Computerized control - Partial exam 1 (20%)

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Instructions

Write your answers clearly and motivate well! You can write by hand or on a computer. If you write by hand, you may scan your pages using, for instance, the app CamScanner that can produce pdf-documents. **Answers should be handed in on Blackboard (assignment "Partial exam 1") no later than midnight 2016-02-11.**

The plant

The dynamic model of a particular industrial induction motor[1] can be written

$$G(s) = \frac{168}{s(s^2 + 25.9s + 168)} \tag{1}$$

Problem 1 (10p)

Determine the poles of the continuous-time model and plot these in the imaginary plane.

Problem 2 (50p)

Obtain a sampled model of the plant in (1) using zero-order-hold sampling for arbitrary h (sample the model symbolically). You may express the pulse-transfer function as a sum of rational functions in z (you do not have to write it on a common fraction line).

Problem 3 (10p)

Choose a reasonable value for the sampling period h, considering the poles of the plant $G(s)$, and using the rule-of-thumb in Åström & Wittenmark section 2.9.

Problem 4 (20p)

Determine the poles of the discrete-time system using the sampling period you decided in Problem 3. Plot the poles in the imaginary plane. For each pole, indicate which continuous-time pole it corresponds to.

[1] Jung, Seul, and Richard C. Dorf. "Analytic PIDA controller design technique for a third order system." Decision and Control, 1996., Proceedings of the 35th IEEE Conference on. Vol. 3. IEEE, 1996.

Problem 5 (10p)

The unit delay in discrete-time has pulse-transfer function

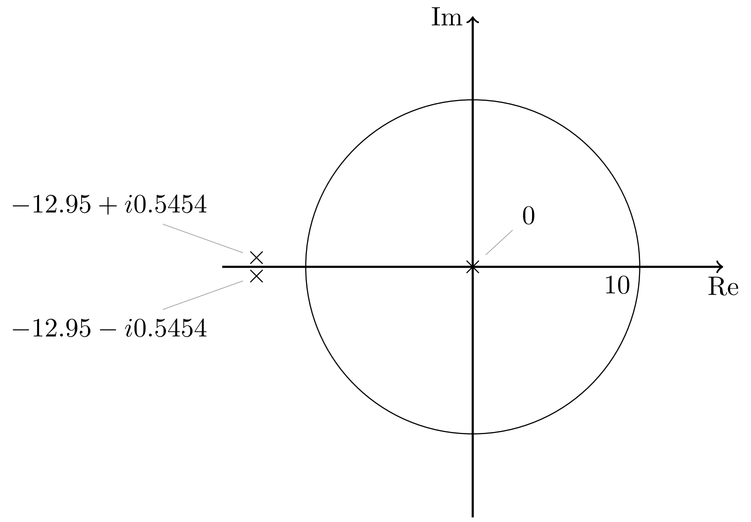
$$H(z) = \frac{1}{z}$$

with a single pole in the origin. Where is the pole of the corresponding continuous-time system?

Solutions

Problem 1

The continuous-time poles are



Problem 2 - Sampled model

1

Find the step-response in the Laplace-domain by partial fraction decomposition. This gives (with the help of a symbolic math tool)

$$\begin{aligned} Y(s) = \frac{G(s)}{s} &= \frac{168}{s^2(s^2 + 25.9s + 168)} \\ &= \frac{1}{s^2} - \frac{0.15417}{s} + \frac{0.15417s + 2.993}{s^2 + 25.9s + 168}. \end{aligned}$$

The last term can be written on the form

$$\frac{0.15417s + 2.993}{s^2 + 25.9s + 168} = b_1 \frac{s+a}{(s+a)^2 + \omega^2} + b_2 \frac{\omega}{(s+a)^2 + \omega^2},$$

with

$$a = 12.95, \quad \omega = 0.5454.$$

$$b_1 = 0.15417, \quad b_2 = 1.8269$$

We can then use the formulas for the laplace-transform of exponentially decaying sine- and cosines

$$\mathrm{e}^{-at}\cos(\omega t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{s+a}{(s+a)^2 + \omega^2}$$

$$\mathrm{e}^{-at}\sin(\omega t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{\omega}{(s+a)^2 + \omega^2}$$

2.

Apply the inverse laplace transform on each of the terms to get $y(t)$:

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{0.15417}{s}\right\} \\
&+ \mathcal{L}^{-1}\left\{0.15417\frac{s+12.95}{(s+12.95)^2+0.5454^2}\right\} \\
&+ \mathcal{L}^{-1}\left\{1.8269\frac{0.5454}{(s+12.95)^2+0.5454^2}\right\} \\
&= tu(t) - 0.15417u(t) + 0.15417\mathrm{e}^{-12.95t}\cos(0.5454t) + 1.8269\mathrm{e}^{-12.95t}\sin(0.5454t).
\end{aligned}$$

3.

Sample and apply z-transform The sampled step-response becomes (note that $u(kh) = u(k)$)

$$y(kh) = hku(k) - 0.15417u(k) + 0.15417\left(\mathrm{e}^{-12.95h}\right)^k\cos(0.5454hk) + 1.8269\left(\mathrm{e}^{-12.95h}\right)^k\sin(0.5454hk).$$

From the table of z-transforms we get

$$\begin{aligned}
Y(z) &= \frac{hz}{(z-1)^2} - \frac{0.15417z}{z-1} \\
&+ 0.15417\frac{z(z-\mathrm{e}^{-12.95h}\cos(0.5454h))}{z^2-2\mathrm{e}^{-12.95h}\cos(0.5454h)z+\mathrm{e}^{-2\cdot 12.95h}} \\
&+ 1.8269\frac{z\mathrm{e}^{-12.95h}\sin(.5454h)}{z^2-2\mathrm{e}^{-12.95h}\cos(0.5454h)z+\mathrm{e}^{-2\cdot 12.95h}}.
\end{aligned}$$

4

Divide by $U(z) = \frac{z}{z-1}$

$$\begin{aligned}
H(z) = \frac{Y(z)}{U(z)} &= \frac{h}{z-1} - 0.15417 \\
&+ 0.15417\frac{(z-1)(z-\mathrm{e}^{-12.95h}\cos(0.5454h))}{z^2-2\mathrm{e}^{-12.95h}\cos(0.5454h)z+\mathrm{e}^{-2\cdot 12.95h}} \\
&+ 1.8269\frac{(z-1)\mathrm{e}^{-12.95h}\sin(.5454h)}{z^2-2\mathrm{e}^{-12.95h}\cos(0.5454h)z+\mathrm{e}^{-2\cdot 12.95h}} \\
&= \frac{h-0.15417z+0.15417}{z-1} \\
&+ \frac{(z-1)\big(0.15417(z-\mathrm{e}^{-12.95h}\cos(0.5454h)) + 1.8296\mathrm{e}^{-12.95h}\sin(.5454h)\big)}{(z-\mathrm{e}^{-12.95h}\cos(0.5454h))^2+\mathrm{e}^{-12.95h}\sin^2(0.5454h)}.
\end{aligned}$$

Problem 3 - choose sampling period

The dynamics of the system is governed by the two complex-conjugated poles. These are almost completely damped, with

$$\zeta = \frac{12.95}{\sqrt{12.95^2+0.5454^2}} \approx 0.999.$$

and

$$\phi = \cos^{-1}(\zeta) = 0.042.$$

This gives for the epxression for rise time of a second order system (Å&W 2.9)

$$T_r \approx \omega_0^{-1} \mathrm{e}^{\phi/\tan(\phi)} = \omega_0^{-1} 2.72.$$

It is suitable with 4-10 sampling periods per rise time. This gives

$$(4-10)h \approx \omega_0^{-1} 2.72$$

or

$$h \approx \omega_0^{-1}(0.272 - 0.68)$$

with

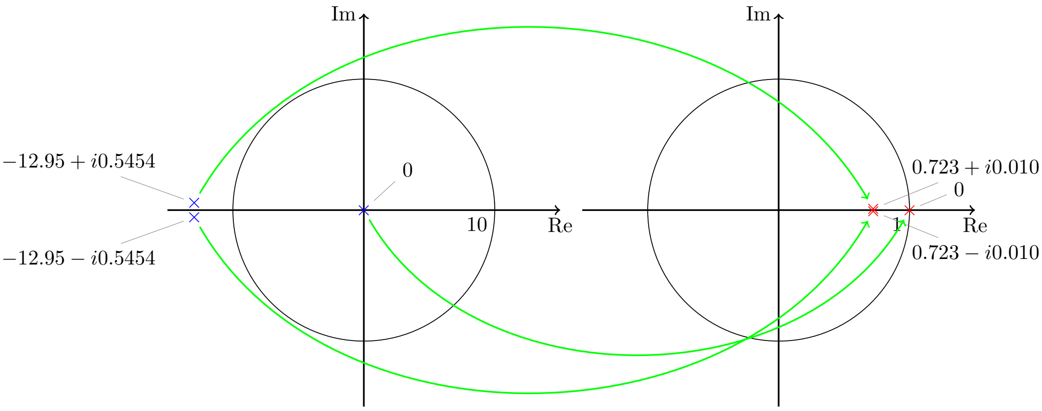
$$\omega_0 = \sqrt{0.5454^2 + 12.95^2} = 12.96$$
$$h \approx 0.021 - 0.052.$$

Problem 4 - plot the discrete-time poles

With $h = 0.025$ the poles are

$$\begin{aligned} z_1 &= \mathrm{e}^{-12.95h}\cos(0.5454h) + i\mathrm{e}^{-12.95h}\sin(0.5454h) \approx= 0.723 + i0.010 \\ z_2 &= \mathrm{e}^{-12.95h}\cos(0.5454h) - i\mathrm{e}^{-12.95h}\sin(0.5454h) \approx= 0.723 - i0.010 \\ z_3 &= 1 \end{aligned}$$

and plotted below



Problem 5 - pole of the continuous-time delay

The pole of a transfer function $G(s)$ is a value of s for which the value of the transfer function goes to infinity. The continuous-time transfer function corresponding to a unit delay in discrete time (delay of h) is

$$G(s) = \mathrm{e}^{-sh}.$$

The value of the transfer function goes to infinity as $s \to -\infty$. Hence the pole is at minus infinity.

Or, in other words a pole at minus infinity is mapped to zero through the mapping

$$z = \mathrm{e}^{sh}$$

that relates continuous- and discrete-time poles.