A STUDY ON PARTIAL DYNAMIC EQUATION ON TIME SCALES INVOLVING DERIVATIVES OF POLYNOMIALS

PETRO KOLOSOV

Abstract. Let $\mathbf{P}^m_b(x)$ be a $2m+1-$degree integer-valued polynomial in x, b. Let be a two-dimensional time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x, b)\colon\ x \in \mathbb{T}_1,\ b \in \mathbb{T}_2\}$. Let be $\mathbb{T}_1 = \mathbb{T}_2$. In this manuscript we derive and discuss the following partial dynamic equation on time scales. For every $t \in \mathbb{T}_1,\ x, b \in \Lambda^2,\ m \in \mathbb{N}$

$$(t^{2m+1})^{\Delta}_t = \frac{\partial \mathbf{P}^m_b(x)}{\Delta x}\bigg|_{x=t,\ b=\sigma(t)} + \frac{\partial \mathbf{P}^m_b(x)}{\Delta b}\bigg|_{x=t,\ b=t}, \quad m = 0, 1, 2, \ldots$$

where $\sigma(t) > t$ is forward jump operator. In addition, we discuss various derivative operators in context of partial cases of above equation, we show finite difference, classical derivative, $q-$derivative, $q-$power derivative on behalf of it.

Contents

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1. Definitions

We now set the following notation, which remains fixed for the remainder of this paper:

- Let be a function $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$ then $f^{\Delta}(t)$ is delta time-scale derivative [BP01] of f
$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$
where $\mu(t) = \sigma(t) - t,\ \mu(t) \neq 0$ and $\sigma(t) > t$ is forward jump operator.

- $\dfrac{\partial f(t_1, \ldots, t_n)}{\Delta_i t_i}$, $f_{t_i}^{\Delta_i}(t)$ is delta partial derivative of $f\colon \Lambda^n \to \mathbb{R}$ on $n-$dimensional time scale Λ^n [BG04, AM02, Jac06], defined as a limit
$$f_{t_i}^{\Delta_i}(t) = \lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, \ldots, t_{i-1}, \sigma_i(t_i), t_{t+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, s_i, t_{t+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i},$$
where $\sigma_i(t_i) > t_i$ and $\sigma_i(t_i) - s_i \neq 0$.

- $D_q f(x)$ is $q-$derivative [Jac09, Ern00, Ern08, KC01]
$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x},$$
where $x \neq 0,\ x \in \mathbb{R},\ q \in \mathbb{R}$.

- $D_{n,q} f(t)$ is $q-$power derivative [AMT11]
$$D_{n,q} f(t) = \frac{f(qt^n) - f(t)}{qt^n - t},$$
where $qt^n - t \neq 0$ and n is odd positive integer and $0 < q < 1$.

- $\mathcal{D}_q f(x)$ is $q-$power derivative
$$\mathcal{D}_q f(x) = \frac{f(x^q) - f(x)}{x^q - x},$$
where $x^q \neq x,\ x \in \mathbb{R},\ q \in \mathbb{R}$.

- $\mathbf{P}_b^m(x),\ x, b \in \mathbb{R},\ m \in \mathbb{N}$ is $2m+1-$degree integer-valued polynomial [Kol16]
$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r, \tag{1.1}$$

where $\mathbf{A}_{m,r},\ m \in \mathbb{N}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m, \\ (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}, & \text{if } 0 \leq r < m, \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

where B_t are Bernoulli numbers [Wei]. It is assumed that $B_1 = \frac{1}{2}$.

- $\mathbb{Z}$ is an integer time scale such that $\sigma(t) = t+1$ and $\mu(t) = 1$.
- $\mathbb{R}$ is a real time scale such that $\sigma(t) = t + \Delta t$ and $\mu(t) = \Delta t,\ \Delta t \to 0$.
- $q^{\mathbb{R}}$ is a quantum time scale such that $\sigma(t) = qt$ and $\mu(t) = qt - t$, [page 18 [BP01]].
- $\mathbb{R}^q$ is a quantum power time scale such that $\sigma(t) = t^q$ and $\mu(t) = t^q - t$.
- $q^{\mathbb{R}^n}$ is a pure quantum power time scale such that $\sigma(t) = qt^n > t,\ 0 < q < 1,\ \mu(t) = qt^n - t$ and n is positive odd integer [AMT11].

2. Introduction

Time-scale calculus is quite graceful generalization and unification of the theory of differential equations. Firstly being introduced by Hilger [Hil88] in his Ph.D thesis in 1988 and thereafter greatly extended by Bohner and Peterson [BP01] in 2001, the calculus on time scales became a sharp tool in the world on differential equations. Various derivative operators like classical derivative $\frac{d}{dx}f(x)$, $q-$derivative $D_q f(x)$, $q-$power derivative $\mathcal{D}_q f(x)$, finite difference $\Delta f(x)$ etc, may be simply expressed in terms of time-scale derivative over particular time scale $\mathbb{T}$. For instance,

$$\begin{aligned} f'(x) &= f^{\Delta}(x), \quad x \in \mathbb{T} = \mathbb{R} \\ \Delta f(x) &= f^{\Delta}(x), \quad x \in \mathbb{T} = \mathbb{Z} \\ D_{n,q} f(x) &= f^{\Delta}(x), \quad x \in \mathbb{T} = q^{\mathbb{R}^n} \\ D_q f(x) &= f^{\Delta}(x), \quad x \in \mathbb{T} = q^{\mathbb{R}} \\ \mathcal{D}_q f(x) &= f^{\Delta}(x), \quad x \in \mathbb{T} = \mathbb{R}^q \end{aligned}$$

In context of Computer Science, namely object oriented programming paradigm, the time scale calculus may be thought as unified interface of derivative operator. Furthermore, the idea of time-scale calculus was slightly extended in [BHT17, BHT16, Cap09, MT09].

3. Main results

Time scale derivative of the polynomial t^{2m+1} may be expressed as follows

Theorem 3.1. *Let $\mathbf{P}_b^m(x)$ be a $2m+1-$degree integer-valued polynomial defined by (1.1). Let be a two-dimensional time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x,b)\colon\ x \in \mathbb{T}_1,\ b \in \mathbb{T}_2\}$. Let be $\mathbb{T}_1 = \mathbb{T}_2$. For every $t \in \mathbb{T}_1,\ x,b \in \Lambda^2,\ m \in \mathbb{N},\ m = const$*

$$(t^{2m+1})^{\Delta} = \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta x}\right|_{x=t,\ b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}\right|_{x=t,\ b=t}, \quad m = 0,1,2,\dots,$$

where

- *$\sigma(t) > t$ is forward jump operator,*
- *$\left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta x}\right|_{x=t,\ b=\sigma(t)}$ is the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable x, evaluated at $x = t,\ b = \sigma(t)$,*
- *$\left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}\right|_{x=t,\ b=t}$ is the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable b, evaluated at $x = t,\ b = t$.*

In other words, theorem 3.1 says

> For every odd-exponent polynomial t^{2m+1}, its derivative on time scales equals to the sum of the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable x, evaluated at $x = t,\ b = \sigma(t)$ and the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable b, evaluated at $x = t,\ b = t$.

In extended form theorem 3.1 may be written as

$$(t^{2m+1})^{\Delta}_t = \frac{\partial}{\Delta x}\left(\sum_{k=0}^{b-1}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r\right)\Bigg|_{x=t,\ b=\sigma(t)} + \frac{\partial}{\Delta b}\left(\sum_{k=0}^{b-1}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r\right)\Bigg|_{x=t,\ b=t}$$

4. Discussion and examples

To understand the nature of theorem 3.1, let's discuss an example of some popular time scales, like integer time scale $\mathbb{Z}$, real time scale $\mathbb{R}$, quantum time scale $q^{\mathbb{R}}$, quantum-power time scale $\mathbb{R}^q$. We use the principle *Divide et Impera !* in order to understand entire behavior of theorem 3.1.

4.1. Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$.

Corollary 4.1. *(Finite difference.) Let be a two-dimensional time scale $\Lambda^2 = \mathbb{Z} \times \mathbb{Z} := \{t = (x,b)\colon\ x \in \mathbb{Z},\ b \in \mathbb{Z}\}$. For every $t \in \mathbb{Z},\ x,b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z},\ m \in \mathbb{N}$*

$$\Delta t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x}(t,\sigma(t)) + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t,t),$$

where the forward jump oprerator $\sigma(t)$ is defined as $\sigma(t) = t + 1$.

Example 4.2. *Let be $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z},\ m \in \mathbb{N}$ and let $m = 1$, then*

$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} = 3b + 3b^2$$

$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) = 3t + 3t^2$$

$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta b} = 1 - 6b^2 + 6bx$$

$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = 1$$

Summing up previously obtained partial time-scale derivatives, we get the ordinary finite difference of odd polynomial t^{2m+1}, $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z},\ m \in \mathbb{N}$

$$\Delta t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = 3t^2 + 3t + 1.$$

Example 4.3. *Let be $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z},\ m \in \mathbb{N}$ and let $m = 2$, then*

$$\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} = 5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$$

$$\frac{\partial \mathbf{P}_b^2(x)}{\Delta b} = 1 + 30b^4 - 60b^3x + 30b^2x^2$$

$$\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}(t, \sigma(t)) = 5t + 10t^2 + 10t^3 + 5t^4$$

$$\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}(t, t) = 1$$

Summing up previously obtained partial time-scale derivatives, we get time ordinary finite difference of odd polynomial t^{2m+1}, $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z},\ m \in \mathbb{N}$

$$\Delta t^5 = \frac{\partial \mathbf{P}_b^2(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^2(x)}{\Delta b}(t, t) = 1 + 5t + 10t^2 + 10t^3 + 5t^4.$$

Corollary 4.4. *For every $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z},\ m \in \mathbb{N}$*

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta x}(t, \sigma(t)) = \sum_{r=1}^{2m} \binom{2m+1}{r} t^r$$

Corollary 4.5. *For every $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z},\ m \in \mathbb{N}$*

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t, t) = 1$$

4.2. **Time scale of real numbers $\mathbb{T} = \mathbb{R} \times \mathbb{R}$.**

Corollary 4.6. *(Classical derivative.) Let be a two-dimensional time scale $\Lambda^2 = \mathbb{R} \times \mathbb{R} := \{t = (x, b) \colon\ x \in \mathbb{R},\ b \in \mathbb{R}\}$. For every $t \in \mathbb{R},\ x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R},\ m \in \mathbb{N}$*

$$\frac{d}{dt} t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\partial x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^m(x)}{\partial b}(t, t),$$

where $\sigma(t) = t + \Delta t,\ \Delta t \to 0.$

Example 4.7. *Let be* $t \in \mathbb{R},\ x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R},\ m \in \mathbb{N}$ *and let* $m = 1$*, then*

$$\frac{\partial \mathbf{P}_b^1(x)}{\partial x} = -3b + 3b^2$$
$$\frac{\partial \mathbf{P}_b^1(x)}{\partial b} = 6b - 6b^2 - 3x + 6bx$$

$$\frac{\partial \mathbf{P}_b^1(x)}{\partial x}(t, \sigma(t)) = -3t + 3t^2$$
$$\frac{\partial \mathbf{P}_b^1(x)}{\partial b}(t, t) = 3t$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of odd polynomial $t^{2m+1}, t \in \mathbb{R},\ x \in \Lambda^2 = \mathbb{R} \times \mathbb{R},\ m \in \mathbb{N}$

$$\frac{d}{dt}t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\partial x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^1(x)}{\partial b}(t, t) = 3t^2.$$

Example 4.8. *Let be* $t \in \mathbb{R},\ x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R},\ m \in \mathbb{N}$ *and let* $m = 2$*, then*

$$\frac{\partial \mathbf{P}_b^2(x)}{\partial x} = -15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x,$$
$$\frac{\partial \mathbf{P}_b^2(x)}{\partial b} = 30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2$$
$$\frac{\partial \mathbf{P}_b^2(x)}{\partial x}(t, \sigma(t)) = -5t^2 + 5t^4$$
$$\frac{\partial \mathbf{P}_b^2(x)}{\partial b}(t, t) = 5t^2$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of an odd polynomial $t^{2m+1},\ t \in \mathbb{R},\ x \in \Lambda^2 = \mathbb{R} \times \mathbb{R},\ m \in \mathbb{N}$

$$\frac{d}{dt}t^5 = \frac{\partial \mathbf{P}_b^2(x)}{\partial x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^2(x)}{\partial b}(t, t) = 5t^4.$$

4.3. **Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$.**

Corollary 4.9. *(Q-derivative [Jac09].) Let be a two-dimensional time scale* $\Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}} := \{t = (x, b):\ x \in q^{\mathbb{R}},\ b \in q^{\mathbb{R}}\}$*. For every* $t \in q^{\mathbb{R}},\ x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}},\ m \in \mathbb{N}$

$$D_q t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t, t),$$

where $\sigma(t) = qt,\ q > 1.$

Example 4.10. *Let be $t \in q^{\mathbb{R}},\ x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}},\ m \in \mathbb{N}$ and let $m = 1$, then*

$$\begin{aligned}
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} = -3b + 3b^2\\
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta b} = 3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx\\
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) = -3qt + 3q^2t^2\\
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = 3qt + t^2 + qt^2 - 2q^2t^2
\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get $q-$derivative of odd polynomial $t^{2m+1},\ t \in q^{\mathbb{R}},\ x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}},\ m \in \mathbb{N}$

$$D_q t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = t^2 + qt^2 + q^2t^2.$$

For every $t \in q^{\mathbb{R}},\ x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}},\ m \in \mathbb{N}$ the following polynomial identity holds as q tends to zero

$$\lim_{q \to 0} \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = t^2$$

However, it would be generalized as follows

Corollary 4.11. *For every $t \in q^{\mathbb{R}},\ x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}},\ m \in \mathbb{N}$*

$$\lim_{q \to 0} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t, t) = t^{2m}.$$

Example 4.12. *Let be $t \in q^{\mathbb{R}},\ x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}},\ m \in \mathbb{N}$ and let $m = 2$, then*

$$\begin{aligned}
\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx - 15b^2qx + 10b^3qx\\
\frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 + 6b^4q^2 - 15b^3q^3\\
&+ 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x\\
&- 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}(t, \sigma(t)) = 5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4\\
&\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}(t, t) = -5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4
\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get the $q-$derivative of odd polynomial $t^{2m+1},\ t \in q^{\mathbb{R}},\ x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}},\ m \in \mathbb{N}$

$$D_q t^5 = \frac{\partial \mathbf{P}_b^2(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^2(x)}{\Delta b}(t, t) = t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4.$$

4.4. **Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$.**

Corollary 4.13. *(Q-power derivative [AMT11].) Let be a two-dimensional time scale* $\Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q := \{t = (x, b) : \ b \in \mathbb{R}^q, \ x \in \mathbb{R}^q\}$. *For every* $t \in \mathbb{R}^q, \ x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q, \ m \in \mathbb{N}$

$$\mathcal{D}_q t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t, t),$$

where $\sigma(t) = t^q, \ q > 1$.

Example 4.14. *Let be* $t \in \mathbb{R}^q, \ x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q, \ m \in \mathbb{N}$ *and let* $m = 1$, *then*

$$\begin{aligned}
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} = 3b + 3b^2 \\
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta b} = 3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^q x \\
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) = -3t^q + 3t^{2q} \\
&\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = t^2 + 3t^q - 2t^{2q} + t^{1+q}
\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get $q-$*power derivative of odd polynomial* $t^{2m+1}, \ t \in \mathbb{R}^q, \ x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q, \ m \in \mathbb{N}$

$$\mathcal{D}_q t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = t^2 + t^{2q} + t^{1+q}.$$

Example 4.15. *Let be* $t \in \mathbb{R}^q, \ x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q, \ m \in \mathbb{N}$ *and let* $m = 2$, *then*

$$\begin{aligned}
\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bx^q - 15b^2x^q + 10b^3x^q \\
\frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{2q} - 15b^{3q} + 6b^{4q} + 10b^{1+q} - 15b^{2+q} + 6b^{3+q} \\
&- 15b^{1+2q} + 6b^{2+2q} + 6b^{1+3q} - 15bx + 30b^2x - 15b^3x - 15b^qx + 30b^{2q}x \\
&- 15b^{3q}x + 30b^{1+q}x - 15b^{2+q}x - 15b^{1+2q}x + 5x^2 - 15bx^2 + 10b^2x^2 \\
&- 15b^qx^2 + 10b^{2q}x^2 + 10b^{1+q}x^2
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}(t, \sigma(t)) = -10t^{2q} + 15t^{3q} - 5t^{4q} + 5t^{1+q} - 15t^{1+2q} + 10t^{1+3q} \\
&\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}(t, t) = t^4 + 10t^{2q} - 15t^{3q} + 6t^{4q} - 5t^{1+q} + t^{3+q} + 15t^{1+2q} + t^{2+2q} - 9t^{1+3q}
\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get $q-$*power derivative of odd polynomial* $t^{2m+1}, \ t \in \mathbb{R}^q, \ x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q, \ m \in \mathbb{N}$

$$\mathcal{D}_q t^5 = \frac{\partial \mathbf{P}_b^2(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^2(x)}{\Delta b}(t, t) = t^4 + t^{4q} + t^{3+q} + t^{2+2q} + t^{1+3q}.$$

Another polynomial identity, that is exponential sum holds

Corollary 4.16. *For every* $t \in \mathbb{R}^q,\ x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q,\ t \in \mathbb{R},\ m \in \mathbb{N}$

$$\lim_{q \to 0} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t,t) = \sum_{k=0}^{2m} t^k$$

4.5. **Pure quantum power time scale** $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$**.** In this subsection we discuss a pure quantum power time scale $q^{\mathbb{R}^j}$ provided by Aldwoah, Malinowska and Torres in [AMT11], among with the $q-$power derivative operator $D_{n,q}f(t)$ defined by

$$D_{n,q}f(t) = \frac{f(qt^n) - f(t)}{qt^n - t},$$

where n is odd positive integer and $0 < q < 1$.

Corollary 4.17. *(Quantum power derivative [AMT11].) Let be a two-dimensional time scale* $\Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j} := \{t = (x,b)\colon\ b \in q^{\mathbb{R}^j},\ x \in q^{\mathbb{R}^j}\}$*. For every* $t \in q^{\mathbb{R}^j},\ x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j},\ m \in \mathbb{N}$

$$D_{n,q}t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t,t),$$

where $\sigma(t) = qt^n,\ \sigma(t) > t$*.*

Example 4.18. *Let be* $t \in q^{\mathbb{R}^j},\ x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j},\ m \in \mathbb{N}$ *and let* $m = 1$*, then*

$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} = -3b + 3b^2$$
$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta b} = 3b - 2b^2 + 3b^j q - 2b^{1+j}q - 2b^{2j}q^2 - 3x + 3bx + 3b^j qx$$
$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) = -3qt^j + 3q^2t^{2j}$$
$$\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t,t) = t^2 + 3qt^j - 2q^2t^{2j} + qt^{1+j}$$

Summing up previously obtained partial time-scale derivatives, we get $q-$*power derivative of odd polynomial* $t^{2m+1},\ t \in q^{\mathbb{R}^j},\ x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j},\ m \in \mathbb{N}$

$$D_{n,q}t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t,t) = t^2 + q^2t^{2j} + qt^{1+j}.$$

Another polynomial identity, that is exponential sum holds

Corollary 4.19. *For every* $t \in q^{\mathbb{R}^j},\ x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j},\ t \in \mathbb{R},\ m \in \mathbb{N}$

$$\lim_{j \to 0} \lim_{q \to 1} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t,t) = \sum_{k=0}^{2m} t^k$$

An identity in even polynomials holds too

Corollary 4.20. *For every* $t \in q^{\mathbb{R}^j},\ x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j},\ t \in \mathbb{R},\ m \in \mathbb{N}$

$$\lim_{j \to 0} \lim_{q \to 0} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t,t) = t^{2m}$$

Example 4.21. *Let be $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $m \in \mathbb{N}$ and let $m = 2$, then*

$$\begin{aligned}
\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx^j - 15b^2qx^j + 10b^3qx^j \\
\frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{1+j}q - 15b^{2+j}q + 6b^{3+j}q + 10b^{2j}q^2 - 15b^{1+2j}q^2 \\
&+ 6b^{2+2j}q^2 - 15b^{3j}q^3 + 6b^{1+3j}q^3 + 6b^{4j}q^4 - 15bx + 30b^2x - 15b^3x - 15b^jqx \\
&+ 30b^{1+j}qx - 15b^{2+j}qx + 30b^{2j}q^2x - 15b^{1+2j}q^2x - 15b^{3j}q^3x + 5x^2 - 15bx^2 \\
&+ 10b^2x^2 - 15b^jqx^2 + 10b^{1+j}qx^2 + 10b^{2j}q^2x^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}(t, \sigma(t)) &= -10q^2t^{2j} + 15q^3t^{3j} - 5q^4t^{4j} + 5qt^{1+j} - 15q^2t^{1+2j} + 10q^3t^{1+3j} \\
\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}(t, t) &= t^4 + 10q^2t^{2j} - 15q^3t^{3j} + 6q^4t^{4j} - 5qt^{1+j} + qt^{3+j} + 15q^2t^{1+2j} + q^2t^{2+2j} - 9q^3t^{1+3j}
\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we $q-$power derivative of odd polynomial t^{2m+1}, $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $m \in \mathbb{N}$

$$D_{n,q}t^5 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) = t^4 + q^4t^{4j} + qt^{3+j} + q^2t^{2+2j} + q^3t^{1+3j}.$$

5. Proof of main theorem

By [Lemma 3.1 [Kol16]], for every $x \in \mathbb{R}$, $m \in \mathbb{N}$ it is true that

$$\mathbf{P}_x^m(x) = x^{2m+1} \tag{5.1}$$

Proof of theorem 3.1. Let be $x, b \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 := \{t = (x, b) \colon\ x \in \mathbb{T}_1,\ b \in \mathbb{T}_2\}$. Let be $\mathbb{T}_1 = \mathbb{T}_2$. Assume that time-scale derivative $(x^{2m+1})^\Delta$ is

$$(x^{2m+1})^\Delta = \lim_{b \to x} \lim_{t \to x} \frac{\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(t)}{\sigma(x) - t}, \tag{5.2}$$

where $\sigma(x) > x$ is forward jump operator. However, equation (5.2) is not a timescale derivative of $\mathbf{P}_b^m(x)$ over x how it might seem because of denominator $\sigma(x) - t$. Parameter b of $\mathbf{P}_b^m(x)$ is implicitly incremented as well. Let's try to express nominator of (5.2) in terms of partial derivative $\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}$ on timescales. Let be the following equation

$$\mathbf{P}_{\sigma(b)}^m(x) - \mathbf{P}_b^m(x) = \mathbf{P}_b^m(x)_b^\Delta \cdot \Delta b$$

Let $t \to x$ in (5.2). Then nominator of (5.2) equals to

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(x) = \mathbf{P}_{\sigma(b)}^m(x) - \mathbf{P}_b^m(x) + A$$

where A is yet implicit term. Let's now collapse the terms $f_m(x, b)$ from both sides of above equation, such that

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) = \mathbf{P}_{\sigma(b)}^m(\sigma(x)) + A$$

Therefore,

$$A = \mathbf{P}^m_{\sigma(b)}(\sigma(x)) - \mathbf{P}^m_{\sigma(b)}(\sigma(x)) = \mathbf{P}^m_b(x)^{\Delta}_x(x,\sigma(b)) \cdot \Delta x$$

Now, let's express the nominator of (5.2) as follows

$$\mathbf{P}^m_{\sigma(b)}(\sigma(x)) - \mathbf{P}^m_b(x) = \mathbf{P}^m_b(x)^{\Delta}_x(x,\sigma(b)) \cdot \Delta x + \mathbf{P}^m_b(x)^{\Delta}_b(x,b) \cdot \Delta b$$

$$\mathbf{P}^m_{\sigma(b)}(\sigma(x)) - \mathbf{P}^m_b(x) = \mathbf{P}^m_b(x)^{\Delta}_x(x,\sigma(b)) \cdot (\sigma(x)-x) + \mathbf{P}^m_b(x)^{\Delta}_b(x,b) \cdot (\sigma(b)-b)$$

We can collapse the terms $(\sigma(x)-x)$, $(\sigma(b)-b)$ in above expressions, as $b \to x$. Therefore,

$$\frac{\mathbf{P}^m_{\sigma(x)}(\sigma(x)) - \mathbf{P}^m_x(x)}{\sigma(x)-x} = \mathbf{P}^m_b(x)^{\Delta}_x(x,\sigma(x)) + \mathbf{P}^m_b(x)^{\Delta}_b(x,x)$$

Finally, by the identity (5.1) we can express timescale derivative of x^{2m+1}, $x \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$, $m \in \mathbb{N}$ as

$$(x^{2m+1})^{\Delta}(t) = \frac{\partial \mathbf{P}^m_b(x)}{\Delta x}(t,\sigma(t)) + \frac{\partial \mathbf{P}^m_b(x)}{\Delta b}(t,t)$$

This completes the proof. □

6. Mathematica scripts

To fulfill our study, we attach here a link to the set of *Mathematica* programs, designed to verify the results of current manuscript. To reach these programs follow the link [Kol20].

7. Conclusion and future research

In this manuscript we have discussed partial time scale differential equation involving derivatives of polynomials in context of time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$ where $\mathbb{T}_1 = \mathbb{T}_2$. Future research can be conducted to study the case $\mathbb{T}_1 \neq \mathbb{T}_2$, which makes the theorem 3.1 to be generalised

$$\frac{\partial \mathbf{P}^m_b(x)}{\Delta x} + \frac{\partial \mathbf{P}^m_b(x)}{\Delta b} = \alpha_m(x,b)(x^{2m+1})^{\Delta},$$

where $\alpha_m(x,b)$ is arbitrary differentiable function. Also, it is worth to discuss the theorem 3.1 in context of high order derivatives on time scales. We have established a few power identities, and shown the theorem 3.1 for different 2-dimensional time scales Λ^2 like integer time scale $\mathbb{Z} \times \mathbb{Z}$, real time scale $\mathbb{R} \times \mathbb{R}$, quantum time scale $q^{\mathbb{R}} \times q^{\mathbb{R}}$ and quantum power time scale $\mathbb{R}^q \times \mathbb{R}^q$.

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Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io