

Relative stability

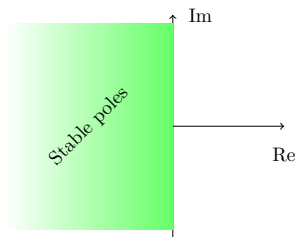
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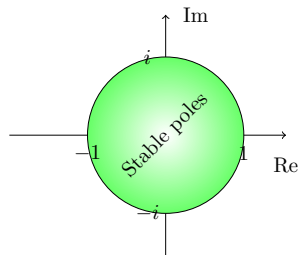
Root locus

The rules for drawing the root locus is the same in continuous as in discrete time. But the interpretation differs.

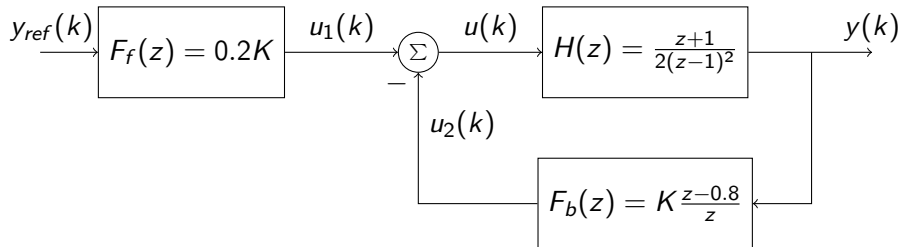
Continuous time



Discrete time



Example: Position servo for the hard disk drive arm



Characteristic equation

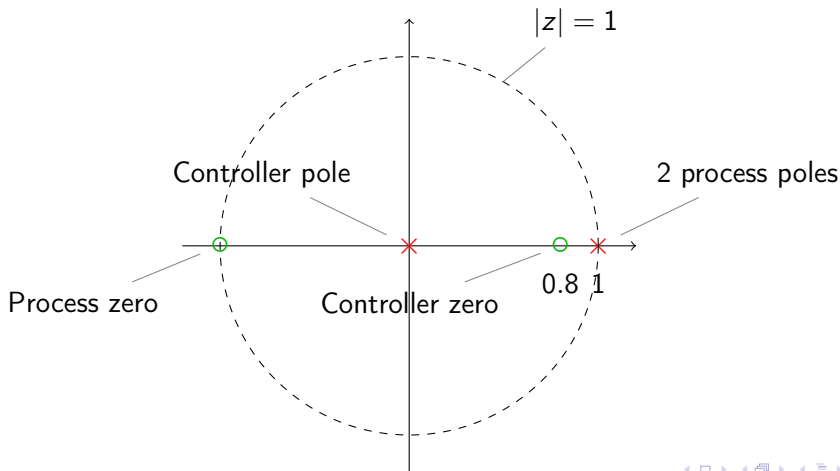
$$\begin{aligned} 1 + H(z)F_b(z) &= 0 \\ 1 + \frac{z+1}{2(z-1)^2} K \frac{z-0.8}{z} &= 0 \\ (z-1)^2 z + \frac{K}{2} (z+1)(z-0.8) &= 0 \end{aligned}$$

For which values of K will the system be stable?

Example: Position servo for the hard disk drive arm

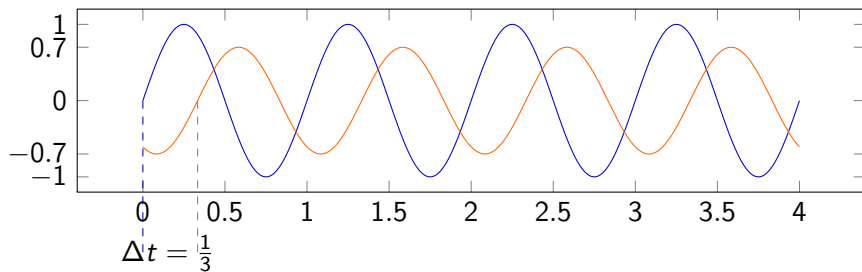
Activity Complete the root locus!

$$(z - 1)^2 z + \frac{K}{2}(z + 1)(z - 0.8) = 0$$



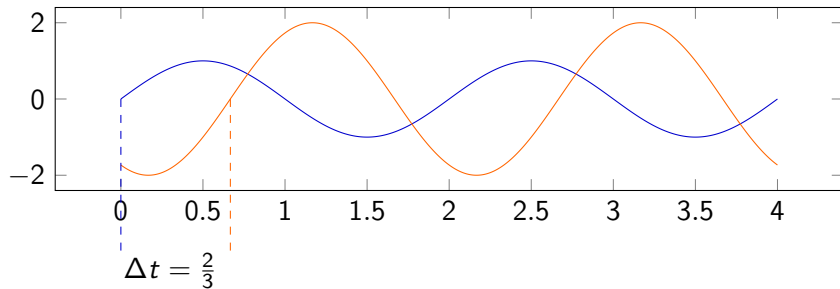
Relative stability

Sinusoid in - sinusoid out



$$\omega_1 = \frac{2\pi}{T} = 2\pi, |G(i\omega_1)| = 0.7, \arg G(i\omega_1) = -\omega_1 \Delta t = -2\pi \frac{1}{3} = -\frac{2\pi}{3}$$

Sinusoid in - sinusoid out

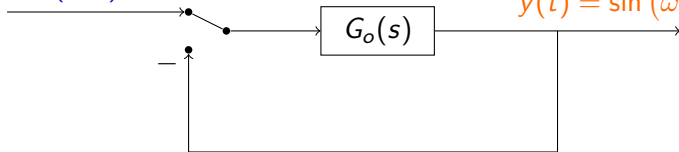


$$\omega_1 = \frac{2\pi}{T} = \quad , |G(i\omega_1)| = \quad , \arg G(i\omega_1) = -\omega_1 \Delta t =$$

If the phase shift is π

$$G_o(i\omega_1) = -1, |G_o(i\omega_1)| = 1, \arg G_o(i\omega_1) = -\pi$$

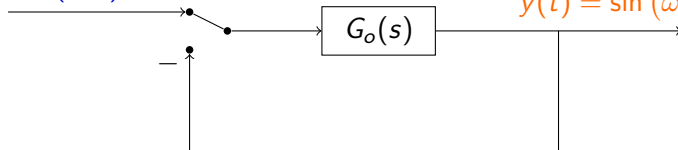
$$u(t) = \sin(\omega_1 t)$$



If the phase shift is π

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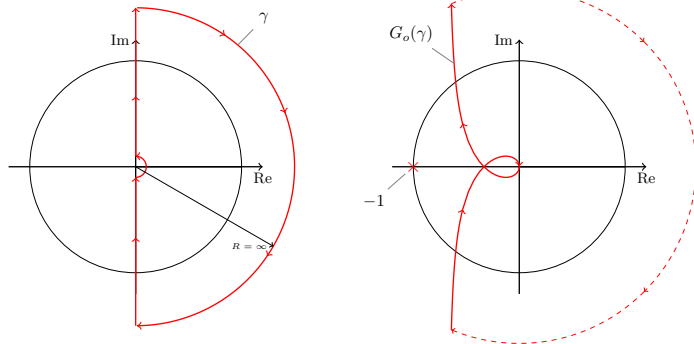
$$\text{Closed-loop transfer function: } G_c(s) = \frac{G_o(s)}{1+G_o(s)}$$

We want

$$1 + G_o(i\omega) \neq 0, \quad \forall \omega$$

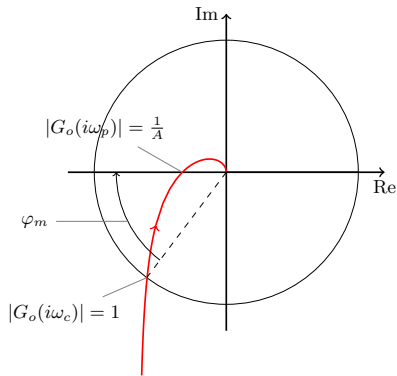
If not, then the closed-loop system will have poles on the imaginary axis (in the s-domain).

The simplified Nyquist criterion in the s-plane



If the open-loop system (the loop gain) is not unstable, i.e. $G_o(s)$ has no poles in the right-half plane, then the closed-loop system will be stable if the Nyquist curve **do not encircle the point** $s = -1$. The point $s = -1$ should stay on the left side of the Nyquist curve when we go along the curve from low to high frequencies.

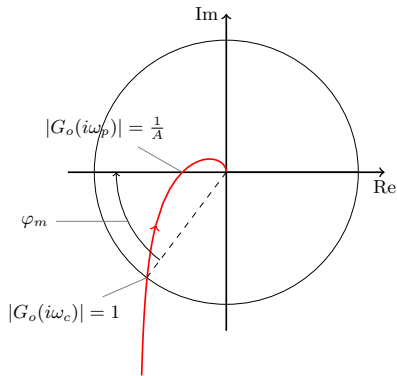
Stability margins



- ▶ Cross-over frequency: The frequency ω_c for which $|G_o(i\omega)| = 1$.
- ▶ Phase margin: The angle φ_m to the negative real axis for the point where the Nyquist curve intersects the unit circle.

$$\varphi_m = \arg G_o(i\omega_c) - (-180^\circ) = \arg G_o(i\omega_c) + 180^\circ$$

Stability margins



- ▶ phase-cross-over frequency: The frequency ω_p for which $\arg G_o(i\omega) = -180^\circ$.
- ▶ Gain margin: The gain $K = A$ that would make the Nyquist curve of $KG_o(i\omega h)$ go through the point $-1 + i0$. This means that

$$|G_o(i\omega_p h)| = \frac{1}{A}.$$

The effect of sampling on the stability margins

$$G(s) = \frac{1}{s^2 + 1.4s + 1} \xrightarrow{h=0.4} H(z) = \frac{0.066z + 0.055}{z^2 - 1.450z + 0.571}$$

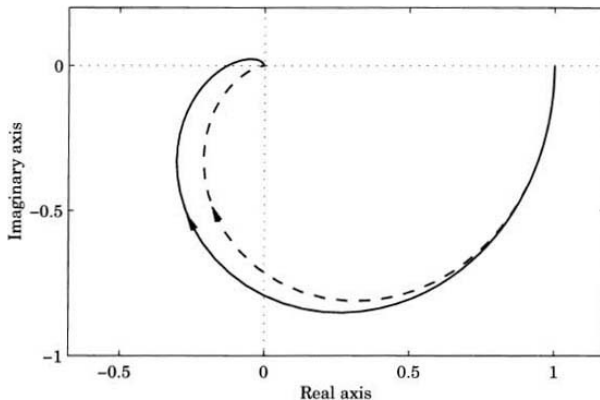


Figure 3.3 The frequency curve of (3.6) (dashed) and for (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

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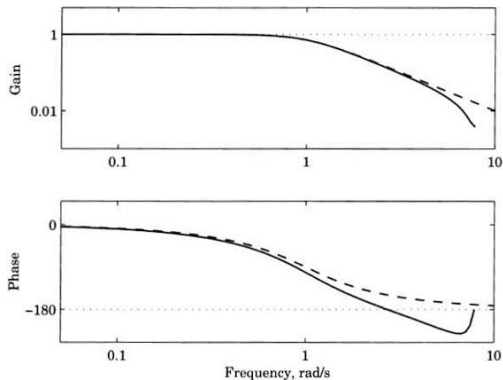


Figure 3.4 The Bode diagram of (3.6) (dashed) and of (3.6) sampled with zero-order hold when $h = 0.4$ (solid).

Selecting the sampling period

One can use the phase margin to determine a suitable sampling period. Given a desired cross-over frequency ω_c and a maximum acceptable negative change in the phase margin $\Delta\varphi \approx 5^\circ - 15^\circ \approx 0.09\text{rad} - 0.26\text{rad}$ (a rule-of-thumb).

$$\begin{array}{c} \text{ZOH} \\ \begin{array}{c} u_s(t) \longrightarrow \boxed{G_{\text{ZOH}}(s) = \frac{1-e^{-sh}}{s} \approx e^{-s\frac{h}{2}}} \longrightarrow u(t) \end{array} \end{array}$$
$$\arg G_{\text{ZOH}}(i\omega_c) \approx \arg e^{-i\omega_c \frac{h}{2}} = -\omega_c \frac{h}{2} \approx -0.09\text{rad} - -0.26\text{rad}$$

Activity Use the rule-of-thumb above to calculate a sampling period for the case $\omega_c = 20 \text{ rad/s}$ and $\Delta\varphi = 0.2 \text{ rad}$.