Lecture 11: Front Door Criterion

POL-GA 1251
Quantitative Political Analysis II
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NYU Politics

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Front Door Criterion

- Back-door criterion and IV-based identification are common.
- Pearl (1995) demonstrated a third type of identifying condition: "front-door criterion."
- Not been applied much, yet.
- Interesting comparison to IV.
- Raises some issues we will see again with mediation analysis.

Front Door Criterion

$$\begin{tikzcd}
T \arrow[r] & M \arrow[r] & Y \\
& U \arrow[ul, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

- Recall IV exclusion restriction: endogenous treatment mediates *all* of effect of instrument on outcome.
- Consider graph above. Similar exclusion restriction.
- Difference is that outer variable T is endogenous.
- Nonetheless, graph offers potential to identify effect of T on Y.
- "Front door criterion" (Pearl 1995).

Front Door Criterion

$$\begin{tikzcd} T \arrow[r] & M \arrow[r] & Y \\ & U \arrow[ul, bend left] \arrow[ur, bend right] & \end{tikzcd}$$

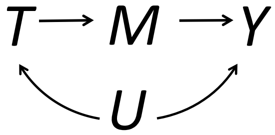
Intuition using linear structural equations:

- $M = \alpha_0 + \alpha_1 T + \varepsilon$.
- $Y = \beta_0 + \beta_1 M + U + \nu$.

Now, this implies for the relationship between Y and T:

- $Y = (\beta_0 + \beta_1 \alpha_0) + \beta_1 \alpha_1 T + (\beta_1 \varepsilon + U + \nu)$
- Effect of T on Y is $\beta_1 \alpha_1$.
- Regressing Y on T suffers from omitted variable bias (due to U).
- Nonetheless, can estimate β_1 and α_1 separately based on the structural equations.

Front Door Criterion



Non-parametric generalization:

- Suppose a sample indexed by i.
- Realized and potential outcomes for unit i:
  - $Y_i = Y_i(t, m)$ when $T_i = t$ and $M_i = m$.
  - $M_i = M_i(t)$ when $T_i = t$.
- For simplicity, suppose $T_i \in \{0, 1\}$ and $M_i(t) \in \{0, 1\}$, while $Y_i(t, m) \in \mathbb{R}$.

Front Door Criterion

- Generally, effect of T_i can be written

$$\tau_i = Y_i(1, M_i(1)) - Y_i(0, M_i(0)).$$

Front Door Criterion

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- Decomposition into mediated and direct effect (clever zeroes):

$$\begin{aligned}\tau_i &= Y_i(1, M_i(1)) - Y_i(1, M_i(0)) + Y_i(1, M_i(0)) - Y_i(0, M_i(0)) \\ &= Y_i(1, M_i(1)) - Y_i(0, M_i(1)) + Y_i(0, M_i(1)) - Y_i(0, M_i(0))\end{aligned}$$

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- Front door criterion consists of two assumptions:
  1. No direct effect of T on Y (exclusion restriction):

     $$Y_i(t, m) = Y_i(1-t, m) \equiv Y_i(m).$$

     Then, $\tau_i = Y_i(M_i(1)) - Y_i(M_i(0))$.
  2. Two unconfoundedness assumptions:

     $$\begin{aligned} M_i(t) \perp\!\!\!\perp T_i &\quad \text{for } t \in \{0,1\} \\ Y_i(m) \perp\!\!\!\perp M_i(t) \mid T_i &\quad \text{for } t \in \{0,1\}, m \in \{0,1\}.\end{aligned}$$

- Analysis could be conditional on X_i as well.

Front Door Estimand

- We can use these assumptions to identify the ATE.
- By assumption 1, the ATE is given by

$$\begin{aligned}
\mathrm{E}[\tau_i] &= \mathrm{E}[Y_i(M_i(1)) - Y_i(M_i(0))] \\
&= \mathrm{E}[Y_i(M_i(1)) - Y_i(M_i(0)) \mid T_i = 1] \Pr[T_i = 1] \\
&\qquad + \mathrm{E}[Y_i(M_i(1)) - Y_i(M_i(0)) \mid T_i = 0](1 - \Pr[T_i = 1]) \\
&= \mathrm{ATT} * \Pr[T_i = 1] + \mathrm{ATC} * (1 - \Pr[T_i = 1])
\end{aligned}$$

- $\Pr[T_i = 1]$ is obviously identified.
- So, need to identify the ATT and ATC, then we are done.
- Let's focus on the ATT.
- We use assumption 2 to identify the counterfactual part.

Front Door Estimand

$$\text{ATT} = \underbrace{\mathrm{E}\left[Y_i(M_i(1)) \mid T_i = 1\right]}_{A} - \underbrace{\mathrm{E}\left[Y_i(M_i(0)) \mid T_i = 1\right]}_{B}.$$

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$$\begin{aligned}
A &= \mathrm{E}\left[Y_i(1) \mid M_i(1) = 1, T_i = 1\right] \Pr[M_i(1) = 1 \mid T_i = 1] \\
&\quad + \mathrm{E}\left[Y_i(0) \mid M_i(1) = 0, T_i = 1\right] \Pr[M_i(1) = 0 \mid T_i = 1] \\
&= \mathrm{E}\left[Y_i \mid M_i = 1, T_i = 1\right] \Pr[M_i = 1 \mid T_i = 1] \\
&\quad + \mathrm{E}\left[Y_i \mid M_i = 0, T_i = 1\right] (1 - \Pr[M_i = 1 \mid T_i = 1]).
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$$\begin{aligned}
\text{ATT} = &(\Pr[M = 1 \mid T = 1] - \Pr[M = 1 \mid T = 0]) \\
&\times (\mathrm{E}\left[Y_i \mid M_i = 1, T_i = 1\right] - \mathrm{E}\left[Y_i \mid M_i = 0, T_i = 1\right]).
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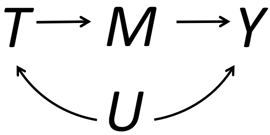
\end{aligned}$$

Similarly,

$$\begin{aligned}
\text{ATC} =& \left(\Pr[M = 1 \mid T = 1] - \Pr[M = 1 \mid T = 0]\right) \\
& \times \left(\mathrm{E}\left[Y_i \mid M_i = 1, T_i = 0\right] - \mathrm{E}\left[Y_i \mid M_i = 0, T_i = 0\right]\right).
\end{aligned}$$

Front Door Estimand



- $\text{ATE} = \text{ATT} * \Pr[T_i = 1] + \text{ATC} * \Pr[T_i = 0]$
- Sometimes people assume "no interaction":

$$\begin{aligned}&\mathrm{E}[Y_i \mid M_i = 1, T_i = 1] - \mathrm{E}[Y_i \mid M_i = 0, T_i = 1] \\ &\qquad = \mathrm{E}[Y_i \mid M_i = 1, T_i = 0] - \mathrm{E}[Y_i \mid M_i = 0, T_i = 0]\end{aligned}$$

- *If* no interactions holds, then we can take an inverse-variance-weighted (rather than $\Pr[T_i = 1]$-weighted) average of the ATT and ATC estimates to get a more efficient estimate of the ATE. (An OLS regression of Y on M and T does this automatically.)

Estimation

- Estimand:

$$\text{ATE} = \text{ATT} * \Pr[T_i = 1] + \text{ATC} * \Pr[T_i = 0]$$

- Non-parametric estimation of components with matching or weighting, semi-parametric with regressions. Then take products of component estimates.
- Inference using delta method or bootstrap.

Application: Age-Period-Cohort Effects

MHE "fundamentally unidentified question" (FUQ)

- Causal system in which different treatments are deterministic functions of each other.
  - $Y = A + B + C$ where $A = B - C$ deterministically.
  - Effect of school starting age on test scores.
    Starting age = Age - Years in school.
  - Effect of age on political opinion ("younger generation thinks ...").
    Age = Period - Cohort.
- Cannot imagine an experiment that varies A while fixing B or C.

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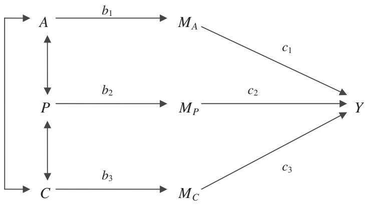
Winship and Harding (2008) proposal:

- Leverage front-door criterion.
- Not necessarily *FU*Qd.
- Application: age-period-cohort models.

Application: Age-Period-Cohort Effects

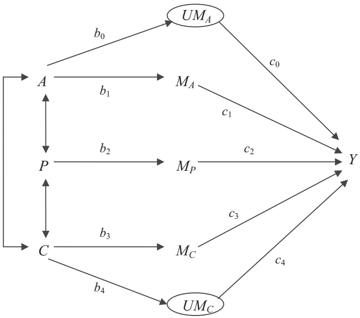
Figure 4
Hypothetical Age–Period–Cohort Model
With Intervening Mechanisms



- $A = P - C$.
- Regression specification $Y = \beta_0 + \beta_1 A + \beta_2 P + \beta_3 C + \varepsilon$ unidentifiable (not full rank; collider problems).
- Proposed solution: front door criterion identification.
- Requires finding M variables for which exclusion and unconfoundedness hold.
- Winship & Harding analyze with linear models for all effects.

Application: Age-Period-Cohort Effects

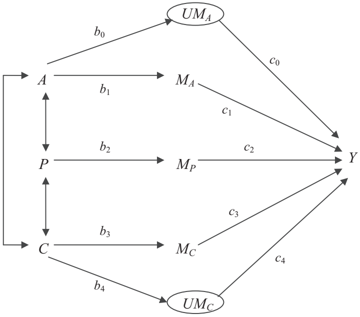
Figure 6
Hypothetical Age–Period–Cohort Model
With Unobserved Mechanisms



- We actually need fewer restrictions, which may be more realistic.

Application: Age-Period-Cohort Effects

Figure 6
Hypothetical Age–Period–Cohort Model
With Unobserved Mechanisms



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Application: Age-Period-Cohort Effects

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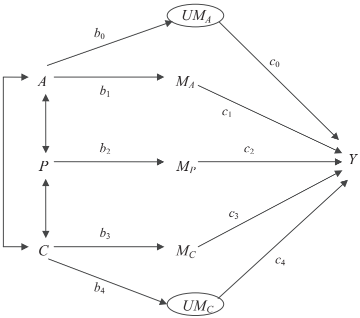
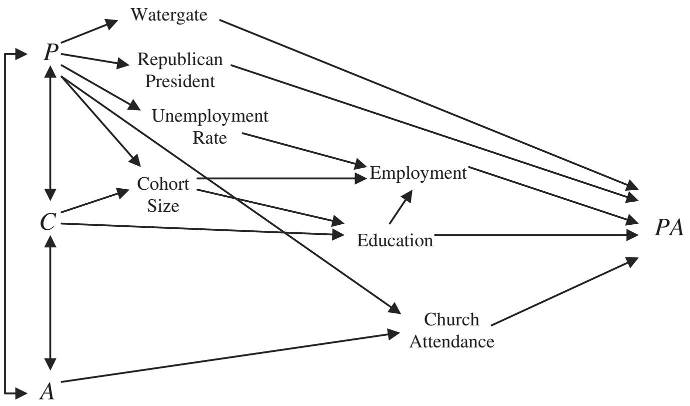


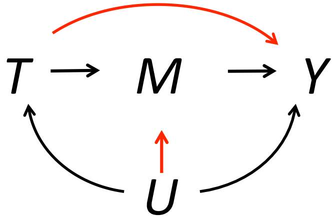
- We actually need fewer restrictions, which may be more realistic.
- $P \to Y$ with front door. But others?
- Consider $A \to Y$.
- Cannot condition on P and C because of rank deficiency.
- But can condition on M_P and C.
- Similar available for $C \to Y$.

Application: Age-Period-Cohort Effects

Figure 8
Full Age–Period–Cohort Model for Political Alienation With Multiple Mechanisms

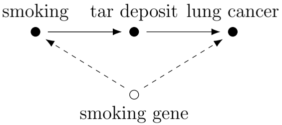


Violations

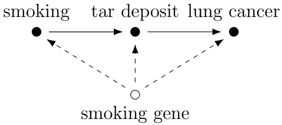


- Violation of either FDC assumption means FDC estimates would be biased.

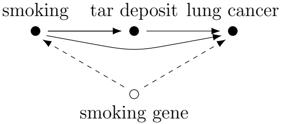
Violations



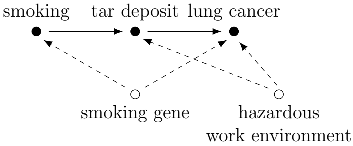
(a) Original Pearl DAG for front-door criterion



(b) Freedman Concern 1: smoking gene → tar deposits



(c) Freedman Concern 2: smoking → lung cancer



(d) Imbens Concern: hazardous work environment

Figure 5: Front-Door Criterion

(Imbens 2019)

Sensitivity Analysis

- If violations suspected, one way to proceed is sensitivity analysis.

Sensitivity Analysis

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- Glynn & Kashin (2014) derive the formula for the bias of the FDC ATT estimand when FDC assumptions are violated.
- Sensitivity analysis for ATT requires specifying values for three parameters:
  1. The size of the effect of U on Y when $T = 0$ and $M = 0$.
  2. How U differs for the treated when $M = 0$.
  3. The size of the direct effect of T on Y, holding M fixed.

More on "Bad Control"

- This analysis is also useful for diagnosing "bad control" problems.
- We saw an example of post-treatment bias due to colliders opening back door paths.
- Colliders can also distort paths that run from treatment to outcome of interest.

A Crash Course in Good and Bad Controls

Carlos Cinelli* Andrew Forney† Judea Pearl ‡

March 21, 2022

More on "Bad Control"

Jeffrey Wooldridge
@jmwooldridge

Suppose I have two 4th grade test scores, math4 and read4. I want to estimate the causal effect of class size on performance. Assume I have convincing controls. Is there a way to use a DAG to illustrate why I shouldn't include read4 in the equation for math4?

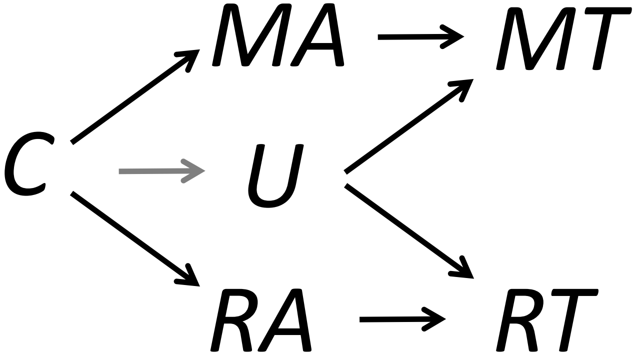
2:40 PM · Mar 12, 2022 · Twitter Web App

14 Retweets **2** Quote Tweets **118** Likes

More on "Bad Control"

- Suppose class size (C) *randomly assigned*.
- Math test scores (MT) and reading test scores (RT) are determined by some common factors (U) like general aptitude, testing conditions, etc., possibly affected by class size (although they don't have to be), as well as math and reading specific factors (MA and RA).
- What happens if we condition on (that is, include as a control) RT?

More on “Bad Control”



More on “Bad Control”

```
# C is class size
# MA is math-specific aptitude
# RA is reading-specific aptitude
# U is other factors determining both test scores, possibly endogenous to C
# MT is math test score
# RT is reading test score
# Structural equations
n <- 1000
C = rbinom(n, 1, p=.5)
MA = C + rnorm(n)
RA = C + rnorm(n)
U = C + rnorm(n)
MT = MA + U + rnorm(n)
RT = RA + U + rnorm(n)
# Thus effect of C on MT is 1*1 + 1*1 = 2
coef(lm(MT~C))
```

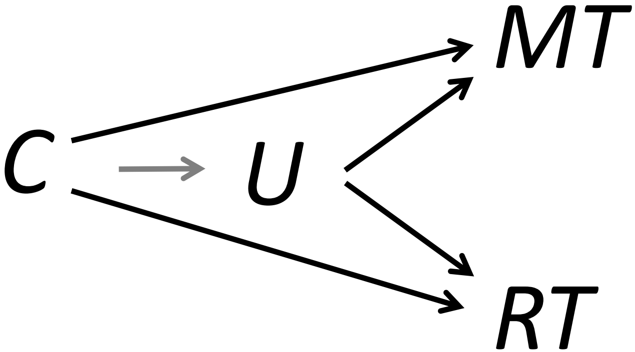
```
(Intercept)           C
-0.02278858  2.01950119
```

```
# Conditioning on RT induces negative dependency between MA and U.
# This creates a new path coefficient between C ans RT inducing bias:
coef(lm(MT~C+RT))
```

```
(Intercept)           C          RT
 -0.0424226   1.3357233   0.3351787
```

More on "Bad Control"

Even simpler:



Discussion

- Front-door analyses are still less well-understood. DAGs are a helpful tool.
- Front-door criterion is rarely applied, but growing area.
- Possible solution to “surrogate measures” problem.
- More connections between front-door identification and mediation analysis later in semester.