Example: Converge To Mean

Let G be an undirected connected graph. Because G is a connected graph there is a path from every vertex to every vertex. Let N be the number of vertices in G. We are given that $N > 1$.

A vertex of the graph represents an agent. An agent j can communicate with an agent k if and only if there is an edge between vertices j and k in the graph. Associated with each agent v is a floating point number, $v.m$; these values change as the algorithm proceeds. Let $v.M$ be the initial value of $v.m$. Let avg be the average of the $v.M$ over all v:

$avg = \sum_v v.M/N$

Consider the following algorithm by which agents collaborate to compute an approximation to avg. The algorithm is a *do-od* loop with a guarded command for every edge (v, w). Because the graph is undirected the guarded commands for (v, w) and (w, v) are the same.

In the program, ϵ is a positive, small constant and $|x|$ stands for the magnitude of x. The guarded command for edge $\{v, w\}$ is:

$g(v, w) \rightarrow c(v, w)$

where $g(v, w)$ is $|v.m - mid(v, w)| >= \epsilon$

and where $mid(v, w) = (v.m + w.m)/2.0$,

This guard is True if $v.m$ and $w.m$ are further than ϵ away from their midpoint, and the command moves them towards their midpoints.

We want the algorithm to terminate, and at termination we want $v.m$, for each v, to be approximately avg. We want to identify a positive constant $delta$ such that at termination $|v.m - avg| < \delta$. You don't need to identify the smallest δ.

Part 1

Is the algorithm correct?

Part 2

If the algorithm is correct, then:

a. Give the invariant I that you use to prove correctness. If the algorithm is incorrect, give an example.

To show that the algorithm is correct enter the following steps:

b. Show that I holds initially, and

c. Show that I is stable.

Part 3

If the algorithm is correct then

a. Give a variant function.

Carry out the following steps.

b. Show that the function is well-founded, i.e., show that the value of the function cannot decrease forever.

c. Show that execution of a guarded command with a true guard reduces the value of the variant function.

Part 4

If the algorithm is correct, then show that if the invariant holds and all guards are False then the desired postcondition of the algorithm holds. Give a value of δ.

ANSWERS

Part 1

Is the algorithm correct?

Answer

Yes

$\square$

1. Invariant I:

   $$\sum_v v.m = \sum_v v.M$$

2. Prove I holds initially.

   Follows from $v.m = v.M$, all v.

3. Prove that I is stable.

   For a transition due to the execution of a guarded command for an edge (v, w):

   for all k:
   $\{(v.m + w.m) = k\}\ g(v, w)\ \rightarrow\ c(v, w)\{(v.m + w.m) = k\}$

The above Hoare triple follows from the definition of mid.

Part 3

If the algorithm is correct then

a. Give a variant function:

Answer

We will identify a positive constant γ, and use the following function as the variant function:

$\lceil \sum_v v.m * v.m)/\gamma \rceil$

The notation $\lceil x \rceil$ stands for the smallest integer greater than or equal to x.

Carry out the following steps.

b. Show that the function is well-founded, i.e., show that the value of the function cannot decrease forever.

Answer

The function is bounded below by 0.

The function is a variant function because it takes on integer values and is bounded below.

Therefore the function can decrease only a finite number of times.

c. Show that execution of a guarded command with a true guard reduces the value of the variant function.

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If the guard $g(v, w)$ is True then

$v.m^2 + w.m^2 \geq (mid + \epsilon)^2 + (mid - \epsilon)^2$

So, each execution of the guarded command reduces the variant function by at least

$(mid + \epsilon)^2 + (mid - \epsilon)^2 - 2 * mid^2 \quad \geq \quad 2 * \epsilon^2$

Defining γ as a positive value less than $2 * \epsilon^2$, reduces the variant function at each step.

Part 4

If the algorithm is correct, then show that if the invariant holds and all guards are False then the desired postcondition of the algorithm holds. Give a value of δ.

An Introduction to Distributed Algorithms by K. Mani Chandy,
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