NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

PETRO KOLOSOV

Abstract. In this manuscript we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind and Eulerian numbers.

Contents

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1. Introduction and main results

In this manuscript we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind and Eulerian numbers.

Allow us to start from the definition of multifold sums of powers. We utilize the recurrence proposed by Donald Knuth in his article *Johann Faulhaber and sums of powers*, see [1]

$$\begin{aligned}\Sigma^0\, n^m &= n^m\\ \Sigma^1\, n^m &= \Sigma^0\, 1^m + \Sigma^0\, 2^m + \cdots + \Sigma^0\, n^m\\ \Sigma^{r+1}\, n^m &= \Sigma^r\, 1^m + \Sigma^r\, 2^m + \cdots + \Sigma^r\, n^m\end{aligned}$$

Throughout the paper, we utilize the Newton's interpolation formula as stated below

Proposition 1.1. *(Newton's series around arbitrary point [2, Lemma V].)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

where $\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j)$ *is* k*-degree forward finite difference of* f*.*

Which indeed holds, because

$$\begin{aligned}n^3 &= 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}\\ n^3 &= 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}\\ n^3 &= 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}\end{aligned}$$

Proposition 1.2 (Newton's series for power)**.** *For non-negative integers* m, n *and an arbitrary integer* t

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

Thus, for an arbitrary integer t, the ordinary sum of powers is

$$\Sigma^1 n^m = \sum_{k=1}^{n}\sum_{j=0}^{m}\binom{-t+k}{j}\Delta^j t^m = \sum_{j=0}^{m}\Delta^j t^m \sum_{k=1}^{n}\binom{-t+k}{j}$$

Proposition 1.3 (Segmented Hockey stick identity)**.** *For integers n, t and j*

$$\sum_{k=0}^{n}\binom{-t+k}{j} = (-1)^j\binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Therefore,

Proposition 1.4 (Ordinary sums of powers via Newton's series)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m}\Delta^j t^m \left[(-1)^j\binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}\right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1 n^m = \sum_{j=0}^{m}\Delta^j t^m \sum_{k=1}^{n}\binom{-t+k}{j}$, where $\sum_{k=1}^{n}\binom{-t+k}{j} = (-1)^j\binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.3). $\square$

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1 n^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$

which was discussed in [3, p. 190] and in [4]. The coefficients $0, 1, 6, 6, 0, 1, 14, 36, 24, \ldots$ are given by the sequence A131689 in the OEIS [5]. The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [6]. For instance,

$$\Sigma^1 n^3 = 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$

$$\Sigma^1 n^4 = 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}$$

The coefficients $1, 7, 12, 6, 1, 15, \ldots$ are given by the sequence A028246 in the OEIS [5]. Interestingly enough that the paper [6] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^{k} j!\left[\binom{n+1-r}{j+1} + (-1)^j\binom{r+j-1}{j+1}\right]\left\{{k \atop j}\right\}_r$$

where $\left\{ {k \atop j} \right\}_r$ are generalized Stirling numbers of the second kind. The formula above is identical to the proposition (1.4), which implies that finite differences can be expressed in terms of generalized Stirling numbers of the second kind, that is $\Delta^j t^m = j! \left\{ {m \atop j} \right\}_t$.

By considering the special cases of the proposition (1.4) for $t = 4$, we observe rather unexpected formulas for sums of powers, namely

$$\Sigma^1\, n^0 = 1\left(\binom{n-3}{1} + \binom{3}{1}\right)$$
$$\Sigma^1\, n^1 = 4\left(\binom{n-3}{1} + \binom{3}{1}\right) + 1\left(\binom{n-3}{2} - \binom{4}{2}\right)$$
$$\Sigma^1\, n^2 = 16\left(\binom{n-3}{1} + \binom{3}{1}\right) + 9\left(\binom{n-3}{2} - \binom{4}{2}\right) + 2\left(\binom{n-2}{3} + \binom{5}{3}\right)$$
$$\Sigma^1\, n^3 = 64\left(\binom{n-3}{1} + \binom{3}{1}\right) + 61\left(\binom{n-3}{2} - \binom{4}{2}\right) + 30\left(\binom{n-3}{3} + \binom{5}{3}\right) + 6\left(\binom{n-3}{4} - \binom{6}{4}\right)$$

The coefficients $1, 4, 1, 16, 9, \ldots$ are given by the sequence A391633 in the OEIS [5]. In general,

$$\Sigma^1\, n^m = \sum_{j=0}^{m} \Delta^j 4^m \left[\binom{n-3}{j+1} + (-1)^j \binom{j+3}{j+1}\right].$$

To obtain the formula for double sum of powers, we apply the summation operator over the ordinary sum of powers again, thus

$$\Sigma^2\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \sum_{k=1}^{n} \binom{j+t-1}{j+1} + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$$

which yields

$$\Sigma^2\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$$

Thus,

Proposition 1.5 (Double sums of powers via Newton's series)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2} \right]$$

Proof. We have $\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$, where $\sum_{k=1}^{n} \binom{k-t+1}{j+1} = (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}$ by means of segmented hockey stick identity (1.3). □

For example, given $t = 5$, the double sums of powers are

$$\begin{aligned}
\Sigma^2 n^0 &= 1 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) \\
\Sigma^2 n^1 &= 5 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 1 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\
\Sigma^2 n^2 &= 25 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 11 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\
&\quad + 2 \left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4} \right) \\
\Sigma^2 n^3 &= 125 \left(\binom{n-3}{2} + \binom{4}{1} n - \binom{4}{2} \right) + 91 \left(\binom{n-3}{3} - \binom{5}{2} n + \binom{5}{3} \right) \\
&\quad + 36 \left(\binom{n-3}{4} + \binom{6}{3} n - \binom{6}{4} \right) + 6 \left(\binom{n-3}{5} - \binom{7}{4} n + \binom{7}{5} \right)
\end{aligned}$$

The coefficients $1, 5, 1, 25, 11, 2, \ldots$ are given by the sequence A391635 in the OEIS [5]. In general,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j 5^m \left[\binom{n-3}{j+2} + (-1)^j \binom{j+4}{j+1} n^1 + (-1)^{j+1} \binom{j+4}{j+2} n^0 \right]$$

Similarly, we obtain the formula for the triple sums of powers

Proposition 1.6 (Triple sums of powers via Newton's series)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\begin{aligned}
\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \Bigg[& (-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + \\
& + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3} \Bigg]
\end{aligned}$$

Proof. By summing up the double powers sums, we get

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[(-1)^j \binom{j+t-1}{j+1} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} k^0 + \binom{k-t+2}{j+2} \right]$$

$$= \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \sum_{k=1}^{n} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} \sum_{k=1}^{n} k^0 + \sum_{k=1}^{n} \binom{k-t+2}{j+2} \right]$$

Note that $\sum_{k=1}^{n} k^1 = \Sigma^2 n^0$ and $\sum_{k=1}^{n} k^0 = \Sigma^1 n^0$. Thus,

$$\sum_{k=1}^{n} \binom{k-t+2}{j+2} = (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3}$$

by segmented hockey stick identity (1.3). This completes the proof. $\square$

For example, given $t = 4$, the triple sums of powers are

$$\Sigma^3 n^0 = 1 \left(\binom{n-1}{3} + \binom{3}{1} \Sigma^2 n^0 - \binom{3}{2} \Sigma^1 n^0 + \binom{3}{3} \Sigma^0 n^0 \right)$$

$$\Sigma^3 n^1 = 4 \left(\binom{n-1}{3} + \binom{3}{1} \Sigma^2 n^0 - \binom{3}{2} \Sigma^1 n^0 + \binom{3}{3} \Sigma^0 n^0 \right)$$

$$+ 1 \left(\binom{n-1}{4} - \binom{4}{2} \Sigma^2 n^0 + \binom{4}{3} \Sigma^1 n^0 - \binom{4}{4} \Sigma^0 n^0 \right)$$

$$\Sigma^3 n^2 = 16 \left(\binom{n-1}{3} + \binom{3}{1} \Sigma^2 n^0 - \binom{3}{2} \Sigma^1 n^0 + \binom{3}{3} \Sigma^0 n^0 \right)$$

$$+ 9 \left(\binom{n-1}{4} - \binom{4}{2} \Sigma^2 n^0 + \binom{4}{3} \Sigma^1 n^0 - \binom{4}{4} \Sigma^0 n^0 \right)$$

$$+ 2 \left(\binom{n-1}{5} + \binom{5}{3} \Sigma^2 n^0 - \binom{5}{4} \Sigma^1 n^0 + \binom{5}{5} \Sigma^0 n^0 \right)$$

In general,

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j 4^m \left[(-1)^j \binom{j+3}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+3}{j+2} \Sigma^1 n^0 + \right.$$

$$\left. + (-1)^{j+2} \binom{j+3}{j+3} \Sigma^0 n^0 + \binom{n-1}{j+3} \right]$$

Continuing similarly, we are able to derive the formula for multifold sums of powers, which is

Theorem 1.7 (Multifold sums of powers via Newton's series). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By Newton's series for power (1.2) and repeated applications of the segmented hockey stick identity (1.3). □

In its explicit form

Example 1.8 (Explicit expansion of the s–sum in the r-fold case). *For non-negative integers r, n, m and an arbitrary integer t, the r-fold sum $\Sigma^r n^m$ can be written as*

$$\begin{aligned}\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \Big[& (-1)^j \binom{j+t-1}{j+1} \Sigma^{r-1} n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^{r-2} n^0 \\ & + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^{r-3} n^0 + \cdots + (-1)^{j+r-1} \binom{j+t-1}{j+r} \Sigma^0 n^0 \\ & + \binom{n-t+r}{j+r} \Big]\end{aligned}$$

We may observe that

Proposition 1.9 (Multifold sum of zero powers). *For integers r and n*

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. By hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. □

Which yields the following binomial variations of the multifold sums of powers (1.7)

Proposition 1.10 (Multifold sums of powers binomial form). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right) + \binom{n-t+r}{j+r} \right]$$

Proposition 1.11 (Multifold sums of powers binomial form re-indexed)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right) + \binom{n-t+r}{j+r} \right]$$

Finite differences of powers are closely related to Stirling numbers of the second kind

Lemma 1.12 (Finite differences via Stirling numbers)**.** *For non-negative integers j, m and an arbitrary integer t*

$$\Delta^j t^m = \sum_{k=0}^{m} \binom{t}{k} \left\{ {m \atop j+k} \right\} (j+k)!$$

Which implies variations of the formulas for sums of powers

Proposition 1.13 (Ordinary sums of powers via Stirling numbers)**.** *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right] \binom{t}{k} \left\{ {m \atop j+k} \right\} (j+k)!$$

Proof. By ordinary sums of powers via Newton's series (1.4) and finite difference via Stirling numbers of the second kind (1.12). □

In general,

Proposition 1.14 (Multifold sums of powers via Stirling numbers)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right] \binom{t}{k} \left\{ {m \atop j+k} \right\} (j+k)!$$

Proof. By multifold sums of powers via Newton's series (1.7) and finite difference via Stirling numbers of the second kind (1.12). □

The proposition above can be presented in a pure binomial form as well, by means of the identity (1.9): $\Sigma^r n^0 = \binom{r+n-1}{r}$.

In addition, we are able to express multifold sums of powers via Eulerian numbers, by expressing the forward finite difference via the Worpitzky identity [7]

Lemma 1.15 (Worpitzky identity). *For non-negative integers t, m*

$$t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k}{m}$$

where $\left\langle {n \atop k} \right\rangle$ are Eulerian numbers. Thus,

Lemma 1.16 (Finite difference via Eulerian numbers). *For non-negative integers j, m and an arbitrary integer t*

$$\Delta^j t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k}{m-j}$$

Therefore,

Proposition 1.17 (Multifold sums of powers via Eulerian numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right] \left\langle {m \atop k} \right\rangle \binom{t+k}{m-j}$$

Proof. By multifold sums of powers via Newton's series (1.7) and finite difference via Eulerian numbers of the second kind (1.16). ☐

2. Backward difference form

The formula for multifold sums of powers via Newton's series (1.7) can be altered to be in terms of backward differences easily, because

Proposition 2.1. *For integers j, m, t*

$$\Delta^j t^m = \nabla^j (t+j)^m$$

Thus,

Proposition 2.2 (Multifold sums of powers via backward differences)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \nabla^j (t+j)^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By multifold sums of powers via Newton's series (1.7) and by proposition (2.1). □

3. Central difference form

The formula for multifold sums of powers via Newton's series (1.7) can be altered to be in terms of central differences easily, because

Proposition 3.1. *For integers j, m, t*

$$\Delta^j t^m = \delta^j \left(t + \frac{j}{2} \right)^m$$

Thus,

Proposition 3.2 (Multifold sums of powers via central differences)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \delta^j \left(t + \frac{j}{2} \right)^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By multifold sums of powers via Newton's series (1.7) and by proposition (3.1). □

4. Future research

In this manuscript we focus on the idea to combine the Newton's interpolation formula and the hockey-stick family identities for binomial coefficients to express the sums of powers seamlessly.

This particular idea is great, however it can be generalized even further, so that the main aim becomes to utilize an interpolation formula for power n^m in terms of an *abstract difference operator* $D(n^m)$ and binomial coefficients $\binom{f(n)}{k}$ such that n indicates the variable of power function. The difference operator can be arbitrary, for example: forward, backward, or central differences. For instance, the abstract interpolation formula is

$$n^m = \sum_k \binom{f(n)}{k} D(n^m, k)$$

Thus, the formula of sums of powers involves the abstract difference operator D evaluated at some point k and the hockey-stick family identity over the binomial coefficients $\binom{n}{k}$

$$\Sigma^1\, n^m = \sum_k D(n^m, k) \sum_{j \leq n} \binom{f(j)}{k}$$

Similarly, for multifold sums of powers

$$\Sigma^r\, n^m = \sum_k D(n^m, k) \binom{f(n+r)}{k}$$

Many interpolation approaches involve rising factorials $x_{(n)}$, falling factorials $(x)_n$, or factorials $n!$, and thus can be expressed in terms of binomial coefficients, because

$$\frac{(x)_n}{n!} = \binom{x}{n}; \quad \frac{x_{(n)}}{n!} = \binom{x+n-1}{n}.$$

In particular, Donald Knuth provides a formula multifold sums of odd powers [1] based on the operator of central finite differences of power evaluated in zero, that is

Proposition 4.1 (Multifold sums of odd powers)**.**

$$\begin{aligned} \Sigma^r\, n^{2m-1} &= \sum_{k=1}^{m} (2k-1)! T(2m, 2k) \binom{n+k-1+r}{2k-1+r} \\ &= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} \frac{1}{2k} \delta^{2k} 0^{2m} \end{aligned}$$

where $T(n, k)$ are thr central factorial numbers of the second kind, see [8, section 58] and [9, formula (10a)], such that

$$T(n, k) = \frac{1}{k!} \delta^k 0^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n$$

In general, the central factorial numbers of the second kind $T(n, k)$ were defined by Riordan in his fundamental work *Combinatorial identities* [10, ch. 6.5, formula (24)], via the polynomial identity

Lemma 4.2 (Riordan power identity)**.**

$$n^m = \sum_{k=1}^{m} T(m,k) n^{[k]}$$

where $n^{[k]}$ are central factorials $n^{[k]} = n\prod_{j=0}^{k-1}\left(n+\frac{k}{2}-j\right)$. The sequence A008957 in the OEIS [5] provides non-zero central factorial numbers of the second kind $T(2n,2k)$.

Knuth's formula (4.1) utilizes the operator of central finite differences of power evaluated in zero. Thus, it is worth to investigate the existence of sums of odd powers involving the central differences evaluated at an arbitrary integer point t, similar to multifold sums of powers via Newton's series (1.7).

5. Proof of Segmented hockey stick identity

First we split the sum $\sum_{k=0}^{n}\binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n}\binom{-t+k}{j} = \sum_{k=0}^{t-1}\binom{-t+k}{j} + \sum_{k=t}^{n}\binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0,1,2,\cdots,t,\cdots,n\}$, with $t<n$. Considering the sum $\sum_{k=0}^{t-1}\binom{-t+k}{j}$ we notice that

$$\sum_{k=0}^{t-1}\binom{-t+k}{j} = \binom{-t}{j} + \binom{-t+1}{j} + \binom{-t+2}{j} + \cdots + \\ + \binom{-t+t-2}{j} + \binom{-t+t-1}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j} = \sum_{k=1}^{t}\binom{-k}{j} = \sum_{k=0}^{t-1}\binom{-k-1}{j}$$

By using the identity $\binom{-k}{j} = (-1)^j\binom{j+k-1}{j}$, we obtain

$$\binom{-k-1}{j} = \binom{-(k+1)}{j} = (-1)^j\binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j} = (-1)^j\sum_{k=0}^{t-1}\binom{j+k}{j} = (-1)^j\binom{j+t}{j+1}$$

by the hockey-stick identity $\sum_{k=0}^{t}\binom{j+k}{j}=\binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n}\binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}$$

Thus

$$\sum_{k=t}^{n}\binom{-t+k}{j}=\sum_{k=0}^{n-t}\binom{k}{j}=\binom{n-t+1}{j+1}$$

again by the hockey-stick identity $\sum_{k=0}^{t}\binom{j+k}{j}=\binom{j+t+1}{j+1}$. Combining the two parts, we obtain

$$\sum_{k=0}^{n}\binom{-t+k}{j}=(-1)^{j}\binom{j+t}{j+1}+\binom{n-t+1}{j+1}$$

This completes the proof.

6. Conclusions

In this manuscript we have derived formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers. In addition, in (4) we discuss the future research directions that may lead to a complete framework for sums of powers, by means of combining interpolation approaches, binomial coefficients and variations of the hockey-stick identity for binomial coefficients. The most important results of this manuscript are validated using Mathematica programs, see (8).

7. Acknowledgements

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Version: 1.3.1+main.02219a4

Sources: github.com/kolosovpetro/NewtonsInterpolationFormulaAndSumsOfPowers

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Email: kolosovp94@gmail.com

8. Mathematica programs

Use the *Mathematica* package [12] to validate the results

Mathematica Function	**Validates / Prints**
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateMultifoldSumsOfPowersViaNewtonsSeries[r]`	Validates Theorem (1.7)
`ValidateFiniteDifferenceViaStirlingNumbers[t]`	Validates Lemma (1.12)
`ValidateFiniteDifferenceViaEulerianNumbers[t]`	Validates Lemma (1.16)
`ValidateMultifoldSumsOfPowersViaCentralDifferences[r]`	Validates Proposition (3.2)
`ValidateMultifoldSumsOfPowersViaBackwardDifferences[r]`	Validates Proposition (2.2)
`ValidateMultifoldSumsOfPowersViaStirlingNumbers[r]`	Validates Proposition (1.14)
`ValidateMultifoldSumsOfPowersViaEulerianNumbers[r]`	Validates Proposition (1.17)
`ValidateMultifoldSumsOfPowersBinomialForm[r]`	Validates Proposition (1.10)
`ValidateMultifoldSumsOfPowersBinomialFormReindexed[r]`	Validates Proposition (1.11)

DevOps Engineer

URL: https://kolosovpetro.github.io