Goals:

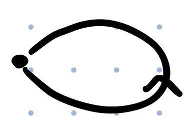
" Holes = boundaries of things that aren't there

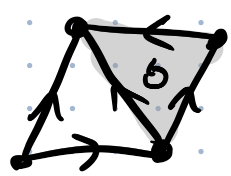
:= pseudo-boundaries / actual boundaries "

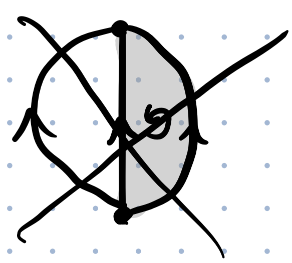
" general X hard to get handle on to compute.
approximate with more combinatorial spaces:
spaces made of triangles. "

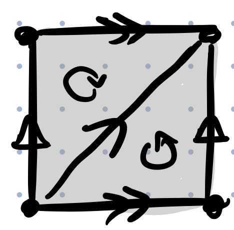
"Spaces modeled on triangles ⟺ something that triangles map into "

"Yoneda:
maps from test space into gen space
⟺ maps from test space as a gen space into gen space "



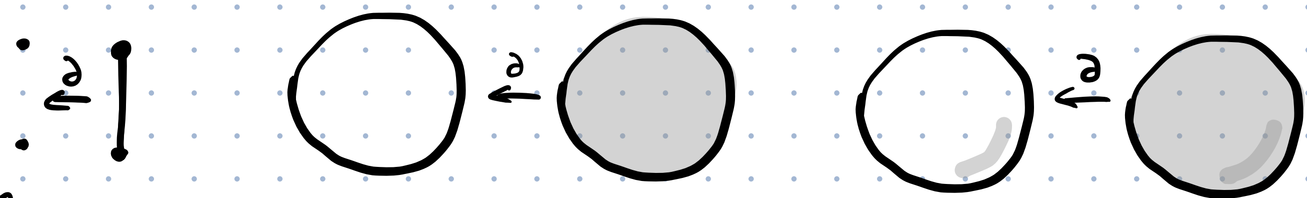






§ Holes [3:00]

"taking boundary, ∂ : n-dim subsp $\to$ (n-1) dim subsp"



"Shld be" ∂D^1 but isn't

"Shld be" ∂D^2 but isn't

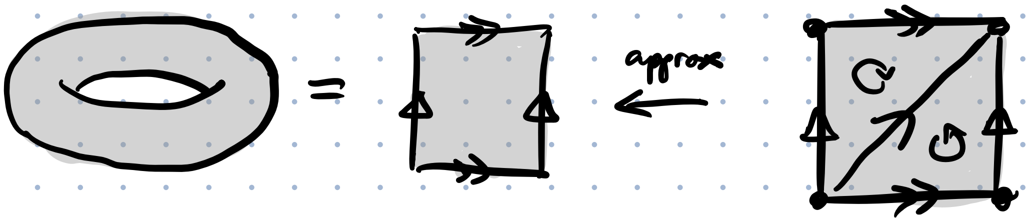
"Shld be" ∂D^3 but isn't

When it is, $\partial\partial D^2 = \emptyset$. Sim, $\partial\partial D^3 = \emptyset$. $\boxed{\partial \circ \partial = 0}$

Holes = "boundaries of things that aren't there"

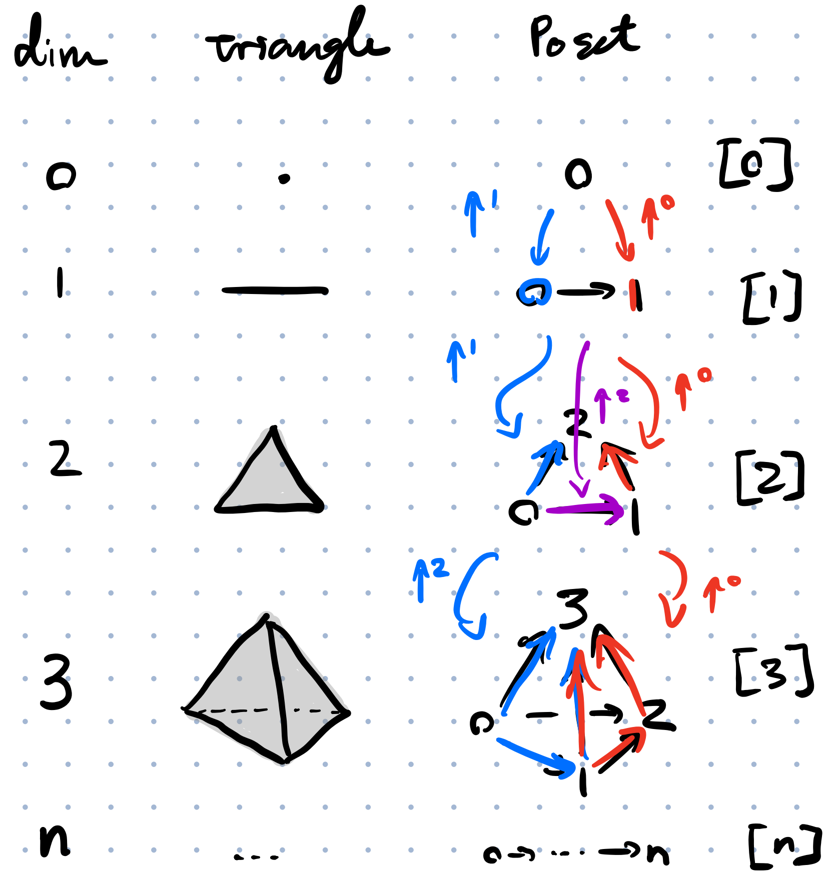
:= "potential boundaries" / "boundaries"

General X $\xleftarrow{\text{approx}}$ "Spaces modelled on triangles"



Combinatorial, better for computation

§ "Spaces modelled on Triangles" – ΔSets [7:00]



"category of triangles" $\Delta :=$

| $\mathrm{obj}(\Delta) = \{[n] \mid n \in \mathbb{N}\}$

| $\forall\, [n],[m] \in \Delta$,
$\Delta([n],[m]) :=$ monotonic injections

"Spaces modeled on triangles" := "something that triangles map into"

$X \in \Delta\mathrm{Set} :=$

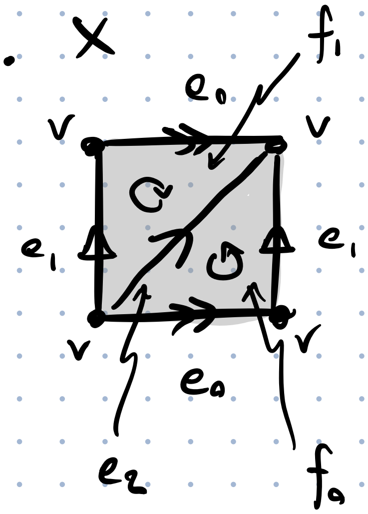
| $\forall\, n \in \mathbb{N}$, a set $X([n])$ "maps $[n] \to X$" (c.f.) Yoneda

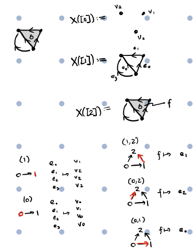
| $\forall\, n \in \mathbb{N}$, $\forall\, \varphi : [n] \to [n+1]$,
a map $\downarrow_\varphi : X([n+1]) \to X([n])$ "Restricting maps $[n] \xrightarrow{\varphi} [n+1] \longrightarrow X$"

| Composition, identity respected

"$X : \Delta^{op} \to \mathrm{Set}$ presheaf on Δ"

Eg. X





$X([0]) := \{v\}$

$X([1]) := \{e_0, e_1, e_2\}$

$X([2]) := \{f_0, f_1\}$

$X([n]) := \varnothing \quad , n \geq 3$

$\uparrow^0$

$$0 \to 1 \qquad \begin{matrix} e_0 \\ e_1 \\ e_2 \end{matrix} \mapsto v$$

$\uparrow^1$

$$0 \to 1 \qquad \begin{matrix} e_0 \\ e_1 \\ e_2 \end{matrix} \mapsto v$$

$\uparrow^0$

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} \qquad \begin{matrix} f_0 \\ f_1 \end{matrix} \mapsto \begin{matrix} e_1 \\ e_0 \end{matrix}$$

$\uparrow^1$

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} \qquad \begin{matrix} f_0 \\ f_1 \end{matrix} \mapsto \begin{matrix} e_2 \\ e_2 \end{matrix}$$

$\uparrow^2$

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} \qquad \begin{matrix} f_0 \\ f_1 \end{matrix} \mapsto \begin{matrix} e_0 \\ e_1 \end{matrix}$$

For $X, Y \in \Delta Set$,

$\varphi \in \Delta Set(X, Y) :=$

| $\forall n \in \mathbb{N}, \quad \varphi_n : X([n]) \to Y([n])$ "pushing forward maps $[n] \longrightarrow X \xrightarrow[\varphi]{} Y$"

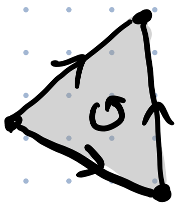
| $\forall n \in \mathbb{N}, \forall \alpha \in \Delta([n], [n+1]),$ " $[n] \xrightarrow{\alpha} [n+1] \longrightarrow X \xrightarrow{\varphi} Y$

$$\begin{tikzcd} X([n+1]) \arrow[r, "\varphi_{n+1}"] \arrow[d, "\alpha"'] & Y([n+1]) \arrow[d, "\alpha"] \\ X([n]) \arrow[r, "\varphi_n"'] & Y([n]) \end{tikzcd}$$

Pushforward by φ then restrict along α = restrict along α then pushforward by φ "

§ Yoneda [4:00]

Eg. Δ_2



$$\Delta_2([0]) := \{\uparrow^0, \uparrow^1, \uparrow^2\} = \Delta([0],[2])$$

$$\Delta_2([1]) := \{\uparrow^0, \uparrow^1, \uparrow^2\} = \Delta([1],[2])$$

$$\Delta_2([2]) := \{\mathbb{1}_{[2]}\} = \Delta([2],[2])$$

$$\Delta_2([n]) := \varnothing \;,\; n \geq 3 = \Delta([n],[2])$$

i.e. $\Delta_2 := \Delta(-,[2])$. "Yoneda embedding of [2]"

$\Delta_n := \Delta(-,[n])$ "[2] viewed as a space modelled on Δ"

Lem. Yoneda (applied to ΔSet).

"morphisms from [n] into X
as spaces modelled on Δ
$\leftrightarrow$ morphisms from [n] into X
in def of X."

$$\Delta Set(\Delta_n, X) \cong X([n])$$

Pf. "the data of $\varphi : \Delta_n \to X$
is eqv to where
the unique n-dim face goes"

$$\varphi \mapsto \varphi_{[n]}(\mathbb{1}_{[n]})$$

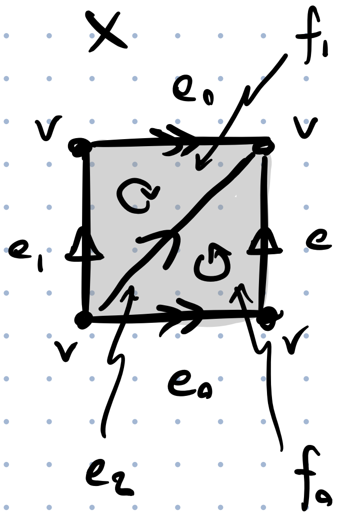
$\square$

Cor. $\Delta_- : \Delta \to \Delta Set$ embedding.

§ Homology [6:00]

$\boxed{\partial \circ \partial = 0}$

Eg. X

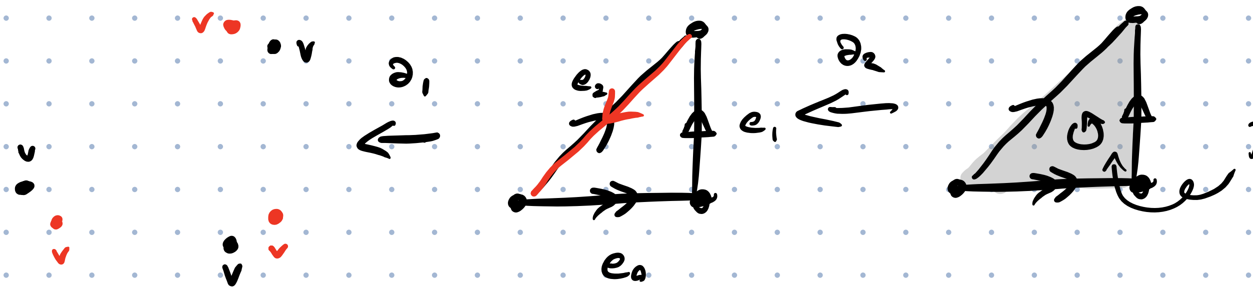


$$\emptyset \xleftarrow{\partial_0} \cdot\, v$$

$\partial_0 v =$ "nothing" $= 0$

v "=" $v \xleftarrow{\partial_1}$

$\partial_1 e_i = v - v = 0$



$\partial_2 f_0 = e_1 - e_2 + e_0$. Sim, $\partial_2 f_1 = e_0 - e_2 + e_1$.

$$\rightsquigarrow \quad 0 \xleftarrow{\partial_0} \mathbb{Z}X([0]) \xleftarrow{\partial_1} \mathbb{Z}X([1]) \xleftarrow{\partial_2} \mathbb{Z}X([2]) \xleftarrow{\partial_3} 0 \leftarrow \cdots$$

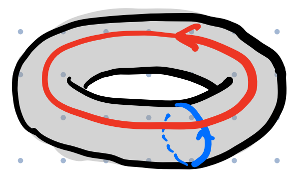
$$\mathbb{Z}X([0]) = \mathbb{Z}v, \quad \mathbb{Z}X([1]) = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \quad \mathbb{Z}X([2]) = \mathbb{Z}f_0 \oplus \mathbb{Z}f_2$$

"chain complex"

$H_k(X) :=$ "potential boundaries" / "boundaries" $:= \operatorname{Ker} \partial_k / \operatorname{Im} \partial_{k+1}$

$H_0(X) = \dfrac{\mathbb{Z}v}{0} \cong \mathbb{Z}$ "X path conn"

$$H_1(X) = \frac{\mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}(\cancel{e_0 - e_2} + e_1)}{\mathbb{Z}(e_1 \cancel{- e_2} + e_0)} \cong \mathbb{Z}^2$$



$\partial_2(f_0 - f_1) = 0 \Rightarrow H_2(X) = \dfrac{\mathbb{Z}(f_0 - f_1)}{0} \cong \mathbb{Z}$ "the donut is empty on the inside"