

A STUDY ON PARTIAL DYNAMIC EQUATION ON TIME SCALES INVOLVING DERIVATIVES OF POLYNOMIALS

PETRO KOLOSOV

ABSTRACT. Let $\mathbf{P}_b^m(x)$ be a $2m + 1$ -degree integer-valued polynomial in x, b . Let be a two-dimensional time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x, b) : x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$. Let be $\mathbb{T}_1 = \mathbb{T}_2$. In this manuscript we derive and discuss the following partial dynamic equation on time scales. For every $t \in \mathbb{T}_1$, $x, b \in \Lambda^2$, $m = \text{const}$, $m \in \mathbb{N}$

$$(t^{2m+1})^\Delta = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t},$$

where $\sigma(t) > t$ is forward jump operator. In addition, we discuss various derivative operators in context of partial cases of above equation, we show finite difference, classical derivative, q -derivative, q -power derivative on behalf of it.

CONTENTS

1. Definitions	2
2. Introduction	3
3. Main results	3
4. Discussion and examples	4
4.1. Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$	4
4.2. Time scale of real numbers $\mathbb{T} = \mathbb{R} \times \mathbb{R}$	5
4.3. Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$	6
4.4. Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$	8
4.5. Pure quantum power time scale $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$	9
5. Proof of main theorem	10
Proof of theorem 3.1	11
6. Mathematica scripts	12
7. Conclusion and future research	15
References	16

Date: September 16, 2024.

2010 Mathematics Subject Classification. 26E70, 05A30.

Key words and phrases. Dynamic equations on time scales, Partial differential equations on time scales, Partial dynamic equations on time scales, Partial differentiation on time scales, Dynamical systems .

1. DEFINITIONS

We now set the following notation, which remains fixed for the remainder of this paper:

- Let be a function $f: \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$ then $f^\Delta(t)$ is delta time-scale derivative [1] of f

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

where $\mu(t) = \sigma(t) - t$, $\mu(t) \neq 0$ and $\sigma(t) > t$ is forward jump operator.

- $\frac{\partial f(t_1, \dots, t_n)}{\Delta_i t_i}$, $f_{t_i}^{\Delta_i}(t)$ is delta partial derivative of $f: \Lambda^n \rightarrow \mathbb{R}$ on n -dimensional time scale Λ^n [2, 3, 4], defined as a limit

$$f_{t_i}^{\Delta_i}(t) = \lim_{\substack{s_i \rightarrow t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, \dots, t_{i-1}, \sigma_i(t_i), t_{t+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, s_i, t_{t+1}, \dots, t_n)}{\sigma_i(t_i) - s_i},$$

where $\sigma_i(t_i) > t_i$ and $\sigma_i(t_i) - s_i \neq 0$.

- $D_q f(x)$ is q -derivative [5, 6, 7, 8]

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x},$$

where $x \neq 0$, $x \in \mathbb{R}$, $q \in \mathbb{R}$.

- $D_{n,q} f(t)$ is q -power derivative [9]

$$D_{n,q} f(t) = \frac{f(qt^n) - f(t)}{qt^n - t},$$

where $qt^n - t \neq 0$ and n is odd positive integer and $0 < q < 1$.

- $\mathcal{D}_q f(x)$ is q -power derivative

$$\mathcal{D}_q f(x) = \frac{f(x^q) - f(x)}{x^q - x},$$

where $x^q \neq x$, $x \in \mathbb{R}$, $q \in \mathbb{R}$.

- $\mathbf{P}_b^m(x)$, $x, b \in \mathbb{R}$, $m \in \mathbb{N}$ is $2m + 1$ -degree integer-valued polynomial [10]

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x - k)^r, \quad (1.1)$$

where $\mathbf{A}_{m,r}$, $m \in \mathbb{N}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m, \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m, \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

where B_t are Bernoulli numbers [11]. It is assumed that $B_1 = \frac{1}{2}$.

- \mathbb{Z} is an integer time scale such that $\sigma(t) = t + 1$ and $\mu(t) = 1$.
- \mathbb{R} is a real time scale such that $\sigma(t) = t + \Delta t$ and $\mu(t) = \Delta t$, $\Delta t \rightarrow 0$.
- $q^{\mathbb{R}}$ is a quantum time scale such that $\sigma(t) = qt$ and $\mu(t) = qt - t$, [page 18 [1]].
- \mathbb{R}^q is a quantum power time scale such that $\sigma(t) = t^q$ and $\mu(t) = t^q - t$.
- $q^{\mathbb{R}^n}$ is a pure quantum power time scale such that $\sigma(t) = qt^n > t$, $0 < q < 1$, $\mu(t) = qt^n - t$ and n is positive odd integer [9].

2. INTRODUCTION

Time-scale calculus is quite graceful generalization and unification of the theory of differential equations. Firstly being introduced by Hilger [12] in his Ph.D thesis in 1988 and thereafter greatly extended by Bohner and Peterson [1] in 2001, the calculus on time scales became a sharp tool in the world on differential equations. Various derivative operators like classical derivative $\frac{d}{dx}f(x)$, q -derivative $D_q f(x)$, q -power derivative $\mathcal{D}_q f(x)$, finite difference $\Delta f(x)$ etc, may be simply expressed in terms of time-scale derivative over particular time scale \mathbb{T} . For instance,

$$\begin{aligned} f'(x) &= f^\Delta(x), & x \in \mathbb{T} = \mathbb{R} \\ \Delta f(x) &= f^\Delta(x), & x \in \mathbb{T} = \mathbb{Z} \\ D_{n,q} f(x) &= f^\Delta(x), & x \in \mathbb{T} = q^{\mathbb{R}^n} \\ D_q f(x) &= f^\Delta(x), & x \in \mathbb{T} = q^{\mathbb{R}} \\ \mathcal{D}_q f(x) &= f^\Delta(x), & x \in \mathbb{T} = \mathbb{R}^q \end{aligned}$$

In context of Computer Science, namely object oriented programming paradigm, the time scale calculus may be thought as unified interface of derivative operator. Furthermore, the idea of time-scale calculus was slightly extended in [13, 14, 15, 16].

3. MAIN RESULTS

Time scale derivative of the polynomial t^{2m+1} may be expressed as follows

Theorem 3.1. Let $\mathbf{P}_b^m(x)$ be a $2m + 1$ -degree integer-valued polynomial defined by (1.1). Let be a two-dimensional time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x, b) : x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$. Let be $\mathbb{T}_1 = \mathbb{T}_2$. For every $t \in \mathbb{T}_1$, $x, b \in \Lambda^2$, $m \in \mathbb{N}$, $m = \text{const}$

$$(t^{2m+1})^\Delta = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t},$$

where

- $\sigma(t) > t$ is forward jump operator,
- $\frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)}$ is the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable x , evaluated at $x = t$, $b = \sigma(t)$,
- $\frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t}$ is the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable b , evaluated at $x = t$, $b = t$.

In other words, theorem 3.1 says

For every odd-exponent polynomial t^{2m+1} , its derivative on time scales equals to the sum of the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable x , evaluated at $x = t$, $b = \sigma(t)$ and the value of the partial derivative on time scales of $\mathbf{P}_b^m(x)$ with respect to the variable b , evaluated at $x = t$, $b = t$.

In extended form theorem 3.1 may be written as

$$(t^{2m+1})_t^\Delta = \frac{\partial}{\Delta x} \left(\sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \right) \Big|_{x=t, b=\sigma(t)} + \frac{\partial}{\Delta b} \left(\sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \right) \Big|_{x=t, b=t}$$

4. DISCUSSION AND EXAMPLES

To understand the nature of theorem 3.1, let's discuss an example of some popular time scales, like integer time scale \mathbb{Z} , real time scale \mathbb{R} , quantum time scale $q^{\mathbb{R}}$, quantum-power time scale \mathbb{R}^q . We use the principle *Divide et Impera !* in order to understand entire behavior of theorem 3.1.

4.1. Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$.

Corollary 4.1. (Finite difference.) Let be a two-dimensional time scale $\Lambda^2 = \mathbb{Z} \times \mathbb{Z} := \{t = (x, b) : x \in \mathbb{Z}, b \in \mathbb{Z}\}$. For every $t \in \mathbb{Z}$, $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, $m \in \mathbb{N}$

$$\Delta t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} (t, t) \Big|_{x=t, b=t},$$

where the forward jump operator $\sigma(t)$ is defined as $\sigma(t) = t + 1$.

Example 4.2. Let be $t \in \mathbb{Z}$, $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, $m \in \mathbb{N}$ and let $m = 1$, then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= 3t + 3t^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 1 - 6b^2 + 6bx \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} &= 1\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get the ordinary finite difference of odd polynomial t^{2m+1} , $t \in \mathbb{Z}$, $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, $m \in \mathbb{N}$

$$\Delta t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = 3t^2 + 3t + 1.$$

Example 4.3. Let be $t \in \mathbb{Z}$, $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, $m \in \mathbb{N}$ and let $m = 2$, then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= 5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 1 + 30b^4 - 60b^3x + 30b^2x^2 \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= 5t + 10t^2 + 10t^3 + 5t^4 \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} \Big|_{x=t, b=t} &= 1\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get time ordinary finite difference of odd polynomial t^{2m+1} , $t \in \mathbb{Z}$, $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, $m \in \mathbb{N}$

$$\Delta t^5 = \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} \Big|_{x=t, b=t} = 1 + 5t + 10t^2 + 10t^3 + 5t^4.$$

Corollary 4.4. For every $t \in \mathbb{Z}$, $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, $m \in \mathbb{N}$

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} = \sum_{r=1}^{2m} \binom{2m+1}{r} t^r$$

Corollary 4.5. For every $t \in \mathbb{Z}$, $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, $m \in \mathbb{N}$

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = 1$$

4.2. Time scale of real numbers $\mathbb{T} = \mathbb{R} \times \mathbb{R}$.

Corollary 4.6. (Classical derivative.) Let be a two-dimensional time scale $\Lambda^2 = \mathbb{R} \times \mathbb{R} := \{t = (x, b) : x \in \mathbb{R}, b \in \mathbb{R}\}$. For every $t \in \mathbb{R}$, $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$, $m \in \mathbb{N}$

$$\frac{d}{dt}t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\partial x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\partial b} \Big|_{x=t, b=t},$$

where $\sigma(t) = t + \Delta t$, $\Delta t \rightarrow 0$.

Example 4.7. Let be $t \in \mathbb{R}$, $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$, $m \in \mathbb{N}$ and let $m = 1$, then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^1(x)}{\partial x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\partial b} &= 6b - 6b^2 - 3x + 6bx \\ \frac{\partial \mathbf{P}_b^1(x)}{\partial x} \Big|_{x=t, b=\sigma(t)} &= -3t + 3t^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\partial b} \Big|_{x=t, b=t} &= 3t \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of odd polynomial t^{2m+1} , $t \in \mathbb{R}$, $x \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$, $m \in \mathbb{N}$

$$\frac{d}{dt}t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\partial x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\partial b} \Big|_{x=t, b=t} = 3t^2.$$

Example 4.8. Let be $t \in \mathbb{R}$, $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$, $m \in \mathbb{N}$ and let $m = 2$, then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\partial x} &= -15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x, \\ \frac{\partial \mathbf{P}_b^2(x)}{\partial b} &= 30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2 \\ \frac{\partial \mathbf{P}_b^2(x)}{\partial x} \Big|_{x=t, b=\sigma(t)} &= -5t^2 + 5t^4 \\ \frac{\partial \mathbf{P}_b^2(x)}{\partial b} \Big|_{x=t, b=t} &= 5t^2 \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of an odd polynomial t^{2m+1} , $t \in \mathbb{R}$, $x \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$, $m \in \mathbb{N}$

$$\frac{d}{dt}t^5 = \frac{\partial \mathbf{P}_b^2(x)}{\partial x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^2(x)}{\partial b} \Big|_{x=t, b=t} = 5t^4.$$

4.3. Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$.

Corollary 4.9. (*Q-derivative [5].*) Let be a two-dimensional time scale $\Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}} := \{t = (x, b) : x \in q^{\mathbb{R}}, b \in q^{\mathbb{R}}\}$. For every $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$

$$D_q t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t},$$

where $\sigma(t) = qt, q > 1$.

Example 4.10. Let be $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$ and let $m = 1$, then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= -3qt + 3q^2t^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} &= 3qt + t^2 + qt^2 - 2q^2t^2 \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get q -derivative of odd polynomial $t^{2m+1}, t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$

$$D_q t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^2 + qt^2 + q^2t^2.$$

For every $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ the following polynomial identity holds as q tends to zero

$$\lim_{q \rightarrow 0} \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^2$$

However, it would be generalized as follows

Corollary 4.11. For every $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$

$$\lim_{q \rightarrow 0} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = t^{2m}.$$

Example 4.12. Let be $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$ and let $m = 2$, then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx - 15b^2qx + 10b^3qx \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 + 6b^4q^2 - 15b^3q^3 \\ &\quad + 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x \\ &\quad - 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2 \end{aligned}$$

$$\begin{aligned}\left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} &= 5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4 \\ \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} &= -5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get the q -derivative of odd polynomial t^{2m+1} , $t \in q^{\mathbb{R}}$, $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$, $m \in \mathbb{N}$

$$D_q t^5 = \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} = t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4.$$

4.4. Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$.

Corollary 4.13. (Q -power derivative [9].) Let be a two-dimensional time scale $\Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q := \{t = (x, b) : b \in \mathbb{R}^q, x \in \mathbb{R}^q\}$. For every $t \in \mathbb{R}^q$, $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, $m \in \mathbb{N}$

$$\mathcal{D}_q t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t, t),$$

where the forward jump operator is defined as $\sigma(t) = t^q$, $q > 1$.

Example 4.14. Let be $t \in \mathbb{R}^q$, $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, $m \in \mathbb{N}$ and let $m = 1$, then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^q x \\ \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} &= -3t^q + 3t^{2q} \\ \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}\right|_{x=t, b=t} &= t^2 + 3t^q - 2t^{2q} + t^{1+q}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get q -power derivative of odd polynomial t^{2m+1} , $t \in \mathbb{R}^q$, $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, $m \in \mathbb{N}$

$$\mathcal{D}_q t^3 = \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}\right|_{x=t, b=t} = t^2 + t^{2q} + t^{1+q}.$$

Example 4.15. Let be $t \in \mathbb{R}^q$, $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, $m \in \mathbb{N}$ and let $m = 2$, then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bx^q - 15b^2x^q + 10b^3x^q \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{2q} - 15b^{3q} + 6b^{4q} + 10b^{1+q} - 15b^{2+q} + 6b^{3+q} \\ &\quad - 15b^{1+2q} + 6b^{2+2q} + 6b^{1+3q} - 15bx + 30b^2x - 15b^3x - 15b^q x + 30b^{2q}x \\ &\quad - 15b^{3q}x + 30b^{1+q}x - 15b^{2+q}x - 15b^{1+2q}x + 5x^2 - 15bx^2 + 10b^2x^2 \\ &\quad - 15b^q x^2 + 10b^{2q}x^2 + 10b^{1+q}x^2\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} &= -10t^{2q} + 15t^{3q} - 5t^{4q} + 5t^{1+q} - 15t^{1+2q} + 10t^{1+3q} \\ \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} &= t^4 + 10t^{2q} - 15t^{3q} + 6t^{4q} - 5t^{1+q} + t^{3+q} + 15t^{1+2q} + t^{2+2q} - 9t^{1+3q}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get q -power derivative of odd polynomial t^{2m+1} , $t \in \mathbb{R}^q$, $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, $m \in \mathbb{N}$

$$\mathcal{D}_q t^5 = \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} = t^4 + t^{4q} + t^{3+q} + t^{2+2q} + t^{1+3q}.$$

Another polynomial identity, that is exponential sum holds

Corollary 4.16. *For every $t \in \mathbb{R}^q$, $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, $t \in \mathbb{R}$, $m \in \mathbb{N}$*

$$\lim_{q \rightarrow 0} \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}\right|_{x=t, b=t} = \sum_{k=0}^{2m} t^k$$

4.5. Pure quantum power time scale $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$. In this subsection we discuss a pure quantum power time scale $q^{\mathbb{R}^j}$ provided by Aldwoah, Malinowska and Torres in [9], among with the q -power derivative operator $D_{n,q}f(t)$ defined by

$$D_{n,q}f(t) = \frac{f(qt^n) - f(t)}{qt^n - t},$$

where n is odd positive integer and $0 < q < 1$.

Corollary 4.17. *(Quantum power derivative [9].) Let be a two-dimensional time scale $\Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j} := \{t = (x, b) : b \in q^{\mathbb{R}^j}, x \in q^{\mathbb{R}^j}\}$. For every $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $m \in \mathbb{N}$*

$$D_{n,q}t^{2m+1} = \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}\right|_{x=t, b=t},$$

where $\sigma(t) = qt^n$, $\sigma(t) > t$.

Example 4.18. *Let be $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $m \in \mathbb{N}$ and let $m = 1$, then*

$$\begin{aligned}\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 3b - 2b^2 + 3b^j q - 2b^{1+j} q - 2b^{2j} q^2 - 3x + 3bx + 3b^j qx \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) &= -3qt^j + 3q^2 t^{2j} \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) &= t^2 + 3qt^j - 2q^2 t^{2j} + qt^{1+j}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get q -power derivative of odd polynomial t^{2m+1} , $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $m \in \mathbb{N}$

$$D_{n,q}t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^2 + q^2 t^{2j} + qt^{1+j}.$$

Another polynomial identity, that is exponential sum holds

Corollary 4.19. For every $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $t \in \mathbb{R}$, $m \in \mathbb{N}$

$$\lim_{j \rightarrow 0} \lim_{q \rightarrow 1} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = \sum_{k=0}^{2m} t^k$$

An identity in even polynomials holds too

Corollary 4.20. For every $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $t \in \mathbb{R}$, $m \in \mathbb{N}$

$$\lim_{j \rightarrow 0} \lim_{q \rightarrow 0} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = t^{2m}$$

Example 4.21. Let be $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $m \in \mathbb{N}$ and let $m = 2$, then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx^j - 15b^2qx^j + 10b^3qx^j \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{1+j}q - 15b^{2+j}q + 6b^{3+j}q + 10b^{2j}q^2 - 15b^{1+2j}q^2 \\ &\quad + 6b^{2+2j}q^2 - 15b^{3j}q^3 + 6b^{1+3j}q^3 + 6b^{4j}q^4 - 15bx + 30b^2x - 15b^3x - 15b^j qx \\ &\quad + 30b^{1+j}qx - 15b^{2+j}qx + 30b^{2j}q^2x - 15b^{1+2j}q^2x - 15b^{3j}q^3x + 5x^2 - 15bx^2 \\ &\quad + 10b^2x^2 - 15b^j qx^2 + 10b^{1+j}qx^2 + 10b^{2j}q^2x^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= -10q^2t^{2j} + 15q^3t^{3j} - 5q^4t^{4j} + 5qt^{1+j} - 15q^2t^{1+2j} + 10q^3t^{1+3j} \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} \Big|_{x=t, b=t} &= t^4 + 10q^2t^{2j} - 15q^3t^{3j} + 6q^4t^{4j} - 5qt^{1+j} + qt^{3+j} + 15q^2t^{1+2j} + q^2t^{2+2j} - 9q^3t^{1+3j} \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get q -power derivative of odd polynomial t^{2m+1} , $t \in q^{\mathbb{R}^j}$, $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $m \in \mathbb{N}$

$$D_{n,q}t^5 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^4 + q^4 t^{4j} + qt^{3+j} + q^2 t^{2+2j} + q^3 t^{1+3j}.$$

5. PROOF OF MAIN THEOREM

By [Lemma 3.1 [10]], for every $x \in \mathbb{R}$, $m \in \mathbb{N}$ it is true that

$$\mathbf{P}_x^m(x) = x^{2m+1} \tag{5.1}$$

Proof of theorem 3.1. Let be $x, b \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 := \{t = (x, b) : x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$. Let be $\mathbb{T}_1 = \mathbb{T}_2$. Assume that time-scale derivative $(x^{2m+1})^\Delta$ is

$$(x^{2m+1})^\Delta = \lim_{b \rightarrow x} \lim_{t \rightarrow x} \frac{\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(t)}{\sigma(x) - t}, \quad (5.2)$$

where $\sigma(x) > x$ is forward jump operator. However, equation (5.2) is not a timescale derivative of $\mathbf{P}_b^m(x)$ over x how it might seem because of denominator $\sigma(x) - t$. Parameter b of $\mathbf{P}_b^m(x)$ is implicitly incremented as well. Let's try to express nominator of (5.2) in terms of partial derivative $\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}$ on timescales. Let be the following equation

$$\mathbf{P}_{\sigma(b)}^m(x) - \mathbf{P}_b^m(x) = \mathbf{P}_b^m(x)_b^\Delta \cdot \Delta b$$

Let $t \rightarrow x$ in (5.2). Then nominator of (5.2) equals to

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(x) = \mathbf{P}_{\sigma(b)}^m(x) - \mathbf{P}_b^m(x) + A$$

where A is yet implicit term. Let's now collapse the terms $f_m(x, b)$ from both sides of above equation, such that

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) = \mathbf{P}_{\sigma(b)}^m(x) + A$$

Therefore,

$$A = \mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_{\sigma(b)}^m(x) = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(b)) \cdot \Delta x$$

Now, let's express the nominator of (5.2) as follows

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(x) = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(b)) \cdot \Delta x + \mathbf{P}_b^m(x)_b^\Delta(x, b) \cdot \Delta b$$

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(x) = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(b)) \cdot (\sigma(x) - x) + \mathbf{P}_b^m(x)_b^\Delta(x, b) \cdot (\sigma(b) - b)$$

We can collapse the terms $(\sigma(x) - x)$, $(\sigma(b) - b)$ in above expressions, as $b \rightarrow x$. Therefore,

$$\frac{\mathbf{P}_{\sigma(x)}^m(\sigma(x)) - \mathbf{P}_x^m(x)}{\sigma(x) - x} = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(x)) + \mathbf{P}_b^m(x)_b^\Delta(x, x)$$

Finally, by the identity (5.1) we can express timescale derivative of x^{2m+1} , $x \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$, $m \in \mathbb{N}$ as

$$(x^{2m+1})^\Delta(t) = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t}$$

This completes the proof. □

6. MATHEMATICA SCRIPTS

To fulfill our study, we attach here a link to the set of *Mathematica* programs, designed to verify the results of current manuscript. To reach these programs follow the link [17]. To reproduce results, proceed as follows:

- Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$:
 - Example 4.2: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x + 1` in Mathematica package and execute definition.
 - * Execute `timeScaleDerivativeX[1, x, b]` which produces $-3b + 3b^2$.
 - * Execute `Expand[timeScaleDerivativeX[1, t, sigma[t]]]` which produces $3t + 3t^2$.
 - * Execute `timeScaleDerivativeB[1, x, b]` which produces $1 - 6b^2 + 6bx$.
 - * Execute `timeScaleDerivativeB[1, t, t]` which produces 1.
 - * Execute `mainTheorem[1]` which produces $1 + 3t + 3t^2$.
 - Example 4.3: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x + 1` in Mathematica package and execute definition.
 - * `timeScaleDerivativeX[2, x, b]` which produces $5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$.
 - * `Expand[timeScaleDerivativeX[2, t, sigma[t]]]` which produces $5t + 10t^2 + 10t^3 + 5t^4$.
 - * `timeScaleDerivativeB[2, x, b]` which produces $1 + 30b^4 - 60b^3x + 30b^2x^2$.
 - * `timeScaleDerivativeB[2, t, t]` which produces 1.
 - * `mainTheorem[2]` which produces $1 + 5t + 10t^2 + 10t^3 + 5t^4$.
- Time scale of real numbers $\mathbb{T} = \mathbb{R} \times \mathbb{R}$:
 - Example 4.7: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x + Global`dx` in Mathematica package and execute definition.
 - * Execute `timeScaleDerivativeX[1, x, b]` which produces $-3b + 3b^2$.
 - * Execute `Limit[Expand[timeScaleDerivativeB[1, x, b]], dx -> 0]` which produces $6b - 6b^2 - 3x + 6bx$.
 - * Execute `timeScaleDerivativeX[1, t, t]` which produces $-3t + 3t^2$.

- * Execute `Limit[Expand[timeScaleDerivativeB[1, t, t]], dx -> 0]` which produces $3t$.
- * Execute `Limit[mainTheorem[1], dx -> 0]` which produces $3t^2$.
- Example 4.8: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x + Global`dx` in Mathematica package and execute definition.
 - * Execute `Limit[Expand[timeScaleDerivativeX[2, x, b]], dx -> 0]` which produces $-15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$.
 - * Execute `Limit[Expand[timeScaleDerivativeB[2, x, b]], dx -> 0]` which produces $30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2$.
 - * Execute `Limit[Expand[timeScaleDerivativeX[2, t, sigma[t]]], dx -> 0]` which produces $-5t^2 + 5t^4$.
 - * Execute `Limit[Expand[timeScaleDerivativeB[2, t, t]], dx -> 0]` which produces $5t^2$.
 - * Execute `Limit[mainTheorem[2], dx -> 0]` which produces $5t^4$.

• Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$:

- Example 4.10: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x * Global`q` in Mathematica package and execute definition.
 - * Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces $-3b + 3b^2$.
 - * Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces $3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx$.
 - * Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]]` which produces $-3qt + 3q^2t^2$.
 - * Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces $3qt + t^2 + qt^2 - 2q^2t^2$.
 - * Execute `Expand[Simplify[mainTheorem[1]]]` which produces $t^2 + qt^2 + q^2t^2$.
- Example 4.12: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x * Global`q` in Mathematica package and execute definition.
 - * Execute `Expand[Simplify[timeScaleDerivativeX[2, x, b]]]` which produces $-15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx - 15b^2qx + 10b^3qx$.
 - * Execute `Expand[Simplify[timeScaleDerivativeB[2, x, b]]]` which produces $10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 +$

$$6b^4q^2 - 15b^3q^3 + 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x - 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2.$$

- * Execute `Expand[Simplify[timeScaleDerivativeX[2, t, sigma[t]]]` which produces $5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4$.
- * Execute `Expand[Simplify[timeScaleDerivativeB[2, t, t]]]` which produces $-5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4$.
- * Execute `Expand[Simplify[mainTheorem[2]]]` which produces $t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4$.
- Corollary 4.11: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x * Global'q` in Mathematica package and execute definition.
 - * Execute `Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 0]` for various values of m .
- Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$:
 - Example 4.14: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x ∧ Global'q` in Mathematica package and execute definition.
 - * Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces $-3b + 3b^2$.
 - * Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces $3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^qx$.
 - * Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]` which produces $-3t^q + 3t^{2q}$.
 - * Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces $t^2 + 3t^q - 2t^{2q} + t^{1+q}$.
 - * Execute `Expand[Simplify[mainTheorem[1]]]` which produces $t^2 + t^{2q} + t^{1+q}$.
 - Example 4.15: Similarly to Example 4.14 with $m = 2$.
 - Corollary 4.16: Execute the commands of Mathematica package
 - * Set `sigma[x_] := x ∧ Global'q` in Mathematica package and execute definition.
 - * Execute `Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 0]` for various values of m .
- Pure quantum power time scale $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$:
 - Example 4.18: Execute the commands of Mathematica package

- * Set `sigma[x_] := Global`q * x ^ Global`j` in Mathematica package and execute definition.
 - * Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces $-3b + 3b^2$.
 - * Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces $3b - 2b^2 + 3b^j q - 2b^{1+j} q - 2b^{2j} q^2 - 3x + 3bx + 3b^j qx$.
 - * Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]` which produces $-3qt^j + 3q^2 t^{2j}$.
 - * Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces $t^2 + 3qt^j - 2q^2 t^{2j} + qt^{1+j}$.
 - * Execute `Expand[Simplify[mainTheorem[1]]]` which produces $t^2 + q^2 t^{2j} + qt^{1+j}$.
- Example 4.21: Similarly as Example 4.18 for $m = 2$.
 - Corollary 4.19: Execute the commands of Mathematica package
 - * Set `sigma[x_] := Global`q * x ^ Global`j` in Mathematica package and execute definition.
 - * Execute `Limit[Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 1], j -> 0]` for various values of m .
 - Corollary 4.20: Execute the commands of Mathematica package
 - * Set `sigma[x_] := Global`q * x ^ Global`j` in Mathematica package and execute definition.
 - * Execute `Limit[Limit[Expand[Simplify[timeScaleDerivativeB[5, t, t]]], q -> 0], j -> 0]` for various values of m .

7. CONCLUSION AND FUTURE RESEARCH

In this manuscript we have discussed partial time scale differential equation involving derivatives of polynomials in context of time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$ where $\mathbb{T}_1 = \mathbb{T}_2$. Future research can be conducted to study the case $\mathbb{T}_1 \neq \mathbb{T}_2$, which makes the theorem 3.1 to be generalised

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta x} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} = \alpha_m(x, b)(x^{2m+1})^\Delta,$$

where $\alpha_m(x, b)$ is arbitrary differentiable function. Also, it is worth to discuss the theorem 3.1 in context of high order derivatives on time scales. We have established a few power identities, and shown the theorem 3.1 for different 2-dimensional time scales Λ^2 like integer time scale $\mathbb{Z} \times \mathbb{Z}$, real time scale $\mathbb{R} \times \mathbb{R}$, quantum time scale $q^{\mathbb{R}} \times q^{\mathbb{R}}$ and quantum power time scale $\mathbb{R}^q \times \mathbb{R}^q$.

REFERENCES

- [1] Martin Bohner and Allan Peterson. *Dynamic equations on time scales: An introduction with applications*. Springer Science & Business Media, 2001. <https://web.mst.edu/~bohner/sample.pdf>.
- [2] Martin Bohner and Gusein Sh Guseinov. Partial differentiation on time scales. *Dynamic systems and applications*, 13(3-4):351–379, 2004. <http://web.mst.edu/~bohner/papers/pdots.pdf>.
- [3] Calvin D Ahlbrandt and Christina Morian. Partial differential equations on time scales. *Journal of Computational and Applied Mathematics*, 141(1-2):35–55, 2002. [https://doi.org/10.1016/S0377-0427\(01\)00434-4](https://doi.org/10.1016/S0377-0427(01)00434-4).
- [4] B. Jackson. Partial dynamic equations on time scales. *Journal of Computational and Applied Mathematics*, 186(2):391 – 415, 2006. <https://doi.org/10.1016/j.cam.2005.02.011>.
- [5] Frederick H Jackson. Xi.—on q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, 46(2):253–281, 1909.
- [6] Thomas Ernst. *The history of q-calculus and a new method*. Citeseer, 2000.
- [7] Thomas Ernst. The different tongues of q-calculus. *Proceedings of the Estonian Academy of Sciences*, 57(2), 2008.
- [8] Victor Kac and Pokman Cheung. *Quantum calculus*. Springer Science & Business Media, 2001.
- [9] Khaled A Aldwoah, Agnieszka B Malinowska, and Delfim FM Torres. The power quantum calculus and variational problems. *arXiv preprint arXiv:1107.0344*, 2011. <https://arxiv.org/abs/1107.0344>.
- [10] Petro Kolosov. On the link between Binomial Theorem and Discrete Convolution of Polynomials. *arXiv preprint arXiv:1603.02468*, 2016. <https://arxiv.org/abs/1603.02468>.
- [11] Eric W Weisstein. "Bernoulli Number." From MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/BernoulliNumber.html>.
- [12] S Hilger. *Eine neue Kettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeit*. PhD thesis, Ph. D. Thesis, Universität Würzburg, 1988.
- [13] Benaoumeur Bayour, Ahmed Hammoudi, and Delfim FM Torres. A truly conformable calculus on time scales. *arXiv preprint arXiv:1705.08928*, 2017. <https://arxiv.org/abs/1705.08928>.
- [14] Nadia Benkhettou, Salima Hassani, and Delfim FM Torres. A conformable fractional calculus on arbitrary time scales. *Journal of King Saud University-Science*, 28(1):93–98, 2016.
- [15] M Cristina Caputo. Time scales: from nabla calculus to delta calculus and vice versa via duality. *arXiv preprint arXiv:0910.0085*, 2009.
- [16] Natália Martins and Delfim FM Torres. Calculus of variations on time scales with nabla derivatives. *Nonlinear Analysis: Theory, Methods & Applications*, 71(12):e763–e773, 2009.
- [17] Petro Kolosov. Supplementary Mathematica Programs. 2020. <https://github.com/kolosovpetro/AStudyOnDynamicEquations/tree/develop/mathematica>.

Version: Local-0.1.0

Email address: kolosovp94@gmail.com

URL: <https://kolosovpetro.github.io>