ML Reading Group (Fall 2015) **Lecture: 1**

Interesting Linear Algebra Problem

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1 Today

– Compute linear algebra arithmetic using theory

2 Theory

Theorem 2.11 from Friedberg, which is true by the definitions of composition of functions and matrix multiplication:

Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

An explicit example of the above is the following:

Consider $U(f(x)) = f'(x)$ where $U : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $T(f(x)) = \displaystyle\int_0^x f(t)dt$ where $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$. $UT = I$, by calculus.

Let the standard bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ be α and β respectively. As an easy exercise, verify **Theorem** 2.11 with the above example.

Theorem 2.23 from Friedberg:

Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. In other words, $Q = [I]_{\beta'}^{\beta}$.

Then

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q$$

Proof:

Let I be the identity transformation on V. Then $T = IT = TI$. So, by Theorem 2.11,

$Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [IT]_{\beta'}^{\beta} = [TI]_{\beta'}^{\beta} = [T]_\beta^\beta [I]_{\beta'}^{\beta} = [T]_\beta Q$.

Now multiply the left hand side of the first and last terms by Q^{-1}.

A subtle detail of this proof is that Q is *always* invertible. Why is this the case? Refer to `http://math.stackexchange.com/questions/25779/rigorously-proving-that-a-change-of-basis-matrix-is-always-invertible` for a great explanation.

3 Problem

Calculate $Q^{-1}AQ$ without inverting Q, entirely by hand, where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

3.1 Solution

Recognize that Q is the change of basis matrix where ϵ is the standard basis of $\mathbb{R}^5$ and $\gamma = \{e_2, e_4, e_5, e_1, e_3\} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\gamma_5\}$. Consequently we may write Q as $[I]_\gamma^\epsilon$. By theorem 2.23, $Q^{-1}AQ = [A]_\gamma$.

So, we would like to compute $A\gamma_1, \ldots, A\gamma_5$ with respect to γ. In order to compute this easily, recognize that $A\gamma_1 = Ae_2$. Multiplying, note we get the second column of A, or $c - e_3$ where $c = e_1 + e_2 + e_3 + e_4 + e_5$. Clearly, $c = \gamma_1 + \ldots + \gamma_5$ as well. And remember that $e_3 = \gamma_5$. So, $A\gamma_1 = c - \gamma_5$. Repeating for the others, we get the following matrix:

$$[A]_\gamma = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

3.2 Alternate Solution

Observe that Q is orthogonal. So $Q^{-1} = Q^T$. Also observe that Q and Q^T are compositions of elementary row and composition operations. Consequently, you can compute Q^TAQ without performing a single matrix multiplication – it can be done solely using row/column switches!