Tools for primal degenerate linear programs

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Dual-guided pivot rules for LP

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Vector Space Decomposition for LP

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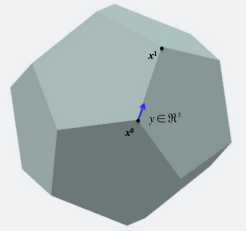
What are the links between PS, DCA, IPS, and MMCC ?

LP in standard form

$$\begin{array}{rrcll} \min & \mathbf{c}^{\mathsf{T}}\mathbf{x} & & & \\ & \mathbf{A}\mathbf{x} & = & \mathbf{b} & \quad [\pi] \\ & \mathbf{x} & \geq & \mathbf{0} & \end{array}$$



From $\mathbf{x}^0$ to $\mathbf{x}^1$

1. Find a *potential* improving direction $\mathbf{y}^0 \in \mathbb{R}^n$.
2. Determine step-size $\rho^0 \in \mathbb{R}$.
3. Compute $\mathbf{x}^1 := \mathbf{x}^0 + \rho^0 \mathbf{y}^0$.

$\bar{c}_j = 0, \forall j \in B$: Pricing for $j \in N$ (non-basic variables)

Selection of an entering variable into basis $\mathbf{A}_B$ relies on the minimum reduced cost of non-basic variables

$$\bar{c}_j = c_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j, \quad \forall j \in N. \qquad \boldsymbol{\pi}^\intercal = \mathbf{c}_B^\intercal \mathbf{A}_B^{-1}$$

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FIND THE MINIMUM REDUCED COST VALUE μ (OPTIMAL IF $\mu \geq 0$)

$$\begin{aligned} \max \quad & \mu \\ & \mu \leq c_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j, \quad \forall j \in N \qquad [y_j] \end{aligned}$$

*** μ is the smallest reduced cost (given $\boldsymbol{\pi}$). ***

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EQUIVALENT TO FINDING A CONVEX COMBINATION OF NON-BASIC VARIABLES

$$\begin{aligned} \mu = \quad \min \quad & \sum_{j \in N} \bar{c}_j y_j \\ & \sum_{j \in N} y_j = 1 \qquad [\mu] \\ & y_j \geq 0, \quad \forall j \in N \end{aligned}$$

$\mathbf{y}^0$

Entering variable only impacts the current basic variables, at most m variables.
All but one non-basic variables remain at zero.
$\mathbf{y}^0 \in \mathbb{R}^n$ makes changes on at most $m+1$ components.

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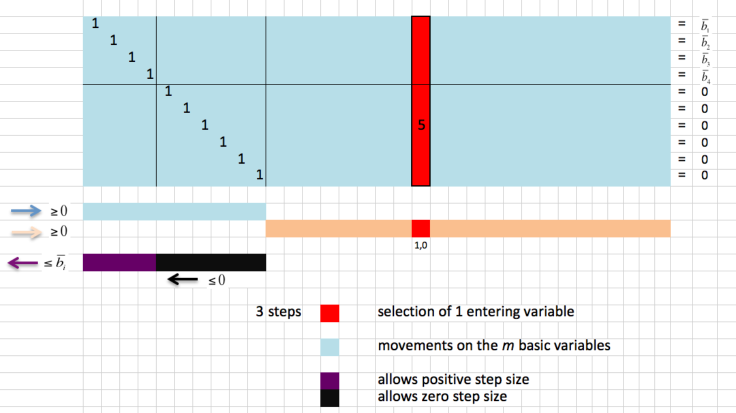
$\mathbf{y}^0$ is formed by a combination of variables :
the basic ones and the entering variable.

Step size computed such that $\mathbf{x}^1 := \mathbf{x}^0 + \rho^0 \, \mathbf{y}^0 \geq \mathbf{0}$.

Left multiplication by basis inverse $\mathbf{A}_B^{-1}$

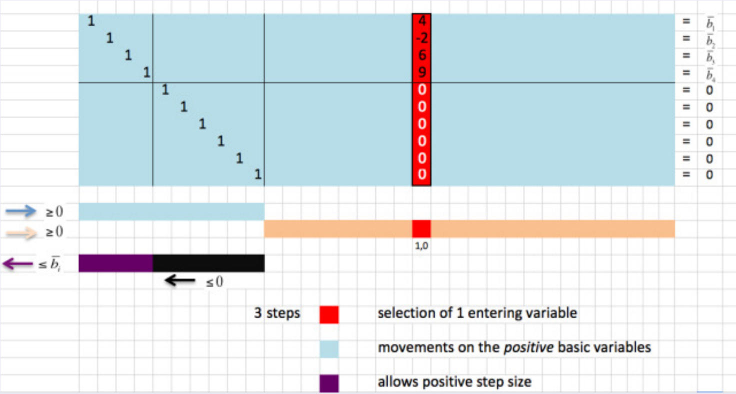
	$x1$	$x2$	$x3$	$\lambda4$	$\lambda5$	$\lambda6$	$x4$	$x5$	$x6$	$x7$	$x8$			ψ	θ_ω
c	2	3	1				10	17	-20	14	-4				-5
	1						2	2	1	-5	7	=	30	2	1
		1					4	3	-5	10	-10	=	25	3	-2
			1				-3	1	2	3	11	=	50	1	5
				1					6	5	-13	=	0	?	0
					1				3	4	-8	=	0	?	0
						1			3	-4	0	=	0	?	0
x^0	30	25	50	0	0	0	0	0	0	0	0		185 $= c^T x^0$		
									0.4	0.3	0.3				

Primal Simplex tableau



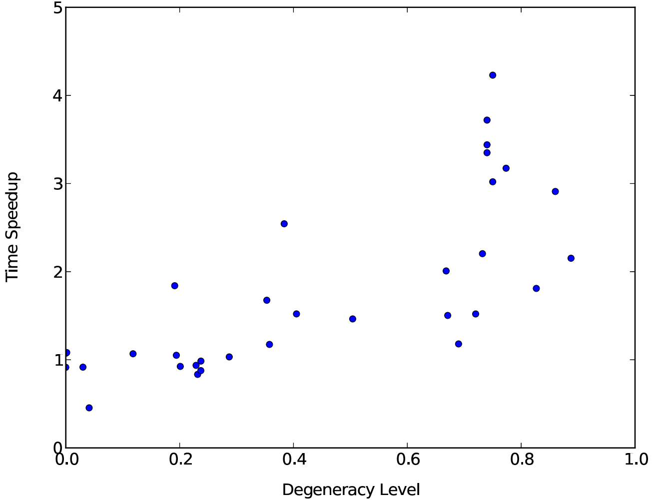
*** Changes on at most $m + 1$ components. ***

POSITIVE EDGE STRATEGY



*** Changes on at most $p + 1$ components. ***
Non-degenerate pivot.

Speedup of 2.31 for LPs with a degeneracy level above 25%



The *Positive Edge* is a pricing rule for the Primal Simplex :
it identifies variables $x_j, j \notin \mathrm{B}$ such that $\bar{\mathbf{a}}_{Zj} = \mathbf{0}$ ...
without computing $\bar{\mathbf{a}}_{Zj}$.

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If such a variable x_j has a negative reduced cost $\bar{c}_j < 0$,
it strictly improves the objective function when entered into the basis.

Let $\mathbf{v} \neq \mathbf{0}$ be a random vector of $|Z|$ non-zero components.

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Pre-compute once $\mathbf{w}^\mathsf{T} := \mathbf{v}^\mathsf{T}\ [\mathbf{A}_B^{-1}]_Z$, then compute $\mathbf{w}^\mathsf{T}\mathbf{a}_{Zj}$, on the **original data**.

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Determining if x_j satisfies the Positive Edge rule is done in $O(m)$, as for $\bar{c}_j = c_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j, j \notin B$, where $\boldsymbol{\pi} = \mathbf{c}_B^\top A_B^{-1}$.

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On large-scale instances ($m \approx 100\ 000$, $n \approx 450\ 000$),
cpu time to identify all variables satisfying the PE rule :

usual definition	Positive Edge
$O(m^2 n)$	$O(mn)$

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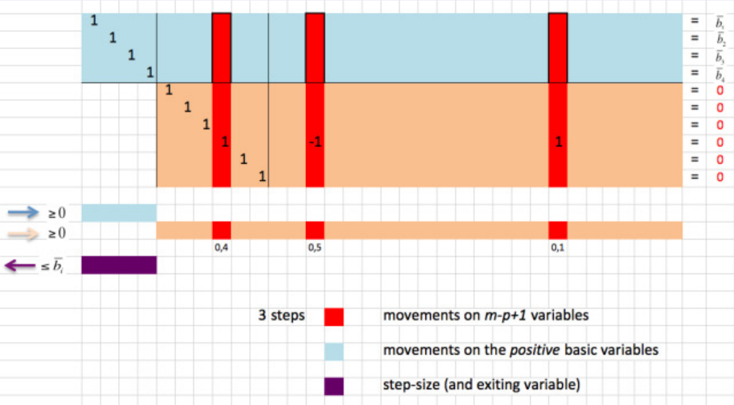
usual definition	Positive Edge
$O(m^2 n)$	$O(mn)$
2500 seconds	**0.5 seconds**

BLOCK DIAGONAL : KEEP DEGENERATE VARIABLES IN FIRST STEP (IPS)



*** Changes on at most $m + 1$ components. ***

Vectors and matrices are written in **bold face**.

$\mathbf{I}_\ell$: the $\ell \times \ell$ identity matrix.
$\mathbf{0}$ $(\mathbf{1})$: a vector/matrix with all zeros (ones) entries of appropriate dimensions.
$\mathbf{A}_{RC}$: sub-matrix of $\mathbf{A}$ containing the rows and columns indexed by R and C.

Standard notation : basis $\mathbf{A}_B$, inverse $\mathbf{A}_B^{-1}$, $\mathbf{c}_B^\intercal \mathbf{x}_B, \mathbf{A}_B \mathbf{x}_B$, $\boldsymbol{\pi} = \mathbf{c}_B \mathbf{A}_B^{-1}$ ...
$\mathbf{l}_F < \mathbf{x}_F < \mathbf{u}_F$, $\mathbf{x}_L = \mathbf{l}_L$, $\mathbf{x}_U = \mathbf{u}_U$

Useful decomposition of $\mathbf{x} \in \mathbb{R}^n$ in $\mathbf{Ax} = \mathbf{b}$, $\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_F \\ \mathbf{x}_L \\ \mathbf{x}_U \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{x}_F \\ \mathbf{x}_{B_L} \\ \mathbf{x}_{B_U} \\ \mathbf{x}_{N_L} \\ \mathbf{x}_{N_U} \end{bmatrix}$$

$$\text{For } \emptyset \subseteq S \subseteq B : \mathbf{x} = \begin{bmatrix} \mathbf{x}_S \\ \mathbf{x}_{\bar{S}} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{S_F} \\ \mathbf{x}_{S_L} \\ \mathbf{x}_{S_F} \\ \mathbf{x}_{\bar{S}_U} \\ \mathbf{x}_{\bar{S}_L} \\ \mathbf{x}_{\bar{S}_U} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \mathbf{x}_\emptyset \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_F \\ \mathbf{x}_L \\ \mathbf{x}_U \end{bmatrix}$$

Linear program *LP*

$$\begin{aligned} z^\star := \quad & \min \quad \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{st.} \quad & \quad \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{l} \le \mathbf{x} \le \mathbf{u} \end{aligned}$$

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\end{aligned}
$$

Generic Algorithm with single parameter set S, $\quad \emptyset \subseteq S \subseteq B$

1. Let $k = 0$ and assume a feasible basic solution $\mathbf{x}^k$ to LP.
2. For $\emptyset \subseteq S \subseteq B$, construct the residual problem $\mathrm{LP}_S(\mathbf{x}^k)$.
3. Fix a subset of the dual variables in row-set R : $\quad \boldsymbol{\pi}_R^\mathsf{T} = \mathbf{c}_S^\mathsf{T}\mathbf{A}_{RS}^{-1}$.
4. Determine the value of the smallest reduced cost μ_S^k.
   If $\mu_S^k \ge 0$, STOP. Current solution $\mathbf{x}^k$ is optimal for LP.
5. Retrieve direction $\mathbf{y}_S^k \in \mathbb{R}^n$ and compute its maximum step-size ρ_S^k.
6. Update $\quad \mathbf{x}^{k+1} := \mathbf{x}^k + \rho_S^k\,\mathbf{y}_S^k$;
   $z^{k+1} := z^k + \rho_S^k\,\mu_S^k$;
   $k := k + 1$.
7. Goto Step 2.

Step 2 Construct residual problem $LP(\mathbf{x}^k)$

Assume a feasible solution $\mathbf{x}^k = \begin{bmatrix} \mathbf{x}_F^k \\ \mathbf{x}_L^k \\ \mathbf{x}_U^k \end{bmatrix}$

$\mathbf{x} := \mathbf{x}^k + \mathbf{y}, \quad \mathbf{y} \in \mathbb{R}^n$

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$$\begin{aligned} \mathbf{x} &:= \mathbf{x}^k + \mathbf{y}, && \mathbf{y} \in \mathbb{R}^n \\ &= \mathbf{x}^k + (\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}), && \vec{\mathbf{y}}, \overleftarrow{\mathbf{y}} \geq \mathbf{0}, \quad \vec{\mathbf{y}}^\intercal \overleftarrow{\mathbf{y}} = 0, \quad \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k, \quad \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k \end{aligned}$$

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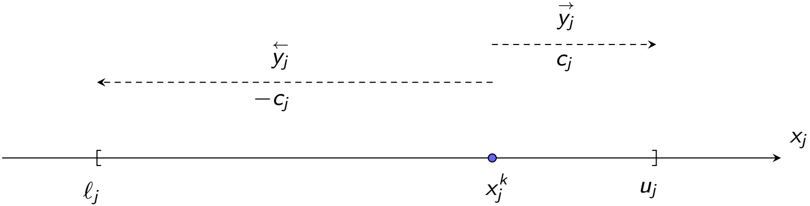
\begin{aligned}
\mathbf{x} &:= \mathbf{x}^k + \mathbf{y}, \quad \mathbf{y} \in \mathbb{R}^n \\
&= \mathbf{x}^k + (\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}), \qquad \vec{\mathbf{y}}, \overleftarrow{\mathbf{y}} \geq \mathbf{0}, \quad \vec{\mathbf{y}}^\top \overleftarrow{\mathbf{y}} = 0, \quad \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k, \quad \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k \\
&= \mathbf{x}^k + \left(\begin{bmatrix} \vec{\mathbf{y}}_F \\ \vec{\mathbf{y}}_L \\ \vec{\mathbf{y}}_U \end{bmatrix} - \begin{bmatrix} \overleftarrow{\mathbf{y}}_F \\ \overleftarrow{\mathbf{y}}_L \\ \overleftarrow{\mathbf{y}}_U \end{bmatrix} \right); \quad \vec{\mathbf{y}}, \overleftarrow{\mathbf{y}} \geq \mathbf{0}, \quad \vec{\mathbf{y}}^\top \overleftarrow{\mathbf{y}} = 0, \quad \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^k, \quad \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^k
\end{aligned}
$$



Residual problem $LP(\mathbf{x}^k)$; $\mathbf{x} := \mathbf{x}^k + (\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}})$ (change of variables)

$$
\begin{aligned}
z^\star := \min \quad & \mathbf{c}^\mathsf{T}\mathbf{x}^k + \mathbf{c}^\mathsf{T}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) \\
\text{st.} \quad & \mathbf{A}\mathbf{x}^k + \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{b} \\
& \mathbf{0} \le \vec{\mathbf{y}} \le \vec{\mathbf{r}}^k \\
& \mathbf{0} \le \overleftarrow{\mathbf{y}} \le \overleftarrow{\mathbf{r}}^k
\end{aligned}
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\begin{array}{rcccl}
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\text{st.} & \mathbf{A}\mathbf{x}^k & + & \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) & = \mathbf{b} \\
 & & & \mathbf{0} \le \vec{\mathbf{y}} \le \vec{\mathbf{r}}^k & \\
 & & & \mathbf{0} \le \overleftarrow{\mathbf{y}} \le \overleftarrow{\mathbf{r}}^k &
\end{array}
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Residual problem $LP(\mathbf{x}^k)$

$$
\begin{array}{rccl}
z^\star := \mathbf{c}^\mathsf{T}\mathbf{x}^k + \min & \mathbf{c}^\mathsf{T}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) & & \\
\text{st.} & \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) & = & \mathbf{0} \\
 & \mathbf{0} \le \vec{\mathbf{y}} \le \vec{\mathbf{r}}^k & & \\
 & \mathbf{0} \le \overleftarrow{\mathbf{y}} \le \overleftarrow{\mathbf{r}}^k & &
\end{array}
$$

Residual problem with $S = F, \quad \bar{S} = L \cup U$

$z^\star := \mathbf{c}^\intercal \mathbf{x}^k +$

$$
\begin{array}{lrcrcrcl}
\min & \mathbf{c}_F^\intercal(\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \mathbf{c}_L^\intercal(\vec{\mathbf{y}}_L) & - & \mathbf{c}_U^\intercal(\overleftarrow{\mathbf{y}}_U) & & \\
\text{st.} & \mathbf{A}_F(\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \mathbf{A}_L(\vec{\mathbf{y}}_L) & - & \mathbf{A}_U(\overleftarrow{\mathbf{y}}_U) & = & \mathbf{0} \\
 & \vec{\mathbf{y}}_F \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_F \geq \mathbf{0}, & & \vec{\mathbf{y}}_L \geq \mathbf{0}, & & \overleftarrow{\mathbf{y}}_U \geq \mathbf{0} & & \\
 & \vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F,\ \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F, & & \vec{\mathbf{y}}_L \leq \vec{\mathbf{r}}_L, & & \overleftarrow{\mathbf{y}}_U \leq \overleftarrow{\mathbf{r}}_U & &
\end{array}
$$

Residual problem – General case : $\emptyset \subseteq S \subseteq B$

$$
\begin{array}{llll}
\mathbf{c}^\mathsf{T}\mathbf{x}^k + & \min\ \mathbf{c}_S^\mathsf{T}(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \ + & \mathbf{c}_{\bar{S}}^\mathsf{T}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & \\
& \text{st.}\quad \mathbf{A}_S(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \ + & \mathbf{A}_{\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \\
& \vec{\mathbf{y}}_S \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_S \geq \mathbf{0} & \vec{\mathbf{y}}_{\bar{S}} \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0} & \\
& & \overleftarrow{\mathbf{y}}_{\bar{S}_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{\bar{S}_U} \leq \mathbf{0} & \\
& \vec{\mathbf{y}}_{S_F} \leq \vec{\mathbf{r}}_{S_F},\ \overleftarrow{\mathbf{y}}_{S_F} \leq \overleftarrow{\mathbf{r}}_{S_F} & \vec{\mathbf{y}}_{\bar{S}_F} \leq \vec{\mathbf{r}}_{\bar{S}_F},\ \overleftarrow{\mathbf{y}}_{\bar{S}_F} \leq \overleftarrow{\mathbf{r}}_{\bar{S}_F} & \\
& \overleftarrow{\mathbf{y}}_{S_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{S_U} \leq \mathbf{0} & \vec{\mathbf{y}}_{\bar{S}_L} \leq \vec{\mathbf{r}}_{\bar{S}_L}, \overleftarrow{\mathbf{y}}_{\bar{S}_U} \leq \overleftarrow{\mathbf{r}}_{\bar{S}_U} &
\end{array}
$$

$$S = S_F \cup S_L \cup S_U; \qquad \bar{S} = \bar{S}_F \cup \bar{S}_L \cup \bar{S}_U$$

Residual problem with $S = B$ (basic), $\bar{S} = N$ (non-basic)

$$
\begin{array}{llll}
\mathbf{c}^\mathsf{T}\mathbf{x}^k + & \min\ \mathbf{c}_B^\mathsf{T}(\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) + & \mathbf{c}_N^\mathsf{T}(\vec{\mathbf{y}}_N - \overleftarrow{\mathbf{y}}_N) & \\
& \text{st.}\quad \mathbf{A}_B(\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) + & \mathbf{A}_N(\vec{\mathbf{y}}_N - \overleftarrow{\mathbf{y}}_N) & = \mathbf{0} \\
& \vec{\mathbf{y}}_B \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_B \geq \mathbf{0} & \vec{\mathbf{y}}_N \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_N \geq \mathbf{0} & \\
& & \overleftarrow{\mathbf{y}}_{N_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{N_U} \leq \mathbf{0} & \\
& \vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F,\ \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F & & \\
& \overleftarrow{\mathbf{y}}_{B_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{B_U} \leq \mathbf{0} & \vec{\mathbf{y}}_{N_L} \leq \vec{\mathbf{r}}_{N_L}, \overleftarrow{\mathbf{y}}_{N_U} \leq \overleftarrow{\mathbf{r}}_{N_U} &
\end{array}
$$

$$B = F \cup B_L \cup B_U; \qquad N = N_L \cup N_U$$

For $\emptyset \subseteq S \subseteq B$, find $\mathbf{y}^k = \begin{bmatrix} \mathbf{y}_S^k \\ \mathbf{y}_{\bar{S}}^k \end{bmatrix} = \begin{bmatrix} (\vec{\mathbf{y}}_S^k - \overleftarrow{\mathbf{y}}_S^k) \\ (\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \end{bmatrix}$ of min reduced cost μ^k.

For basic columns $\mathbf{A}_S$,
select a $s \times s$ square matrix $\mathbf{A}_{RS}$,
a set of s independent rows.

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}_{RS} & \mathbf{0} \\ \mathbf{A}_{ZS} & \mathbf{I}_{m-s} \end{bmatrix} \qquad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{RS}^{-1} & \mathbf{0} \\ -\mathbf{A}_{ZS}\mathbf{A}_{RS}^{-1} & \mathbf{I}_{m-s} \end{bmatrix}$$

$$\emptyset \subseteq S \subseteq B$$

Perform $\mathbf{T}^{-1}\mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{0}$. General case $\boldsymbol{\pi}^\intercal = \psi^\intercal \mathbf{T}^{-1}$

$$\begin{array}{llll}
\mathbf{c}^\intercal \mathbf{x}^k + \min & \mathbf{c}_S^\intercal(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \; + & \mathbf{c}_{\bar{S}}^\intercal(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & \\
\text{st.} & (\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \; + & \bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \quad [\psi_R = \mathbf{c}_S] \\
 & \vec{\mathbf{y}}_S \geq \mathbf{0}, \; \overleftarrow{\mathbf{y}}_S \geq \mathbf{0} & & \\
 & & \bar{\mathbf{A}}_{Z\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \quad [\psi_Z \; \textit{unknown}] \\
 & & \vec{\mathbf{y}}_{\bar{S}} \geq \mathbf{0}, \; \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0} & \\
 & \vec{\mathbf{y}}_S \leq \vec{\mathbf{r}}_S, \; \overleftarrow{\mathbf{y}}_S \leq \overleftarrow{\mathbf{r}}_S & \vec{\mathbf{y}}_{\bar{S}} \leq \vec{\mathbf{r}}_{\bar{S}}, \; \overleftarrow{\mathbf{y}}_{\bar{S}} \leq \overleftarrow{\mathbf{r}}_{\bar{S}} &
\end{array}$$

Observe $\bar{\mathbf{c}}_S = \mathbf{0}$ (basic variables).

PERFORM $\mathbf{T}^{-1}\mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{0}$. GENERAL CASE : DETAILS $\qquad \boldsymbol{\pi}^{\mathsf{T}} = \boldsymbol{\psi}^{\mathsf{T}}\mathbf{T}^{-1}$

$$
\begin{array}{rlll}
\mathbf{c}^{\mathsf{T}}\mathbf{x}^{k} + \min & \mathbf{c}_S^{\mathsf{T}}(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \ + & \mathbf{c}_{\bar{S}}^{\mathsf{T}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & \\
\text{st.} & (\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \ + & \bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \quad [\psi_R = \mathbf{c}_S] \\
& \vec{\mathbf{y}}_S \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_S \geq \mathbf{0} & & \\
& & \bar{\mathbf{A}}_{Z\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \quad [\psi_Z \ \mathit{unknown}] \\
& & \vec{\mathbf{y}}_{\bar{S}} \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0} & \\
& & \overleftarrow{\mathbf{y}}_{\bar{S}_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{\bar{S}_U} \leq \mathbf{0} & \\
& \vec{\mathbf{y}}_{S_F} \leq \vec{\mathbf{r}}_{S_F},\ \overleftarrow{\mathbf{y}}_{S_F} \leq \overleftarrow{\mathbf{r}}_{S_F} & \vec{\mathbf{y}}_{\bar{S}_F} \leq \vec{\mathbf{r}}_{\bar{S}_F},\ \overleftarrow{\mathbf{y}}_{\bar{S}_F} \leq \overleftarrow{\mathbf{r}}_{\bar{S}_F} & \\
& \overleftarrow{\mathbf{y}}_{S_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{S_U} \leq \mathbf{0} & \vec{\mathbf{y}}_{\bar{S}_L} \leq \vec{\mathbf{r}}_{\bar{S}_L}, \overleftarrow{\mathbf{y}}_{\bar{S}_U} \leq \overleftarrow{\mathbf{r}}_{\bar{S}_U} &
\end{array}
$$

$$S = S_F \cup S_L \cup S_U; \qquad \bar{S} = \bar{S}_F \cup \bar{S}_L \cup \bar{S}_U$$

Step 4 Find minimum reduced cost μ_S^k

Pricing of the variables

$S \subseteq B$, hence $\bar{\mathbf{c}}_S = \mathbf{0}$.
Therefore pricing of $\vec{\mathbf{y}}_{\bar{S}}$ and $\overleftarrow{\mathbf{y}}_{\bar{S}}$ needed to get partial direction $(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k)$ followed by impact on $(\vec{\mathbf{y}}_S^k - \overleftarrow{\mathbf{y}}_S^k)$ to complete direction $\mathbf{y}^k = \begin{bmatrix} \mathbf{y}_S^k \\ \mathbf{y}_{\bar{S}}^k \end{bmatrix}$.

Primal/dual formulations of the pricing $\psi_R^\mathsf{T} = \mathbf{c}_S^\mathsf{T} \mathbf{A}_{RS}^{-1}$

$$
\begin{array}{llrll}
\max \ \mu & \text{st.} \ \mu \mathbf{1}^\mathsf{T} & \leq & \mathbf{c}_{\bar{S}}^\mathsf{T} - \psi_R^\mathsf{T} \mathbf{A}_{R\bar{S}} - \psi_Z^\mathsf{T} \bar{\mathbf{A}}_{Z\bar{S}} & [\vec{\mathbf{y}}_{\bar{S}}] \\
& \mu \mathbf{1}^\mathsf{T} & \leq & -(\mathbf{c}_{\bar{S}}^\mathsf{T} - \psi_R^\mathsf{T} \mathbf{A}_{R\bar{S}} - \psi_Z^\mathsf{T} \bar{\mathbf{A}}_{Z\bar{S}}) & [\overleftarrow{\mathbf{y}}_{\bar{S}}]
\end{array}
$$

A convex combination of the variables $\vec{\mathbf{y}}_{\bar{S}}$ and $\overleftarrow{\mathbf{y}}_{\bar{S}}$

$$
\begin{array}{lrll}
\equiv & \mu = \min \ (\mathbf{c}_{\bar{S}}^\mathsf{T} - \psi_R^\mathsf{T} \mathbf{A}_{R\bar{S}})(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & & \\
\text{st.} & \mathbf{1}^\mathsf{T} \vec{\mathbf{y}}_{\bar{S}} + \mathbf{1}^\mathsf{T} \overleftarrow{\mathbf{y}}_{\bar{S}} & = 1 & [\mu] \\
& \mathbf{A}_{Z\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} & [\psi_Z] \\
& \vec{\mathbf{y}}_{\bar{S}}, \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0}, \overleftarrow{\mathbf{y}}_{\bar{S}_L} \leq \mathbf{0}, \ \vec{\mathbf{y}}_{\bar{S}_U} \leq \mathbf{0} & &
\end{array}
$$

*** Optimal solution : μ_S^k, $\vec{\mathbf{y}}_{\bar{S}}^k$ and $\overleftarrow{\mathbf{y}}_{\bar{S}}^k$. ***

... COMPLETE DIRECTION $\mathbf{y}_S^k$

Given $\mu^k < 0$ and $(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k)$, find impacts on other variables $(\vec{\mathbf{y}}_S^k - \overleftarrow{\mathbf{y}}_S^k)$.

$$(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) \quad + \quad \bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \quad = \quad \mathbf{0}$$

$$\vec{\mathbf{y}}_S \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_S \geq \mathbf{0} \qquad \vec{y}_j \overleftarrow{y}_j = 0, \quad \forall j \in S_F$$

DIRECTION $\mathbf{y}_S^k$

$$\mathbf{y}_S^k = \begin{bmatrix} (\vec{\mathbf{y}}_S^k - \overleftarrow{\mathbf{y}}_S^k) \\ (\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \end{bmatrix} = \begin{bmatrix} -\bar{\mathbf{A}}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \\ (\vec{\mathbf{y}}_{\bar{S}}^k - \overleftarrow{\mathbf{y}}_{\bar{S}}^k) \end{bmatrix} = \begin{bmatrix} \textit{Impact level} \\ \textit{Pricing level} \end{bmatrix}$$

Positive part; *Negative part*.

STEP 5 COMPUTE MAXIMUM STEP-SIZE ρ_S^k

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_{S_F}^k \\ \vec{\mathbf{y}}_{S_L}^k \\ \vec{\mathbf{y}}_{S_U} \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_{S_F}^k \\ \vec{\mathbf{r}}_{S_L}^k \\ \mathbf{0} \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_{S_F}^k \\ \overleftarrow{\mathbf{y}}_{S_L}^k \\ \overleftarrow{\mathbf{y}}_{S_U}^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_{S_F}^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{r}}_{S_U}^k \end{bmatrix}$$

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_{\bar{S}_F}^k \\ \vec{\mathbf{y}}_{\bar{S}_L}^k \\ \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_{\bar{S}_F}^k \\ \vec{\mathbf{r}}_{\bar{S}_L}^k \\ \mathbf{0} \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_{\bar{S}_F}^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{y}}_{\bar{S}_U}^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_{\bar{S}_F}^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{r}}_{\bar{S}_U}^k \end{bmatrix}$$

10 out of 12 types of residual upper bonds to verify. $\emptyset \subseteq S \subseteq B$

STEP 5 COMPUTE MAXIMUM STEP-SIZE ρ_S^k

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_F^k \\ \vec{\mathbf{y}}_{B_L}^k \\ \vec{\mathbf{y}}_{B_U} \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_F^k \\ \vec{\mathbf{r}}_{B_L}^k \\ \mathbf{0} \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_F^k \\ \overleftarrow{\mathbf{y}}_{B_L}^k \\ \overleftarrow{\mathbf{y}}_{B_U}^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_F^k \\ \mathbf{0} \\ \overleftarrow{\mathbf{r}}_{B_U}^k \end{bmatrix}$$

$$\rho \left[\ \vec{\mathbf{y}}_{N_L}^k \ \right] \leq \left[\ \vec{\mathbf{r}}_{N_L}^k \ \right]; \quad \rho \left[\ \overleftarrow{\mathbf{y}}_{N_U}^k \ \right] \leq \left[\ \overleftarrow{\mathbf{r}}_{N_U}^k \ \right]$$

8 types of residual upper bonds to verify for Primal Simplex. $S = B$

Step 5 Compute maximum step-size ρ_S^k

$$\rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_P^k \\ \overleftarrow{\mathbf{y}}_Z^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_P^k \\ \mathbf{0} \end{bmatrix}$$

Only 2 types of residual upper bonds to verify for PS in standard form.

$S = B = P \cup Z$ (positive and zero variables)

Step 5 Compute maximum step-size ρ_S^k

$$\rho \begin{bmatrix} \vec{\mathbf{y}}_F^k \\ \vec{\mathbf{y}}_L^k \end{bmatrix} \leq \begin{bmatrix} \vec{\mathbf{r}}_F^k \\ \vec{\mathbf{r}}_L^k \end{bmatrix}; \quad \rho \begin{bmatrix} \overleftarrow{\mathbf{y}}_F^k \\ \overleftarrow{\mathbf{y}}_U^k \end{bmatrix} \leq \begin{bmatrix} \overleftarrow{\mathbf{r}}_F^k \\ \overleftarrow{\mathbf{r}}_U^k \end{bmatrix}$$

4 types of strictly positive residual upper bonds to verify in MMCC. $S = \emptyset$

STEP 5 COMPUTE MAXIMUM STEP-SIZE ρ_S^k

$$\rho \left[\ \vec{\mathbf{y}}_F^k \ \right] \leq \left[\ \vec{\mathbf{r}}_F^k \ \right]; \quad \rho \left[\ \overleftarrow{\mathbf{y}}_F^k \ \right] \leq \left[\ \overleftarrow{\mathbf{r}}_F^k \ \right]$$

$$\rho \left[\ \vec{\mathbf{y}}_L^k \ \right] \leq \left[\ \vec{\mathbf{r}}_L^k \ \right]; \quad \rho \left[\ \overleftarrow{\mathbf{y}}_U^k \ \right] \leq \left[\ \overleftarrow{\mathbf{r}}_U^k \ \right]$$

4 types of strictly positive residual upper bonds to verify in IPS. $S = F$

$$\mathbf{T} = [\mathbf{A}_B \;\; \emptyset], \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_B^{-1} \\ \emptyset \end{bmatrix}$$

$$B = F \cup B_L \cup B_U; \quad N = N_L \cup N_U$$

Special case #1 : $S = B$

$$\mathbf{T} = [\mathbf{A}_B \;\; \emptyset], \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_B^{-1} \\ \emptyset \end{bmatrix}$$

$$B = F \cup B_L \cup B_U; \quad N = N_L \cup N_U$$

Primal simplex method (Dantzig 1945)

$$\begin{array}{llcll}
\mathbf{c}^\mathsf{T}\mathbf{x}^k + \min & \mathbf{c}_B^\mathsf{T}(\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) & + & \mathbf{c}_N^\mathsf{T}(\vec{\mathbf{y}}_N - \overleftarrow{\mathbf{y}}_N) & \\
\text{st.} & (\vec{\mathbf{y}}_B - \overleftarrow{\mathbf{y}}_B) & + & \bar{\mathbf{A}}_N(\vec{\mathbf{y}}_N - \overleftarrow{\mathbf{y}}_N) & = \mathbf{0} \quad [\psi^\mathsf{T} = \mathbf{c}_B] \\
& \vec{\mathbf{y}}_B \geq \mathbf{0}, \; \overleftarrow{\mathbf{y}}_B \geq \mathbf{0} & & & \\
& & & \vec{\mathbf{y}}_N \geq \mathbf{0}, \; \overleftarrow{\mathbf{y}}_N \geq \mathbf{0} & \\
& & & \overleftarrow{\mathbf{y}}_{N_L} = \mathbf{0}, \; \vec{\mathbf{y}}_{N_U} = \mathbf{0} & \\
& \vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F, \; \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F & & & \\
& \overleftarrow{\mathbf{y}}_{B_L} \leq \mathbf{0}, \; \vec{\mathbf{y}}_{B_U} \leq \mathbf{0} & & \vec{\mathbf{y}}_{N_L} \leq \vec{\mathbf{r}}_{N_L}, \overleftarrow{\mathbf{y}}_{N_U} \leq \overleftarrow{\mathbf{r}}_{N_U} & \quad *\rho_B \geq 0*
\end{array}$$

Properties of PS

No equality constraints in the pricing problem.
Pricing contains • convex combination of the non-basic variables
• non-negativity restrictions (a cone).
Due to the step-size $*\rho_B \geq 0*$, possible degenerate pivots.
Oscillation of μ_B ; it may even not converge towards 0.

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}_{RF} & \mathbf{0} \\ \mathbf{A}_{ZF} & \mathbf{I}_{m-r} \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{RF}^{-1} & \mathbf{0} \\ -\mathbf{A}_{ZS}\mathbf{A}_{RF}^{-1} & \mathbf{I}_{m-r} \end{bmatrix}.$$

Special case #2 : $S = F \Rightarrow \bar{S} = L \cup U$

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}_{RF} & \mathbf{0} \\ \mathbf{A}_{ZF} & \mathbf{I}_{m-r} \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{RF}^{-1} & \mathbf{0} \\ -\mathbf{A}_{ZS}\mathbf{A}_{RF}^{-1} & \mathbf{I}_{m-r} \end{bmatrix}.$$

Improved Primal Simplex method (Elhallaoui et al. 2011)

$$\begin{array}{llcllll}
\mathbf{c}^\intercal \mathbf{x}^k + \min & \mathbf{c}_F^\intercal(\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \mathbf{c}_L^\intercal \vec{\mathbf{y}}_L - \mathbf{c}_U^\intercal \overleftarrow{\mathbf{y}}_U & & & \\
\text{st.} & (\vec{\mathbf{y}}_F - \overleftarrow{\mathbf{y}}_F) & + & \bar{\mathbf{A}}_{RL}\vec{\mathbf{y}}_L - \bar{\mathbf{A}}_{RU}\overleftarrow{\mathbf{y}}_U & = & \mathbf{0} & [\psi_R = \mathbf{c}_F] \\
& \vec{\mathbf{y}}_F \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_F \geq \mathbf{0} & & & & & \\
& & & \bar{\mathbf{A}}_{ZL}\vec{\mathbf{y}}_L - \bar{\mathbf{A}}_{ZU}\overleftarrow{\mathbf{y}}_U & = & \mathbf{0} & [\psi_Z] \\
& & & \vec{\mathbf{y}}_L \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_U \geq \mathbf{0} & & & \\
& \vec{\mathbf{y}}_F \leq \vec{\mathbf{r}}_F,\ \overleftarrow{\mathbf{y}}_F \leq \overleftarrow{\mathbf{r}}_F, & & \vec{\mathbf{y}}_L \leq \vec{\mathbf{r}}_L, \overleftarrow{\mathbf{y}}_U \leq \overleftarrow{\mathbf{r}}_U & & & *\rho > 0*
\end{array}$$

Properties of IPS

f equality constraints in the master problem, $m - f$ in the pricing problem.

Non-degenerate pivots only ($\rho_F > 0$).

$z^0 > z^1 > z^2 > \cdots = z^\star$ cost **strictly decreasing** at each iteration ($\rho_F > 0$).

Oscillation of μ_F but converging towards 0.

Select $S = \emptyset$. $\mathbf{T} = \begin{bmatrix} \emptyset & \mathbf{I}_m \end{bmatrix}, \mathbf{T}^{-1} = \begin{bmatrix} \emptyset \\ \mathbf{I}_m \end{bmatrix}$

Special case #3 : $S = \emptyset$

Select $S = \emptyset$. $\mathbf{T} = \begin{bmatrix} \emptyset & \mathbf{I}_m \end{bmatrix}, \mathbf{T}^{-1} = \begin{bmatrix} \emptyset \\ \mathbf{I}_m \end{bmatrix}$

Minimum mean cycle-canceling algorithm adapted for LP

$$
\begin{array}{llrll}
\mathbf{c}^\intercal \mathbf{x}^k + & \min & \mathbf{c}^\intercal(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) & & \\
& \text{st.} & \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) & = \mathbf{0} & \textit{Directions} \\
& & \vec{\mathbf{y}}, \overleftarrow{\mathbf{y}} & \geq \mathbf{0} & \textit{in the cone} \\
& & \overleftarrow{\mathbf{y}}_L, \vec{\mathbf{y}}_U & \leq \mathbf{0} & \textit{at vertex } \mathbf{x}^k \\
& & \vec{\mathbf{y}}_F, \overleftarrow{\mathbf{y}}_F, \vec{\mathbf{y}}_L, \overleftarrow{\mathbf{y}}_U & \leq \vec{\mathbf{r}}^k_F, \overleftarrow{\mathbf{r}}^k_F, \vec{\mathbf{r}}^k_L, \overleftarrow{\mathbf{r}}^k_U & *\textit{Step size } \rho_\emptyset > 0*
\end{array}
$$

Properties of MMCC

All equality constraints in the pricing problem.

Upper bounds in the master problem.

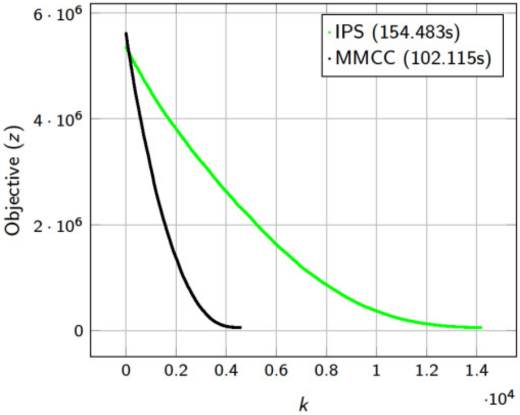
$z^0 > z^1 > z^2 > \cdots = z^\star$ cost **strictly decreasing** at each iteration ($\rho_\emptyset > 0$).

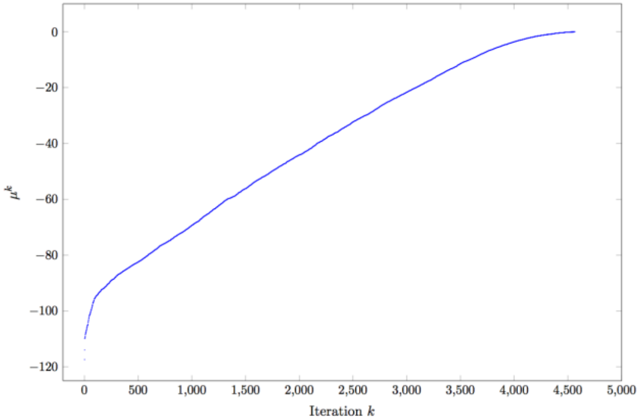
$\mu^0 \leq \mu^1 \leq \mu^2 \leq \cdots = 0$ smallest reduced cost **non decreasing**.

MMCC is strongly polynomial for network flow problems in $O(mn)$ phases.

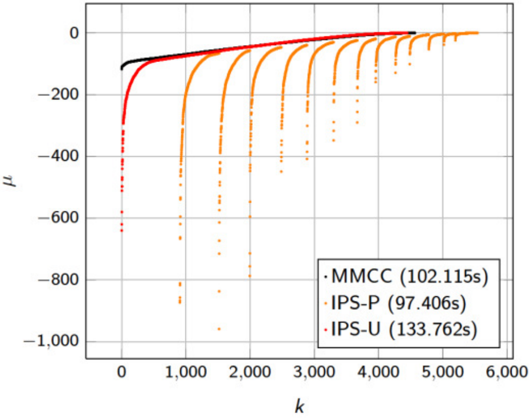
Goldberg and Tarjan (1989), Radzick and Goldberg (1994)

COMPUTATIONAL RESULTS

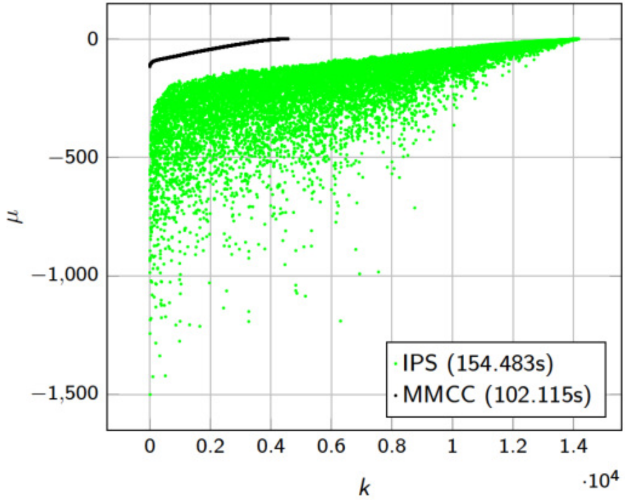



0
−20
−40
−60
−80
−100
−120
μ^k
0
500
1,000
1,500
2,000
2,500
3,000
3,500
4,000
4,500
5,000
Iteration k

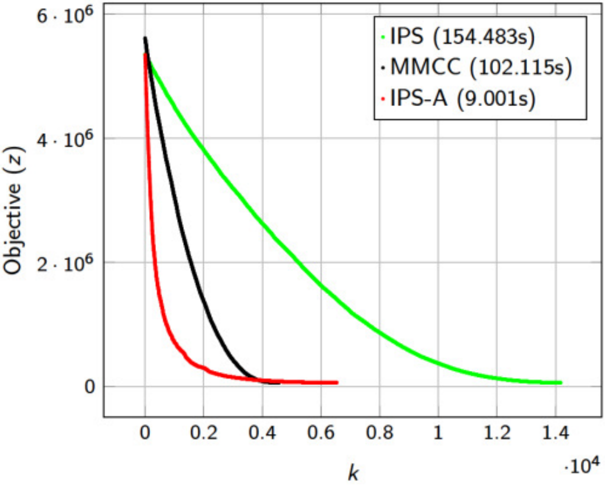
COMPUTATIONAL RESULTS



COMPUTATIONAL RESULTS



Computational results

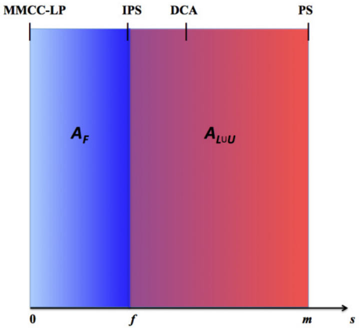


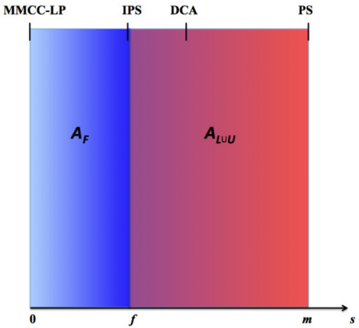
Linear program LP

$$
\begin{aligned}
z^\star := \quad & \min \quad \mathbf{c}^\mathsf{T}\mathbf{x} \\
\text{st.} \quad & \qquad \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{l} \le \mathbf{x} \le \mathbf{u}
\end{aligned}
$$

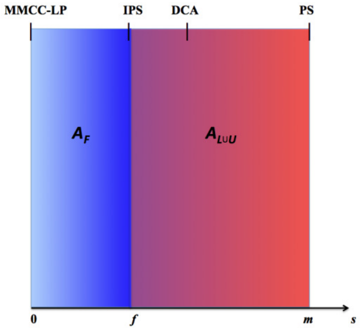
Generic Algorithm with single parameter set S, $\emptyset \subseteq S \subseteq B$

1. Let $k = 0$ and assume a feasible basic solution $\mathbf{x}^k$ to LP.
2. For $\emptyset \subseteq S \subseteq B$, construct the residual problem $\mathrm{LP}_S(\mathbf{x}^k)$.
3. Fix a subset of the dual variables in row-set R : $\boldsymbol{\pi}_R^\mathsf{T} = \mathbf{c}_S^\mathsf{T}\mathbf{A}_{RS}^{-1}$.
4. Determine the value of the smallest reduced cost μ_S^k.
   If $\mu_S^k \ge 0$, STOP. Current solution $\mathbf{x}^k$ is optimal for LP.
5. Retrieve direction $\mathbf{y}_S^k \in \mathbb{R}^n$ and compute its maximum step-size ρ_S^k.
6. Update $\mathbf{x}^{k+1} := \mathbf{x}^k + \rho_S^k\,\mathbf{y}_S^k$;
   $z^{k+1} := z^k + \rho_S^k\,\mu_S^k$;
   $k := k + 1$.
7. Goto Step 2.

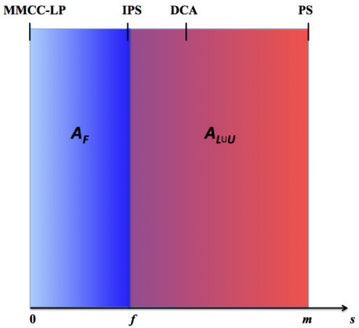

MMCC-LP
IPS
DCA
PS
A_F
$A_{L\cup U}$
0
f
m
s



- $S \cap \{L \cup U\} \neq \emptyset$: It may come up with degenerate pivots and not converge.
  - Primal Simplex method (PS). $S = B, * \rho_B \geq 0 *$



- $S \cap \{L \cup U\} \neq \emptyset$: It may come up with degenerate pivots and not converge.
  - Primal Simplex method (PS). $S = B$, $*\ \rho_B \geq 0\ *$
- $\emptyset \subseteq S \subseteq F$: It ensures a non-degenerate pivot at every iteration.
  - Improved Primal Simplex algorithm (IPS). $S = F$, $*\ \rho_F > 0\ *$



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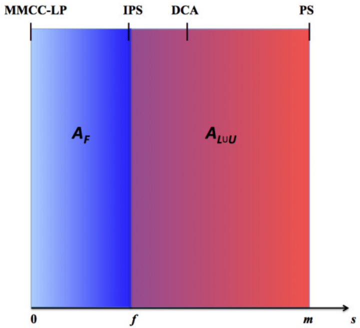

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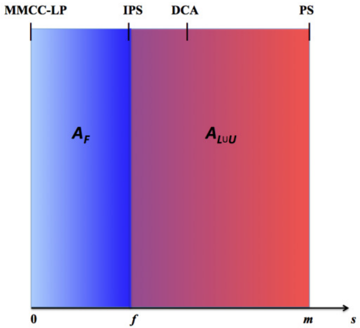
    Strongly polynomial for network flow problems.



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    Strongly polynomial for network flow problems.

* $S \subset F$: Optimal direction $\mathbf{y}_S^k$ can be an interior ray.

*** Imagine the same transformation is kept for a while...

- $\mathbf{T} := [\,\mathbf{\Lambda},\ \mathbf{\Lambda}^{\perp}]$
- $\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_R \\ \mathbf{\Lambda}_Z \end{bmatrix}$, where $\mathbf{\Lambda}_R$ is a set of s independent rows.
- $\mathbf{T} = \begin{bmatrix} \mathbf{\Lambda}_R & \mathbf{0} \\ \mathbf{\Lambda}_Z & \mathbf{I}_{m-s} \end{bmatrix}$ $\qquad$ $\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{\Lambda}_R^{-1} & \mathbf{0} \\ \mathbf{\Lambda}_Z\mathbf{\Lambda}_R^{-1} & \mathbf{I}_{m-s} \end{bmatrix}$
- $\mathbf{T}^{-1}$ splits row-space $\mathbb{R}^m$ of $LP(\mathbf{x}^k)$ into two vector subspaces $\mathbf{V}$ and $\mathbf{V}^{\perp}$.
- Vector subspace basis $\mathbf{\Lambda}$ spans $\mathbf{V}$ of dimension $0 \leq s \leq m$.
- Vector $\mathbf{a} \in \mathbf{V}$ if and only if $\bar{\mathbf{a}}_Z = \mathbf{0}$, where $\bar{\mathbf{a}} = \mathbf{T}^{-1}\mathbf{a} = \begin{bmatrix} \bar{\mathbf{a}}_R \\ \mathbf{0} \end{bmatrix}$.
- $\emptyset \subseteq S \subseteq B$: index subset of basic columns spanned by $\mathbf{\Lambda}$.
- Algorithmic properties derived according to subset S.

At iteration k, a dynamic Dantzig-Wolfe decomposition

$$
\begin{array}{llllll}
\mathbf{c}^{\intercal}\mathbf{x}^{k} + & \min & \mathbf{c}_S^{\intercal}(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) & + & \mathbf{c}_{\bar{S}}^{\intercal}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & \\
& \text{st.} & \mathbf{A}_{RS}(\vec{\mathbf{y}}_S - \overleftarrow{\mathbf{y}}_S) & + & \mathbf{A}_{R\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \quad [\boldsymbol{\pi}_R] \\
& & & & \bar{\mathbf{A}}_{Z\bar{S}}(\vec{\mathbf{y}}_{\bar{S}} - \overleftarrow{\mathbf{y}}_{\bar{S}}) & = \mathbf{0} \\
& & \vec{\mathbf{y}}_S \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_S \geq \mathbf{0} & & \vec{\mathbf{y}}_{\bar{S}} \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0} & \\
& & & & \overleftarrow{\mathbf{y}}_{\bar{S}_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{\bar{S}_U} \leq \mathbf{0} & \\
& & \vec{\mathbf{y}}_{S_F} \leq \vec{\mathbf{r}}_{S_F},\ \overleftarrow{\mathbf{y}}_{S_F} \leq \overleftarrow{\mathbf{r}}_{S_F} & & \vec{\mathbf{y}}_{\bar{S}_F} \leq \vec{\mathbf{r}}_{\bar{S}_F},\ \overleftarrow{\mathbf{y}}_{\bar{S}_F} \leq \overleftarrow{\mathbf{r}}_{\bar{S}_F} & \\
& & \overleftarrow{\mathbf{y}}_{S_L} \leq \mathbf{0},\ \vec{\mathbf{y}}_{S_U} \leq \mathbf{0} & & \vec{\mathbf{y}}_{\bar{S}_L} \leq \vec{\mathbf{r}}_{\bar{S}_L}, \overleftarrow{\mathbf{y}}_{\bar{S}_U} \leq \overleftarrow{\mathbf{r}}_{\bar{S}_U} &
\end{array}
$$

- **Pricing domain** keeps the $m - s$ equality rows corresponding to $\mathbf{V}^{\perp}$ and $\vec{\mathbf{y}}_{\bar{S}} \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_{\bar{S}} \geq \mathbf{0},\ \overleftarrow{\mathbf{y}}_{\bar{S}_L} = \mathbf{0},\ \vec{\mathbf{y}}_{\bar{S}_U} = \mathbf{0}$.
- **Master domain** keeps the s equality rows corresponding to $\mathbf{V}$, and all remaining lower and upper bounds of variables.

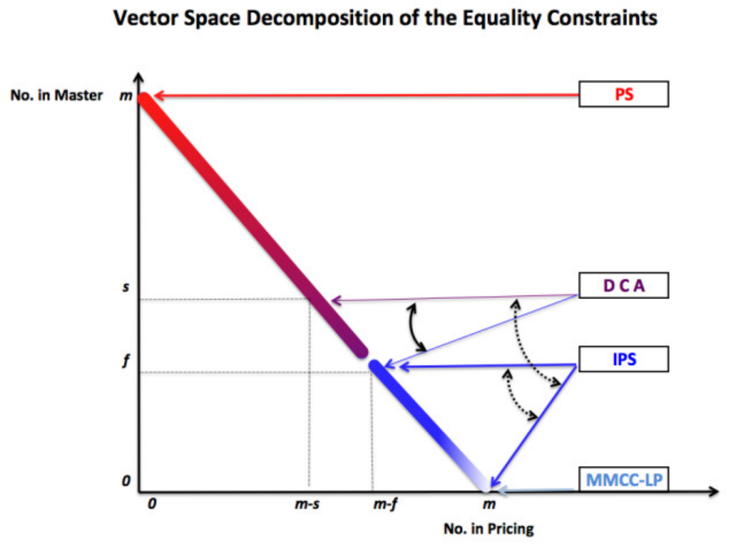
Dynamic Constraint Aggregation for Set Partitioning (Elhallaoui et al. 2005)

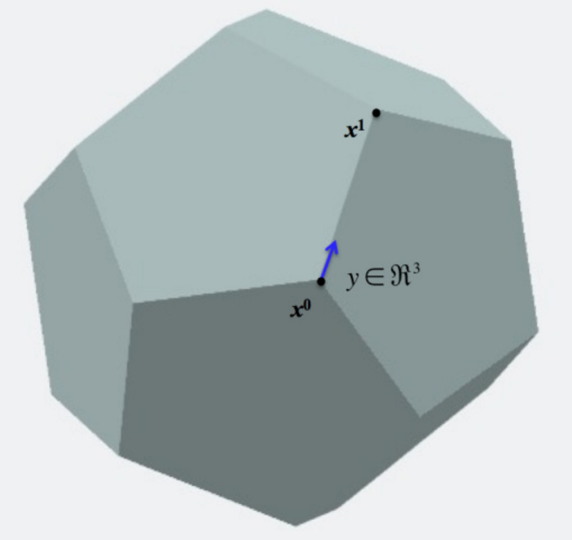
The partition of the row-set is derived from the f groups of identical rows of A_F.

$$\overset{\mathbf{A}_F}{\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 1 & & 1 & \\ 1 & & & 1 \\ 1 & 1 & & \\ & 1 & 1 & \\ 1 & & & 1 \end{pmatrix}} \mapsto \overset{\boldsymbol{\Lambda}}{\left(\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ \hline 1 & & & & \\ & 1 & & & \\ & & & & 1 \end{array}\right)}.$$

Properties

$F \subseteq S \subseteq B$ for fractional solutions but $S = F$ (or $S = \emptyset$) for binary solutions.
Degenerate pivots may occur, $\rho \geq 0$, if $r > f$.


Vector Space Decomposition of the Equality Constraints
No. in Master
m
s
f
0
0
m-s
m-f
m
No. in Pricing
PS
D C A
IPS
MMCC-LP


x^1
$y \in \Re^3$
x^0

MMCC:

non-extreme ray a-c-d