FINDING THE DERIVATIVE OF POLYNOMIALS VIA DOUBLE LIMIT

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Abstract. Differentiation is process of finding the derivative, or rate of change, of a function. Derivative itself is defined by the limit of function's change divided by the function's argument change as change tends to zero. In particular, for polynomials the function's change is calculated via Binomial expansion. This manuscript provides another approach to reach polynomial's function change as a limit of certain polynomial identity, and therefore expressing the derivative of polynomial as double limit.

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1. Introduction

Differentiation is process of finding the derivative, or rate of change, of a function. Derivative of a function $f(x)$ over domain x is defined by the limit of function's change divided by the function's argument change as change tends to zero, i.e

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

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Given the polynomial function $f(x) = x^n,\ n \in \mathbb{N}$ its derivative expressed as follows

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{(x+h)^n - x^n}{h}\right] \tag{1.1}$$

Therefore, the change of polynomial function from the nominator of (1.1) is being expressed applying Binomial theorem [1] so that

$$(x+h)^n - x^n = \sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k$$

Hence, arriving to well-known identity

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k\right] = nx^{n-1}$$

More precisely, consider the case $f(x) = x^5,\ x \in \mathbb{R}$

$$\frac{\mathrm{d}x^5}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{5h^4x + 10h^3x^2 + 10h^2x^3 + 5hx^4}{h}\right] = 5x^4$$

However, there is another approach to express the polynomial function's change $(x+h)^n - x^n$ using polynomial identity [2], that is

$$\mathbf{P}_b^m(x) = x^{2m+1}, \quad \text{as } b \to x$$

Polynomials $\mathbf{P}_b^m(x)$ are polynomials in $(x,b) \in \mathbb{R}$, for example

$$\begin{aligned}
&\mathbf{P}_b^0(x) = b,\\
&\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x,\\
&\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x + 5bx^2 - 15b^2x^2 + 10b^3x^2,\\
&\mathbf{P}_b^3(x) = -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x\\
&\qquad + 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3
\end{aligned}$$

Further explanations on the topic of polynomials $\mathbf{P}_b^m(x)$ are available at [3, 4]. Now we can express the polynomial function's change in terms of $\mathbf{P}_b^m(x)$ for odd power polynomials as limit

$$(x+h)^{2m+1} - x^{2m+1} = \lim_{b\to x+h}\left[\mathbf{P}_b^m(x+h) - x^{2m+1}\right]$$

For instance, let be $m = 2$ then x^5 polynomial function's change is

$$
\begin{aligned}
(x+h)^5 - x^5 &= \lim_{b \to x+h} \left[\mathbf{P}_b^2(x+h) - x^5 \right] \\
&= \lim_{b \to x+h} \left[5b^2x - 15bx^2 - 15b^2x^2 + 10x^3 + 30bx^3 + 10b^2x^3 - 15x^4 - 15bx^4 + 5x^5 \right] \\
&= h^5 + 5h^4x + 10h^3x^2 + 10h^2x^3 + 5hx^4
\end{aligned}
$$

Therefore, the derivative of odd-power polynomial $x^{2m+1},\ x \in \mathbb{R},\ m \in \mathbb{N}$ can be expressed in terms of double limit as follows

$$
\frac{\mathrm{d}x^{2m+1}}{\mathrm{d}x} = \lim_{h \to 0} \lim_{b \to x+h} \left[\frac{\mathbf{P}_b^m(x+h) - x^{2m+1}}{h} \right] \tag{1.2}
$$

For example, given $m = 1$ and therefore $f(x) = x^3$ we get

$$
\begin{aligned}
\frac{\mathrm{d}x^3}{\mathrm{d}x} &= \lim_{h \to 0} \lim_{b \to x+h} \left[\frac{\mathbf{P}_b^1(x+h) - x^3}{h} \right] = \lim_{h \to 0} \left[3h - 2h^2 + 6x - 6hx - 6x^2 + \frac{3x^2}{h} \right. \\
&\quad \left. - \frac{3x^3}{h} - 3(h+x) + 3h(h+x) + 6x(h+x) - \frac{3x(h+x)}{h} + \frac{3x^2(h+x)}{h} \right] \\
&= \lim_{h \to 0} \left[h^2 + 3hx + 3x^2 \right] = 3x^2
\end{aligned}
$$

Even-powered $2m+2,\ m \geq 0$ polynomials can be differentiated similarly, expressing the function's gain in terms of limit of the polynomial $\mathbf{P}_b^m(x)$, i.e

$$
(x+h)^{2m+2} - x^{2m+2} = \lim_{b \to x+h} \left[x\mathbf{P}_b^m(x+h) + h\mathbf{P}_b^m(x+h) - x^{2m+2} \right]
$$

Given $m = 1$, the gain of even-powered polynomial x^{2m+2} in its extended form is

$$
\begin{aligned}
(x+h)^4 - x^4 &= \lim_{b \to x+h} \left[x\mathbf{P}_b^1(x+h) + h\mathbf{P}_b^1(x+h) - x^4 \right] \\
&= 3h^3 - 2h^4 + 9h^2x - 8h^3x + 9hx^2 - 12h^2x^2 + 3x^3 \\
&\quad - 8hx^3 - 3x^4 - 3h^2(h+x) + 3h^3(h+x) - 6hx(h+x) \\
&\quad + 9h^2x(h+x) - 3x^2(h+x) + 9hx^2(h+x) + 3x^3(h+x)
\end{aligned}
$$

So that generally speaking of even-powered polynomial $x^{2m+2},\ m \geq 1$ we can conclude that its derivative can be expressed as double limit similarly to (1.2)

$$
\frac{\mathrm{d}x^{2m+2}}{\mathrm{d}x} = \lim_{h \to 0} \lim_{b \to x+h} \left[\frac{x\mathbf{P}_b^m(x+h) + h\mathbf{P}_b^m(x+h) - x^{2m+2}}{h} \right] \tag{1.3}
$$

So that given $m = 1$, the derivative $\frac{\mathrm{d}x^4}{\mathrm{d}x}$ of even-powered polynomial x^{2m+2} in its extended form is

$$\begin{aligned}\frac{\mathrm{d}x^4}{\mathrm{d}x} &= \lim_{h\to 0}\lim_{b\to x+h}\left[\frac{x\mathbf{P}_b^1(x+h)+h\mathbf{P}_b^1(x+h)-x^4}{h}\right]\\ &= \lim_{h\to 0}\big[3h^3-2h^4+9h^2x-8h^3x+9hx^2-12h^2x^2+3x^3\\ &-8hx^3-3x^4-3h^2(h+x)+3h^3(h+x)-6hx(h+x)\\ &+9h^2x(h+x)-3x^2(h+x)+9hx^2(h+x)+3x^3(h+x)\big]=4x^3\end{aligned}$$

Therefore, in its general form the derivative of polynomial function $f(x) = x^n$ can be expressed via

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = \begin{cases} \lim_{h\to 0}\lim_{b\to x+h}\left[\frac{x\mathbf{P}_b^m(x+h)+h\mathbf{P}_b^m(x+h)-x^{2m+2}}{h}\right] & n=2m+2\\ \lim_{h\to 0}\lim_{b\to x+h}\left[\frac{\mathbf{P}_b^m(x+h)-x^{2m+1}}{h}\right] & n=2m+1\end{cases} \tag{1.4}$$

2. Conclusions

In this manuscript we have discussed and proposed the way of finding of the derivative of polynomials via double limit as it is stated in expression (1.4).

3. Verification of the results

The main results of the manuscript, i.e the expression (1.4) can be validated using supplementary Mathematica programs at `this link`.

References

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