# Chapter 2

# Granular flows

## 2.1 Introduction

A granular material is a conglomeration of a large number of discrete solid grains of sizes greater than $1\mu m$ whose behaviour is governed by frictional contact and inelastic collisions. A schematic representation of the size range of the granular materials is presented in figure 2.1. Granular materials, characterized by interaction between individual grains, lie between two extremes scales: the molecular-scale range predominated by electrostatic force, i.e. Van der Waals forces, and the continuum scale which is described by the bulk property of the material. In various soil classification systems adopted in soil mechanics, sand is classified as a granular material having grain sizes greater than $75\mu m$. A grain size of $75\mu m$ is an important transition point, where the frictional effect starts to dominate the material behaviour and the effect of the electrostatic Van der Waals forces diminishes. The extent of the grain size range of the granular materials from the molecular size to a continuum scale indicates that they have a complex behaviour, demonstrating a mix of grain-like and continuum-like behaviour.

The physics of non-cohesive granular assemblies is intriguing. Despite being ubiquitous in nature, granular materials are the most poorly understood materials from a theoretical standpoint. For such a poorly understood area, the flow of granular materials has a surprising range of geo-hazard predictions and industrial applications. For years, granular materials have resisted theoretical development, demonstrating non-trivial behaviour that resembles solid and/or fluid-like behaviour under different circumstances. Even in the simplest of situations, granular materials can exhibit surprisingly complex behaviour. Macroscopically, the complex mix of solid and fluid-like behaviour can be illustrated by a simple example; while one walks on the beach, the solid-like behaviour of soil becomes evident as it supports one's weight, but if we scoop a handful of soil and allow it to run through the fingers, the fluid-like nature becomes obvious.

Microscopically this complex behaviour has various reasons. The range of the grain size gives rise to complex interactions between grains constituting the granular media. Unlike other micro-scale particles, soil grains are insensitive to thermal energy dissipation (Mehta, 2011), because the thermal energy dissipation in a granular material is several orders of magnitude smaller in comparison with the energy dissipation due to interaction between the grains. The thermal energy scales are small when compared to the energy required to move the grains. The granular material reaches the static equilibrium quickly due to its dissipative nature, unless an external source of energy is constantly applied (Choi, 2005).

Our knowledge of the behaviour of granular assemblies is restricted to two extremes: the solid-like behaviour of dense granular assemblies that resist the shearing force by undergoing plastic deformations, and the fluid-like flow behaviour characterized by high shear rates. Granular media are *a priori* simple systems made of solid grains interacting through their contacts. However, they still resist our understanding and no theoretical framework is available to describe their behaviour (Pouliquen et al., 2006). The strong dependency of the behaviour of granular material on its surrounding environment makes it difficult to have a unified theoretical framework. When strongly agitated, the granular material behaves like a dissipative gas, and kinetic theories have been developed to describe this regime (Popken and Cleary, 1999; Xu et al., 2003). On the other hand, during slow deformations, the quasi-static regime is dominated by steric hindrance and friction forces are often described using plasticity theories. In between the two regimes, the material flows like a fluid, and the grains experience enduring contacts, which is incompatible with the assumptions of the kinetic theory (Pouliquen et al., 2006) that describes the dilute regime of a granular flow. Typical granular flows are dense and hence a fundamental statistical theory is not appropriate to describe their properties. Moreover, during the process of granular flow, the material can exist in all the above-mentioned states, which further complicates our understanding of granular flows.

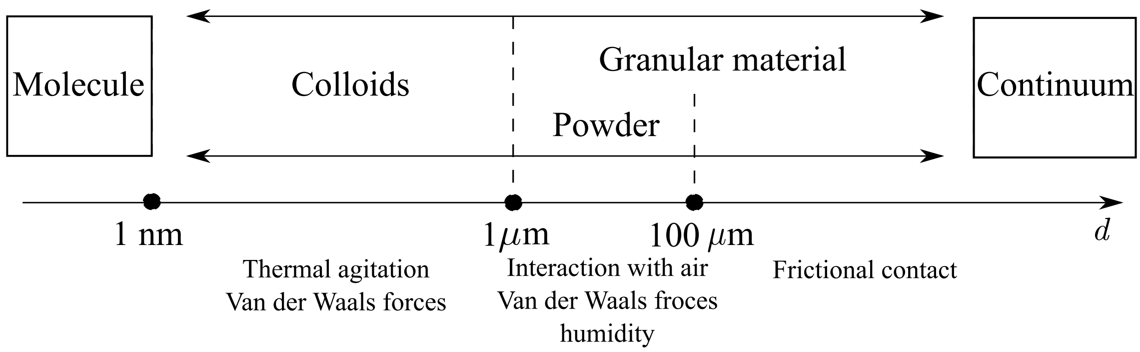


Figure 2.1 Particle size range and their predominant characteristics

## 2.2 Modelling the granular flows

Granular flows can be classified into three different regimes (Jaeger et al., 1996): the dense slow quasi-static regime characterized by long duration between contacts and grain interaction via frictional contact (Roux and Combe, 2002); the rapid and dilute flow regime characterized by grains moving freely between successive collisions (Goldhirsch, 2003); and an intermediate fluid-like regime in which the material is dense but still flows like a fluid and the grains interact both by collision and through friction (Midi, 2004; Pouliquen and Forterre, 2002). Transfer of grain kinetic energy and momentum within a rapidly flowing granular medium occurs during these collisions (Popken and Cleary, 1999).

Different approaches have been used to model the granular flows at different scales of description. The dynamics of a homogeneous granular flow involve at least three distinct scales: (1) the *Microscopic scale* characterized by small time and length scales representing contact/grain interactions, (2) the *Mesoscopic scale*, where grain rearrangements, development of micro-structures and shear rates have a dominant influence on the granular flow behaviour, and (3) the *Macroscopic scale* which involves large length scales that are related to geometric correlations at even larger scales. The interesting issue is whether one should consider or neglect a particular scale while modelling the granular dynamics (Radjai and Richefeu, 2009). However, the difficulty in modelling the granular flows originates from the fundamental characteristics of the granular matter such as negligible thermal fluctuations, highly-dissipative interactions, and a lack of separation between the microscopic grain scale and the macroscopic scale of the flow (Goldhirsch, 2003).

Granular flow modelling began as early as 1776 with Coulomb's paper describing the yielding of granular material as a frictional process. Although it was not about granular flows, *per se*, the prediction of soil failure for Civil engineering applications describes the onset of structural collapse leading to catastrophe (Campbell, 2006). Mohr-Coulomb's yield criterion along with a flow rule from metal plasticity is sufficient to describe the behaviour of granular flow as a continuum process, without considering the interaction of individual grains.

Advanced models based on the critical state concept (Schofield and Wroth, 1968) provides further insight into continuum description of granular flows. According to the critical state theory, the 'under consolidated' or loose soil tends to increase in density upon shearing, while the dense 'over consolidated' soil dilates when sheared, until it reaches the critical state. As dense granular flow involves large shear stresses, it is reasonable to assume that the shearing occurs at the critical state. Large applied stress can cause the granular solids to deform at the grain scale and squeeze them into the inter-grain pores. However, the critical state is independent of the applied stress. In granular flows, the applied stress during the actual flow is less in comparison with the stress on the soil underneath a structure, where the soil is subjected to large shear strains. It is therefore reasonable to assume that the flow is incompressible and takes place at the critical state (Campbell, 2006).

The main limitation of the continuum approach is the assumption that the friction angle, $\phi$ is a constant material parameter, which is found to vary by a factor of 3, violating the fundamental assumption of quasi-static flow theories (Potapov and Campbell, 1996). Although the mechanism of dense granular flow is attributed to the bulk friction, it is the formation of force chains and the rearrangement of internal structure of the granular assembly that causes friction-like behaviour. Experiments (Savage and Sayed, 1984; Savage, 1984) and computer simulations (Campbell and Brennan, 1985) indicate a weak relation between the bulk friction and the packing density, due to the micro-structural rearrangement of grains (Campbell, 1986). As the packing density increases, the grains tend to arrange themselves in a regular order when sheared. In order to understand the development of micro-structure, it is important to look at the grain-level interactions. Bagnold (1954) was the first to try and model granular materials as individual grains. Bagnold's theory of motion of individual grains in a shear flow and inter-grain friction inducing random velocities is reminiscent of the thermal motion of molecules in the kinetic theory of gases. The two common approaches in modelling the granular flow: the kinetic theory and the shallow water theory, are discussed below.

### 2.2.1 Kinetic theory

The gas kinetic theory assumes that the particles interact by instantaneous collisions, which implies only binary (two-particle) collisions. The particles are modelled using a single coefficient of restitution, to represent the energy dissipated by the impact normal to the point of contact between the particles, and for the most part, the surface friction or any other particle interactions tangential to the point of contact are ignored (Campbell, 1990). Jenkins and Savage (1983) extended the kinetic theory for thermal fluids to idealized granular mixtures to predict the rapid deformation of granular material by including energy dissipation during collision for nearly inelastic particles. Savage and Jeffrey (1981) extended the kinetic theory to predict simple shear flow behaviour for a wide range of coefficients of restitution. The kinetic theory is capable of predicting the shear flow behaviour only for mixtures composed of particles with identical density and size (Iddir and Arastoopour, 2005), however real systems are composed of particles that vary in size, and segregation of particles can occur.

The kinetic theory is valid for dispersed granular flows (Ng et al., 2008), however Van Wachem et al. (2001) observed that the numerical simulations of dense granular flow based on kinetic theory were poor in comparison with the experimental data on fluidized bed expansion. Confined granular flows are usually dense, because of their mechanism of energy dissipation and their tendency to form clusters. The dense granular flows lie in an intermediate regime, where both the grain inertia and the contact network has significant influence on the flow behaviour (Pouliquen and Forterre, 2002). Thus, a part of the force is transmitted through the force network, which contradicts the two basic assumptions in the kinetic theory, i.e. binary collision and the molecular chaos. For dense granular flow conditions, the total stress transmission in the flow regime is the sum of the rate-dependent (collision-transition) and the rate-independent (friction) components (Ng et al., 2008). Addition of frictional stress component (Schaeffer, 1987) to the kinetic theory improves the ability of the model to predict the dense granular flows. The main advantage of kinetic theories is that they can be used to derive deterministic constitutive laws to describe the behaviour of granular flows in a theoretical framework (Jenkins and Savage, 1983). Kinetic theories formulated on the assumption of solid phase stress as a viscous response have limitations when applied to granular flows. A viscous material produces no force unless it is in motion, hence the kinetic theory based on viscous solid phase cannot explain the static force exerted by the granular materials on the walls, as observed in experiments. The addition of a frictional component to the kinetic theory improves its prediction of granular flows. However, the frictional component that is based on long-duration contact is added to the instantaneous collision contact term. Also, the rapid-flow models based on gas kinetic theory assume that the molecular collisions are elastic, which means that they do not dissipate energy (Campbell, 2006), which is in contrast to the reality. Finally, the important assumption of gas kinetic theory is molecular chaos, which assumes no correlation between the velocities or positions of the colliding particles, which is not true especially in a dense granular flow where the particles interact many times with their neighbours and a strong correlation between their velocities is inevitable.

### 2.2.2 Shallow-water approximation

The Navier-Stokes equation in fluid mechanics is capable of describing the dynamics of fluid flow under different conditions. However, the proposed models for granular flows tend to be specialized for a particular situation. By drawing a simple analogy from fluid dynamics one can model granular flows as non-Newtonian fluids using a variant of the Navier-Stokes equation; one such approach is the depth-averaged shallow-water equation, which has been applied to solve granular flow dynamics with a reasonable amount of success. The Savage-Hutter model (Savage and Hutter, 1991), is a depth-average continuum-mechanics based approach which consists of hyperbolic partial differential equations to describe the distribution of the depth and the topography of an avalanching mass of cohesion-less granular media (Hutter et al., 2005). This approach is based on the assumption that the horizontal length scale is very large in comparison with the vertical length scale, which allows us to neglect the horizontal partial derivatives relative to the vertical partial derivatives. Field observations indicate an aspect ratio of $10^{-3}$ to $10^{-4}$ for natural avalanches (Cawthor, 2006). By neglecting the vertical length scale, the continuum equation for conservation of mass and momentum can be written as

$$\partial_x u + \partial_y v = 0\,, \tag{2.1}$$

$$\partial_t u + u\partial_x u + v\partial_y u = (\nabla\cdot\boldsymbol{\sigma})_x + \boldsymbol{F}_x\,. \tag{2.2}$$

The continuum equation requires determining the components of the stress tensor and a suitable constitutive law. *Savage-Hutter (SH) model* uses the Mohr-Coulomb law to describe the constitutive relation. The conservation of mass and momentum in the SH model is based on the assumption of granular flow as an incompressible fluid flow, which means that throughout the avalanche, the density of the avalanching material remains constant. Although Hutter et al. (1995) observed the density of the granular flow to remain almost constant in a flow down a curved chute, the destructive nature of landslides and avalanches restricts us from inferring a conclusive result. The SH model involves the following assumptions: (1) Coulomb-type sliding takes place with a bed friction angle $\delta$, (2) Mohr-Coulomb frictional behaviour occurs inside the material with internal angle of friction, $\phi \geq \delta$, and (3) the velocity profile is assumed to be uniform throughout the avalanche depth. The granular flow over a rigid plane inclined at an angle, $\theta$ is shown in figure 2.2. The mass and momentum balance in the SH model is written as

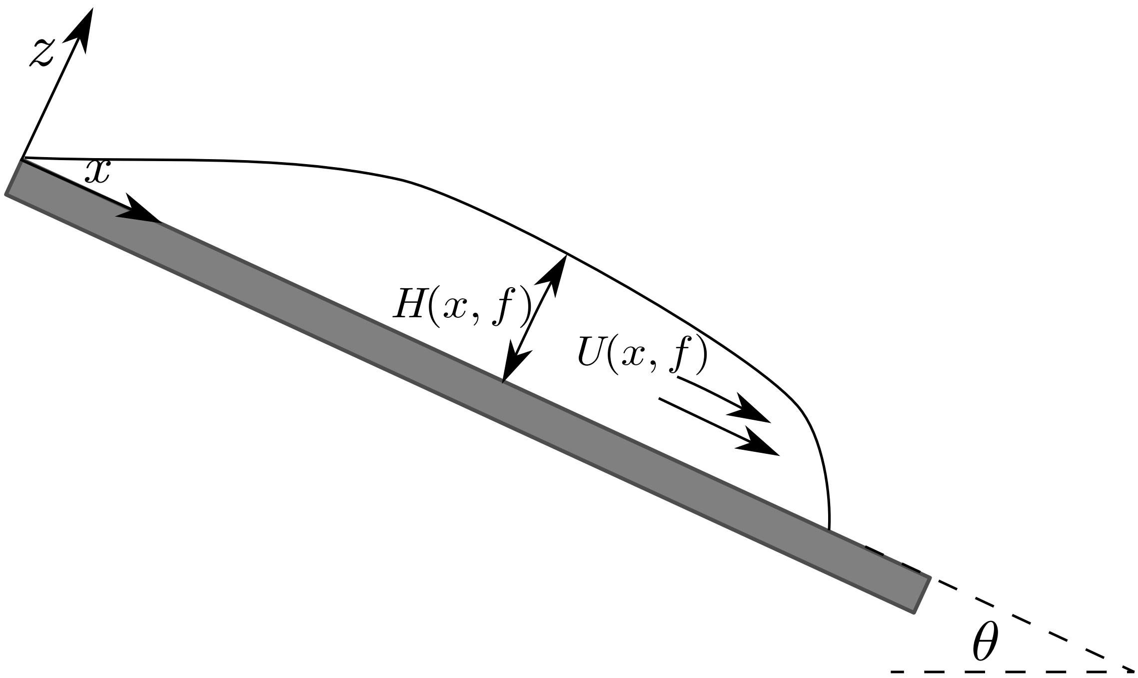


Figure 2.2 Illustration of the Savage-Hutter model

$$\frac{\partial H}{\partial T} + \frac{\partial}{\partial X}(HU) = 0\,, \tag{2.3}$$

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} = (\sin\theta - \tan\delta sgn(U)\cos\theta) - \beta\frac{\partial H}{\partial X}\,, \tag{2.4}$$

where capital letters denote non-dimensional quantities with respect to the typical horizontal and vertical length scales $(L^*, H^*)$ and the time scale $\sqrt{L^*/g}$. The key feature in the shallow water approximation is the Mohr-Coulomb constitutive law, which is applied at the free surface and at the base, to describe the granular flow. Comparison of the model with the post-calculation of Madlein avalanche in Austria indicates that the Coulomb basal friction is insufficient and requires an additional viscous component. The SH model's predictions were not satisfactory for granular flows down gentle slopes of inclination angle $\leq 30^o$, where granular materials exhibit a different behaviour (Hutter et al., 2005).

### 2.2.3 Rheology

Rheology is the science of flow of materials with solid and fluid characteristics. In practice, rheology is principally concerned with describing the mechanical behaviour of those materials that cannot be described by the classical theories, by establishing an empirical relation between deformation and stresses. Consider a granular assembly of grains having diameter $d$ and density $\rho_d$ under a confining pressure $P$ (see figure 2.3). If the material is sheared at a constant shear rate, $\dot{\gamma} = V_w/L$ is imposed by the relative movement of the top plate with a velocity $V_w$. In the absence of gravity, the force balance implies that the shear stress, $\tau = \sigma_{xy}$, and normal stress, $P = \sigma_{xy}$, are homogeneous across the cell. This configuration is the simplest configuration to study the rheology of granular flow, i.e. to study the effect of the strain rate, $\dot{\gamma}$, and pressure, $P$ on the volume and shear stress, $\tau$.

Even though the granular materials have been extensively researched at microscopic level, the continuum representation of granular materials in terms of conservation of mass and momentum is still an area of concern (Daniel et al., 2007; Midi, 2004). The prediction of rheology of granular materials even in the simplest case is complicated as they exhibit rate-dependent behaviour and no single constitutive equation is able to describe the behaviour over a range of shear stress rates. Da Cruz et al. (2005) developed a very famous rheology for granular flows, that is based on the simple two-dimensional shear in the absence of gravity and establishes that the flow regime and rheological parameters scale with a dimensionless number that represents the relative strength of inertia forces with respect to the confining pressure (Daniel et al., 2007), along the lines of Savage and Hutter (1991). The shear stress, $\tau$, is proportional to the confining pressure, $P$, and is written as

$$\tau = P\mu(I)\,. \tag{2.5}$$

The friction coefficient $\mu$ depends on the single non-dimensional parameter $I$, expressed as

$$I = \frac{\dot{\gamma} d}{\sqrt{P\rho_p}}\,. \tag{2.6}$$

The parameter $I$ can be interpreted in terms of different time scales controlling the grain flow. If the grains are rigid, i.e. neglecting the elastic properties of the grains, then $I$ is the only non-dimensional parameter in the problem. Hence, the shear stress, $\tau$, has to be proportional to the pressure, $P$, times a function of $I$. Comparing the shape of the function $\mu(I)$ with the experimental results of flow down an inclined plane, Jop et al. (2006) observed that the frictional coefficient increases from a minimal value of $\mu_s$ to an asymptotic value of $\mu_2$, when the value of $I$ increases. The variation of friction coefficient with $I$ is shown in figure 2.4. To formulate a complete constitutive model, it is essential to describe the volumetric behaviour. Based on the dimensional analysis, it can be argued that the volume change is also a function of dimensionless parameter $I$ and that it also depends on the maximum and the minimum possible void ratios and the time for microscopic rearrangement of grains.

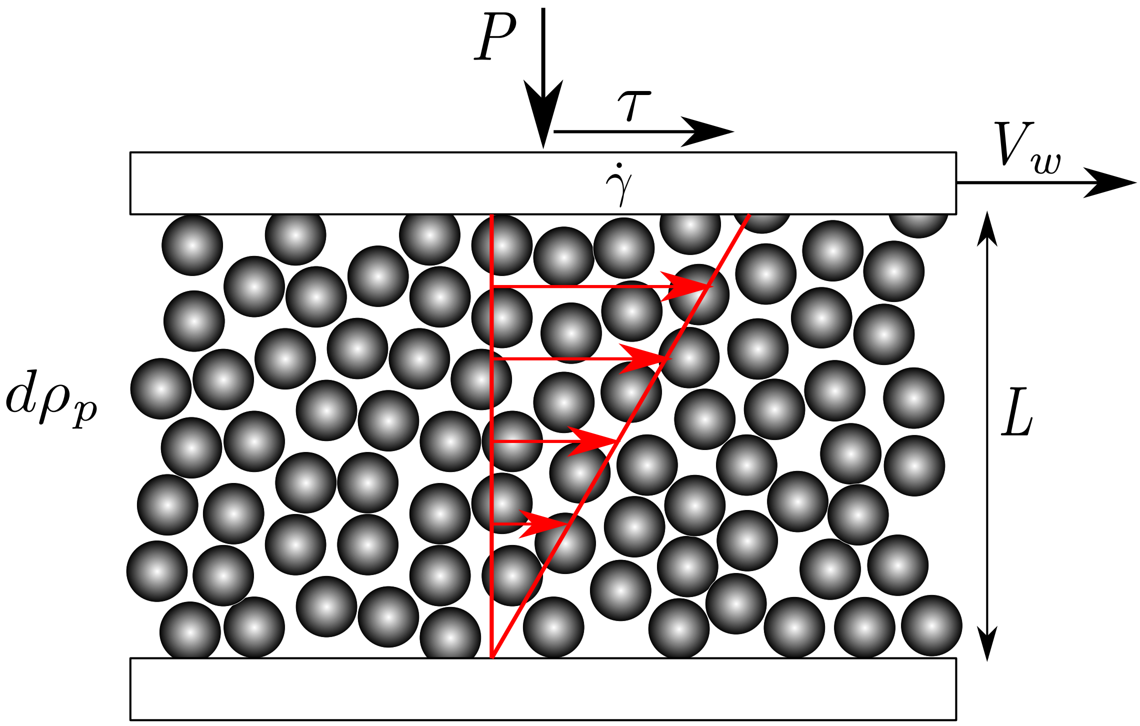


Figure 2.3 Plane shear stress distribution under a constant pressure and shear rate for a granular assembly

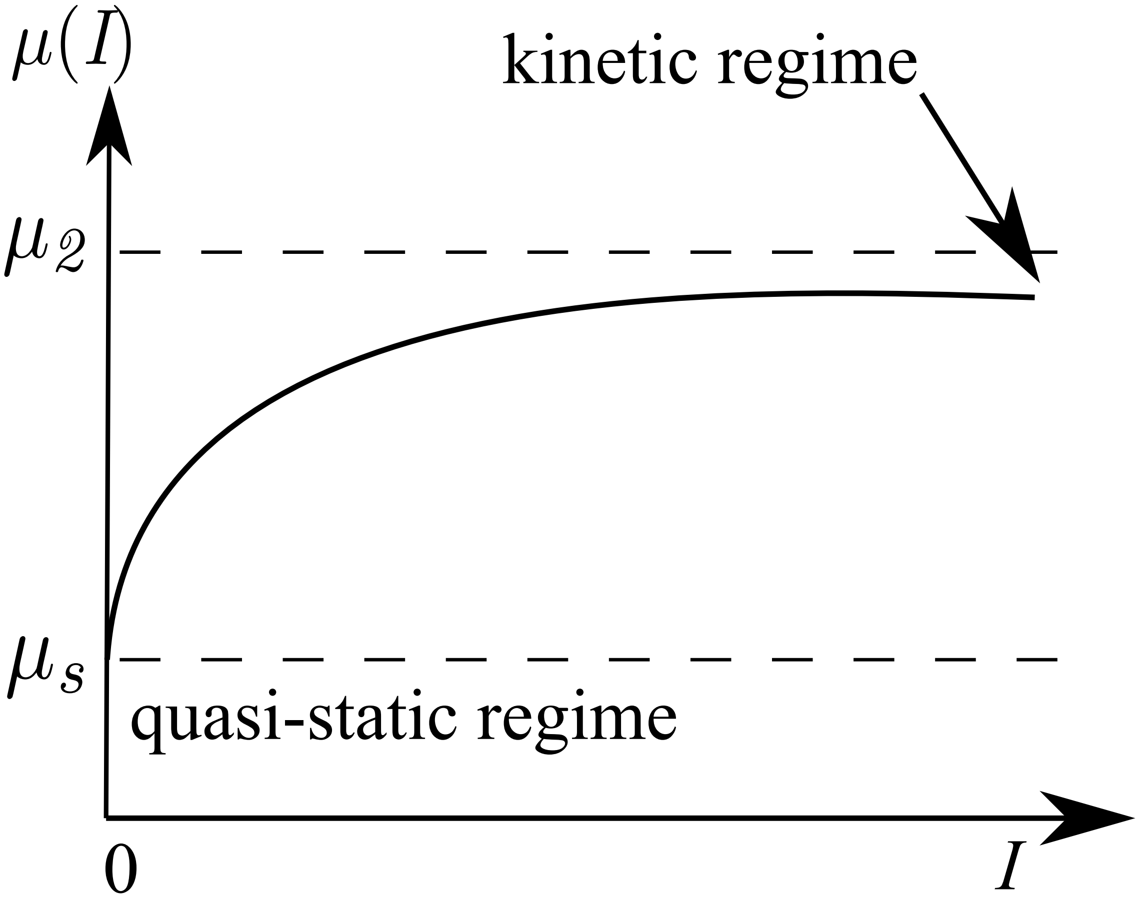


Figure 2.4 Sketch of dependence of frictional coefficient $\mu$ with dimensionless shear rate $I$, reproduced after Pouliquen et al. (2006)

In general, the flow regimes can be classified based on the dimensionless number $I$ (Da Cruz et al., 2005). Dilute or "collisional" flow occurs for $I > 10^{-1}$ and the grain collision is chiefly binary, accompanied by additional "bounce-back" akin to gases (Kamrin, 2008). In the dilute flow regime, the grains are rarely in long-duration contacts and can be described by dissipative Boltzmann kinetics. The "quasi-static" regime occurs at the other extreme of the spectrum, $I < 10^{-3}$, where the intermittent motion is prevalent. The inertial time is always small enough for the grains to align to a dense compaction, without significant collisional dissipation. The frictional sliding and stick-slip dynamics dominate the dissipation mechanism. The moderate-flow regime is observed for $I$ between $10^{-3}$and $10^{-1}$, characterized by faster flows, with a high rate of contact formation and more energy dissipation per impact. In this regime, $I$ has a one-one relationship with $\mu$ and is large enough for rate dependence, but small enough for the flow to remain dense. Moderate flows also exhibit the property of *shearing dilation*, where an increase in the normalized flow rate causes the steady-state packing fraction to decrease, which is different from *shear dilation*, which refers to a decrease in the packing density as a function of total shear. Flows which are too slow to be moderate still undergo shear dilation due to geometric packing constraints, but shearing dilation occurs only in faster flows due to rate effects (Kamrin, 2008). Figure 2.5 shows the variation of frictional coefficient $\mu$ and packing fraction $\phi$ with dimensionless number $I$ for various flow regimes.

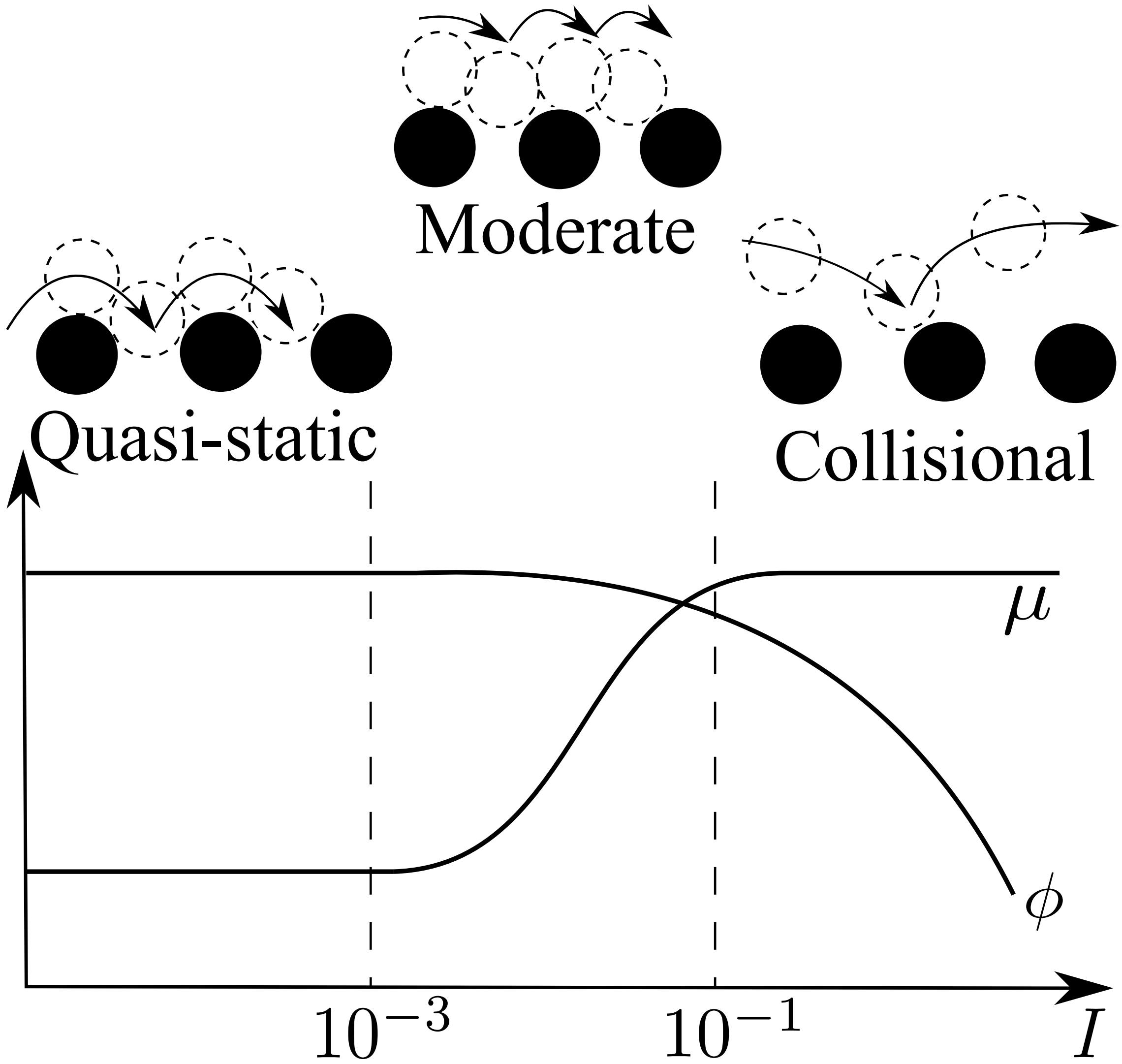


Figure 2.5 Variation of dimensionless parameter $I$ through the various flow regimes under simple shearing, reproduced after (Kamrin, 2008)

Campbell (2002) described the "Moderate regime" as an elastic granular flow regime, where the inter-grain stiffness governs the overall flow behaviour of the granular assembly. At high concentration, the stresses are proportional to the contact stiffness, and the streaming stiffness is negligible. When a dense granular assembly is sheared, the force chains that transmit the forces continue to rotate until it becomes unstable and collapses. As the force-chain rotates, the granular material tend to dilate, however it is restricted due to the constant volume constraint; instead, the rotation compresses the chain, generating an elastic response (Campbell, 2006). Campbell (2002) divided the flow into the elastic and the inertial regimes. In the Elastic regime, the force is transmitted principally through the deformation of force chains with a natural stress scaling of $\tau d/k$. The force chain forms when the grains are sheared at the rate of $\dot{\gamma}$, and hence the rate of chain formation is proportional to the shear rate $\dot{\gamma}$. This transition regime can be explained using the force-chain concept. The lifetime of a force chain is proportional to $1/\dot{\gamma}$, consequently the product of rate of formation and the lifetime of the force chain is independent of $\dot{\gamma}$, and the stresses generated are quasi-static. However at higher shear rates, the elastic forces in the chain have to absorb the additional inertial force of the grains, requiring extra force to rotate the chain proportional to the shear rate. Even though the grains are locked in force chains, the forces generated must reflect the grain inertia. The ratio of elastic to inertial effects is governed by a dimensionless parameter

$$k^* = \frac{k}{\rho d^3 \dot{\gamma}^2}, \tag{2.7}$$

where $k/\rho d^3\dot{\gamma}^2 = (\tau/\rho d^2\gamma^2)/(\tau d/k)$ is the ratio of Bagnold's inertial to the elastic stress scaling. The important dimensionless parameter is $k^*$, which is a measure of inertially-induced deformation, reflects the relative effects of elastic to inertial forces, i.e. at large $k^*$, the elastic forces dominate and at small $k^*$, inertial forces dominate (Campbell, 2006). Campbell (2002) observed a strong correlation between the coefficient of friction and the dimensionless parameter, $k$ as the flow progresses through different regimes, similar to the observation made by Kamrin (2010).

Constitutive laws, which describe the dilatancy and friction, allow us to deduce the dependency of pressure and shear stress on shear rate and solid fraction. In contrast to the observation of Campbell (2002), Da Cruz et al. (2005) found that the elastic stiffness has little effect on the constitutive law, for values greater than $10^4$, however that it does affect the coordination number. Da Cruz et al. (2005) also observed that the microscopic friction coefficient, $\mu$, has a significant influence on the dilatancy, and the solid fraction remains a linearly-decreasing function of $I$. The frictional properties of the material are found to control the solid fraction, from the critical state to the collisional regime (Da Cruz et al., 2005).

Although the rheology tends to describe the behaviour of granular flows, the mechanism of granular flows was found to vary with time, position and feedback mechanism (Iverson, 2003). Rheology summarizes the mechanical behaviour at scales smaller in comparison with the Representative Elemental Volume (REV), for a substance modelled as a continuum. Rheology-based descriptions are generally restricted to homogeneous materials that exhibit time-independent behaviour, hence are unsuitable for describing granular flows where the stress history has a significant effect on the flow dynamics. The estimation of debris flow yield strength highlights the limitation of rheologies which do not consider the development of strength with evolution of time and space. Johnson (1965) emphasized that debris yield strength is predominantly a frictional phenomenon analogous to the Coulomb strength of granular soils, and that strength consequently varies with effective normal stress. Treatment of yield strength as an adjustable rheological property contradicts the basic understanding that the strength evolves as the debris-flow motion progresses. Frictional behaviour implies no explicit dependence of shear resistance on shear rate, whereas rheological formulas commonly used to model debris flows generally include a viscous component that specifies a fixed functional relationship between shear resistances and shear rate. Although rate-dependent shear resistance is observed in debris flows, its magnitude and origin indicate that it is ancillary rather than essential (Iverson, 2003).

The two main modelling techniques that are commonly employed to describe the granular flow are the continuum approach and the discrete element approach. The continuum approach involves treating granular assembly as a continuum and describing its response using constitutive laws, while the discrete approach involves considering the individual grains of the granular material and applying Newton's laws of motion to describe the deformation of the granular material. These approaches are adopted in the present study and detailed discussions are provided in **??**.

## 2.3 Studies on granular flows

The flow of dense granular material is a common phenomenon in engineering predictions, such as avalanche, landslides, and debris-flow modelling. Despite the huge amount of research that has gone into describing the behaviour of granular flow, a constitutive equation that describes the overall behaviour of a flowing granular material is still lacking. To circumvent this difficulty, depth-averaged constitutive equations have been employed along with an empirical friction coefficient and a velocity profile deduced from experiments (Iverson, 2003; Midi, 2004; Pouliquen, 1999). Although this approach has been successful to a certain extent in predicting geophysical flows (Hutter et al., 1995; Pouliquen and Chevoir, 2002), it presents two important shortcomings (Lajeunesse et al., 2005): first, the depth-average method is true only if the thickness of the flowing layer is thin in comparison with the lateral dimension, and second, the empirical laws are deduced from experiments performed under steady-flow conditions. These cast doubts on the validity of the depth-averaged approach. Two simple granular flow studies, granular column collapse and granular flow down an inclined plane, have been carried out by various researchers to understand the flow behaviour.

### 2.3.1 Granular column collapse

Lube et al. (2005) and Lajeunesse et al. (2004) have carried out experimental investigation on the collapse behaviour of a granular column on a horizontal plane. Both the experiments involved filling a cylinder of height $H_0$ and radius $L_0$ with granular material of mass $m$. The granular column is then released *en masse* by quickly removing the cylinder, thus allowing the granular material to collapse onto the horizontal surface, forming a deposit having a final height $H_f$ and radius $L_f$. Although the experiment is simple and attractive allowing us to explore the limitations of depth-average modelling techniques, a constitutive law that could describe the entire flow behaviour is still lacking. The primary aim of these experiments was to determine the scaling laws for the run-out distance.

#### Deposit morphology

Lajeunesse et al. (2005) observed that the flow dynamics and the final deposit remain independent of the volume of granular material that is released, but depend only on the initial aspect ratio $a$ of the granular column. The experiment was carried out to understand the effect of the geometrical configuration on the run-out, the mechanism of initiation of the flow, the evolution of flow with time, and to understand how such complex flow dynamics could produce deposits obeying simple power laws. Lube et al. (2005) explored the effect of density and shape of grains on flow dynamics, whereas Lajeunesse et al. (2004) worked with glass beads to study the influence of bead size and substrate properties on the deposit morphology. Surprisingly, both drew the striking conclusion that the flow duration, the spreading velocity, the final extent of the deposit, and the fraction of energy dissipated during the flow can be scaled in a quantitative way independent of substrate properties, bead size, density, and shape of the granular material and released mass, $m$ (Lajeunesse et al., 2005). Lube et al. (2005) scaled the run-out distance as

$$\frac{L_f - L_0}{L_f} \approx \begin{cases} 1.24a, & a \lesssim 1.7 \\ 1.6a^{1.2}, & a \gtrsim 1.7 \end{cases} \tag{2.8}$$

while Lajeunesse et al. (2004) scaled run-out as

$$\frac{L_f - L_0}{L_f} \approx \begin{cases} 1.35a, & a \lesssim 0.74 \\ 2.0a^{1.2}, & a \gtrsim 0.74 \end{cases} \tag{2.9}$$

Quasi-two-dimensional collapse of a granular column on a horizontal surface (Lajeunesse et al., 2005) reveals that the geometric configuration influences the scaling of the run-out distance. The run-out in a quasi-two-dimensional collapse of a granular column in a rectangular channel, is scaled as

$$\frac{L_f - L_0}{L_f} \approx \begin{cases} 1.2a, & a \lesssim 2.3 \\ 1.9a^{2/3}, & a \gtrsim 2.3 \end{cases} \tag{2.10}$$

At large aspect ratios, the run-out is well represented by a simple power-law dependence. The exponent is found to vary with the channel width: $\Delta L/L_0 \approx \lambda a^{0.65}$ for narrow channels and $\Delta L/L_0 \approx \lambda a^{0.9}$ for wide channels. The constant of proportionality $\lambda$ is found to vary with the internal friction angle of the granular material, which contradicts the findings of previous authors, especially Lube et al. (2005) who found that the scaling of run-out is independent of the granular material, perhaps due to a narrow range of experimental materials (Staron et al., 2005). Balmforth and Kerswell (2005) observed that the material properties have almost no influence on the exponent of the normalised run-out as a function of the initial aspect ratio. The numerical constant of proportionality, however, showed clear material dependence. This corroborates the conclusions of Lajeunesse et al. (2004) and refutes that of Lube et al. (2005). Daerr and Douady (1999) also observed strong influence of initial packing density and the internal structure on the behaviour of granular flows.

The scaling found for quasi-two-dimensional experiments in the narrow gap configuration gives similar results as Lube et al. (2005) and roughly a scaling of $L_f/(L_f - L_0) \propto a^{2/3}$. Numerical simulations of granular column collapse by Zenit (2005) and Staron et al. (2005) yielded similar scaling of run-out with aspect ratio $a$, unlike other authors, Zenit (2005) did not observe any transition in the run-out behaviour of a granular column collapse with the aspect ratio $a$. The origin of the exponents is still under discussion. No model has yet achieved a comprehensive explanation of the dependence of the complex-collapse dynamics on simple power laws. However, it was observed that a simple friction model cannot effectively describe the collapse dynamics. Staron et al. (2005) explained the mechanism of spreading using an initial potential energy approach.

For higher aspect ratios, the free fall of the column controls the dynamics of the collapse and the energy dissipation at the base is attributed to the coefficient of restitution. Thus, the initial potential energy stored in the system is dissipated by sideways flow of material and the mass ejected sideways is found to play a significant role in the spreading process, i.e. as $a$ increases, the same fraction of initial potential energy drives an increasing proportion of initial mass against friction, thus explaining the power-law dependence of the run-out distance on $a$. Taking advantage of the similarity between granular slumping and the classical "*dam break*" problem in fluid mechanics, Kerswell (2005) solved both the axis-symmetric and two-dimensional granular-collapse problem using the shallow-water approximation. Although the results of the shallow-water approximation have good agreement with experimental results, the shallow-water approximation overestimates the run-out distance for columns with aspect ratio $a$ greater than unity. The shallow-water equation does not take into account the effect of vertical acceleration (Lajeunesse et al., 2005), which has been found to play a significant role in controlling the collapse dynamics Staron et al. (2005), thus resulting in overestimation of run-out. Tall columns showed significantly longer run-out distances when continuum approaches like material point method is used (Bandara, 2013; Mast et al., 2014).

The evolution of the scaled deposit height $H_f/L_0$ with the aspect ratio $a$ for axis-symmetric collapse (Lajeunesse et al., 2005) is given as

$$H_f/L_0 \approx \begin{cases} a, & a \lesssim 0.74 \\ 0.74, & a \gtrsim 0.74 \end{cases} \tag{2.11}$$

and for two-dimensional collapse

$$H_f/L_0 \approx \begin{cases} a, & a \lesssim 0.7 \\ a^{1/3}, & a \gtrsim 0.7 \end{cases} \tag{2.12}$$

The scaling of the final collapse height is found to be similar with the experimental results of Lube et al. (2005) and Balmforth and Kerswell (2005), and the numerical simulation of Staron et al. (2005). Numerical simulation of granular column collapse (Lacaze et al., 2008; Staron et al., 2005) showed a transition in the flow behaviour at $a \geq 10$, which was not observed in granular collapse experiments (Balmforth and Kerswell, 2005; Lajeunesse et al., 2004; Lube et al., 2005). In the depth-averaged shallow-water model, which integrates over the depth, the emphasis was on capturing the scaling of the final deposit, rather than trying to reproduce the internal structure of the flow. The shallow-water model captures well the final deposit scaling for lower aspect ratios, however fails to capture the flow dynamics for granular columns with higher aspect ratios, where the flow is governed mainly by the vertical collapse of the granular column as a whole. The run-out distance predicted is clearly erroneous in the collapse regime where there is a sudden drop in efficiency by which the initial potential energy of the system is converted into the kinetic energy for spreading. Even a more sophisticated basal drag law will not be sufficient to model the mechanism of granular column collapse realistically using the shallow-water approximation (Kerswell, 2005).

### Flow dynamics

The variation of final scaled deposit with aspect ratio $a$ shows a transition in the run-out behaviour for an aspect ratio of 1.7, indicating a transformation in the spreading process or the collapse mechanism. To understand the collapse mechanism, it is insufficient to study only the final scaled profile, and hence the entire flow process should be analysed. Lajeunesse et al. (2005) observed the flow regime and deposit morphology for a quasi-two-dimensional granular collapse in a rectangular channel. The flow phenomenology of a granular column collapse in a rectangular channel was surprisingly similar to that observed in the axis-symmetric collapse (Lajeunesse et al., 2004; Lube et al., 2005), depending mainly on the initial aspect ratio $a$.

The flow dynamics involve spreading of granular mass by avalanching of flanks producing a truncated cone for $a \lesssim 0.74$ and a cone for $a \gtrsim 0.74$; the transition of flow dynamics occurs as the value of $a$ is increased. The evolution of the deposit height remains independent of the flow for $a \lesssim 0.7$, however it exhibits significant dependence on the geometrical configuration for $a \gtrsim 0.7$. In rectangular channels, the effect of sidewall on the run-out behaviour was observed; the surface velocity profile between the sidewalls is that of a plug flow with a high slip velocity at the wall and low shear along the direction transverse to the flow. Systematic measurements indicate that the ratio of the maximum surface velocity to the surface velocity at the wall is between 1.2 and 1.4. Lajeunesse et al. (2005) observed that the difference between the evolution of $H_f$ in the axis-symmetric geometry and in the rectangular channel is not an experimental artefact due to the side wall friction, but is a *geometrical effect*.

Understanding the internal flow structure will provide an insight into the complex collapse dynamics. For smaller values of aspect ratio $a \leq 0.7$, the flow is initiated by a failure at the edge of the pile along a well-defined fracture surface above which material slides down and below which the grains remain static. The grains located above the fracture move "*en masse*" and most of the shear is concentrated along this surface forming a "*truncated-cone-like*" deposit with a central motionless plateau. For columns with larger aspect ratios, the flow is still initiated by failure along a well-defined surface, an inclined plane in two-dimensional geometry or a cone in the axis-symmetric case. However, the initial height of the column is much higher than the top of the failure surface, causing a vertical fall of grains until they reach the summit where they diverge along the horizontal direction, dissipating a lot of kinetic energy, resulting in a final conical deposit. Interestingly, the final deposit height coincides with the summit of the failure surface in the axis-symmetric geometry, whereas in the rectangular channel, the deposit summit always lies above the top of the failure surface (Lajeunesse et al., 2005).

Identification of the static region is an important task, as it is a prime component in describing the collapse mechanism. Regardless of the experimental configuration, for all values of $a$ the flow is initiated by rupture along a well-defined failure surface and the failure angle remains of the order of $50^o$ to $55^o$. The failure angle is consistent with an interpretation of *active Coulomb failure*, which leads to a failure angle $\phi_f = 45^o + \delta/2$, where $\delta$ is the internal friction angle of the granular material. Estimating the internal friction angle of glass beads from the angle of repose as $22^o$, the failure angle is estimated as $56^o$, which is in good agreement with the experimental findings. Contrary to the suggestion of Lajeunesse et al. (2004), the fracture angle was found to have no direct effect on the transition between truncated cone and conical deposit occurring at aspect ratio $a$ of 0.7 (Lajeunesse et al., 2005). Schaefer (1990) observed the onset of instability, when the neutrally stable plane waves makes an angle of $\phi_I$ with the major principle stress axis. The onset of instability triggers unstable plane waves in a narrow wedge of $56^o$ to $65^o$, which corresponds to the angle of shear bands: this observation matches well with the failure angle observed in the granular flow. A rate-dependent constitutive relationship (Jop et al., 2006) for dense granular flows indicates the angle of shear-band orientation depends on the inertial number $I$. For small to moderate values of $I$, the orientation of shear bands is found to vary from the Roscoe and the Coulomb solutions to a unique admissible angle (Lemiale et al., 2011). Daerr and Douady (1999) observed active Coulomb-type yielding in transient surface flows for granular materials having a packing density of 0.62 to 0.65. The comparison of initial and final areas indicates a change in the packing; the initial area is systematically smaller than the final area. This change in packing is typical of granular slumping in a channel and reflects that the pile, which initially had a relatively close packing, expanded (by about 10%) as the flow progressed, to form the final deposit (Balmforth and Kerswell, 2005).

A critical time $\tau_c$ is defined as the transition time at which the flow is fully developed. The velocity field then depends on the position of grains along the pile, see Lajeunesse et al. (2005). In the front, the flow involves the entire thickness of the pile and corresponds to a plug flow in the horizontal direction. In the region above the static core, the flow is locally parallel to the failure surface and has an upper linear part and a lower exponential tail near the static bed (Lajeunesse et al., 2005). The velocity flow profile is similar to that of a steady granular flow (Midi, 2004). As the pile spreads, interface separation occurs as the flow diverges and the static region starts to move inwards; this effect is predominant in the case of granular flows in a rectangular channel. The typical velocity observed at the front of the ejecting mass is $v = \sqrt{2gL_0}$.

The flow evolution as per Staron et al. (2005) involves three stages. The first stage involves conversion of the initial potential energy of the grains into vertical motion, resulting in downwards acceleration of grains. In the second stage, the grains undergo collision with the base and/or neighbouring grains, and their vertical motion is converted into horizontal motion. In the final stage, the grains eventually leave the base area of the column and flow sideways. The typical time required for the flow to cease and form the final deposit, from the instant of its release, is $T = \sqrt{2H_0/g}$ (Staron et al., 2005). While plotting the variation of normalized potential and kinetic energy with normalized time, Staron et al. (2005) observed that the flow ceases when the normalized time $t/T_0$ is 2.5, i.e. the flow is assumed to have stopped when the total normalized energy is almost zero. This observation is consistent with the experimental results of Lube et al. (2005) and Lajeunesse et al. (2005). The transition of the flow occurs when the normalized time $t/T_0$ is 1.0 or at critical time $\tau_c$, which is defined as the time at which the flow is fully mobilized.

### Comments on modelling

In order to have a detailed understanding of the final profile of the collapsed granular column, it is important to solve the collapse problem as an *initial-value problem* (Balmforth and Kerswell, 2005), beginning from the instant of release and extending to the time when the material finally ceases to flow, forming the final deposit. As the process of granular collapse involves collective dynamics of collisions and momentum transfer, the prediction of the trajectory of a single grain is difficult. In fact, there are quantitative disagreements between theory and experiments; the final shapes are reproducible, but not perfectly. Some of the disagreement arises because the experiments did not have exactly the same amount of materials; it is indeed difficult to fill the pile with exactly the same amount of material, which results in differences in packing. However, the theoretical errors are due to the incapability of the models to capture the physics that governs the flow dynamics (Balmforth and Kerswell, 2005). Shallow water models fail to account for the vertical acceleration, which is responsible for the momentum transfer and, in turn, the spreading process. This restricts the shallow water model to capture the mechanism of collapse until the critical time $\tau_c$. Surprisingly, shallow-water models capture certain experimental aspects for columns with lower aspect ratios (Balmforth and Kerswell, 2005; Kerswell, 2005; Mangeney et al., 2010), even though the contrast between surface flows and the static region is important in this range of aspect ratio. Thus, the assumption of plug flow in the horizontal direction is not critical in capturing the run-out behaviour, especially if the basal friction coefficient is used as a fitting parameter (Lajeunesse et al., 2005).

Simple mathematical models based on conservation of horizontal momentum capture the scaling laws of the final deposit, however they fail to describe the initial transition regime, indicating that the initial transition has negligible effect on the run-out, which is incorrect. Models based on the initial potential energy show promise, but the effect of material properties, such as basal friction and coefficient of restitution, on the run-out behaviour is still unclear and produces non-physical run-outs. The famous $\mu(I)$ rheology predicts well the normalized run-out behaviour in comparison with the experimental results, for lower aspect ratios. The spreading dynamics is found to be similar for the continuum and grain approaches; however, the rheology falls short in predicting the run-out distance for higher aspect ratios. Unlike Lube et al. (2005), many researchers (Balmforth and Kerswell, 2005; Kerswell, 2005) observed strong dependency of material properties on the run-out distance, moistening the materials or the sides of the channel even by a small amount leads to markedly different results. Staron and Hinch (2007) observed that the friction has little effect on the run-out for granular column collapse for high aspect ratios, which are driven mainly by the free vertical fall of grains. The initial conditions have a significant impact on the overall behaviour of the granular system, indicating the significance of the triggering mechanism in case of the natural flows (Staron and Hinch, 2007). Numerical investigations, such as Discrete Element Method techniques, allow us to evaluate quantities which are not accessible experimentally, thus providing useful insight into the flow dynamics. Subsequent chapters discuss the methodology and modelling of granular column by continuum- and discrete-element approaches.

### 2.3.2 Flow down an inclined plane

Studies on the flow of granular materials down inclined planes are important to understand the mechanism of geophysical hazards, such as granular avalanches, debris flows and submarine landslides. Large scale field tests on dry and saturated granular materials were carried out to capture the mechanism of granular flows down an inclined plane (Denlinger and Iverson, 2001; Okada and Ochiai, 2008). The granular material stored in a reservoir at the top of the inclined plane is released by opening a gate; the material flows down and develops into a dense granular flow.

A variant of the above experiment is the experiment on fluidized beds which involves an initial bed of granular material of thickness '$h$' which is inclined gradually until the granular material flows. When the plane inclination reaches a critical angle, $\theta_{start}$, the material starts to flow and reaches a sustained flow until the inclination is decreased down to a second critical angle, $\theta_{stop}$ (Midi, 2004). The occurrence of two critical angles indicates the hysteretic nature of granular materials. Reciprocally, the critical angle thresholds can be interpreted in terms of critical layer thicknesses $h_{stop}(\theta)$ and $h_{start}(\theta)$. The measurement of $h_{stop}(\phi)$ is easier as it corresponds to the thickness of the deposit remaining on the plane once the flow has ceased.

Three regions can be observed: a region where no flow occurs, $(h < h_{stop}(\theta))$, a sub-critical region where both static and flowing layers can exist $(h_{stop}(\theta) < h < h_{start}(\theta)$ and a region where flow always occurs, $(h > h_{start}(\theta))$. In the flow regime, i.e. $(h > h_{start}(\theta))$, the flow is steady and uniform for moderate inclination, but accelerates along the plane for large inclinations (Midi, 2004). The critical angle controlling the flow behaviour tends to increase when the thickness of the bed decreases (Daerr and Douady, 1999; Pouliquen and Chevoir, 2002), which can be attributed to the non-trivial finite-size effects and/or boundary effects that are not well understood (Forterre and Pouliquen, 2008).

Fast moving granular flows can undergo a motion-induced self-fluidization process under the combined effects of front instabilities setting on at large values of the Froude number which are responsible for extensive air entrainment, and small deflation rates associated with small incipient fluidization velocities and longer collapse time of the bed solids. Self-fluidization results in enhanced mobility of the solids, causing an inviscid flow to an extent that may largely exceed the establishment of a "*granular liquid*" state in purely granular flow (Bareschino et al., 2008). It is understood that, for a granular material to flow, it has to exceed a certain critical threshold, i.e. the friction criterion: the ratio of shear stress to normal stress. As there is no internal stress scale for a granular material, granular materials exhibit solid-fluid transition behaviour based on the friction criterion (Forterre and Pouliquen, 2008). The stress ratio in the flowing regime above the static bed indicates the solid-to-fluid transition is a yielding phenomenon and can be described by Mohr-Coulomb-like failure criterion (Zhang and Campbell, 1992). This is in contrast to the mechanism of behaviour of other complex fluids, where there is an internal stress scale linked to the breakage of microscopic structure. From a microscopic point of view, the strength of the granular materials is due to the internal friction between grains, but packed frictionless materials still exhibit macroscopic friction.

Constitutive laws based on Plasticity-theories relate the micro-structure to the macroscopic behaviour (Roux and Combe, 2002), which provide useful insight into the mechanism of granular flow. However, at present they are limited to initiation of deformation and do not predict the quasi-static flow. Material Point Method simulation of granular flow down an inclined plane (Abe et al., 2006; Bandara, 2013) captures the flow behaviour in the initial stages, however it exhibits inconsistent behaviour when the granular material ceases to flow. This may be due to the application of small deformation theory to a large deformation problem and to the use of zero dilation. In the case of flow down an inclined plane, the only control parameter is the flow rate, $Q$, and uniform, steady flows are possible if the system is confined between walls. The additional friction induced by the lateral walls has a significant effect on the flow and causes localization at the free surface. The $\mu(I)$ friction law captures the velocity profile and the localization at the free surface. However, the model fails to capture the transition from a continuous flow to an avalanching regime as the flow rate is decreased (Pouliquen et al., 2006). The flow can cause strong Coulomb shear stresses to develop on a plane normal to the basal flow boundary. The stresses dissipate energy as the flow encounters obstructions: models that lack multi-dimensional momentum transport or Coulomb friction cannot represent this energy dissipation and lodging.

### 2.3.3 Saturated and submerged granular flows

Geophysical hazards, such as debris flows and submarine landslides, usually involve flow of granular solids and water as a single-phase system. The momentum transfer between the discrete and the continuous phases significantly affects the dynamics of the flow as a whole (Topin et al., 2011). Although certain macroscopic models are able to capture simple mechanical behaviours (Peker and Helvacı, 2007), the complex physical mechanisms occurring at the grain scale, such as hydrodynamic instabilities, formation of clusters, collapse, and transport (Topin et al., 2011), have largely been ignored. In particular, when the solid phase reaches a high volume fraction, the strong heterogeneity arising from the contact forces between the grains, and the hydrodynamic forces, are difficult to integrate into the homogenization process involving global averages (Topin et al., 2011).

In two-phase models (Pitman and Le, 2005), the momentum transfer between the grains and the suspension fluid depends on the momentum equations of both the phases. In case of mixture theory based models (Meruane et al., 2010), the shear-induced migration and grains collisions are considered in an average sense. In order to describe the mechanism of saturated and/or immersed granular flows, it is important to consider both the dynamics of the solid phase and the role of the ambient fluid (Denlinger and Iverson, 2001; Iverson, 1997). The dynamics of the solid phase alone are insufficient to describe the mechanism of granular flow in a fluid; it is important to consider the effect of hydrodynamic forces that reduce the weight of the solids inducing a transition from dense-compacted to dense-suspended flows, and the drag interactions which counteract the movement of the solids (Meruane et al., 2010).

Iverson (2000) observed in the large-scale field tests that the soil prepared in a loose state on a slope and subjected to a rainfall flows rapidly like a liquid when it breaks, whereas a dense soil only slowly creeps. The underlying mechanism is related to the dilation or contraction character of the granular material, which the authors described as the “pore pressure feedback.” The compaction or dilation of grains can cause additional stress in the grains which can stabilise or destabilise the soil. The flow is controlled by the coupling between the dilatancy of the granular layer and the development of pore pressure in the fluid phase (Pailha et al., 2008).

The collapse of a granular column, which mimics the collapse of a cliff, has been extensively studied in the case of dry granular material, when the interstitial fluid plays no role. The case of the collapse in presence of an interstitial fluid has been less studied (Topin et al., 2012). Rondon et al. (2011) performed granular column collapse experiments in fluid to understand the role of initial volume fraction. They observed that, contrary to the dense case, the whole loose column is mobilised immediately. The loose column in fluid spreads almost twice than the dense case. The collapse of a granular column in a viscous fluid is found to be mainly controlled by the initial volume fraction and not by the aspect ratio of the column. The role of the initial volume fraction observed in the viscous collapse can be understood by the pore pressure feedback mechanism proposed by (Iverson, 2000; Schaeffer and Iverson, 2008) in the context of landslides. The dense column needs to dilate in order to flow. When it starts to fall, liquid is then sucked into the column, which is then stabilized by the additional viscous drag (Rondon et al., 2011; Topin et al., 2012). By opposition the loose column when it starts flowing expands and ejects liquid, leading to a partial fluidization of the material. In the dry case, inertia is responsible for the enhance of mobility at high aspect ratio. In the viscous regime, the inertial effects remain negligible, which could explain why the important parameter controlling the dynamics is the initial volume fraction and not the initial aspect ratio. Topin et al. (2011) observed that the run-out exhibit a power-law relation with the peak kinetic energy and the largest value of run-out in fluid inertial regime, and the lowest in grain-inertial and intermediate value for the viscous-regime. The viscous regime is where the grain reaches the viscous limit velocity, the Stoke's number $S_t << 1$ and the density ratio $r >> S_t$ (Courrech du Pont et al., 2003). However, the role of the volume fraction on dry granular collapse has not been precisely studied and the preparation of the pile may also play a role Daerr and Douad (1999).

Cassar et al. (2005) carried out experimental investigation on the flow of dense granular material down an inclined plane fully-immersed in water. The velocities observed in the submarine case were found to be a magnitude smaller than the dry condition. The run-out behaviour collapse on to a single friction law, which shows that the major role of the fluid is to change the time it takes for a grain to fall into a void-space. The $\mu(I)$ rheology for dry dense flows captures the behaviour of dense submarine granular flows, if the inertial time scale in the rheology is replaced with a viscous time scale. In the case of dry granular flows, the parameter $I$ is defined as the ratio between the time taken for a grain to fall into the hole, $t_{micro}$, and the meantime, $t_{mean}$, which is inversely related to the shear rate. Pitman and Le (2005) observed that if the fluid inertial effects are small enough, then a simpler model can be adopted. Hence, assuming that the fluid velocity is low enough for the contact interaction between grains to be significant, the time taken by the grain to fall into a hole, $t_{micro}$, is then controlled by the viscosity of the ambient fluid. Thus, the dimensionless parameter can be modified to incorporate the viscous time to describe granular flow in a fluid (Pouliquen et al., 2005).

The constitutive law is found to be valid only for the steady uniform regime; unsteady phenomena such as the triggering of avalanches result in coupling between the granular grains and the ambient fluid and are much more complex to model. Transient regimes characterized by change in solid fraction, dilation at the onset of flow and development of excess pore pressure, result in altering the balance between the stress carried by the fluid and that carried by the grains, thereby changing the overall behaviour of the flow (Denlinger and Iverson, 2001). The $\mu(I)$

rheology seems to predict well the flow of granular materials in the dense regime. However, the transition to the quasi-static regime where the shear rate vanishes is not captured by the simple model. Also, shear band formation observed under certain flow configurations is not predicted. The flow threshold or the hysteresis characterizing the flow or no-flow condition is not correctly captured by the model, which can be due to the discrepancies between the physical mechanism controlling the grain level interactions, clustering, and vortex formations. When the scale of the system is larger than the size of the structure, a simple rheology is expected to capture the overall flow behaviour, however the size of the correlated motion is the same as that of the system, causing difficulties in modelling the flow behaviour (Pouliquen et al., 2005). Hence, it is essential to study the behaviour of granular flows at various scales, i.e. microscopic, meso-scale and at continuum level, in order to develop a constitutive model that captures the entire flow process.

## 2.4 Summary

Granular flow involves three distinct regimes: the dense quasi-static regime, the rapid and dilute flow regime, and an intermediate regime. Many models, such as the shallow-water approximation, kinetic theory approach, and rheologies based on shear rate, have captured the basic flow dynamics, but have failed to describe the complete mechanics of the granular flow. The dynamics of homogeneous granular flow involve at least three different scales, making it difficult to describe the mechanics of granular flow by simple theories. It is important to describe the granular dynamics as an initial-value problem, beginning from the instance of its initiation to the time at which the material ceases to flow, forming a final deposit. During this process, the granular materials undergo phase transition, in addition to the transition in their flow dynamics. Most theoretical models are incapable of capturing these transition regimes. Experimental conditions are too difficult to reproduce precisely, resulting in inherent inconsistencies in the results. Numerical approaches, such as Discrete Element Method, allow us to evaluate quantities which are not accessible experimentally, thus providing useful insight into the flow dynamics, thereby enabling us to develop better constitutive laws.

# References


Abe, K., Johansson, J., and Konagi, K. (2006). A new method for the run-out analysis and motion prediction of rapid and long travelling landslides with MPM. *Doboku Gakkai Ronbunshuu*, 63:93–109 (in Japanese).

Bagnold, R. (1954). Experiments on a gravity-free dispersion of large solid spheres in a Newtonian fluid under shear. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 225(1160):49.

Balmforth, N. J. and Kerswell, R. R. (2005). Granular collapse in two dimensions. *Journal of Fluid Mechanics*, 538:399–428.

Bandara, S. (2013). *Material Point Method to simulate Large Deformation Problems in Fluid-saturated Granular Medium*. PhD thesis, University of Cambridge.

Bareschino, P., Lirer, L., Marzocchella, A., Petrosino, P., and Salatino, P. (2008). Self-fluidization of subaerial rapid granular flows. *Powder Technology*, 182(3):323–333.

Campbell, C. (1986). The effect of microstructure development on the collisional stress tensor in a granular flow. *Acta Mechanica*, 63(1):61–72.

Campbell, C. and Brennan, C. (1985). Computer simulation of granular shear flows. *Journal of Fluid Mechanics*, 151:167–88.

Campbell, C. S. (1990). Rapid Granular Flows. *Annual Review of Fluid Mechanics*, 22(1):57–90.

Campbell, C. S. (2002). Granular shear flows at the elastic limit. *Journal of Fluid Mechanics*, 465:261–291.

Campbell, C. S. (2006). Granular material flows - An overview. *Powder Technology*, 162(3):208–229.

Cassar, C., Nicolas, M., and Pouliquen, O. (2005). Submarine granular flows down inclined planes. *Physics of Fluids*, 17(10):103301–11.

Cawthor, C. J. (2006). The flow of granular media. Technical report, University of Cambridge, Deparment of Applied Mathematics and Theoretical Physics.

Choi, J. (2005). *Transport-limited aggregation and dense granular flow*. Phd, Massachusetts Institute of Technology.

Courrech du Pont, S., Gondret, P., Perrin, B., and Rabaud, M. (2003). Granular Avalanches in Fluids. *Physical Review Letters*, 90(4):044301.

Da Cruz, F., Emam, S., Prochnow, M., Roux, J. N., and Chevoir, F. (2005). Rheophysics of dense granular materials: Discrete simulation of plane shear flows. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, 72(2):1–17.

Daerr, A. and Douad, S. (1999). Sensitivity of granular surface flows to preparation. *Europhys. Lett.*, 47(3):324–330.

Daerr, A. and Douady, S. (1999). Two types of avalanche behaviour in granular media. *Nature*, 399(6733):241–243.

Daniel, R. C., Poloski, A. P., and Eduardo Saez, A. (2007). A continuum constitutive model for cohesionless granular flows. *Chemical Engineering Science*, 62(5):1343–1350.

Denlinger, R. and Iverson, R. (2001). Flow of variably fluidized granular masses across three-dimensional terrain, ii: Numerical predictions and experimental tests. *J. Geophys. Res*, 106(B1):553–566.

Forterre, Y. and Pouliquen, O. (2008). Flows of Dense Granular Media. *Annual Review of Fluid Mechanics*, 40(1):1–24.

Goldhirsch, I. (2003). Rapid granular flows. *Annual Review of Fluid Mechanics*, 35:267–293.

Hutter, K., Koch, T., Pluuss, C., and Savage, S. (1995). The dynamics of avalanches of granular materials from initiation to runout. Part II. Experiments. *Acta Mechanica*, 109(1):127–165.

Hutter, K., Wang, Y., and Pudasaini, S. P. (2005). The Savage: Hutter Avalanche Model: How Far Can It be Pushed? *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, 363(1832):1507–1528.

Iddir, H. and Arastoopour, H. (2005). Modeling of multitype particle flow using the kinetic theory approach. *AIChE Journal*, 51(6):1620–1632.

Iverson, R. (2003). The debris-flow rheology myth. In Rickenmann and Chen, editors, *Debris-Flow Hazards Mitigation: Mechanics, Prediction, and Assessment*, pages 303–314, Rotterdam.

Iverson, R. M. (1997). The physics of debris flows. *Rev. Geophys.*, 35(3):245–296.

Iverson, R. M. (2000). Acute Sensitivity of Landslide Rates to Initial Soil Porosity. *Science*, 290(5491):513–516.

Jaeger, H., Nagel, S., and Behringer, R. (1996). Granular solids, liquids, and gases. *Reviews of Modern Physics*, 68(4):1259–1273.

Jenkins, J. T. and Savage, S. B. (1983). A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles. *Journal of Fluid Mechanics*, 130:187–202.

Johnson, A. (1965). *A model for debris flow*. PhD thesis, The Pennsylvania State University.

Jop, P., Forterre, Y., and Pouliquen, O. (2006). A constitutive law for dense granular flows. *Nature*, 441(7094):727–730.

Kamrin, K. (2008). *Stochastic and deterministic models for dense granular flow*. PhD thesis, Massachusetts Institute of Technology.

Kamrin, K. (2010). Nonlinear elasto-plastic model for dense granular flow. *International Journal of Plasticity*, 26(2):167–188.

Kerswell, R. (2005). Dam break with Coulomb friction: A model for granular slumping? *Physics of Fluids*, 17:057101.

Lacaze, L., Phillips, J. C., and Kerswell, R. R. (2008). Planar collapse of a granular column: Experiments and discrete element simulations. *Physics of Fluids*, 20(6).

Lajeunesse, E., Mangeney-Castelnau, A., and Vilotte, J. P. (2004). Spreading of a granular mass on a horizontal plane. *Physics of Fluids*, 16(7):2371.

Lajeunesse, E., Monnier, J. B., and Homsy, G. M. (2005). Granular slumping on a horizontal surface. *Physics of Fluids*, 17(10).

Lemiale, V., Muhlhaus, H. B., Meriaux, C., Moresi, L., and Hodkinson, L. (2011). Rate effects in dense granular materials: Linear stability analysis and the fall of granular columns. *International Journal for Numerical and Analytical Methods in Geomechanics*, 35(2):293–308.

Lube, G., Huppert, H. E., Sparks, R. S. J., and Freundt, A. (2005). Collapses of two-dimensional granular columns. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, 72(4):1–10.

Mangeney, A., Roche, O., and Hungr, O. (2010). Erosion and mobility in granular collapse over sloping beds. *Journal of Geophysical Research*, 115(F3):F03040.

Mast, C. M., Arduino, P., Mackenzie-Helnwein, P., and Miller, G. R. (2014). Simulating granular column collapse using the Material Point Method. *Acta Geotechnica*.

Mehta, A. (2011). *Granular Physics*. Cambridge University Press.

Meruane, C., Tamburrino, A., and Roche, O. (2010). On the role of the ambient fluid on gravitational granular flow dynamics. *Journal of Fluid Mechanics*, 648:381–404.

Midi, G. D. R. (2004). On dense granular flows. *European Physical Journal E*, 14(4):341–365.

Ng, B. H., Ding, Y., and Ghadiri, M. (2008). Assessment of the kinetic-frictional model for dense granular flow. *Particuology*, 6(1):50–58.

Okada, Y. and Ochiai, H. (2008). Flow characteristics of 2-phase granular mass flows from model flume tests. *Engineering Geology*, 97(1-2):1–14.

Pailha, M., Pouliquen, O., and Nicolas, M. (2008). Initiation of Submarine Granular Avalanches: Role of the Initial Volume Fraction. *AIP Conference Proceedings*, 1027(1):935–937.

Peker, S. and Helvacı, S. (2007). *Solid-liquid two phase flow*. Elsevier.

Pitman, E. B. and Le, L. (2005). A Two-Fluid Model for Avalanche and Debris Flows. *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, 363(1832):1573–1601.

Popken, L. and Cleary, P. W. (1999). Comparison of Kinetic Theory and Discrete Element Schemes for Modelling Granular Couette Flows. *Journal of Computational Physics*, 155(1):1–25.

Potapov, A. and Campbell, C. (1996). Computer simulation of hopper flow. *Physics of Fluids*, 8:2884.

Pouliquen, O. (1999). Scaling laws in granular flows down rough inclined planes. *Physics of Fluids*, 11(3):542–548.

Pouliquen, O., Cassar, C., Forterre, Y., Jop, P., and Nicolas, M. (2005). How do grains flow: Towards a simple rheology of dense granular flows. In *Powders and Grains*.

Pouliquen, O., Cassar, C., Jop, P., Forterre, Y., and Nicolas, M. (2006). Flow of dense granular material: towards simple constitutive laws. *Journal of Statistical Mechanics: Theory and Experiment*, 2006(07):P07020–P07020.

Pouliquen, O. and Chevoir, F. (2002). Dense flows of dry granular material. *Comptes Rendus Physique*, 3(2):163–175.

Pouliquen, O. and Forterre, Y. (2002). Friction law for dense granular flows: Application to the motion of a mass down a rough inclined plane. *Journal of Fluid Mechanics*, 453:133–151.

Radjai, F. and Richefeu, V. (2009). Contact dynamics as a nonsmooth discrete element method. *Mechanics of Materials*, 41(6):715–728.

Rondon, L., Pouliquen, O., and Aussillous, P. (2011). Granular collapse in a fluid: Role of the initial volume fraction. *Physics of Fluids*, 23(7):073301–073301–7.

Roux, J. and Combe, G. (2002). Quasistatic rheology and the origins of strain. *Comptes Rendus Physique*, 3(2):131–140.

Savage, S. and Hutter, K. (1991). The dynamics of avalanches of granular materials from initiation to runout. Part I: Analysis. *Acta Mechanica*, 86(1):201–223.

Savage, S. and Jeffrey, D. (1981). The stress tensor in a granular flow at high shear rates. *Journal of Fluid Mechanics*, 110(1):255–272.

Savage, S. and Sayed, M. (1984). Stresses developed by dry cohesionless granular materials sheared in an annular shear cell. *Journal of Fluid Mechanics*, 142:391–430.

Savage, S. B. (1984). The Mechanics of Rapid Granular Flows. *Advances in Applied Mechanics*, 24(C):289–366.

Schaefer, D. G. (1990). Instability and ill-posedness in the deformation of granular materials. *International Journal for Numerical and Analytical Methods in Geomechanics*, 14(4):253–278.

Schaeffer, D. G. (1987). Instability in the evolution equations describing incompressible granular flow. *Journal of Differential Equations*, 66(1):19–50.

Schaeffer, D. G. and Iverson, R. M. (2008). Steady and Intermittent Slipping in a Model of Landslide Motion Regulated by Pore-Pressure Feedback. *SIAM Journal on Applied Mathematics*, 69(3):769–786.

Schofield, A. and Wroth, P. (1968). *Critical state soil mechanics*. European civil engineering series. McGraw-Hill.

Staron, L. and Hinch, E. J. (2007). The spreading of a granular mass: Role of grain properties and initial conditions. *Granular Matter*, 9(3-4):205–217.

Staron, L., Radjai, F., and Vilotte, J. P. (2005). Multi-scale analysis of the stress state in a granular slope in transition to failure. *The European physical journal. E, Soft matter*, 18(3):311–20.

Topin, V., Dubois, F., Monerie, Y., Perales, F., and Wachs, A. (2011). Micro-rheology of dense particulate flows: Application to immersed avalanches. *Journal of Non-Newtonian Fluid Mechanics*, 166(1-2):63–72.

Topin, V., Monerie, Y., Perales, F., and Radjaï, F. (2012). Collapse Dynamics and Runout of Dense Granular Materials in a Fluid. *Physical Review Letters*, 109(18):188001.

Van Wachem, B., Schouten, J., Van den Bleek, C., Krishna, R., and Sinclair, J. (2001). Comparative analysis of CFD models of dense gas–solid systems. *AIChE Journal*, 47(5):1035–1051.

Xu, H., Louge, M., and Reeves, A. (2003). Solutions of the kinetic theory for bounded collisional granular flows. *Continuum Mechanics and Thermodynamics*, 15(4):321–349.

Zenit, R. (2005). Computer simulations of the collapse of a granular column. *Physics of Fluids*, 17(Compendex):031703–1–031703–4.

Zhang, Y. and Campbell, C. S. (1992). The interface between fluid-like and solid-like behaviour in two-dimensional granular flows. *Journal of Fluid Mechanics*, 237:541–568.