

# Formalization of "Knight's Tour Revisited"

Lukas Koller

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## Abstract

This is a formalization of [1]. In [1] the existence of Knight's paths and Knight's circuits are proved for arbitrary  $n \times m$ -boards with  $\min(n, m) \geq 5$  and for the Knight's circuit  $n \cdot m$  is even.

A Knight's path is a sequence of squares on a chessboard s.t. every step in sequence is a valid move for a Knight. A Knight is a chess figure that is only able to move two squares vertically and one square horizontally or two squares horizontally and one square vertically. Finding a Knight's path is an instance of the Hamiltonian Path Problem. A Knight's circuit is a Knight's path, where additionally the Knight can move from the last square to the first square of the path, forming a loop.

The main idea for the proof of the existence of a Knight's path on a  $n \times m$ -board, is to inductively construct a Knight's paths for the  $n \times m$ -board from a few pre-computed Knight's paths for small boards, i.e.  $5 \times 5$ ,  $5 \times 6$ , ...,  $8 \times 9$ . The paths for small boards are transformed (i.e. transpose, mirror, translate) and concatenated to create paths for larger boards.

While formalizing the proofs I have noticed two mistakes in the original proof in [1]: (i) the pre-computed path for the  $6 \times 6$ -board that ends in the upper-left (in Figure 2) and (ii) the pre-computed path for the  $8 \times 8$ -board that ends in the upper-left (in Figure 5) are incorrect. I.e. on the  $6 \times 6$ -board the Knight cannot step from square 26 to square 27; in the  $8 \times 8$ -board the Knight cannot step from square 27 to square 28. In this formalization I have replaced the two incorrect paths with correct paths.

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```

theory KnightsTour
  imports Main
begin

```

## 1 Definitions

```

type-synonym square = int  $\times$  int
type-synonym board = square set

```

A  $(n \times m)$ -board is the set of all squares  $(i, j)$  where  $1 \leq i \leq n$  and  $1 \leq j \leq m$ .  $(1, 1)$  is the lower-left corner.

```

definition board :: nat  $\Rightarrow$  nat  $\Rightarrow$  board where
  board n m =  $\{(i, j) \mid i \ j. \ 1 \leq i \wedge i \leq \text{int } n \wedge 1 \leq j \wedge j \leq \text{int } m\}$ 

```

A path is a sequence of steps on a board. A path is represented by the list of visited squares on the board. Each square on the  $(n \times m)$ -board is identified by its coordinates  $(i, j)$ .

```

type-synonym path = square list

```

A Knight can only move two squares vertically and one square horizontally or two squares horizontally and one square vertically. Thus, a knight at position  $(i, j)$  can only move to  $(i \pm 1, j \pm 2)$  or  $(i \pm 2, j \pm 1)$ .

**definition** *valid-step* :: *square*  $\Rightarrow$  *square*  $\Rightarrow$  *bool* **where**  
*valid-step*  $s_i s_j \equiv (\text{case } s_i \text{ of } (i,j) \Rightarrow s_j \in \{(i+1,j+2),(i-1,j+2),(i+1,j-2),(i-1,j-2),$   
 $(i+2,j+1),(i-2,j+1),(i+2,j-1),(i-2,j-1)\})$

Now we define an inductive predicate that characterizes a Knight's path. A square  $s_i$  can be pre-pended to a current Knight's path  $s_j \# ps$  if (i) there is a valid step from the square  $s_i$  to the first square  $s_j$  of the current path and (ii) the square  $s_i$  has not been visited yet.

**inductive** *knight's-path* :: *board*  $\Rightarrow$  *path*  $\Rightarrow$  *bool* **where**  
*knight's-path*  $\{s_i\} [s_i]$   
 $| s_i \notin b \Rightarrow \text{valid-step } s_i s_j \Rightarrow \text{knight's-path } b (s_j \# ps) \Rightarrow \text{knight's-path } (b \cup \{s_i\})$   
 $(s_i \# s_j \# ps)$

**code-pred** *knight's-path* .

A sequence is a Knight's circuit iff the sequence is a Knight's path and there is a valid step from the last square to the first square.

**definition** *knight's-circuit*  $b ps \equiv (\text{knight's-path } b ps \wedge \text{valid-step } (\text{last } ps) (\text{hd } ps))$

## 2 Executable Checker for a Knight's Path

This section gives the implementation and correctness-proof for an executable checker for a knight's-path wrt. the definition *knight's-path*.

### 2.1 Implementation of an Executable Checker

**fun** *row-exec* :: *nat*  $\Rightarrow$  *int set* **where**  
*row-exec* 0 = {}  
 $| \text{row-exec } m = \text{insert } (\text{int } m) (\text{row-exec } (m-1))$

**fun** *board-exec-aux* :: *nat*  $\Rightarrow$  *int set*  $\Rightarrow$  *board* **where**  
*board-exec-aux* 0  $M = \{\}$   
 $| \text{board-exec-aux } k M = \{(\text{int } k, j) \mid j. j \in M\} \cup \text{board-exec-aux } (k-1) M$

Compute a board.

**fun** *board-exec* :: *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *board* **where**  
*board-exec*  $n m = \text{board-exec-aux } n (\text{row-exec } m)$

**fun** *step-checker* :: *square*  $\Rightarrow$  *square*  $\Rightarrow$  *bool* **where**  
*step-checker*  $(i,j) (i',j') =$   
 $((i+1,j+2) = (i',j') \vee (i-1,j+2) = (i',j') \vee (i+1,j-2) = (i',j') \vee (i-1,j-2)$   
 $= (i',j')$   
 $\vee (i+2,j+1) = (i',j') \vee (i-2,j+1) = (i',j') \vee (i+2,j-1) = (i',j') \vee (i-2,j-1)$   
 $= (i',j'))$

**fun** *path-checker* :: *board*  $\Rightarrow$  *path*  $\Rightarrow$  *bool* **where**  
*path-checker*  $b [] = \text{False}$

| *path-checker* *b* [*s<sub>i</sub>*] = ( $\{s_i\} = b$ )  
 | *path-checker* *b* (*s<sub>i</sub>*#*s<sub>j</sub>*#*ps*) = (*s<sub>i</sub>* ∈ *b* ∧ *step-checker* *s<sub>i</sub>* *s<sub>j</sub>* ∧ *path-checker* (*b* − {*s<sub>i</sub>*}) (*s<sub>j</sub>*#*ps*))

**fun** *circuit-checker* :: *board* ⇒ *path* ⇒ *bool* **where**  
*circuit-checker* *b* *ps* = (*path-checker* *b* *ps* ∧ *step-checker* (*last* *ps*) (*hd* *ps*))

## 2.2 Correctness Proof of the Executable Checker

**lemma** *row-exec-leq*:  $j \in \text{row-exec } m \longleftrightarrow 1 \leq j \wedge j \leq \text{int } m$   
**by** (*induction* *m*) *auto*

**lemma** *board-exec-aux-leq-mem*:  $(i,j) \in \text{board-exec-aux } k \ M \longleftrightarrow 1 \leq i \wedge i \leq \text{int } k \wedge j \in M$   
**by** (*induction* *k* *M* *rule*: *board-exec-aux.induct*) *auto*

**lemma** *board-exec-leq*:  $(i,j) \in \text{board-exec } n \ m \longleftrightarrow 1 \leq i \wedge i \leq \text{int } n \wedge 1 \leq j \wedge j \leq \text{int } m$   
**using** *board-exec-aux-leq-mem* *row-exec-leq* **by** *auto*

**lemma** *board-exec-correct*: *board* *n* *m* = *board-exec* *n* *m*  
**unfolding** *board-def* **using** *board-exec-leq* **by** *auto*

**lemma** *step-checker-correct*: *step-checker* *s<sub>i</sub>* *s<sub>j</sub>*  $\longleftrightarrow$  *valid-step* *s<sub>i</sub>* *s<sub>j</sub>*

**proof**

**assume** *step-checker* *s<sub>i</sub>* *s<sub>j</sub>*  
**then show** *valid-step* *s<sub>i</sub>* *s<sub>j</sub>*  
**unfolding** *valid-step-def*  
**apply** (*cases* *s<sub>i</sub>*)  
**apply** (*cases* *s<sub>j</sub>*)  
**apply** *auto*  
**done**

**next**

**assume** *assms*: *valid-step* *s<sub>i</sub>* *s<sub>j</sub>*  
**then show** *step-checker* *s<sub>i</sub>* *s<sub>j</sub>*  
**unfolding** *valid-step-def* **by** *auto*

**qed**

**lemma** *step-checker-rev*: *step-checker* (*i,j*) (*i',j'*)  $\implies$  *step-checker* (*i',j'*) (*i,j*)  
**apply** (*simp* *only*: *step-checker.simps*)  
**by** (*elim* *disjE*) *auto*

**lemma** *knight-path-intro-rev*:

**assumes** *s<sub>i</sub>* ∈ *b* *valid-step* *s<sub>i</sub>* *s<sub>j</sub>* *knight-path* (*b* − {*s<sub>i</sub>*}) (*s<sub>j</sub>*#*ps*)  
**shows** *knight-path* *b* (*s<sub>i</sub>*#*s<sub>j</sub>*#*ps*)  
**using** *assms*

**proof** −

**assume** *assms*: *s<sub>i</sub>* ∈ *b* *valid-step* *s<sub>i</sub>* *s<sub>j</sub>* *knight-path* (*b* − {*s<sub>i</sub>*}) (*s<sub>j</sub>*#*ps*)  
**then have** *s<sub>i</sub>* ∉ (*b* − {*s<sub>i</sub>*}) *b* − {*s<sub>i</sub>*} ∪ {*s<sub>i</sub>*} = *b*

```

    by auto
  then show ?thesis
    using assms knights-path.intros(2)[of  $s_i$   $b - \{s_i\}$ ] by auto
qed

```

Final correctness corollary for the executable checker *path-checker*.

```

lemma path-checker-correct: path-checker  $b$   $ps \longleftrightarrow$  knights-path  $b$   $ps$ 
proof
  assume path-checker  $b$   $ps$ 
  then show knights-path  $b$   $ps$ 
proof (induction rule: path-checker.induct)
  case ( $\exists s_i s_j xs$   $b$ )
  then show ?case using step-checker-correct knights-path-intro-rev by auto
qed (auto intro: knights-path.intros)
next
  assume knights-path  $b$   $ps$ 
  then show path-checker  $b$   $ps$ 
    using step-checker-correct
    by (induction rule: knights-path.induct) (auto elim: knights-path.cases)
qed

```

```

corollary knights-path-exec-simp: knights-path (board  $n$   $m$ )  $ps \longleftrightarrow$  path-checker
(board-exec  $n$   $m$ )  $ps$ 
  using board-exec-correct path-checker-correct[symmetric] by simp

```

```

lemma circuit-checker-correct: circuit-checker  $b$   $ps \longleftrightarrow$  knights-circuit  $b$   $ps$ 
  unfolding knights-circuit-def using path-checker-correct step-checker-correct by
  auto

```

```

corollary knights-circuit-exec-simp:
  knights-circuit (board  $n$   $m$ )  $ps \longleftrightarrow$  circuit-checker (board-exec  $n$   $m$ )  $ps$ 
  using board-exec-correct circuit-checker-correct[symmetric] by simp

```

### 3 Basic Properties of *knights-path* and *knights-circuit*

```

lemma board-leq-subset:  $n_1 \leq n_2 \wedge m_1 \leq m_2 \implies$  board  $n_1$   $m_1 \subseteq$  board  $n_2$   $m_2$ 
  unfolding board-def by auto

```

```

lemma finite-row-exec: finite (row-exec  $m$ )
  by (induction  $m$ ) auto

```

```

lemma finite-board-exec-aux: finite  $M \implies$  finite (board-exec-aux  $n$   $M$ )
  by (induction  $n$ ) auto

```

```

lemma board-finite: finite (board  $n$   $m$ )
  using finite-board-exec-aux finite-row-exec by (simp only: board-exec-correct) auto

```

```

lemma card-row-exec: card (row-exec  $m$ ) =  $m$ 
proof (induction  $m$ )

```

```

case (Suc m)
have int (Suc m) ∉ row-exec m
  using row-exec-leq by auto
then have card (insert (int (Suc m)) (row-exec m)) = 1 + card (row-exec m)
  using card-Suc-eq by (metis Suc plus-1-eq-Suc row-exec.simps(1))
then have card (row-exec (Suc m)) = 1 + card (row-exec m)
  by auto
then show ?case using Suc.IH by auto
qed auto

```

**lemma** *set-comp-ins*:

```

 $\{(k,j) \mid j. j \in \text{insert } x \ M\} = \text{insert } (k,x) \ \{(k,j) \mid j. j \in M\}$  (is ?Mi = ?iM)
proof
  show ?Mi ⊆ ?iM
  proof
    fix y assume y ∈ ?Mi
    then obtain j where [simp]: y = (k,j) and j ∈ insert x M by blast
    then have j = x ∨ j ∈ M by auto
    then show y ∈ ?iM by (elim disjE) auto
  qed
next
  show ?iM ⊆ ?Mi
  proof
    fix y assume y ∈ ?iM
    then obtain j where [simp]: y = (k,j) and j ∈ insert x M by blast
    then have j = x ∨ j ∈ M by auto
    then show y ∈ ?Mi by (elim disjE) auto
  qed
qed

```

**lemma** *finite-card-set-comp*: *finite M ⟹ card {(k,j) | j. j ∈ M} = card M*

**proof** (*induction M* **rule**: *finite-induct*)

```

  case (insert x M)
  then show ?case using set-comp-ins[of k x M] by auto
qed auto

```

**lemma** *card-board-exec-aux*: *finite M ⟹ card (board-exec-aux k M) = k \* card M*

**proof** (*induction k*)

```

  case (Suc k)
  let ?M' = {(int (Suc k),j) | j. j ∈ M}
  let ?rec-k = board-exec-aux k M

```

```

have finite: finite ?M' finite ?rec-k
  using Suc finite-board-exec-aux by auto
then have card-Un-simp: card (?M' ∪ ?rec-k) = card ?M' + card ?rec-k
  using board-exec-aux-leq-mem card-Un-Int[of ?M' ?rec-k] by auto

```

```

have card-M: card ?M' = card M
  using Suc finite-card-set-comp by auto

```

```

have card (board-exec-aux (Suc k) M) = card ?M' + card ?rec-k
  using card-Un-simp by auto
also have ... = card M + k * card M
  using Suc card-M by auto
also have ... = (Suc k) * card M
  by auto
finally show ?case .
qed auto

```

```

lemma card-board: card (board n m) = n * m
proof -
  have card (board n m) = card (board-exec-aux n (row-exec m))
    using board-exec-correct by auto
  also have ... = n * m
    using card-row-exec card-board-exec-aux finite-row-exec by auto
  finally show ?thesis .
qed

```

```

lemma knights-path-board-non-empty: knights-path b ps  $\implies$  b  $\neq$  {}
  by (induction arbitrary: ps rule: knights-path.induct) auto

```

```

lemma knights-path-board-m-n-geq-1: knights-path (board n m) ps  $\implies$  min n m  $\geq$ 
1
  unfolding board-def using knights-path-board-non-empty by fastforce

```

```

lemma knights-path-non-nil: knights-path b ps  $\implies$  ps  $\neq$  []
  by (induction arbitrary: b rule: knights-path.induct) auto

```

```

lemma knights-path-set-eq: knights-path b ps  $\implies$  set ps = b
  by (induction rule: knights-path.induct) auto

```

```

lemma knights-path-subset:
  knights-path b1 ps1  $\implies$  knights-path b2 ps2  $\implies$  set ps1  $\subseteq$  set ps2  $\longleftrightarrow$  b1  $\subseteq$  b2
  using knights-path-set-eq by auto

```

```

lemma knights-path-board-unique: knights-path b1 ps  $\implies$  knights-path b2 ps  $\implies$ 
b1 = b2
  using knights-path-set-eq by auto

```

```

lemma valid-step-neq: valid-step si sj  $\implies$  si  $\neq$  sj
  unfolding valid-step-def by auto

```

```

lemma valid-step-non-transitive: valid-step si sj  $\implies$  valid-step sj sk  $\implies$   $\neg$ valid-step
si sk
proof -
  assume assms: valid-step si sj valid-step sj sk
  obtain ii ji ij jj ik jk where [simp]: si = (ii,ji) sj = (ij,jj) sk = (ik,jk) by
force

```

```

then have step-checker (ii,ji) (ij,jj) step-checker (ij,jj) (ik,jk)
  using assms step-checker-correct by auto
then show ¬valid-step si sk
  apply (simp add: step-checker-correct[symmetric])
  apply (elim disjE)
  apply auto
done
qed

lemma knights-path-distinct: knights-path b ps ⇒ distinct ps
proof (induction rule: knights-path.induct)
  case (2 si b sj ps)
  then have si ∉ set (sj # ps)
    using knights-path-set-eq valid-step-neq by blast
  then show ?case using 2 by auto
qed auto

lemma knights-path-length: knights-path b ps ⇒ length ps = card b
  using knights-path-set-eq knights-path-distinct by (metis distinct-card)

lemma knights-path-take:
  assumes knights-path b ps 0 < k k < length ps
  shows knights-path (set (take k ps)) (take k ps)
  using assms
proof (induction arbitrary: k rule: knights-path.induct)
  case (2 si b sj ps)
  then have k = 1 ∨ k = 2 ∨ 2 < k by force
  then show ?case
    using 2
  proof (elim disjE)
    assume k = 2
    then have take k (si # sj # ps) = [si, sj] si ∉ {sj} using 2 valid-step-neq by
      auto
    then show ?thesis using 2 knights-path.intros by auto
  next
    assume 2 < k
    then have k-simps: k-2 = k-1-1 0 < k-2 k-2 < length ps and
      take-simp1: take k (si # sj # ps) = si # take (k-1) (sj # ps) and
      take-simp2: take k (si # sj # ps) = si # sj # take (k-1-1) ps
      using assms 2 take-Cons'[of k si sj # ps] take-Cons'[of k-1 sj ps] by auto
    then have knights-path (set (take (k-1) (sj # ps))) (take (k-1) (sj # ps))
      using 2 k-simps by auto
    then have kp: knights-path (set (take (k-1) (sj # ps))) (sj # take (k-2) ps)
      using take-Cons'[of k-1 sj ps] by (auto simp: k-simps elim: knights-path.cases)

    have no-mem: si ∉ set (take (k-1) (sj # ps))
      using 2 set-take-subset[of k-1 sj # ps] knights-path-set-eq by blast
    have knights-path (set (take (k-1) (sj # ps)) ∪ {si}) (si # sj # take (k-2) ps)
      using knights-path.intros(2)[OF no-mem ⟨valid-step si sj⟩ kp] by auto
  end
end

```



```

    then show ?thesis using k-simps take-simp2 knights-path-set-eq by metis
  qed (auto intro: knights-path.intros)
qed auto

```

```

lemma knights-path-drop:
  assumes knights-path b ps 0 < k k < length ps
  shows knights-path (set (drop k ps)) (drop k ps)
  using assms
proof (induction arbitrary: k rule: knights-path.induct)
  case (2 si b sj ps)
  then have (k = 1 ∧ ps = []) ∨ (k = 1 ∧ ps ≠ []) ∨ 1 < k by force
  then show ?case
    using 2
  proof (elim disjE)
    assume k = 1 ∧ ps ≠ []
    then show ?thesis using 2 knights-path-set-eq by force
  next
    assume 1 < k
    then have 0 < k-1 k-1 < length (sj#ps) drop k (si#sj#ps) = drop (k-1)
      (sj#ps)
    using assms 2 drop-Cons'[of k si sj#ps] by auto
    then show ?thesis
      using 2 by auto
  qed
qed (auto intro: knights-path.intros)
qed auto

```

A Knight's path can be split to form two new disjoint Knight's paths.

```

corollary knights-path-split:
  assumes knights-path b ps 0 < k k < length ps
  shows
    ∃ b1 b2. knights-path b1 (take k ps) ∧ knights-path b2 (drop k ps) ∧ b1 ∪ b2 = b
    ∧ b1 ∩ b2 = {}
  using assms
proof -
  let ?b1=set (take k ps)
  let ?b2=set (drop k ps)
  have kp1: knights-path ?b1 (take k ps) and kp2: knights-path ?b2 (drop k ps)
    using assms knights-path-take knights-path-drop by auto
  have union: ?b1 ∪ ?b2 = b
    using assms knights-path-set-eq by (metis append-take-drop-id set-append)
  have inter: ?b1 ∩ ?b2 = {}
    using assms knights-path-distinct by (metis append-take-drop-id distinct-append)
  show ?thesis using kp1 kp2 union inter by auto
qed

```

Append two disjoint Knight's paths.

```

corollary knights-path-append:
  assumes knights-path b1 ps1 knights-path b2 ps2 b1 ∩ b2 = {} valid-step (last
    ps1) (hd ps2)

```

```

shows knights-path (b1 ∪ b2) (ps1 @ ps2)
using assms
proof (induction arbitrary: ps2 b2 rule: knights-path.induct)
case (1 si)
then have si ∉ b2 ps2 ≠ [] valid-step si (hd ps2) knights-path b2 (hd ps2#tl ps2)

    using knights-path-non-nil by auto
    then have knights-path (b2 ∪ {si}) (si#hd ps2#tl ps2)
    using knights-path.intros by blast
    then show ?case using ⟨ps2 ≠ []⟩ by auto
next
case (2 si b1 sj ps1)
then have si ∉ b1 ∪ b2 valid-step si sj knights-path (b1 ∪ b2) (sj#ps1@ps2) by
auto
then have knights-path (b1 ∪ b2 ∪ {si}) (si#sj#ps1@ps2)
    using knights-path.intros by auto
then show ?case by auto
qed

lemma valid-step-rev: valid-step si sj ⇒ valid-step sj si
using step-checker-correct step-checker-rev by (metis prod.exhaust-sel)

```

Reverse a Knight's path.

```

corollary knights-path-rev:
assumes knights-path b ps
shows knights-path b (rev ps)
using assms
proof (induction rule: knights-path.induct)
case (2 si b sj ps)
then have knights-path {si} [si] b ∩ {si} = {} valid-step (last (rev (sj # ps)))
(hd [si])
    using valid-step-rev by (auto intro: knights-path.intros)
then have knights-path (b ∪ {si}) ((rev (sj#ps))@[si])
    using 2 knights-path-append by blast
then show ?case by auto
qed (auto intro: knights-path.intros)

```

Reverse a Knight's circuit.

```

corollary knights-circuit-rev:
assumes knights-circuit b ps
shows knights-circuit b (rev ps)
using assms knights-path-rev valid-step-rev
unfolding knights-circuit-def by (auto simp: hd-rev last-rev)

```

```

lemma knights-circuit-rotate1:
assumes knights-circuit b (si#ps)

```

```

    shows knight-circuit b (ps@[si])
  proof (cases ps = [])
    case True
    then show ?thesis using assms by auto
  next
    case False
    have kp1: knight-path b (si#ps) valid-step (last (si#ps)) (hd (si#ps))
      using assms unfolding knight-circuit-def by auto
    then have kp-elim: si ∉ (b - {si}) valid-step si (hd ps) knight-path (b - {si})
  ps
    using ⟨ps ≠ []⟩ by (auto elim: knight-path.cases)
    then have vs': valid-step (last (ps@[si])) (hd (ps@[si]))
      using ⟨ps ≠ []⟩ valid-step-rev by auto

    have kp2: knight-path {si} [si] (b - {si}) ∩ {si} = {}
      by (auto intro: knight-path.intros)

    have vs: valid-step (last ps) (hd [si])
      using ⟨ps ≠ []⟩ ⟨valid-step (last (si#ps)) (hd (si#ps))⟩ by auto

    have (b - {si}) ∪ {si} = b
      using kp1 kp-elim knight-path-set-eq by force
    then show ?thesis
      unfolding knight-circuit-def
      using vs knight-path-append[OF ⟨knight-path (b - {si}) ps⟩ kp2] vs' by auto
  qed

```

A Knight's circuit can be rotated to start at any square on the board.

```

lemma knight-circuit-rotate-to:
  assumes knight-circuit b ps hd (drop k ps) = si k < length ps
  shows ∃ ps'. knight-circuit b ps' ∧ hd ps' = si
  using assms
proof (induction k arbitrary: b ps)
  case (Suc k)
  let ?sj = hd ps
  let ?ps' = tl ps
  show ?case
proof (cases si = ?sj)
  case True
  then show ?thesis using Suc by auto
next
  case False
  then have ?ps' ≠ []
    using Suc by (metis drop-Nil drop-Suc drop-eq-Nil2 le-antisym nat-less-le)
  then have knight-circuit b (?sj#?ps')
    using Suc by (metis list.exhaust-sel tl-Nil)
  then have knight-circuit b (?ps'@[?sj]) hd (drop k (?ps'@[?sj])) = si
    using Suc knight-circuit-rotate1 by (auto simp: drop-Suc)
  then show ?thesis using Suc by auto

```

qed  
qed *auto*

For positive boards (1,1) can only have (2,3) and (3,2) as a neighbour.

**lemma** *valid-step-1-1*:  
**assumes** *valid-step* (1,1) (i,j)  $i > 0$   $j > 0$   
**shows**  $(i,j) = (2,3) \vee (i,j) = (3,2)$   
**using** *assms* **unfolding** *valid-step-def* **by** *auto*

**lemma** *list-len-g-1-split*:  $\text{length } xs > 1 \implies \exists x_1 x_2 xs'. xs = x_1 \# x_2 \# xs'$   
**proof** (*induction xs*)  
**case** (*Cons x xs*)  
**then have**  $\text{length } xs > 0$  **by** *auto*  
**then have**  $\text{length } xs \geq 1$  **by** *presburger*  
**then have**  $\text{length } xs = 1 \vee \text{length } xs > 1$  **by** *auto*  
**then show** ?*case*  
**proof** (*elim disjE*)  
**assume**  $\text{length } xs = 1$   
**then obtain**  $x_1$  **where** [*simp*]:  $xs = [x_1]$   
**using** *length-Suc-conv*[*of xs 0*] **by** *auto*  
**then show** ?*thesis* **by** *auto*  
**next**  
**assume**  $1 < \text{length } xs$   
**then show** ?*thesis* **using** *Cons* **by** *auto*  
qed  
qed *auto*

**lemma** *list-len-g-3-split*:  $\text{length } xs > 3 \implies \exists x_1 x_2 xs' x_3. xs = x_1 \# x_2 \# xs' @ [x_3]$   
**proof** (*induction xs*)  
**case** (*Cons x xs*)  
**then have**  $\text{length } xs = 3 \vee \text{length } xs > 3$  **by** *auto*  
**then show** ?*case*  
**proof** (*elim disjE*)  
**assume**  $\text{length } xs = 3$   
**then obtain**  $x_1 xs_1$  **where** [*simp*]:  $xs = x_1 \# xs_1$   $\text{length } xs_1 = 2$   
**using** *length-Suc-conv*[*of xs 2*] **by** *auto*  
**then obtain**  $x_2 xs_2$  **where** [*simp*]:  $xs_1 = x_2 \# xs_2$   $\text{length } xs_2 = 1$   
**using** *length-Suc-conv*[*of xs\_1 1*] **by** *auto*  
**then obtain**  $x_3$  **where** [*simp*]:  $xs_2 = [x_3]$   
**using** *length-Suc-conv*[*of xs\_2 0*] **by** *auto*  
**then show** ?*thesis* **by** *auto*  
**next**  
**assume**  $\text{length } xs > 3$   
**then show** ?*thesis* **using** *Cons* **by** *auto*  
qed  
qed *auto*

Any Knight's circuit on a positive board can be rotated to start with (1,1) and end with (3,2).

**corollary** *rotate-knights-circuit*:

**assumes** *knights-circuit* (board  $n$   $m$ )  $ps$   $\min n\ m \geq 5$

**shows**  $\exists ps. \text{knights-circuit (board } n\ m)\ ps \wedge \text{hd } ps = (1,1) \wedge \text{last } ps = (3,2)$

**using** *assms*

**proof** –

**let**  $?b = \text{board } n\ m$

**have** *knights-path*  $?b\ ps$

**using** *assms* **unfolding** *knights-circuit-def* **by** *auto*

**then have**  $(1,1) \in \text{set } ps$

**using** *assms* *knights-path-set-eq* **by** (*auto simp: board-def*)

**then obtain**  $k$  **where**  $\text{hd (drop } k\ ps) = (1,1)$   $k < \text{length } ps$

**by** (*metis hd-drop-conv-nth in-set-conv-nth*)

**then obtain**  $ps_r$  **where**  $ps_r\text{-prems: knights-circuit } ?b\ ps_r\ \text{hd } ps_r = (1,1)$

**using** *assms* *knights-circuit-rotate-to* **by** *blast*

**then have**  $kp: \text{knights-path } ?b\ ps_r$  **and**  $\text{valid-step (last } ps_r)\ (1,1)$

**unfolding** *knights-circuit-def* **by** *auto*

**have**  $(1,1) \in ?b\ (1,2) \in ?b\ (1,3) \in ?b$

**using** *assms* **unfolding** *board-def* **by** *auto*

**then have**  $(1,1) \in \text{set } ps_r\ (1,2) \in \text{set } ps_r\ (1,3) \in \text{set } ps_r$

**using**  $kp$  *knights-path-set-eq* **by** *auto*

**have**  $3 < \text{card } ?b$

**using** *assms* *board-leq-subset card-board*[*of* 5 5]

*card-mono*[*OF* *board-finite*[*of*  $n\ m$ ], *of* board 5 5] **by** *auto*

**then have**  $3 < \text{length } ps_r$

**using** *knights-path-length*  $kp$  **by** *auto*

**then obtain**  $s_j\ ps'\ s_k$  **where** [*simp*]:  $ps_r = (1,1) \# s_j \# ps' @ [s_k]$

**using**  $\langle \text{hd } ps_r = (1,1) \rangle \text{list-len-g-3-split}$ [*of*  $ps_r$ ] **by** *auto*

**have**  $s_j \neq s_k$

**using**  $kp$  *knights-path-distinct* **by** *force*

**have**  $vs-s_k: \text{valid-step } s_k\ (1,1)$

**using**  $\langle \text{valid-step (last } ps_r)\ (1,1) \rangle$  **by** *simp*

**have**  $vs-s_j: \text{valid-step } (1,1)\ s_j$  **and**  $kp': \text{knights-path } (?b - \{(1,1)\})\ (s_j \# ps' @ [s_k])$

**using**  $kp$  **by** (*auto elim: knights-path.cases*)

**have**  $s_j \in \text{set } ps_r\ s_k \in \text{set } ps_r$  **by** *auto*

**then have**  $s_j \in ?b\ s_k \in ?b$

**using**  $kp$  *knights-path-set-eq* **by** *blast+*

**then have**  $0 < \text{fst } s_j \wedge 0 < \text{snd } s_j\ 0 < \text{fst } s_k \wedge 0 < \text{snd } s_k$

**unfolding** *board-def* **by** *auto*

**then have**  $s_k = (2,3) \vee s_k = (3,2)\ s_j = (2,3) \vee s_j = (3,2)$

**using**  $vs-s_k\ vs-s_j\ \text{valid-step-1-1}\ \text{valid-step-rev}$  **by** (*metis prod.collapse*)**+**

**then have**  $s_k = (3,2) \vee s_j = (3,2)$

**using**  $\langle s_j \neq s_k \rangle$  **by** *auto*

**then show** *?thesis*

**proof** (*elim disjE*)

```

    assume  $s_k = (3,2)$ 
    then have  $\text{last } ps_r = (3,2)$  by auto
    then show  $?thesis$  using  $ps_r\text{-prems}$  by auto
next
    assume  $s_j = (3,2)$ 
    then have  $vs: \text{valid-step } (\text{last } ((1,1)\#rev (s_j\#ps'\@[s_k]))) \text{ (hd } ((1,1)\#rev$ 
 $(s_j\#ps'\@[s_k])))$ 
      unfolding  $\text{valid-step-def}$  by auto

    have  $\text{rev-simp: } rev (s_j\#ps'\@[s_k]) = s_k\#(rev ps')\@[s_j]$  by auto

    have  $\text{knight-path } (?b - \{(1,1)\}) (rev (s_j\#ps'\@[s_k]))$ 
      using  $\text{knight-path-rev}[OF kp']$  by auto
    then have  $(1,1) \notin (?b - \{(1,1)\}) \text{ valid-step } (1,1) s_k$ 
       $\text{knight-path } (?b - \{(1,1)\}) (s_k\#(rev ps')\@[s_j])$ 
      using  $\text{assms } vs\text{-}s_k \text{ valid-step-rev}$  by (auto simp:  $\text{rev-simp}$ )
    then have  $\text{knight-path } (?b - \{(1,1)\} \cup \{(1,1)\}) ((1,1)\#s_k\#(rev ps')\@[s_j])$ 
      using  $\text{knight-path.intros}(2)[of (1,1) ?b - \{(1,1)\} s_k (rev ps')\@[s_j]]$  by auto
    then have  $\text{knight-path } ?b ((1,1)\#rev (s_j\#ps'\@[s_k]))$ 
      using  $\text{assms}$  by (simp add:  $\text{board-def insert-absorb rev-simp}$ )
    then have  $\text{knight-circuit } ?b ((1,1)\#rev (s_j\#ps'\@[s_k]))$ 
      unfolding  $\text{knight-circuit-def}$  using  $vs$  by auto
    then show  $?thesis$ 
      using  $\langle s_j = (3,2) \rangle$  by auto
qed
qed

```

## 4 Transposing Paths and Boards

### 4.1 Implementation of Path and Board Transposition

**definition**  $\text{transpose-square } s_i = (\text{case } s_i \text{ of } (i,j) \Rightarrow (j,i))$

**fun**  $\text{transpose} :: \text{path} \Rightarrow \text{path}$  **where**

```

   $\text{transpose } [] = []$ 
  |  $\text{transpose } (s_i\#ps) = (\text{transpose-square } s_i)\#\text{transpose } ps$ 

```

**definition**  $\text{transpose-board} :: \text{board} \Rightarrow \text{board}$  **where**

$\text{transpose-board } b \equiv \{(j,i) \mid i\ j. (i,j) \in b\}$

### 4.2 Correctness of Path and Board Transposition

**lemma**  $\text{transpose2: } \text{transpose-square } (\text{transpose-square } s_i) = s_i$

unfolding  $\text{transpose-square-def}$  by (auto split:  $\text{prod.splits}$ )

**lemma**  $\text{transpose-nil: } ps = [] \longleftrightarrow \text{transpose } ps = []$

using  $\text{transpose.elims}$  by blast

**lemma**  $\text{transpose-length: } \text{length } ps = \text{length } (\text{transpose } ps)$

```

    by (induction ps) auto

lemma hd-transpose: ps ≠ [] ⇒ hd (transpose ps) = transpose-square (hd ps)
  by (induction ps) (auto simp: transpose-square-def)

lemma last-transpose: ps ≠ [] ⇒ last (transpose ps) = transpose-square (last ps)
proof (induction ps)
  case (Cons si ps)
  then show ?case
  proof (cases ps = [])
    case True
    then show ?thesis using Cons by (auto simp: transpose-square-def)
  next
    case False
    then show ?thesis using Cons transpose-nil by auto
  qed
qed auto

lemma take-transpose:
  shows take k (transpose ps) = transpose (take k ps)
proof (induction ps arbitrary: k)
  case Nil
  then show ?case by auto
next
  case (Cons si ps)
  then obtain i j where si = (i,j) by force
  then have k = 0 ∨ k > 0 by auto
  then show ?case
  proof (elim disjE)
    assume k > 0
    then show ?thesis using Cons.IH by (auto simp: ⟨si = (i,j)⟩ take-Cons')
  qed auto
qed

lemma drop-transpose:
  shows drop k (transpose ps) = transpose (drop k ps)
proof (induction ps arbitrary: k)
  case Nil
  then show ?case by auto
next
  case (Cons si ps)
  then obtain i j where si = (i,j) by force
  then have k = 0 ∨ k > 0 by auto
  then show ?case
  proof (elim disjE)
    assume k > 0
    then show ?thesis using Cons.IH by (auto simp: ⟨si = (i,j)⟩ drop-Cons')
  qed auto
qed

```

**lemma** *transpose-board-correct*:  $s_i \in b \longleftrightarrow (\text{transpose-square } s_i) \in \text{transpose-board } b$   
**unfolding** *transpose-board-def* *transpose-square-def* **by** (*auto split: prod.splits*)

**lemma** *transpose-board*:  $\text{transpose-board } (\text{board } n \ m) = \text{board } m \ n$   
**unfolding** *board-def* **using** *transpose-board-correct* **by** (*auto simp: transpose-square-def*)

**lemma** *insert-transpose-board*:  
 $\text{insert } (\text{transpose-square } s_i) (\text{transpose-board } b) = \text{transpose-board } (\text{insert } s_i \ b)$   
**unfolding** *transpose-board-def* *transpose-square-def* **by** (*auto split: prod.splits*)

**lemma** *transpose-board2*:  $\text{transpose-board } (\text{transpose-board } b) = b$   
**unfolding** *transpose-board-def* **by** *auto*

**lemma** *transpose-union*:  $\text{transpose-board } (b_1 \cup b_2) = \text{transpose-board } b_1 \cup \text{transpose-board } b_2$   
**unfolding** *transpose-board-def* **by** *auto*

**lemma** *transpose-valid-step*:  
 $\text{valid-step } s_i \ s_j \longleftrightarrow \text{valid-step } (\text{transpose-square } s_i) (\text{transpose-square } s_j)$   
**unfolding** *valid-step-def* *transpose-square-def* **by** (*auto split: prod.splits*)

**lemma** *transpose-knights-path'*:  
**assumes** *knights-path* *b ps*  
**shows** *knights-path* ( $\text{transpose-board } b$ ) ( $\text{transpose } ps$ )  
**using** *assms*  
**proof** (*induction rule: knights-path.induct*)  
**case** (1  $s_i$ )  
**then have**  $\text{transpose-board } \{s_i\} = \{\text{transpose-square } s_i\}$   $\text{transpose } [s_i] = [\text{transpose-square } s_i]$   
**using** *transpose-board-correct* **by** (*auto simp: transpose-square-def split: prod.splits*)  
**then show** ?*case* **by** (*auto intro: knights-path.intros*)  
**next**  
**case** (2  $s_i \ b \ s_j \ ps$ )  
**then have** *prems*:  $\text{transpose-square } s_i \notin \text{transpose-board } b$   
 $\text{valid-step } (\text{transpose-square } s_i) (\text{transpose-square } s_j)$   
**and**  $\text{transpose } (s_j \# ps) = \text{transpose-square } s_j \# \text{transpose } ps$   
**using** 2 *transpose-board-correct* *transpose-valid-step* **by** *auto*  
**then show** ?*case*  
**using** 2 *knights-path.intros*(2)[*OF prems*] *insert-transpose-board* **by** *auto*  
**qed**

**corollary** *transpose-knights-path*:  
**assumes** *knights-path* ( $\text{board } n \ m$ ) *ps*  
**shows** *knights-path* ( $\text{board } m \ n$ ) ( $\text{transpose } ps$ )  
**using** *assms* *transpose-knights-path'*[*of board n m ps*] **by** (*auto simp: transpose-board*)



**corollary** *transpose-knights-circuit*:  
**assumes** *knights-circuit* (board  $n$   $m$ )  $ps$   
**shows** *knights-circuit* (board  $m$   $n$ ) (*transpose*  $ps$ )  
**using** *assms*  
**proof** –  
 have *knights-path* (board  $n$   $m$ )  $ps$  **and** *vs*: *valid-step* (last  $ps$ ) (hd  $ps$ )  
 using *assms* **unfolding** *knights-circuit-def* **by** *auto*  
 then have *kp-t*: *knights-path* (board  $m$   $n$ ) (*transpose*  $ps$ ) **and**  $ps \neq []$   
 using *transpose-knights-path* *knights-path-non-nil* **by** *auto*  
 then have *valid-step* (last (*transpose*  $ps$ )) (hd (*transpose*  $ps$ ))  
 using *vs* *hd-transpose* *last-transpose* *transpose-valid-step* **by** *auto*  
 then show *?thesis* **using** *kp-t* **by** (*auto simp: knights-circuit-def*)  
**qed**

## 5 Mirroring Paths and Boards

### 5.1 Implementation of Path and Board Mirroring

**abbreviation** *min1*  $ps \equiv \text{Min } ((fst) \text{ ' set } ps)$   
**abbreviation** *max1*  $ps \equiv \text{Max } ((fst) \text{ ' set } ps)$   
**abbreviation** *min2*  $ps \equiv \text{Min } ((snd) \text{ ' set } ps)$   
**abbreviation** *max2*  $ps \equiv \text{Max } ((snd) \text{ ' set } ps)$

**definition** *mirror1-square*  $:: int \Rightarrow square \Rightarrow square$  **where**  
*mirror1-square*  $n \ s_i = (\text{case } s_i \text{ of } (i,j) \Rightarrow (n-i,j))$

**fun** *mirror1-aux*  $:: int \Rightarrow path \Rightarrow path$  **where**  
*mirror1-aux*  $n \ [] = []$   
 $| \text{ mirror1-aux } n \ (s_i \# ps) = (\text{mirror1-square } n \ s_i) \# \text{mirror1-aux } n \ ps$

**definition** *mirror1*  $ps = \text{mirror1-aux } (\text{max1 } ps + \text{min1 } ps) \ ps$

**definition** *mirror1-board*  $:: int \Rightarrow board \Rightarrow board$  **where**  
*mirror1-board*  $n \ b \equiv \{ \text{mirror1-square } n \ s_i \mid s_i. s_i \in b \}$

**definition** *mirror2-square*  $:: int \Rightarrow square \Rightarrow square$  **where**  
*mirror2-square*  $m \ s_i = (\text{case } s_i \text{ of } (i,j) \Rightarrow (i,m-j))$

**fun** *mirror2-aux*  $:: int \Rightarrow path \Rightarrow path$  **where**  
*mirror2-aux*  $m \ [] = []$   
 $| \text{ mirror2-aux } m \ (s_i \# ps) = (\text{mirror2-square } m \ s_i) \# \text{mirror2-aux } m \ ps$

**definition** *mirror2*  $ps = \text{mirror2-aux } (\text{max2 } ps + \text{min2 } ps) \ ps$

**definition** *mirror2-board*  $:: int \Rightarrow board \Rightarrow board$  **where**  
*mirror2-board*  $m \ b \equiv \{ \text{mirror2-square } m \ s_i \mid s_i. s_i \in b \}$

## 5.2 Correctness of Path and Board Mirroring

**lemma** *mirror1-board-id*: *mirror1-board* (int  $n+1$ ) (*board*  $n$   $m$ ) = *board*  $n$   $m$  (is -  
= ?b)

**proof**

show *mirror1-board* (int  $n+1$ ) ?b  $\subseteq$  ?b

**proof**

fix  $s_i'$

assume *assms*:  $s_i' \in \text{mirror1-board } (int\ n+1)\ ?b$

then obtain  $i'\ j'$  where [simp]:  $s_i' = (i', j')$  **by** force

then have  $(i', j') \in \text{mirror1-board } (int\ n+1)\ ?b$

using *assms* **by** auto

then obtain  $i\ j$  where  $(i, j) \in ?b$  *mirror1-square* (int  $n+1$ )  $(i, j) = (i', j')$

unfolding *mirror1-board-def* **by** auto

then have  $1 \leq i \wedge i \leq int\ n\ 1 \leq j \wedge j \leq int\ m\ i' = (int\ n+1) - i\ j' = j$

unfolding *board-def mirror1-square-def* **by** auto

then have  $1 \leq i' \wedge i' \leq int\ n\ 1 \leq j' \wedge j' \leq int\ m$

**by** auto

then show  $s_i' \in ?b$

unfolding *board-def* **by** auto

**qed**

**next**

show ?b  $\subseteq \text{mirror1-board } (int\ n+1)\ ?b$

**proof**

fix  $s_i$

assume *assms*:  $s_i \in ?b$

then obtain  $i\ j$  where [simp]:  $s_i = (i, j)$  **by** force

then have  $(i, j) \in ?b$

using *assms* **by** auto

then have  $1 \leq i \wedge i \leq int\ n\ 1 \leq j \wedge j \leq int\ m$

unfolding *board-def* **by** auto

then obtain  $i'\ j'$  where  $i' = (int\ n+1) - i\ j' = j$  **by** auto

then have  $(i', j') \in ?b$  *mirror1-square* (int  $n+1$ )  $(i', j') = (i, j)$

using  $\langle 1 \leq i \wedge i \leq int\ n \rangle\ \langle 1 \leq j \wedge j \leq int\ m \rangle$

unfolding *mirror1-square-def* **by** (auto simp: *board-def*)

then show  $s_i \in \text{mirror1-board } (int\ n+1)\ ?b$

unfolding *mirror1-board-def* **by** force

**qed**

**qed**

**lemma** *mirror2-board-id*: *mirror2-board* (int  $m+1$ ) (*board*  $n$   $m$ ) = *board*  $n$   $m$  (is -  
= ?b)

**proof**

show *mirror2-board* (int  $m+1$ ) ?b  $\subseteq$  ?b

**proof**

fix  $s_i'$

assume *assms*:  $s_i' \in \text{mirror2-board } (int\ m+1)\ ?b$

then obtain  $i'\ j'$  where [simp]:  $s_i' = (i', j')$  **by** force

then have  $(i', j') \in \text{mirror2-board } (int\ m+1)\ ?b$

using *assms* **by** auto

```

then obtain  $i\ j$  where  $(i,j) \in ?b$  mirror2-square  $(\text{int } m+1)$   $(i,j) = (i',j')$ 
  unfolding mirror2-board-def by auto
then have  $1 \leq i \wedge i \leq \text{int } n$   $1 \leq j \wedge j \leq \text{int } m$   $i'=i$   $j'=(\text{int } m+1)-j$ 
  unfolding board-def mirror2-square-def by auto
then have  $1 \leq i' \wedge i' \leq \text{int } n$   $1 \leq j' \wedge j' \leq \text{int } m$ 
  by auto
then show  $s_i' \in ?b$ 
  unfolding board-def by auto
qed
next
show  $?b \subseteq \text{mirror2-board } (\text{int } m+1)$   $?b$ 
proof
  fix  $s_i$ 
  assume  $\text{assms}: s_i \in ?b$ 
  then obtain  $i\ j$  where  $[\text{simp}]: s_i = (i,j)$  by force
  then have  $(i,j) \in ?b$ 
    using  $\text{assms}$  by auto
  then have  $1 \leq i \wedge i \leq \text{int } n$   $1 \leq j \wedge j \leq \text{int } m$ 
    unfolding board-def by auto
  then obtain  $i'\ j'$  where  $i'=i$   $j'=(\text{int } m+1)-j$  by auto
  then have  $(i',j') \in ?b$  mirror2-square  $(\text{int } m+1)$   $(i',j') = (i,j)$ 
    using  $\langle 1 \leq i \wedge i \leq \text{int } n \rangle$   $\langle 1 \leq j \wedge j \leq \text{int } m \rangle$ 
    unfolding mirror2-square-def by (auto simp: board-def)
  then show  $s_i \in \text{mirror2-board } (\text{int } m+1)$   $?b$ 
    unfolding mirror2-board-def by force
qed
qed

```

**lemma** *knights-path-min1: knights-path (board  $n\ m$ )  $ps \implies \text{min1 } ps = 1$*

**proof** –

```

assume  $\text{assms}: \text{knights-path } (\text{board } n\ m)$   $ps$ 
then have  $\text{min } n\ m \geq 1$ 
  using knights-path-board-m-n-geq-1 by auto
then have  $(1,1) \in \text{board } n\ m$  and  $\text{ge-1}: \forall (i,j) \in \text{board } n\ m. i \geq 1$ 
  unfolding board-def by auto
then have  $\text{finite}: \text{finite } ((\text{fst}) \text{ ` board } n\ m)$  and
   $\text{non-empty}: (\text{fst}) \text{ ` board } n\ m \neq \{\}$  and
   $\text{mem-1}: 1 \in (\text{fst}) \text{ ` board } n\ m$ 
  using board-finite by auto (metis fstI image-eqI)
then have  $\text{Min } ((\text{fst}) \text{ ` board } n\ m) = 1$ 
  using  $\text{ge-1}$  by (auto simp: Min-eq-iff)
then show  $?thesis$ 
  using  $\text{assms}$  knights-path-set-eq by auto
qed

```

**lemma** *knights-path-min2: knights-path (board  $n\ m$ )  $ps \implies \text{min2 } ps = 1$*

**proof** –

```

assume  $\text{assms}: \text{knights-path } (\text{board } n\ m)$   $ps$ 
then have  $\text{min } n\ m \geq 1$ 

```

```

    using knights-path-board-m-n-geq-1 by auto
  then have  $(1,1) \in \text{board } n \ m$  and  $ge-1: \forall (i,j) \in \text{board } n \ m. j \geq 1$ 
    unfolding board-def by auto
  then have finite: finite  $((snd) \text{ ' board } n \ m)$  and
    non-empty:  $(snd) \text{ ' board } n \ m \neq \{\}$  and
    mem-1:  $1 \in (snd) \text{ ' board } n \ m$ 
    using board-finite by auto (metis sndI image-eqI)
  then have Min  $((snd) \text{ ' board } n \ m) = 1$ 
    using ge-1 by (auto simp: Min-eq-iff)
  then show ?thesis
    using assms knights-path-set-eq by auto
qed

```

```

lemma knights-path-max1: knights-path (board n m) ps  $\implies$  max1 ps = int n
proof -
  assume assms: knights-path (board n m) ps
  then have min n m  $\geq 1$ 
    using knights-path-board-m-n-geq-1 by auto
  then have  $(int \ n, 1) \in \text{board } n \ m$  and leq-n:  $\forall (i,j) \in \text{board } n \ m. i \leq int \ n$ 
    unfolding board-def by auto
  then have finite: finite  $((fst) \text{ ' board } n \ m)$  and
    non-empty:  $(fst) \text{ ' board } n \ m \neq \{\}$  and
    mem-1:  $int \ n \in (fst) \text{ ' board } n \ m$ 
    using board-finite by auto (metis fstI image-eqI)
  then have Max  $((fst) \text{ ' board } n \ m) = int \ n$ 
    using leq-n by (auto simp: Max-eq-iff)
  then show ?thesis
    using assms knights-path-set-eq by auto
qed

```

```

lemma knights-path-max2: knights-path (board n m) ps  $\implies$  max2 ps = int m
proof -
  assume assms: knights-path (board n m) ps
  then have min n m  $\geq 1$ 
    using knights-path-board-m-n-geq-1 by auto
  then have  $(1, int \ m) \in \text{board } n \ m$  and leq-m:  $\forall (i,j) \in \text{board } n \ m. j \leq int \ m$ 
    unfolding board-def by auto
  then have finite: finite  $((snd) \text{ ' board } n \ m)$  and
    non-empty:  $(snd) \text{ ' board } n \ m \neq \{\}$  and
    mem-1:  $int \ m \in (snd) \text{ ' board } n \ m$ 
    using board-finite by auto (metis sndI image-eqI)
  then have Max  $((snd) \text{ ' board } n \ m) = int \ m$ 
    using leq-m by (auto simp: Max-eq-iff)
  then show ?thesis
    using assms knights-path-set-eq by auto
qed

```

```

lemma mirror1-aux-nil: ps = []  $\longleftrightarrow$  mirror1-aux m ps = []
  using mirror1-aux.elims by blast

```

**lemma** *mirror1-nil*:  $ps = [] \longleftrightarrow \text{mirror1 } ps = []$   
**unfolding** *mirror1-def* **using** *mirror1-aux-nil* **by** *blast*

**lemma** *mirror2-aux-nil*:  $ps = [] \longleftrightarrow \text{mirror2-aux } m \text{ } ps = []$   
**using** *mirror2-aux.elims* **by** *blast*

**lemma** *mirror2-nil*:  $ps = [] \longleftrightarrow \text{mirror2 } ps = []$   
**unfolding** *mirror2-def* **using** *mirror2-aux-nil* **by** *blast*

**lemma** *length-mirror1-aux*:  $\text{length } ps = \text{length } (\text{mirror1-aux } n \text{ } ps)$   
**by** (*induction ps*) *auto*

**lemma** *length-mirror1*:  $\text{length } ps = \text{length } (\text{mirror1 } ps)$   
**unfolding** *mirror1-def* **using** *length-mirror1-aux* **by** *auto*

**lemma** *length-mirror2-aux*:  $\text{length } ps = \text{length } (\text{mirror2-aux } n \text{ } ps)$   
**by** (*induction ps*) *auto*

**lemma** *length-mirror2*:  $\text{length } ps = \text{length } (\text{mirror2 } ps)$   
**unfolding** *mirror2-def* **using** *length-mirror2-aux* **by** *auto*

**lemma** *mirror1-board-iff*:  $s_i \notin b \longleftrightarrow \text{mirror1-square } n \text{ } s_i \notin \text{mirror1-board } n \text{ } b$   
**unfolding** *mirror1-board-def* *mirror1-square-def* **by** (*auto split: prod.splits*)

**lemma** *mirror2-board-iff*:  $s_i \notin b \longleftrightarrow \text{mirror2-square } n \text{ } s_i \notin \text{mirror2-board } n \text{ } b$   
**unfolding** *mirror2-board-def* *mirror2-square-def* **by** (*auto split: prod.splits*)

**lemma** *insert-mirror1-board*:  
 $\text{insert } (\text{mirror1-square } n \text{ } s_i) (\text{mirror1-board } n \text{ } b) = \text{mirror1-board } n (\text{insert } s_i \text{ } b)$   
**unfolding** *mirror1-board-def* *mirror1-square-def* **by** (*auto split: prod.splits*)

**lemma** *insert-mirror2-board*:  
 $\text{insert } (\text{mirror2-square } n \text{ } s_i) (\text{mirror2-board } n \text{ } b) = \text{mirror2-board } n (\text{insert } s_i \text{ } b)$   
**unfolding** *mirror2-board-def* *mirror2-square-def* **by** (*auto split: prod.splits*)

**lemma**  $(i::\text{int}) = i' + 1 \implies n - i = n - (i' + 1)$   
**by** *auto*

**lemma** *valid-step-mirror1*:  
 $\text{valid-step } s_i \text{ } s_j \longleftrightarrow \text{valid-step } (\text{mirror1-square } n \text{ } s_i) (\text{mirror1-square } n \text{ } s_j)$   
**proof**  
**assume** *assms*: *valid-step*  $s_i \text{ } s_j$   
**obtain**  $i \text{ } j \text{ } i' \text{ } j'$  **where** [*simp*]:  $s_i = (i, j) \text{ } s_j = (i', j')$  **by** *force*  
**then have** *valid-step*  $(n - i, j) (n - i', j')$   
**using** *assms* **unfolding** *valid-step-def*  
**apply** *simp*  
**apply** (*elim disjE*)  
**apply** *auto*

```

    done
  then show valid-step (mirror1-square n si) (mirror1-square n sj)
    unfolding mirror1-square-def by auto
next
assume assms: valid-step (mirror1-square n si) (mirror1-square n sj)
obtain i j i' j' where [simp]: si = (i,j) sj = (i',j') by force
then have valid-step (i,j) (i',j')
  using assms unfolding valid-step-def mirror1-square-def
  apply simp
  apply (elim disjE)
  apply auto
done
then show valid-step si sj
  unfolding mirror1-square-def by auto
qed

```

```

lemma valid-step-mirror2:
  valid-step si sj  $\longleftrightarrow$  valid-step (mirror2-square m si) (mirror2-square m sj)
proof
  assume assms: valid-step si sj
  obtain i j i' j' where [simp]: si = (i,j) sj = (i',j') by force
  then have valid-step (i,m-j) (i',m-j')
    using assms unfolding valid-step-def
    apply simp
    apply (elim disjE)
    apply auto
  done
  then show valid-step (mirror2-square m si) (mirror2-square m sj)
    unfolding mirror2-square-def by auto
next
assume assms: valid-step (mirror2-square m si) (mirror2-square m sj)
obtain i j i' j' where [simp]: si = (i,j) sj = (i',j') by force
then have valid-step (i,j) (i',j')
  using assms unfolding valid-step-def mirror2-square-def
  apply simp
  apply (elim disjE)
  apply auto
done
then show valid-step si sj
  unfolding mirror1-square-def by auto
qed

```

```

lemma hd-mirror1:
  assumes knights-path (board n m) ps hd ps = (i,j)
  shows hd (mirror1 ps) = (int n+1-i,j)
  using assms
proof -
  have hd (mirror1 ps) = hd (mirror1-aux (int n+1) ps)
    unfolding mirror1-def using assms knights-path-min1 knights-path-max1 by

```

```

auto
  also have ... = hd (mirror1-aux (int n+1) ((hd ps)#(tl ps)))
    using assms knights-path-non-nil by (metis list.collapse)
  also have ... = (int n+1-i,j)
    using assms by (auto simp: mirror1-square-def)
  finally show ?thesis .
qed

lemma last-mirror1-aux:
  assumes ps ≠ [] last ps = (i,j)
  shows last (mirror1-aux n ps) = (n-i,j)
  using assms
proof (induction ps)
  case (Cons si ps)
  then show ?case
    using mirror1-aux-nil Cons by (cases ps = []) (auto simp: mirror1-square-def)
qed auto

lemma last-mirror1:
  assumes knights-path (board n m) ps last ps = (i,j)
  shows last (mirror1 ps) = (int n+1-i,j)
  unfolding mirror1-def using assms last-mirror1-aux knights-path-non-nil
  by (simp add: knights-path-max1 knights-path-min1)

lemma hd-mirror2:
  assumes knights-path (board n m) ps hd ps = (i,j)
  shows hd (mirror2 ps) = (i,int m+1-j)
  using assms
proof -
  have hd (mirror2 ps) = hd (mirror2-aux (int m+1) ps)
    unfolding mirror2-def using assms knights-path-min2 knights-path-max2 by
auto
  also have ... = hd (mirror2-aux (int m+1) ((hd ps)#(tl ps)))
    using assms knights-path-non-nil by (metis list.collapse)
  also have ... = (i,int m+1-j)
    using assms by (auto simp: mirror2-square-def)
  finally show ?thesis .
qed

lemma last-mirror2-aux:
  assumes ps ≠ [] last ps = (i,j)
  shows last (mirror2-aux m ps) = (i,m-j)
  using assms
proof (induction ps)
  case (Cons si ps)
  then show ?case
    using mirror2-aux-nil Cons by (cases ps = []) (auto simp: mirror2-square-def)
qed auto

```

```

lemma last-mirror2:
  assumes knights-path (board n m) ps last ps = (i,j)
  shows last (mirror2 ps) = (i,int m+1-j)
  unfolding mirror2-def using assms last-mirror2-aux knights-path-non-nil
  by (simp add: knights-path-max2 knights-path-min2)

lemma mirror1-aux-knights-path:
  assumes knights-path b ps
  shows knights-path (mirror1-board n b) (mirror1-aux n ps)
  using assms
proof (induction rule: knights-path.induct)
  case (1 si)
  then have mirror1-board n {si} = {mirror1-square n si}
    unfolding mirror1-board-def by blast
  then show ?case by (auto intro: knights-path.intros)
next
  case (2 si b sj ps)
  then have prems: mirror1-square n si ∉ mirror1-board n b
    valid-step (mirror1-square n si) (mirror1-square n sj)
    and mirror1-aux n (sj#ps) = mirror1-square n sj#mirror1-aux n ps
    using 2 mirror1-board-iff valid-step-mirror1 by auto
  then show ?case
    using 2 knights-path.intros(2)[OF prems] insert-mirror1-board by auto
qed

corollary mirror1-knights-path:
  assumes knights-path (board n m) ps
  shows knights-path (board n m) (mirror1 ps)
  using assms
proof –
  have [simp]: min1 ps = 1 max1 ps = int n
    using assms knights-path-min1 knights-path-max1 by auto
  then have mirror1-board (int n+1) (board n m) = (board n m)
    using mirror1-board-id by auto
  then have knights-path (board n m) (mirror1-aux (int n+1) ps)
    using assms mirror1-aux-knights-path[of board n m ps int n+1] by auto
  then show ?thesis unfolding mirror1-def by auto
qed

lemma mirror2-aux-knights-path:
  assumes knights-path b ps
  shows knights-path (mirror2-board n b) (mirror2-aux n ps)
  using assms
proof (induction rule: knights-path.induct)
  case (1 si)
  then have mirror2-board n {si} = {mirror2-square n si}
    unfolding mirror2-board-def by blast
  then show ?case by (auto intro: knights-path.intros)
next

```



```

case (2 si b sj ps)
then have prems: mirror2-square n si ∉ mirror2-board n b
      valid-step (mirror2-square n si) (mirror2-square n sj)
      and mirror2-aux n (sj#ps) = mirror2-square n sj#mirror2-aux n ps
      using 2 mirror2-board-iff valid-step-mirror2 by auto
then show ?case
      using 2 knights-path.intros(2)[OF prems] insert-mirror2-board by auto
qed

```

**corollary** *mirror2-knights-path*:

```

assumes knights-path (board n m) ps
shows knights-path (board n m) (mirror2 ps)
proof -
have [simp]: min2 ps = 1 max2 ps = int m
  using assms knights-path-min2 knights-path-max2 by auto
then have mirror2-board (int m+1) (board n m) = (board n m)
  using mirror2-board-id by auto
then have knights-path (board n m) (mirror2-aux (int m+1) ps)
  using assms mirror2-aux-knights-path[of board n m ps int m+1] by auto
then show ?thesis unfolding mirror2-def by auto
qed

```

### 5.3 Rotate Knight's Paths

Transposing (*transpose*) and mirroring (along first axis *mirror1*) a Knight's path preserves the Knight's path's property. Tranpose+Mirror1 equals a 90deg-clockwise turn.

**corollary** *rot90-knights-path*:

```

assumes knights-path (board n m) ps
shows knights-path (board m n) (mirror1 (transpose ps))
using assms transpose-knights-path mirror1-knights-path by auto

```

**lemma** *hd-rot90-knights-path*:

```

assumes knights-path (board n m) ps hd ps = (i,j)
shows hd (mirror1 (transpose ps)) = (int m+1-j,i)
using assms
proof -
have hd (transpose ps) = (j,i) knights-path (board m n) (transpose ps)
  using assms knights-path-non-nil hd-transpose transpose-knights-path
  by (auto simp: transpose-square-def)
then show ?thesis using hd-mirror1 by auto
qed

```

**lemma** *last-rot90-knights-path*:

```

assumes knights-path (board n m) ps last ps = (i,j)
shows last (mirror1 (transpose ps)) = (int m+1-j,i)
using assms
proof -
have last (transpose ps) = (j,i) knights-path (board m n) (transpose ps)

```

```

    using assms knights-path-non-nil last-transpose transpose-knights-path
    by (auto simp: transpose-square-def)
  then show ?thesis using last-mirror1 by auto
qed

```

## 6 Translating Paths and Boards

When constructing knight's paths for larger boards multiple knight's paths for smaller boards are concatenated. To concatenate paths the coordinates in the path need to be translated. Therefore, simple auxiliary functions are provided.

### 6.1 Implementation of Path and Board Translation

Translate the coordinates for a path by  $(k_1, k_2)$ .

```

fun trans-path :: int × int ⇒ path ⇒ path where
  trans-path (k1, k2) [] = []
| trans-path (k1, k2) ((i, j) # xs) = (i + k1, j + k2) # (trans-path (k1, k2) xs)

```

Translate the coordinates of a board by  $(k_1, k_2)$ .

```

definition trans-board :: int × int ⇒ board ⇒ board where
  trans-board t b ≡ (case t of (k1, k2) ⇒ {(i + k1, j + k2) | i j. (i, j) ∈ b})

```

### 6.2 Correctness of Path and Board Translation

```

lemma trans-path-length: length ps = length (trans-path (k1, k2) ps)
  by (induction ps) auto

```

```

lemma trans-path-non-nil: ps ≠ [] ⟹ trans-path (k1, k2) ps ≠ []
  by (induction ps) auto

```

```

lemma trans-path-correct: (i, j) ∈ set ps ⟷ (i + k1, j + k2) ∈ set (trans-path (k1, k2) ps)

```

```

proof (induction ps)
  case (Cons si ps)
  then show ?case by (cases si) auto
qed auto

```

```

lemma trans-path-non-nil-last:
  ps ≠ [] ⟹ last (trans-path (k1, k2) ps) = last (trans-path (k1, k2) ((i, j) # ps))
  using trans-path-non-nil by (induction ps) auto

```

```

lemma hd-trans-path:
  assumes ps ≠ []
  shows hd (trans-path (k1, k2) ps) = (i + k1, j + k2)
  using assms by (induction ps) auto

```

```

lemma last-trans-path:
  assumes  $ps \neq []$   $last\ ps = (i,j)$ 
  shows  $last\ (trans-path\ (k_1,k_2)\ ps) = (i+k_1,j+k_2)$ 
  using assms
proof (induction ps)
  case (Cons si ps)
  then show ?case
    using trans-path-non-nil-last[symmetric]
    apply (cases si)
    apply (cases ps = [])
    apply auto
    done
qed (auto)

lemma take-trans:
  shows  $take\ k\ (trans-path\ (k_1,k_2)\ ps) = trans-path\ (k_1,k_2)\ (take\ k\ ps)$ 
proof (induction ps arbitrary: k)
  case Nil
  then show ?case by auto
next
  case (Cons si ps)
  then obtain i j where  $s_i = (i,j)$  by force
  then have  $k = 0 \vee k > 0$  by auto
  then show ?case
  proof (elim disjE)
    assume  $k > 0$ 
    then show ?thesis using Cons.IH by (auto simp:  $\langle s_i = (i,j) \rangle take-Cons'$ )
  qed auto
qed

lemma drop-trans:
  shows  $drop\ k\ (trans-path\ (k_1,k_2)\ ps) = trans-path\ (k_1,k_2)\ (drop\ k\ ps)$ 
proof (induction ps arbitrary: k)
  case Nil
  then show ?case by auto
next
  case (Cons si ps)
  then obtain i j where  $s_i = (i,j)$  by force
  then have  $k = 0 \vee k > 0$  by auto
  then show ?case
  proof (elim disjE)
    assume  $k > 0$ 
    then show ?thesis using Cons.IH by (auto simp:  $\langle s_i = (i,j) \rangle drop-Cons'$ )
  qed auto
qed

lemma trans-board-correct:  $(i,j) \in b \longleftrightarrow (i+k_1,j+k_2) \in trans-board\ (k_1,k_2)\ b$ 
  unfolding trans-board-def by auto

```

**lemma** *board-subset*:  $n_1 \leq n_2 \implies m_1 \leq m_2 \implies \text{board } n_1 \ m_1 \subseteq \text{board } n_2 \ m_2$   
**unfolding** *board-def* **by** *auto*

Board concatenation

**corollary** *board-concat*:

**shows**  $\text{board } n \ m_1 \cup \text{trans-board } (0, \text{int } m_1) (\text{board } n \ m_2) = \text{board } n \ (m_1 + m_2)$   
**(is**  $?b1 \cup ?b2 = ?b$ )

**proof**

**show**  $?b1 \cup ?b2 \subseteq ?b$  **unfolding** *board-def trans-board-def* **by** *auto*

**next**

**show**  $?b \subseteq ?b1 \cup ?b2$

**proof**

**fix**  $x$

**assume**  $x \in ?b$

**then obtain**  $i \ j$  **where**  $x\text{-split}$ :  $x = (i, j) \ 1 \leq i \wedge i \leq \text{int } n \ 1 \leq j \wedge j \leq \text{int } (m_1 + m_2)$

**unfolding** *board-def* **by** *auto*

**then have**  $j \leq \text{int } m_1 \vee (\text{int } m_1 < j \wedge j \leq \text{int } (m_1 + m_2))$  **by** *auto*

**then show**  $x \in ?b1 \cup ?b2$

**proof**

**assume**  $j \leq \text{int } m_1$

**then show**  $x \in ?b1 \cup ?b2$  **using**  $x\text{-split}$  **unfolding** *board-def* **by** *auto*

**next**

**assume**  $asm$ :  $\text{int } m_1 < j \wedge j \leq \text{int } (m_1 + m_2)$

**then have**  $(i, j - \text{int } m_1) \in \text{board } n \ m_2$  **using**  $x\text{-split}$  **unfolding** *board-def* **by** *auto*

**then show**  $x \in ?b1 \cup ?b2$

**using**  $x\text{-split } asm \text{ trans-board-correct}[of \ i \ j - \text{int } m_1 \ \text{board } n \ m_2 \ 0 \ \text{int } m_1]$  **by**

*auto*

**qed**

**qed**

**qed**

**lemma** *transpose-trans-board*:

$\text{transpose-board } (\text{trans-board } (k_1, k_2) \ b) = \text{trans-board } (k_2, k_1) (\text{transpose-board } b)$

**unfolding** *transpose-board-def trans-board-def* **by** *blast*

**corollary** *board-concatT*:

**shows**  $\text{board } n_1 \ m \cup \text{trans-board } (\text{int } n_1, 0) (\text{board } n_2 \ m) = \text{board } (n_1 + n_2) \ m$  **(is**  $?b_1 \cup ?b_2 = ?b$ )

**proof** –

**let**  $?b_1 T = \text{board } m \ n_1$

**let**  $?b_2 T = \text{trans-board } (0, \text{int } n_1) (\text{board } m \ n_2)$

**have**  $?b_1 \cup ?b_2 = \text{transpose-board } (?b_1 T \cup ?b_2 T)$

**using** *transpose-board2 transpose-union transpose-board transpose-trans-board*

**by** *auto*

**also have**  $\dots = \text{transpose-board } (\text{board } m \ (n_1 + n_2))$

**using** *board-concat* **by** *auto*

**also have**  $\dots = \text{board } (n_1 + n_2) \ m$

using transpose-board by auto  
 finally show ?thesis .  
 qed

**lemma** trans-valid-step:  
 valid-step (i,j) (i',j')  $\implies$  valid-step (i+k<sub>1</sub>,j+k<sub>2</sub>) (i'+k<sub>1</sub>,j'+k<sub>2</sub>)  
 unfolding valid-step-def by auto

Translating a path and a boards preserves the validity.

**lemma** trans-knights-path:  
 assumes knights-path b ps  
 shows knights-path (trans-board (k<sub>1</sub>,k<sub>2</sub>) b) (trans-path (k<sub>1</sub>,k<sub>2</sub>) ps)  
 using assms  
**proof** (induction rule: knights-path.induct)  
 case (2 s<sub>i</sub> b s<sub>j</sub> xs)  
 then obtain i j i' j' where split: s<sub>i</sub> = (i,j) s<sub>j</sub> = (i',j') by force  
 let ?s<sub>i</sub>=(i+k<sub>1</sub>,j+k<sub>2</sub>)  
 let ?s<sub>j</sub>=(i'+k<sub>1</sub>,j'+k<sub>2</sub>)  
 let ?xs=trans-path (k<sub>1</sub>,k<sub>2</sub>) xs  
 let ?b=trans-board (k<sub>1</sub>,k<sub>2</sub>) b  
 have simps: trans-path (k<sub>1</sub>,k<sub>2</sub>) (s<sub>i</sub>#s<sub>j</sub>#xs) = ?s<sub>i</sub>#?s<sub>j</sub>#?xs  
               ?b ∪ {?s<sub>i</sub>} = trans-board (k<sub>1</sub>,k<sub>2</sub>) (b ∪ {s<sub>i</sub>})  
 unfolding trans-board-def using split by auto  
 have ?s<sub>i</sub> ∉ ?b valid-step ?s<sub>i</sub> ?s<sub>j</sub> knights-path ?b (?s<sub>j</sub>#?xs)  
 using 2 split trans-valid-step by (auto simp: trans-board-def)  
 then have knights-path (?b ∪ {?s<sub>i</sub>}) (?s<sub>i</sub>#?s<sub>j</sub>#?xs)  
 using knights-path.intros by auto  
 then show ?case using simps by auto  
**qed** (auto simp: trans-board-def intro: knights-path.intros)

Predicate that indicates if two squares s<sub>i</sub> and s<sub>j</sub> are adjacent in ps.

**definition** step-in :: path  $\Rightarrow$  square  $\Rightarrow$  square  $\Rightarrow$  bool **where**  
 step-in ps s<sub>i</sub> s<sub>j</sub>  $\equiv$  ( $\exists k. 0 < k \wedge k < \text{length } ps \wedge \text{last } (\text{take } k \text{ } ps) = s_i \wedge \text{hd } (\text{drop } k \text{ } ps) = s_j$ )

**lemma** step-in-Cons: step-in ps s<sub>i</sub> s<sub>j</sub>  $\implies$  step-in (s<sub>k</sub>#ps) s<sub>i</sub> s<sub>j</sub>  
**proof** –  
 assume step-in ps s<sub>i</sub> s<sub>j</sub>  
 then obtain k where 0 < k  $\wedge$  k < length ps last (take k ps) = s<sub>i</sub> hd (drop k ps) = s<sub>j</sub>  
 unfolding step-in-def by auto  
 then have 0 < k+1  $\wedge$  k+1 < length (s<sub>k</sub>#ps)  
           last (take (k+1) (s<sub>k</sub>#ps)) = s<sub>i</sub> hd (drop (k+1) (s<sub>k</sub>#ps)) = s<sub>j</sub>  
 by auto  
 then show ?thesis  
 by (auto simp: step-in-def)  
**qed**

**lemma** step-in-append: step-in ps s<sub>i</sub> s<sub>j</sub>  $\implies$  step-in (ps@ps') s<sub>i</sub> s<sub>j</sub>

**proof** –  
 assume *step-in ps s<sub>i</sub> s<sub>j</sub>*  
 then obtain *k* where  $0 < k \wedge k < \text{length } ps \text{ last } (\text{take } k \text{ } ps) = s_i \text{ hd } (\text{drop } k \text{ } ps) = s_j$   
 unfolding *step-in-def* by *auto*  
 then have  $0 < k \wedge k < \text{length } (ps@ps')$   
 $\text{last } (\text{take } k \text{ } (ps@ps')) = s_i \text{ hd } (\text{drop } k \text{ } (ps@ps')) = s_j$   
 by *auto*  
 then show *?thesis*  
 by (*auto simp: step-in-def*)  
**qed**

**lemma** *step-in-prepend: step-in ps s<sub>i</sub> s<sub>j</sub>  $\implies$  step-in (ps'@ps) s<sub>i</sub> s<sub>j</sub>*  
 using *step-in-Cons* by (*induction ps' arbitrary: ps*) *auto*

**lemma** *step-in-valid-step: knights-path b ps  $\implies$  step-in ps s<sub>i</sub> s<sub>j</sub>  $\implies$  valid-step s<sub>i</sub> s<sub>j</sub>*

**proof** –  
 assume *assms: knights-path b ps step-in ps s<sub>i</sub> s<sub>j</sub>*  
 then obtain *k* where *k-prems*:  $0 < k \wedge k < \text{length } ps \text{ last } (\text{take } k \text{ } ps) = s_i \text{ hd } (\text{drop } k \text{ } ps) = s_j$   
 unfolding *step-in-def* by *auto*  
 then have  $k = 1 \vee k > 1$  by *auto*  
 then show *?thesis*  
**proof** (*elim disjE*)  
 assume  $k = 1$   
 then obtain *ps'* where  $ps = s_i \# s_j \# ps'$   
 using *k-prems list-len-g-1-split* by *fastforce*  
 then show *?thesis*  
 using *assms* by (*auto elim: knights-path.cases*)  
**next**  
 assume  $k > 1$   
 then have  $0 < k-1 \wedge k-1 < \text{length } ps$   
 using *k-prems* by *auto*  
 then obtain *b* where *knights-path b* ( $\text{drop } (k-1) \text{ } ps$ )  
 using *assms knights-path-split* by *blast*  
  
 obtain *ps'* where  $\text{drop } (k-1) \text{ } ps = s_i \# s_j \# ps'$   
 using *k-prems*  $\langle 0 < k-1 \wedge k-1 < \text{length } ps \rangle$   
 by (*metis Cons-nth-drop-Suc Suc-diff-1 hd-drop-conv-nth last-snoc take-hd-drop*)  
 then show *?thesis*  
 using  $\langle \text{knights-path } b \text{ } (\text{drop } (k-1) \text{ } ps) \rangle$  by (*auto elim: knights-path.cases*)  
**qed**  
**qed**

**lemma** *trans-step-in:*  
*step-in ps (i,j) (i',j')  $\implies$  step-in (trans-path (k<sub>1</sub>,k<sub>2</sub>) ps) (i+k<sub>1</sub>,j+k<sub>2</sub>) (i'+k<sub>1</sub>,j'+k<sub>2</sub>)*  
**proof** –  
 let *?ps'=trans-path (k<sub>1</sub>,k<sub>2</sub>) ps*

**assume** *step-in*  $ps$   $(i,j)$   $(i',j')$   
**then obtain**  $k$  **where**  $0 < k \wedge k < \text{length } ps$   $\text{last } (take\ k\ ps) = (i,j)$   $\text{hd } (drop\ k\ ps) = (i',j')$   
**unfolding** *step-in-def* **by** *auto*  
**then have**  $take\ k\ ps \neq []$   $drop\ k\ ps \neq []$  **by** *fastforce+*  
**then have**  $0 < k \wedge k < \text{length } ?ps'$   
 $\text{last } (take\ k\ ?ps') = (i+k_1, j+k_2)$   $\text{hd } (drop\ k\ ?ps') = (i'+k_1, j'+k_2)$   
**using** *trans-path-length*  
 $\text{last-trans-path}[OF\ \langle take\ k\ ps \neq [] \rangle\ \langle \text{last } (take\ k\ ps) = (i,j) \rangle]\ \text{take-trans}$   
 $\text{hd-trans-path}[OF\ \langle drop\ k\ ps \neq [] \rangle\ \langle \text{hd } (drop\ k\ ps) = (i',j') \rangle]\ \text{drop-trans}$   
**by** *auto*  
**then show** *?thesis*  
**by** (*auto simp: step-in-def*)  
**qed**

**lemma** *transpose-step-in*:  
 $\text{step-in } ps\ s_i\ s_j \implies \text{step-in } (\text{transpose } ps)\ (\text{transpose-square } s_i)\ (\text{transpose-square } s_j)$   
**(is**  $- \implies \text{step-in } ?psT\ ?s_iT\ ?s_jT)$   
**proof**  $-$   
**assume** *step-in*  $ps\ s_i\ s_j$   
**then obtain**  $k$  **where**  
 $k\text{-prems: } 0 < k \wedge k < \text{length } ps$   $\text{last } (take\ k\ ps) = s_i$   $\text{hd } (drop\ k\ ps) = s_j$   
**unfolding** *step-in-def* **by** *auto*  
**then have** *non-nil*:  $take\ k\ ps \neq []$   $drop\ k\ ps \neq []$  **by** *fastforce+*  
**have**  $take\ k\ ?psT = \text{transpose } (take\ k\ ps)$   $drop\ k\ ?psT = \text{transpose } (drop\ k\ ps)$   
**using** *take-transpose drop-transpose* **by** *auto*  
**then have**  $\text{last } (take\ k\ ?psT) = ?s_iT$   $\text{hd } (drop\ k\ ?psT) = ?s_jT$   
**using** *non-nil k-prems hd-transpose last-transpose* **by** *auto*  
**then show** *step-in*  $?psT\ ?s_iT\ ?s_jT$   
**unfolding** *step-in-def* **using** *k-prems transpose-length* **by** *auto*  
**qed**

**lemma** *hd-take*:  $0 < k \implies \text{hd } xs = \text{hd } (take\ k\ xs)$   
**by** (*induction xs*) *auto*

**lemma** *last-drop*:  $k < \text{length } xs \implies \text{last } xs = \text{last } (drop\ k\ xs)$   
**by** (*induction xs*) *auto*

### 6.3 Concatenate Knight's Paths and Circuits

Concatenate two knight's path on a  $n \times m$ -board along the 2nd axis if the first path contains the step  $s_i \rightarrow s_j$  and there are valid steps  $s_i \rightarrow \text{hd } ps_2'$  and  $s_j \rightarrow \text{last } ps_2'$ , where  $ps_2'$  is  $ps_2$  is translated by  $m_1$ . An arbitrary step in  $ps_2$  is preserved.

**corollary** *knights-path-split-concat-si-prev*:

**assumes** *knights-path* (*board*  $n\ m_1$ )  $ps_1$  *knights-path* (*board*  $n\ m_2$ )  $ps_2$   
 $\text{step-in } ps_1\ s_i\ s_j$   $\text{hd } ps_2 = (i_h, j_h)$   $\text{last } ps_2 = (i_l, j_l)$  *step-in*  $ps_2\ (i,j)\ (i',j')$

$valid\_step\ s_i\ (i_h, int\ m_1+j_h)\ valid\_step\ (i_l, int\ m_1+j_l)\ s_j$   
**shows**  $\exists ps. knights\_path\ (board\ n\ (m_1+m_2))\ ps \wedge hd\ ps = hd\ ps_1$   
 $\wedge last\ ps = last\ ps_1 \wedge step\_in\ ps\ (i, int\ m_1+j)\ (i', int\ m_1+j')$   
**using** *assms*  
**proof** –  
**let**  $?b_1 = board\ n\ m_1$   
**let**  $?b_2 = board\ n\ m_2$   
**let**  $?ps_2' = trans\_path\ (0, int\ m_1)\ ps_2$   
**let**  $?b' = trans\_board\ (0, int\ m_1)\ ?b_2$   
**have**  $kp2': knights\_path\ ?b'\ ?ps_2'$  **using** *assms trans-knights-path* **by** *auto*  
**then have**  $?ps_2' \neq []$  **using** *knights-path-non-nil* **by** *auto*  
  
**obtain**  $k$  **where**  $k$ -prems:  
 $0 < k < length\ ps_1\ last\ (take\ k\ ps_1) = s_i\ hd\ (drop\ k\ ps_1) = s_j$   
**using** *assms unfolding step-in-def* **by** *auto*  
**let**  $?ps = (take\ k\ ps_1) @ ?ps_2' @ (drop\ k\ ps_1)$   
**obtain**  $b_1\ b_2$  **where**  $b$ -prems:  $knights\_path\ b_1\ (take\ k\ ps_1)\ knights\_path\ b_2\ (drop\ k\ ps_1)$   
 $b_1 \cup b_2 = ?b_1\ b_1 \cap b_2 = \{\}$   
**using** *assms*  $\langle 0 < k \rangle \langle k < length\ ps_1 \rangle knights\_path\_split$  **by** *blast*  
  
**have**  $hd\ ?ps_2' = (i_h, int\ m_1+j_h)\ last\ ?ps_2' = (i_l, int\ m_1+j_l)$   
**using** *assms knights-path-non-nil hd-trans-path last-trans-path* **by** *auto*  
**then have**  $hd\ ?ps_2' = (i_h, int\ m_1+j_h)\ last\ ((take\ k\ ps_1) @ ?ps_2') = (i_l, int\ m_1+j_l)$   
**using**  $\langle ?ps_2' \neq [] \rangle$  **by** *auto*  
**then have**  $vs: valid\_step\ (last\ (take\ k\ ps_1))\ (hd\ ?ps_2')$   
 $valid\_step\ (last\ ((take\ k\ ps_1) @ ?ps_2'))\ (hd\ (drop\ k\ ps_1))$   
**using** *assms k-prems* **by** *auto*  
  
**have**  $?b_1 \cap ?b' = \{\}$  **unfolding** *board-def trans-board-def* **by** *auto*  
**then have**  $b_1 \cap ?b' = \{\} \wedge (b_1 \cup ?b') \cap b_2 = \{\}$  **using**  $b$ -prems **by** *blast*  
**then have** *inter-empty*:  $b_1 \cap ?b' = \{\} \wedge (b_1 \cup ?b') \cap b_2 = \{\}$  **by** *auto*  
  
**have**  $knights\_path\ (b_1 \cup ?b')\ ((take\ k\ ps_1) @ ?ps_2')$   
**using**  $kp2'\ b$ -prems *inter-empty vs knights-path-append* **by** *auto*  
**then have**  $knights\_path\ (b_1 \cup ?b' \cup b_2)\ ?ps$   
**using**  $b$ -prems *inter-empty vs knights-path-append* **where**  $ps_1 = (take\ k\ ps_1) @ ?ps_2'$  **by** *auto*  
**then have**  $knights\_path\ (?b_1 \cup ?b')\ ?ps$   
**using**  $b$ -prems *Un-commute Un-assoc* **by** *metis*  
**then have**  $kp: knights\_path\ (board\ n\ (m_1+m_2))\ ?ps$   
**using** *board-concat*  $[of\ n\ m_1\ m_2]$  **by** *auto*  
  
**have**  $hd: hd\ ?ps = hd\ ps_1$   
**using** *assms*  $\langle 0 < k \rangle knights\_path\_non\_nil\ hd\_take$  **by** *auto*  
  
**have**  $last: last\ ?ps = last\ ps_1$   
**using** *assms*  $\langle k < length\ ps_1 \rangle knights\_path\_non\_nil\ last\_drop$  **by** *auto*



**have**  $m\text{-simps}$ :  $j + \text{int } m_1 = \text{int } m_1 + j$   $j' + \text{int } m_1 = \text{int } m_1 + j'$  **by** *auto*  
**have**  $si$ :  $\text{step-in } ?ps (i, \text{int } m_1 + j) (i', \text{int } m_1 + j')$   
**using**  $\text{assms step-in-append}[OF \text{ step-in-prepend}[OF \text{ trans-step-in}],$   
 $\text{of } ps_2 \ i \ j \ i' \ j' \ \text{take } k \ ps_1 \ 0 \ \text{int } m_1 \ \text{drop } k \ ps_1]$   
**by** (*auto simp: m-simps*)

**show**  $?thesis$  **using**  $kp \ hd \ last \ si$  **by** *auto*  
**qed**

**lemma**  $\text{len1-hd-last}$ :  $\text{length } xs = 1 \implies \text{hd } xs = \text{last } xs$   
**by** (*induction xs*) *auto*

Weaker version of  $\llbracket \text{knight-path } (\text{board } ?n \ ?m_1) \ ?ps_1; \text{knight-path } (\text{board } ?n \ ?m_2) \ ?ps_2; \text{step-in } ?ps_1 \ ?s_i \ ?s_j; \text{hd } ?ps_2 = (?i_h, ?j_h); \text{last } ?ps_2 = (?i_l, ?j_l); \text{step-in } ?ps_2 \ (?i, ?j) \ (?i', ?j'); \text{valid-step } ?s_i \ (?i_h, \text{int } ?m_1 + ?j_h); \text{valid-step } (?i_l, \text{int } ?m_1 + ?j_l) \ ?s_j \rrbracket \implies \exists ps. \text{knight-path } (\text{board } ?n \ (?m_1 + ?m_2)) \ ps \wedge \text{hd } ps = \text{hd } ?ps_1 \wedge \text{last } ps = \text{last } ?ps_1 \wedge \text{step-in } ps \ (?i, \text{int } ?m_1 + ?j) \ (?i', \text{int } ?m_1 + ?j')$ .

**corollary**  $\text{knight-path-split-concat}$ :

**assumes**  $\text{knight-path } (\text{board } n \ m_1) \ ps_1 \ \text{knight-path } (\text{board } n \ m_2) \ ps_2$   
 $\text{step-in } ps_1 \ s_i \ s_j \ \text{hd } ps_2 = (i_h, j_h) \ \text{last } ps_2 = (i_l, j_l)$   
 $\text{valid-step } s_i \ (i_h, \text{int } m_1 + j_h) \ \text{valid-step } (i_l, \text{int } m_1 + j_l) \ s_j$   
**shows**  $\exists ps. \text{knight-path } (\text{board } n \ (m_1 + m_2)) \ ps \wedge \text{hd } ps = \text{hd } ps_1 \wedge \text{last } ps = \text{last } ps_1$

**proof** –

**have**  $\text{length } ps_2 = 1 \vee \text{length } ps_2 > 1$   
**using**  $\text{assms knight-path-non-nil}$  **by** (*meson length-0-conv less-one linorder-neqE-nat*)  
**then show**  $?thesis$   
**proof** (*elim disjE*)  
**let**  $?s_k = (i_h, \text{int } m_1 + j_h)$   
**assume**  $\text{length } ps_2 = 1$

**then have**  $(i_h, j_h) = (i_l, j_l)$   
**using**  $\text{assms len1-hd-last}$  **by** *metis*  
**then have**  $\text{valid-step } s_i \ ?s_k \ \text{valid-step } ?s_k \ s_j \ \text{valid-step } s_i \ s_j$   
**using**  $\text{assms step-in-valid-step}$  **by** *auto*  
**then show**  $?thesis$   
**using**  $\text{valid-step-non-transitive}$  **by** *blast*

**next**

**assume**  $\text{length } ps_2 > 1$   
**then obtain**  $i_1 \ j_1 \ i_2 \ j_2 \ ps_2'$  **where**  $ps_2 = (i_1, j_1) \# (i_2, j_2) \# ps_2'$   
**using**  $\text{list-len-g-1-split}$  **by** *fastforce*  
**then have**  $\text{last } (\text{take } 1 \ ps_2) = (i_1, j_1) \ \text{hd } (\text{drop } 1 \ ps_2) = (i_2, j_2)$  **by** *auto*  
**then have**  $\text{step-in } ps_2 \ (i_1, j_1) \ (i_2, j_2)$  **using**  $\langle \text{length } ps_2 > 1 \rangle$  **by** (*auto simp: step-in-def*)  
**then show**  $?thesis$   
**using**  $\text{assms knight-path-split-concat-si-prev}$  **by** *blast*  
**qed**  
**qed**

Concatenate two knight's path on a  $n \times m$ -board along the 1st axis.

**corollary** *knight's-path-split-concat* $T$ :

```

assumes knight's-path (board  $n_1$   $m$ )  $ps_1$  knight's-path (board  $n_2$   $m$ )  $ps_2$ 
          step-in  $ps_1$   $s_i$   $s_j$  hd  $ps_2 = (i_h, j_h)$  last  $ps_2 = (i_l, j_l)$ 
          valid-step  $s_i$  (int  $n_1 + i_h, j_h$ ) valid-step (int  $n_1 + i_l, j_l$ )  $s_j$ 
shows  $\exists ps. \textit{knight's-path}$  (board  $(n_1 + n_2)$   $m$ )  $ps \wedge \textit{hd}$   $ps = \textit{hd}$   $ps_1 \wedge \textit{last}$   $ps =$ 
last  $ps_1$ 
using assms
proof –
  let  $?ps_1 T = \textit{transpose}$   $ps_1$ 
  let  $?ps_2 T = \textit{transpose}$   $ps_2$ 
  have  $kps: \textit{knight's-path}$  (board  $m$   $n_1$ )  $?ps_1 T$  knight's-path (board  $m$   $n_2$ )  $?ps_2 T$ 
    using assms transpose-knight's-path by auto

  let  $?s_i T = \textit{transpose-square}$   $s_i$ 
  let  $?s_j T = \textit{transpose-square}$   $s_j$ 
  have  $si: \textit{step-in}$   $?ps_1 T$   $?s_i T$   $?s_j T$ 
    using assms transpose-step-in by auto

  have  $ps_1 \neq []$   $ps_2 \neq []$ 
    using assms knight's-path-non-nil by auto
  then have  $hd\text{-}last2: \textit{hd}$   $?ps_2 T = (j_h, i_h)$  last  $?ps_2 T = (j_l, i_l)$ 
    using assms hd-transpose last-transpose by (auto simp: transpose-square-def)

  have  $vs: \textit{valid-step}$   $?s_i T$   $(j_h, \textit{int } n_1 + i_h)$  valid-step  $(j_l, \textit{int } n_1 + i_l)$   $?s_j T$ 
    using assms transpose-valid-step by (auto simp: transpose-square-def split: prod.splits)

  then obtain  $ps$  where
     $ps\text{-prems}: \textit{knight's-path}$  (board  $m$   $(n_1 + n_2)$ )  $ps$  hd  $ps = \textit{hd}$   $?ps_1 T$  last  $ps = \textit{last}$ 
     $?ps_1 T$ 
    using knight's-path-split-concat[OF  $kps$   $si$   $hd\text{-}last2$   $vs$ ] by auto
  then have  $ps \neq []$  using knight's-path-non-nil by auto
  let  $?ps T = \textit{transpose}$   $ps$ 
  have knight's-path (board  $(n_1 + n_2)$   $m$ )  $?ps T$  hd  $?ps T = \textit{hd}$   $ps_1$  last  $?ps T = \textit{last}$ 
     $ps_1$ 
    using  $\langle ps_1 \neq [] \rangle \langle ps \neq [] \rangle ps\text{-prems}$  transpose-knight's-path hd-transpose last-transpose
    by (auto simp: transpose2)
  then show thesis by auto
qed

```

Concatenate two Knight's path along the 2nd axis. There is a valid step from the last square in the first Knight's path  $ps_1$  to the first square in the second Knight's path  $ps_2$ .

**corollary** *knight's-path-concat*:

```

assumes knight's-path (board  $n$   $m_1$ )  $ps_1$  knight's-path (board  $n$   $m_2$ )  $ps_2$ 
          hd  $ps_2 = (i_h, j_h)$  valid-step (last  $ps_1$ )  $(i_h, \textit{int } m_1 + j_h)$ 
shows knight's-path (board  $n$   $(m_1 + m_2)$ )  $(ps_1 @ (\textit{trans-path}$   $(0, \textit{int } m_1)$   $ps_2))$ 

```

**proof** –  
 let  $?ps_2' = \text{trans-path } (0, \text{int } m_1) \text{ } ps_2$   
 let  $?b = \text{trans-board } (0, \text{int } m_1) \text{ (board } n \text{ } m_2)$   
 have  $\text{inter-empty: board } n \text{ } m_1 \cap ?b = \{\}$   
 unfolding  $\text{board-def trans-board-def}$  **by** *auto*  
 have  $\text{hd } ?ps_2' = (i_h, \text{int } m_1 + j_h)$   
 using  $\text{assms knights-path-non-nil hd-trans-path}$  **by** *auto*  
 then have  $kp: \text{knights-path (board } n \text{ } m_1) \text{ } ps_1 \text{ knights-path } ?b \text{ } ?ps_2'$  **and**  
 $vs: \text{valid-step (last } ps_1) \text{ (hd } ?ps_2')$   
 using  $\text{assms trans-knights-path}$  **by** *auto*  
 then show  $\text{knights-path (board } n \text{ } (m_1 + m_2)) \text{ } (ps_1 @ ?ps_2')$   
 using  $\text{knights-path-append[OF kp inter-empty vs] board-concat}$  **by** *auto*  
**qed**

Concatenate two Knight's path along the 2nd axis. The first Knight's path end in  $(2, m_1 - 1)$  (lower-right) and the second Knight's paths start in  $(1, 1)$  (lower-left).

**corollary** *knights-path-lr-concat:*

assumes  $\text{knights-path (board } n \text{ } m_1) \text{ } ps_1 \text{ knights-path (board } n \text{ } m_2) \text{ } ps_2$   
 $\text{last } ps_1 = (2, \text{int } m_1 - 1) \text{ hd } ps_2 = (1, 1)$   
 shows  $\text{knights-path (board } n \text{ } (m_1 + m_2)) \text{ } (ps_1 @ (\text{trans-path } (0, \text{int } m_1) \text{ } ps_2))$

**proof** –  
 have  $\text{valid-step (last } ps_1) \text{ (1, int } m_1 + 1)$   
 using  $\text{assms unfolding valid-step-def}$  **by** *auto*  
 then show  $?thesis$   
 using  $\text{assms knights-path-concat}$  **by** *auto*  
**qed**

Concatenate two Knight's circuits along the 2nd axis. In the first Knight's path the squares  $(2, m_1 - 1)$  and  $(4, m_1)$  are adjacent and the second Knight's circuit starts in  $(1, 1)$  (lower-left) and end in  $(3, 2)$ .

**corollary** *knights-circuit-lr-concat:*

assumes  $\text{knights-circuit (board } n \text{ } m_1) \text{ } ps_1 \text{ knights-circuit (board } n \text{ } m_2) \text{ } ps_2$   
 $\text{step-in } ps_1 \text{ (2, int } m_1 - 1) \text{ (4, int } m_1)$   
 $\text{hd } ps_2 = (1, 1) \text{ last } ps_2 = (3, 2) \text{ step-in } ps_2 \text{ (2, int } m_2 - 1) \text{ (4, int } m_2)$   
 shows  $\exists ps. \text{knights-circuit (board } n \text{ } (m_1 + m_2)) \text{ } ps \wedge \text{step-in } ps \text{ (2, int } (m_1 + m_2) - 1) \text{ (4, int } (m_1 + m_2))$

**proof** –  
 have  $kp1: \text{knights-path (board } n \text{ } m_1) \text{ } ps_1$  **and**  $kp2: \text{knights-path (board } n \text{ } m_2) \text{ } ps_2$   
  
 and  $vs: \text{valid-step (last } ps_1) \text{ (hd } ps_1)$   
 using  $\text{assms unfolding knights-circuit-def}$  **by** *auto*  
  
 have  $m\text{-simps: int } m_1 + (\text{int } m_2 - 1) = \text{int } (m_1 + m_2) - 1 \text{ int } m_1 + \text{int } m_2 = \text{int } (m_1 + m_2)$  **by** *auto*  
  
 have  $\text{valid-step (2, int } m_1 - 1) \text{ (1, int } m_1 + 1) \text{ valid-step (3, int } m_1 + 2) \text{ (4, int } m_1)$   
 unfolding  $\text{valid-step-def}$  **by** *auto*

```

then obtain ps where knight-path (board n (m1+m2)) ps hd ps = hd ps1 last
ps = last ps1 and
  si: step-in ps (2,int (m1+m2)-1) (4,int (m1+m2))
using assms kp1 kp2
  knight-path-split-concat-si-prev[of n m1 ps1 m2 ps2 (2,int m1-1)
    (4,int m1) 1 1 3 2 2 int m2-1 4 int m2]
by (auto simp only: m-simps)
then have knight-circuit (board n (m1+m2)) ps
using vs by (auto simp: knight-circuit-def)
then show ?thesis
using si by auto
qed

```

## 7 Parsing Paths

In this section functions are implemented to parse and construct paths. The parser converts the matrix representation ((*nat list*) *list*) used in [1] to a path (*path*).

for debugging

```

fun test-path :: path ⇒ bool where
  test-path (si#sj#xs) = (step-checker si sj ∧ test-path (sj#xs))
| test-path - = True

```

```

fun f-opt :: ('a ⇒ 'a) ⇒ 'a option ⇒ 'a option where
  f-opt - None = None
| f-opt f (Some a) = Some (f a)

```

```

fun add-opt-fst-sq :: int ⇒ square option ⇒ square option where
  add-opt-fst-sq - None = None
| add-opt-fst-sq k (Some (i,j)) = Some (k+i,j)

```

```

fun find-k-in-col :: nat ⇒ nat list ⇒ int option where
  find-k-in-col k [] = None
| find-k-in-col k (c#cs) = (if c = k then Some 1 else f-opt ((+) 1) (find-k-in-col k
cs))

```

```

fun find-k-sqr :: nat ⇒ (nat list) list ⇒ square option where
  find-k-sqr k [] = None
| find-k-sqr k (r#rs) = (case find-k-in-col k r of
  None ⇒ f-opt (λ(i,j). (i+1,j)) (find-k-sqr k rs)
| Some j ⇒ Some (1,j))

```

Auxiliary function to easily parse pre-computed boards from paper.

```

fun to-sqrs :: nat ⇒ (nat list) list ⇒ path option where
  to-sqrs 0 rs = Some []
| to-sqrs k rs = (case find-k-sqr k rs of
  None ⇒ None

```

| *Some*  $s_i \Rightarrow f\text{-opt } (\lambda ps. ps@[s_i]) (to\text{-sqrs } (k-1) \text{ } rs))$

**fun** *num-elems* :: (*nat list*) *list*  $\Rightarrow$  *nat* **where**  
*num-elems* (*r#rs*) = *length* *r* \* *length* (*r#rs*)

Convert a matrix (*nat list list*) to a path (*path*). With this function we implicitly define the lower-left corner to be (1,1) and the upper-right corner to be (*n,m*).

**definition** *to-path* *rs*  $\equiv$  *to-sqrs* (*num-elems* *rs*) (*rev* *rs*)

Example

**value** *to-path*  
[[3,22,13,16,5],  
[12,17,4,21,14],  
[23,2,15,6,9],  
[18,11,8,25,20],  
[1,24,19,10,7::nat]]

## 8 Knight's Paths for $5 \times m$ -Boards

Given here are knight's paths, *kp5xmlr* and *kp5xmur*, for the  $(5 \times m)$ -board that start in the lower-left corner for  $m \in \{5, 6, 7, 8, 9\}$ . The path *kp5xmlr* ends in the lower-right corner, whereas the path *kp5xmur* ends in the upper-right corner. The tables show the visited squares numbered in ascending order.

**abbreviation** *b5x5*  $\equiv$  *board* 5 5

A Knight's path for the  $(5 \times 5)$ -board that starts in the lower-left and ends in the lower-right.

3	22	13	16	5
12	17	4	21	14
23	2	15	6	9
18	11	8	25	20
1	24	19	10	7

**abbreviation** *kp5x5lr*  $\equiv$  *the* (*to-path*

[[3,22,13,16,5],  
[12,17,4,21,14],  
[23,2,15,6,9],  
[18,11,8,25,20],  
[1,24,19,10,7]])

**lemma** *kp-5x5-lr*: *knight's-path* *b5x5* *kp5x5lr*

**by** (*simp* *only*: *knight's-path-exec-simp*) *eval*

**lemma** *kp-5x5-lr-hd*: *hd kp5x5lr = (1,1)* **by** *eval*

**lemma** *kp-5x5-lr-last*: *last kp5x5lr = (2,4)* **by** *eval*

**lemma** *kp-5x5-lr-non-nil*: *kp5x5lr ≠ []* **by** *eval*

A Knight's path for the  $(5 \times 5)$ -board that starts in the lower-left and ends in the upper-right.

7	12	15	20	5
16	21	6	25	14
11	8	13	4	19
22	17	2	9	24
1	10	23	18	3

**abbreviation** *kp5x5ur*  $\equiv$  *the (to-path*

*[[7,12,15,20,5],*  
*[16,21,6,25,14],*  
*[11,8,13,4,19],*  
*[22,17,2,9,24],*  
*[1,10,23,18,3]]*)

**lemma** *kp-5x5-ur*: *knight's-path b5x5 kp5x5ur*  
**by** *(simp only: knight's-path-exec-simp)* *eval*

**lemma** *kp-5x5-ur-hd*: *hd kp5x5ur = (1,1)* **by** *eval*

**lemma** *kp-5x5-ur-last*: *last kp5x5ur = (4,4)* **by** *eval*

**lemma** *kp-5x5-ur-non-nil*: *kp5x5ur ≠ []* **by** *eval*

**abbreviation** *b5x6*  $\equiv$  *board 5 6*

A Knight's path for the  $(5 \times 6)$ -board that starts in the lower-left and ends in the lower-right.

7	14	21	28	5	12
22	27	6	13	20	29
15	8	17	24	11	4
26	23	2	9	30	19
1	16	25	18	3	10

**abbreviation** *kp5x6lr*  $\equiv$  *the (to-path*

*[[7,14,21,28,5,12],*  
*[22,27,6,13,20,29],*  
*[15,8,17,24,11,4],*  
*[26,23,2,9,30,19],*  
*[1,16,25,18,3,10]]*)

**lemma** *kp-5x6-lr: knights-path b5x6 kp5x6lr*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x6-lr-hd: hd kp5x6lr = (1,1)* **by** *eval*

**lemma** *kp-5x6-lr-last: last kp5x6lr = (2,5)* **by** *eval*

**lemma** *kp-5x6-lr-non-nil: kp5x6lr ≠ []* **by** *eval*

A Knight's path for the  $(5 \times 6)$ -board that starts in the lower-left and ends in the upper-right.

3	10	29	20	5	12
28	19	4	11	30	21
9	2	17	24	13	6
18	27	8	15	22	25
1	16	23	26	7	14

**abbreviation** *kp5x6ur ≡ the (to-path*

*[[3,10,29,20,5,12],*  
*[28,19,4,11,30,21],*  
*[9,2,17,24,13,6],*  
*[18,27,8,15,22,25],*  
*[1,16,23,26,7,14]])*

**lemma** *kp-5x6-ur: knights-path b5x6 kp5x6ur*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x6-ur-hd: hd kp5x6ur = (1,1)* **by** *eval*

**lemma** *kp-5x6-ur-last: last kp5x6ur = (4,5)* **by** *eval*

**lemma** *kp-5x6-ur-non-nil: kp5x6ur ≠ []* **by** *eval*

**abbreviation** *b5x7 ≡ board 5 7*

A Knight's path for the  $(5 \times 7)$ -board that starts in the lower-left and ends in the lower-right.

3	12	21	30	5	14	23
20	29	4	13	22	31	6
11	2	19	32	7	24	15
28	33	10	17	26	35	8
1	18	27	34	9	16	25

**abbreviation** *kp5x7lr ≡ the (to-path*

*[[3,12,21,30,5,14,23],*  
*[20,29,4,13,22,31,6],*

$[11, 2, 19, 32, 7, 24, 15],$   
 $[28, 33, 10, 17, 26, 35, 8],$   
 $[1, 18, 27, 34, 9, 16, 25]]$   
**lemma** *kp-5x7-lr: knights-path b5x7 kp5x7lr*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x7-lr-hd: hd kp5x7lr = (1,1)* **by** *eval*

**lemma** *kp-5x7-lr-last: last kp5x7lr = (2,6)* **by** *eval*

**lemma** *kp-5x7-lr-non-nil: kp5x7lr ≠ []* **by** *eval*

A Knight's path for the  $(5 \times 7)$ -board that starts in the lower-left and ends in the upper-right.

3	32	11	34	5	26	13
10	19	4	25	12	35	6
31	2	33	20	23	14	27
18	9	24	29	16	7	22
1	30	17	8	21	28	15

**abbreviation** *kp5x7ur ≡ the (to-path*

$[[3, 32, 11, 34, 5, 26, 13],$   
 $[10, 19, 4, 25, 12, 35, 6],$   
 $[31, 2, 33, 20, 23, 14, 27],$   
 $[18, 9, 24, 29, 16, 7, 22],$   
 $[1, 30, 17, 8, 21, 28, 15]]$

**lemma** *kp-5x7-ur: knights-path b5x7 kp5x7ur*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x7-ur-hd: hd kp5x7ur = (1,1)* **by** *eval*

**lemma** *kp-5x7-ur-last: last kp5x7ur = (4,6)* **by** *eval*

**lemma** *kp-5x7-ur-non-nil: kp5x7ur ≠ []* **by** *eval*

**abbreviation** *b5x8 ≡ board 5 8*

A Knight's path for the  $(5 \times 8)$ -board that starts in the lower-left and ends in the lower-right.

3	12	37	26	5	14	17	28
34	23	4	13	36	27	6	15
11	2	35	38	25	16	29	18
22	33	24	9	20	31	40	7
1	10	21	32	39	8	19	30

**abbreviation** *kp5x8lr ≡ the (to-path*



$[[3,12,37,26,5,14,17,28],$   
 $[34,23,4,13,36,27,6,15],$   
 $[11,2,35,38,25,16,29,18],$   
 $[22,33,24,9,20,31,40,7],$   
 $[1,10,21,32,39,8,19,30]]$   
**lemma** *kp-5x8-lr: knights-path b5x8 kp5x8lr*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x8-lr-hd: hd kp5x8lr = (1,1)* **by** *eval*

**lemma** *kp-5x8-lr-last: last kp5x8lr = (2,7)* **by** *eval*

**lemma** *kp-5x8-lr-non-nil: kp5x8lr ≠ []* **by** *eval*

A Knight's path for the  $(5 \times 8)$ -board that starts in the lower-left and ends in the upper-right.

33	8	17	38	35	6	15	24
18	37	34	7	16	25	40	5
9	32	29	36	39	14	23	26
30	19	2	11	28	21	4	13
1	10	31	20	3	12	27	22

**abbreviation** *kp5x8ur ≡ the (to-path*

$[[33,8,17,38,35,6,15,24],$   
 $[18,37,34,7,16,25,40,5],$   
 $[9,32,29,36,39,14,23,26],$   
 $[30,19,2,11,28,21,4,13],$   
 $[1,10,31,20,3,12,27,22]]$

**lemma** *kp-5x8-ur: knights-path b5x8 kp5x8ur*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x8-ur-hd: hd kp5x8ur = (1,1)* **by** *eval*

**lemma** *kp-5x8-ur-last: last kp5x8ur = (4,7)* **by** *eval*

**lemma** *kp-5x8-ur-non-nil: kp5x8ur ≠ []* **by** *eval*

**abbreviation** *b5x9 ≡ board 5 9*

A Knight's path for the  $(5 \times 9)$ -board that starts in the lower-left and ends in the lower-right.

9	4	11	16	23	42	33	36	25
12	17	8	3	32	37	24	41	34
5	10	15	20	43	22	35	26	29
18	13	2	7	38	31	28	45	40
1	6	19	14	21	44	39	30	27

**abbreviation**  $kp5x9lr \equiv$  the (to-path

$[[9,4,11,16,23,42,33,36,25],$   
 $[12,17,8,3,32,37,24,41,34],$   
 $[5,10,15,20,43,22,35,26,29],$   
 $[18,13,2,7,38,31,28,45,40],$   
 $[1,6,19,14,21,44,39,30,27]]$ )

**lemma**  $kp-5x9-lr$ : knights-path b5x9  $kp5x9lr$

**by** (simp only: knights-path-exec-simp) eval

**lemma**  $kp-5x9-lr-hd$ :  $hd\ kp5x9lr = (1,1)$  **by** eval

**lemma**  $kp-5x9-lr-last$ :  $last\ kp5x9lr = (2,8)$  **by** eval

**lemma**  $kp-5x9-lr-non-nil$ :  $kp5x9lr \neq []$  **by** eval

A Knight's path for the  $(5 \times 9)$ -board that starts in the lower-left and ends in the upper-right.

9	4	11	16	27	32	35	40	25
12	17	8	3	36	41	26	45	34
5	10	15	20	31	28	33	24	39
18	13	2	7	42	37	22	29	44
1	6	19	14	21	30	43	38	23

**abbreviation**  $kp5x9ur \equiv$  the (to-path

$[[9,4,11,16,27,32,35,40,25],$   
 $[12,17,8,3,36,41,26,45,34],$   
 $[5,10,15,20,31,28,33,24,39],$   
 $[18,13,2,7,42,37,22,29,44],$   
 $[1,6,19,14,21,30,43,38,23]]$ )

**lemma**  $kp-5x9-ur$ : knights-path b5x9  $kp5x9ur$

**by** (simp only: knights-path-exec-simp) eval

**lemma**  $kp-5x9-ur-hd$ :  $hd\ kp5x9ur = (1,1)$  **by** eval

**lemma**  $kp-5x9-ur-last$ :  $last\ kp5x9ur = (4,8)$  **by** eval

**lemma**  $kp-5x9-ur-non-nil$ :  $kp5x9ur \neq []$  **by** eval

**lemmas**  $kp-5xm-lr =$

$kp-5x5-lr\ kp-5x5-lr-hd\ kp-5x5-lr-last\ kp-5x5-lr-non-nil$   
 $kp-5x6-lr\ kp-5x6-lr-hd\ kp-5x6-lr-last\ kp-5x6-lr-non-nil$   
 $kp-5x7-lr\ kp-5x7-lr-hd\ kp-5x7-lr-last\ kp-5x7-lr-non-nil$   
 $kp-5x8-lr\ kp-5x8-lr-hd\ kp-5x8-lr-last\ kp-5x8-lr-non-nil$   
 $kp-5x9-lr\ kp-5x9-lr-hd\ kp-5x9-lr-last\ kp-5x9-lr-non-nil$

**lemmas**  $kp-5xm-ur =$

$kp-5x5-ur\ kp-5x5-ur-hd\ kp-5x5-ur-last\ kp-5x5-ur-non-nil$

$kp-5x6-ur$   $kp-5x6-ur-hd$   $kp-5x6-ur-last$   $kp-5x6-ur-non-nil$   
 $kp-5x7-ur$   $kp-5x7-ur-hd$   $kp-5x7-ur-last$   $kp-5x7-ur-non-nil$   
 $kp-5x8-ur$   $kp-5x8-ur-hd$   $kp-5x8-ur-last$   $kp-5x8-ur-non-nil$   
 $kp-5x9-ur$   $kp-5x9-ur-hd$   $kp-5x9-ur-last$   $kp-5x9-ur-non-nil$

For every  $5 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1,1)$  (bottom-left) and ends in  $(2,m-1)$  (bottom-right).

**lemma** *knight's-path-5xm-lr-exists*:

**assumes**  $m \geq 5$

**shows**  $\exists ps. \text{knight's-path } (board\ 5\ m)\ ps \wedge hd\ ps = (1,1) \wedge last\ ps = (2, \text{int } m-1)$

**using** *assms*

**proof** (*induction m rule: less-induct*)

**case** (*less m*)

**then have**  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  **by** *auto*

**then show** *?case*

**proof** (*elim disjE*)

**assume**  $m \in \{5,6,7,8,9\}$

**then show** *?thesis* **using** *kp-5xm-lr* **by** *fastforce*

**next**

**assume** *m-ge*:  $5 \leq m-5$

**then obtain**  $ps_1$  **where**  $ps_1-IH$ : *knight's-path* (*board 5 (m-5)*)  $ps_1$  *hd*  $ps_1 = (1,1)$

$last\ ps_1 = (2, \text{int } (m-5)-1)$   $ps_1 \neq []$

**using** *less.IH*[*of m-5*] *knight's-path-non-nil* **by** *auto*

**let**  $?ps_2 = kp5x5lr$

**let**  $?ps_2' = ps_1 @ trans\text{-}path\ (0, \text{int } (m-5))\ ?ps_2$

**have** *knight's-path* *b5x5*  $?ps_2$  *hd*  $?ps_2 = (1, 1)$   $?ps_2 \neq []$   $last\ ?ps_2 = (2,4)$

**using** *kp-5xm-lr* **by** *auto*

**then have** 1: *knight's-path* (*board 5 m*)  $?ps_2'$

**using** *m-ge*  $ps_1-IH$  *knight's-path-lr-concat*[*of 5 m-5 ps\_1 5 ?ps\_2*] **by** *auto*

**have** 2: *hd*  $?ps_2' = (1,1)$  **using**  $ps_1-IH$  **by** *auto*

**have** *last* (*trans-path* ( $0, \text{int } (m-5)$ )  $?ps_2$ ) =  $(2, \text{int } m-1)$

**using** *m-ge* *last-trans-path*[*OF*  $\langle ?ps_2 \neq [] \rangle \langle last\ ?ps_2 = (2,4) \rangle$ ] **by** *auto*

**then have** 3:  $last\ ?ps_2' = (2, \text{int } m-1)$

**using** *last-appendR*[*OF trans-path-non-nil*[*OF*  $\langle ?ps_2 \neq [] \rangle$ , *symmetric*]] **by**

*metis*

**show** *?thesis* **using** 1 2 3 **by** *auto*

**qed**

**qed**

For every  $5 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1,1)$  (bottom-left) and ends in  $(4,m-1)$  (top-right).

**lemma** *knight's-path-5xm-ur-exists*:

**assumes**  $m \geq 5$

**shows**  $\exists ps. \text{knight's-path } (board\ 5\ m)\ ps \wedge hd\ ps = (1,1) \wedge last\ ps = (4, \text{int } m-1)$

```

using assms
proof -
  have  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  using assms by auto
  then show ?thesis
  proof (elim disjE)
    assume  $m \in \{5,6,7,8,9\}$ 
    then show ?thesis using kp-5xm-ur by fastforce
  next
    assume  $m-ge: 5 \leq m-5$ 
    then obtain  $ps_1$  where  $ps-prems: knights-path (board\ 5\ (m-5))\ ps_1\ hd\ ps_1 =$ 
       $(1,1)$ 
       $last\ ps_1 = (2, int\ (m-5)-1)\ ps_1 \neq []$ 
    using knights-path-5xm-lr-exists[of  $(m-5)$ ] knights-path-non-nil by auto
    let  $?ps_2 = kp5x5ur$ 
    let  $?ps' = ps_1 @ trans-path\ (0, int\ (m-5))\ ?ps_2$ 
    have  $knights-path\ b5x5\ ?ps_2\ hd\ ?ps_2 = (1, 1)\ ?ps_2 \neq []$ 
       $last\ ?ps_2 = (4,4)$ 
    using kp-5xm-ur by auto
    then have 1:  $knights-path (board\ 5\ m)\ ?ps'$ 
      using m-ge ps-prems knights-path-lr-concat[of  $5\ m-5\ ps_1\ 5\ ?ps_2$ ] by auto

    have 2:  $hd\ ?ps' = (1,1)$  using ps-prems by auto

    have  $last\ (trans-path\ (0, int\ (m-5))\ ?ps_2) = (4, int\ m-1)$ 
      using m-ge last-trans-path[OF  $\langle ?ps_2 \neq [] \rangle \langle last\ ?ps_2 = (4,4) \rangle$ ] by auto
    then have 3:  $last\ ?ps' = (4, int\ m-1)$ 
      using last-appendR[OF  $trans-path-non-nil[OF\ \langle ?ps_2 \neq [] \rangle, symmetric]$ ] by
metis

    show ?thesis using 1 2 3 by auto
  qed
qed

 $5 \leq ?m \implies \exists ps. knights-path (board\ 5\ ?m)\ ps \wedge hd\ ps = (1, 1) \wedge last\ ps =$ 
 $(2, int\ ?m - 1)$  and  $5 \leq ?m \implies \exists ps. knights-path (board\ 5\ ?m)\ ps \wedge$ 
 $hd\ ps = (1, 1) \wedge last\ ps = (2, int\ ?m - 1)$  formalize Lemma 1 from [1].

lemmas knights-path-5xm-exists = knights-path-5xm-lr-exists knights-path-5xm-ur-exists

```

## 9 Knight's Paths and Circuits for $6 \times m$ -Boards

abbreviation  $b6x5 \equiv board\ 6\ 5$

A Knight's path for the  $(6 \times 5)$ -board that starts in the lower-left and ends in the upper-left.

10	19	4	29	12
3	30	11	20	5
18	9	24	13	28
25	2	17	6	21
16	23	8	27	14
1	26	15	22	7

**abbreviation**  $kp6x5ul \equiv$  the (to-path

[[10,19,4,29,12],  
 [3,30,11,20,5],  
 [18,9,24,13,28],  
 [25,2,17,6,21],  
 [16,23,8,27,14],  
 [1,26,15,22,7]])

**lemma**  $kp-6x5-ul$ : knights-path  $b6x5$   $kp6x5ul$   
 by (simp only: knights-path-exec-simp) eval

**lemma**  $kp-6x5-ul-hd$ :  $hd\ kp6x5ul = (1,1)$  by eval

**lemma**  $kp-6x5-ul-last$ :  $last\ kp6x5ul = (5,2)$  by eval

**lemma**  $kp-6x5-ul-non-nil$ :  $kp6x5ul \neq []$  by eval

A Knight's circuit for the  $(6 \times 5)$ -board.

16	9	6	27	18
7	26	17	14	5
10	15	8	19	28
25	30	23	4	13
22	11	2	29	20
1	24	21	12	3

**abbreviation**  $kc6x5 \equiv$  the (to-path

[[16,9,6,27,18],  
 [7,26,17,14,5],  
 [10,15,8,19,28],  
 [25,30,23,4,13],  
 [22,11,2,29,20],  
 [1,24,21,12,3]])

**lemma**  $kc-6x5$ : knights-circuit  $b6x5$   $kc6x5$   
 by (simp only: knights-circuit-exec-simp) eval

**lemma**  $kc-6x5-hd$ :  $hd\ kc6x5 = (1,1)$  by eval

**lemma**  $kc-6x5-non-nil$ :  $kc6x5 \neq []$  by eval

**abbreviation**  $b6x6 \equiv$  board 6 6

The path given for the  $6 \times 6$ -board that ends in the upper-left is wrong. The Knight cannot move from square 26 to square 27.

14	23	6	28	12	21
7	36	13	22	5	27
24	15	29	35	20	11
30	8	17	26	34	4
16	25	2	32	10	19
1	31	9	18	3	33

**abbreviation**  $kp6x6ul\text{-}false \equiv the (to\text{-}path$

$[[14,23,6,28,12,21],$   
 $[7,36,13,22,5,27],$   
 $[24,15,29,35,20,11],$   
 $[30,8,17,26,34,4],$   
 $[16,25,2,32,10,19],$   
 $[1,31,9,18,3,33]])$

**lemma**  $\neg knights\text{-}path\ b6x6\ kp6x6ul\text{-}false$   
**by** (*simp only: knights-path-exec-simp*) *eval*

I have computed a correct Knight's path for the  $6 \times 6$ -board that ends in the upper-left. A Knight's path for the  $(6 \times 6)$ -board that starts in the lower-left and ends in the upper-left.

8	25	10	21	6	23
11	36	7	24	33	20
26	9	34	3	22	5
35	12	15	30	19	32
14	27	2	17	4	29
1	16	13	28	31	18

**abbreviation**  $kp6x6ul \equiv the (to\text{-}path$

$[[8,25,10,21,6,23],$   
 $[11,36,7,24,33,20],$   
 $[26,9,34,3,22,5],$   
 $[35,12,15,30,19,32],$   
 $[14,27,2,17,4,29],$   
 $[1,16,13,28,31,18]])$

**lemma**  $kp\text{-}6x6\text{-}ul\text{: knights-path}\ b6x6\ kp6x6ul$   
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma**  $kp\text{-}6x6\text{-}ul\text{-}hd\text{: hd}\ kp6x6ul = (1,1)$  **by** *eval*

**lemma**  $kp\text{-}6x6\text{-}ul\text{-}last\text{: last}\ kp6x6ul = (5,2)$  **by** *eval*

**lemma** *kp-6x6-ul-non-nil*:  $kp6x6ul \neq []$  **by** *eval*

A Knight's circuit for the  $(6 \times 6)$ -board.

4	25	34	15	18	7
35	14	5	8	33	16
24	3	26	17	6	19
13	36	23	30	9	32
22	27	2	11	20	29
1	12	21	28	31	10

**abbreviation** *kc6x6*  $\equiv$  *the (to-path*

*[[4,25,34,15,18,7],*  
*[35,14,5,8,33,16],*  
*[24,3,26,17,6,19],*  
*[13,36,23,30,9,32],*  
*[22,27,2,11,20,29],*  
*[1,12,21,28,31,10]]*)

**lemma** *kc-6x6*: *knight's-circuit b6x6 kc6x6*  
**by** (*simp only: knight's-circuit-exec-simp*) *eval*

**lemma** *kc-6x6-hd*:  $hd\ kc6x6 = (1,1)$  **by** *eval*

**lemma** *kc-6x6-non-nil*:  $kc6x6 \neq []$  **by** *eval*

**abbreviation** *b6x7*  $\equiv$  *board 6 7*

A Knight's path for the  $(6 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

18	23	8	39	16	25	6
9	42	17	24	7	40	15
22	19	32	41	38	5	26
33	10	21	28	31	14	37
20	29	2	35	12	27	4
1	34	11	30	3	36	13

**abbreviation** *kp6x7ul*  $\equiv$  *the (to-path*

*[[18,23,8,39,16,25,6],*  
*[9,42,17,24,7,40,15],*  
*[22,19,32,41,38,5,26],*  
*[33,10,21,28,31,14,37],*  
*[20,29,2,35,12,27,4],*  
*[1,34,11,30,3,36,13]]*)

**lemma** *kp-6x7-ul*: *knight's-path b6x7 kp6x7ul*  
**by** (*simp only: knight's-path-exec-simp*) *eval*

**lemma** *kp-6x7-ul-hd*:  $hd\ kp6x7ul = (1,1)$  **by** *eval*

**lemma** *kp-6x7-ul-last*:  $last\ kp6x7ul = (5,2)$  **by** *eval*

**lemma** *kp-6x7-ul-non-nil*:  $kp6x7ul \neq []$  **by** *eval*

A Knight's circuit for the  $(6 \times 7)$ -board.

26	37	8	17	28	31	6
9	18	27	36	7	16	29
38	25	10	19	30	5	32
11	42	23	40	35	20	15
24	39	2	13	22	33	4
1	12	41	34	3	14	21

**abbreviation** *kc6x7*  $\equiv$  *the (to-path*

$[[26,37,8,17,28,31,6],$   
 $[9,18,27,36,7,16,29],$   
 $[38,25,10,19,30,5,32],$   
 $[11,42,23,40,35,20,15],$   
 $[24,39,2,13,22,33,4],$   
 $[1,12,41,34,3,14,21]]]$

**lemma** *kc-6x7*: *knight's-circuit b6x7 kc6x7*  
**by** (*simp only: knight's-circuit-exec-simp*) *eval*

**lemma** *kc-6x7-hd*:  $hd\ kc6x7 = (1,1)$  **by** *eval*

**lemma** *kc-6x7-non-nil*:  $kc6x7 \neq []$  **by** *eval*

**abbreviation** *b6x8*  $\equiv$  *board 6 8*

A Knight's path for the  $(6 \times 8)$ -board that starts in the lower-left and ends in the upper-left.

18	31	8	35	16	33	6	45
9	48	17	32	7	46	15	26
30	19	36	47	34	27	44	5
37	10	21	28	43	40	25	14
20	29	2	39	12	23	4	41
1	38	11	22	3	42	13	24

**abbreviation** *kp6x8ul*  $\equiv$  *the (to-path*

$[18,31,8,35,16,33,6,45],$   
 $[9,48,17,32,7,46,15,26],$   
 $[30,19,36,47,34,27,44,5],$   
 $[37,10,21,28,43,40,25,14],$   
 $[20,29,2,39,12,23,4,41],$



$[1,38,11,22,3,42,13,24]]$ )  
**lemma** *kp-6x8-ul: knights-path b6x8 kp6x8ul*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-6x8-ul-hd: hd kp6x8ul = (1,1)* **by** *eval*

**lemma** *kp-6x8-ul-last: last kp6x8ul = (5,2)* **by** *eval*

**lemma** *kp-6x8-ul-non-nil: kp6x8ul ≠ []* **by** *eval*

A Knight's circuit for the  $(6 \times 8)$ -board.

30	35	8	15	28	39	6	13
9	16	29	36	7	14	27	38
34	31	10	23	40	37	12	5
17	48	33	46	11	22	41	26
32	45	2	19	24	43	4	21
1	18	47	44	3	20	25	42

**abbreviation** *kc6x8 ≡ the (to-path*  
 $[[30,35,8,15,28,39,6,13],$   
 $[9,16,29,36,7,14,27,38],$   
 $[34,31,10,23,40,37,12,5],$   
 $[17,48,33,46,11,22,41,26],$   
 $[32,45,2,19,24,43,4,21],$   
 $[1,18,47,44,3,20,25,42]]$ )  
**lemma** *kc-6x8: knights-circuit b6x8 kc6x8*  
**by** (*simp only: knights-circuit-exec-simp*) *eval*

**lemma** *kc-6x8-hd: hd kc6x8 = (1,1)* **by** *eval*

**lemma** *kc-6x8-non-nil: kc6x8 ≠ []* **by** *eval*

**abbreviation** *b6x9 ≡ board 6 9*

A Knight's path for the  $(6 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

22	45	10	53	20	47	8	35	18
11	54	21	46	9	36	19	48	7
44	23	42	37	52	49	32	17	34
41	12	25	50	27	38	29	6	31
24	43	2	39	14	51	4	33	16
1	40	13	26	3	28	15	30	5

**abbreviation** *kp6x9ul ≡ the (to-path*  
 $[[22,45,10,53,20,47,8,35,18],$

$[11, 54, 21, 46, 9, 36, 19, 48, 7],$   
 $[44, 23, 42, 37, 52, 49, 32, 17, 34],$   
 $[41, 12, 25, 50, 27, 38, 29, 6, 31],$   
 $[24, 43, 2, 39, 14, 51, 4, 33, 16],$   
 $[1, 40, 13, 26, 3, 28, 15, 30, 5]]$   
**lemma** *kp-6x9-ul: knights-path b6x9 kp6x9ul*  
**by** (*simp only: knights-path-exec-simp*) *eval*  
**lemma** *kp-6x9-ul-hd: hd kp6x9ul = (1,1) by eval*  
**lemma** *kp-6x9-ul-last: last kp6x9ul = (5,2) by eval*  
**lemma** *kp-6x9-ul-non-nil: kp6x9ul ≠ [] by eval*

A Knight's circuit for the  $(6 \times 9)$ -board.

14	49	4	51	24	39	6	29	22
3	52	13	40	5	32	23	42	7
48	15	50	25	38	41	28	21	30
53	2	37	12	33	26	31	8	43
16	47	54	35	18	45	10	27	20
1	36	17	46	11	34	19	44	9

**abbreviation** *kc6x9 ≡ the (to-path*  
 $[[14, 49, 4, 51, 24, 39, 6, 29, 22],$   
 $[3, 52, 13, 40, 5, 32, 23, 42, 7],$   
 $[48, 15, 50, 25, 38, 41, 28, 21, 30],$   
 $[53, 2, 37, 12, 33, 26, 31, 8, 43],$   
 $[16, 47, 54, 35, 18, 45, 10, 27, 20],$   
 $[1, 36, 17, 46, 11, 34, 19, 44, 9]]$   
**lemma** *kc-6x9: knights-circuit b6x9 kc6x9*  
**by** (*simp only: knights-circuit-exec-simp*) *eval*  
**lemma** *kc-6x9-hd: hd kc6x9 = (1,1) by eval*  
**lemma** *kc-6x9-non-nil: kc6x9 ≠ [] by eval*

**lemmas** *kp-6xm-ul =*  
*kp-6x5-ul kp-6x5-ul-hd kp-6x5-ul-last kp-6x5-ul-non-nil*  
*kp-6x6-ul kp-6x6-ul-hd kp-6x6-ul-last kp-6x6-ul-non-nil*  
*kp-6x7-ul kp-6x7-ul-hd kp-6x7-ul-last kp-6x7-ul-non-nil*  
*kp-6x8-ul kp-6x8-ul-hd kp-6x8-ul-last kp-6x8-ul-non-nil*  
*kp-6x9-ul kp-6x9-ul-hd kp-6x9-ul-last kp-6x9-ul-non-nil*

**lemmas** *kc-6xm =*  
*kc-6x5 kc-6x5-hd kc-6x5-non-nil*  
*kc-6x6 kc-6x6-hd kc-6x6-non-nil*  
*kc-6x7 kc-6x7-hd kc-6x7-non-nil*

*kc-6x8 kc-6x8-hd kc-6x8-non-nil*  
*kc-6x9 kc-6x9-hd kc-6x9-non-nil*

For every  $6 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1,1)$  (bottom-left) and ends in  $(5,2)$  (top-left).

**lemma** *knight's-path-6xm-ul-exists*:

**assumes**  $m \geq 5$   
**shows**  $\exists ps. \text{ knights-path } (\text{board } 6 \ m) \ ps \wedge \text{hd } ps = (1,1) \wedge \text{last } ps = (5,2)$   
**using** *assms*  
**proof** (*induction m rule: less-induct*)  
**case** (*less m*)  
**then have**  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  **by** *auto*  
**then show** *?case*  
**proof** (*elim disjE*)  
**assume**  $m \in \{5,6,7,8,9\}$   
**then show** *?thesis* **using** *kp-6xm-ul* **by** *fastforce*  
**next**  
**let**  $?ps_1 = \text{kp6x5ul}$   
**let**  $?b_1 = \text{board } 6 \ 5$   
**have**  $?ps_1\text{-prems: knights-path } ?b_1 \ ?ps_1 \ \text{hd } ?ps_1 = (1,1) \ \text{last } ?ps_1 = (5,2)$   
**using** *kp-6xm-ul* **by** *auto*  
**assume**  $m\text{-ge: } 5 \leq m-5$   
**then obtain**  $ps_2$  **where**  $ps_2\text{-IH: knights-path } (\text{board } 6 \ (m-5)) \ ps_2 \ \text{hd } ps_2 = (1,1)$   
 $\text{last } ps_2 = (5,2)$   
**using** *less.IH[of m-5] knights-path-non-nil* **by** *auto*  
**have**  $27 < \text{length } ?ps_1 \ \text{last } (\text{take } 27 \ ?ps_1) = (2,4) \ \text{hd } (\text{drop } 27 \ ?ps_1) = (4,5)$   
**by** *eval+*  
**then have**  $\text{step-in } ?ps_1 \ (2,4) \ (4,5)$   
**unfolding** *step-in-def* **using** *zero-less-numeral* **by** *blast*  
**then have**  $\text{step-in } ?ps_1 \ (2,4) \ (4,5)$   
 $\text{valid-step } (2,4) \ (1, \text{int } 5+1)$   
 $\text{valid-step } (5, \text{int } 5+2) \ (4,5)$   
**unfolding** *valid-step-def* **by** *auto*  
**then show** *?thesis*  
**using**  $\langle 5 \leq m-5 \rangle \ ps_1\text{-prems } ps_2\text{-IH knights-path-split-concat[of } 6 \ 5 \ ?ps_1 \ m-5 \ ps_2]$  **by** *auto*  
**qed**  
**qed**

For every  $6 \times m$ -board with  $m \geq 5$  there exists a knight's circuit.

**lemma** *knight's-circuit-6xm-exists*:

**assumes**  $m \geq 5$   
**shows**  $\exists ps. \text{ knights-circuit } (\text{board } 6 \ m) \ ps$   
**using** *assms*  
**proof** –  
**have**  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  **using** *assms* **by** *auto*  
**then show** *?thesis*

```

proof (elim disjE)
  assume  $m \in \{5,6,7,8,9\}$ 
  then show ?thesis using kc-6xm by fastforce
next
  let ?ps1 = rev kc6x5
  have knight-circuit b6x5 ?ps1 last ?ps1 = (1,1)
    using kc-6xm knight-circuit-rev by (auto simp: last-rev)
  then have ps1-prems: knight-path b6x5 ?ps1 valid-step (last ?ps1) (hd ?ps1)
    unfolding knight-circuit-def using valid-step-rev by auto
  assume m-ge: 5 ≤ m-5
  then obtain ps2 where ps2-prems: knight-path (board 6 (m-5)) ps2 hd ps2
= (1,1)
      last ps2 = (5,2)
    using knight-path-6xm-ul-exists[of (m-5)] knight-path-non-nil by auto

  have  $2 < \text{length } ?ps_1$  last (take 2 ?ps1) = (2,4) hd (drop 2 ?ps1) = (4,5) by
eval+
  then have step-in ?ps1 (2,4) (4,5)
    unfolding step-in-def using zero-less-numeral by blast
  then have step-in ?ps1 (2,4) (4,5)
    valid-step (2,4) (1,int 5+1)
    valid-step (5,int 5+2) (4,5)
    unfolding valid-step-def by auto
  then have  $\exists ps. \text{knight-path (board 6 } m) ps \wedge \text{hd } ps = \text{hd } ?ps_1 \wedge \text{last } ps = \text{last } ?ps_1$ 
    using m-ge ps1-prems ps2-prems knight-path-split-concat[of 6 5 ?ps1 m-5 ps2] by auto
  then show ?thesis using ps1-prems by (auto simp: knight-circuit-def)
qed
qed

 $5 \leq ?m \implies \exists ps. \text{knight-path (board 6 } ?m) ps \wedge \text{hd } ps = (1, 1) \wedge \text{last } ps = (5, 2)$ 
and  $5 \leq ?m \implies \exists ps. \text{knight-circuit (board 6 } ?m) ps$  formalize
Lemma 2 from [1].

```

**lemmas** *knight-path-6xm-exists = knight-path-6xm-ul-exists knight-circuit-6xm-exists*

## 10 Knight's Paths and Circuits for $8 \times m$ -Boards

**abbreviation** *b8x5*  $\equiv$  *board 8 5*

A Knight's path for the  $(8 \times 5)$ -board that starts in the lower-left and ends in the upper-left.

28	7	22	39	26
23	40	27	6	21
8	29	38	25	14
37	24	15	20	5
16	9	30	13	34
31	36	33	4	19
10	17	2	35	12
1	32	11	18	3

**abbreviation**  $kp8x5ul \equiv$  the (to-path

[[28,7,22,39,26],  
 [23,40,27,6,21],  
 [8,29,38,25,14],  
 [37,24,15,20,5],  
 [16,9,30,13,34],  
 [31,36,33,4,19],  
 [10,17,2,35,12],  
 [1,32,11,18,3]])

**lemma**  $kp-8x5-ul$ : *knight's-path*  $b8x5$   $kp8x5ul$   
 by (simp only: *knight's-path-exec-simp*) eval

**lemma**  $kp-8x5-ul-hd$ :  $hd\ kp8x5ul = (1,1)$  by eval

**lemma**  $kp-8x5-ul-last$ :  $last\ kp8x5ul = (7,2)$  by eval

**lemma**  $kp-8x5-ul-non-nil$ :  $kp8x5ul \neq []$  by eval

A Knight's circuit for the  $(8 \times 5)$ -board.

26	7	28	15	24
31	16	25	6	29
8	27	30	23	14
17	32	39	34	5
38	9	18	13	22
19	40	33	4	35
10	37	2	21	12
1	20	11	36	3

**abbreviation**  $kc8x5 \equiv$  the (to-path

[[26,7,28,15,24],  
 [31,16,25,6,29],  
 [8,27,30,23,14],  
 [17,32,39,34,5],  
 [38,9,18,13,22],  
 [19,40,33,4,35],  
 [10,37,2,21,12],

```

[1,20,11,36,3]])
lemma kc-8x5: knights-circuit b8x5 kc8x5
  by (simp only: knights-circuit-exec-simp) eval

lemma kc-8x5-hd: hd kc8x5 = (1,1) by eval

lemma kc-8x5-last: last kc8x5 = (3,2) by eval

lemma kc-8x5-non-nil: kc8x5 ≠ [] by eval

lemma kc-8x5-si: step-in kc8x5 (2,4) (4,5) (is step-in ?ps -)
proof -
  have  $0 < (21::nat)$   $21 < \text{length } ?ps$   $\text{last } (\text{take } 21 ?ps) = (2,4)$   $\text{hd } (\text{drop } 21 ?ps) = (4,5)$ 
  by eval+
  then show ?thesis unfolding step-in-def by blast
qed

abbreviation b8x6 ≡ board 8 6

```

A Knight's path for the  $(8 \times 6)$ -board that starts in the lower-left and ends in the upper-left.

42	11	26	9	34	13
25	48	43	12	27	8
44	41	10	33	14	35
47	24	45	20	7	28
40	19	32	3	36	15
23	46	21	6	29	4
18	39	2	31	16	37
1	22	17	38	5	30

```

abbreviation kp8x6ul ≡ the (to-path
  [[42,11,26,9,34,13],
  [25,48,43,12,27,8],
  [44,41,10,33,14,35],
  [47,24,45,20,7,28],
  [40,19,32,3,36,15],
  [23,46,21,6,29,4],
  [18,39,2,31,16,37],
  [1,22,17,38,5,30]])
lemma kp-8x6-ul: knights-path b8x6 kp8x6ul
  by (simp only: knights-path-exec-simp) eval

```

```

lemma kp-8x6-ul-hd: hd kp8x6ul = (1,1) by eval

```

```

lemma kp-8x6-ul-last: last kp8x6ul = (7,2) by eval

```

**lemma** *kp-8x6-ul-non-nil*: *kp8x6ul*  $\neq []$  **by** *eval*

A Knight's circuit for the  $(8 \times 6)$ -board. I have reversed circuit s.t. the circuit steps from  $(2,5)$  to  $(4,6)$  and not the other way around. This makes the proofs easier.

8	29	24	45	12	37
25	46	9	38	23	44
30	7	28	13	36	11
47	26	39	10	43	22
6	31	4	27	14	35
3	48	17	40	21	42
32	5	2	19	34	15
1	18	33	16	41	20

**abbreviation** *kc8x6*  $\equiv$  *the (to-path*

*[[8,29,24,45,12,37],*  
*[25,46,9,38,23,44],*  
*[30,7,28,13,36,11],*  
*[47,26,39,10,43,22],*  
*[6,31,4,27,14,35],*  
*[3,48,17,40,21,42],*  
*[32,5,2,19,34,15],*  
*[1,18,33,16,41,20]]*)

**lemma** *kc-8x6*: *knight-circuit b8x6 kc8x6*

**by** (*simp only: knight-circuit-exec-simp*) *eval*

**lemma** *kc-8x6-hd*: *hd kc8x6* =  $(1,1)$  **by** *eval*

**lemma** *kc-8x6-non-nil*: *kc8x6*  $\neq []$  **by** *eval*

**lemma** *kc-8x6-si*: *step-in kc8x6*  $(2,5)$   $(4,6)$  (**is** *step-in ?ps* - -)

**proof** -

**have**  $0 < (34::nat)$   $34 < \text{length } ?ps$

*last (take 34 ?ps)* =  $(2,5)$  *hd (drop 34 ?ps)* =  $(4,6)$  **by** *eval+*

**then show** *?thesis* **unfolding** *step-in-def* **by** *blast*

**qed**

**abbreviation** *b8x7*  $\equiv$  *board 8 7*

A Knight's path for the  $(8 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

38	19	6	55	46	21	8
5	56	39	20	7	54	45
18	37	4	47	34	9	22
3	48	35	40	53	44	33
36	17	52	49	32	23	10
51	2	29	14	41	26	43
16	13	50	31	28	11	24
1	30	15	12	25	42	27

**abbreviation**  $kp8x7ul \equiv$  the (to-path

[[38,19,6,55,46,21,8],  
 [5,56,39,20,7,54,45],  
 [18,37,4,47,34,9,22],  
 [3,48,35,40,53,44,33],  
 [36,17,52,49,32,23,10],  
 [51,2,29,14,41,26,43],  
 [16,13,50,31,28,11,24],  
 [1,30,15,12,25,42,27]])

**lemma**  $kp\text{-}8x7\text{-}ul$ : *knight's-path*  $b8x7$   $kp8x7ul$   
 by (simp only: *knight's-path-exec-simp*) eval

**lemma**  $kp\text{-}8x7\text{-}ul\text{-}hd$ :  $hd\ kp8x7ul = (1,1)$  by eval

**lemma**  $kp\text{-}8x7\text{-}ul\text{-}last$ :  $last\ kp8x7ul = (7,2)$  by eval

**lemma**  $kp\text{-}8x7\text{-}ul\text{-}non\text{-}nil$ :  $kp8x7ul \neq []$  by eval

A Knight's circuit for the  $(8 \times 7)$ -board. I have reversed circuit s.t. the circuit steps from  $(2,6)$  to  $(4,7)$  and not the other way around. This makes the proofs easier.

36	31	18	53	20	29	44
17	54	35	30	45	52	21
32	37	46	19	8	43	28
55	16	7	34	27	22	51
38	33	26	47	6	9	42
3	56	15	12	25	50	23
14	39	2	5	48	41	10
1	4	13	40	11	24	49

**abbreviation**  $kc8x7 \equiv$  the (to-path

[[36,31,18,53,20,29,44],  
 [17,54,35,30,45,52,21],  
 [32,37,46,19,8,43,28],  
 [55,16,7,34,27,22,51],  
 [38,33,26,47,6,9,42],



```

[3,56,15,12,25,50,23],
[14,39,2,5,48,41,10],
[1,4,13,40,11,24,49]])
lemma kc-8x7: knights-circuit b8x7 kc8x7
  by (simp only: knights-circuit-exec-simp) eval

lemma kc-8x7-hd: hd kc8x7 = (1,1) by eval

lemma kc-8x7-non-nil: kc8x7 ≠ [] by eval

lemma kc-8x7-si: step-in kc8x7 (2,6) (4,7) (is step-in ?ps -)
proof -
  have  $0 < (41::nat) \ 41 < \text{length } ?ps$ 
    last (take 41 ?ps) = (2,6) hd (drop 41 ?ps) = (4,7) by eval+
  then show ?thesis unfolding step-in-def by blast
qed

abbreviation b8x8 ≡ board 8 8

```

The path given for the  $8 \times 8$ -board that ends in the upper-left is wrong. The Knight cannot move from square 27 to square 28.

24	11	37	9	26	21	39	7
36	64	24	22	38	8	27	20
12	23	10	53	58	49	6	28
63	35	61	50	55	52	19	40
46	13	54	57	48	59	29	5
34	62	47	60	51	56	41	18
14	45	2	32	16	43	4	30
1	33	15	44	3	31	17	42

```

abbreviation kp8x8ul-false ≡ the (to-path
  [[24,11,37,9,26,21,39,7],
  [36,64,25,22,38,8,27,20],
  [12,23,10,53,58,49,6,28],
  [63,35,61,50,55,52,19,40],
  [46,13,54,57,48,59,29,5],
  [34,62,47,60,51,56,41,18],
  [14,45,2,32,16,43,4,30],
  [1,33,15,44,3,31,17,42]])

```

```

lemma ¬knights-path b8x8 kp8x8ul-false
  by (simp only: knights-path-exec-simp) eval

```

I have computed a correct Knight's path for the  $8 \times 8$ -board that ends in the upper-left.

38	41	36	27	32	43	20	25
35	64	39	42	21	26	29	44
40	37	6	33	28	31	24	19
5	34	63	14	7	22	45	30
62	13	4	9	58	49	18	23
3	10	61	52	15	8	57	46
12	53	2	59	48	55	50	17
1	60	11	54	51	16	47	56

**abbreviation**  $kp8x8ul \equiv$  the (to-path  
 $[[38,41,36,27,32,43,20,25],$   
 $[35,64,39,42,21,26,29,44],$   
 $[40,37,6,33,28,31,24,19],$   
 $[5,34,63,14,7,22,45,30],$   
 $[62,13,4,9,58,49,18,23],$   
 $[3,10,61,52,15,8,57,46],$   
 $[12,53,2,59,48,55,50,17],$   
 $[1,60,11,54,51,16,47,56]]$ )

**lemma**  $kp\text{-}8x8\text{-}ul$ : knights-path  $b8x8$   $kp8x8ul$   
**by** (simp only: knights-path-exec-simp) eval

**lemma**  $kp\text{-}8x8\text{-}ul\text{-}hd$ :  $hd\ kp8x8ul = (1,1)$  **by** eval

**lemma**  $kp\text{-}8x8\text{-}ul\text{-}last$ :  $last\ kp8x8ul = (7,2)$  **by** eval

**lemma**  $kp\text{-}8x8\text{-}ul\text{-}non\text{-}nil$ :  $kp8x8ul \neq []$  **by** eval

A Knight's circuit for the  $(8 \times 8)$ -board.

48	13	30	9	56	45	28	7
31	10	47	50	29	8	57	44
14	49	12	55	46	59	6	27
11	32	37	60	51	54	43	58
36	15	52	63	38	61	26	5
33	64	35	18	53	40	23	42
16	19	2	39	62	21	4	25
1	34	17	20	3	24	41	22

**abbreviation**  $kc8x8 \equiv$  the (to-path  
 $[[48,13,30,9,56,45,28,7],$   
 $[31,10,47,50,29,8,57,44],$   
 $[14,49,12,55,46,59,6,27],$   
 $[11,32,37,60,51,54,43,58],$   
 $[36,15,52,63,38,61,26,5],$   
 $[33,64,35,18,53,40,23,42],$

```

[16,19,2,39,62,21,4,25],
[1,34,17,20,3,24,41,22]])
lemma kc-8x8: knights-circuit b8x8 kc8x8
  by (simp only: knights-circuit-exec-simp) eval

lemma kc-8x8-hd: hd kc8x8 = (1,1) by eval

lemma kc-8x8-non-nil: kc8x8 ≠ [] by eval

lemma kc-8x8-si: step-in kc8x8 (2,7) (4,8) (is step-in ?ps -)
proof –
  have  $0 < (4::nat) \ 4 < length \ ?ps$ 
     $last \ (take \ 4 \ ?ps) = (2,7) \ hd \ (drop \ 4 \ ?ps) = (4,8)$  by eval+
  then show ?thesis unfolding step-in-def by blast
qed

abbreviation b8x9 ≡ board 8 9

```

A Knight's path for the  $(8 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

32	47	6	71	30	45	8	43	26
5	72	31	46	7	70	27	22	9
48	33	4	29	64	23	44	25	42
3	60	35	62	69	28	41	10	21
34	49	68	65	36	63	24	55	40
59	2	61	16	67	56	37	20	11
50	15	66	57	52	13	18	39	54
1	58	51	14	17	38	53	12	19

```

abbreviation kp8x9ul ≡ the (to-path
   $[[32,47,6,71,30,45,8,43,26],$ 
   $[5,72,31,46,7,70,27,22,9],$ 
   $[48,33,4,29,64,23,44,25,42],$ 
   $[3,60,35,62,69,28,41,10,21],$ 
   $[34,49,68,65,36,63,24,55,40],$ 
   $[59,2,61,16,67,56,37,20,11],$ 
   $[50,15,66,57,52,13,18,39,54],$ 
   $[1,58,51,14,17,38,53,12,19]])$ 
lemma kp-8x9-ul: knights-path b8x9 kp8x9ul
  by (simp only: knights-path-exec-simp) eval

```

```

lemma kp-8x9-ul-hd: hd kp8x9ul = (1,1) by eval

```

```

lemma kp-8x9-ul-last: last kp8x9ul = (7,2) by eval

```

```

lemma kp-8x9-ul-non-nil: kp8x9ul ≠ [] by eval

```

A Knight's circuit for the  $(8 \times 9)$ -board.

42	19	38	5	36	21	34	7	60
39	4	41	20	63	6	59	22	33
18	43	70	37	58	35	68	61	8
3	40	49	64	69	62	57	32	23
50	17	44	71	48	67	54	9	56
45	2	65	14	27	12	29	24	31
16	51	72	47	66	53	26	55	10
1	46	15	52	13	28	11	30	25

**abbreviation**  $kc8x9 \equiv$  the (to-path

[[42,19,38,5,36,21,34,7,60],  
 [39,4,41,20,63,6,59,22,33],  
 [18,43,70,37,58,35,68,61,8],  
 [3,40,49,64,69,62,57,32,23],  
 [50,17,44,71,48,67,54,9,56],  
 [45,2,65,14,27,12,29,24,31],  
 [16,51,72,47,66,53,26,55,10],  
 [1,46,15,52,13,28,11,30,25]])

**lemma**  $kc\text{-}8x9$ : knights-circuit  $b8x9$   $kc8x9$

**by** (simp only: knights-circuit-exec-simp) eval

**lemma**  $kc\text{-}8x9\text{-}hd$ :  $hd\ kc8x9 = (1,1)$  **by** eval

**lemma**  $kc\text{-}8x9\text{-}non\text{-}nil$ :  $kc8x9 \neq []$  **by** eval

**lemma**  $kc\text{-}8x9\text{-}si$ : step-in  $kc8x9$  (2,8) (4,9) (is step-in ?ps - -)

**proof** -

**have**  $0 < (55::nat)$   $55 < length\ ?ps$

$last\ (take\ 55\ ?ps) = (2,8)$   $hd\ (drop\ 55\ ?ps) = (4,9)$  **by** eval+

**then show** ?thesis **unfolding** step-in-def **by** blast

**qed**

**lemmas**  $kp\text{-}8xm\text{-}ul =$

$kp\text{-}8x5\text{-}ul\ kp\text{-}8x5\text{-}ul\text{-}hd\ kp\text{-}8x5\text{-}ul\text{-}last\ kp\text{-}8x5\text{-}ul\text{-}non\text{-}nil$   
 $kp\text{-}8x6\text{-}ul\ kp\text{-}8x6\text{-}ul\text{-}hd\ kp\text{-}8x6\text{-}ul\text{-}last\ kp\text{-}8x6\text{-}ul\text{-}non\text{-}nil$   
 $kp\text{-}8x7\text{-}ul\ kp\text{-}8x7\text{-}ul\text{-}hd\ kp\text{-}8x7\text{-}ul\text{-}last\ kp\text{-}8x7\text{-}ul\text{-}non\text{-}nil$   
 $kp\text{-}8x8\text{-}ul\ kp\text{-}8x8\text{-}ul\text{-}hd\ kp\text{-}8x8\text{-}ul\text{-}last\ kp\text{-}8x8\text{-}ul\text{-}non\text{-}nil$   
 $kp\text{-}8x9\text{-}ul\ kp\text{-}8x9\text{-}ul\text{-}hd\ kp\text{-}8x9\text{-}ul\text{-}last\ kp\text{-}8x9\text{-}ul\text{-}non\text{-}nil$

**lemmas**  $kc\text{-}8xm =$

$kc\text{-}8x5\ kc\text{-}8x5\text{-}hd\ kc\text{-}8x5\text{-}last\ kc\text{-}8x5\text{-}non\text{-}nil\ kc\text{-}8x5\text{-}si$   
 $kc\text{-}8x6\ kc\text{-}8x6\text{-}hd\ kc\text{-}8x6\text{-}non\text{-}nil\ kc\text{-}8x6\text{-}si$   
 $kc\text{-}8x7\ kc\text{-}8x7\text{-}hd\ kc\text{-}8x7\text{-}non\text{-}nil\ kc\text{-}8x7\text{-}si$   
 $kc\text{-}8x8\ kc\text{-}8x8\text{-}hd\ kc\text{-}8x8\text{-}non\text{-}nil\ kc\text{-}8x8\text{-}si$   
 $kc\text{-}8x9\ kc\text{-}8x9\text{-}hd\ kc\text{-}8x9\text{-}non\text{-}nil\ kc\text{-}8x9\text{-}si$

For every  $8 \times m$ -board with  $m \geq 5$  there exists a knight's circuit.

**lemma** *knight-circuit-8xm-exists*:

**assumes**  $m \geq 5$

**shows**  $\exists ps. \text{knight-circuit } (\text{board } 8 \ m) \ ps \wedge \text{step-in } ps \ (2, \text{int } m-1) \ (4, \text{int } m)$

**using** *assms*

**proof** (*induction m rule: less-induct*)

**case** (*less m*)

**then have**  $m \in \{5, 6, 7, 8, 9\} \vee 5 \leq m-5$  **by** *auto*

**then show** *?case*

**proof** (*elim disjE*)

**assume**  $m \in \{5, 6, 7, 8, 9\}$

**then show** *?thesis* **using** *kc-8xm* **by** *fastforce*

**next**

**let**  $?ps_2 = \text{kc8x5}$

**let**  $?b_2 = \text{board } 8 \ 5$

**have**  $ps_2\text{-prems: knight-circuit } ?b_2 \ ?ps_2 \ \text{hd } ?ps_2 = (1, 1) \ \text{last } ?ps_2 = (3, 2)$

**using** *kc-8xm* **by** *auto*

**have**  $21 < \text{length } ?ps_2 \ \text{last } (\text{take } 21 \ ?ps_2) = (2, \text{int } 5-1) \ \text{hd } (\text{drop } 21 \ ?ps_2) = (4, \text{int } 5)$

**by** *eval+*

**then have** *si: step-in ?ps<sub>2</sub> (2, int 5-1) (4, int 5)*

**unfolding** *step-in-def* **using** *zero-less-numeral* **by** *blast*

**assume** *m-ge: 5 ≤ m-5*

**then obtain**  $ps_1$  **where**  $ps_1\text{-IH: knight-circuit } (\text{board } 8 \ (m-5)) \ ps_1$   
 $\text{step-in } ps_1 \ (2, \text{int } (m-5)-1) \ (4, \text{int } (m-5))$

**using** *less.IH[of m-5] knight-path-non-nil* **by** *auto*

**then show** *?thesis*

**using** *m-ge ps<sub>2</sub>-prems si knight-circuit-lr-concat[of 8 m-5 ps<sub>1</sub> 5 ?ps<sub>2</sub>]* **by**

*auto*

**qed**

**qed**

For every  $8 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1, 1)$  (bottom-left) and ends in  $(7, 2)$  (top-left).

**lemma** *knight-path-8xm-ul-exists*:

**assumes**  $m \geq 5$

**shows**  $\exists ps. \text{knight-path } (\text{board } 8 \ m) \ ps \wedge \text{hd } ps = (1, 1) \wedge \text{last } ps = (7, 2)$

**using** *assms*

**proof** –

**have**  $m \in \{5, 6, 7, 8, 9\} \vee 5 \leq m-5$  **using** *assms* **by** *auto*

**then show** *?thesis*

**proof** (*elim disjE*)

**assume**  $m \in \{5, 6, 7, 8, 9\}$

**then show** *?thesis* **using** *kp-8xm-ul* **by** *fastforce*

**next**

**let**  $?ps_1 = \text{kp8x5ul}$

**have**  $ps_1\text{-prems: knight-path } b8x5 \ ?ps_1 \ \text{hd } ?ps_1 = (1, 1) \ \text{last } ?ps_1 = (7, 2)$

**using** *kp-8xm-ul* **by** *auto*

**assume** *m-ge: 5 ≤ m-5*

**then have**  $b\text{-prems}$ :  $5 \leq \min 8 (m-5)$   
**unfolding**  $board\text{-}def$  **by**  $auto$   
  
**obtain**  $ps_2$  **where**  $knight\text{-}circuit (board\ 8\ (m-5))\ ps_2$   
**using**  $m\text{-}ge\ knight\text{-}circuit\text{-}8xm\text{-}exists[of\ (m-5)]\ knight\text{-}path\text{-}non\text{-}nil$  **by**  $auto$   
**then obtain**  $ps_2'$  **where**  $ps_2'\text{-prems}'$ :  $knight\text{-}circuit (board\ 8\ (m-5))\ ps_2'$   
 $hd\ ps_2' = (1,1)\ last\ ps_2' = (3,2)$   
**using**  $b\text{-prems}$   $\langle 5 \leq \min 8 (m-5) \rangle$   $rotate\text{-}knight\text{-}circuit$  **by**  $blast$   
**then have**  $ps_2'\text{-path}$ :  $knight\text{-}path (board\ 8\ (m-5))\ (rev\ ps_2')$   
 $valid\text{-}step (last\ ps_2') (hd\ ps_2')\ hd\ (rev\ ps_2') = (3,2)\ last\ (rev\ ps_2') = (1,1)$   
**unfolding**  $knight\text{-}circuit\text{-}def$  **using**  $knight\text{-}path\text{-}rev$  **by**  $(auto\ simp: hd\text{-}rev\ last\text{-}rev)$

**have**  $34 < length\ ?ps_1\ last\ (take\ 34\ ?ps_1) = (4,5)\ hd\ (drop\ 34\ ?ps_1) = (2,4)$   
**by**  $eval+$   
**then have**  $step\text{-}in\ ?ps_1\ (4,5)\ (2,4)$   
**unfolding**  $step\text{-}in\text{-}def$  **using**  $zero\text{-}less\text{-}numeral$  **by**  $blast$   
**then have**  $step\text{-}in\ ?ps_1\ (4,5)\ (2,4)$   
 $valid\text{-}step\ (4,5)\ (3, int\ 5+2)$   
 $valid\text{-}step\ (1, int\ 5+1)\ (2,4)$   
**unfolding**  $valid\text{-}step\text{-}def$  **by**  $auto$   
**then have**  $\exists ps. knight\text{-}path (board\ 8\ m)\ ps \wedge hd\ ps = hd\ ?ps_1 \wedge last\ ps = last\ ?ps_1$   
**using**  $m\text{-}ge\ ps_1\text{-prems}\ ps_2'\text{-prems}'\ ps_2'\text{-path}$   
 $knight\text{-}path\text{-}split\text{-}concat[of\ 8\ 5\ ?ps_1\ m-5\ rev\ ps_2']$  **by**  $auto$   
**then show**  $?thesis$  **using**  $ps_1\text{-prems}$  **by**  $auto$   
**qed**  
**qed**

$5 \leq ?m \implies \exists ps. knight\text{-}circuit (board\ 8\ ?m)\ ps \wedge step\text{-}in\ ps\ (2, int\ ?m - 1)\ (4, int\ ?m)$  and  $5 \leq ?m \implies \exists ps. knight\text{-}path (board\ 8\ ?m)\ ps \wedge hd\ ps = (1, 1) \wedge last\ ps = (7, 2)$  formalize Lemma 3 from [1].

**lemmas**  $knight\text{-}path\text{-}8xm\text{-}exists = knight\text{-}circuit\text{-}8xm\text{-}exists\ knight\text{-}path\text{-}8xm\text{-}ul\text{-}exists$

## 11 Knight's Paths and Circuits for $n \times m$ -Boards

In this section the desired theorems are proved. The proof uses the previous lemmas to construct paths and circuits for arbitrary  $n \times m$ -boards.

A Knight's path for the  $(5 \times 5)$ -board that starts in the lower-left and ends in the upper-left.

7	20	9	14	5
10	25	6	21	16
19	8	15	4	13
24	11	2	17	22
1	18	23	12	3

**abbreviation**  $kp5x5ul \equiv$  the (to-path

$[[7,20,9,14,5],$   
 $[10,25,6,21,16],$   
 $[19,8,15,4,13],$   
 $[24,11,2,17,22],$   
 $[1,18,23,12,3]])$

**lemma**  $kp-5x5-ul$ : knights-path  $b5x5$   $kp5x5ul$

**by** (simp only: knights-path-exec-simp) eval

A Knight's path for the  $(5 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

17	14	25	6	19	8	29
26	35	18	15	28	5	20
13	16	27	24	7	30	9
34	23	2	11	32	21	4
1	12	33	22	3	10	31

**abbreviation**  $kp5x7ul \equiv$  the (to-path

$[[17,14,25,6,19,8,29],$   
 $[26,35,18,15,28,5,20],$   
 $[13,16,27,24,7,30,9],$   
 $[34,23,2,11,32,21,4],$   
 $[1,12,33,22,3,10,31]])$

**lemma**  $kp-5x7-ul$ : knights-path  $b5x7$   $kp5x7ul$

**by** (simp only: knights-path-exec-simp) eval

A Knight's path for the  $(5 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

7	12	37	42	5	18	23	32	27
38	45	6	11	36	31	26	19	24
13	8	43	4	41	22	17	28	33
44	39	2	15	10	35	30	25	20
1	14	9	40	3	16	21	34	29

**abbreviation**  $kp5x9ul \equiv$  the (to-path

$[[7,12,37,42,5,18,23,32,27],$   
 $[38,45,6,11,36,31,26,19,24],$   
 $[13,8,43,4,41,22,17,28,33],$   
 $[44,39,2,15,10,35,30,25,20],$   
 $[1,14,9,40,3,16,21,34,29]])$

**lemma**  $kp-5x9-ul$ : knights-path  $b5x9$   $kp5x9ul$

**by** (simp only: knights-path-exec-simp) eval

**abbreviation**  $b7x7 \equiv$  board 7 7

A Knight's path for the  $(7 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

9	30	19	42	7	32	17
20	49	8	31	18	43	6
29	10	41	36	39	16	33
48	21	38	27	34	5	44
11	28	35	40	37	26	15
22	47	2	13	24	45	4
1	12	23	46	3	14	25

**abbreviation**  $kp7x7ul \equiv$  the (to-path

$[[9,30,19,42,7,32,17],$   
 $[20,49,8,31,18,43,6],$   
 $[29,10,41,36,39,16,33],$   
 $[48,21,38,27,34,5,44],$   
 $[11,28,35,40,37,26,15],$   
 $[22,47,2,13,24,45,4],$   
 $[1,12,23,46,3,14,25]])$

**lemma**  $kp-7x7-ul$ : *knight's-path*  $b7x7$   $kp7x7ul$   
**by** (*simp only: knight's-path-exec-simp*) *eval*

**abbreviation**  $b7x9 \equiv$  board 7 9

A Knight's path for the  $(7 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

59	4	17	50	37	6	19	30	39
16	63	58	5	18	51	38	7	20
3	60	49	36	57	42	29	40	31
48	15	62	43	52	35	56	21	8
61	2	13	26	45	28	41	32	55
14	47	44	11	24	53	34	9	22
1	12	25	46	27	10	23	54	33

**abbreviation**  $kp7x9ul \equiv$  the (to-path

$[[59,4,17,50,37,6,19,30,39],$   
 $[16,63,58,5,18,51,38,7,20],$   
 $[3,60,49,36,57,42,29,40,31],$   
 $[48,15,62,43,52,35,56,21,8],$   
 $[61,2,13,26,45,28,41,32,55],$   
 $[14,47,44,11,24,53,34,9,22],$   
 $[1,12,25,46,27,10,23,54,33]])$

**lemma**  $kp-7x9-ul$ : *knight's-path*  $b7x9$   $kp7x9ul$   
**by** (*simp only: knight's-path-exec-simp*) *eval*



**abbreviation**  $b9x7 \equiv \text{board } 9 \ 7$

A Knight's path for the  $(9 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

5	20	53	48	7	22	31
52	63	6	21	32	55	8
19	4	49	54	47	30	23
62	51	46	33	56	9	58
3	18	61	50	59	24	29
14	43	34	45	28	57	10
17	2	15	60	35	38	25
42	13	44	27	40	11	36
1	16	41	12	37	26	39

**abbreviation**  $kp9x7ul \equiv \text{the (to-path}$

$[[5,20,53,48,7,22,31],$   
 $[52,63,6,21,32,55,8],$   
 $[19,4,49,54,47,30,23],$   
 $[62,51,46,33,56,9,58],$   
 $[3,18,61,50,59,24,29],$   
 $[14,43,34,45,28,57,10],$   
 $[17,2,15,60,35,38,25],$   
 $[42,13,44,27,40,11,36],$   
 $[1,16,41,12,37,26,39]])$

**lemma**  $kp-9x7-ul$ : *knight's-path*  $b9x7$   $kp9x7ul$

**by** (*simp only: knight's-path-exec-simp*) *eval*

**abbreviation**  $b9x9 \equiv \text{board } 9 \ 9$

A Knight's path for the  $(9 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

13	26	39	52	11	24	37	50	9
40	81	12	25	38	51	10	23	36
27	14	53	58	63	68	73	8	49
80	41	64	67	72	57	62	35	22
15	28	59	54	65	74	69	48	7
42	79	66	71	76	61	56	21	34
29	16	77	60	55	70	75	6	47
78	43	2	31	18	45	4	33	20
1	30	17	44	3	32	19	46	5

**abbreviation**  $kp9x9ul \equiv \text{the (to-path}$

$[[13,26,39,52,11,24,37,50,9],$   
 $[40,81,12,25,38,51,10,23,36],$

```

[27,14,53,58,63,68,73,8,49],
[80,41,64,67,72,57,62,35,22],
[15,28,59,54,65,74,69,48,7],
[42,79,66,71,76,61,56,21,34],
[29,16,77,60,55,70,75,6,47],
[78,43,2,31,18,45,4,33,20],
[1,30,17,44,3,32,19,46,5]])
lemma kp-9x9-ul: knights-path b9x9 kp9x9ul
  by (simp only: knights-path-exec-simp) eval

The following lemma is a sub-proof used in Lemma 4 in [1]. I moved the
sub-proof out to a separate lemma.

lemma knights-circuit-exists-even-n-gr10:
  assumes even n n ≥ 10 m ≥ 5
     $\exists ps. \text{knights-path } (\text{board } (n-5) \ m) \ ps \wedge \text{hd } ps = (\text{int } (n-5), 1)$ 
     $\wedge \text{last } ps = (\text{int } (n-5)-1, \text{int } m-1)$ 
  shows  $\exists ps. \text{knights-circuit } (\text{board } m \ n) \ ps$ 
  using assms
proof –
  let  $?b_2 = \text{board } (n-5) \ m$ 
  assume  $n \geq 10$ 
  then obtain  $ps_2$  where ps2-prems: knights-path ?b2 ps2 hd ps2 = (int (n-5), 1)

     $\text{last } ps_2 = (\text{int } (n-5)-1, \text{int } m-1)$ 
    using assms by auto
  let  $?ps_2\text{-}m2 = \text{mirror2 } ps_2$ 
  have ps2-m2-prems: knights-path ?b2 ?ps2-m2 hd ?ps2-m2 = (int (n-5), int m)
     $\text{last } ?ps_2\text{-}m2 = (\text{int } (n-5)-1, 2)$ 
    using ps2-prems mirror2-knights-path hd-mirror2 last-mirror2 by auto

  obtain  $ps_1$  where ps1-prems: knights-path (board 5 m) ps1 hd ps1 = (1, 1) last
ps1 = (2, int m-1)
    using assms knights-path-5xm-exists by auto
  let  $?ps_1' = \text{trans-path } (\text{int } (n-5), 0) \ ps_1$ 
  let  $?b_1' = \text{trans-board } (\text{int } (n-5), 0) \ (\text{board } 5 \ m)$ 
  have ps1'-prems: knights-path ?b1' ?ps1' hd ?ps1' = (int (n-5)+1, 1)
     $\text{last } ?ps_1' = (\text{int } (n-5)+2, \text{int } m-1)$ 
    using ps1-prems trans-knights-path knights-path-non-nil hd-trans-path last-trans-path
by auto

  let  $?ps = ?ps_1' @ ?ps_2\text{-}m2$ 
  let  $?psT = \text{transpose } ?ps$ 

  have  $n-5 \geq 5$  using  $\langle n \geq 10 \rangle$  by auto
  have inter: ?b1' ∩ ?b2 = {}
    unfolding trans-board-def board-def using  $\langle n-5 \geq 5 \rangle$  by auto
  have union: ?b1' ∪ ?b2 = board n m
    using  $\langle n-5 \geq 5 \rangle$  board-concatT[of n-5 m 5] by auto

```

```

have vs: valid-step (last ?ps1) (hd ?ps2-m2) and valid-step (last ?ps2-m2) (hd
?ps1)
  unfolding valid-step-def using ps1'-prems ps2-m2-prems by auto
then have vs-c: valid-step (last ?ps) (hd ?ps)
  using ps1'-prems ps2-m2-prems knights-path-non-nil by auto

have knights-path (board n m) ?ps
  using ps1'-prems ps2-m2-prems inter vs union knights-path-append[of ?b1' ?ps1'
?b2 ?ps2-m2]
  by auto
then have knights-circuit (board n m) ?ps
  unfolding knights-circuit-def using vs-c by auto
then show ?thesis using transpose-knights-circuit by auto
qed

```

For every  $n \times m$ -board with  $\min n m \geq 5$  and odd  $n$  there exists a Knight's path that starts in  $(n, 1)$  (top-left) and ends in  $(n-1, m-1)$  (top-right).

This lemma formalizes Lemma 4 from [1]. Formalizing the proof of this lemma was quite challenging as a lot of details on how to exactly combine the boards are left out in the original proof in [1].

**lemma** *knights-path-odd-n-exists*:

```

assumes odd n min n m ≥ 5
shows  $\exists ps. \text{knights-path } (\text{board } n \ m) \ ps \wedge \text{hd } ps = (\text{int } n, 1) \wedge \text{last } ps = (\text{int } n-1, \text{int } m-1)$ 
using assms
proof –
  obtain x where  $x = n + m$  by auto
  then show ?thesis
    using assms
  proof (induction x arbitrary: n m rule: less-induct)
    case (less x)
    then have  $m = 5 \vee m = 6 \vee m = 7 \vee m = 8 \vee m = 9 \vee m \geq 10$  by auto
    then show ?case
    proof (elim disjE)
      assume [simp]:  $m = 5$ 
      have  $\text{odd } n \ n \geq 5$  using less by auto
      then have  $n = 5 \vee n = 7 \vee n = 9 \vee n-5 \geq 5$  by presburger
      then show ?thesis
      proof (elim disjE)
        assume [simp]:  $n = 5$ 
        let ?ps=mirror1 (transpose kp5x5ul)
        have kp: knights-path (board n m) ?ps
          using kp-5x5-ul rot90-knights-path by auto
        have  $\text{hd } ?ps = (\text{int } n, 1) \ \text{last } ?ps = (\text{int } n-1, \text{int } m-1)$ 
          by (simp only: <m = 5> <n = 5> | eval)+
        then show ?thesis using kp by auto
      next
        assume [simp]:  $n = 7$ 

```

```

let ?ps=mirror1 (transpose kp5x7ul)
have kp: knights-path (board n m) ?ps
  using kp-5x7-ul rot90-knights-path by auto
have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
  by (simp only: ⟨m = 5⟩ ⟨n = 7⟩ | eval)+
then show ?thesis using kp by auto
next
assume [simp]: n = 9
let ?ps=mirror1 (transpose kp5x9ul)
have kp: knights-path (board n m) ?ps
  using kp-5x9-ul rot90-knights-path by auto
have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
  by (simp only: ⟨m = 5⟩ ⟨n = 9⟩ | eval)+
then show ?thesis using kp by auto
next
let ?b2=board m (n-5)
assume n-5 ≥ 5
then have ∃ ps. knights-circuit ?b2 ps
proof -
  have n-5 = 6 ∨ n-5 = 8 ∨ n-5 ≥ 10
    using ⟨n-5 ≥ 5⟩ less by presburger
  then show ?thesis
  proof (elim disjE)
    assume n-5 = 6
    then obtain ps where knights-circuit (board (n-5) m) ps
      using knights-path-6xm-exists[of m] by auto
    then show ?thesis
      using transpose-knights-circuit by auto
  next
    assume n-5 = 8
    then obtain ps where knights-circuit (board (n-5) m) ps
      using knights-path-8xm-exists[of m] by auto
    then show ?thesis
      using transpose-knights-circuit by auto
  next
    assume n-5 ≥ 10
    then show ?thesis
      using less less.IH[of n-10+m n-10 m]
        knights-circuit-exists-even-n-gr10[of n-5 m] by auto
  qed
qed
then obtain ps2 where knights-circuit ?b2 ps2 hd ps2 = (1,1) last ps2 =
(3,2)
  using ⟨n-5 ≥ 5⟩ rotate-knights-circuit[of m n-5] by auto
then have rev-ps2-prems: knights-path ?b2 (rev ps2) valid-step (last ps2) (hd
ps2)
  hd (rev ps2) = (3,2) last (rev ps2) = (1,1)
  unfolding knights-circuit-def using knights-path-rev by (auto simp: hd-rev
last-rev)

```

```

    let ?ps1=kp5x5ul
    have ps1-prems: knights-path (board 5 5) ?ps1 hd ?ps1 = (1,1) last ?ps1 =
(4,2)
      using kp-5x5-ul by simp eval+

      have 16 < length ?ps1 last (take 16 ?ps1) = (4,5) hd (drop 16 ?ps1) =
(2,4) by eval+
      then have si: step-in ?ps1 (4,5) (2,4)
        unfolding step-in-def using zero-less-numeral by blast

      have vs: valid-step (4,5) (3,int 5+2) valid-step (1,int 5+1) (2,4)
        unfolding valid-step-def by auto

      obtain ps where knights-path (board m n) ps hd ps = (1,1) last ps = (4,2)
        using ⟨n-5 ≥ 5⟩ ps1-prems rev-ps2-prems si vs
        knights-path-split-concat[of 5 5 ?ps1 n-5 rev ps2 (4,5) (2,4)] by auto
      then show ?thesis
        using rot90-knights-path hd-rot90-knights-path last-rot90-knights-path by
fastforce
      qed
    next
      assume [simp]: m = 6
      then obtain ps where
        ps-prems: knights-path (board m n) ps hd ps = (1,1) last ps = (int m-1,2)
        using less knights-path-6xm-exists[of n] by auto
      let ?ps'=mirror1 (transpose ps)
      have knights-path (board n m) ?ps' hd ?ps' = (int n,1) last ?ps' = (int n-1,int
m-1)
        using ps-prems rot90-knights-path hd-rot90-knights-path last-rot90-knights-path
by auto
      then show ?thesis by auto
    next
      assume [simp]: m = 7
      have odd n n ≥ 5 using less by auto
      then have n = 5 ∨ n = 7 ∨ n = 9 ∨ n-5 ≥ 5 by presburger
      then show ?thesis
      proof (elim disjE)
        assume [simp]: n = 5
        let ?ps=mirror1 kp5x7lr
        have kp: knights-path (board n m) ?ps
          using kp-5x7-lr mirror1-knights-path by auto
        have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
          by (simp only: ⟨m = 7⟩ ⟨n = 5⟩ | eval)+
        then show ?thesis using kp by auto
      next
        assume [simp]: n = 7
        let ?ps=mirror1 (transpose kp7x7ul)
        have kp: knights-path (board n m) ?ps

```

```

    using kp-7x7-ul rot90-knights-path by auto
  have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
    by (simp only: ⟨m = 7⟩ ⟨n = 7⟩ | eval)+
  then show ?thesis using kp by auto
next
  assume [simp]: n = 9
  let ?ps=mirror1 (transpose kp7x9ul)
  have kp: knights-path (board n m) ?ps
    using kp-7x9-ul rot90-knights-path by auto
  have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
    by (simp only: ⟨m = 7⟩ ⟨n = 9⟩ | eval)+
  then show ?thesis using kp by auto
next
  let ?b2=board m (n-5)
  let ?b2T=board (n-5) m
  assume n-5 ≥ 5
  then have ∃ ps. knights-circuit ?b2 ps
  proof -
    have n-5 = 6 ∨ n-5 = 8 ∨ n-5 ≥ 10
      using ⟨n-5 ≥ 5⟩ less by presburger
    then show ?thesis
    proof (elim disjE)
      assume n-5 = 6
      then obtain ps where knights-circuit (board (n-5) m) ps
        using knights-path-6xm-exists[of m] by auto
      then show ?thesis
        using transpose-knights-circuit by auto
    next
      assume n-5 = 8
      then obtain ps where knights-circuit (board (n-5) m) ps
        using knights-path-8xm-exists[of m] by auto
      then show ?thesis
        using transpose-knights-circuit by auto
    next
      assume n-5 ≥ 10
      then show ?thesis
        using less less.IH[of n-10+m n-10 m]
          knights-circuit-exists-even-n-gr10[of n-5 m] by auto
    qed
  qed
  then obtain ps2 where ps2-prems: knights-circuit ?b2 ps2 hd ps2 = (1,1)
    last ps2 = (3,2)
    using ⟨n-5 ≥ 5⟩ rotate-knights-circuit[of m n-5] by auto
  let ?ps2T=transpose ps2
  have ps2T-prems: knights-path ?b2T ?ps2T hd ?ps2T = (1,1) last ?ps2T =
    (2,3)
    using ps2-prems transpose-knights-path knights-path-non-nil hd-transpose
    last-transpose
    unfolding knights-circuit-def transpose-square-def by auto

```

```

let ?ps1=kp5x7lr
have ps1-prems: knights-path b5x7 ?ps1 hd ?ps1 = (1,1) last ?ps1 = (2,6)
  using kp-5x7-lr by simp eval+

  have 29 < length ?ps1 last (take 29 ?ps1) = (4,2) hd (drop 29 ?ps1) =
(5,4) by eval+
  then have si: step-in ?ps1 (4,2) (5,4)
    unfolding step-in-def using zero-less-numeral by blast

  have vs: valid-step (4,2) (int 5+1,1) valid-step (int 5+2,3) (5,4)
    unfolding valid-step-def by auto

  obtain ps where knights-path (board n m) ps hd ps = (1,1) last ps = (2,6)
    using ⟨n-5 ≥ 5⟩ ps1-prems ps2T-prems si vs
    knights-path-split-concatT[of 5 m ?ps1 n-5 ?ps2T (4,2) (5,4)] by auto
  then show ?thesis
    using mirror1-knights-path hd-mirror1 last-mirror1 by fastforce
qed
next
assume [simp]: m = 8
then obtain ps where ps-prems: knights-path (board m n) ps hd ps = (1,1)
  last ps = (int m-1,2)
  using less knights-path-8xm-exists[of n] by auto
let ?ps'=mirror1 (transpose ps)
have knights-path (board n m) ?ps' hd ?ps' = (int n,1) last ?ps' = (int n-1,int
m-1)
  using ps-prems rot90-knights-path hd-rot90-knights-path last-rot90-knights-path
by auto
  then show ?thesis by auto
next
assume [simp]: m = 9
have odd n n ≥ 5 using less by auto
then have n = 5 ∨ n = 7 ∨ n = 9 ∨ n-5 ≥ 5 by presburger
then show ?thesis
proof (elim disjE)
  assume [simp]: n = 5
  let ?ps=mirror1 kp5x9lr
  have kp: knights-path (board n m) ?ps
    using kp-5x9-lr mirror1-knights-path by auto
  have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
    by (simp only: ⟨m = 9⟩ ⟨n = 5⟩ | eval)+
  then show ?thesis using kp by auto
next
assume [simp]: n = 7
let ?ps=mirror1 (transpose kp9x7ul)
have kp: knights-path (board n m) ?ps
  using kp-9x7-ul rot90-knights-path by auto
have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)

```

```

    by (simp only: ⟨m = 9⟩ ⟨n = 7⟩ | eval)+
  then show ?thesis using kp by auto
next
  assume [simp]: n = 9
  let ?ps=mirror1 (transpose kp9x9ul)
  have kp: knights-path (board n m) ?ps
    using kp-9x9-ul rot90-knights-path by auto
  have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
    by (simp only: ⟨m = 9⟩ ⟨n = 9⟩ | eval)+
  then show ?thesis using kp by auto
next
  let ?b2=board m (n-5)
  let ?b2T=board (n-5) m
  assume n-5 ≥ 5
  then have ∃ ps. knights-circuit ?b2 ps
  proof -
    have n-5 = 6 ∨ n-5 = 8 ∨ n-5 ≥ 10
      using ⟨n-5 ≥ 5⟩ less by presburger
    then show ?thesis
    proof (elim disjE)
      assume n-5 = 6
      then obtain ps where knights-circuit (board (n-5) m) ps
        using knights-path-6xm-exists[of m] by auto
      then show ?thesis
        using transpose-knights-circuit by auto
    next
      assume n-5 = 8
      then obtain ps where knights-circuit (board (n-5) m) ps
        using knights-path-8xm-exists[of m] by auto
      then show ?thesis
        using transpose-knights-circuit by auto
    next
      assume n-5 ≥ 10
      then show ?thesis
        using less less.IH[of n-10+m n-10 m]
          knights-circuit-exists-even-n-gr10[of n-5 m] by auto
    qed
  qed
  then obtain ps2 where ps2-prems: knights-circuit ?b2 ps2 hd ps2 = (1,1)
    last ps2 = (3,2)
    using ⟨n-5 ≥ 5⟩ rotate-knights-circuit[of m n-5] by auto
  let ?ps2T=transpose (rev ps2)
  have ps2T-prems: knights-path ?b2T ?ps2T hd ?ps2T = (2,3) last ?ps2T =
    (1,1)
    using ps2-prems knights-path-rev transpose-knights-path knights-path-non-nil
      hd-transpose last-transpose
      unfolding knights-circuit-def transpose-square-def by (auto simp: hd-rev
last-rev)

```



```

let ?ps1=kp5x9lr
have ps1-prems: knights-path b5x9 ?ps1 hd ?ps1 = (1,1) last ?ps1 = (2,8)
  using kp-5x9-lr by simp eval+

  have 16 < length ?ps1 last (take 16 ?ps1) = (5,4) hd (drop 16 ?ps1) =
(4,2) by eval+
  then have si: step-in ?ps1 (5,4) (4,2)
    unfolding step-in-def using zero-less-numeral by blast

  have vs: valid-step (5,4) (int 5+2,3) valid-step (int 5+1,1) (4,2)
    unfolding valid-step-def by auto

  obtain ps where knights-path (board n m) ps hd ps = (1,1) last ps = (2,8)
    using ⟨n-5 ≥ 5⟩ ps1-prems ps2 T-prems si vs
    knights-path-split-concat T[of 5 m ?ps1 n-5 ?ps2 T (5,4) (4,2)] by auto
  then show ?thesis
    using mirror1-knights-path hd-mirror1 last-mirror1 by fastforce
qed
next
let ?b1=board n 5
let ?b2=board n (m-5)
assume m ≥ 10
then have n+5 < x 5 ≤ min n 5 n+(m-5) < x 5 ≤ min n (m-5)
  using less by auto
then obtain ps1 ps2 where kp-prems:
  knights-path ?b1 ps1 hd ps1 = (int n,1) last ps1 = (int n-1,4)
  knights-path (board n (m-5)) ps2 hd ps2 = (int n,1) last ps2 = (int n-1,int
(m-5)-1)
  using less.prems less.IH[of n+5 n 5] less.IH[of n+(m-5) n m-5] by auto
let ?ps=ps1@trans-path (0,int 5) ps2
have valid-step (last ps1) (int n,int 5+1)
  unfolding valid-step-def using kp-prems by auto
then have knights-path (board n m) ?ps hd ?ps = (int n,1) last ?ps = (int
n-1,int m-1)
  using ⟨m ≥ 10⟩ kp-prems knights-path-concat[of n 5 ps1 m-5 ps2]
  knights-path-non-nil trans-path-non-nil last-trans-path by auto
then show ?thesis by auto
qed
qed
qed

```

Auxiliary lemma that constructs a Knight's circuit if  $m \geq 5$  and  $n \geq 10 \wedge$  even  $n$ .

```

lemma knights-circuit-exists-n-even-gr-10:
  assumes n ≥ 10 ∧ even n m ≥ 5
  shows ∃ ps. knights-circuit (board n m) ps
  using assms
proof -

```

```

obtain  $ps_1$  where  $ps_1$ -prems: knight-path (board 5  $m$ )  $ps_1$  hd  $ps_1 = (1,1)$ 
  last  $ps_1 = (2, \text{int } m-1)$ 
  using assms knight-path-5xm-exists by auto
let  $?ps_1' = \text{trans-path } (\text{int } (n-5), 0) ps_1$ 
let  $?b5xm' = \text{trans-board } (\text{int } (n-5), 0) (\text{board } 5 m)$ 
have  $ps_1'$ -prems: knight-path  $?b5xm' ?ps_1'$  hd  $?ps_1' = (\text{int } (n-5)+1, 1)$ 
  last  $?ps_1' = (\text{int } (n-5)+2, \text{int } m-1)$ 
  using  $ps_1$ -prems trans-knight-path knight-path-non-nil hd-trans-path last-trans-path
by auto

assume  $n \geq 10 \wedge \text{even } n$ 
then have odd  $(n-5)$  min  $(n-5)$   $m \geq 5$  using assms by auto
then obtain  $ps_2$  where  $ps_2$ -prems: knight-path (board  $(n-5)$   $m$ )  $ps_2$  hd  $ps_2 =$ 
 $(\text{int } (n-5), 1)$ 
  last  $ps_2 = (\text{int } (n-5)-1, \text{int } m-1)$ 
  using knight-path-odd-n-exists[of  $n-5$   $m$ ] by auto
let  $?ps_2' = \text{mirror2 } ps_2$ 
have  $ps_2'$ -prems: knight-path (board  $(n-5)$   $m$ )  $?ps_2'$  hd  $?ps_2' = (\text{int } (n-5), \text{int } m)$ 
  last  $?ps_2' = (\text{int } (n-5)-1, 2)$ 
  using  $ps_2$ -prems mirror2-knight-path hd-mirror2 last-mirror2 by auto

have inter:  $?b5xm' \cap \text{board } (n-5) m = \{\}$ 
  unfolding trans-board-def board-def by auto

have union:  $\text{board } n m = ?b5xm' \cup \text{board } (n-5) m$ 
  using  $\langle n \geq 10 \wedge \text{even } n \rangle$  board-concatT[of  $n-5$   $m$  5] by auto

have vs: valid-step (last  $?ps_1'$ ) (hd  $?ps_2'$ ) valid-step (last  $?ps_2'$ ) (hd  $?ps_1'$ )
  using  $ps_1'$ -prems  $ps_2'$ -prems unfolding valid-step-def by auto

let  $?ps = ?ps_1' @ ?ps_2'$ 
have last  $?ps = \text{last } ?ps_2'$  hd  $?ps = \text{hd } ?ps_1'$ 
  using  $ps_1'$ -prems  $ps_2'$ -prems knight-path-non-nil by auto
then have vs-c: valid-step (last  $?ps$ ) (hd  $?ps$ )
  using vs by auto

have knight-path (board  $n$   $m$ )  $?ps$ 
  using  $ps_1'$ -prems  $ps_2'$ -prems inter union vs knight-path-append by auto
then show ?thesis
  using vs-c unfolding knight-circuit-def by blast
qed

```

Final Theorem 1: For every  $n \times m$ -board with  $\min n m \geq 5$  and  $n*m$  even there exists a Knight's circuit.

```

theorem knight-circuit-exists:
  assumes  $\min n m \geq 5$  even  $(n*m)$ 
  shows  $\exists ps. \text{knight-circuit } (\text{board } n m) ps$ 
  using assms

```

```

proof –
  have  $n = 6 \vee m = 6 \vee n = 8 \vee m = 8 \vee (n \geq 10 \wedge \text{even } n) \vee (m \geq 10 \wedge \text{even } m)$ 
  using assms by auto
  then show ?thesis
  proof (elim disjE)
    assume  $n = 6$ 
    then show ?thesis
    using assms knights-path-6xm-exists by auto
  next
    assume  $m = 6$ 
    then obtain  $ps$  where knights-circuit (board  $m$   $n$ )  $ps$ 
    using assms knights-path-6xm-exists by auto
    then show ?thesis
    using transpose-knights-circuit by auto
  next
    assume  $n = 8$ 
    then show ?thesis
    using assms knights-path-8xm-exists by auto
  next
    assume  $m = 8$ 
    then obtain  $ps$  where knights-circuit (board  $m$   $n$ )  $ps$ 
    using assms knights-path-8xm-exists by auto
    then show ?thesis
    using transpose-knights-circuit by auto
  next
    assume  $n \geq 10 \wedge \text{even } n$ 
    then show ?thesis
    using assms knights-circuit-exists-n-even-gr-10 by auto
  next
    assume  $m \geq 10 \wedge \text{even } m$ 
    then obtain  $ps$  where knights-circuit (board  $m$   $n$ )  $ps$ 
    using assms knights-circuit-exists-n-even-gr-10 by auto
    then show ?thesis
    using transpose-knights-circuit by auto
  qed
qed

```

Final Theorem 2: for every  $n \times m$ -board with  $\min n \ m \geq 5$  there exists a Knight's path.

```

theorem knights-path-exists:
  assumes  $\min n \ m \geq 5$ 
  shows  $\exists ps. \text{knights-path } (\text{board } n \ m) \ ps$ 
  using assms
proof –
  have  $\text{odd } n \vee \text{odd } m \vee \text{even } (n*m)$  by simp
  then show ?thesis
  proof (elim disjE)
    assume  $\text{odd } n$ 

```

```

    then show ?thesis
      using assms knights-path-odd-n-exists by auto
  next
    assume odd m
    then obtain ps where knights-path (board m n) ps
      using assms knights-path-odd-n-exists by auto
    then show ?thesis
      using transpose-knights-path by auto
  next
    assume even (n*m)
    then show ?thesis
      using assms knights-circuit-exists by (auto simp: knights-circuit-def)
qed
qed

THE END

end

```

## References

- [1] P. Cull and J. D. Curtins. Knight’s tour revisited. *Fibonacci Quarterly*, 16:276–285, 1978.