

# A STUDY ON PARTIAL DYNAMIC EQUATION ON TIME SCALES INVOLVING DERIVATIVES OF POLYNOMIALS

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ABSTRACT. Let  $\mathbf{P}_b^m(x)$  be a  $2m + 1$ -degree integer-valued polynomial in  $x, b$ . Let be a two-dimensional time scale  $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x, b) : x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$ . Let be  $\mathbb{T}_1 = \mathbb{T}_2$ . In this manuscript we derive and discuss the following partial dynamic equation on time scales. For every  $t \in \mathbb{T}_1$ ,  $x, b \in \Lambda^2$ ,  $m = \text{const}$ ,  $m \in \mathbb{N}$

$$(t^{2m+1})^\Delta = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t},$$

where  $\sigma(t) > t$  is forward jump operator. In addition, we discuss various derivative operators in context of partial cases of above equation, we show finite difference, classical derivative,  $q$ -derivative,  $q$ -power derivative on behalf of it.

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## 1. DEFINITIONS

We now set the following notation, which remains fixed for the remainder of this paper:

- Let be a function  $f: \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}^\kappa$  then  $f^\Delta(t)$  is delta time-scale derivative [BP01] of  $f$

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

where  $\mu(t) = \sigma(t) - t$ ,  $\mu(t) \neq 0$  and  $\sigma(t) > t$  is forward jump operator.

- $\frac{\partial f(t_1, \dots, t_n)}{\Delta_i t_i}$ ,  $f_{t_i}^{\Delta_i}(t)$  is delta partial derivative of  $f: \Lambda^n \rightarrow \mathbb{R}$  on  $n$ -dimensional time scale  $\Lambda^n$  [BG04, AM02, Jac06], defined as a limit

$$f_{t_i}^{\Delta_i}(t) = \lim_{\substack{s_i \rightarrow t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, \dots, t_{i-1}, \sigma_i(t_i), t_{t+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, s_i, t_{t+1}, \dots, t_n)}{\sigma_i(t_i) - s_i},$$

where  $\sigma_i(t_i) > t_i$  and  $\sigma_i(t_i) - s_i \neq 0$ .

- $D_q f(x)$  is  $q$ -derivative [Jac09, Ern00, Ern08, KC01]

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x},$$

where  $x \neq 0$ ,  $x \in \mathbb{R}$ ,  $q \in \mathbb{R}$ .

- $D_{n,q} f(t)$  is  $q$ -power derivative [AMT11]

$$D_{n,q} f(t) = \frac{f(qt^n) - f(t)}{qt^n - t},$$

where  $qt^n - t \neq 0$  and  $n$  is odd positive integer and  $0 < q < 1$ .

- $\mathcal{D}_q f(x)$  is  $q$ -power derivative

$$\mathcal{D}_q f(x) = \frac{f(x^q) - f(x)}{x^q - x},$$

where  $x^q \neq x$ ,  $x \in \mathbb{R}$ ,  $q \in \mathbb{R}$ .

- $\mathbf{P}_b^m(x)$ ,  $x, b \in \mathbb{R}$ ,  $m \in \mathbb{N}$  is  $2m + 1$ -degree integer-valued polynomial [Kol16]

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x - k)^r, \quad (1.1)$$

where  $\mathbf{A}_{m,r}$ ,  $m \in \mathbb{N}$  is a real coefficient defined recursively

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m, \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}, & \text{if } 0 \leq r < m, \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

where  $B_t$  are Bernoulli numbers [Wei]. It is assumed that  $B_1 = \frac{1}{2}$ .

- $\mathbb{Z}$  is an integer time scale such that  $\sigma(t) = t + 1$  and  $\mu(t) = 1$ .
- $\mathbb{R}$  is a real time scale such that  $\sigma(t) = t + \Delta t$  and  $\mu(t) = \Delta t$ ,  $\Delta t \rightarrow 0$ .
- $q^{\mathbb{R}}$  is a quantum time scale such that  $\sigma(t) = qt$  and  $\mu(t) = qt - t$ , [page 18 [BP01]].
- $\mathbb{R}^q$  is a quantum power time scale such that  $\sigma(t) = t^q$  and  $\mu(t) = t^q - t$ .
- $q^{\mathbb{R}^n}$  is a pure quantum power time scale such that  $\sigma(t) = qt^n > t$ ,  $0 < q < 1$ ,  $\mu(t) = qt^n - t$  and  $n$  is positive odd integer [AMT11].

## 2. INTRODUCTION

Time-scale calculus is quite graceful generalization and unification of the theory of differential equations. Firstly being introduced by Hilger [Hil88] in his Ph.D thesis in 1988 and thereafter greatly extended by Bohner and Peterson [BP01] in 2001, the calculus on time scales became a sharp tool in the world on differential equations. Various derivative operators like classical derivative  $\frac{d}{dx}f(x)$ ,  $q$ -derivative  $D_q f(x)$ ,  $q$ -power derivative  $\mathcal{D}_q f(x)$ , finite difference  $\Delta f(x)$  etc, may be simply expressed in terms of time-scale derivative over particular time scale  $\mathbb{T}$ . For instance,

$$\begin{aligned} f'(x) &= f^\Delta(x), & x \in \mathbb{T} = \mathbb{R} \\ \Delta f(x) &= f^\Delta(x), & x \in \mathbb{T} = \mathbb{Z} \\ D_{n,q} f(x) &= f^\Delta(x), & x \in \mathbb{T} = q^{\mathbb{R}^n} \\ D_q f(x) &= f^\Delta(x), & x \in \mathbb{T} = q^{\mathbb{R}} \\ \mathcal{D}_q f(x) &= f^\Delta(x), & x \in \mathbb{T} = \mathbb{R}^q \end{aligned}$$

In context of Computer Science, namely object oriented programming paradigm, the time scale calculus may be thought as unified interface of derivative operator. Furthermore, the idea of time-scale calculus was slightly extended in [BHT17, BHT16, Cap09, MT09].

## 3. MAIN RESULTS

Time scale derivative of the polynomial  $t^{2m+1}$  may be expressed as follows

**Theorem 3.1.** Let  $\mathbf{P}_b^m(x)$  be a  $2m + 1$ -degree integer-valued polynomial defined by (1.1). Let be a two-dimensional time scale  $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x, b) : x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$ . Let be  $\mathbb{T}_1 = \mathbb{T}_2$ . For every  $t \in \mathbb{T}_1$ ,  $x, b \in \Lambda^2$ ,  $m \in \mathbb{N}$ ,  $m = \text{const}$

$$(t^{2m+1})^\Delta = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t},$$

where

- $\sigma(t) > t$  is forward jump operator,
- $\frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)}$  is the value of the partial derivative on time scales of  $\mathbf{P}_b^m(x)$  with respect to the variable  $x$ , evaluated at  $x = t$ ,  $b = \sigma(t)$ ,
- $\frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t}$  is the value of the partial derivative on time scales of  $\mathbf{P}_b^m(x)$  with respect to the variable  $b$ , evaluated at  $x = t$ ,  $b = t$ .

In other words, theorem 3.1 says

For every odd-exponent polynomial  $t^{2m+1}$ , its derivative on time scales equals to the sum of the value of the partial derivative on time scales of  $\mathbf{P}_b^m(x)$  with respect to the variable  $x$ , evaluated at  $x = t$ ,  $b = \sigma(t)$  and the value of the partial derivative on time scales of  $\mathbf{P}_b^m(x)$  with respect to the variable  $b$ , evaluated at  $x = t$ ,  $b = t$ .

In extended form theorem 3.1 may be written as

$$(t^{2m+1})_t^\Delta = \frac{\partial}{\Delta x} \left( \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \right) \Big|_{x=t, b=\sigma(t)} + \frac{\partial}{\Delta b} \left( \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \right) \Big|_{x=t, b=t}$$

#### 4. DISCUSSION AND EXAMPLES

To understand the nature of theorem 3.1, let's discuss an example of some popular time scales, like integer time scale  $\mathbb{Z}$ , real time scale  $\mathbb{R}$ , quantum time scale  $q^{\mathbb{R}}$ , quantum-power time scale  $\mathbb{R}^q$ . We use the principle *Divide et Impera !* in order to understand entire behavior of theorem 3.1.

##### 4.1. Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$ .

**Corollary 4.1.** (Finite difference.) Let be a two-dimensional time scale  $\Lambda^2 = \mathbb{Z} \times \mathbb{Z} := \{t = (x, b) : x \in \mathbb{Z}, b \in \mathbb{Z}\}$ . For every  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$

$$\Delta t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} (t, t) \Big|_{x=t, b=t},$$

where the forward jump operator  $\sigma(t)$  is defined as  $\sigma(t) = t + 1$ .

**Example 4.2.** Let be  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= 3t + 3t^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 1 - 6b^2 + 6bx \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} &= 1\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get the ordinary finite difference of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$

$$\Delta t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = 3t^2 + 3t + 1.$$

**Example 4.3.** Let be  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= 5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 1 + 30b^4 - 60b^3x + 30b^2x^2 \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= 5t + 10t^2 + 10t^3 + 5t^4 \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} \Big|_{x=t, b=t} &= 1\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get time ordinary finite difference of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$

$$\Delta t^5 = \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} \Big|_{x=t, b=t} = 1 + 5t + 10t^2 + 10t^3 + 5t^4.$$

**Corollary 4.4.** For every  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} = \sum_{r=1}^{2m} \binom{2m+1}{r} t^r$$

**Corollary 4.5.** For every  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = 1$$

4.2. Time scale of real numbers  $\mathbb{T} = \mathbb{R} \times \mathbb{R}$ .

**Corollary 4.6.** (Classical derivative.) Let be a two-dimensional time scale  $\Lambda^2 = \mathbb{R} \times \mathbb{R} := \{t = (x, b) : x \in \mathbb{R}, b \in \mathbb{R}\}$ . For every  $t \in \mathbb{R}$ ,  $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$

$$\frac{d}{dt}t^{2m+1} = \left. \frac{\partial \mathbf{P}_b^m(x)}{\partial x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial \mathbf{P}_b^m(x)}{\partial b} \right|_{x=t, b=t},$$

where  $\sigma(t) = t + \Delta t$ ,  $\Delta t \rightarrow 0$ .

**Example 4.7.** Let be  $t \in \mathbb{R}$ ,  $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^1(x)}{\partial x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\partial b} &= 6b - 6b^2 - 3x + 6bx \\ \left. \frac{\partial \mathbf{P}_b^1(x)}{\partial x} \right|_{x=t, b=\sigma(t)} &= -3t + 3t^2 \\ \left. \frac{\partial \mathbf{P}_b^1(x)}{\partial b} \right|_{x=t, b=t} &= 3t \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}$ ,  $x \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$

$$\frac{d}{dt}t^3 = \left. \frac{\partial \mathbf{P}_b^1(x)}{\partial x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial \mathbf{P}_b^1(x)}{\partial b} \right|_{x=t, b=t} = 3t^2.$$

**Example 4.8.** Let be  $t \in \mathbb{R}$ ,  $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\partial x} &= -15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x, \\ \frac{\partial \mathbf{P}_b^2(x)}{\partial b} &= 30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2 \\ \left. \frac{\partial \mathbf{P}_b^2(x)}{\partial x} \right|_{x=t, b=\sigma(t)} &= -5t^2 + 5t^4 \\ \left. \frac{\partial \mathbf{P}_b^2(x)}{\partial b} \right|_{x=t, b=t} &= 5t^2 \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of an odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}$ ,  $x \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$

$$\frac{d}{dt}t^5 = \left. \frac{\partial \mathbf{P}_b^2(x)}{\partial x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial \mathbf{P}_b^2(x)}{\partial b} \right|_{x=t, b=t} = 5t^4.$$

4.3. Quantum time scale  $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$ .

**Corollary 4.9.** (*Q-derivative* [Jac09].) *Let be a two-dimensional time scale  $\Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}} := \{t = (x, b) : x \in q^{\mathbb{R}}, b \in q^{\mathbb{R}}\}$ . For every  $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$*

$$D_q t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t},$$

where  $\sigma(t) = qt, q > 1$ .

**Example 4.10.** *Let be  $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$  and let  $m = 1$ , then*

$$\begin{aligned} \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= -3qt + 3q^2t^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} &= 3qt + t^2 + qt^2 - 2q^2t^2 \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -derivative of odd polynomial  $t^{2m+1}, t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$

$$D_q t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^2 + qt^2 + q^2t^2.$$

For every  $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$  the following polynomial identity holds as  $q$  tends to zero

$$\lim_{q \rightarrow 0} \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^2$$

However, it would be generalized as follows

**Corollary 4.11.** *For every  $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$*

$$\lim_{q \rightarrow 0} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = t^{2m}.$$

**Example 4.12.** *Let be  $t \in q^{\mathbb{R}}, x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}, m \in \mathbb{N}$  and let  $m = 2$ , then*

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx - 15b^2qx + 10b^3qx \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 + 6b^4q^2 - 15b^3q^3 \\ &\quad + 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x \\ &\quad - 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2 \end{aligned}$$

$$\begin{aligned}\left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} &= 5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4 \\ \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} &= -5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get the  $q$ -derivative of odd polynomial  $t^{2m+1}$ ,  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ ,  $m \in \mathbb{N}$

$$D_q t^5 = \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} = t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4.$$

#### 4.4. Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$ .

**Corollary 4.13.** ( $Q$ -power derivative [AMT11].) Let be a two-dimensional time scale  $\Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q := \{t = (x, b) : b \in \mathbb{R}^q, x \in \mathbb{R}^q\}$ . For every  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$

$$\mathcal{D}_q t^{2m+1} = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x}(t, \sigma(t)) + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b}(t, t),$$

where the forward jump operator is defined as  $\sigma(t) = t^q$ ,  $q > 1$ .

**Example 4.14.** Let be  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= 3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^q x \\ \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} &= -3t^q + 3t^{2q} \\ \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}\right|_{x=t, b=t} &= t^2 + 3t^q - 2t^{2q} + t^{1+q}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$

$$\mathcal{D}_q t^3 = \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^1(x)}{\Delta b}\right|_{x=t, b=t} = t^2 + t^{2q} + t^{1+q}.$$

**Example 4.15.** Let be  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then

$$\begin{aligned}\frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bx^q - 15b^2x^q + 10b^3x^q \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{2q} - 15b^{3q} + 6b^{4q} + 10b^{1+q} - 15b^{2+q} + 6b^{3+q} \\ &\quad - 15b^{1+2q} + 6b^{2+2q} + 6b^{1+3q} - 15bx + 30b^2x - 15b^3x - 15b^q x + 30b^{2q}x \\ &\quad - 15b^{3q}x + 30b^{1+q}x - 15b^{2+q}x - 15b^{1+2q}x + 5x^2 - 15bx^2 + 10b^2x^2 \\ &\quad - 15b^q x^2 + 10b^{2q}x^2 + 10b^{1+q}x^2\end{aligned}$$



$$\begin{aligned}\left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} &= -10t^{2q} + 15t^{3q} - 5t^{4q} + 5t^{1+q} - 15t^{1+2q} + 10t^{1+3q} \\ \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} &= t^4 + 10t^{2q} - 15t^{3q} + 6t^{4q} - 5t^{1+q} + t^{3+q} + 15t^{1+2q} + t^{2+2q} - 9t^{1+3q}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$

$$\mathcal{D}_q t^5 = \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^2(x)}{\Delta b}\right|_{x=t, b=t} = t^4 + t^{4q} + t^{3+q} + t^{2+2q} + t^{1+3q}.$$

Another polynomial identity, that is exponential sum holds

**Corollary 4.16.** *For every  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $t \in \mathbb{R}$ ,  $m \in \mathbb{N}$*

$$\lim_{q \rightarrow 0} \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}\right|_{x=t, b=t} = \sum_{k=0}^{2m} t^k$$

**4.5. Pure quantum power time scale  $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$ .** In this subsection we discuss a pure quantum power time scale  $q^{\mathbb{R}^j}$  provided by Aldwoah, Malinowska and Torres in [AMT11], among with the  $q$ -power derivative operator  $D_{n,q}f(t)$  defined by

$$D_{n,q}f(t) = \frac{f(qt^n) - f(t)}{qt^n - t},$$

where  $n$  is odd positive integer and  $0 < q < 1$ .

**Corollary 4.17.** *(Quantum power derivative [AMT11].) Let be a two-dimensional time scale  $\Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j} := \{t = (x, b) : b \in q^{\mathbb{R}^j}, x \in q^{\mathbb{R}^j}\}$ . For every  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$*

$$D_{n,q} t^{2m+1} = \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta x}\right|_{x=t, b=\sigma(t)} + \left.\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}\right|_{x=t, b=t},$$

where  $\sigma(t) = qt^n$ ,  $\sigma(t) > t$ .

**Example 4.18.** *Let be  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then*

$$\begin{aligned}\frac{\partial \mathbf{P}_b^1(x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} &= 3b - 2b^2 + 3b^j q - 2b^{1+j} q - 2b^{2j} q^2 - 3x + 3bx + 3b^j qx \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta x}(t, \sigma(t)) &= -3qt^j + 3q^2 t^{2j} \\ \frac{\partial \mathbf{P}_b^1(x)}{\Delta b}(t, t) &= t^2 + 3qt^j - 2q^2 t^{2j} + qt^{1+j}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$

$$D_{n,q}t^3 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^2 + q^2 t^{2j} + qt^{1+j}.$$

Another polynomial identity, that is exponential sum holds

**Corollary 4.19.** For every  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $t \in \mathbb{R}$ ,  $m \in \mathbb{N}$

$$\lim_{j \rightarrow 0} \lim_{q \rightarrow 1} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = \sum_{k=0}^{2m} t^k$$

An identity in even polynomials holds too

**Corollary 4.20.** For every  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $t \in \mathbb{R}$ ,  $m \in \mathbb{N}$

$$\lim_{j \rightarrow 0} \lim_{q \rightarrow 0} \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t} = t^{2m}$$

**Example 4.21.** Let be  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx^j - 15b^2qx^j + 10b^3qx^j \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{1+j}q - 15b^{2+j}q + 6b^{3+j}q + 10b^{2j}q^2 - 15b^{1+2j}q^2 \\ &\quad + 6b^{2+2j}q^2 - 15b^{3j}q^3 + 6b^{1+3j}q^3 + 6b^{4j}q^4 - 15bx + 30b^2x - 15b^3x - 15b^j qx \\ &\quad + 30b^{1+j}qx - 15b^{2+j}qx + 30b^{2j}q^2x - 15b^{1+2j}q^2x - 15b^{3j}q^3x + 5x^2 - 15bx^2 \\ &\quad + 10b^2x^2 - 15b^j qx^2 + 10b^{1+j}qx^2 + 10b^{2j}q^2x^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{P}_b^2(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} &= -10q^2t^{2j} + 15q^3t^{3j} - 5q^4t^{4j} + 5qt^{1+j} - 15q^2t^{1+2j} + 10q^3t^{1+3j} \\ \frac{\partial \mathbf{P}_b^2(x)}{\Delta b} \Big|_{x=t, b=t} &= t^4 + 10q^2t^{2j} - 15q^3t^{3j} + 6q^4t^{4j} - 5qt^{1+j} + qt^{3+j} + 15q^2t^{1+2j} + q^2t^{2+2j} - 9q^3t^{1+3j} \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$

$$D_{n,q}t^5 = \frac{\partial \mathbf{P}_b^1(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^1(x)}{\Delta b} \Big|_{x=t, b=t} = t^4 + q^4 t^{4j} + qt^{3+j} + q^2 t^{2+2j} + q^3 t^{1+3j}.$$

## 5. PROOF OF MAIN THEOREM

By [Lemma 3.1 [Kol16]], for every  $x \in \mathbb{R}$ ,  $m \in \mathbb{N}$  it is true that

$$\mathbf{P}_x^m(x) = x^{2m+1} \tag{5.1}$$

**Proof of theorem 3.1.** Let be  $x, b \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 := \{t = (x, b) : x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$ . Let be  $\mathbb{T}_1 = \mathbb{T}_2$ . Assume that time-scale derivative  $(x^{2m+1})^\Delta$  is

$$(x^{2m+1})^\Delta = \lim_{b \rightarrow x} \lim_{t \rightarrow x} \frac{\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(t)}{\sigma(x) - t}, \quad (5.2)$$

where  $\sigma(x) > x$  is forward jump operator. However, equation (5.2) is not a timescale derivative of  $\mathbf{P}_b^m(x)$  over  $x$  how it might seem because of denominator  $\sigma(x) - t$ . Parameter  $b$  of  $\mathbf{P}_b^m(x)$  is implicitly incremented as well. Let's try to express nominator of (5.2) in terms of partial derivative  $\frac{\partial \mathbf{P}_b^m(x)}{\Delta b}$  on timescales. Let be the following equation

$$\mathbf{P}_{\sigma(b)}^m(x) - \mathbf{P}_b^m(x) = \mathbf{P}_b^m(x)_b^\Delta \cdot \Delta b$$

Let  $t \rightarrow x$  in (5.2). Then nominator of (5.2) equals to

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(x) = \mathbf{P}_{\sigma(b)}^m(x) - \mathbf{P}_b^m(x) + A$$

where  $A$  is yet implicit term. Let's now collapse the terms  $f_m(x, b)$  from both sides of above equation, such that

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) = \mathbf{P}_{\sigma(b)}^m(x) + A$$

Therefore,

$$A = \mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_{\sigma(b)}^m(x) = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(b)) \cdot \Delta x$$

Now, let's express the nominator of (5.2) as follows

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(x) = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(b)) \cdot \Delta x + \mathbf{P}_b^m(x)_b^\Delta(x, b) \cdot \Delta b$$

$$\mathbf{P}_{\sigma(b)}^m(\sigma(x)) - \mathbf{P}_b^m(x) = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(b)) \cdot (\sigma(x) - x) + \mathbf{P}_b^m(x)_b^\Delta(x, b) \cdot (\sigma(b) - b)$$

We can collapse the terms  $(\sigma(x) - x)$ ,  $(\sigma(b) - b)$  in above expressions, as  $b \rightarrow x$ . Therefore,

$$\frac{\mathbf{P}_{\sigma(x)}^m(\sigma(x)) - \mathbf{P}_x^m(x)}{\sigma(x) - x} = \mathbf{P}_b^m(x)_x^\Delta(x, \sigma(x)) + \mathbf{P}_b^m(x)_b^\Delta(x, x)$$

Finally, by the identity (5.1) we can express timescale derivative of  $x^{2m+1}$ ,  $x \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$ ,  $m \in \mathbb{N}$  as

$$(x^{2m+1})^\Delta(t) = \frac{\partial \mathbf{P}_b^m(x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} \Big|_{x=t, b=t}$$

This completes the proof. □

## 6. MATHEMATICA SCRIPTS

To fulfill our study, we attach here a link to the set of *Mathematica* programs, designed to verify the results of current manuscript. To reach these programs follow the link [\[Kol20\]](#). To reproduce results, proceed as follows:

- Time scale of integers  $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$ :
  - Example 4.2: Execute the commands of Mathematica package
    - \* Set `sigma[x_] := x + 1`
    - \* `timeScaleDerivativeX[1, x, b]` which produces  $-3b + 3b^2$ .
    - \* `Expand[timeScaleDerivativeX[1, t, sigma[t]]]` which produces  $3t + 3t^2$ .
    - \* `timeScaleDerivativeB[1, x, b]` which produces  $1 - 6b^2 + 6bx$ .
    - \* `timeScaleDerivativeB[1, t, t]` which produces 1.
    - \* `mainTheorem[1]` which produces  $1 + 3t + 3t^2$ .
  - Example 4.3: Execute the commands of Mathematica package
    - \* Set `sigma[x_] := x + 1`
    - \* `timeScaleDerivativeX[2, x, b]` which produces  $5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$ .
    - \* `Expand[timeScaleDerivativeX[2, t, sigma[t]]]` which produces  $5t + 10t^2 + 10t^3 + 5t^4$ .
    - \* `timeScaleDerivativeB[2, x, b]` which produces  $1 + 30b^4 - 60b^3x + 30b^2x^2$ .
    - \* `timeScaleDerivativeB[2, t, t]` which produces 1.
    - \* `mainTheorem[2]` which produces  $1 + 5t + 10t^2 + 10t^3 + 5t^4$ .
- Time scale of real numbers  $\mathbb{T} = \mathbb{R} \times \mathbb{R}$ :
  - Example 4.7: Execute the commands of Mathematica package
    - \* Set `sigma[x_] := x + Global`dx`
    - \* Execute `timeScaleDerivativeX[1, x, b]` which produces  $-3b + 3b^2$ .
    - \* Execute `Limit[Expand[timeScaleDerivativeB[1, x, b]], dx -> 0]` which produces  $6b - 6b^2 - 3x + 6bx$ .
    - \* Execute `timeScaleDerivativeX[1, t, t]` which produces  $-3t + 3t^2$ .
    - \* Execute `Limit[Expand[timeScaleDerivativeB[1, t, t]], dx -> 0]` which produces  $3t$ .

\* Execute `Limit[mainTheorem[1], dx -> 0]` which produces  $3t^2$ .

– Example 4.8:

- Quantum time scale  $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$ :
  - Example 4.10:
  - Example 4.12:
  - Corollary 4.11:
- Quantum power time scale  $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$ :
  - Example 4.14:
  - Example 4.15:
  - Corollary 4.16:
- Pure quantum power time scale  $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$ :
  - Example 4.18:
  - Example 4.21:
  - Corollary 4.19:
  - Corollary 4.20:

## 7. CONCLUSION AND FUTURE RESEARCH

In this manuscript we have discussed partial time scale differential equation involving derivatives of polynomials in context of time scale  $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$  where  $\mathbb{T}_1 = \mathbb{T}_2$ . Future research can be conducted to study the case  $\mathbb{T}_1 \neq \mathbb{T}_2$ , which makes the theorem 3.1 to be generalised

$$\frac{\partial \mathbf{P}_b^m(x)}{\Delta x} + \frac{\partial \mathbf{P}_b^m(x)}{\Delta b} = \alpha_m(x, b)(x^{2m+1})^\Delta,$$

where  $\alpha_m(x, b)$  is arbitrary differentiable function. Also, it is worth to discuss the theorem 3.1 in context of high order derivatives on time scales. We have established a few power identities, and shown the theorem 3.1 for different 2-dimensional time scales  $\Lambda^2$  like integer time scale  $\mathbb{Z} \times \mathbb{Z}$ , real time scale  $\mathbb{R} \times \mathbb{R}$ , quantum time scale  $q^{\mathbb{R}} \times q^{\mathbb{R}}$  and quantum power time scale  $\mathbb{R}^q \times \mathbb{R}^q$ .

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