Graphical Models Coursework 4

Razvan Valentin Marinescu
Student Number: 14060166
`razvan.marinescu.14@ucl.ac.uk`

Konstantinos Georgiadis
Student Number: 14110861
`konstantinos.georgiadis.14@ucl.ac.uk`

January 7, 2015

Exercise 12.2

We want to solve the following optimization problem:

$$w^* = \arg\max_w p(y_{1:T}|\mathbf{x}_{1:T}, \mathbf{w})$$

We rewrite the optimization problem in the following way:

$$\begin{aligned}
\mathbf{w}^* &= \arg\max_w p(y_{1:T}|\mathbf{x}_{1:T}, \mathbf{w}) \\
&= \arg\max_{\mathbf{w}} \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{(y_t - \mathbf{w}^\top \mathbf{x}_{t-1})^2}{2\sigma_t^2}}
\end{aligned}$$

The normalization denominators are constants which do not change the solution:

$$= \arg\max_{\mathbf{w}} \prod_{t=2}^{T} e^{-\frac{(y_t - \mathbf{w}^\top \mathbf{x}_{t-1})^2}{2\sigma_t^2}}$$

We can also take the log of the function:

$$\begin{aligned}
&= \arg\max_{\mathbf{w}} \log\left(\prod_{t=2}^{T} e^{-\frac{(y_t - \mathbf{w}^\top \mathbf{x}_{t-1})^2}{2\sigma_t^2}}\right) \\
&= \arg\max_{\mathbf{w}} \log\left(e^{\sum_{t=2}^{T} -\frac{(y_t - \mathbf{w}^\top \mathbf{x}_{t-1})^2}{2\sigma_t^2}}\right) \\
&= \arg\max_{\mathbf{w}} \sum_{t=2}^{T} -\frac{(y_t - \mathbf{w}^\top \mathbf{x}_{t-1})^2}{2\sigma_t^2}
\end{aligned}$$

We can remove the 1/2 factors and the minus signs by changing it into an argmin:

$$\begin{aligned}
&= \arg\min_{\mathbf{w}} \sum_{t=2}^{T} \frac{(y_t - \mathbf{w}^\top \mathbf{x}_{t-1})^2}{\sigma_t^2} \\
&= \arg\min_{\mathbf{w}} \sum_{t=2}^{T} \left(\frac{y_t}{\sigma_t} - \frac{\mathbf{w}^\top \mathbf{x}_{t-1}}{\sigma_t}\right)^2
\end{aligned}$$

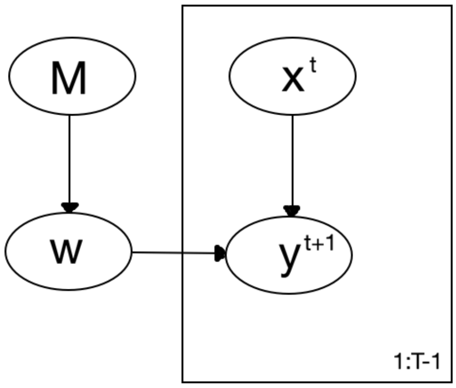
We can rewrite this as a least squares problem as such:

$$= \arg\min_{\mathbf{w}} ||\mathbf{X}\mathbf{w} - \mathbf{y}||_2^2, \text{ where: } \mathbf{X} = \begin{bmatrix} \mathbf{x}_1^\top/\sigma_2 \\ ... \\ \mathbf{x}_{T-1}^\top/\sigma_T \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_2/\sigma_2 \\ ... \\ y_T/\sigma_T \end{bmatrix}$$

The common solution to which, is well known as the following:

$$\mathbf{w}^* = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}$$

The hierarchical belief network is the following:



We initially start with equation 12.4.5:

$$p(M|\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}) = \frac{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}|M)p(M)}{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})}$$

The idea is to calculate the above probability for every different model M and then we will choose the one with the highest probability. Let us take for example two of them:

$$\frac{p(M=i|\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})}{p(M=j|\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})} = \frac{\frac{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}|M=i)p(M=i)}{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})}}{\frac{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}|M=j)p(M=j)}{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})}} = \frac{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}|M=i)p(M=i)}{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}|M=j)p(M=j)}$$

However, we are given that the prior p(M) is flat, therefore we need only:

$$\frac{p(M=i|\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})}{p(M=j|\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})} = \frac{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}|M=i)}{p(\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T}|M=j)} = \frac{\prod_{t=1}^{T-1} p(x_t)p(\mathbf{y}_{2:T}|\mathbf{x}_{1:T-1}, M=i)}{\prod_{t=1}^{T-1} p(x_t)p(\mathbf{y}_{2:T}|\mathbf{x}_{1:T-1}, M=j)}$$

By adjusting equation 12.4.6 we are now left with:

$$\frac{p(M=i|\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})}{p(M=j|\mathbf{x}_{1:T-1}, \mathbf{y}_{2:T})} = \frac{p(\mathbf{y}_{2:T}|\mathbf{x}_{1:T-1}, M=i)}{p(\mathbf{y}_{2:T}|\mathbf{x}_{1:T-1}, M=j)} = \frac{\int_{\mathbf{w}} p(\mathbf{w}|M=i)\prod_{t=1}^{T-1} p(\mathbf{y}_{t+1}|\mathbf{x}_t, \mathbf{w}, M=i)}{\int_{\mathbf{w}} p(\mathbf{w}|M=j)\prod_{t=1}^{T-1} p(\mathbf{y}_{t+1}|\mathbf{x}_t, \mathbf{w}, M=j)}$$

Since we are only interested in model comparison, we can equivalently use equation 12.4.7, after adjusting it to our current problem(we set $\alpha = 1$ right away and $\phi(x_t) = x_t$):

$$2\log p(\mathbf{y}_{2:T}|\mathbf{x}_{1:T-1}, M=i) = -\left(\sum_{t=2}^{T} \log\left(2\pi\sigma_t^2\right) + \frac{y_t^2}{\sigma_t^2}\right) + \mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b} - \log\det(\mathbf{A})$$

where:

$$\mathbf{A} = \mathbf{I} + \sum_{t=1}^{T-1} \frac{1}{\sigma_{t+1}^2}\mathbf{x}_t\mathbf{x}_t^\top, \mathbf{b} = \sum_{t=1}^{T-1} \frac{1}{\sigma_{t+1}^2} y_{t+1}\mathbf{x}_t$$

We also note that the first term is not affected by the model choice and therefore is not required in the computation. Therefore, we only need to calculate for each model the following and then find the maximum value:

$$\mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b} - \log\det(\mathbf{A})$$

The MATLAB code used:

```
clear all; clc; close all;
import brml.*
load('dodder.mat');
ModelLikelihoods = zeros(2^6-1,1);
for M = 1:2^6-1
    binary = logical(de2bi(M,6));
    A = eye(sum(binary));
    b = zeros(sum(binary),1);
    for t = 1:T-1
        A = A + (x(binary,t)*x(binary,t)')/(sigma(t+1)^2);
        b = b + (y(t+1)*x(binary,t))/(sigma(t+1)^2);
        %ModelLikelihoods(M) = ModelLikelihoods(M) + log(2*pi*sigma(t)^2) + (y(t)
    ^2)/(sigma(t)^2);
    end
    ModelLikelihoods(M) = - ModelLikelihoods(M) + b'*(A\b)-log(det(A));
end
[~,BestModel] = max(ModelLikelihoods);
de2bi(BestModel,6)
```

MATLAB prints out that the best model is the one using only the first four variables:
ans =
1 1 1 1 0 0

Exercise 12.3

We want to show that:

$$\frac{1}{(2\pi\alpha^{-1})^{K/2}}e^{-\frac{\alpha}{2}\mathbf{w}^\top\mathbf{w}}\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum_n(y^n-\mathbf{w}^\top\phi(x^n))^2} = \frac{1}{(2\pi\alpha^{-1})^{K/2}}\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{1}{2\sigma^2}\sum_n(y^n)^2}e^{-\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}}$$

By continual equivalences we have:

Removing the fractions:

$$e^{-\frac{\alpha}{2}\mathbf{w}^\top\mathbf{w}}e^{-\frac{1}{2\sigma^2}\sum_n(y^n-\mathbf{w}^\top\phi(x^n))^2} = e^{-\frac{1}{2\sigma^2}\sum_n(y^n)^2}e^{-\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}}$$

Taking the logarithm:

$$-\frac{\alpha}{2}\mathbf{w}^\top\mathbf{w}-\frac{1}{2\sigma^2}\sum_n(y^n-\mathbf{w}^\top\phi(x^n))^2 = -\frac{1}{2\sigma^2}\sum_n(y^n)^2-\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}$$

Expanding the quadratic:

$$-\frac{\alpha}{2}\mathbf{w}^\top\mathbf{w}-\frac{1}{2\sigma^2}\left(\sum_n(y^n)^2-2y^n\mathbf{w}^\top\phi(x^n)+(\mathbf{w}^\top\phi(x^n))^2\right) = -\frac{1}{2\sigma^2}\sum_n(y^n)^2-\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}$$

Distributing the sum and rearranging:

$$-\frac{\alpha}{2}\mathbf{w}^\top\mathbf{w}-\frac{1}{2\sigma^2}\sum_n(y^n)^2+\frac{1}{\sigma^2}\sum_n y^n\mathbf{w}^\top\phi(x^n)-\frac{1}{2\sigma^2}\sum_n(\mathbf{w}^\top\phi(x^n)\mathbf{w}^\top\phi(x^n)) = -\frac{1}{2\sigma^2}\sum_n(y^n)^2-\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}$$

Removing the sum over y's and rearranging:

$$-\frac{\alpha}{2}\mathbf{w}^\top\mathbf{I}\mathbf{w}+\mathbf{w}^\top\frac{1}{\sigma^2}\sum_n y^n\phi(x^n)-\mathbf{w}^\top\frac{1}{2\sigma^2}\sum_n(\phi(x^n)\phi(x^n)^\top)\mathbf{w} = -\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}$$

Substituting with the definition of b and rearranging:

$$-\frac{1}{2}\mathbf{w}^\top\left(\alpha\mathbf{I}+\frac{1}{\sigma^2}\sum_n\phi(x^n)\phi(x^n)^\top\right)\mathbf{w}+\mathbf{w}^\top\mathbf{b} = -\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}$$

Substituting with the definition of A and rearranging:

$$-\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{w}^\top\mathbf{b} = -\frac{1}{2}\mathbf{w}^\top\mathbf{A}\mathbf{w}+\mathbf{w}^\top\mathbf{b}$$

We now want to find out the expression for $2\log p(y^1,...,y^N|x^1,...,x^N,K)$:

$$2\log p(y^1,...,y^N|x^1,...,x^N,K) =$$

$$2\log\left(\int_{\mathbf{w}} \frac{1}{(2\pi\alpha^{-1})^{K/2}}\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (y^n)^2} e^{-\frac{1}{2}\mathbf{w}^\top \mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}} d\mathbf{w}\right) =$$

$$2\log\left(\frac{1}{(2\pi\alpha^{-1})^{K/2}}\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (y^n)^2} \int_{\mathbf{w}} e^{-\frac{1}{2}\mathbf{w}^\top \mathbf{A}\mathbf{w}+\mathbf{b}^\top\mathbf{w}} d\mathbf{w}\right) =$$

Based on 8.4.13 :

$$2\log\left(\frac{1}{(2\pi\alpha^{-1})^{K/2}}\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (y^n)^2} \sqrt{\det(2\pi\mathbf{A}^{-1})} e^{\frac{1}{2}\mathbf{b}^\top\mathbf{A}^{-1}\mathbf{b}}\right) =$$

$$2\left(-\log(2\pi\alpha^{-1})^{K/2} - \log(2\pi\sigma^2)^{N/2} - \frac{1}{2\sigma^2}\sum_n (y^n)^2 + \log(\sqrt{\det(2\pi\mathbf{A}^{-1})}) + \frac{1}{2}\mathbf{b}^\top\mathbf{A}^{-1}\mathbf{b}\right) =$$

$$2\left(-\frac{K}{2}\log(2\pi\alpha^{-1}) - \frac{N}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_n (y^n)^2 + \log(\sqrt{(2\pi)^K\det(\mathbf{A}^{-1})}) + \frac{1}{2}\mathbf{b}^\top\mathbf{A}^{-1}\mathbf{b}\right) =$$

$$-K\log(2\pi\alpha^{-1}) - N\log(2\pi\sigma^2) - \frac{1}{\sigma^2}\sum_n (y^n)^2 + \log((2\pi)^K\det(\mathbf{A})^{-1}) + \mathbf{b}^\top\mathbf{A}^{-1}\mathbf{b} =$$

$$-K\log(2\pi) - K\log(\alpha^{-1}) - N\log(2\pi\sigma^2) - \frac{1}{\sigma^2}\sum_n (y^n)^2 + K\log(2\pi) + \log(\det(\mathbf{A})^{-1}) + \mathbf{b}^\top\mathbf{A}^{-1}\mathbf{b} =$$

$$K\log(\alpha) - N\log(2\pi\sigma^2) - \frac{1}{\sigma^2}\sum_n (y^n)^2 - \log(\det(\mathbf{A})) + \mathbf{b}^\top\mathbf{A}^{-1}\mathbf{b}$$

Exercise 23.4

Here we only made two changes to the code. We changed the input string variable s and also at the end, when the most likely set of words (that are in the dictionary) are found, we make MATLAB output the log likelihood: if val;
disp([num2str(t) ':' str]);
log(maxval.tablet)
end

MATLAB outputs for the most likely state:

659:the monkey is on the branch

And the log likelihood of that state is:

-94.5250

```
clear all; clc; close all;
import brml.*;
load freq; % http://www.data-compression.com/english.shtml
l = {'a','b','c','d','e','f','g','h','i','j','k','l','m','n','o','p','q','r','s',
    't','u','v','w','x','y','z',' '};
load typing; % get the A transition and B emission matrices
figure(1); imagesc(A); set(gca,'xtick',1:27); set(gca,'xticklabel',l); set(gca,'
    ytick',1:27); set(gca,'yticklabel',l)
colorbar; colormap hot; title('transition')
```

```
figure(2); imagesc(B); set(gca,'xtick',1:27); set(gca,'xticklabel',l); set(gca,'ytick',1:27); set(gca,'yticklabel',l)
colorbar; colormap hot; title('emission')
ph1=condp(ones(27,1)); % uniform first hidden state distribution

%s = 'kezrninh'; Nmax=200; % observed sequence
s = 'rgenmonleunosbpnntje vrancg';  Nmax=1200; % observed sequence (brilliant is the answer)
v=double(s)-96; v=replace(v,-64,27); % convert to numbers

% find the most likely hidden sequences by defining a Factor Graph:
T = length(s);
hh=1:T; vv=T+1:2*T;
empot=array([vv(1) hh(1)],B);
prior=array(hh(1),ph1);
pot{1} = multpots([setpot(empot,vv(1),v(1)) prior]);
for t=2:T
    tranpot=array([hh(t) hh(t-1)],A);
    empot=array([vv(t) hh(t)],B);
    pot{t} = multpots([setpot(empot,vv(t),v(t)) tranpot]);
end
FG = FactorGraph(pot);

[maxstate, maxval, mess]=maxNprodFG(pot,FG,Nmax);
for n=1:Nmax
    maxstatearray(n,:)= horzcat(maxstate(n,1:length(s)).state);
end
strs=char(replace(maxstatearray+96,123,32)) % make strings from the decodings
fid=fopen('brit-a-z.txt','r'); % see http://www.curlewcommunications.co.uk/wordlist.html for Disclaimer and Copyright
w=textscan(fid,'%s'); w=w{1}; % get the words from the dictionary

% discard those decodings that are not in the dictionary:
% (An alternative would be to just compute the probability of each word in
% the dictionary to generate the observed sequence.)
for t=1:Nmax
    str = strs(t,:); % current string
    spac = strfind(str,' '); % chop the string into words
    spac = [spac length(str)+1]; % find the spaces
    start=1; val=1;
    for i=1:length(spac) % go through all the words in the string
        wd{i} = str(start:(spac(i)-1));
        start=spac(i)+1;
        if isempty(find(strcmp(wd{i},w))) % check if word is in the dictionary
            val=0; break
        end
    end
    if val;
        disp([num2str(t) ':' str]);
        log(maxval.table{t})
    end
end
```

Exercise 23.11

Just as in the first order HMM we have that:

$$\arg\max_{h_{1:T}} p(h_{1:T}|v_{1:T}) = \arg\max_{h_{1:T}} p(h_{1:T}, v_{1:T})$$

We now proceed to create the first message, with the main difference being that it is now a message of two variables, instead of one:

$$\max_{h_T} p(h_1)p(v_1|h_1)p(h_2|h_1)p(v_2|h_2)\prod_{t=3}^{T} p(v_t|h_t)p(h_t|h_{t-1},h_{t-2}) =$$
$$p(h_1)p(v_1|h_1)p(h_2|h_1)p(v_2|h_2)\prod_{t=3}^{T-1} p(v_t|h_t)p(h_t|h_{t-1},h_{t-2}) \max_{h_T} p(v_T|h_T)p(h_T|h_{T-1},h_{T-2}) =$$
$$p(h_1)p(v_1|h_1)p(h_2|h_1)p(v_2|h_2)\prod_{t=3}^{T-1} p(v_t|h_t)p(h_t|h_{t-1},h_{t-2})\mu(h_{T-1},h_{T-2})$$

Where the first message was:

$$\mu(h_{T-1},h_{T-2}) = \max_{h_T} p(v_T|h_T)p(h_T|h_{T-1},h_{T-2})$$

Likewise, for $3 \le t \le T-1$, we get the messages:

$$\mu(h_{t-1},h_{t-2}) = \max_{h_t} p(v_t|h_t)p(h_t|h_{t-1},h_{t-2})\mu(h_t,h_{t-1})$$

Lastly, we have the message:

$$\mu(h_1) = \max_{h_2} p(v_2|h_2)p(h_2|h_1)\mu(h_2,h_1)$$

And now we can start the backtracking to find the optimal states:

$$h_1^* = \arg\max_{h_1} p(h_1)p(v_1|h_1)\mu(h_1)$$

$$h_2^* = \arg\max_{h_2} p(h_2|h_1^*)p(v_2|h_2)\mu(h_2,h_1^*)$$

For $3 \le t \le T-1$, we have:

$$h_t^* = \arg\max_{h_t} p(h_t|h_{t-1}^*,h_{t-2}^*)p(v_t|h_t)\mu(h_t,h_{t-1}^*)$$

And lastly we have:

$$h_T^* = \arg\max_{h_T} p(h_T|h_{T-1}^*,h_{T-2}^*)p(v_T|h_T)$$

Exercise 27.5

We have:

$$\langle \log \frac{p(\mathbf{x}')}{p(\mathbf{x})} \rangle_{\tilde{q}(\mathbf{x}'|\mathbf{x})} =$$

$$\int_{-\inf}^{+\inf} \log \frac{\mathcal{N}(\mathbf{x}'|0, \sigma_p^2 \mathbf{I})}{\mathcal{N}(\mathbf{x}|0, \sigma_p^2 \mathbf{I})} \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' =$$

$$\int_{-\inf}^{+\inf} \log \frac{\frac{1}{(\sqrt{2\pi})^N \det(\sigma_p^2 \mathbf{I})} e^{-\frac{1}{2}\mathbf{x}'^\top (\sigma_p^2 \mathbf{I})^{-1} \mathbf{x}'}}{\frac{1}{(\sqrt{2\pi})^N \det(\sigma_p^2 \mathbf{I})} e^{-\frac{1}{2}\mathbf{x}^\top (\sigma_p^2 \mathbf{I})^{-1} \mathbf{x}}} \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' =$$

$$\int_{-\inf}^{+\inf} \log \frac{e^{-\frac{1}{2}\mathbf{x}'^\top (\sigma_p^2 \mathbf{I})^{-1} \mathbf{x}'}}{e^{-\frac{1}{2}\mathbf{x}^\top (\sigma_p^2 \mathbf{I})^{-1} \mathbf{x}}} \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' =$$

$$\int_{-\inf}^{+\inf} \log \frac{e^{-\frac{1}{2\sigma_p^2}\mathbf{x}'^\top \mathbf{x}'}}{e^{-\frac{1}{2\sigma_p^2}\mathbf{x}^\top \mathbf{x}}} \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' =$$

$$\int_{-\inf}^{+\inf} \left(-\frac{1}{2\sigma_p^2}\mathbf{x}'^\top \mathbf{x}' + \frac{1}{2\sigma_p^2}\mathbf{x}^\top \mathbf{x} \right) \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' =$$

$$\int_{-\inf}^{+\inf} -\frac{1}{2\sigma_p^2}\mathbf{x}'^\top \mathbf{x}' \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' + \int_{-\inf}^{+\inf} \frac{1}{2\sigma_p^2}\mathbf{x}^\top \mathbf{x} \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' =$$

$$-\frac{1}{2\sigma_p^2} \int_{-\inf}^{+\inf} \mathbf{x}'^\top \mathbf{x}' \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' + \frac{1}{2\sigma_p^2}\mathbf{x}^\top \mathbf{x} \int_{-\inf}^{+\inf} \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' =$$

However, we know that $\mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I})$ is a distribution, therefore :

$$-\frac{1}{2\sigma_p^2} \int_{-\inf}^{+\inf} \mathbf{x}'^\top \mathbf{x}' \mathcal{N}(\mathbf{x}'|\mathbf{x}, \sigma_q^2 \mathbf{I}) d\mathbf{x}' + \frac{1}{2\sigma_p^2}\mathbf{x}^\top \mathbf{x} =$$

Lastly, based on the result 8.5, with $\mathbf{A} = \mathbf{I}, \mu = \mathbf{x}, \mathbf{\Sigma} = \sigma_q^2 \mathbf{I}$ we have :

$$-\frac{1}{2\sigma_p^2} \left(\mathbf{x}^\top \mathbf{I} \mathbf{x} + trace(\mathbf{I}\sigma_q^2 \mathbf{I}) \right) + \frac{1}{2\sigma_p^2}\mathbf{x}^\top \mathbf{x} =$$

$$-\frac{1}{2\sigma_p^2} \left(\mathbf{x}^\top \mathbf{x} + N\sigma_q^2 \right) + \frac{1}{2\sigma_p^2}\mathbf{x}^\top \mathbf{x} =$$

$$-\frac{N\sigma_q^2}{2\sigma_p^2}$$

Exercise 27.6

The modified code:

```
clear all; clc; close all;
import brml.*
H=2; V=2; T=10;
% make a HMM
for totaliterator = 1:20
    Astart = rand(H,H);
    Bstart = rand(V,H);
    astart = rand(H,1);
    lambdaiterator = 0;
    for lambda = [0.1 1 10 20]
        lambdaiterator = lambdaiterator + 1;
        A=condp(Astart.^lambda);
```

```
        B=condp(Bstart);
        a=condp(astart);

        % draw some samples for v:
        h(1)=randgen(a); v(1)=randgen(B(:,h(1)));
        for t=2:T
            h(t)=randgen(A(:,h(t-1)));
            v(t)=randgen(B(:,h(t)));
        end
        [logalpha,~] = HMMforward(v,A,a,B);
        logbeta = HMMbackward(v,A,B);
        gamma = HMMsmooth(logalpha,logbeta,B,A,v); % exact marginal

        % single site Gibbs updating
        hsamp(:,1)=randgen(1:H,1,T);
        hv=1:T; vv=T+1:2*T; % hidden and visible variable indices

        num_samples=100;
        for sample=2:num_samples
            h = hsamp(:,sample-1);
            emiss=array([vv(1) hv(1)],B);
            trantm=array(hv(1),a);
            trant=array([hv(2) hv(1)],A);
            h(1) = randgen(table(setpot(multpots([trantm trant emiss]),[vv(1) hv(2)],[v(1) h(2)])));

            for t=2:T-1
                trantm.table=A; trantm.variables=[hv(t) hv(t-1)];
                trant.table=A; trant.variables=[hv(t+1) hv(t)];
                emiss.table=B; emiss.variables=[vv(t) hv(t)];
                h(t) = randgen(table(setpot(multpots([trantm trant emiss]),[vv(t) hv(t-1) hv(t+1)],[v(t) h(t-1) h(t+1)])));
            end

            trantm.table=A; trantm.variables=[hv(T) hv(T-1)];
            emiss.table=B; emiss.variables=[vv(T) hv(T)];
            h(T) = randgen(table(setpot(multpots([trantm emiss]),[vv(T) hv(T-1)],[v(T) h(T-1)])));

            hsamp(:,sample)=h; % take the sample after a forward sweep through time
        end
        for t=1:T
            gamma_samp(:,t) = count(hsamp(t,:),H)/num_samples;
        end
        %gamma_samp % sample marginal
        %fprintf('mean absolute error in the marginal estimate=%g\n',mean(abs(gamma(:)-gamma_samp(:))))
        Errors(lambdaiterator,totaliterator) = mean(abs(gamma(:)-gamma_samp(:)));
    end
end
fprintf('Mean absolute errors in the marginal estimate for:\nlambda = 0.1: %f\nlambda = 1: %f\nlambda = 10: %f\nlambda = 20: %f\n', mean(Errors(1,:)), mean(Errors(2,:)), mean(Errors(3,:)), mean(Errors(4,:)));
```

And MATLAB outputs:

Mean absolute errors in the marginal estimate for:
lambda = 0.1: 0.035778
lambda = 1: 0.035945
lambda = 10: 0.173067

lambda = 20: 0.149106

We can see that with the higher lambda values, the error has increased by a fair amount. This is due to an extreme case of what appears in figure 27.7(b) of the book. When the transition matrix is near deterministic, it is extremely overpowering and has the effect of 'locking' into a state. For example, with the transition matrix:

$$\begin{bmatrix} 1-e & e \\ e & 1-e \end{bmatrix}$$

where e is a very small value, the trend will be that it might lock to either all h being in state 1 or all h in state 2 and then it is very unlikely that it will stop being in this state.

Exercise 27.9