

# A STUDY ON PARTIAL DYNAMIC EQUATION ON TIME SCALES INVOLVING DERIVATIVES OF POLYNOMIALS

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ABSTRACT. Let  $P(m, b, x)$  be a  $2m + 1$ -degree polynomial in  $x, b$ . Let be a two-dimensional timescale  $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x, b): x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$  such that  $\mathbb{T}_1 = \mathbb{T}_2$ . In this manuscript we derive and discuss an identity that connects the timescale derivative of odd-power polynomial with partial derivatives of polynomial  $P(m, b, x)$  evaluated in particular points. For every  $t \in \mathbb{T}_1$  and  $x, b \in \Lambda^2$

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(t), t) + \frac{\partial P(m, b, x)}{\Delta b}(m, t, t)$$

such that  $\sigma(t) > t$  is forward jump operator. In addition, we discuss various derivative operators in context of partial cases of above equation, we show finite difference, classical derivative,  $q$ -derivative,  $q$ -power derivative on behalf of it.

## CONTENTS

1. Definitions	2
2. Introduction	3
3. Main results	4
4. Discussion and examples	5
4.1. Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$	5
4.2. Time scale of real numbers $\mathbb{T} = \mathbb{R} \times \mathbb{R}$	7
4.3. Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$	8
4.4. Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$	9

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4.5. Pure quantum power time scale $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$	11
5. Proof of main theorem	12
Proof of theorem 3.1	13
6. Conclusion and future research	14
References	14
7. Addendum 1: Mathematica scripts	16

## 1. DEFINITIONS

We now set the following notation such that remains fixed for the remainder of this manuscript

- Let be a function  $f: \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}^\kappa$  then  $f^\Delta(t)$  is delta timescale derivative [1]

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

where  $\sigma(t) - t \neq 0$  and  $\sigma(t) > t$  is forward jump operator.

- $\frac{\partial f(t_1, \dots, t_n)}{\Delta_i t_i}$  is the delta partial derivative of  $f: \Lambda^n \rightarrow \mathbb{R}$  on  $n$ -dimensional timescale  $\Lambda^n$  defined via the limit [2, 3, 4]

$$\frac{\partial f(t_1, \dots, t_n)}{\Delta_i t_i} = \lim_{s_i \rightarrow t_i} \frac{f(t_1, \dots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n)}{\sigma_i(t_i) - s_i}$$

where  $\sigma_i(t_i) > t_i$  and  $\sigma_i(t_i) - s_i \neq 0$ .

- $D_q f(x)$  is  $q$ -derivative [5, 6, 7, 8]

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

where  $x \neq 0$ ,  $x \in \mathbb{R}$ ,  $q \in \mathbb{R}$ .

- $D_{n,q}f(t)$  is  $q$ -power derivative [9]

$$D_{n,q}f(t) = \frac{f(qt^n) - f(t)}{qt^n - t}$$

where  $qt^n - t \neq 0$  and  $n$  is odd positive integer and  $0 < q < 1$ .

- $\mathcal{D}_q f(x)$  is  $q$ -power derivative

$$\mathcal{D}_q f(x) = \frac{f(x^q) - f(x)}{x^q - x}$$

where  $x^q \neq x$ ,  $x \in \mathbb{R}$ ,  $q \in \mathbb{R}$ .

- $P(m, b, x)$  is  $2m + 1$ -degree polynomial in  $x, b$

$$P(m, b, x) = \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x - k)^r \quad (1.1)$$

where  $\mathbf{A}_{m,r}$  is a real coefficient defined recursively, see [10].

- $\mathbb{Z}$  is an integer timescale such that  $\sigma(t) = t + 1$  and  $\mu(t) = 1$ .
- $\mathbb{R}$  is a real timescale such that  $\sigma(t) = t + \Delta t$  and  $\mu(t) = \Delta t$ ,  $\Delta t \rightarrow 0$ .
- $q^{\mathbb{R}}$  is a quantum timescale such that  $\sigma(t) = qt$  and  $\mu(t) = qt - t$ , [page 18 [1]].
- $\mathbb{R}^q$  is a quantum power timescale such that  $\sigma(t) = t^q$  and  $\mu(t) = t^q - t$ .
- $q^{\mathbb{R}^n}$  is a pure quantum power timescale such that  $\sigma(t) = qt^n > t$ ,  $0 < q < 1$ ,  $\mu(t) = qt^n - t$  and  $n$  is positive odd integer [9].

## 2. INTRODUCTION

Time-scale calculus is quite graceful generalization and unification of the theory of differential equations. Firstly being introduced by Hilger [12] in his Ph.D thesis in 1988 and thereafter greatly extended by Bohner and Peterson [1] in 2001, the calculus on time scales

became a sharp tool in the world on differential equations. Various derivative operators like classical derivative  $\frac{d}{dx}f(x)$ ,  $q$ -derivative  $D_qf(x)$ ,  $q$ -power derivative  $\mathcal{D}_qf(x)$ , finite difference  $\Delta f(x)$  etc, may be simply expressed in terms of time-scale derivative over particular time scale  $\mathbb{T}$ . For instance,

$$f'(x) = f^\Delta(x), \quad x \in \mathbb{T} = \mathbb{R}$$

$$\Delta f(x) = f^\Delta(x), \quad x \in \mathbb{T} = \mathbb{Z}$$

$$D_{n,q}f(x) = f^\Delta(x), \quad x \in \mathbb{T} = q^{\mathbb{R}^n}$$

$$D_qf(x) = f^\Delta(x), \quad x \in \mathbb{T} = q^{\mathbb{R}}$$

$$\mathcal{D}_qf(x) = f^\Delta(x), \quad x \in \mathbb{T} = \mathbb{R}^q$$

In context of Computer Science, namely object oriented programming paradigm, the time scale calculus may be thought as unified interface of derivative operator. Furthermore, the idea of time-scale calculus was slightly extended in [13, 14, 15, 16].

### 3. MAIN RESULTS

Timescale derivative of odd-powered polynomial  $t^{2m+1}$  may be expressed as follows

**Theorem 3.1.** *Let  $P(m, b, x)$  be a  $2m+1$ -degree polynomial in  $x, b$ . Let be a two-dimensional timescale  $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (x, b): x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$  such that  $\mathbb{T}_1 = \mathbb{T}_2$ . For every  $t \in \mathbb{T}_1$  and  $x, b \in \Lambda^2$*

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(t), t) + \frac{\partial P(m, b, x)}{\Delta b}(m, t, t)$$

where

- $\sigma(t) > t$  – is forward jump operator
- $\frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(t), t)$  – is the value of the partial derivative on time scales of  $P(m, b, x)$  with respect to the variable  $x$  evaluated in point  $x = t, b = \sigma(t)$

- $\frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$  – is the value of the partial derivative on time scales of  $P(m,b,x)$  with respect to the variable  $b$ , evaluated at  $x = t$ ,  $b = t$

In simpler words, the theorem 3.1 says

For every odd-powered polynomial  $x^{2m+1}$ , the derivative on time scales  $\frac{\Delta x^{2m+1}}{\Delta x}$  evaluated in point  $t \in \mathbb{T}_1$  equals to partial derivative on time scales of the polynomial  $P(m,b,x)$  with respect to  $x$  evaluated in point  $(x,b) = (t,\sigma(t))$  plus the value of the partial derivative on time scales of the polynomial  $P(m,b,x)$  with respect to  $b$ , evaluated in point  $(x,b) = (t,t)$ .

In its extended form the theorem 3.1 is as follows

$$\begin{aligned} \frac{\Delta x^{2m+1}}{\Delta x}(t) &= \frac{\partial}{\Delta x} \left( \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \right) (m, \sigma(t), t) \\ &\quad + \frac{\partial}{\Delta b} \left( \sum_{k=0}^{b-1} \sum_{r=0}^m \mathbf{A}_{m,r} k^r (x-k)^r \right) (m, t, t) \end{aligned}$$

#### 4. DISCUSSION AND EXAMPLES

To understand the nature of the theorem 3.1, we discuss a few examples involving widely-known time scales including integer timescale  $\mathbb{Z}$ , real timescale  $\mathbb{R}$ , quantum timescale  $q^{\mathbb{R}}$  and quantum-power timescale  $\mathbb{R}^q$ .

##### 4.1. Time scale of integers $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$ .

**Corollary 4.1.** *(Finite difference.) Let be a two-dimensional timescale  $\Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ . For every  $t \in \mathbb{Z}$  and  $x, b \in \Lambda^2$*

$$\Delta x^{2m+1}(t) = \frac{\partial P(m,b,x)}{\Delta x}(m, \sigma(t), t) + \frac{\partial P(m,b,x)}{\Delta b}(m, \sigma(t), t)$$

where  $\sigma(t)$  is the forward jump operator defined as  $\sigma(t) = t + 1$ .

**Example 4.2.** Let be  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then

$$\begin{aligned}\frac{\partial P(1, b, x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial P(1, b, x)}{\Delta b} &= 1 - 6b^2 + 6bx\end{aligned}$$

*Evaluating in points yields*

$$\begin{aligned}\frac{\partial P(1, b, x)}{\Delta x}(\sigma(t), t) &= 3t + 3t^2 \\ \frac{\partial P(1, b, x)}{\Delta b}(t, t) &= 1\end{aligned}$$

*Summing up previously obtained partial timescale derivatives, we get ordinary finite difference of odd-powered polynomial  $x^3$  evaluated in point  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$*

$$\Delta x^3(t) = \frac{\partial P(1, b, x)}{\Delta x}(\sigma(t), t) + \frac{\partial P(1, b, x)}{\Delta b}(t, t) = 3t + 3t^2 + 1$$

**Example 4.3.** Let be  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then

$$\begin{aligned}\frac{\partial P(2, b, x)}{\Delta x} &= 5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x \\ \frac{\partial P(2, b, x)}{\Delta b} &= 1 + 30b^4 - 60b^3x + 30b^2x^2\end{aligned}$$

*Evaluating in points yields*

$$\begin{aligned}\frac{\partial P(2, b, x)}{\Delta x}(\sigma(t), t) &= 5t + 10t^2 + 10t^3 + 5t^4 \\ \frac{\partial P(2, b, x)}{\Delta b}(t, t) &= 1\end{aligned}$$

*Summing up previously obtained partial timescale derivatives, we get time ordinary finite difference of odd-powered polynomial  $x^{2m+1}$  and  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$*

$$\Delta x^5(t) = \frac{\partial P(2, b, x)}{\Delta x}(t, \sigma(t)) + \frac{\partial P(2, b, x)}{\Delta b}(t, t) = 1 + 5t + 10t^2 + 10t^3 + 5t^4$$

**Corollary 4.4.** For every  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$

$$\frac{\partial P(m, b, x)}{\Delta x}(m, \sigma(t), t) = \sum_{r=1}^{2m} \binom{2m+1}{r} t^r$$

**Corollary 4.5.** *For every  $t \in \mathbb{Z}$ ,  $x, b \in \Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ ,  $m \in \mathbb{N}$*

$$\frac{\partial P(m, b, x)}{\Delta b}(m, t, t) = 1$$

4.2. **Time scale of real numbers**  $\mathbb{T} = \mathbb{R} \times \mathbb{R}$ .

**Corollary 4.6.** *(Classical derivative.) Let be a two-dimensional time scale  $\Lambda^2 = \mathbb{R} \times \mathbb{R} := \{t = (x, b) : x \in \mathbb{R}, b \in \mathbb{R}\}$ . For every  $t \in \mathbb{R}$ ,  $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$*

$$\frac{d}{dt}t^{2m+1} = \frac{\partial P(m, b, x)}{\partial x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial P(m, b, x)}{\partial b} \Big|_{x=t, b=t},$$

where  $\sigma(t) = t + \Delta t$ ,  $\Delta t \rightarrow 0$ .

**Example 4.7.** *Let be  $t \in \mathbb{R}$ ,  $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then*

$$\begin{aligned} \frac{\partial P(1, b, x)}{\partial x} &= -3b + 3b^2 \\ \frac{\partial P(1, b, x)}{\partial b} &= 6b - 6b^2 - 3x + 6bx \\ \frac{\partial P(1, b, x)}{\partial x} \Big|_{x=t, b=\sigma(t)} &= -3t + 3t^2 \\ \frac{\partial P(1, b, x)}{\partial b} \Big|_{x=t, b=t} &= 3t \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}$ ,  $x \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$

$$\frac{d}{dt}t^3 = \frac{\partial P(1, b, x)}{\partial x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial P(1, b, x)}{\partial b} \Big|_{x=t, b=t} = 3t^2.$$

**Example 4.8.** *Let be  $t \in \mathbb{R}$ ,  $x, b \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then*

$$\begin{aligned} \frac{\partial P(2, b, x)}{\partial x} &= -15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x, \\ \frac{\partial P(2, b, x)}{\partial b} &= 30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2 \\ \frac{\partial P(2, b, x)}{\partial x} \Big|_{x=t, b=\sigma(t)} &= -5t^2 + 5t^4 \\ \frac{\partial P(2, b, x)}{\partial b} \Big|_{x=t, b=t} &= 5t^2 \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get classical derivative of an odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}$ ,  $x \in \Lambda^2 = \mathbb{R} \times \mathbb{R}$ ,  $m \in \mathbb{N}$

$$\frac{d}{dt}t^5 = \left. \frac{\partial P(2, b, x)}{\partial x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(2, b, x)}{\partial b} \right|_{x=t, b=t} = 5t^4.$$

#### 4.3. Quantum time scale $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$ .

**Corollary 4.9.** (*Q-derivative* [5].) *Let be a two-dimensional time scale  $\Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}} := \{t = (x, b) : x \in q^{\mathbb{R}}, b \in q^{\mathbb{R}}\}$ . For every  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ ,  $m \in \mathbb{N}$*

$$D_q t^{2m+1} = \left. \frac{\partial P(m, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(m, b, x)}{\Delta b} \right|_{x=t, b=t},$$

where  $\sigma(t) = qt$ ,  $q > 1$ .

**Example 4.10.** *Let be  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then*

$$\begin{aligned} \frac{\partial P(1, b, x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial P(1, b, x)}{\Delta b} &= 3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx \\ \left. \frac{\partial P(1, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} &= -3qt + 3q^2t^2 \\ \left. \frac{\partial P(1, b, x)}{\Delta b} \right|_{x=t, b=t} &= 3qt + t^2 + qt^2 - 2q^2t^2 \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -derivative of odd polynomial  $t^{2m+1}$ ,  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ ,  $m \in \mathbb{N}$

$$D_q t^3 = \left. \frac{\partial P(1, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(1, b, x)}{\Delta b} \right|_{x=t, b=t} = t^2 + qt^2 + q^2t^2.$$

For every  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$  the following polynomial identity holds as  $q$  tends to zero

$$\lim_{q \rightarrow 0} \left. \frac{\partial P(1, b, x)}{\Delta b} \right|_{x=t, b=t} = t^2$$

However, it would be generalized as follows



**Corollary 4.11.** *For every  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ ,  $m \in \mathbb{N}$*

$$\lim_{q \rightarrow 0} \frac{\partial P(m, b, x)}{\Delta b} \Big|_{x=t, b=t} = t^{2m}.$$

**Example 4.12.** *Let be  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then*

$$\begin{aligned} \frac{\partial P(2, b, x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx - 15b^2qx + 10b^3qx \\ \frac{\partial P(2, b, x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 + 6b^4q^2 - 15b^3q^3 \\ &\quad + 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x \\ &\quad - 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2 \end{aligned}$$

$$\frac{\partial P(2, b, x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} = 5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4$$

$$\frac{\partial P(2, b, x)}{\Delta b} \Big|_{x=t, b=t} = -5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4$$

*Summing up previously obtained partial time-scale derivatives, we get the  $q$ -derivative of odd polynomial  $t^{2m+1}$ ,  $t \in q^{\mathbb{R}}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}} \times q^{\mathbb{R}}$ ,  $m \in \mathbb{N}$*

$$D_q t^5 = \frac{\partial P(2, b, x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial P(2, b, x)}{\Delta b} \Big|_{x=t, b=t} = t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4.$$

#### 4.4. Quantum power time scale $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$ .

**Corollary 4.13.** ( *$Q$ -power derivative [9].*) *Let be a two-dimensional time scale  $\Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q := \{t = (x, b) : b \in \mathbb{R}^q, x \in \mathbb{R}^q\}$ . For every  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$*

$$\mathcal{D}_q t^{2m+1} = \frac{\partial P(m, b, x)}{\Delta x}(t, \sigma(t)) + \frac{\partial P(m, b, x)}{\Delta b}(t, t),$$

*where the forward jump operator is defined as  $\sigma(t) = t^q$ ,  $q > 1$ .*

**Example 4.14.** Let be  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then

$$\begin{aligned}\frac{\partial P(1, b, x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial P(1, b, x)}{\Delta b} &= 3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^q x \\ \left. \frac{\partial P(1, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} &= -3t^q + 3t^{2q} \\ \left. \frac{\partial P(1, b, x)}{\Delta b} \right|_{x=t, b=t} &= t^2 + 3t^q - 2t^{2q} + t^{1+q}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$

$$\mathcal{D}_q t^3 = \left. \frac{\partial P(1, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(1, b, x)}{\Delta b} \right|_{x=t, b=t} = t^2 + t^{2q} + t^{1+q}.$$

**Example 4.15.** Let be  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then

$$\begin{aligned}\frac{\partial P(2, b, x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bx^q - 15b^2x^q + 10b^3x^q \\ \frac{\partial P(2, b, x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{2q} - 15b^{3q} + 6b^{4q} + 10b^{1+q} - 15b^{2+q} + 6b^{3+q} \\ &\quad - 15b^{1+2q} + 6b^{2+2q} + 6b^{1+3q} - 15bx + 30b^2x - 15b^3x - 15b^qx + 30b^{2q}x \\ &\quad - 15b^{3q}x + 30b^{1+q}x - 15b^{2+q}x - 15b^{1+2q}x + 5x^2 - 15bx^2 + 10b^2x^2 \\ &\quad - 15b^qx^2 + 10b^{2q}x^2 + 10b^{1+q}x^2 \\ \left. \frac{\partial P(2, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} &= -10t^{2q} + 15t^{3q} - 5t^{4q} + 5t^{1+q} - 15t^{1+2q} + 10t^{1+3q} \\ \left. \frac{\partial P(2, b, x)}{\Delta b} \right|_{x=t, b=t} &= t^4 + 10t^{2q} - 15t^{3q} + 6t^{4q} - 5t^{1+q} + t^{3+q} + 15t^{1+2q} + t^{2+2q} - 9t^{1+3q}\end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $m \in \mathbb{N}$

$$\mathcal{D}_q t^5 = \left. \frac{\partial P(2, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(2, b, x)}{\Delta b} \right|_{x=t, b=t} = t^4 + t^{4q} + t^{3+q} + t^{2+2q} + t^{1+3q}.$$

Another polynomial identity, that is exponential sum holds

**Corollary 4.16.** *For every  $t \in \mathbb{R}^q$ ,  $x, b \in \Lambda^2 = \mathbb{R}^q \times \mathbb{R}^q$ ,  $t \in \mathbb{R}$ ,  $m \in \mathbb{N}$*

$$\lim_{q \rightarrow 0} \left. \frac{\partial P(m, b, x)}{\Delta b} \right|_{x=t, b=t} = \sum_{k=0}^{2m} t^k$$

**4.5. Pure quantum power time scale**  $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$ . In this subsection we discuss a pure quantum power time scale  $q^{\mathbb{R}^j}$  provided by Aldwoah, Malinowska and Torres in [9], among with the  $q$ -power derivative operator  $D_{n,q}f(t)$  defined by

$$D_{n,q}f(t) = \frac{f(qt^n) - f(t)}{qt^n - t},$$

where  $n$  is odd positive integer and  $0 < q < 1$ .

**Corollary 4.17.** *(Quantum power derivative [9].) Let be a two-dimensional time scale  $\Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j} := \{t = (x, b) : b \in q^{\mathbb{R}^j}, x \in q^{\mathbb{R}^j}\}$ . For every  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$*

$$D_{n,q}t^{2m+1} = \left. \frac{\partial P(m, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(m, b, x)}{\Delta b} \right|_{x=t, b=t},$$

where  $\sigma(t) = qt^n$ ,  $\sigma(t) > t$ .

**Example 4.18.** *Let be  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$  and let  $m = 1$ , then*

$$\begin{aligned} \frac{\partial P(1, b, x)}{\Delta x} &= -3b + 3b^2 \\ \frac{\partial P(1, b, x)}{\Delta b} &= 3b - 2b^2 + 3b^j q - 2b^{1+j} q - 2b^{2j} q^2 - 3x + 3bx + 3b^j qx \\ \frac{\partial P(1, b, x)}{\Delta x}(t, \sigma(t)) &= -3qt^j + 3q^2 t^{2j} \\ \frac{\partial P(1, b, x)}{\Delta b}(t, t) &= t^2 + 3qt^j - 2q^2 t^{2j} + qt^{1+j} \end{aligned}$$

Summing up previously obtained partial time-scale derivatives, we get  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$

$$D_{n,q}t^3 = \left. \frac{\partial P(1, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(1, b, x)}{\Delta b} \right|_{x=t, b=t} = t^2 + q^2 t^{2j} + qt^{1+j}.$$

Another polynomial identity, that is exponential sum holds

**Corollary 4.19.** *For every  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $t \in \mathbb{R}$ ,  $m \in \mathbb{N}$*

$$\lim_{j \rightarrow 0} \lim_{q \rightarrow 1} \frac{\partial P(m, b, x)}{\Delta b} \Big|_{x=t, b=t} = \sum_{k=0}^{2m} t^k$$

An identity in even polynomials holds too

**Corollary 4.20.** *For every  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $t \in \mathbb{R}$ ,  $m \in \mathbb{N}$*

$$\lim_{j \rightarrow 0} \lim_{q \rightarrow 0} \frac{\partial P(m, b, x)}{\Delta b} \Big|_{x=t, b=t} = t^{2m}$$

**Example 4.21.** *Let be  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$  and let  $m = 2$ , then*

$$\begin{aligned} \frac{\partial P(2, b, x)}{\Delta x} &= -15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx^j - 15b^2qx^j + 10b^3qx^j \\ \frac{\partial P(2, b, x)}{\Delta b} &= 10b^2 - 15b^3 + 6b^4 + 10b^{1+j}q - 15b^{2+j}q + 6b^{3+j}q + 10b^{2j}q^2 - 15b^{1+2j}q^2 \\ &\quad + 6b^{2+2j}q^2 - 15b^{3j}q^3 + 6b^{1+3j}q^3 + 6b^{4j}q^4 - 15bx + 30b^2x - 15b^3x - 15b^j qx \\ &\quad + 30b^{1+j}qx - 15b^{2+j}qx + 30b^{2j}q^2x - 15b^{1+2j}q^2x - 15b^{3j}q^3x + 5x^2 - 15bx^2 \\ &\quad + 10b^2x^2 - 15b^j qx^2 + 10b^{1+j}qx^2 + 10b^{2j}q^2x^2 \end{aligned}$$

$$\frac{\partial P(2, b, x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} = -10q^2t^{2j} + 15q^3t^{3j} - 5q^4t^{4j} + 5qt^{1+j} - 15q^2t^{1+2j} + 10q^3t^{1+3j}$$

$$\frac{\partial P(2, b, x)}{\Delta b} \Big|_{x=t, b=t} = t^4 + 10q^2t^{2j} - 15q^3t^{3j} + 6q^4t^{4j} - 5qt^{1+j} + qt^{3+j} + 15q^2t^{1+2j} + q^2t^{2+2j} - 9q^3t^{1+3j}$$

Summing up previously obtained partial time-scale derivatives, we  $q$ -power derivative of odd polynomial  $t^{2m+1}$ ,  $t \in q^{\mathbb{R}^j}$ ,  $x, b \in \Lambda^2 = q^{\mathbb{R}^j} \times q^{\mathbb{R}^j}$ ,  $m \in \mathbb{N}$

$$D_{n,q}t^5 = \frac{\partial P(1, b, x)}{\Delta x} \Big|_{x=t, b=\sigma(t)} + \frac{\partial P(1, b, x)}{\Delta b} \Big|_{x=t, b=t} = t^4 + q^4t^{4j} + qt^{3+j} + q^2t^{2+2j} + q^3t^{1+3j}.$$

## 5. PROOF OF MAIN THEOREM

By [Lemma 3.1 [10]], for every  $x \in \mathbb{R}$ ,  $m \in \mathbb{N}$  it is true that

$$P(m, x, x) = x^{2m+1} \tag{5.1}$$

**Proof of theorem 3.1.** Let be  $x, b \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 := \{t = (x, b) : x \in \mathbb{T}_1, b \in \mathbb{T}_2\}$ . Let be  $\mathbb{T}_1 = \mathbb{T}_2$ . Assume that time-scale derivative  $(x^{2m+1})^\Delta$  is

$$(x^{2m+1})^\Delta = \lim_{b \rightarrow x} \lim_{t \rightarrow x} \frac{P(m, \sigma(b), \sigma(x)) - P(m, b, t)}{\sigma(x) - t}, \quad (5.2)$$

where  $\sigma(x) > x$  is forward jump operator. However, equation (5.2) is not a timescale derivative of  $P(m, b, x)$  over  $x$  how it might seem because of denominator  $\sigma(x) - t$ . Parameter  $b$  of  $P(m, b, x)$  is implicitly incremented as well. Let's try to express nominator of (5.2) in terms of partial derivative  $\frac{\partial P(m, b, x)}{\Delta b}$  on timescales. Let be the following equation

$$P(m, \sigma(b), x) - P(m, b, x) = P(m, b, x)_b^\Delta \cdot \Delta b$$

Let  $t \rightarrow x$  in (5.2). Then nominator of (5.2) equals to

$$P(m, \sigma(b), \sigma(x)) - P(m, b, x) = P(m, \sigma(b), x) - P(m, b, x) + A$$

where  $A$  is yet implicit term. Let's now collapse the terms  $f_m(x, b)$  from both sides of above equation, such that

$$P(m, \sigma(b), \sigma(x)) = P(m, \sigma(b), \sigma(x)) + A$$

Therefore,

$$A = P(m, \sigma(b), \sigma(x)) - P(m, \sigma(b), \sigma(x)) = P(m, b, x)_x^\Delta(x, \sigma(b)) \cdot \Delta x$$

Now, let's express the nominator of (5.2) as follows

$$P(m, \sigma(b), \sigma(x)) - P(m, b, x) = P(m, b, x)_x^\Delta(x, \sigma(b)) \cdot \Delta x + P(m, b, x)_b^\Delta(x, b) \cdot \Delta b$$

$$P(m, \sigma(b), \sigma(x)) - P(m, b, x) = P(m, b, x)_x^\Delta(x, \sigma(b)) \cdot (\sigma(x) - x) + P(m, b, x)_b^\Delta(x, b) \cdot (\sigma(b) - b)$$

We can collapse the terms  $(\sigma(x) - x)$ ,  $(\sigma(b) - b)$  in above expressions, as  $b \rightarrow x$ . Therefore,

$$\frac{P(m, \sigma(x), \sigma(x)) - P(m, x, x)}{\sigma(x) - x} = P(m, b, x)_x^\Delta(x, \sigma(x)) + P(m, b, x)_b^\Delta(x, x)$$

Finally, by the identity (5.1) we can express timescale derivative of  $x^{2m+1}$ ,  $x \in \Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$ ,  $m \in \mathbb{N}$  as

$$(x^{2m+1})^\Delta(t) = \left. \frac{\partial P(m, b, x)}{\Delta x} \right|_{x=t, b=\sigma(t)} + \left. \frac{\partial P(m, b, x)}{\Delta b} \right|_{x=t, b=t}$$

This completes the proof.  $\square$

## 6. CONCLUSION AND FUTURE RESEARCH

In this manuscript we have discussed partial time scale differential equation involving derivatives of polynomials in context of time scale  $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$  where  $\mathbb{T}_1 = \mathbb{T}_2$ . Future research can be conducted to study the case  $\mathbb{T}_1 \neq \mathbb{T}_2$ , which makes the theorem 3.1 to be generalised

$$\frac{\partial P(m, b, x)}{\Delta x} + \frac{\partial P(m, b, x)}{\Delta b} = \alpha_m(x, b)(x^{2m+1})^\Delta,$$

where  $\alpha_m(x, b)$  is arbitrary differentiable function. Also, it is worth to discuss the theorem 3.1 in context of high order derivatives on time scales. We have established a few power identities, and shown the theorem 3.1 for different 2-dimensional time scales  $\Lambda^2$  like integer time scale  $\mathbb{Z} \times \mathbb{Z}$ , real time scale  $\mathbb{R} \times \mathbb{R}$ , quantum time scale  $q^\mathbb{R} \times q^\mathbb{R}$  and quantum power time scale  $\mathbb{R}^q \times \mathbb{R}^q$ .

## REFERENCES

- [1] Martin Bohner and Allan Peterson. *Dynamic equations on time scales: An introduction with applications*. Springer Science & Business Media, 2001. <https://web.mst.edu/~bohner/sample.pdf>.
- [2] Martin Bohner and Gusein Sh Guseinov. Partial differentiation on time scales. *Dynamic systems and applications*, 13(3-4):351–379, 2004. <http://web.mst.edu/~bohner/papers/pdots.pdf>.
- [3] Calvin D Ahlbrandt and Christina Morian. Partial differential equations on time scales. *Journal of Computational and Applied Mathematics*, 141(1-2):35–55, 2002. [https://doi.org/10.1016/S0377-0427\(01\)00434-4](https://doi.org/10.1016/S0377-0427(01)00434-4).
- [4] B. Jackson. Partial dynamic equations on time scales. *Journal of Computational and Applied Mathematics*, 186(2):391 – 415, 2006. <https://doi.org/10.1016/j.cam.2005.02.011>.
- [5] Frederick H Jackson. Xi.—on q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, 46(2):253–281, 1909.

- [6] Thomas Ernst. *The history of q-calculus and a new method*. Citeseer, 2000.
- [7] Thomas Ernst. The different tongues of q-calculus. *Proceedings of the Estonian Academy of Sciences*, 57(2), 2008.
- [8] Victor Kac and Pokman Cheung. *Quantum calculus*. Springer Science & Business Media, 2001.
- [9] Khaled A Aldwoah, Agnieszka B Malinowska, and Delfim FM Torres. The power quantum calculus and variational problems. *arXiv preprint arXiv:1107.0344*, 2011. <https://arxiv.org/abs/1107.0344>.
- [10] Petro Kolosov. On the link between Binomial Theorem and Discrete Convolution of Polynomials. *arXiv preprint arXiv:1603.02468*, 2016. <https://arxiv.org/abs/1603.02468>.
- [11] Eric W Weisstein. "Bernoulli Number." From MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/BernoulliNumber.html>.
- [12] S Hilger. *Eine Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeit*. PhD thesis, Ph. D. Thesis, Universität Würzburg, 1988.
- [13] Benaoumeur Bayour, Ahmed Hammoudi, and Delfim FM Torres. A truly conformable calculus on time scales. *arXiv preprint arXiv:1705.08928*, 2017. <https://arxiv.org/abs/1705.08928>.
- [14] Nadia Benkhattou, Salima Hassani, and Delfim FM Torres. A conformable fractional calculus on arbitrary time scales. *Journal of King Saud University-Science*, 28(1):93–98, 2016.
- [15] M Cristina Caputo. Time scales: from nabla calculus to delta calculus and vice versa via duality. *arXiv preprint arXiv:0910.0085*, 2009.
- [16] Natália Martins and Delfim FM Torres. Calculus of variations on time scales with nabla derivatives. *Nonlinear Analysis: Theory, Methods & Applications*, 71(12):e763–e773, 2009.
- [17] Petro Kolosov. Supplementary Mathematica Programs. 2020. <https://github.com/kolosovpetro/AStudyOnDynamicEquations/tree/develop/mathematica>.

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## 7. ADDENDUM 1: MATHEMATICA SCRIPTS

To fulfill our study, we attach here a link to the set of *Mathematica* programs, designed to verify the results of current manuscript. To reach these programs follow the link [17]. To reproduce results, proceed as follows:

- Time scale of integers  $\mathbb{T} = \mathbb{Z} \times \mathbb{Z}$ :
  - Example 4.2: Execute the commands of Mathematica package
    - \* Set `sigma[x_] := x + 1` in Mathematica package and execute definition.
    - \* Execute `timeScaleDerivativeX[1, x, b]` which produces  $-3b + 3b^2$ .
    - \* Execute `Expand[timeScaleDerivativeX[1, t, sigma[t]]]` which produces  $3t + 3t^2$ .
    - \* Execute `timeScaleDerivativeB[1, x, b]` which produces  $1 - 6b^2 + 6bx$ .
    - \* Execute `timeScaleDerivativeB[1, t, t]` which produces 1.
    - \* Execute `mainTheorem[1]` which produces  $1 + 3t + 3t^2$ .
  - Example 4.3: Execute the commands of Mathematica package
    - \* Set `sigma[x_] := x + 1` in Mathematica package and execute definition.
    - \* `timeScaleDerivativeX[2, x, b]` which produces  $5b - 30b^2 + 40b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$ .
    - \* `Expand[timeScaleDerivativeX[2, t, sigma[t]]]` which produces  $5t + 10t^2 + 10t^3 + 5t^4$ .
    - \* `timeScaleDerivativeB[2, x, b]` which produces  $1 + 30b^4 - 60b^3x + 30b^2x^2$ .
    - \* `timeScaleDerivativeB[2, t, t]` which produces 1.
    - \* `mainTheorem[2]` which produces  $1 + 5t + 10t^2 + 10t^3 + 5t^4$ .
- Time scale of real numbers  $\mathbb{T} = \mathbb{R} \times \mathbb{R}$ :



– Example 4.7: Execute the commands of Mathematica package

- \* Set `sigma[x_] := x + Global`dx` in Mathematica package and execute definition.
- \* Execute `timeScaleDerivativeX[1, x, b]` which produces  $-3b + 3b^2$ .
- \* Execute `Limit[Expand[timeScaleDerivativeB[1, x, b]], dx -> 0]` which produces  $6b - 6b^2 - 3x + 6bx$ .
- \* Execute `timeScaleDerivativeX[1, t, t]` which produces  $-3t + 3t^2$ .
- \* Execute `Limit[Expand[timeScaleDerivativeB[1, t, t]], dx -> 0]` which produces  $3t$ .
- \* Execute `Limit[mainTheorem[1], dx -> 0]` which produces  $3t^2$ .

– Example 4.8: Execute the commands of Mathematica package

- \* Set `sigma[x_] := x + Global`dx` in Mathematica package and execute definition.
- \* Execute `Limit[Expand[timeScaleDerivativeX[2, x, b]], dx -> 0]` which produces  $-15b^2 + 30b^3 - 15b^4 + 10bx - 30b^2x + 20b^3x$ .
- \* Execute `Limit[Expand[timeScaleDerivativeB[2, x, b]], dx -> 0]` which produces  $30b^2 - 60b^3 + 30b^4 - 30bx + 90b^2x - 60b^3x + 5x^2 - 30bx^2 + 30b^2x^2$ .
- \* Execute `Limit[Expand[timeScaleDerivativeX[2, t, sigma[t]]], dx -> 0]` which produces  $-5t^2 + 5t^4$ .
- \* Execute `Limit[Expand[timeScaleDerivativeB[2, t, t]], dx -> 0]` which produces  $5t^2$ .
- \* Execute `Limit[mainTheorem[2], dx -> 0]` which produces  $5t^4$ .

• Quantum time scale  $\mathbb{T} = q^{\mathbb{R}} \times q^{\mathbb{R}}$ :

– Example 4.10: Execute the commands of Mathematica package

- \* Set `sigma[x_] := x * Global`q` in Mathematica package and execute definition.

- \* Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces  $-3b + 3b^2$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces  $3b - 2b^2 + 3bq - 2b^2q - 2b^2q^2 - 3x + 3bx + 3bqx$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]` which produces  $-3qt + 3q^2t^2$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces  $3qt + t^2 + qt^2 - 2q^2t^2$ .
  - \* Execute `Expand[Simplify[mainTheorem[1]]]` which produces  $t^2 + qt^2 + q^2t^2$ .
- Example 4.12: Execute the commands of Mathematica package
- \* Set `sigma[x_] := x * Global`q` in Mathematica package and execute definition.
  - \* Execute `Expand[Simplify[timeScaleDerivativeX[2, x, b]]]` which produces  $-15b^2 + 30b^3 - 15b^4 + 5bx - 15b^2x + 10b^3x + 5bqx - 15b^2qx + 10b^3qx$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeB[2, x, b]]]` which produces  $10b^2 - 15b^3 + 6b^4 + 10b^2q - 15b^3q + 6b^4q + 10b^2q^2 - 15b^3q^2 + 6b^4q^2 - 15b^3q^3 + 6b^4q^3 + 6b^4q^4 - 15bx + 30b^2x - 15b^3x - 15bqx + 30b^2qx - 15b^3qx + 30b^2q^2x - 15b^3q^2x - 15b^3q^3x + 5x^2 - 15bx^2 + 10b^2x^2 - 15bqx^2 + 10b^2qx^2 + 10b^2q^2x^2$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeX[2, t, sigma[t]]]` which produces  $5qt^2 - 10q^2t^2 - 15q^2t^3 + 15q^3t^3 + 10q^3t^4 - 5q^4t^4$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeB[2, t, t]]]` which produces  $-5qt^2 + 10q^2t^2 + 15q^2t^3 - 15q^3t^3 + t^4 + qt^4 + q^2t^4 - 9q^3t^4 + 6q^4t^4$ .
  - \* Execute `Expand[Simplify[mainTheorem[2]]]` which produces  $t^4 + qt^4 + q^2t^4 + q^3t^4 + q^4t^4$ .
- Corollary 4.11: Execute the commands of Mathematica package

- \* Set `sigma[x_] := x * Global`q` in Mathematica package and execute definition.
  - \* Execute `Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 0]` for various values of `m`.
- Quantum power time scale  $\mathbb{T} = \mathbb{R}^q \times \mathbb{R}^q$ :
    - Example 4.14: Execute the commands of Mathematica package
      - \* Set `sigma[x_] := x ^ Global`q` in Mathematica package and execute definition.
      - \* Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces  $-3b + 3b^2$ .
      - \* Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces  $3b - 2b^2 + 3b^q - 2b^{2q} - 2b^{1+q} - 3x + 3bx + 3b^q x$ .
      - \* Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]` which produces  $-3t^q + 3t^{2q}$ .
      - \* Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces  $t^2 + 3t^q - 2t^{2q} + t^{1+q}$ .
      - \* Execute `Expand[Simplify[mainTheorem[1]]]` which produces  $t^2 + t^{2q} + t^{1+q}$ .
    - Example 4.15: Similarly to Example 4.14 with  $m = 2$ .
    - Corollary 4.16: Execute the commands of Mathematica package
      - \* Set `sigma[x_] := x ^ Global`q` in Mathematica package and execute definition.
      - \* Execute `Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 0]` for various values of `m`.
  - Pure quantum power time scale  $\mathbb{T} = q^{\mathbb{R}^n} \times q^{\mathbb{R}^n}$ :
    - Example 4.18: Execute the commands of Mathematica package

- \* Set `sigma[x_] := Global`q * x ^ Global`j` in Mathematica package and execute definition.
  - \* Execute `Expand[Simplify[timeScaleDerivativeX[1, x, b]]]` which produces  $-3b + 3b^2$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeB[1, x, b]]]` which produces  $3b - 2b^2 + 3b^j q - 2b^{1+j} q - 2b^{2j} q^2 - 3x + 3bx + 3b^j qx$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeX[1, t, sigma[t]]]]` which produces  $-3qt^j + 3q^2 t^{2j}$ .
  - \* Execute `Expand[Simplify[timeScaleDerivativeB[1, t, t]]]` which produces  $t^2 + 3qt^j - 2q^2 t^{2j} + qt^{1+j}$ .
  - \* Execute `Expand[Simplify[mainTheorem[1]]]` which produces  $t^2 + q^2 t^{2j} + qt^{1+j}$ .
- Example 4.21: Similarly as Example 4.18 for  $m = 2$ .
- Corollary 4.19: Execute the commands of Mathematica package
- \* Set `sigma[x_] := Global`q * x ^ Global`j` in Mathematica package and execute definition.
  - \* Execute `Limit[Limit[Expand[Simplify[timeScaleDerivativeB[m, t, t]]], q -> 1], j -> 0]` for various values of  $m$ .
- Corollary 4.20: Execute the commands of Mathematica package
- \* Set `sigma[x_] := Global`q * x ^ Global`j` in Mathematica package and execute definition.
  - \* Execute `Limit[Limit[Expand[Simplify[timeScaleDerivativeB[5, t, t]]], q -> 0], j -> 0]` for various values of  $m$ .

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