Nomenclature

Roman Symbols

$\mathbf{a}$	Acceleration of a grain or a material point
a	Initial aspect ratio of a granular column
C	Concentration in Kinetic theory
c	Pressure wave velocity (or) Lattice speed in LBM
d	Grain diameter
E_D	Energy lost through drag
E_f	Energy lost through friction
E_k	Kinetic energy
E_p	Potential energy
E_r	Energy lost due to remolding and transformation
E_v	Energy lost through viscous dissipation
$\mathbf{f}$ / F	Applied force (or) Vector of fluxes in the Navier-Stokes equation
g	Acceleration due to gravity
H	Height of a granular column
h	Vertical length scale in shallow water approximation (or) Length of a LBM cell
I	Inertial number (or) Moment of Inertia
k	Stiffness of a grain (or) time step (or) Permeability

k_f	Dimensionless parameter defined as the ratio of normal stiffness k_n to the applied normal pressure σ_n'
k^*	Dimensionless parameter defined as the ratio of elastic to inertial effects
L	Length of a granular column (or) background cell size
l	Horizontal length scale in shallow water approximation (or) particle spacing in the Material Point Method
m	Mass
p	Pressure
T	Temperature
t	Time
u	Velocity component along x-direction
v	Velocity component along y-direction
W	Width of the channel

Greek Symbols

α	Conductivity
δ	Overlap between grains (or) Dirac function
Γ	Dissipation through inelastic collision
$\dot{\gamma}$	Shear rate
λ	Constant of proportionality in the power-law relationship (or) The time step factor in DEM
μ	Coefficient of friction (or) Dynamic viscosity in fluid
ϕ	Friction angle in degrees
ρ	Density
σ	Stress or confining pressure
τ	Shear stress (or) Surface traction (or) Relaxation time in LBM

τ_c	Critical time
θ	Slope angle

Superscripts

$'$	Effective component of the normal stress

Subscripts

0	Initial state
ext	External
f	Final state
int	Internal
n	Normal component
t	Tangential component

Chapter 2

Granular flows

2.1 Introduction

A granular material is a conglomeration of a large number of discrete solid grains of sizes greater than $1\mu m$ whose behaviour is governed by frictional contacts and inelastic collisions. Figure 2.1 provides a schematic representation of the size range of granular materials. Characterized by the interaction between individual grains, granular materials lie between two extremes scales: the molecular-scale range predominated by the electrostatic force, i.e. Van der Waals forces, and the continuum scale which is described by the bulk property of the material. In various soil classification systems, sand is classified as a granular material with grain sizes greater than $75\mu m$. A grain size of $75\mu m$ demarcates an important transition: the point at which the frictional effect begins to dominate the material behaviour and the effect of the electrostatic Van de Waals forces diminishes. The wide range of grain size for granular materials, from the molecular size to the continuum scale, indicates the complexity of granular material behaviour. Such complexity encompasses both grain-like and continuum-like behaviour.

The physics of non-cohesive granular assemblies is intriguing. Despite being ubiquitous in nature and having a wide range of applications, including geo-hazard predictions, granular materials are the most poorly understood materials from a theoretical standpoint. For years, granular materials have resisted theoretical development, demonstrating non-trivial behaviour that resembles solid and/or fluid-like behaviour under different circumstances. Even in very simplistic situations, granular materials exhibit surprisingly complex behaviour at the macroscopic level. For instance, walking along the beach and bending over to scoop up a handful of sand demonstrates both its solid-like behaviour, from the firm support of one's weight as one walks, and its fluid-like behaviour, from the handful of sand running through one's fingers.

The range of grain size gives rise to complex interactions between grains which constitute the particular granular media. Unlike other micro-scale particles, soil grains are insensitive to thermal energy dissipation (Mehta, 2011), because the thermal energy dissipation in a granular material is several orders of magnitude smaller in comparison to the energy dissipation due to the interaction between the grains. Consequently, the thermal energy scales are small when compared to the energy required to move the grains. The granular material reaches the static equilibrium quickly due to its dissipative nature, unless an external source of energy is constantly applied (Choi, 2005).

Knowledge of the behaviour of granular assemblies is restricted to two extremes: the solid-like behaviour of dense granular assemblies that resist the shearing force by undergoing plastic deformations and the fluid-like flow behaviour characterized by high shear rates. Granular media are *a priori* simple systems made of solid grains interacting through their contacts. However, they still resist our understanding and no theoretical framework is available to describe their behaviour (Pouliquen et al., 2006). Because the behaviour of the granular material is highly dependent on its surrounding environment, it is difficult to establish a unified theoretical framework. When strongly agitated, the granular material behaves like a dissipative gas, and kinetic theories have been developed to describe this regime (Popken and Cleary, 1999; Xu et al., 2003). During slow deformations, on the other hand, the quasi-static regime is dominated by steric hindrance, and friction forces are often described using plasticity theories. In between the two regimes, the material flows like a fluid, and the grains experience enduring contacts, a behaviour which is incompatible with the assumptions of the kinetic theory (Pouliquen et al., 2006) that describes the dilute regime of a granular flow. Typical granular flows are dense and hence a fundamental statistical theory is not appropriate to describe their properties. Moreover, during the process of granular flow, the material can exist in all the above-mentioned states, further complicating our understanding of granular flows.

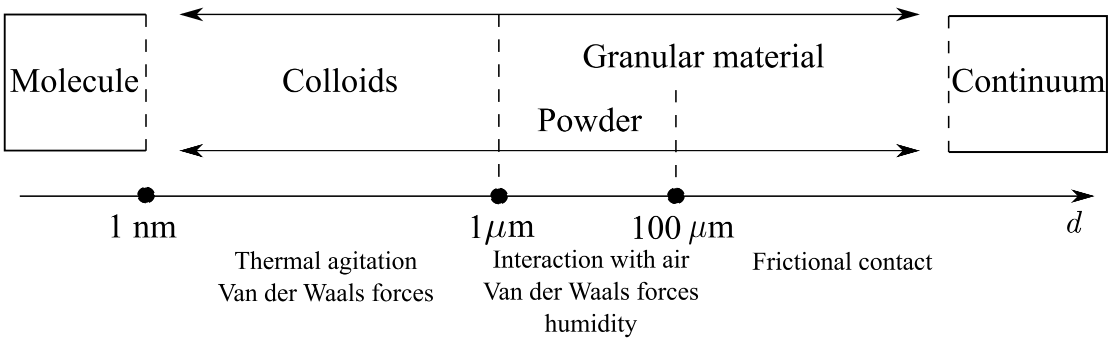


Figure 2.1 Particle size range and characteristics

2.2 Modelling granular flow

Different approaches have been used to model granular flows at different scales of description. The dynamics of a homogeneous granular flow involve at least three distinct scales: (1) the *microscopic scale* characterized by small time and length scales representing contact/grain interactions, (2) the *mesoscopic scale*, where grain rearrangements, developments of micro-structures and shear rates have a dominant influence on the granular flow behaviour, and (3) the *macroscopic scale* which involves large length scales that are related to geometric correlations at even larger scales. With such distinct scales, an immediate issue arises with modelling granular flow dynamics: whether one should consider or neglect a particular scale (Radjai and Richefeu, 2009). The difficulty in modelling the granular flows originates from the fundamental characteristics of the granular matter such as negligible thermal fluctuations, highly-dissipative interactions, and a lack of separation between the microscopic grain scale and the macroscopic scale of the flow (Goldhirsch, 2003).

Granular flows display a large span of grain concentrations, and therefore exhibit different behaviour at different concentrations (figure 2.2). Granular flows can be classified into three different regimes (Jaeger et al., 1996): *(a)* kinetic regime, *(b)* collisional regime, and *(c)* frictional regime. In the dilute part of the flow, grains randomly fluctuate and translate, this form of viscous dissipation and stress is named the *kinetic regime*. This regime is characterized by grains moving freely between successive collisions (Goldhirsch, 2003). At higher concentrations, in addition to the previous dissipation form, grains can collide quickly, this gives rise to further dissipation and stress, called the *collisional regime*. This intermediate fluid-like regime is dense but still flows like a fluid, and the grains interact interact through both collision and friction (Midi, 2004; Pouliquen and Forterre, 2002). Transfer of grain kinetic energy and momentum within a rapidly flowing granular medium occurs during these collisions (Popken and Cleary, 1999). At very high concentrations (more than 50% in volume), grains start to endure long, sliding and rubbing contacts, giving rise to a different from of dissipation and stress, *frictional regime*. This dense slow quasi-static regime is characterized by long duration between contacts and grain interaction via frictional contact (Roux and Combe, 2002).

The momentum and energy transfer are different in accordance with differing granular regimes. For the granular phase, it is imperative that any mathematical model attempting to model the granular flow must account for the above mechanisms, at any time and anywhere within the flow. The mathematical models require a comprehensive unified stress tensor able to adequately describe stress within the flow for any of these regimes, and this must be achieved without imposing which regime will dominate over the others. Several theoretical frameworks have been used to describe the granular flow behaviour. The predominant behaviour in most granular flows is friction. Hence, continuum models based on frictional properties of the granular mass have been widely adopted to discuss granular flow behaviour. Alternatively, dilute granular flow behaviour is conventionally modelled using the kinetic theory, while the dense granular flows are described using the $\mu(I)$ rheology. In certain conditions, where the lateral extent of the flow is significantly larger than the vertical component, shallow-water approximation is used. The capability and limitation of the various frameworks are discussed below.

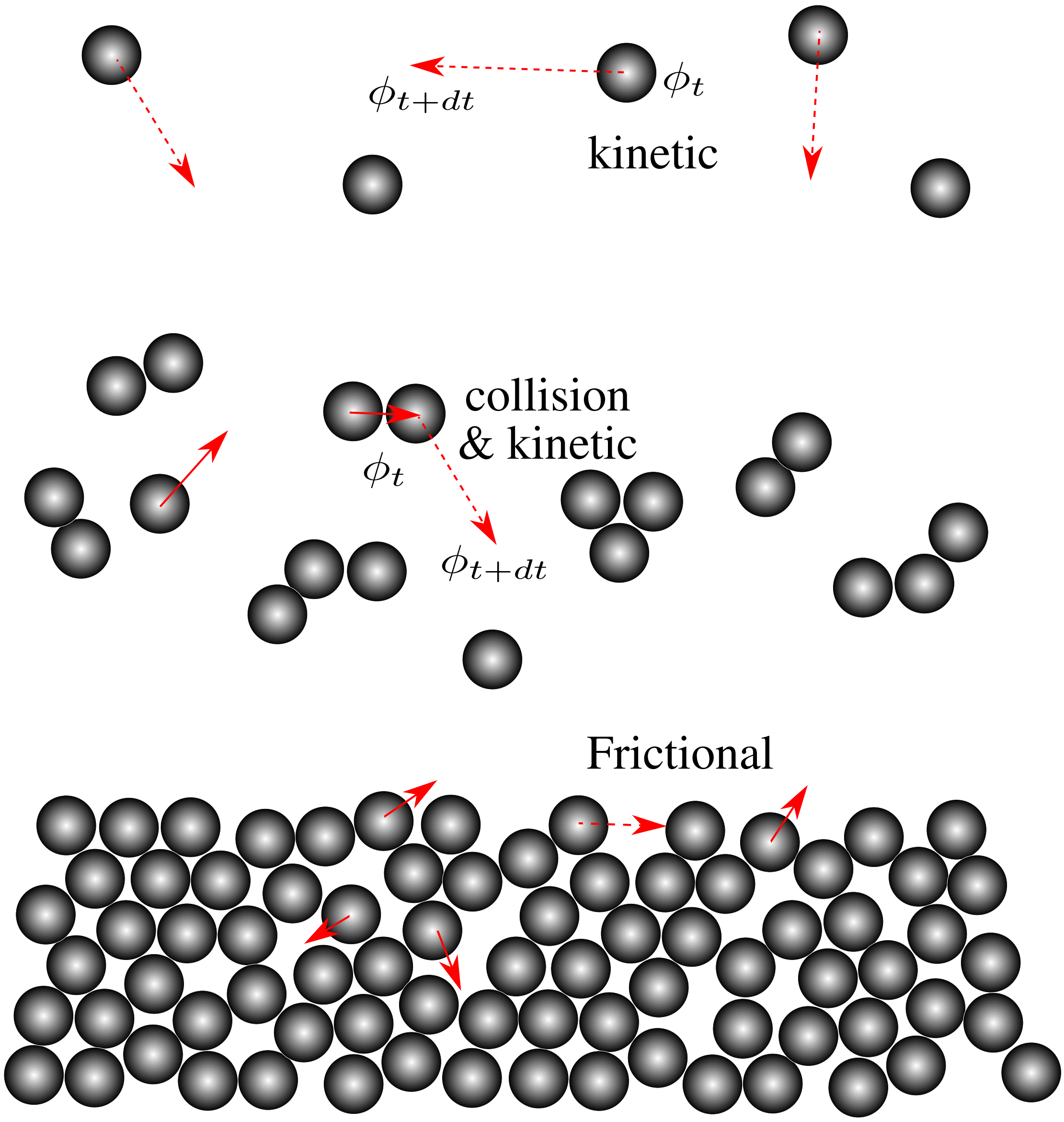


Figure 2.2 The modes of viscous dissipation in a granular flow

2.2.1 Continuum models

In the frictional regime, grains experience long and permanent contacts when they rub and roll against each other. Hence, a stress tensor based on the mechanical law of friction must be developed. Granular flow modelling began as early as 1776 with Coulomb's seminal paper describing the yielding of granular material as a frictional process. Although it was not about granular flows *per se*, the prediction of soil failure for civil engineering applications describes the onset of structural collapse leading to catastrophe (Campbell, 2006). Mohr-Coulomb's yield criterion, along with a flow rule from metal plasticity, is sufficient to describe the behaviour of granular flow as a continuum process, without considering the interaction of individual grains.

Advanced models based on the critical state concept (Schofield and Wroth, 1968) provide further insight into continuum description of granular flows. According to critical state theory, the 'under consolidated', or loose, soil tends to increase in density upon shearing, while the dense 'over consolidated' soil dilates when sheared, until reaching the critical state. Since dense granular flow involves large shear stresses, it is reasonable to assume that the shearing occurs at the critical state. Large applied stress can cause the granular solids to deform at the grain scale and squeeze them into the inter-grain pores. Granular flows experience rapid shearing, and therefore, it is also reasonable to assume that the flow is incompressible and occurs at the critical state (Campbell, 2006).

The main limitation of the continuum approach is the assumption that the friction angle, ϕ, is a constant material parameter (Potapov and Campbell, 1996). Although the mechanism of dense granular flow is attributed to bulk friction, it is the formation of force chains and the rearrangement of internal structure of the granular assembly that causes friction-like behaviour. Experiments (Savage and Sayed, 1984; Savage, 1984) and computer simulations (Campbell and Brennan, 1985) indicate a weak relation between the bulk friction and the packing density, due to the micro-structural rearrangement of grains (Campbell, 1986). As the packing density increases, the grains tend to arrange themselves in a regular order when sheared. In order to understand the development of micro-structure, it is important to examine the grain-level interactions. Bagnold (1954) was the first to attempt to model granular materials as individual grains. Bagnold's theory of motion of individual grains in a shear flow and inter-grain friction inducing random velocities is reminiscent of the thermal motion of molecules in the kinetic theory of gases.

2.2.2 Kinetic theory

The kinetic theory of gases assumes that the particles interact by instantaneous collisions, which implies only binary (two-particle) collisions. The interactions are modelled using a single coefficient of restitution, representing the energy dissipated by the impact normal to the point of contact between the interactions. For the most part, the surface friction or any other particle interactions tangential to the point of contact are ignored (Campbell, 1990). Jenkins and Savage (1983) extended kinetic theory from thermal fluids to idealized granular mixtures to predict the rapid deformation of granular material by including energy dissipation during collision for nearly inelastic particles. Savage and Jeffrey (1981) extended the kinetic theory to predict simple shear flow behaviour for a wide range of coefficients of restitution. While, the kinetic theory is capable of predicting the shear flow behaviour of particles with identical density and size (Iddir and Arastoopour, 2005), real systems are composed of particles that vary in size, and segregation of particles can occur.

The formulation of gas kinetic theory can be used to derive a set of equations for granular flow if the particles are assumed to be rigid. In turn, the rigid particle assumption implies that all contacts occur instantaneously. This assumption creates a vanishing probability of multiple simultaneous contacts and only binary contacts are considered. Kinetic theory formulation yields a set of Navier–Stokes-like equations. Conservation of mass is written as

$$\frac{d\rho c}{dt} + \rho c \nabla \cdot \mathrm{u} = 0\,. \tag{2.1}$$

Conservation of momentum

$$\rho c \frac{d\mathrm{u}}{dt} = \nabla p(\rho, c, T, \varepsilon) + \nabla \cdot (\eta(p, \rho, c, T, \varepsilon) \nabla \mathrm{u})\,. \tag{2.2}$$

Conservation of granular energy (granular temperature)

$$\rho c \frac{dT}{dt} = \nabla \cdot (\alpha(p, \rho, c, T, \varepsilon) \nabla T) + \tau : \nabla \mathrm{u} - \Gamma(p, \rho, c, T, \varepsilon)\,. \tag{2.3}$$

where α is the conductivity, $\tau : \nabla \mathrm{u}$ is the temperature production by shear work, p is the pressure, c is the concentration, ρ is the density, ε is the coefficient of restitution, T is the granular temperature and Γ is the dissipation through inelastic collisions. There are several problems that are immediately apparent with this formulation. The most obvious is that the range of applicability of rapid flow theory is limited. The solid phase stresses are viscous in nature (eq. 2.2), which results in a no-force condition when the granular mass is static (Campbell, 2006).

Kinetic theory is valid for dispersed granular flows (Ng et al., 2008). Van Wachem et al. (2001) observed that numerical simulations of dense granular flow based on kinetic theory did not accurately capture experimental data on fluidised bed expansion. Because of their mechanism of energy dissipation and their tendency to form clusters, confined granular flows are usually dense. Dense granular flows lie in an intermediate regime, where both the grain inertia and the contact network have significant influence on the flow behaviour (Pouliquen and Forterre, 2002). Thus, a part of the force is transmitted through the force network, which contradicts the two basic assumptions in the kinetic theory: binary collision and the molecular chaos.

For dense granular flow conditions, the total stress transmission in the flow regime is the sum of the rate-dependent (collision-transition) and the rate-independent (friction) components (Ng et al., 2008). The addition of a frictional stress component (Schaeffer, 1987) to kinetic theory improves the ability of the model to predict the dense granular flows. The main advantage of kinetic theories is that they can be used to derive deterministic constitutive laws to describe the behaviour of granular flows in a theoretical framework (Jenkins and Savage, 1983). Kinetic theories formulated on the assumption that the solid phase stress has a viscous response have limitations when applied to granular flows. A viscous material produces no force unless it is in motion; hence, kinetic theory based on viscous solid phase cannot explain the static force exerted by the granular materials on the walls, as observed in experiments. The frictional component that is based on long-duration contact is added to the instantaneous collision contacts term, which is self-contradictory. Also, the rapid-flow models based on gas kinetic theory assume that the molecular collisions are elastic, which entails that they do not dissipate energy (Campbell, 2006), an assumption which does not match reality. Finally, the important assumption of gas kinetic theory is molecular chaos, which assumes no correlation between the velocities or positions of the colliding particles. This assumption is in direct contrast to dense granular flow where the particles interact many times with their neighbours and a strong correlation between their velocities is inevitable.

2.2.3 Rheology

Rheology is the science of flow of materials with solid and fluid characteristics. In practice, rheology is principally concerned with describing the mechanical behaviour of those materials that cannot be described by the classical theories. Rheology seeks to establish an empirical relation between deformation and stresses. Consider a granular assembly of grains having diameter d and density ρ_d under a confining pressure σ_n' (see figure 2.3). Assume the material is sheared at a constant shear rate, $\dot{\gamma} = v_w/L$, imposed by the relative movement of the top plate with a velocity v_w. In the absence of gravity, force balance implies that the shear stress,

$\tau = \sigma_{xy}$, and the normal stress, $\sigma_n' = \sigma_{xy}$, are homogeneous across the cell. This configuration is the simplest configuration to study the rheology of granular flow, i.e. to study the effect of strain rate, $\dot{\gamma}$, and pressure, σ_n' on the volume and shear stress, τ.

Even though the granular materials have been extensively researched at the microscopic level, the continuum representation of granular materials in terms of conservation of mass and momentum is still an area of concern (Daniel et al., 2007; Midi, 2004). The prediction of rheology of granular materials, even in the simplest case is complicated, because they exhibit rate-dependent behaviour, and no single constitutive equation is able to describe the behaviour over a range of shear stress rates. Da Cruz et al. (2005) developed a well known rheology for granular flows that is based on a simple two-dimensional shear in the absence of gravity and that establishes the flow regime and rheological parameters scale with a dimensionless number representing the relative strength of inertia forces with respect to the confining pressure (Daniel et al., 2007), along the lines of Savage and Hutter (1991). The shear stress, τ, is proportional to the confining pressure, σ_n', and is written as

$$\tau = \sigma_n' \mu(I)\,. \tag{2.4}$$

The friction coefficient μ depends on the single non-dimensional parameter I, expressed as

$$I = \frac{\dot{\gamma} d}{\sqrt{\sigma_n' \rho_p}}\,. \tag{2.5}$$

The parameter I can be interpreted in terms of different time scales controlling the grain flow. If the grains are rigid, i.e. neglecting the elastic properties of the grains, then I is the only non-dimensional parameter in the problem. Hence, the shear stress, τ, must to be proportional to the pressure, σ_n', multiplied by a function of I. Comparing the shape of the function $\mu(I)$ with the experimental results of flow down an inclined plane, Jop et al. (2006) observed that the frictional coefficient increases from a minimal value of μ_s to an asymptotic value of μ_2, as the value of I increases. The variation of friction coefficient with I is shown in figure 2.4. The friction coefficient can be related to the inertial number I as

$$\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_o/I + 1}\,, \tag{2.6}$$

where I_o is constant, typically in the range of 0.25 - 0.3.

To formulate a complete constitutive model, it is essential to describe the volumetric behaviour. Based on dimensional analysis, it can be argued that the volume change is also a function of dimensionless parameter I and that it also depends on the maximum and the minimum possible void ratios as well as the time for microscopic rearrangement of grains.

Consider two rows of mono-dispersed grains. When a grain is located in the gap formed by two adjacent grains, it is assumed to have a maximum packing fraction. As the grain is sheared along the bottom row of grains, it moves out of the gap, resulting in a minimum packing fraction. The duration required for this rearrangement is directly proportional to the volume fraction. The dimensionless shear rate, expressed as the ratio between the duration of re-arrangement to the mean duration (see figure 2.18), has a linear relationship to the volume fraction.

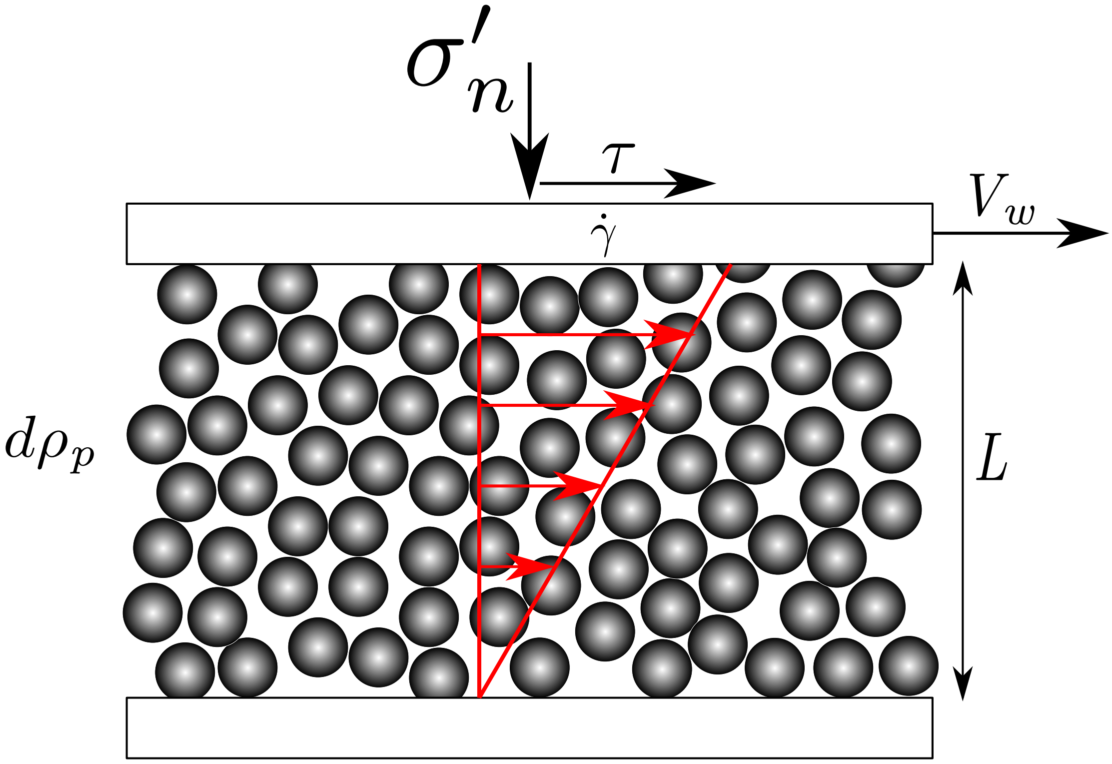


Figure 2.3 Plane shear stress distribution under a constant pressure and shear rate for a granular assembly

In general, the flow regimes can be classified based on the dimensionless number I (Da Cruz et al., 2005). Figure 2.5 shows the variation of frictional coefficient μ and packing fraction with dimensionless number I for different flow regimes under simple shearing. Dilute, or "collisional" flow occurs for $I > 10^{-1}$, and the grain collision is chiefly binary, accompanied by additional "bounce-back" akin to gases (Kamrin, 2008). In the dilute flow regime, the grains are rarely in long-duration contacts and can be described by dissipative Boltzmann kinetics. The "quasi-static" regime occurs at the other extreme of the spectrum, $I < 10^{-3}$, where the intermittent motion is prevalent. The inertial time is always small enough for the grains to align to a dense compaction, without significant collisional dissipation. The frictional sliding and stick-slip dynamics dominate the dissipation mechanism. The

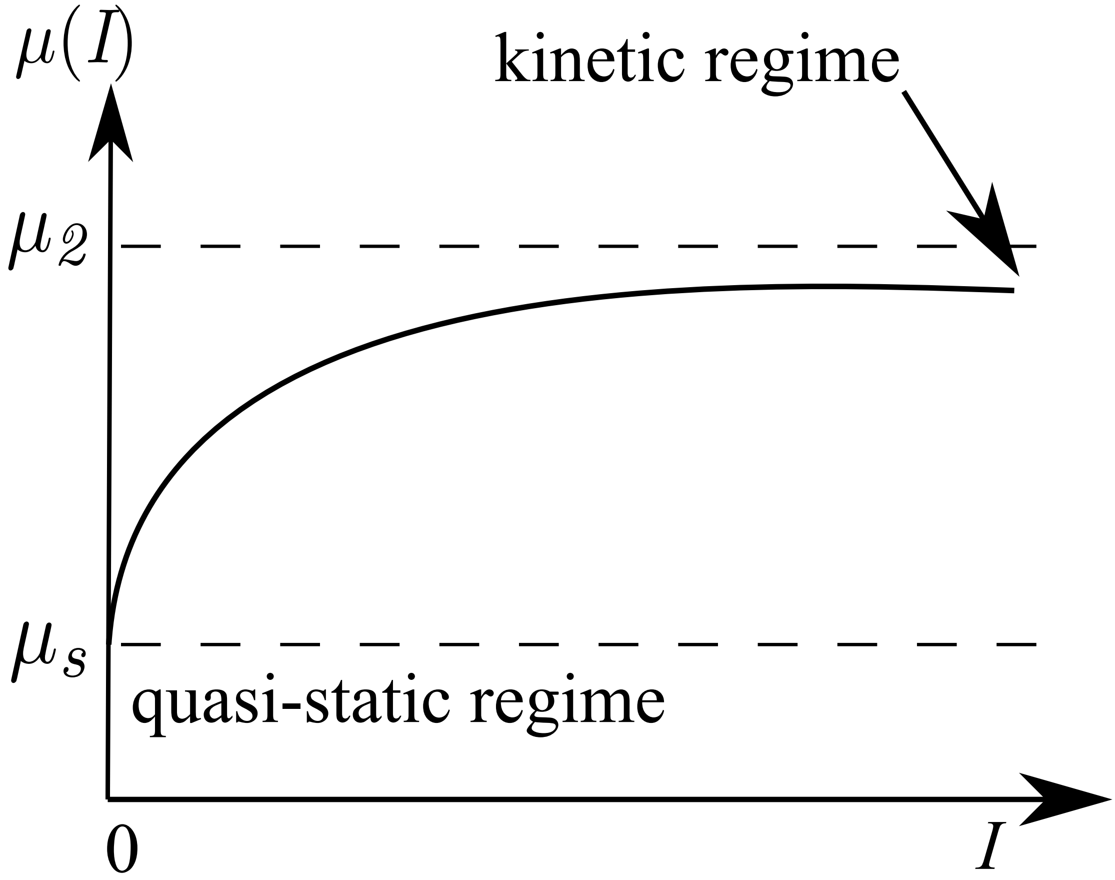


Figure 2.4 Sketch of dependence of frictional coefficient μ with dimensionless shear rate *I*, reproduced from Pouliquen et al. (2006)

stress/strain-rate relationship becomes singular, driving the system with a range of quasi-static normalized shear rates to all give the same time-average value for μ. In this regime, the dissipation is primarily frictional and rate-independent. The packing fraction appears to be independent of I and grain-level interactions control to flow dynamics. The moderate-flow regime is observed for I between $10^{-3}$and 10^{-1} and is characterized by faster flows, with a high rate of contact formation and more energy dissipation per impact. In this regime, I has a one-to-one relationship with μ and is large enough for rate dependence, yet small enough for the flow to remain dense. Moderate flows also exhibit the property of *shear rate dilation*, by which an increase in the normalized flow rate causes the steady-state packing fraction to decrease, which is different from *shear dilation*, which refers to a decrease in the packing density as a function of total shear. Flows which are too slow to be moderate still undergo shear dilation due to geometric packing constraints, but shearing dilation occurs only in faster flows due to rate effects (Kamrin, 2008). In moderate flows, the dissipation is primarily rate-sensitive due to energy loss during contact formation, yet packing remains dense.

Campbell (2002) described the “Moderate regime” as an elastic granular flow regime, in which the inter-grain stiffness governs the overall flow behaviour of the granular assembly. At high concentrations, the stresses are proportional to the contact stiffness, and the streaming stiffness (that represents the momentum carried by the unsteady motions of particles as they move through the system) is negligible. When a dense granular assembly is sheared, the force chains that transmit the forces continue to rotate until they become unstable and collapse. As the force-chains rotate, the granular material tends to dilate. However, it is restricted due to the constant volume constraint; instead, the rotation compresses the chain, generating an elastic response (Campbell, 2006). Campbell (2002) divided the flow into elastic and inertial regimes. In the elastic regime, the force is transmitted principally through the deformation of force chains with a natural stress scaling of $\tau d/k_f$. The dimensionless parameter, k_f, is defined as the ratio of normal stiffness k_n to the applied normal pressure σ_n'. The force chains form when the grains are sheared at the rate of $\dot{\gamma}$, and as a result of the rate of chain formation is proportional to the shear rate $\dot{\gamma}$. This transitional regime can be explained using the force-chain concept. The lifetime of a force chain is proportional to $1/\dot{\gamma}$ and consequently the product of the rate of formation and the lifetime of the force chain is independent of $\dot{\gamma}$, and the stresses generated are quasi-static. However, at higher shear rates, the elastic forces in the chain must absorb the additional inertial force of the grains, requiring extra force to rotate the chain proportional to the shear rate. Even though the grains are locked in force chains, the forces generated must represent the grain inertia. The ratio of elastic to inertial

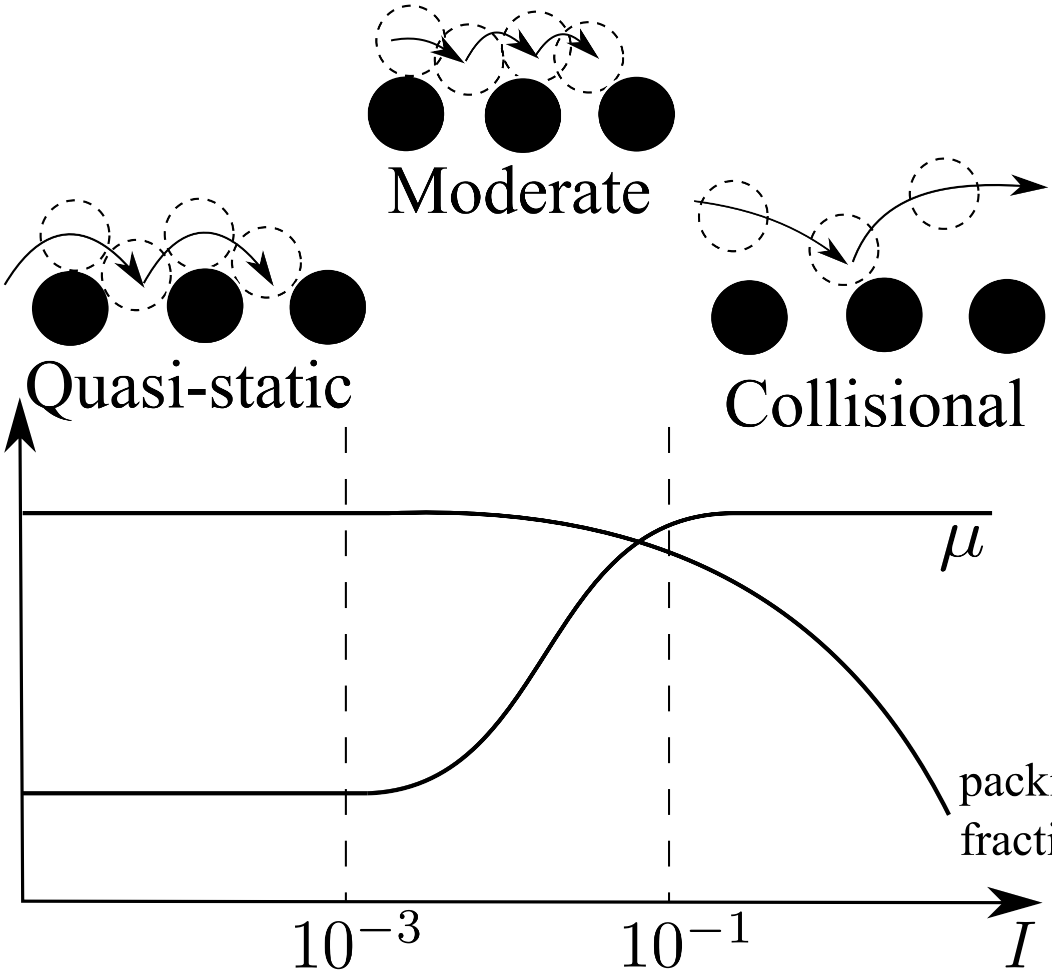


Figure 2.5 Variation of dimensionless parameter I for different flow regimes under simple shearing. (a) *Quasi-static* ($I < 10^{-3}$): Dissipation is primarily frictional and rate-independent. Packing fraction appears independent of I, and grain-level specifics are more important to flow dynamics; (b) *Moderate* ($10^{-3} < I < 10^{-1}$): The dissipation is primarily rate-sensitive due to energy loss during contact formation, yet packing remains dense; (c) *Collisional* ($I > 10^{-1}$): Flow becomes dilute and gas-like. Dynamics modelled best by dissipative Boltzmann kinetics. Redrawn from Kamrin (2008).

effects is governed by a dimensionless parameter

$$k^* = \frac{k_f}{\rho d^3 \dot{\gamma}^2}, \tag{2.7}$$

where $k_f/\rho d^3\dot{\gamma}^2 = (\tau/\rho d^2\gamma^2)/(\tau d/k\_f)$ is the ratio of Bagnold's inertial to the elastic stress scaling. The important dimensionless parameter is k^*, which is a measure of inertially-induced deformation represents the relative effects of elastic to inertial forces, i.e. at large k^*, the elastic forces dominate and at small k^*, inertial forces dominate (Campbell, 2006).

Constitutive laws, which describe dilatancy and friction, allow to deduce the dependency of pressure and shear stress on shear rate and solid fraction. In contrast to the observation of Campbell (2002), Da Cruz et al. (2005) found that the normalised elastic stiffness k_f has little effect on the constitutive law (for values greater than 10^4) however, it does affect the coordination number. Da Cruz et al. (2005) also observed that the microscopic friction coefficient, μ, has a significant influence on the dilatancy, and the solid fraction remains a linearly-decreasing function of *I*. The frictional properties of the material are found to control the solid fraction, from the critical state to the collisional regime (Da Cruz et al., 2005).

Although rheology tends to describe the behaviour of granular flows, the mechanism of granular flows is found to vary with duration, position of granular material in the flow and the pore-pressure feedback mechanism (Iverson, 2003). Rheology summarises the mechanical behaviour at scales that are smaller than the Representative Elemental Volume (REV) for a substance modelled as a continuum. Rheology-based descriptions are generally restricted to homogeneous materials that exhibit time-independent behaviour, and hence are unsuitable for describing granular flows where the stress history has a significant effect on the flow dynamics. The estimation of debris flow yield strength highlights the limitation of rheologies which do not consider the development of strength with evolution of time and space. Johnson (1965) emphasized that debris yield strength is predominantly a frictional phenomenon analogous to the Coulomb strength of granular soils, and that strength consequently varies with effective normal stress. Treatment of yield strength as an adjustable rheological property contradicts the basic understanding that the strength evolves as the debris-flow motion progresses. Frictional behaviour implies no explicit dependence of shear resistance on shear rate, whereas rheological formulas commonly used to model debris flows generally include a viscous component that specifies a fixed functional relationship between shear resistances and shear rate. Although rate-dependent shear resistance is observed in debris flows, its magnitude and origin indicate that it is ancillary rather than essential (Iverson, 2003).

2.2.4 Shallow-water approximation

Developing constitutive laws valid from the quasi-static to dilute regimes remains a serious challenge. A simple elasto-plastic approach fails to model the collisional regimes in a granular flow. On the other hand, the original kinetic theory based on binary collisions does not capture the correct behaviour in the dense regime. In configurations where the flowing layer is thin, a different theoretical framework is adopted. One such approach is the depth-averaged shallow-water equation, which has been applied to solve granular flow dynamics with a reasonable amount of success. The Savage-Hutter model (Savage and Hutter, 1991), is a depth-average continuum-mechanics-based approach which consists of hyperbolic partial differential equations to describe the distribution of the depth and the topography of an avalanching mass of cohesion-less granular media (Hutter et al., 2005). This approach is based on the assumption that the horizontal length scale is very large in comparison with the vertical length scale, which allows to neglect the horizontal partial derivatives relative to the vertical partial derivatives. Field observations of natural avalanches indicate an aspect ratio of 10^{-3} to 10^{-4} (Cawthor, 2006). By neglecting the vertical length scale, the continuum equation for conservation of mass and momentum can be written as

$$\partial_x u + \partial_y v = 0\,, \tag{2.8}$$

$$\partial_t u + u\partial_x u + v\partial_y u = (\nabla \cdot \boldsymbol{\sigma})_x + \boldsymbol{F}_x\,. \tag{2.9}$$

The continuum equation requires determining the components of the stress tensor and a suitable constitutive law. The *Savage-Hutter (SH) model* uses the Mohr-Coulomb law to describe the constitutive relation. The conservation of mass and momentum in the *SH* model is based on the assumption of granular flow as an incompressible fluid flow, which entails that the density of the avalanche, remains constant. Although Hutter et al. (1995) observed the density of the granular flow to remain almost constant in a flow down a curved chute, the destructive nature of landslides and avalanches restricts us from inferring a conclusive result. The SH model involves the following assumptions: (1) Coulomb-type sliding takes place with a bed friction angle δ, (2) Mohr-Coulomb frictional behaviour occurs inside the material with internal angle of friction, $\phi \geq \delta$, and (3) the velocity profile is assumed to be uniform throughout the avalanche depth. The granular flow over a rigid plane inclined at an angle, θ, is shown in figure 2.6. The mass and momentum balance in the SH model is written as

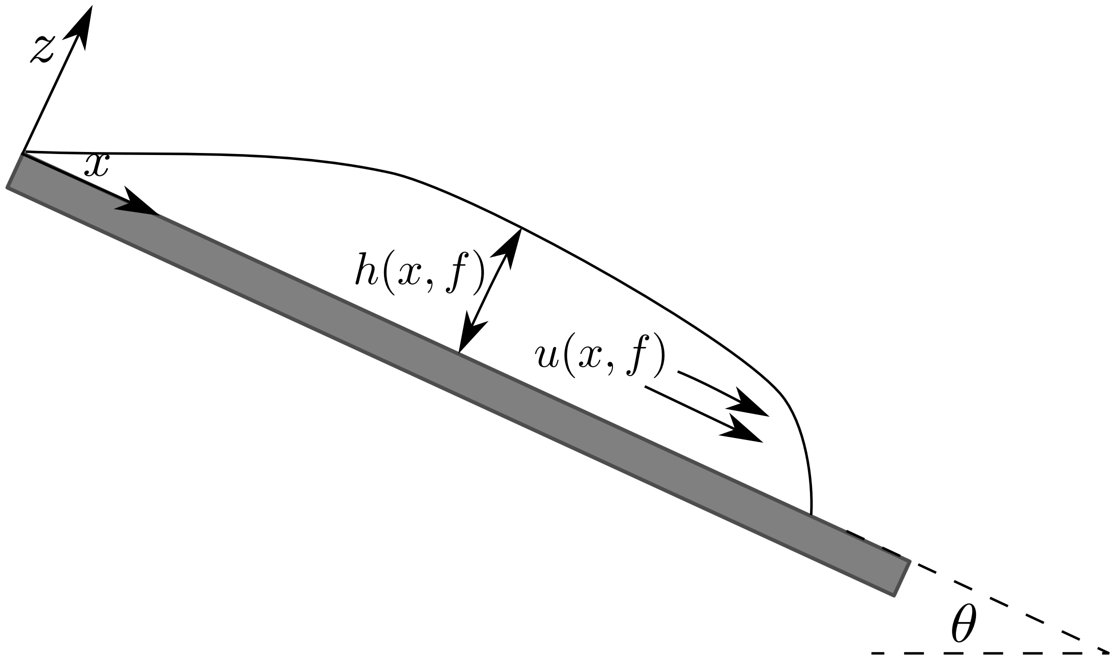


Figure 2.6 Illustration of the Savage-Hutter model

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)=0, \tag{2.10}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=(\sin\theta-\tan\delta sgn(u)\cos\theta)-\beta\frac{\partial h}{\partial x}, \tag{2.11}$$

where capital letters denote non-dimensional quantities with respect to the typical horizontal and vertical length scales (l^*, h^*) and the time scale ($\sqrt{l^*/g}$). The key feature in the shallow water approximation is the Mohr-Coulomb constitutive law, applied at the free surface and at the base, to describe the granular flow. Comparison of the model with the post-calculation of the Madlein avalanche in Austria indicates that the Coulomb basal friction is insufficient and requires an additional viscous component. The SH model's predictions were not satisfactory for granular flows down gentle slopes of inclination angle $\leq 30^o$, where granular materials exhibit premature stops (Hutter et al., 2005). The SH model has not yet been tested in cases where the granular flow interacts with obstacles.

The two main modelling techniques that are commonly employed to describe granular flow are the continuum approach and the discrete element approach. The continuum approach involves treating granular assembly as a continuum and describing its response using constitutive laws, while the discrete approach involves considering the individual grains of the granular material and applying Newton's laws of motion to describe the deformation of the material. These approaches are adopted in the present study and detailed discussions are provided in **??**.

2.3 Studies on granular flows

The flow of dense granular material is a common phenomenon in engineering predictions, such as avalanches, landslides, and debris-flow modelling. Despite the huge amount of research that has gone into describing the behaviour of granular flow, a constitutive equation that describes the overall behaviour of a flowing granular material is still lacking. To model geo-physical scale problems, the depth-averaged constitutive equations have been employed along with an empirical friction coefficient and a velocity profile deduced from experiments (Iverson, 2003; Midi, 2004; Pouliquen, 1999). Although this approach has been successful to a certain extent in predicting geophysical flows (Hutter et al., 1995; Pouliquen and Chevoir, 2002), it presents two important shortcomings (Lajeunesse et al., 2005): firstly, the depth-average method is true only if the thickness of the flowing layer is thin in comparison with the lateral dimension, and secondly, the empirical laws are deduced from experiments performed under steady-flow conditions. The shortcomings cast doubt on the validity of the depth-averaged approach. Two simple granular flow studies, granular column collapse and granular flow down an inclined plane, carried out by various researchers to understand the flow behaviour are discussed in the following subsections.

2.3.1 Granular column collapse

Lube et al. (2005) and Lajeunesse et al. (2004) have carried out experimental investigations on the collapse behaviour of granular columns on a horizontal plane. Both experiments involved filling a column of height H_0 and length L_0 with granular material of mass m. Figure 2.7 shows the schematic view of the experimental configuration of a quasi-two-dimensional granular column collapse in a rectangular channel. The granular column is then released *en masse* by quickly removing the gate, thus allowing the granular material to collapse onto the horizontal surface, forming a deposit with final height H_f and radius L_f. Although the experiment is simple and attractive allowing us to explore the limitations of depth-average modelling techniques, a constitutive law that could describe the entire flow behaviour is still lacking. The primary aim of these experiments was to determine the scaling laws for the run-out distance.

Deposit morphology

Experimental findings

Lajeunesse et al. (2005) observed that the flow dynamics and the final run-out distance remain independent of the volume of granular material that is released, but depend only on

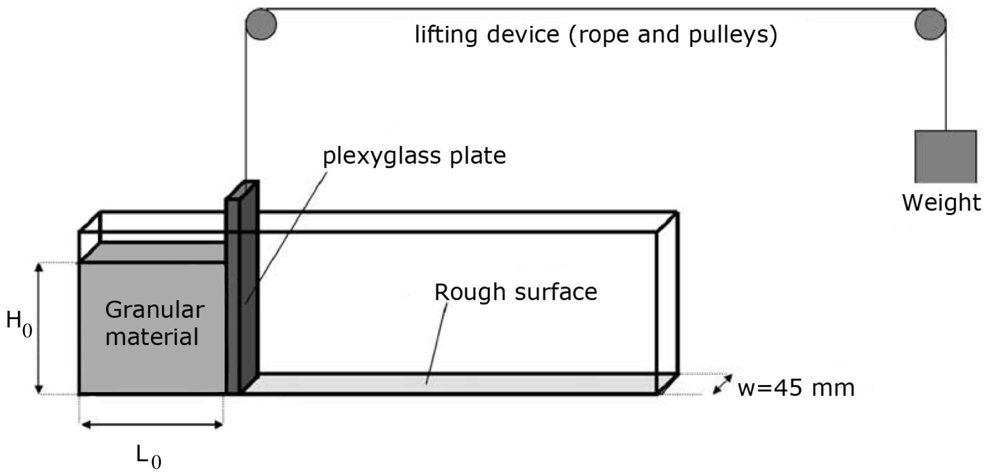


Figure 2.7 Schematic view of the experimental configuration of a quasi-two-dimensional granular column collapse in a rectangular channel (Lajeunesse et al., 2004)

the initial aspect ratio a of the granular column. The experiment was conducted in order to understand the effect of the geometrical configuration on the run-out, the mechanism of initiation of the flow, the evolution of flow with time, and how such complex flow dynamics could produce deposits obeying simple power laws. Lube et al. (2005) explored the effect of density and shape of grains on flow dynamics, whereas Lajeunesse et al. (2004) worked with glass beads to study the influence of bead size and substrate properties on the deposit morphology. Surprisingly, both drew the striking conclusion that the flow duration, the spreading velocity, the final extent of the deposit, and the fraction of energy which is dissipated during the flow can be scaled in a quantitative way independent of substrate properties, bead size, density, and the shape of the granular material and released mass, m (Lajeunesse et al., 2005).

The normalised final run-out distance as a function of the initial aspect ratios of the granular column under plane-strain and axisymmetric conditions is displayed in figure 2.8a. Lube et al. (2005) scaled the run-out distance as

$$\frac{L_f - L_0}{L_0} \approx \begin{cases} 1.24a, & a \lesssim 1.7 \\ 1.6a^{1.2}, & a \gtrsim 1.7 \end{cases} \tag{2.12}$$

while Lajeunesse et al. (2005) scaled the run-out distance as

$$\frac{L_f - L_0}{L_0} \approx \begin{cases} 1.35a, & a \lesssim 0.74 \\ 2.0a^{1.2}, & a \gtrsim 0.74 \end{cases} \tag{2.13}$$

The final run-out distance is found to have a linear relationship for short columns and exhibit a power-law relation with the initial aspect ratio of tall columns.

The normalised final height as a function of the initial aspect ratios of the granular column under plane-strain and axisymmetric conditions is shown in figure 2.8b. The evolution of the final scaled deposit height H_f/L_0, with the initial aspect ratio a for the axisymmetric collapse (Lajeunesse et al., 2005) is given as

$$H_f/L_0 \approx \begin{cases} a, & a \lesssim 0.74 \\ 0.74, & a \gtrsim 0.74 \end{cases} \tag{2.14}$$

and for two-dimensional collapse

$$H_f/L_0 \approx \begin{cases} a, & a \lesssim 0.7 \\ a^{1/3}, & a \gtrsim 0.7 \end{cases} \tag{2.15}$$

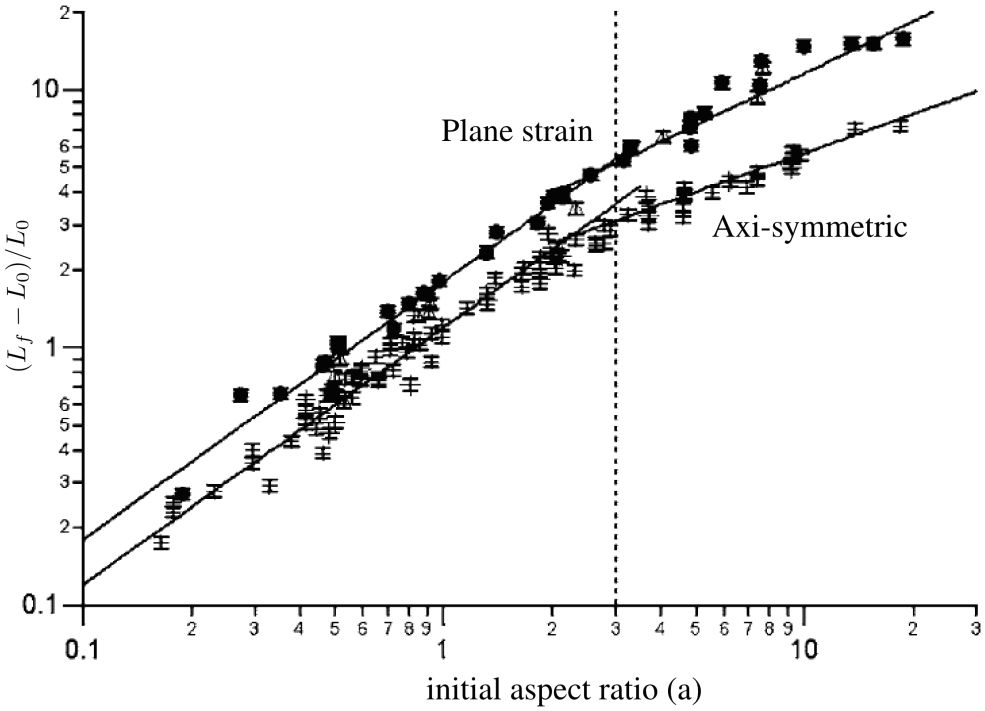
The final height of collapse is unaffected in both 2D collapse and axisymmetric collapse for short columns. In the case of tall columns, axisymmetric collapse exhibits a constant final height, whereas a power-law relation with the initial aspect ratio is observed in 2D collapse.

Axisymmetric versus two-dimensional collapse

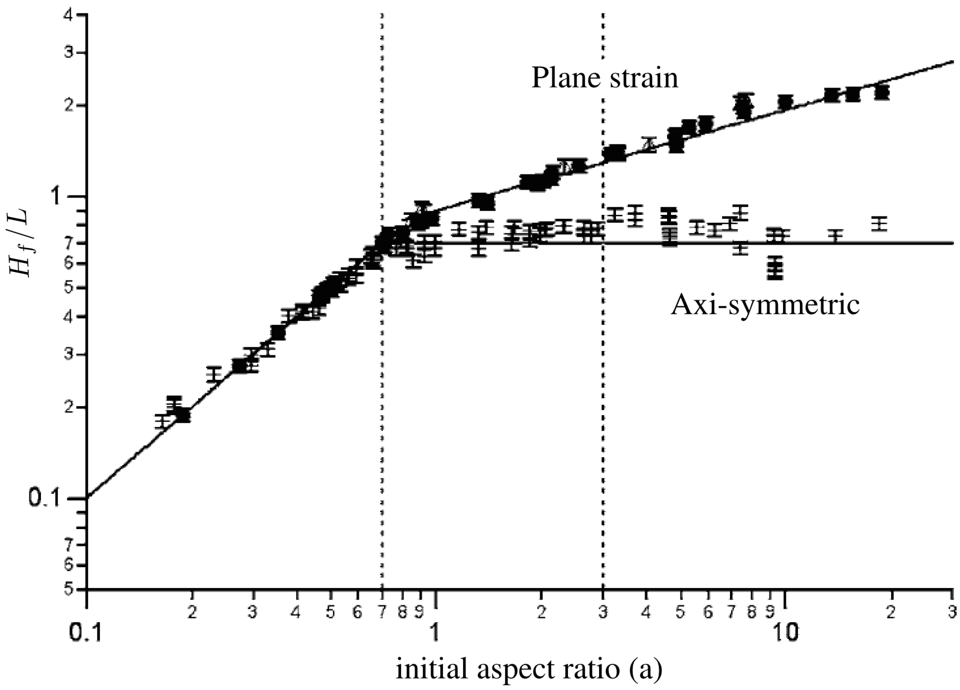
Quasi-two-dimensional collapse of a granular column on a horizontal surface reveals that the geometric configuration influences the scaling of the run-out distance (Lajeunesse et al., 2005). The run-out in a quasi-two-dimensional collapse of a granular column in a rectangular channel scales as

$$\frac{L_f - L_0}{L_0} \approx \begin{cases} 1.2a, & a \lesssim 2.3 \\ 1.9a^{2/3}, & a \gtrsim 2.3 \end{cases} \tag{2.16}$$

Balmforth and Kerswell (2005) studied the collapse of granular columns in rectangular channels with a narrow (width W of the slot = 10 * diameter d) and a wide slot (W = 200 * d), and focused on the deposit shape. Lacaze et al. (2008) observed that slots with width $W \geq 1.2 \times d$ do not crystallise and wider slots $W \geq 2 \times d$ overcome the effect of jamming. As in the axisymmetric case, Balmforth and Kerswell (2005) observed that the run-out is well-represented at large aspect ratios by a simple power-law expression, which depends on the width of the channel. The run-out distance can be scaled as $\Delta L/L_0 \approx \lambda a^{0.65}$ for narrow channels and as $\Delta L/L_0 \approx \lambda a^{0.9}$ for wide channels. The scaling found for quasi-two-dimensional experiments in the narrow gap configuration gives similar results to those



(a) Normalised final run-out distance vs. aspect ratio



(b) Normalised height vs. aspect ratio

Figure 2.8 The normalised final run-out distance and final height as a function of the initial aspect ratios of the granular column under plane-strain and axisymmetric conditions (Lajeunesse et al., 2004).

reported by Lube et al. (2005) and approximately a scaling of $(L_f - L_0)/L_0 \propto a^{2/3}$. However, these laws are influenced by the presence of sidewalls and depend, albeit only in the numerical coefficient of proportionality, on the frictional properties of the granular material.

Balmforth and Kerswell (2005) observed that the constant of proportionality, λ, in the power-law relation is found to vary with the internal friction angle of the granular material. This observation contradicts the findings of previous authors, especially Lube et al. (2005) who found that the scaling of run-out is independent of the granular material. Such a contradiction may perhaps be due to a narrow range of experimental materials and grain size distributions considered in the previous studies. Balmforth and Kerswell (2005) found that the material properties have almost no influence on the exponent of the normalised run-out as a function of the initial aspect ratio. The numerical constant of proportionality, however, showed clear material dependence. There by corroborating the conclusions of Lajeunesse et al. (2004). Daerr and Douady (1999) also observed a strong influence of initial packing density and the internal structure on the behaviour of granular flows. By comparing the initial and final cross-section areas of the pile, Balmforth and Kerswell (2005) observed that the granular material experienced dilation (by about 10%) as the flow progressed to form the final deposit (Balmforth and Kerswell, 2005). Although internal packing structure and density change is found to have an influence on the run-out behaviour, a proper understanding of the effect of density on the run-out and evolution of packing fraction is still lacking.

Numerical modelling

Numerical simulations of granular column collapse by Zenit (2005) and Staron and Hinch (2007) yielded similar scaling of run-out with aspect ratio a. Unlike other researchers, Zenit (2005) did not observe any transition in the run-out behaviour of a granular column collapse with the aspect ratio a. The origin of the exponents is still under discussion. No model has yet achieved a comprehensive explanation of the complex-collapse dynamics. For higher aspect ratios, the free fall of the column controls the dynamics of the collapse, and the energy dissipation at the base is attributed to the coefficient of restitution. Thus, the initial potential energy stored in the system is dissipated by sideways flow of material, and the mass ejected sideways is found to play a significant role in the spreading process, i.e. as a increases, the same fraction of initial potential energy drives an increasing proportion of initial mass against friction. Thus explaining the power-law dependence of the run-out distance on a.

Taking advantage of the similarity between granular slumping and the classical "*dam break*" problem in fluid mechanics, Kerswell (2005) solved both the axisymmetric and two-dimensional granular-collapse problem using the shallow-water approximation. Although the results of the shallow-water approximation have good agreement with experimental

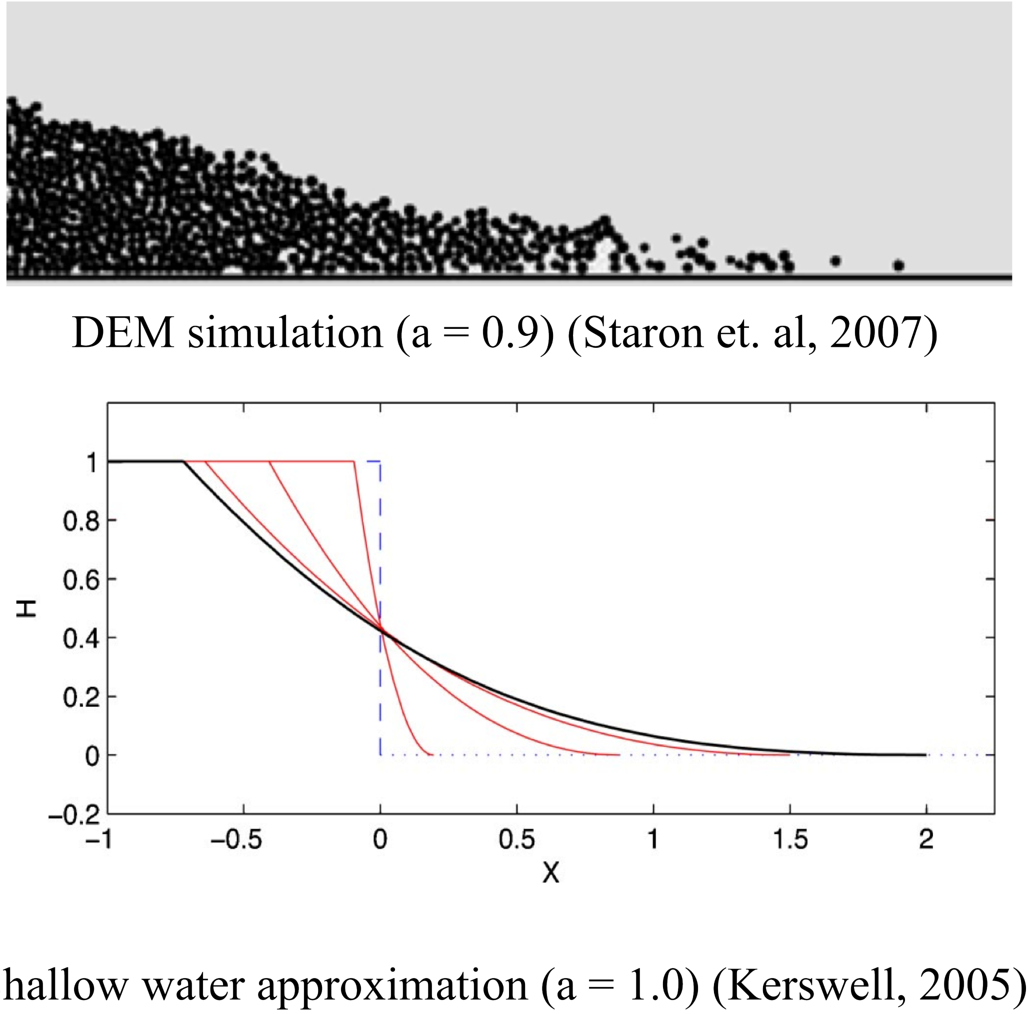


Figure 2.9 Collapse of granular column simulation using DEM (Staron and Hinch, 2007) and Shallow-water approximation (Kerswell, 2005).

results, the shallow-water approximation overestimates the run-out distance for columns with aspect ratio *a* greater than unity. The shallow-water equation does not take into account the effect of vertical acceleration (Lajeunesse et al., 2005), which has been found to play a significant role in controlling the collapse dynamics (Staron and Hinch, 2007), thus resulting in overestimation of the run-out distance. The evolution of run-out predicted from shallow-water approximation and DEM are shown in figure 2.9. Tall columns demonstrated significantly longer run-out distances when using continuum approaches like the material point method (Bandara, 2013; Mast et al., 2014). It was observed that a simple friction model cannot effectively describe the collapse dynamics (Staron and Hinch, 2007). However, the reason for difference in the run-out behaviour is currently not known.

The final collapse height observed in the numerical simulations of granular collapse is similar to that of the experimental results (Balmforth and Kerswell, 2005; Lube et al., 2005). Numerical simulation of granular column collapse (Lacaze et al., 2008; Staron and Hinch, 2007) displayed a transition in the flow behaviour at $a \geq 10$, which was not observed in granular collapse experiments (Balmforth and Kerswell, 2005; Lajeunesse et al., 2004; Lube et al., 2005). In the depth-averaged shallow-water model the emphasis is on capturing the scaling of the final deposit, rather than trying to reproduce the internal structure of the flow. The shallow-water model succeeds in capturing the final deposit scaling for lower aspect ratios, yet it fails to capture the flow dynamics for granular columns with higher aspect ratios, where the flow is governed mainly by the vertical collapse of the granular column as a whole. The run-out distance predicted is clearly erroneous in the collapse regime where there is a sudden drop in efficiency by which the initial potential energy of the system is converted into the kinetic energy for spreading. According to Kerswell (2005), even a more sophisticated basal drag law would not be sufficient to model the mechanism of granular column collapse realistically using the shallow-water approximation. Given the large spectrum of theoretical frameworks, no consensus exists on the origin of the power-law, and finding constitutive laws that maintain validity from the quasi-static to the dilute regimes remains a serious challenge.

Flow dynamics

Experimental findings

The final scaled run-out distance shows a transition from a linear to a power-law relationship with the initial aspect ratio of the column at a of 1.7, indicating either a transformation in the spreading process or the collapse mechanism. To understand the collapse mechanism, it is insufficient to study only the final scaled profile, and hence the entire flow process should be analysed. Lajeunesse et al. (2005) observed the flow regime and deposit morphology for a quasi-two-dimensional granular collapse in a rectangular channel. The flow phenomenology of a granular column collapse in a rectangular channel was surprisingly similar to that observed in the axisymmetric collapse (Lajeunesse et al., 2004; Lube et al., 2005), dependent mainly on the initial aspect ratio a (section 2.3.1).

The flow dynamics involve the spreading of granular mass by avalanching of flanks, producing a truncated cone for $a \lesssim 0.74$ and a cone for $a \gtrsim 0.74$. As the aspect ratio is increased, the transition of flow dynamics occurs (figure 2.10). The evolution of the deposit height remains independent of the flow for $a \lesssim 0.7$, however it exhibits significant dependence on the geometrical configuration for $a \gtrsim 0.7$. In rectangular channels, the effect of side-wall on the run-out behaviour was observed; the surface velocity profile between the side walls

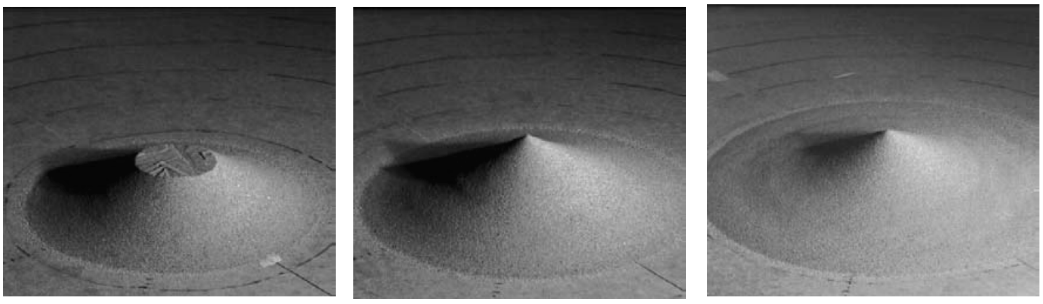


Figure 2.10 Final deposit profiles for granular column collapse experiments with different initial aspect ratios (Lube et al., 2005).

is that of a plug flow with a high slip velocity at the wall and low shear along the direction transverse to the flow. Systematic measurements indicate that the ratio of the maximum surface velocity to the surface velocity at the wall is between 1.2 and 1.4. Lajeunesse et al. (2005) observed that the difference between the evolution of H_f in the axisymmetric geometry and in the rectangular channel is not an experimental artefact due to the side wall friction, rather is a *geometrical effect*.

Understanding the internal flow structure will provide an important insight into the complex collapse dynamics. The failure surface observed at $t = 0.4\tau_c$, where τ_c is the critical time at which the flow is fully mobilised, for granular columns with initial aspect ratio of 0.4 and 3 are shown in figure 2.11. For smaller values of aspect ratio $a \leq 0.7$, the flow is initiated by a failure at the edge of the pile along a well-defined shear band above which material slides down and below which the grains remain static. The grains located above the shear-failure surface move *en masse* and most of the shear is concentrated along this surface, forming a truncated-cone-like deposit with a central motionless plateau in figure 2.11. For columns with larger aspect ratios, the flow is still initiated by failure along a well-defined surface, an inclined plane in two-dimensional geometry or a cone in the axisymmetric case. However, the initial height of the column is much higher than the top of the failure surface, causing a vertical fall of grains until they reach the summit where they diverge along the horizontal direction, dissipating a lot of kinetic energy, resulting in a final conical deposit. Interestingly, the final deposit height coincides with the summit of the failure surface in the axisymmetric geometry, whereas in the rectangular channel, the deposit summit always lies above the top of the failure surface (Lajeunesse et al., 2005). Shallow-water approximations show a truncated cone-like deposit, while DEM simulations show a cone-like deposit for $a \sim 1.0$ (figure 2.9). This shows the inability of the shallow-water approximation to model tall columns.

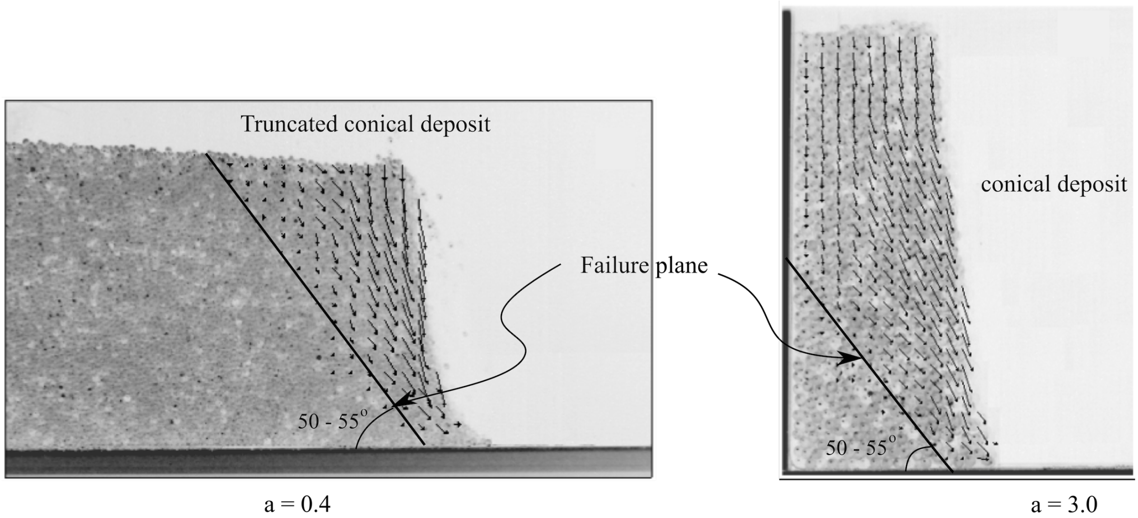


Figure 2.11 The extent of the failure surface for granular columns with initial aspect ratio of 0.4 and 3 at time $t = 0.4\tau_c$ (Lajeunesse et al., 2004). Short columns show truncated conical deposit at the end of the flow, while tall columns exhibit conical deposit.

Identification of the static region is an important task, as the static region is a prime component in describing the collapse mechanism. Regardless of the experimental configuration, the flow is initiated by rupture along a well-defined failure surface, and the failure angle remains of the order of 50° to 55° (figure 2.11). The failure angle is consistent with an interpretation of *active Coulomb failure*, which leads to a failure angle $\phi_f = 45° + \phi'/2$, where ϕ' is the internal friction angle of the granular material. The internal friction angle of glass beads is estimated from the angle of repose as 22°, thereby the failure angle is computed as 56°, which is in good agreement with the experimental findings. Contrary to the assumption of Lajeunesse et al. (2004), the shear-failure angle was found to have no direct effect on the transition between truncated cone and conical deposit occurring at aspect ratio a of 0.7 (Lajeunesse et al., 2005). Schaefer (1990) observed the onset of instability in a narrow wedge of 56° to 65°, which corresponding to the angle of shear bands. A rate-dependent constitutive relationship (Jop et al., 2006) for dense granular flows indicate the angle of shear-band orientation depends on the inertial number I. For small to moderate values of I, the orientation of shear bands is found to vary from the Roscoe and the Coulomb solutions to a unique admissible angle (Lemiale et al., 2011). Daerr and Douady (1999) observed active Coulomb-type yielding in transient surface flows for granular materials with a packing density of 0.62 to 0.65.

Numerical findings

In order to describe the complex flow dynamics, it is necessary to understand the internal structure and the flow behaviour. (Staron and Hinch, 2007) categorised the flow evolution into three stages. The first stage involves conversion of the initial potential energy of the grains into vertical motion, resulting in downwards acceleration of grains. In the second stage, the grains undergo collision with the base and/or neighbouring grains, and their vertical motion is converted into horizontal motion. The velocity field depends on the position of grains along the pile. In the region above the static core, the flow is locally parallel to the failure surface and has an upper linear part and a lower exponential tail near the static bed (figure 2.11). The velocity flow profile is similar to that of a steady granular flow (Midi, 2004). In the third and final stage, the grains eventually leave the base area of the column and flow sideways. At the front, the flow involves the entire thickness of the pile and corresponds to a plug flow in the horizontal direction. The typical velocity observed at the front of the ejecting mass is $v = \sqrt{2gL_0}$. As the pile spreads, the flow diverges resulting in separation of the interface, and the static region starts to move inwards. This particular effect is predominant in the case of granular flows in a rectangular channel.

The typical time required for the flow to cease and form the final deposit, from the instant of its release, is $\tau_c = \sqrt{H_0/g}$ (Staron and Hinch, 2007). While plotting the variation of normalized potential and kinetic energy with normalized time, Staron and Hinch (2007) observed that the flow ceases when the normalized time t/τ_c is 2.5, i.e. the flow is assumed to have stopped when the total normalized energy is almost zero. This observation is consistent with the experimental results of both Lube et al. (2005) and Lajeunesse et al. (2005). The transition of the flow occurs when the normalized time t/τ_c is 1.0, which is defined as the critical time at which the flow is fully mobilized.

Comments on modelling

In order to have a detailed understanding of the final profile of the collapsed granular column, it is important to solve the collapse problem as an *initial-value problem* (Balmforth and Kerswell, 2005), beginning from the instant of release and extending to the time when the material finally ceases to flow, forming the final deposit. Since the process of granular collapse involves collective dynamics of collisions and momentum transfer, the prediction of the trajectory of a single grain is difficult. In fact, there are quantitative disagreements between theory and experiments; the final shapes are reproducible, but not perfectly. Some of the disagreement arises because the experiments did not have exactly the same amount of materials. Understandably, it is indeed difficult to fill the pile with exactly the same amount of material, which inevitably results in differences in packing. However, the theoretical errors are due to the inability of the models to capture the physics that governs the flow dynamics (Balmforth and Kerswell, 2005).

Shallow-water models fail to account for the vertical acceleration, which is responsible for the momentum transfer and, in turn, the spreading process. This failure restricts the shallow water model from capturing the mechanism of collapse until the critical time τ_c. Surprisingly, shallow-water models capture certain experimental aspects for columns with lower aspect ratios (Balmforth and Kerswell, 2005; Kerswell, 2005; Mangeney et al., 2010), even though the contrast between surface flows and the static region is important in this range of aspect ratios. Thus, the assumption of plug flow in the horizontal direction is not critical in capturing the run-out behaviour, particularly if the basal friction coefficient is used as a fitting parameter (Lajeunesse et al., 2005).

Simple mathematical models based on conservation of horizontal momentum capture the scaling laws of the final deposit. However, they fail to describe the initial transition regime, indicating that the initial transition has negligible effect on the run-out, which is incorrect. Models based on the initial potential energy show promise, but the effect of material properties, such as basal friction and coefficient of restitution, on the run-out behaviour is still unclear and produces non-physical run-outs. The $\mu(I)$ rheology predicts the normalized run-out behaviour quite well in comparison with the experimental results, at least for lower aspect ratios. The spreading dynamics are found to be similar for the continuum and grain approaches. Yet, the rheology falls short in predicting the run-out distance for higher aspect ratios.

Unlike Lube et al. (2005), some researchers (Balmforth and Kerswell, 2005; Kerswell, 2005) observed a strong dependency of material properties on the run-out distance. Moistening the materials or the sides of the channel even by a small amount leads to markedly different results. Staron and Hinch (2007) observed that the friction has little effect on the run-out for granular column collapse for high aspect ratios, which are driven mainly by the free vertical fall of grains. The initial conditions have a significant impact on the overall behaviour of the granular system, indicating the significance of the triggering mechanism in the case of the natural flows (Staron and Hinch, 2007). A theoretical framework that is capable of describing the influence of material properties on the run-out behaviour is still lacking. Numerical investigations, such as Discrete Element Method techniques, allow us to evaluate these quantities which are not accessible experimentally, thus providing useful insight into flow dynamics. Subsequent chapters discuss the methodology and modelling of granular columns by continuum- and discrete-element approaches. The effects of initial packing fraction and the internal structure on the run-out behaviour are also discussed. The difference in the mechanism of modelling the granular flows in continuum and discrete approaches are presented in **??**.

2.3.2 Flow down an inclined plane

Most contemporary research on granular materials focuses on with steady-state flow. Transient and inhomogeneous boundary conditions are much less amenable to observation and analysis and have thus been less extensively studied, despite their primary importance in engineering practice. Studies on the flow of granular materials down inclined planes are important to understand the mechanism of geophysical hazards, such as granular avalanches, debris flows and submarine landslides. Large-scale field tests on dry and saturated granular materials have been carried out to capture the mechanism of granular flows down an inclined plane (Denlinger and Iverson, 2001; Okada and Ochiai, 2008).

Experimental findings

Granular material stored in a reservoir at the top of the inclined plane is released by opening a gate (figure 2.12). The flow rate is controlled by the height of the opening. The material flows down and develops into a dense granular flow. An initially static granular layer of uniform thickness, 'h', starts to flow when the plane inclination reaches a critical angle, θ_{start}. The material reaches a sustained flow until the inclination is decreased down to a second critical angle, θ_{stop} (Midi, 2004). The occurrence of two critical angles indicates the hysteretic nature of granular materials. Reciprocally, the critical angle thresholds can be interpreted in terms of critical layer thickness: $h_{stop}(\theta)$ and $h_{start}(\theta)$. The measurement of $h_{stop}(\phi)$ is easier as it corresponds to the thickness of the deposit remaining on the plane once the flow has ceased.

The two curves $h_{stop}(\theta)$ and $h_{start}(\theta)$ divide the phase diagram (h, θ) into three regions: a region where no flow occurs, $(h < h_{stop}(\theta))$; a sub-critical region where both static and flowing layers can exist, $(h_{stop}(\theta) < h < h_{start}(\theta))$; and a region where flow always occurs, $(h > h_{start}(\theta))$. In the flow regime, i.e. $(h > h_{start}(\theta))$, the flow is steady and uniform for moderate inclination, but accelerates along the plane for large inclinations (Midi, 2004). The critical angle controlling the flow behaviour tends to increase when the thickness of the bed decreases (Daerr and Douady, 1999; Pouliquen and Chevoir, 2002). This can be attributed to non-trivial finite-size effects and/or boundary effects that are not well understood (Forterre and Pouliquen, 2008). There exists a value of roughness for which a maximum thickness of deposit is observed, which might correspond to a maximum of effective friction at the bottom (Midi, 2004).

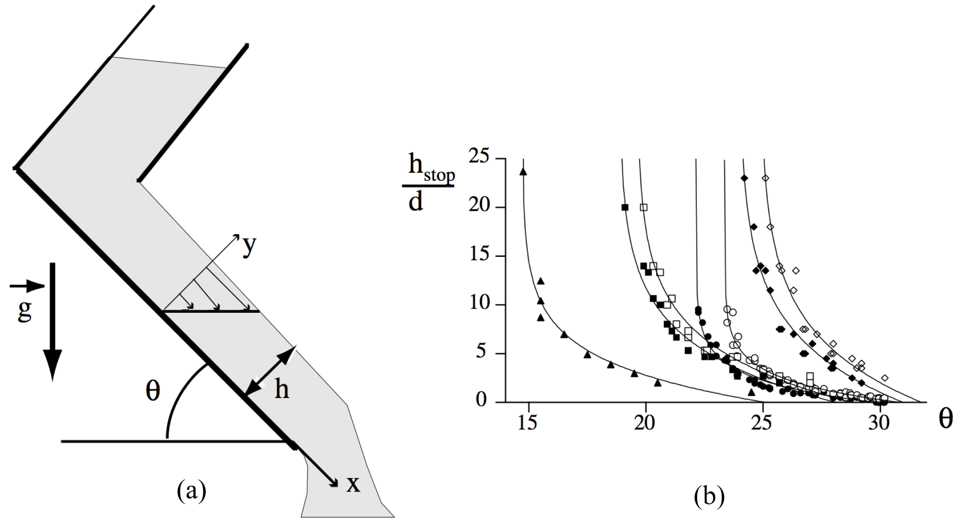


Figure 2.12 Rough inclined plane: (a) Set-up and (b) $h_{stop}(\theta)$ (black symbols) and $h_{start}(\theta)$ (white symbols). Reproduced from (Midi, 2004).

For thick enough piles flowing on a rough inclined plane, $h \geq 20 \times d$, the velocity profiles and rheology follow Bagnold scaling. As the height of the flowing pile reduces, a continuous transition from Bagnold rheology to linear velocity profiles to avalanche-like dynamics occurs, until finally, one reaches the angle of repose θ_r, and the flow ceases. This transition behaviour is difficult to model.

Numerical modelling

Fast moving granular flows can undergo a motion-induced self-fluidisation process under the combined effects of flow front instabilities setting on at large values of the Froude number, which are responsible for extensive entrainment, and longer time between collisions of soil grains. Self-fluidisation results in enhanced mobility of the solids, causing an inviscid flow (Bareschino et al., 2008). It is understood that, for a granular material to flow, it has to exceed a certain critical threshold, i.e. the friction criterion, or the ratio of shear stress to normal stress. Without an internal stress scale for a granular material, granular materials exhibit solid-fluid transition behaviour based on the friction criterion (Forterre and Pouliquen, 2008). The stress ratio in the flowing regime above the static bed indicates that the solid-to-fluid transition is a yielding phenomenon and can be described by Mohr-Coulomb-like failure criterion (Zhang and Campbell, 1992). This is in contrast to the mechanism of behaviour of other complex fluids, where there is an internal stress scale linked to the breakage of microscopic structure. From a microscopic standpoint of view, the strength of granular materials is due to the internal friction between grains, however, packed frictionless materials still exhibit macroscopic friction.

Constitutive laws based on plasticity theories, which relate the micro-structure to the macroscopic behaviour (Roux and Combe, 2002) provide useful insight into the mechanism of granular flow. At present, however, they are limited to the initiation of deformation and do not predict quasi-static flow. Continuum approaches, such as the Material Point Method simulation of granular flow down an inclined plane (Abe et al., 2006; Bandara, 2013), capture the flow behaviour in the initial stages, yet the model exhibits inconsistent behaviour when the granular material ceases to flow. This may be due to the application of small deformation theory to a large deformation problem and the use of zero dilation.

Alternatively, by using the $\mu(I)$ rheology, we can capture the velocity profile and the localization at the free surface. However, the rheology fails to model the transition from a continuous flow to an avalanching regime as the flow rate is decreased (Pouliquen et al., 2006). The rheology predicts no-flow below a critical angle, $\theta_s = \arctan(\mu_s)$, independent of the thickness. Pouliquen (1999) observed that the critical angle increases when the thickness of the flow decreases. Granular flow down rough inclined planes exhibits strong Coulomb shear stresses on a plane normal to the basal flow boundary. The stresses dissipate the energy along the rough surface. Models that lack multi-dimensional momentum transport or Coulomb friction cannot represent this energy dissipation.

Comments on modelling

Granular flow down inclined planes exhibits a transient behaviour. Various velocity profiles and flow behaviours can be obtained not only through changing the height of the flowing granular mass, but also by varying the inclination of the chute, such that there is an overlap region where one can obtain similar flow properties through either procedure. These features can be used to better predict the evolution of an avalanche surface. The experimental configuration is ideal for studying the effectiveness of various theoretical frameworks in modelling granular flows, especially with regards to the transition from solid-like to fluid-like behaviour. The $\mu(I)$ rheology and models that are based on Coulomb friction are able to predict the energy dissipation along the rough surface. However, they fail to capture the transition from static to a flowing behaviour, and when it ceases to flow. In large-scale granular flow down slopes, only those grains that are located on the flow surface will have higher I values, due to lower mean pressures. However, grains located within the main flowing mass experience higher mean pressure, thus resulting in smaller I values. Hence, $\mu(I)$ rheology is less effective in large scale problems and a simple Mohr-Coulomb model will yield a similar response. It can be suggested that the flow is governed by its momentum. It is important to carry out DEM simulations to understand the mechanics of the flow transition behaviour. This will enable us to describe the continuum response of phase transition more efficiently.

2.3.3 Saturated and submerged granular flows

Submarine mass movements pose a significant threat to off-shore structures, especially oil and gas platforms. Geophysical hazards, such as debris flows and submarine landslides, usually involve flow of granular solids and water as a single-phase system. Modelling the multi-phase interaction poses a serious challenge. The momentum transfer between the discrete and the continuous phases significantly affects the dynamics of the flow as a whole (Topin et al., 2011). The complex interactions between the soil and the ambient fluid are shown in figure 2.13. For a given granular mass, the energy balance can be written as

$$E_p + E_s = E_k + E_f + E_D + E_v + E_r\,, \tag{2.17}$$

where E_s is the seismic energy resulting from an earthquake, E_f is the friction loss, E_D is the friction loss due to drag effects on the upper surface of the flow, E_v is the loss due to viscous effects and E_r is the energy used to remold or transform the intact material. During the course of a submarine slide event (and also a sub-aerial slide), there appears to be a process by which there are some changes in the solid to water ratio which provide a sufficiently low strength to allow flow to occur (Locat and Lee, 2002). Whatever the exact nature of the phenomenon, it is embedded in the remoulding energy (E_r). In both sub-aerial and submarine landslides, the triggering energy is the same (initial potential energy). It appears as though the submarine landslides experience more dissipation E_D and E_v than sub-aerial landslides, but run-out in submarine landslide will be shorter due to more dissipation. The effects of hydroplaning and fluidisation of the flowing mass result in complex interactions, complicating the ability to predict to the exact mechanism of run-out and the length of run-out in submarine conditions for a given initial state.

Experimental findings

The collapse of a granular column, which mimics the collapse of a cliff, has been extensively studied in the case of dry granular material, when the interstitial fluid plays no role. The case of the collapse in the presence of an interstitial fluid has not been studied in depth (Topin et al., 2012). Rondon et al. (2011) performed granular column collapse experiments in fluid to understand the role of initial volume fraction. The experimental set-up of granular collapse in fluid performed by Rondon et al. (2011) is shown in figure 2.14.

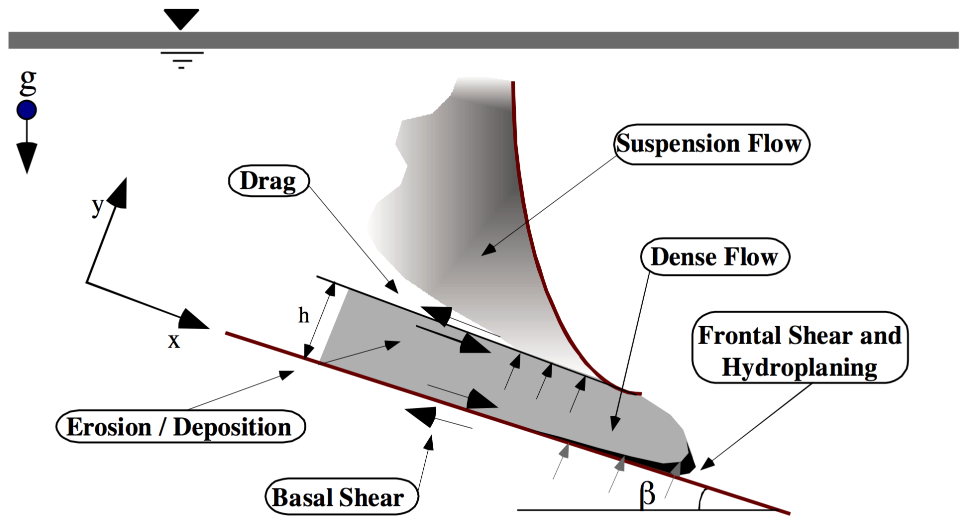


Figure 2.13 Schematic diagram showing the generation of a turbidity current (suspension flow) for drag forces on the surface, potential lifting of frontal lobe leading to the process of hydroplaning, the basal shear stress causing erosion and deposition (Locat and Lee, 2002).

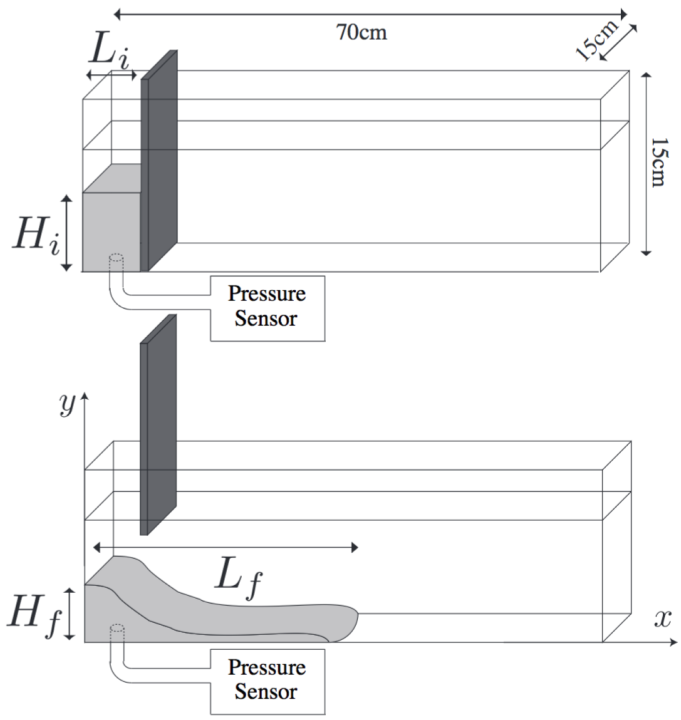


Figure 2.14 Experimental set-up of granular collapse in fluid (Rondon et al., 2011).

Figure 2.15 demonstrates the evolution of run-out and pore-pressure at the bottom of the granular flow. The entire loose column is mobilised immediately, in contrast to the dense case. The loose column in fluid spreads almost twice as long as the dense case. The collapse of a granular column in a viscous fluid is found to be mainly controlled by the initial volume fraction, not by the aspect ratio of the column. The role of the initial volume fraction observed in the viscous collapse can be understood by the pore pressure feedback mechanism proposed by Iverson (2000); Schaeffer and Iverson (2008) in the context of landslides.

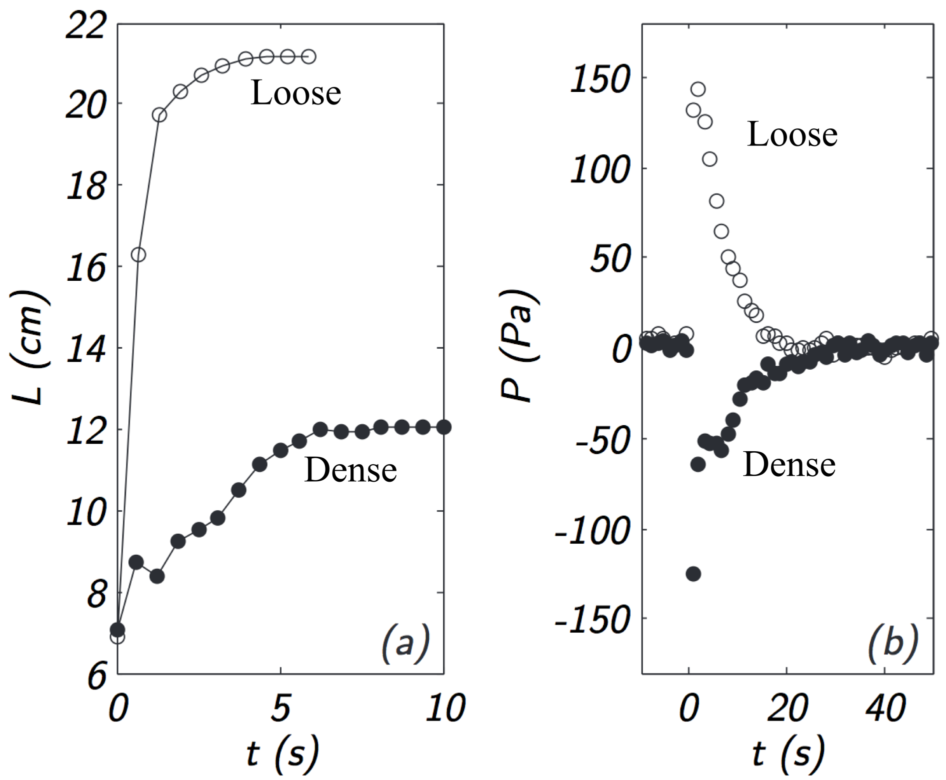


Figure 2.15 Evolution of run-out and pore-water pressure at the bottom of the flow for granular column collapse in fluid for dense and loose conditions (Rondon et al., 2011).

Iverson (2000) observed in the large-scale field tests that soil prepared in a loose state on a slope and subjected to a rainfall flows rapidly like a liquid when it breaks, whereas a dense soil only slowly creeps (figure 2.16). The underlying mechanism is related to the dilation or contraction character of the granular material, which the researchers described as the “pore pressure feedback”. The compaction or dilation of grains can cause additional stress in the grains which can stabilise or destabilise the soil. The loose sediment experiences high positive pore-pressure and low shear stresses resulting in a quicker flow, while the large negative pore-pressure in the dense case delays the run-out. The flow is controlled by the coupling of the dilatancy of the granular layer and the development of pore pressure in the fluid phase (Pailha et al., 2008).

The dense column must dilate in order to flow. When it starts to fall, liquid is then sucked into the column, which is stabilized by the additional viscous drag (Rondon et al., 2011; Topin et al., 2012). In the dense condition, large negative-pore pressures are developed in the initial stages of collapse; they collapse has to overcome the large negative pore-pressure before it starts to flow. This results in a slow flow evolution as demonstrated by the flatness of the initial run-out curve in the dense condition in comparison to the steep slope in the loose case (figure 2.15). When the loose column starts flowing, on the other hand, it expands and ejects liquid, leading to a partial fluidisation of the material. The large positive pore-pressure developed at the bottom of the flow in the loose condition indicates water entrainment and hydroplaning. The entrainment of water at the basal flow front lubricates the frictional effect, and in combination with hydroplaning, results in a longer run-out distance.

Little research has been carried out to understand the difference in mechanisms of dry and submerged granular flow. Cassar et al. (2005) investigated the flow of dense granular material down an inclined plane fully-immersed in water. The velocities observed in the submarine case were found to be a magnitude smaller than the dry condition. This is in contrast to the idea that the submarine landslides tend to flow longer than their sub-aerial counterpart. In order to compare the dry collapse with the submarine collapse, it is important to use the same initial configuration. As discussed previously, packing soil grains to the same initial density is difficult. Hence, it is important to perform numerical studies and develop a theoretical framework that can explain the submarine granular flow behaviour.

Numerical modelling

Although certain macroscopic models are able to capture simple mechanical behaviours, the complex physical mechanisms occurring at the grain scale, such as hydrodynamic instabilities, formation of clusters, collapse, and transport (Peker and Helvacı, 2007; Topin et al., 2011), have largely been ignored. In particular, when the solid phase reaches a high volume fraction, the strong heterogeneity arising from the contact forces between the grains and the hydrodynamic forces, is difficult to integrate into the homogenization process involving global averages (Topin et al., 2011).

In two-phase models (Pitman and Le, 2005), the momentum transfer between the grains and the suspension fluid depends on the momentum equations of both phases. In the case of mixture theory-based models (Meruane et al., 2010), the shear-induced migration and grains collisions are considered in an average sense. In order to describe the mechanism of saturated and/or immersed granular flows, it is necessary to consider both the dynamics of the solid

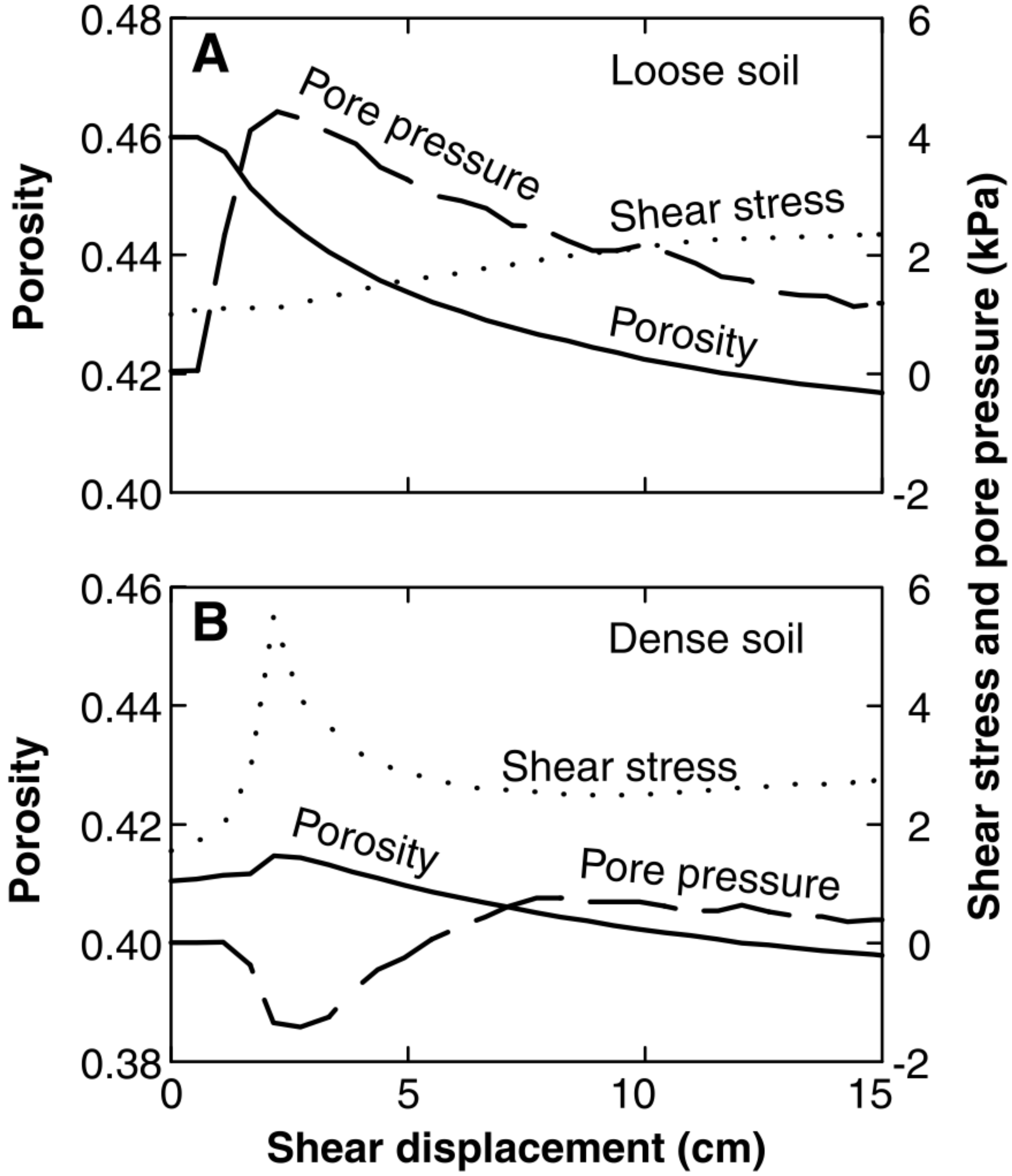


Figure 2.16 Pore pressure feedback mechanism: the effect of density (Iverson, 2000).

phase and the role of the ambient fluid (Denlinger and Iverson, 2001; Iverson, 1997). The dynamics of the solid phase alone are insufficient to describe the mechanism of granular flow in a fluid; it is important to also consider the effect of hydrodynamic forces that reduce the weight of the solids inducing a transition from dense-compacted to dense-suspended flows, and the drag interactions which counteract the movement of the solids (Meruane et al., 2010).

Topin et al. (2011) performed granular collapse in fluid using Non-Smooth Contact Dynamics coupled with the distributed Lagrange multiplier/fictitious domain (DLM/FD) method. The mechanisms for collapse of granular columns in dry and submerged conditions are compared. Topin et al. (2011) observed that for a given initial geometry, the run-out distance in the dry case is significantly higher than the submerged condition, an observation similar to the experimental results of Cassar et al. (2005). In the dry case, inertia is responsible for the enhanced mobility at high aspect ratios. In submerged conditions like the viscous regime, however, the inertial effects remain negligible, a result which could explain why the important parameter controlling the dynamics is the initial volume fraction and not the initial aspect ratio. Topin et al. (2011) observed that the run-out distances exhibit a power-law relation with the peak kinetic energy in all three regimes: fluid inertial, grain inertial and viscous regime (figure 2.17). The viscous regime is where the grain reaches the viscous limit velocity, the Stoke's number, $S_t << 1$, and the density ratio $r >> S_t$ (Courrech du Pont et al., 2003). However, the role of the volume fraction on dry granular collapse has not been precisely studied, and the preparation of the pile may also play a role (Daerr and Douad, 1999).

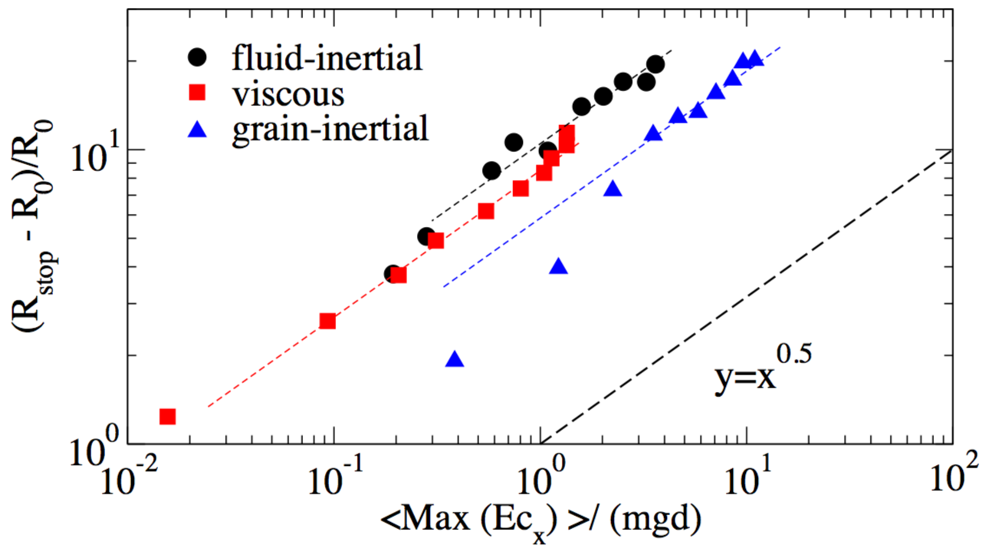


Figure 2.17 Normalised run-out distance as a function of the peak value of the horizontal kinetic energy per grain (Topin et al., 2011).

The $\mu(I)$ rheology relates the non-dimensionless number I to the shear rate through a characteristic time. In the case of dry granular flows, the parameter I is defined as the ratio between the time elapsed for a grain to fall into the hole, t_{micro}, and the meantime, t_{mean}, which is inversely related to the shear rate. Cassar et al. (2005) observed that the run-out behaviour collapse onto a single friction law, demonstrating that the major role of the fluid is to change the time required for a grain to fall into a void-space. If the inertial time scale in the rheology is replaced with a viscous time scale. the $\mu(I)$ rheology for dry dense flows can be modified to capture the behaviour of dense submarine granular flows. Indeed, Pitman and Le (2005) observed that if the fluid inertial effects are small enough, then a simpler model can be adopted. A sketch of the motion of a grain, $z(t)$, during a simple shear, $\dot{\gamma}$, under a confining pressure (P_g) is shown in figure 2.18. Hence, assuming that the fluid velocity is low enough for the contact interaction between grains to be significant, the time required for the grain to fall into a hole, t_{micro}, is then controlled by the viscosity of the ambient fluid. Thus, the dimensionless parameter can thereby be modified to incorporate the viscous time in order to describe granular flow in a fluid (Pouliquen et al., 2005). For short time scales, the grain initially accelerates but as a result of the drag force, the grain eventually reaches a limit viscous velocity, $v_{\infty v}$. The time required to travel a diameter, d, is given as $t_{fall} = d/v_{\infty v}$. This new viscous time is used to define the dimensionless number.

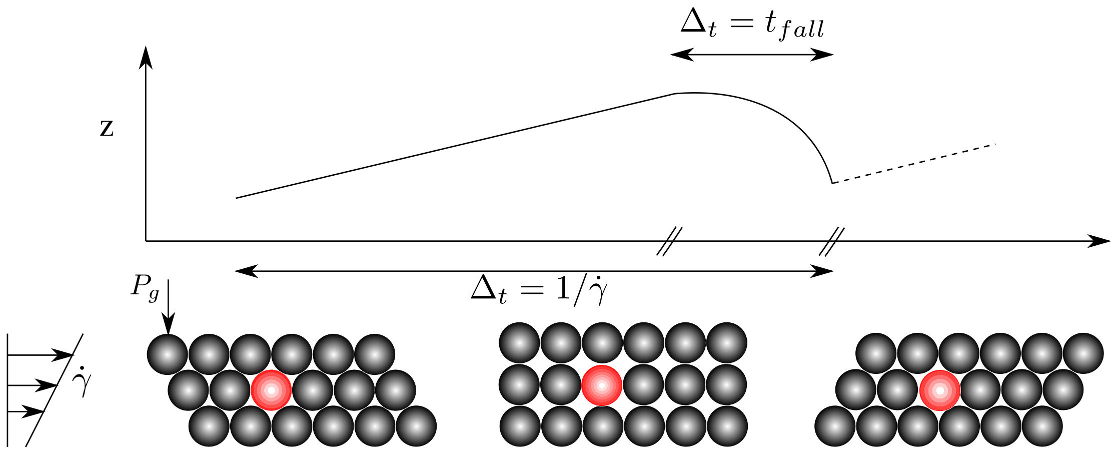


Figure 2.18 Sketch of the motion of a grain, $z(t)$, during a simple shear, $\dot{\gamma}$, under a confining pressure, P_g.

Comments on modelling

The $\mu(I)$ rheology is found to be valid only for the steady uniform regime; unsteady phenomena, such as the triggering of avalanches, result in the coupling between the granular grains and the ambient fluid, a result that is much more complex to model. Transient regimes, characterized by change in solid fraction, dilation at the onset of flow and development of excess pore pressure, result in altering the balance between the stress carried by the fluid and the stress carried by the grains, thereby changing the overall behaviour of the flow (Denlinger and Iverson, 2001). The $\mu(I)$ rheology seems to predict well the flow of granular materials in the dense regime. However, the transition to the quasi-static regime, where the shear rate vanishes, is not captured by the simple model. Furthermore, shear band formation observed under certain flow configurations is not predicted. The flow threshold, or the hysteresis, characterizing the flow or no-flow condition is not correctly captured by the model, either can be due to the discrepancies between the physical mechanism controlling the grain level interactions, clustering, and vortex formations. When the scale of the system is larger than the size of the structure, a simple rheology is expected to capture the overall flow behaviour. Yet, the size of the correlated motion is the same as that of the system, , a similarity which causes difficulties in modelling the flow behaviour (Pouliquen et al., 2005). Hence, it is essential to study the behaviour of granular flows at various scales, i.e. microscopic, meso-scale and continuum level, in order to develop a constitutive model that captures the entire flow process.

2.4 Summary

Granular flow involves three distinct regimes: the dense quasi-static regime, the rapid and dilute flow regime, and an intermediate regime. The dynamics of homogeneous granular flow also involve at least three different scales, making it difficult to describe the mechanics of granular flow by simple theories. It is important to describe the granular dynamics as an initial-value problem. Experimental conditions are too difficult to reproduce precisely, resulting in inherent inconsistencies in the results. Numerical models, such as the shallow-water approximation, kinetic theory approach, and rheologies, have captured the basic flow dynamics but have failed to describe the complete mechanics of the granular flow.

The collapse of a granular column is a simple case of granular flow. Experimental results have shown that the run-out distances exhibit a power-law dependence with the initial aspect ratio of the column. Although numerical simulations and theoretical frameworks were able to recover the power-law behaviour, they were unable to explain the origin of the power law or capture the various stages of the flow. The initial conditions have a significant impact on the overall behaviour of the granular system. However, a theoretical framework that is capable of describing the influence of material properties on the run-out behaviour is still lacking. In the present study, the capability of continuum models, such as the Mohr-Coulomb model and $\mu(I)$ rheology, in simulating the granular column collapse is investigated by comparing the results with the discrete-element simulations. The role of initial packing fraction on the run-out behaviour is also investigated.

Submarine granular flows exhibit complex interactions between the soil grains and the ambient fluid. The presence of fluid causes drag interacts which slows down the run-out, while the entrainment at the flow front causes hydroplaning and longer run-out distance. The run-out distance is found to be controlled by the initial volume rather than the aspect ratio of the column. Loose granular columns flow longer than the dense conditions. However, researchers observed longer run-out in dry conditions compared to the submerged collapse conditions. The difference in the mechanism of dry and submerged granular flows is not well researched. The role of initial volume fraction on the run-out behaviour is not precisely known. This study focuses on the effect of initial packing and permeability on the run-out by performing grain-scale simulations. This study investigates the influence of grain-scale quantities, which are otherwise not accessible experimentally, on the run-out behaviour thus providing useful insight into the flow dynamics, thereby enabling us to develop better constitutive laws.

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