1 - kh$$

which has the Z-transform

$$Y(z) = \frac{z}{z-\mathrm{e}^h} - \frac{z}{z-1} - \frac{hz}{(z-1)^2}$$

Dividing the z-transform of the system response to that of the input (the step) gives

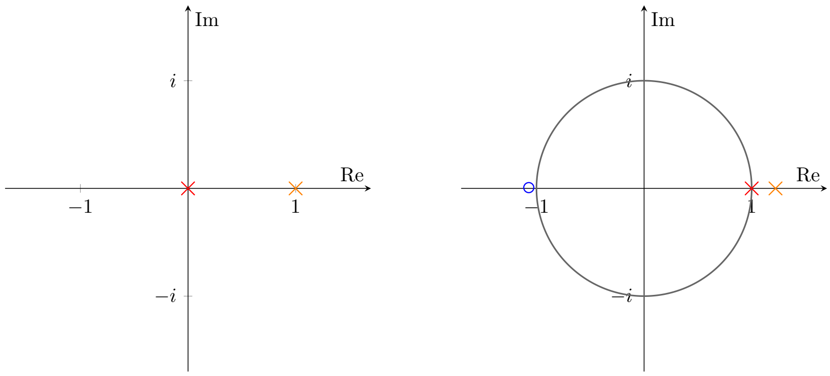
$$\begin{aligned}
H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z}Y(z) &= \frac{z-1}{z-\mathrm{e}^h} - 1 - \frac{h}{z-1} \\
&= \frac{(z-1)^2 - (z-1)(z-\mathrm{e}^h) - h(z-\mathrm{e}^h)}{(z-1)(z-\mathrm{e}^h)} \\
&= \frac{(z-1)(z-1-z+\mathrm{e}^h) - hz + h\mathrm{e}^h}{(z-1)(z-\mathrm{e}^h)} \\
&= \frac{(\mathrm{e}^h - 1 - h)z - (\mathrm{e}^h - 1 - h\mathrm{e}^h)}{(z-1)(z-\mathrm{e}^h)} \\
&= \frac{(\mathrm{e}^h - 1 - h)z - \big((1-h)\mathrm{e}^h - 1\big)}{(z-1)(z-\mathrm{e}^h)}
\end{aligned}$$

The correct pulse transfer function is the third.

Problem 2

The discrete-time poles are in $z = 1$ and $z = \mathrm{e}^{0.2} \approx 1.22$. The zero is in

$$z = -\frac{(1-h)\mathrm{e}^{0.2} - 1}{\mathrm{e}^{0.2} - 1 - h} \approx -1.07.$$



Problem 3

(a)

The root locus starts with two poles at 1 and 0.9. For small values of K we will have poles that are slow and completely damped. The only such response is **B**. As K increases we will have closed-loop poles that follows the unit circle a bit inside it. The poles will have little damping and will increase in speed with K. Finally, one pole move outside the unit circle, and the system becomes unstable. It is easy to see the response that is slow (B) and unstable (C). It is a bit difficult to read off the difference in osciallation period between the other two responses. In summary, we get

K	Step plot			
0.002	A	B	C	D
1.0	A	B	C	D
3.0	A	B	C	D
4.0	A	B	C	D

(b)

The closed-loop system from command signal to the output is

$$H_c(z) = \frac{K\frac{z}{z-1}\frac{1}{z-0.9}}{1 + K\frac{z}{z-1}\frac{1}{z-0.9}} = \frac{Kz}{(z-1)(z-0.9) + Kz}.$$

The characteristic equation is

$$(z-1)(z-0.9) + Kz = z^2 - (1.9 - K)z + 0.9 = 0$$

with solution

$$z = \frac{1.9 - K}{2} \pm \frac{1}{2}\sqrt{(1.9 - K)^2 - 5.6}.$$

We know from the root locus that for a value of K that gives poles with real part 0.7, then the poles are complex-conjugated and so the expression under the root sign must be negative. The real part is given by the first term in the solution to the quadratic equation. We get

$$\frac{1.9 - K}{2} = 0.7 \quad \Rightarrow \quad K = 1.9 - 1.4 = 0.5$$