SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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Abstract. We develop formula for sums of powers using Newton's interpolation formula in terms of backward finite differences of powers.

1. Introduction and main results

Define multifold sums of powers in Knuth's [1] notation

$$\Sigma^0 \, n^m = n^m$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m$$

The book Interpolation by Steffensen [2] gives Newton's formula in terms of backward finite differences

Proposition 1.1 (Newton formula via backward differences)**.**

$$f(x) = \sum_{k=0}^{\infty} \binom{x-a+k-1}{k} \nabla^k f(a)$$

where $\nabla^k f(a) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(a-j)$.

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Thus, by setting $f(n) = n^m$

$$n^m = \sum_{j=0}^{m} \binom{n-t+j-1}{j} \nabla^j t^m$$

Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^{m} \nabla^j t^m \sum_{k=1}^{n} \binom{k-t+j-1}{j}$$

We notice that the sum $\sum_{k=1}^{n} \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j-t$ and $b = j-t-1+n$, we get

$$\sum_{k=1}^{n} \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Because,

$$\sum_{m=a}^{b} \binom{m}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Thus,

$$\sum_{k=1}^{n} \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Applying the identity for binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we obtain

Proposition 1.2 (Ordinary sums of powers via backward differences)**.**

$$\Sigma^1 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right]$$

For example, by setting $t = 2,\ m = 1, 2, 3, 4$, we get formulas for sums of cubes

$$\Sigma^1 n^1 = 2\left[-\binom{2}{1} + \binom{n-2}{1} \right] + 1\left[\binom{2}{2} + \binom{n-1}{2} \right],$$

$$\begin{aligned} \Sigma^1 n^2 = 4\left[-\binom{2}{1} + \binom{n-2}{1} \right] + 3\left[\binom{2}{2} + \binom{n-1}{2} \right] \\ + 2\left[-\binom{2}{3} + \binom{n}{3} \right]. \end{aligned}$$

$$\begin{aligned}\Sigma^1 n^3 &= 8\left[-\binom{2}{1}+\binom{n-2}{1}\right]+7\left[\binom{2}{2}+\binom{n-1}{2}\right]\\&\quad+6\left[-\binom{2}{3}+\binom{n}{3}\right]+6\left[\binom{2}{4}+\binom{n+1}{4}\right].\end{aligned}$$

$$\begin{aligned}\Sigma^1 n^4 &= 16\left[-\binom{2}{1}+\binom{n-2}{1}\right]+15\left[\binom{2}{2}+\binom{n-1}{2}\right]\\&\quad+14\left[-\binom{2}{3}+\binom{n}{3}\right]+12\left[\binom{2}{4}+\binom{n+1}{4}\right]\\&\quad+24\left[-\binom{2}{5}+\binom{n+2}{5}\right].\end{aligned}$$

The coefficients $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \cdots$ for $t = 2$ is the sequence A391068 in the OEIS [3]. For $t = 0$ the coefficients are $1, 0, 1, 0, -1, 2, 0, 1, -6, 6, \cdots$ and registered in the OEIS as A278075. For $t = 1$ the coefficients are $1, 1, 1, 1, 1, 2, 1, 1, 0, 6, \cdots$ and registered in the OEIS as A389570. For $t = 3$ the coefficients are $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \cdots$ and registered in the OEIS as A391210.

References

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Sources: github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences

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