

# Interesting Linear Algebra Problem

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## 1 Today

– Compute linear algebra arithmetic using theory

## 2 Theory

**Theorem 2.11** from Friedberg, which is true by the definitions of composition of functions and matrix multiplication:

Let  $V, W$ , and  $Z$  be finite-dimensional vector spaces with ordered bases  $\alpha, \beta$ , and  $\gamma$ , respectively. Let  $T : V \rightarrow W$  and  $U : W \rightarrow Z$  be linear transformations. Then

$$[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$$

An explicit example of the above is the following:

Consider  $U(f(x)) = f'(x)$  where  $U : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$  and  $T(f(x)) = \int_0^x f(t)dt$  where  $T : P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ .  $UT = I$ , by calculus.

Let the standard bases of  $P_3(\mathbb{R})$  and  $P_2(\mathbb{R})$  be  $\alpha$  and  $\beta$  respectively. As an easy exercise, verify **Theorem 2.11** with the above example.

**Theorem 2.23** from Friedberg:

Let  $T$  be a linear operator on a finite-dimensional vector space  $V$ , and let  $\beta$  and  $\beta'$  be ordered bases for  $V$ . Suppose that  $Q$  is the change of coordinate matrix that changes  $\beta'$ -coordinates into  $\beta$ -coordinates. In other words,  $Q = [I]_{\beta}^{\beta'}$ .

Then

$$[T]_{\beta'} = Q^{-1} [T]_{\beta} Q$$

**Proof:**

Let  $I$  be the identity transformation on  $V$ . Then  $T = IT = TI$ . So, by Theorem 2.11,

$$Q[T]_{\beta'} = [I]_{\beta}^{\beta'} [T]_{\beta'}^{\beta'} = [IT]_{\beta}^{\beta} = [TI]_{\beta}^{\beta} = [T]_{\beta}^{\beta} [I]_{\beta}^{\beta'} = [T]_{\beta} Q.$$

Now multiply the left hand side of the first and last terms by  $Q^{-1}$ .

A subtle detail of this proof is that  $Q$  is *always* invertible. Why is this the case? Refer to <http://math.stackexchange.com/questions/25779/rigorously-proving-that-a-change-of-basis-matrix-is-always-invertible> for a great explanation.

### 3 Problem

Calculate  $Q^{-1}AQ$  without inverting  $Q$ , entirely by hand, where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

#### 3.1 Solution

Recognize that  $Q$  is the change of basis matrix where  $\epsilon$  is the standard basis of  $\mathbb{R}^5$  and  $\gamma = \{e_2, e_4, e_5, e_1, e_3\} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ . Consequently we may write  $Q$  as  $[I]_{\gamma}^{\epsilon}$ . By theorem 2.23,  $Q^{-1}AQ = [A]_{\gamma}$ .

So, we would like to compute  $A\gamma_1, \dots, A\gamma_5$  with respect to  $\gamma$ . In order to compute this easily, recognize that  $A\gamma_1 = Ae_2$ . Multiplying, note we get the second column of  $A$ , or  $c - e_3$  where  $c = e_1 + e_2 + e_3 + e_4 + e_5$ . Clearly,  $c = \gamma_1 + \dots + \gamma_5$  as well. And remember that  $e_3 = \gamma_5$ . So,  $A\gamma_1 = c - \gamma_5$ . Repeating for the others, we get the following matrix:

$$[A]_{\gamma} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$