A STUDY ON PARTIAL DYNAMIC EQUATION ON TIME SCALES INVOLVING DERIVATIVES OF POLYNOMIALS

PETRO KOLOSOV

Abstract. Let $P(m,b,x)$ be a $2m+1$-degree polynomial in x, b. Let be a two-dimensional timescale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x,b)\colon\ x \in \mathbb{T}_1,\ b \in \mathbb{T}_2\}$ such that $\mathbb{T}_1 = \mathbb{T}_2$. In this manuscript we derive and discuss an identity that connects the timescale derivative of odd-power polynomial with partial derivatives of polynomial $P(m,b,x)$ evaluated in particular points. For every $t \in \mathbb{T}_1$ and $(x,b) \in \Lambda^2$

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t) + \frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$$

such that $\sigma(t) > t$ is forward jump operator. In addition, we discuss various derivative operators in context of partial cases of above equation, we show finite difference, classical derivative, $q-$derivative, $q-$power derivative on behalf of it.

Contents

Date: September 18, 2024.

2010 *Mathematics Subject Classification.* 26E70, 05A30.

Key words and phrases. Dynamic equations on time scales, Partial differential equations on time scales, Partial dynamic equations on time scales, Partial differentiation on time scales, Dynamical systems .

1. Definitions

We now set the following notation such that remains fixed for the remainder of this manuscript

- Let be a function $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$ then $f^\Delta(t)$ is delta timescale derivative [1]

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

  where $\sigma(t) - t \neq 0$ and $\sigma(t) > t$ is forward jump operator.

- $\dfrac{\partial f(t_1, \ldots, t_n)}{\Delta_i t_i}$ is the delta partial derivative of $f\colon \Lambda^n \to \mathbb{R}$ on n-dimensional timescale Λ^n defined via the limit [2, 3, 4]

$$\frac{\partial f(t_1, \ldots, t_n)}{\Delta_i t_i} = \lim_{s_i \to t_i} \frac{f(t_1, \ldots, t_{i-1}, \sigma_i(t_i), t_{t+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, s_i, t_{t+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i}$$

  where $\sigma_i(t_i) > t_i$ and $\sigma_i(t_i) - s_i \neq 0$.

- $D_q f(x)$ is $q-$derivative [5, 6, 7, 8]

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

  where $x \neq 0,\ x \in \mathbb{R},\ q \in \mathbb{R}$.

- $D_{n,q}f(t)$ is $q-$power derivative [9]

$$D_{n,q}f(t) = \frac{f(qt^n) - f(t)}{qt^n - t}$$

where $qt^n - t \neq 0$ and n is odd positive integer and $0 < q < 1$.

- $\mathcal{D}_q f(x)$ is $q-$power derivative

$$\mathcal{D}_q f(x) = \frac{f(x^q) - f(x)}{x^q - x}$$

where $x^q \neq x,\ x \in \mathbb{R},\ q \in \mathbb{R}$.

- $P(m, b, x)$ is $2m+1$-degree polynomial in x, b

$$P(m, b, x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.1}$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively, see [10].

- $\mathbb{Z}$ is an integer timescale such that $\sigma(t) = t + 1$ and $\mu(t) = 1$.

- $\mathbb{R}$ is a real timescale such that $\sigma(t) = t + \Delta t$ and $\mu(t) = \Delta t,\ \Delta t \to 0$.

- $q^{\mathbb{R}}$ is a quantum timescale such that $\sigma(t) = qt$ and $\mu(t) = qt - t$, see [1, p. 18].

- $\mathbb{R}^q$ is a quantum power timescale such that $\sigma(t) = t^q$ and $\mu(t) = t^q - t$.

- $q^{\mathbb{R}^n}$ is a pure quantum power timescale such that $\sigma(t) = qt^n > t,\ 0 < q < 1,\ \mu(t) = qt^n - t$ and n is positive odd integer [9].

2. Introduction

Time-scale calculus is quite graceful generalization and unification of the theory of differential equations. Firstly being introduced by Hilger [11] in his Ph.D thesis in 1988 and thereafter greatly extended by Bohner and Peterson [1] in 2001, the calculus on time scales became a sharp tool in the world on differential equations. Various derivative operators like classical derivative $\frac{d}{dx}f(x)$, $q-$derivative $D_q f(x)$, $q-$power derivative $\mathcal{D}_q f(x)$, finite difference $\Delta f(x)$ etc, may be simply expressed in terms of time-scale derivative over particular time scale $\mathbb{T}$. For instance,

$$f'(x) = f^{\Delta}(x), \quad x \in \mathbb{T} = \mathbb{R}$$

$$\Delta f(x) = f^{\Delta}(x), \quad x \in \mathbb{T} = \mathbb{Z}$$

$$D_{n,q} f(x) = f^{\Delta}(x), \quad x \in \mathbb{T} = q^{\mathbb{R}^n}$$

$$D_q f(x) = f^{\Delta}(x), \quad x \in \mathbb{T} = q^{\mathbb{R}}$$

$$\mathcal{D}_q f(x) = f^{\Delta}(x), \quad x \in \mathbb{T} = \mathbb{R}^q$$

In context of Computer Science, namely object oriented programming paradigm, the time scale calculus may be thought as unified interface of derivative operator. Furthermore, the idea of time-scale calculus was slightly extended in [12, 13, 14, 15].

3. Main results

Timescale derivative of odd-powered polynomial t^{2m+1} may be expressed as follows

Theorem 3.1. *Let $P(m,b,x)$ be a $2m+1$-degree polynomial in x, b. Let be a two-dimensional timescale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x,b)\colon\ x \in \mathbb{T}_1,\ b \in \mathbb{T}_2\}$ such that $\mathbb{T}_1 = \mathbb{T}_2$. For every $t \in \mathbb{T}_1$ and $x, b \in \Lambda^2$*

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t) + \frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$$

where

- $\sigma(t) > t$ *– is forward jump operator*
- $\frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t)$ *– is the value of the partial derivative on time scales of $P(m,b,x)$ with respect to the variable x evaluated in point $(x,b) = (t,\sigma(t))$*

- $\frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$ – *is the value of the partial derivative on time scales of* $P(m,b,x)$ *with respect to the variable* b*, evaluated at* $(x,b)=(t,t)$

In simpler words, the theorem 3.1 says

> For every odd-powered polynomial x^{2m+1}, the derivative on time scales $\frac{\Delta x^{2m+1}}{\Delta x}$ evaluated in point $t \in \mathbb{T}_1$ equals to partial derivative on time scales of the polynomial $P(m,b,x)$ with respect to x evaluated in point $(x,b)=(t,\sigma(t))$ plus the value of the partial derivative on time scales of the polynomial $P(m,b,x)$ with respect to b, evaluated in point $(x,b)=(t,t)$.

In its extended form the theorem 3.1 is as follows

$$\begin{aligned}\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial}{\Delta x}\left(\sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r\right)(m,\sigma(t),t)\\ + \frac{\partial}{\Delta b}\left(\sum_{k=0}^{b-1}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(x-k)^r\right)(m,t,t)\end{aligned}$$

4. Discussion and examples

To understand the nature of the theorem 3.1, we discuss a few examples involving widely-known time scales including integer timescale $\mathbb{Z}$, real timescale $\mathbb{R}$, quantum timescale $q^{\mathbb{R}}$ and quantum-power timescale $\mathbb{R}^q$.

4.1. Time scale of integers $\mathbb{T}=\mathbb{Z}\times\mathbb{Z}$.

Corollary 4.1. *(Divided difference.) Let be a two-dimensional timescale* $\Lambda^2=\mathbb{Z}\times\mathbb{Z}$*. For every* $t\in\mathbb{Z}$ *and* $x,b\in\Lambda^2$

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t) + \frac{\partial P(m,b,x)}{\Delta b}(m,\sigma(t),t)$$

where $\sigma(t)$ *is the forward jump operator defined as* $\sigma(t)=t+1$*.*

Example 4.2. *Let be $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, let $m = 1$ then*

$$\frac{\partial P(1,b,x)}{\Delta x} = -3b + 3b^2$$
$$\frac{\partial P(1,b,x)}{\Delta b} = 1 - 6b^2 + 6bx$$

Evaluating in points yields

$$\frac{\partial P(1,b,x)}{\Delta x}(1,\sigma(t),t) = 3t + 3t^2$$
$$\frac{\partial P(1,b,x)}{\Delta b}(1,t,t) = 1$$

Summing up previously obtained partial timescale derivatives, we get ordinary finite difference of odd-powered polynomial x^3 evaluated in point $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$

$$\Delta x^3(t) = \frac{\partial P(1,b,x)}{\Delta x}(1,\sigma(t),t) + \frac{\partial P(1,b,x)}{\Delta b}(1,t,t) = 3t + 3t^2 + 1$$

Example 4.3. *Let be $t \in \mathbb{Z},\ x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$, let $m = 2$ then*

$$\frac{\partial P(2,b,x)}{\Delta x} = 5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$$
$$\frac{\partial P(2,b,x)}{\Delta b} = 1 + 30b^4 - 60b^3x + 30b^2x^2$$

Evaluating in points yields

$$\frac{\partial P(2,b,x)}{\Delta x}(1,\sigma(t),t) = 5t + 10t^2 + 10t^3 + 5t^4$$
$$\frac{\partial P(2,b,x)}{\Delta b}(1,t,t) = 1$$

Summing up previously obtained partial timescale derivatives, we get time ordinary finite difference of odd-powered polynomial x^5 and $t \in \mathbb{Z},\ (x, b) \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$

$$\Delta x^5(t) = \frac{\partial P(2,b,x)}{\Delta x}(1,t,\sigma(t)) + \frac{\partial P(2,b,x)}{\Delta b}(1,t,t) = 1 + 5t + 10t^2 + 10t^3 + 5t^4$$

Corollary 4.4. *For every $t \in \mathbb{Z},\ (x, b) \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$*

$$\frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t) = \sum_{r=1}^{2m} \binom{2m+1}{r} t^r$$

Corollary 4.5. *For every $t \in \mathbb{Z}$, $(x,b) \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$*

$$\frac{\partial P(m,b,x)}{\Delta b}(m,t,t) = 1$$

4.2. **Time scale of real numbers $\mathbb{T} = \mathbb{R} \times \mathbb{R}$.**

Corollary 4.6. *(Classical derivative.) Let be a two-dimensional timescale $\Lambda^2 = \mathbb{R} \times \mathbb{R} := \{t = (x,b)\colon\ x \in \mathbb{R},\ b \in \mathbb{R}\}$. For every $t \in \mathbb{R}$ and $(x,b) \in \Lambda^2$*

$$\frac{\mathrm{d}x^{2m+1}}{\mathrm{d}x}(t) = \frac{\partial P(m,b,x)}{\partial x}(m,\sigma(t),t) + \frac{\partial P(m,b,x)}{\partial b}(m,t,t)$$

where $\sigma(t) = t + \Delta t,\ \Delta t \to 0$.

Example 4.7. *Let be $t \in \mathbb{R}$, $(x,b) \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$, let $m = 1$ then*

$$\frac{\partial P(1,b,x)}{\partial x} = -3b + 3b^2$$
$$\frac{\partial P(1,b,x)}{\partial b} = 6b - 6b^2 - 3x + 6bx$$

Evaluating in points yields

$$\frac{\partial P(1,b,x)}{\partial x}(1,\sigma(t),t) = -3t + 3t^2$$
$$\frac{\partial P(1,b,x)}{\partial b}(1,t,t) = 3t$$

Summing up previously obtained partial timescale derivatives, we get an ordinary derivative of odd polynomial x^3 evaluated in point $t \in \mathbb{R}$.

$$\frac{\mathrm{d}x^3}{\mathrm{d}x}(t) = \frac{\partial P(1,b,x)}{\partial x}(1,\sigma(t),t) + \frac{\partial P(1,b,x)}{\partial b}(1,t,t) = 3t^2.$$

Example 4.8. *Let be $t \in \mathbb{R}$, $(x,b) \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$, let $m = 2$ then*

$$\frac{\partial P(2,b,x)}{\partial x} = -15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x,$$
$$\frac{\partial P(2,b,x)}{\partial b} = 30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2$$

Evaluation in points yields

$$\frac{\partial P(2,b,x)}{\partial x}(2,\sigma(t),t) = -5t^2 + 5t^4$$
$$\frac{\partial P(2,b,x)}{\partial b}(2,\sigma(t),t) = 5t^2$$

Summing up previously obtained partial timescale derivatives, we get classical derivative of an odd polynomial x^5 *evaluated in point* $t \in \mathbb{R}$

$$\frac{\mathrm{d}x^5}{\mathrm{d}x}(t) = \frac{\partial P(2,b,x)}{\partial x}(2,\sigma(t),t) + \frac{\partial P(2,b,x)}{\partial b}(2,\sigma(t),t) = 5t^4.$$

4.3. Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$.

Corollary 4.9. *(Q-derivative [5].) Let be a two-dimensional time scale* $\Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}} := \{t = (x,b)\colon\ x \in q^{\mathbb{R}},\ b \in q^{\mathbb{R}}\}$*. For every* $t \in q^{\mathbb{R}}$ *and* $(x,b) \in \Lambda^2$

$$D_q x^{2m+1}(t) = \frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t) + \frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$$

where $\sigma(t) = qt,\ q > 1$.

Example 4.10. *Let be* $t \in q^{\mathbb{R}},\ x,b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$*, let* $m = 1$ *then*

$$\frac{\partial P(1,b,x)}{\Delta x} = -3b + 3b^2$$
$$\frac{\partial P(1,b,x)}{\Delta b} = 3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx$$

Evaluating in points yields

$$\frac{\partial P(1,b,x)}{\Delta x}(m,\sigma(t),t) = -3qt + 3q^2t^2$$
$$\frac{\partial P(1,b,x)}{\Delta b}(m,t,t) = 3qt + t^2 + qt^2 - 2q^2t^2$$

Summing up previously obtained partial time-scale derivatives, we get the q-derivative of odd-powered polynomial x^3 *evaluated in point* $t \in q^{\mathbb{R}}$

$$D_q x^3(t) = \frac{\partial P(1,b,x)}{\Delta x}(m,\sigma(t),t) + \frac{\partial P(1,b,x)}{\Delta b}(m,t,t) = t^2 + qt^2 + q^2t^2.$$

For every $t \in q^{\mathbb{R}},\ (x,b) \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ the following polynomial identity holds as q tends to zero

$$\lim_{q\to 0} \frac{\partial P(1,b,x)}{\Delta b}(1,t,t) = t^2$$

However, it would be generalized as follows

Corollary 4.11. *For every $t \in q^{\mathbb{R}},\ (x,b) \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$*

$$\lim_{q\to 0} \frac{\partial P(m,b,x)}{\Delta b}(m,t,t) = t^{2m}.$$

Example 4.12. *Let be $t \in q^{\mathbb{R}},\ (x,b) \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$, let $m = 2$ then*

$$\frac{\partial P(2,b,x)}{\Delta x} = -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx - 15b^2qx + 10b^3qx$$

$$\begin{aligned}\frac{\partial P(2,b,x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 + 6b^4q^2 - 15b^3q^3 \\ &+ 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x \\ &- 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2\end{aligned}$$

Evaluating in points yields

$$\frac{\partial P(2,b,x)}{\Delta x}(2,\sigma(t),t) = 5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4$$

$$\frac{\partial P(2,b,x)}{\Delta b}(2,t,t) = -5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4$$

Summing up previously obtained partial time-scale derivatives, we get the $q-$derivative of odd polynomial x^5 evaluated in point $t \in q^{\mathbb{R}}$

$$D_q t^5 = \frac{\partial P(2,b,x)}{\Delta x}(2,\sigma(t),t) + \frac{\partial P(2,b,x)}{\Delta b}(2,t,t) = t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4.$$

4.4. Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$.

Corollary 4.13. *(Q-power derivative [9].) Let be a two-dimensional time scale $\Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q := \{t = (x,b) \colon\ b \in \mathbb{R}^q,\ x \in \mathbb{R}^q\}$. For every $t \in \mathbb{R}^q,\ (x,b) \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$*

$$\mathcal{D}_q t^{2m+1} = \frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t) + \frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$$

where the forward jump operator is defined as $\sigma(t) = t^q,\ q > 1$.

Example 4.14. *Let be $t \in \mathbb{R}^q,\ (x, b) \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, let $m = 1$ then*

$$\frac{\partial P(1,b,x)}{\Delta x} = -3b + 3b^2$$
$$\frac{\partial P(1,b,x)}{\Delta b} = 3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^q x$$

Evaluating in points yields

$$\frac{\partial P(1,b,x)}{\Delta x}(1, \sigma(t), t) = -3t^q + 3t^{2q}$$
$$\frac{\partial P(1,b,x)}{\Delta b}(1, t, t) = t^2 + 3t^q - 2t^{2q} + t^{1+q}$$

Summing up previously obtained partial time-scale derivatives, we get $q-$power derivative of odd polynomial x^3 evaluated in point $t \in \mathbb{R}^q$

$$\mathcal{D}_q t^3 = \frac{\partial P(1,b,x)}{\Delta x}(1, \sigma(t), t) + \frac{\partial P(1,b,x)}{\Delta b}(1, t, t) = t^2 + t^{2q} + t^{1+q}.$$

Example 4.15. *Let be $t \in \mathbb{R}^q,\ (x, b) \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$, let $m = 2$ then*

$$\frac{\partial P(2,b,x)}{\Delta x} = -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bx^q - 15b^2x^q + 10b^3x^q$$
$$\begin{aligned}\frac{\partial P(2,b,x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{2q} - 15b^{3q} + 6b^{4q} + 10b^{1+q} - 15b^{2+q} + 6b^{3+q} \\ &- 15b^{1+2q} + 6b^{2+2q} + 6b^{1+3q} - 15bx + 30b^2x - 15b^3x - 15b^qx + 30b^{2q}x \\ &- 15b^{3q}x + 30b^{1+q}x - 15b^{2+q}x - 15b^{1+2q}x + 5x^2 - 15bx^2 + 10b^2x^2 \\ &- 15b^qx^2 + 10b^{2q}x^2 + 10b^{1+q}x^2\end{aligned}$$

Evaluation in points yields

$$\frac{\partial P(2,b,x)}{\Delta x}(2, \sigma(t), t) = -10t^{2q} + 15t^{3q} - 5t^{4q} + 5t^{1+q} - 15t^{1+2q} + 10t^{1+3q}$$
$$\frac{\partial P(2,b,x)}{\Delta b}(2, t, t) = t^4 + 10t^{2q} - 15t^{3q} + 6t^{4q} - 5t^{1+q} + t^{3+q} + 15t^{1+2q} + t^{2+2q} - 9t^{1+3q}$$

Summing up previously obtained partial time-scale derivatives, we get $q-$power derivative of odd-powered polynomial x^5 evaluated in point $t\in\mathbb{R}^q$

$$\mathcal{D}_q x^5(t)=\frac{\partial P(2,b,x)}{\Delta x}(m,\sigma(t),t)+\frac{\partial P(2,b,x)}{\Delta b}(m,t,t)=t^4+t^{4q}+t^{3+q}+t^{2+2q}+t^{1+3q}.$$

Another polynomial identity, that is exponential sum holds

Corollary 4.16. *For every $t\in\mathbb{R}^q,\ (x,b)\in\Lambda^2=\mathbb{R}^q\times\mathbb{R}^q,\ t\in\mathbb{R}$*

$$\lim_{q\to 0}\frac{\partial P(m,b,x)}{\Delta b}(m,t,t)=\sum_{k=0}^{2m}t^k$$

4.5. **Pure quantum power time scale $\mathbb{T}=q^{\mathbb{R}^n}\times q^{\mathbb{R}^n}$.** In this subsection we discuss a pure quantum power time scale $q^{\mathbb{R}^j}$ provided by Aldwoah, Malinowska and Torres in [9], among with the $q-$power derivative operator $D_{n,q}f(t)$ defined by

$$D_{n,q}f(t)=\frac{f(qt^n)-f(t)}{qt^n-t},$$

where n is odd positive integer and $0<q<1$.

Corollary 4.17. *(Quantum power derivative [9].) Let be a two-dimensional time scale $\Lambda^2=q^{\mathbb{R}^j}\times q^{\mathbb{R}^j}:=\{t=(x,b):\ b\in q^{\mathbb{R}^j},\ x\in q^{\mathbb{R}^j}\}$. For every $t\in q^{\mathbb{R}^j},\ (x,b)\in\Lambda^2=q^{\mathbb{R}^j}\times q^{\mathbb{R}^j}$*

$$D_{n,q}x^{2m+1}(t)=\frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t)+\frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$$

where $\sigma(t)=qt^n,\ \sigma(t)>t$.

Example 4.18. *Let be $t\in q^{\mathbb{R}^j},\ (x,b)\in\Lambda^2=q^{\mathbb{R}^j}\times q^{\mathbb{R}^j}$, let $m=1$ then*

$$\frac{\partial P(1,b,x)}{\Delta x}=-3b+3b^2$$
$$\frac{\partial P(1,b,x)}{\Delta b}=3b-2b^2+3b^jq-2b^{1+j}q-2b^{2j}q^2-3x+3bx+3b^jqx$$

Evaluating in points yields

$$\frac{\partial P(1,b,x)}{\Delta x}(1,\sigma(t),t)=-3qt^j+3q^2t^{2j}$$
$$\frac{\partial P(1,b,x)}{\Delta b}(1,t,t)=t^2+3qt^j-2q^2t^{2j}+qt^{1+j}$$

Summing up previously obtained partial timescale derivatives, we get $q-$power derivative of odd-powered polynomial x^3 evaluated in point $t \in q^{\mathbb{R}^j}$

$$D_{n,q}x^3(t) = \frac{\partial P(1,b,x)}{\Delta x}(1,\sigma(t),t) + \frac{\partial P(1,b,x)}{\Delta b}(1,t,t) = t^2 + q^2t^{2j} + qt^{1+j}.$$

Another polynomial identity, that is exponential sum holds

Corollary 4.19. *For every $t \in q^{\mathbb{R}^j}$, $(x,b) \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, $t \in \mathbb{R}$*

$$\lim_{j\to 0}\lim_{q\to 1} \frac{\partial P(m,b,x)}{\Delta b}(m,t,t) = \sum_{k=0}^{2m} t^k$$

An identity in even polynomials holds too

Corollary 4.20. *For every $t \in q^{\mathbb{R}^j}$, $(x,b) \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$*

$$\lim_{j\to 0}\lim_{q\to 0} \frac{\partial P(m,b,x)}{\Delta b}(m,t,t) = t^{2m}$$

Example 4.21. *Let be $t \in q^{\mathbb{R}^j}$, $(x,b) \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$, let $m = 2$ then*

$$\frac{\partial P(2,b,x)}{\Delta x} = -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx^j - 15b^2qx^j + 10b^3qx^j$$

$$\begin{aligned}\frac{\partial P(2,b,x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{1+j}q - 15b^{2+j}q + 6b^{3+j}q + 10b^{2j}q^2 - 15b^{1+2j}q^2 \\ &+ 6b^{2+2j}q^2 - 15b^{3j}q^3 + 6b^{1+3j}q^3 + 6b^{4j}q^4 - 15bx + 30b^2x - 15b^3x - 15b^jqx \\ &+ 30b^{1+j}qx - 15b^{2+j}qx + 30b^{2j}q^2x - 15b^{1+2j}q^2x - 15b^{3j}q^3x + 5x^2 - 15bx^2 \\ &+ 10b^2x^2 - 15b^jqx^2 + 10b^{1+j}qx^2 + 10b^{2j}q^2x^2\end{aligned}$$

Evaluation in points yields

$$\frac{\partial P(2,b,x)}{\Delta x}(2,\sigma(t),t) = -10q^2t^{2j} + 15q^3t^{3j} - 5q^4t^{4j} + 5qt^{1+j} - 15q^2t^{1+2j} + 10q^3t^{1+3j}$$

$$\frac{\partial P(2,b,x)}{\Delta b}(2,t,t) = t^4 + 10q^2t^{2j} - 15q^3t^{3j} + 6q^4t^{4j} - 5qt^{1+j} + qt^{3+j} + 15q^2t^{1+2j} + q^2t^{2+2j} - 9q^3t^{1+3j}$$

Summing up previously obtained partial timescale derivatives, we $q-$power derivative of odd polynomial x^5 evaluated in point $t \in q^{\mathbb{R}^j}$

$$D_{n,q}x^5(t) = \frac{\partial P(1,b,x)}{\Delta x}(2,\sigma(t),t) + \frac{\partial P(1,b,x)}{\Delta b}(2,t,t) = t^4 + q^4t^{4j} + qt^{3+j} + q^2t^{2+2j} + q^3t^{1+3j}$$

5. Proof of main theorem

By [10, Lemma 3.1], for every $x \in \mathbb{R},\ m \in \mathbb{N}$ it is true that

$$P(m, x, x) = x^{2m+1} \tag{5.1}$$

5.1. **Proof of theorem 3.1.** Let be $x, b \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 := \{t = (x, b)\colon\ x \in \mathbb{T}_1,\ b \in \mathbb{T}_2\}$. Let be $\mathbb{T}_1 = \mathbb{T}_2$. Assume that timescale derivative $(x^{2m+1})^\Delta$ is

$$(x^{2m+1})^\Delta = \lim_{b \to x} \lim_{t \to x} \frac{P(m, \sigma(b), \sigma(x)) - P(m, b, t)}{\sigma(x) - t} \tag{5.2}$$

where $\sigma(x) > x$ is forward jump operator. However, equation (5.2) is not a timescale derivative of $P(m, b, x)$ over x how it might seem because of denominator $\sigma(x) - t$. Parameter b of $P(m, b, x)$ is implicitly incremented as well. Let's try to express nominator of (5.2) in terms of partial derivative $\frac{\partial P(m,b,x)}{\Delta b}$ on timescales. Let be the following equation

$$P(m, \sigma(b), x) - P(m, b, x) = P(m, b, x)^\Delta_b \cdot \Delta b$$

Let $t \to x$ in (5.2). Then nominator of (5.2) equals to

$$P(m, \sigma(b), \sigma(x)) - P(m, b, x) = P(m, \sigma(b), x) - P(m, b, x) + A$$

where A is yet implicit term. Let's now collapse the terms $f_m(x, b)$ from both sides of above equation, such that

$$P(m, \sigma(b), \sigma(x)) = P(m, \sigma(b), \sigma(x)) + A$$

Therefore,

$$A = P(m, \sigma(b), \sigma(x)) - P(m, \sigma(b), \sigma(x)) = P(m, b, x)^\Delta_x (x, \sigma(b)) \cdot \Delta x$$

Now, let's express the nominator of (5.2) as follows

$$P(m, \sigma(b), \sigma(x)) - P(m, b, x) = P(m, b, x)^\Delta_x (x, \sigma(b)) \cdot \Delta x + P(m, b, x)^\Delta_b (x, b) \cdot \Delta b$$

$$P(m, \sigma(b), \sigma(x)) - P(m, b, x) = P(m, b, x)^\Delta_x (x, \sigma(b)) \cdot (\sigma(x) - x) + P(m, b, x)^\Delta_b (x, b) \cdot (\sigma(b) - b)$$

We can collapse the terms $(\sigma(x) - x)$, $(\sigma(b) - b)$ in above expressions, as $b \to x$. Therefore,

$$\frac{P(m, \sigma(x), \sigma(x)) - P(m, x, x)}{\sigma(x) - x} = P(m, b, x)^{\Delta}_{x}(m, \sigma(x), x) + P(m, b, x)^{\Delta}_{b}(m, x, x)$$

Finally, by the identity (5.1) we can express timescale derivative of x^{2m+1}, $x \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$, $m \in \mathbb{N}$ as

$$(x^{2m+1})^{\Delta}(t) = \frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(x), x) + \frac{\partial P(m, b, x)}{\Delta b}(m, x, x)$$

This completes the proof. □

6. Conclusion and future research

In this manuscript we have discussed partial time scale differential equation involving derivatives of polynomials in context of time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$ where $\mathbb{T}_1 = \mathbb{T}_2$. Future research can be conducted to study the case $\mathbb{T}_1 \neq \mathbb{T}_2$, which makes the theorem 3.1 to be generalised

$$\frac{\partial P(m, b, x)}{\Delta x} + \frac{\partial P(m, b, x)}{\Delta b} = \alpha_m(x, b)(x^{2m+1})^{\Delta}$$

where $\alpha_m(x, b)$ is arbitrary differentiable function. Also, it is worth to discuss the theorem 3.1 in context of high order derivatives on time scales. We have established a few power identities, and shown the theorem 3.1 for different 2-dimensional time scales Λ^2 like integer time scale $\mathbb{Z} \times \mathbb{Z}$, real time scale $\mathbb{R} \times \mathbb{R}$, quantum time scale $q^{\mathbb{R}} \times q^{\mathbb{R}}$ and quantum power time scale $\mathbb{R}^q \times \mathbb{R}^q$.

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Version: `Local-0.1.0`

7. Addendum 1: Mathematica scripts

To fulfill our study, we attach here a link to the set of *Mathematica* programs, designed to verify the results of current manuscript. To reach these programs follow the link [16]. To reproduce results, proceed as follows:

- Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$:
  - Example 4.2: Execute the commands of Mathematica package
    - Set `sigma[x_] := x + 1` in Mathematica package and execute definition.
    - Execute `timeScaleDerivativeX[1, x, b]` which produces $-3b + 3b^2$.
    - Execute `Expand[timeScaleDerivativeX[1, t, sigma[t]]]` which produces $3t + 3t^2$.
    - Execute `timeScaleDerivativeB[1, x, b]` which produces $1 - 6b^2 + 6bx$.
    - Execute `timeScaleDerivativeB[1, t, t]` which produces 1.
    - Execute `mainTheorem[1]` which produces $1 + 3t + 3t^2$.
  - Example 4.3: Execute the commands of Mathematica package
    - Set `sigma[x_] := x + 1` in Mathematica package and execute definition.
    - `timeScaleDerivativeX[2, x, b]` which produces $5b-30b^2+40b^3-15b^4+10bx - 30b^2x + 20b^3x$.
    - `Expand[timeScaleDerivativeX[2, t, sigma[t]]]` which produces $5t+10t^2 + 10t^3 + 5t^4$.
    - `timeScaleDerivativeB[2, x, b]` which produces $1 + 30b^4 - 60b^3x + 30b^2x^2$.
    - `timeScaleDerivativeB[2, t, t]` which produces 1.
    - `mainTheorem[2]` which produces $1 + 5t + 10t^2 + 10t^3 + 5t^4$.
- Time scale of real numbers $\mathbb{T} = \mathbb{R} \times \mathbb{R}$:

- Example 4.7: Execute the commands of Mathematica package
  - Set `sigma[x_] := x + Global‘dx` in Mathematica package and execute definition.
  - Execute `timeScaleDerivativeX[1, x, b]` which produces $-3b + 3b^2$.
  - Execute `Limit[Expand[timeScaleDerivativeB[1, x, b]], dx -> 0]` which produces $6b - 6b^2 - 3x + 6bx$.
  - Execute `timeScaleDerivativeX[1, t, t]` which produces $-3t + 3t^2$.
  - Execute `Limit[Expand[timeScaleDerivativeB[1, t, t]], dx -> 0]` which produces $3t$.
  - Execute `Limit[mainTheorem[1], dx -> 0]` which produces $3t^2$.
- Example 4.8: Execute the commands of Mathematica package
  - Set `sigma[x_] := x + Global‘dx` in Mathematica package and execute definition.
  - Execute `Limit[Expand[timeScaleDerivativeX[2, x, b]], dx -> 0]` which produces $-15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$.
  - Execute `Limit[Expand[timeScaleDerivativeB[2, x, b]], dx -> 0]` which produces $30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2$.
  - Execute `Limit[Expand[timeScaleDerivativeX[2, t, sigma[t]]], dx -> 0]` which produces $-5t^2 + 5t^4$.
  - Execute `Limit[Expand[timeScaleDerivativeB[2, t, t]], dx -> 0]` which produces $5t^2$.
  - Execute `Limit[mainTheorem[2], dx -> 0]` which produces $5t^4$.

• Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$:
- Example 4.10: Execute the commands of Mathematica package
  - Set `sigma[x_] := x * Global‘q` in Mathematica package and execute definition.

    * Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces $-3b + 3b^2$.
    * Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces $3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx$.
    * Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]]` which produces $-3qt + 3q^2t^2$.
    * Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces $3qt + t^2 + qt^2 - 2q^2t^2$.
    * Execute `Expand[Simplify[mainTheorem[1]]]` which produces $t^2 + qt^2 + q^2t^2$.
  - Example 4.12: Execute the commands of Mathematica package
    * Set `sigma[x_] := x * Global‘q` in Mathematica package and execute definition.
    * Execute `Expand[Simplify[timeScaleDerivativeX[2, x, b]]]` which produces $-15b^2+30b^3-15b^4+5bx-15b^2x+10b^3x+5bqx-15b^2qx+10b^3qx$.
    * Execute `Expand[Simplify[timeScaleDerivativeB[2, x, b]]]` which produces $10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 + 6b^4q^2 - 15b^3q^3 + 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x - 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2$.
    * Execute `Expand[Simplify[timeScaleDerivativeX[2, t, sigma[t]]]]` which produces $5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4$.
    * Execute `Expand[Simplify[timeScaleDerivativeB[2, t, t]]]` which produces $-5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4$.
    * Execute `Expand[Simplify[mainTheorem[2]]]` which produces $t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4$.
  - Corollary 4.11: Execute the commands of Mathematica package

        * Set `sigma[x_] := x * Global‘q` in Mathematica package and execute definition.
        * Execute `Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 0]` for various values of `m`.

- Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$:
    - Example 4.14: Execute the commands of Mathematica package
        * Set `sigma[x_] := x ∧ Global‘q` in Mathematica package and execute definition.
        * Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces $-3b + 3b^2$.
        * Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces $3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^q x$.
        * Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]]` which produces $-3t^q + 3t^{2q}$.
        * Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces $t^2 + 3t^q - 2t^{2q} + t^{1+q}$.
        * Execute `Expand[Simplify[mainTheorem[1]]]` which produces $t^2 + t^{2q} + t^{1+q}$.
    - Example 4.15: Similarly to Example 4.14 with $m = 2$.
    - Corollary 4.16: Execute the commands of Mathematica package
        * Set `sigma[x_] := x ∧ Global‘q` in Mathematica package and execute definition.
        * Execute `Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 0]` for various values of `m`.

- Pure quantum power time scale $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$:
    - Example 4.18: Execute the commands of Mathematica package

    * Set `sigma[x_] := Global‘q * x ∧ Global‘j` in Mathematica package and execute definition.
    * Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces $-3b + 3b^2$.
    * Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces $3b - 2b^2 + 3b^j q - 2b^{1+j}q - 2b^{2j}q^2 - 3x + 3bx + 3b^j qx$.
    * Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]]` which produces $-3qt^j + 3q^2t^{2j}$.
    * Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces $t^2 + 3qt^j - 2q^2t^{2j} + qt^{1+j}$.
    * Execute `Expand[Simplify[mainTheorem[1]]]` which produces $t^2+q^2t^{2j}+qt^{1+j}$.
- Example 4.21: Similarly as Example 4.18 for $m = 2$.
- Corollary 4.19: Execute the commands of Mathematica package
    * Set `sigma[x_] := Global‘q * x ∧ Global‘j` in Mathematica package and execute definition.
    * Execute `Limit[Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 1], j -> 0]` for various values of `m`.
- Corollary 4.20: Execute the commands of Mathematica package
    * Set `sigma[x_] := Global‘q * x ∧ Global‘j` in Mathematica package and execute definition.
    * Execute `Limit[Limit[Expand[Simplify[timeScaleDerivativeB[5, t, t]]], q -> 0], j -> 0]` for various values of `m`.

Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io