Geometric Langlands Study Group

May 19, 2023

The main focus of this seminar is the book by Beĭlinson and Drinfel'd [BD]. Many references can also be found on Gaitsgory's webpage `https://people.mpim-bonn.mpg.de/gaitsgde/grad_2009/`.

The time is Thursdays 11:05-12:30 in C3, except the 4th May and 11th May, where it is in C4; also the Week 1 talk will be in the junior algebra seminar (on Friday 11:00-12:00 in N3.12).

1 Week 1: Overview

There will be a meeting at 11:00 on Thursday in C3 to allocate talks. Ken will give a talk in the junior algebra seminar on Friday.

2 Week 2: Stacks

This talk should introduce Artin stacks, quotient stacks, fiber products, the stack of G-bundles on scheme X, the classifying stack BG of principal G-bundles, quasi-coherent sheaves on stacks (or on arbitrary functors), associated bundles (example of GL_n and vector bundles), Weil unformisation for $\mathrm{Bun}(\mathrm{GL}_n)$ and the cotangent bundle of $\mathrm{Bun}(G)$. References are [Sor00, Gin23].

3 Week 3: Crystals and $\mathcal{D}$-modules

This talk should introduce the classical definition of $\mathcal{D}$-modules on a smooth scheme, and the equivalence (in smooth case) to crystals in quasi-coherent sheaves. It should also cover twisted differential operators, and some of the six operations including pushforward, tensor, internal Hom, and exceptional pullback (along proper morphisms) of $\mathcal{D}$-modules. It should also define $\mathcal{D}$-modules on stacks, the equivalence to quasi-coherent sheaves on the de Rham stack of the smooth stack, and the description of $\mathcal{D}$-modules on quotient stacks. It should also define $\mathcal{D}$-schemes and crystals valued in schemes, define the jet and conformal block functors, and state the adjunction between them. References are [Gro68], [BD, §7.10], [Neg09b, HTT08, Gin98].

4 Week 4: Hecke actions

This talk should define the loop group and arc group, the affine Grassmannian (but not talk about its geometry yet), the spherical Hecke category $\mathscr{H}$, the monoidal structure on it by convolution, the Beauville-Lazlo theorem (and how it's not needed), the action of loop group on $\mathrm{Bun}(G)$, and the resulting action of the Hecke category. It should also cover the description of the points of the local Hecke correspondence $\mathscr{H}_x$ and strata, and the description in terms of weights and coweights of G (including the example of GL_n). It may end with the statement of geometric Satake equivalence and the definition of Hecke eigensheaves (if there is time). References are [Rei09a, Rei09b, Zhu16], [BD, §7.6].

5 Week 5: Geometric Satake Part 1 - Affine Grassmannian

If there wasn't time last week, the statement of geometric Satake equivalence. Moduli description of the affine Grassmannian, example of $\mathrm{Gr}_{\mathrm{GL}_n}$ being ind-projective, a word on how Gr_{G} is ind-projective for general reductive G. Maybe need to explain how (equivariant) D-modules on ind-schemes work. Explain how studying simple objects of spherical Hecke category reduces to studying Schubert cells of the affine Grassmannian. Prove semi-simplicity of the spherical Hecke category. References are [Zhu16, §1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 5.1].

6 Week 6 : Geometric Satake Part 2 - Fusion Product

Explain how "convolution product = fusion product" gives commutativity constraint for the monoidal structure. Beilinson–Drinfeld Grassmannians as moduli functors and as twisted product of the usual affine Grassmannian over the automorphism group of the formal disk. Factorisation property and the Ran Grassmannian. Fusion product and the fact that it recovers the convolution product. This possibly involves machinery of vanishing and nearby cycles functors. References are [DM82, Chapter 2] [Rei09a][Rei09b][Zhu16].

7 Week 7 : Geometric Satake Part 3 - Tannaka Reconstruction and the Langlands Dual Group

Precise statement of Tannaka reconstruction theorem. If unfinished from last week, show how the fusion product recovers the convolution product. Define the fibre functor and show that it satisfies the conditions of Tannaka reconstruction theorem. Show that the algebraic group obtained has root datum of the Langlands dual group. Reference is [Rei09b].

8 Week 8: Overview of Hitchin's integrable system

This talk should define the classical Hitchin variety, the global Hitchin map, state that image belongs to Poisson commuting functions, and state that the restriction to connected components is an isomorphism. It should also cover the definition of an integrable system and a quantised integrable system, and state the quantisation theorem (to be proven in week 7), and briefly say that the spectrum of the image is related to opers (to be proven in week 8), the point is that this will give us the connection to Hecke eigensheaves. It should also cover the definition of Lie algebroids, connections, their enveloping algebras, the Lie algebroid of infinitesimal symmetries of a ${}^LG$-bundle, the definition of an ${}^LG$-local system and the definition of opers as special kinds of ${}^LG$-local system. References are [BD, §2], [Hit87, Neg09a, BB93, BD05].

9 Week 9: Quantisation of Hitchin's integrable system

The goal of this talk is to construct the quantised system and prove that it is a quantisation, see [Gai09]. Along the way it could also cover: Quantisation schemas for Harish-Chandra pairs $(\mathfrak{g}, K)$, the part of the Feigin-Frenkel theorem related to triviality away from the critical value (to explain the point of twisting), and construction of Pfaffian line bundles. References are [BD, §2] [Gai09, Cla09, Ras09b, Sor00, Ras12, FF92].

10 Week 10: No seminar

11 Week 11: Opers

The goal of this talk is to prove the relation between $\mathfrak{z}(X)$ and opers, called [BD, §5.5] "the birth of opers". It should cover the critical value case of the Feigin-Frankel theorem, and the closed embedding $\mathrm{Spec}(\mathfrak{z}(X)) \hookrightarrow \mathrm{LocSys}_{^LG}$, whose image is identified with opers. It should show that the $\mathcal{D}$-module associated to an oper is a Hecke eigensheaf. It may end with a precise statement of the "Galois to automorphic" geometric Langlands correspondence for opers. References are [Ras12, FF92, Bar09, Ras09a].

References

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