FINDING THE DERIVATIVE OF POLYNOMIALS VIA DOUBLE LIMIT

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Abstract. Differentiation is process of finding the derivative, or rate of change, of a function. Derivative itself is defined by the limit of function's change divided by the function's argument change as change tends to zero. In particular, for polynomials the function's change is calculated via Binomial expansion. This manuscript provides another approach to reach polynomial's function change as a limit of certain polynomial identity, and therefore expressing the derivative of polynomial as double limit.

Contents

1. Introduction

Differentiation is process of finding the derivative, or rate of change, of a function. Derivative of a function $f(x)$ over domain x is defined by the limit of function's change divided by the function's argument change as change tends to zero, i.e

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

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Given the polynomial function $f(x) = x^n,\ n \in \mathbb{N}$ its derivative expressed as follows

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \tag{1.1}$$

Therefore, the change of polynomial function from the nominator of (1.1) is being expressed applying Binomial theorem (citation) so that

$$(x+h)^n - x^n = \sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k$$

Hence, arriving to well-known identity

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k = nx^{n-1}$$

More precisely, consider the case $f(x) = x^5,\ x \in \mathbb{R}$

$$\frac{d}{dx}x^5 = \lim_{h\to 0}\frac{5h^4x + 10h^3x^2 + 10h^2x^3 + 5hx^4}{h} = 5x^4$$

However, there is another approach to express the polynomial function's change $(x+h)^n - x^n$ using polynomial identity (citation), that is

$$\mathbf{P}_b^m(x) = x^{2m+1}, \quad \text{as } b \to x$$

Polynomials $\mathbf{P}_b^m(x)$ are polynomials in $(x,b) \in \mathbb{R}$, for example

$$\mathbf{P}_b^0(x) = b,$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x,$$

$$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x + 5bx^2 - 15b^2x^2 + 10b^3x^2,$$

$$\begin{aligned}\mathbf{P}_b^3(x) = &-7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x \\ &+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3\end{aligned}$$

Now we can express the polynomial function's change in terms of $\mathbf{P}_b^m(x)$ for odd power polynomials as limit

$$(x+h)^{2m+1} - x^{2m+1} = \lim_{b\to x+h}\left[\mathbf{P}_b^m(x+h) - x^{2m+1}\right]$$

For instance, let be $m = 2$ then x^5 polynomial function's change is

$$\begin{aligned}(x+h)^5 - x^5 &= \lim_{b \to x+h} \left[\mathbf{P}_b^2(x+h) - x^5\right] \\ &= \lim_{b \to x+h} \left[5b^2x - 15bx^2 - 15b^2x^2 + 10x^3 + 30bx^3 + 10b^2x^3 - 15x^4 - 15bx^4 + 5x^5\right] \\ &= h^5 + 5h^4x + 10h^3x^2 + 10h^2x^3 + 5hx^4\end{aligned}$$

2. Conclusions

Conclusions of your manuscript.

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