HISTORY AND OVERVIEW OF THE POLYNOMIAL $\mathbf{P}_B^M(X)$

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Abstract. The polynomial $\mathbf{P}_b^m(x)$ is a polynomial of degree $2m+1$ in $(x,b) \in \mathbb{R}$, defined by an identity for odd powers, closely linked to Binomial theorem and Faulhaber's formula. The odd-power identity is derived using certain interpolation techniques, including systems of linear equations, recurrence relations, and finite differences. This manuscript offers a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$, followed by related works based on it. Notable results in related works include the relation between ordinary and partial derivatives for odd powers and finding polynomial derivatives via a double limit. Finally, the manuscript proposes future research directions.

Contents

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1. History and evolution of the polynomial $\mathbf{P}_b^m(x)$

Back then, in 2016 being a student at the faculty of mechanical engineering, I remember myself playing with finite differences of the polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$ having at most $0 \leq n \leq 20$ values. Looking to the values in my finite difference tables, the first and very naive question that came to my mind was

Question 1.1. *Is it possible to re-assemble the value of the polynomial n^3 backwards having its finite differences?*

The answer to this question is definitely *Yes*, by utilizing certain interpolation principles. Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

At that time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. All mathematical laws and relations exist from the very beginning, but we only find and describe them, I thought. That mindset truly inspired me so that my own mathematical journey has been started. Let us begin considering the table of finite differences of the polynomial n^3

Sources: https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed through summation over n, e.g

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots
\end{aligned} \tag{1.1}$$

Finally reaching its generic form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k \tag{1.2}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then our relation (1.2) immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k \tag{1.3}$$

However, as was said, I was not the experienced one mathematician back then, so that I reviewed the relation (1.2) from a little bit different perspective. Not following the convenient solution (1.3), I have rearranged the first order finite differences from the table (1) using (1.1) to get the polynomial n^3

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots$$
$$+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \tag{1.4}$$

Then, rearranging the terms of the equation (1.4) so that it turns into summation in terms of $k(n-k)$

$$n^3 = n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] + \cdots$$
$$\cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)]$$

Gives the interpolation of the polynomial n^3

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.5}$$

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [4]. Sequences such that row sums give the polynomials n^5 and n^7 are also registered in OEIS [5, 6].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that the next curiosity would be

Question 1.2. *Well, if the relation* (1.5) *true for the polynomial* n^3*, then is it true that* (1.5) *can be generalized for higher powers, e.g. for* n^4 *or* n^5 *similarly?*

That was the next question, however without any expectation of the final form of generalized relation. Going a little further, the answer to this question is also *Yes*, by utilizing certain approaches in terms of recurrence relations and systems of linear equations. Let us begin with the background and history review of systems of linear equations approach.

1.1. **System of linear equations approach.** Soon enough the question (1.2) got attention from other people. In 2018, Albert Tkaczyk has published two of his works [7, 8] showing the cases for polynomials n^5, n^7 and n^9 that were obtained similarly as (1.5). In short, it appears that relation (1.5) could be generalized for any non-negative odd power $2m+1$ solving a system of linear equations. It was proposed that the case for n^5 has explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C\right]$$

where A, B, C are yet-unknown coefficients. Denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to reach the form of a compact double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

Observing the equation above, the potential form of generalized odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ it is necessary construct and solve a system of linear equations following the process

$$\begin{aligned} n^5 &= \sum_{r=0}^{2} \mathbf{A}_{2,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2 \end{aligned}$$

Expand the terms $\sum_{k=1}^{n} k^r (n-k)^r$ applying the Faulhaber's formula [2] to get the equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5)\right] - n^5 = 0$$

Multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(-n + n^3) + \mathbf{A}_{2,2}(-n + n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that the odd-power identity holds

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true by arranging the terms $30k^2(n-k)^2 + 1$ over $0 \leq k \leq n$ as tabular. See the OEIS sequence [5]

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $30k^2(n-k)^2+1$. See the OEIS entry A300656.

Now we can see that the relation (1.5) we got via interpolation of cubes can be generalized for all non-negative odd-powers $2m+1$ by constructing and solving a certain system of linear equations. Therefore, the generalized form of odd-power identity has the form

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{1.6}$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, the identity (1.6) is discussed separately in [9, 10].

As one more example, let be $m=3$ so that we have the following relation defined by (1.6)

$$\mathbf{A}_{m,0}n+\mathbf{A}_{m,1}\left[\frac{1}{6}(-n+n^3)\right]+\mathbf{A}_{m,2}\left[\frac{1}{30}(-n+n^5)\right]+\mathbf{A}_{m,3}\left[\frac{1}{420}(-10n+7n^3+3n^7)\right]-n^7=0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n+n^3) + 14\mathbf{A}_{2,2}(-n+n^5) + \mathbf{A}_{3,3}(-10n+7n^3+3n^7) - 420n^7 = 0$$

Expanding brackets and rearranging the terms gives

$$\begin{aligned} 420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5 \\ - 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0 \end{aligned}$$

Combining the common terms yields

$$n(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3})$$

$$+ n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} = 0 \\ 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} = 0 \\ \mathbf{A}_{3,2} - 30 = 0 \\ 3\mathbf{A}_{3,3} - 420 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} = 140 \\ \mathbf{A}_{3,2} = 0 \\ \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} = -14 \\ \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} = 1 \end{cases}$$

So that odd-power identity (1.6) holds

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms $140k^3(n-k)^3 - 14k(n-k) + 1$ over $0 \leq k \leq n$ as the OEIS sequence A300785 [6] shows.

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	127	1					
3	1	1093	1093	1				
4	1	3739	8905	3739	1			
5	1	8905	30157	30157	8905	1		
6	1	17431	71569	101935	71569	17431	1	
7	1	30157	139861	241753	241753	139861	30157	1

Table 4. Values of $140k^3(n-k)^3 - 14k(n-k) + 1$. See the OEIS entry A300785.

However, constructing and solving a system of linear equations for every odd-power $2m+1$ requires a huge effort

Assumption 1.3. *There must be a formula that generates a set of real coefficients* $\mathbf{A}_{m,0}, \mathbf{A}_{m,0}, \ldots \mathbf{A}_{m,m}$ *for each fixed* m *such that*

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r$$

I thought.

1.2. **Recurrence relation approach.** As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly. In [11], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient $\mathbf{A}_{m,r}$ for each natural m, r such that $m \geq 0$ and $0 \leq r \leq m$. The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the $\mathbf{A}_{m,r}$ recursively, taking the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$. Before we consider the derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$, a few words must be said regarding the Faulhaber's formula [2]

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j}$$

it is important to notice that iteration step j is bounded by the value of power p, while the upper index of the binomial coefficient $\binom{p+1}{j}$ is $p+1$. It means that we cannot omit summation bounds letting j run over infinity, unless we perform the following action on the Faulhaber's formula

$$\begin{aligned}\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} &= \left[\frac{1}{p+1}\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1} \\ &= \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}\end{aligned} \tag{1.7}$$

At this point we are good to go through the entire derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$. By applying Binomial theorem $(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}k^t$ and Faulhaber's formula (1.7) to the equation (1.6) we get

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r} \\ &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right] \\ &= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right] \\ &= \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \\ &= \left[\sum_{j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]\end{aligned}$$

Rearranging terms yields

$$\left[\sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \tag{1.8}$$

We can notice that

$$\sum_{t}\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases}\frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0 \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0\end{cases} \tag{1.9}$$

An elegant proof of the binomial identity (1.9) is presented in [12]. In particular, the equation (1.9) is zero for $0 < t \leq j$. So that moving $j = 0$ out of summation in (1.8) we have

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j\geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

Simplifying above equation by using (1.9) yields

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1}\right]}_{(\star)} - \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms in the equation above so that

$$\begin{aligned}\sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by (1.6), we obtain the following relation for polynomials in n

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by d we get

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{d\geq 0} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

Therefore,

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} \right] + 2 \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\sum_{d \geq 0} \frac{(-1)^r}{2r-2d} \binom{r}{2d+1} B_{2r-2d} n^{2d+1} \right] \equiv n^{2m+1} \tag{1.10}$$

Taking the coefficient of n^{2m+1} we set iteration steps $r = m$ and $d = m$ in (1.10) to get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Note that $\binom{r}{2d+1} = 0$ in (1.10) having $r = m$ and $d = m$.

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

because $\binom{r}{2d+1}$ is zero for every $r \leq m$ and $2d+1$ having $\frac{m}{2} \leq d < m$. For example $\binom{r}{m+1} = 0$ having $r \leq m$ and $d = \frac{m}{2}$.

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$, iterating consecutively over $s = 1, 2, \ldots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 1.4. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{1.11}$$

where B_t are Bernoulli numbers [13]. It is assumed that $B_1 = \frac{1}{2}$. Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 5. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [14, 15].

The nominators and denominators of the coefficients $\mathbf{A}_{m,r}$ are also registered as sequences in OEIS [14, 15]. It is as well interesting to notice that row sums of the $\mathbf{A}_{m,r}$ give the powers of 2

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Let be a theorem

Theorem 1.5. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ *is a real coefficient defined recursively by* (1.11)*.*

Finally, we got our road to definition of $2m+1$-degree polynomial $\mathbf{P}_b^m(x)$. Introducing the parameter b to the upper summation bound of the equation (1.5), we have the definition

Definition 1.6. *(Polynomial* $\mathbf{P}_b^m(x)$*of degree* $2m+1$*.)*

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.12}$$

where $\mathbf{A}_{m,r}$ are real coefficients (1.11). A comprehensive discussion on the polynomial $\mathbf{P}_b^m(x)$ as well as its properties can be found at [16]. In 2023, Albert Tkaczyk yet again extended the theorem (1.5) to the so-called three dimension case so that it gives polynomials of the form n^{3l+2} at [17]. For example,

$$\begin{aligned}
\mathbf{P}_b^0(x) &= b, \\
\mathbf{P}_b^1(x) &= 3b^2 - 2b^3 - 3bx + 3b^2x, \\
\mathbf{P}_b^2(x) &= 10b^3 - 15b^4 + 6b^5 \\
&\quad - 15b^2x + 30b^3x - 15b^4x \\
&\quad + 5bx^2 - 15b^2x^2 + 10b^3x^2 \\
\mathbf{P}_b^3(x) &= -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 \\
&\quad + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x \\
&\quad + 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 \\
&\quad + 35b^2x^3 - 70b^3x^3 + 35b^4x^3
\end{aligned}$$

2. Related works

We show related works in format of a directed graph, where node represents a work and edge represents a citation between two works. For example, the node 12 (current manuscript) cites the manuscripts 7 and 8. Nodes of the related work graph are

- 12 is this manuscript

- 7 is *A study on partial dynamic equation on time scales involving derivatives of polynomials* [18]
- 8 is *106.37 An unusual identity for odd-powers* [9]
- 9 is *Another approach to get derivative of odd-power* [19]
- B is *A two-sided Faulhaber-like formula involving Bernoulli polynomials* [20]
- 10 is *Polynomial identity involving Binomial Theorem and Faulhaber's formula* [10]
- 11 is *Finding the derivative of polynomials via double limit* [21]
- 6 is *On the link between binomial theorem and discrete convolution* [16]

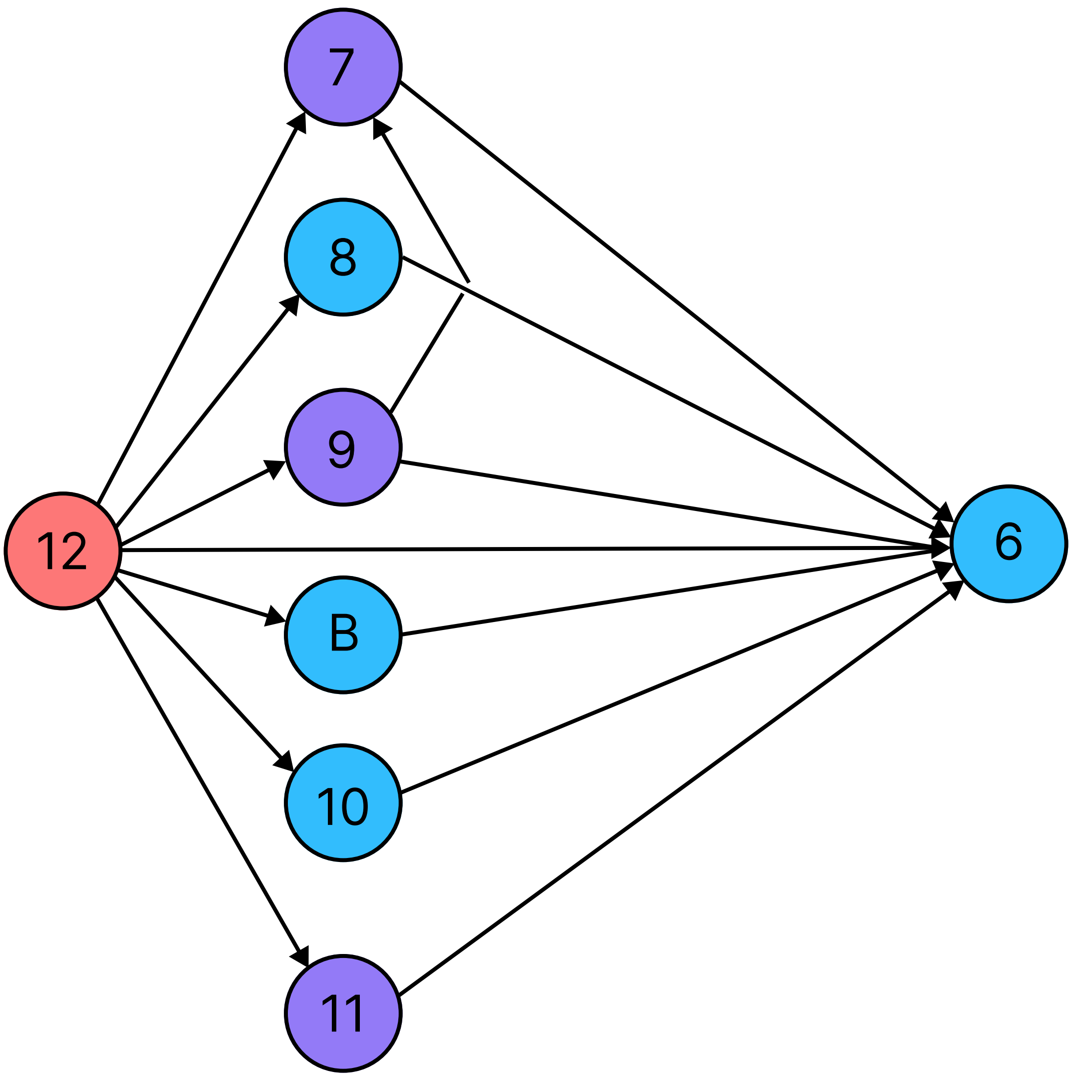

Figure 1. Related works graph.

- 12 This manuscript
- 6 In *On the link between binomial theorem and discrete convolution* [16]: Let $\mathbf{P}_b^m(x)$ be a $2m+1$-degree polynomial in x and $b \in \mathbb{R}$

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r$$

where $\mathbf{A}_{m,r}$ are real coefficients. In this manuscript, we introduce the polynomial $\mathbf{P}_b^m(x)$ and study its properties, establishing a polynomial identity for odd-powers in terms of this polynomial. Based on mentioned polynomial identity for odd-powers, we explore the connection between the Binomial theorem and discrete convolution of odd-powers, further extending this relation to the multinomial case. All findings are verified using Mathematica programs.

- 7 In *A study on partial dynamic equation on time scales involving derivatives of polynomials* [18]: Extends the main results of 6 deriving and discussing an identity that connects the timescale derivative of odd-powered polynomial with partial derivatives of polynomial $\mathbf{P}_b^m(x)$ evaluated in particular points. For every $t \in \mathbb{T}_1$ and $(x,b) \in \Lambda^2$

$$\frac{\Delta x^{2m+1}}{\Delta x}(t) = \frac{\partial P(m,b,x)}{\Delta x}(m,\sigma(t),t) + \frac{\partial P(m,b,x)}{\Delta b}(m,t,t)$$

such that $\sigma(t) > t$ is forward jump operator. In addition, we discuss various derivative operators in the context of the partial cases of above equation, We show finite difference, classical derivative, $q-$derivative, $q-$power derivative on behalf of it.

- 8 In *106.37 An unusual identity for odd-powers* [9]: Explores and proves the partial case of 6 that is the polynomial identity for odd-powers

$$n^{2m+1} = \sum_{k=1}^{n}\sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

- 9 In *Another approach to get derivative of odd-power* [19]: Extends the results of 6 by providing a relation in terms of partial differential equations such that ordinary derivative of odd-power $2m+1$ can be reached in terms of partial derivative of the polynomial $\mathbf{P}_b^m(x)$. Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx}g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)'_x(u,u)$ evaluate in point (u,u) plus partial derivative $(f_v)'_z(u,u)$ evaluate in point (u,u)

$$\frac{d}{dx}g_v(u) = (f_v)'_x(u,u) + (f_v)'_z(u,u) \tag{2.1}$$

where $f_y(x,z) = \sum_{k=1}^{z}\sum_{r=0}^{y}\mathbf{A}_{y,r}k^r(x-k)^r = \mathbf{P}_z^y(x)$.

- B In *A two-sided Faulhaber-like formula involving Bernoulli polynomials* [20]: Based on equation (1.10), the authors give a new identity involving Bernoulli polynomials and combinatorial numbers that provides, in particular, the Faulhaber-like formula for sums of the form $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ for positive integers m and n.
- 10 In *Polynomial identity involving Binomial Theorem and Faulhaber's formula* [10]: proves that for every $n \geq 1,\ n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that the polynomial identity holds

$$n^{2m+1} = \sum_{k=1}^{n}\mathbf{A}_{m,0}k^0(n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m}k^m(n-k)^m$$

which is a direct consequence of the definition of $\mathbf{P}_b^m(x)$ given in 6, reached by utilizing Binomial theorem and Faulhaber's formula.
- 11 In *Finding the derivative of polynomials via double limit* [21]: By applying the results of 6 provides another perspective of ordinary derivatives of polynomials allowing expressing them via a double limit, because

$$\lim_{h\to 0}\mathbf{P}_{x+h}^m(x) = x^{2m+1}$$

- Three sequences were contributed to the OEIS [22, 23, 24] showing the coefficients of the polynomial $\mathbf{P}_b^m(x)$ having fixed points m, b while $x \in \mathbb{R}$.
- OEIS sequences such that row sums give odd-powers [4, 5, 6].
- OEIS sequences related to the coefficients $\mathbf{A}_{m,r}$ [14, 15].

The node indexes in the related works graph are not random, persisting the same values as these works have on my personal website

kolosovpetro.github.io/math

3. Future research and activities

- Differential equation (2.1) can also be expressed in terms of backward and central differential operators, including derivatives on time-scales so that results of [18] could be generalized further.
- Theorem (1.5) provides an opportunity to express odd-power identity in terms of multiplication of certain matrices.
- There are Taylor series and Maclaurin series versions in terms of $\mathbf{P}_b^m(x)$.
- The summation bounds of definition (1.12) can be altered so that k runs over $1 \leq k \leq b$, by symmetry.
- Prove that $\mathbf{P}_b^m(x)$ is an integer valued polynomial in (x, b).
- The definition (1.12) is closely related to discrete convolution because

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$

  where $\sum_{k=0}^{b-1} k^r (x-k)^r$ is the discrete convolution of x^r. It is worth to get a closer look into it so that new relations in terms of discrete convolution may be found.
- All kinds of derivatives e.g. forward, backward and central, including the derivatives on time-scales can be expressed as double limit of $\mathbf{P}_b^m(x)$ extending the results of [21].
- Introducing the definitions of the coefficients $\left[{n \atop k}\right]_m$ and $\left\{{n \atop k}\right\}_m$

$$\begin{bmatrix} n \\ k \end{bmatrix}_m = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

$$\begin{Bmatrix} n \\ r \end{Bmatrix}_m = \sum_{k=0}^{n-1} \mathbf{A}_{m,r} k^r (n-k)^r$$

the novel identities can be reached, for example

$$\begin{bmatrix} 2t+1 \\ 1 \end{bmatrix}_m = \begin{bmatrix} t+2 \\ 2 \end{bmatrix}_m$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_m = \begin{bmatrix} n \\ n-k \end{bmatrix}_m$$
$$\begin{bmatrix} 2t-3r \\ r \end{bmatrix}_m = \begin{bmatrix} t \\ 2r \end{bmatrix}_m = \begin{bmatrix} 2t-3r \\ 2t-4r \end{bmatrix}_m$$

so that combinatorial sense of above is also a topic to research.

- Contribute new OEIS sequences related to $\left[{n \atop k}\right]_m$ and $\left\{{n \atop k}\right\}_m$.
- An identity

$$(x-2a)^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=a+1}^{x-a} (k-a)^r (x-k-a)^r$$

allows to provide a novel proof of power rule in terms of derivatives of polynomials.

- Following the results of https://arxiv.org/pdf/1603.02468v15.pdf, the equation (1.12) approximates the odd-power polynomial x^{2m+1} around given points x_i as it may be observed from the following plots

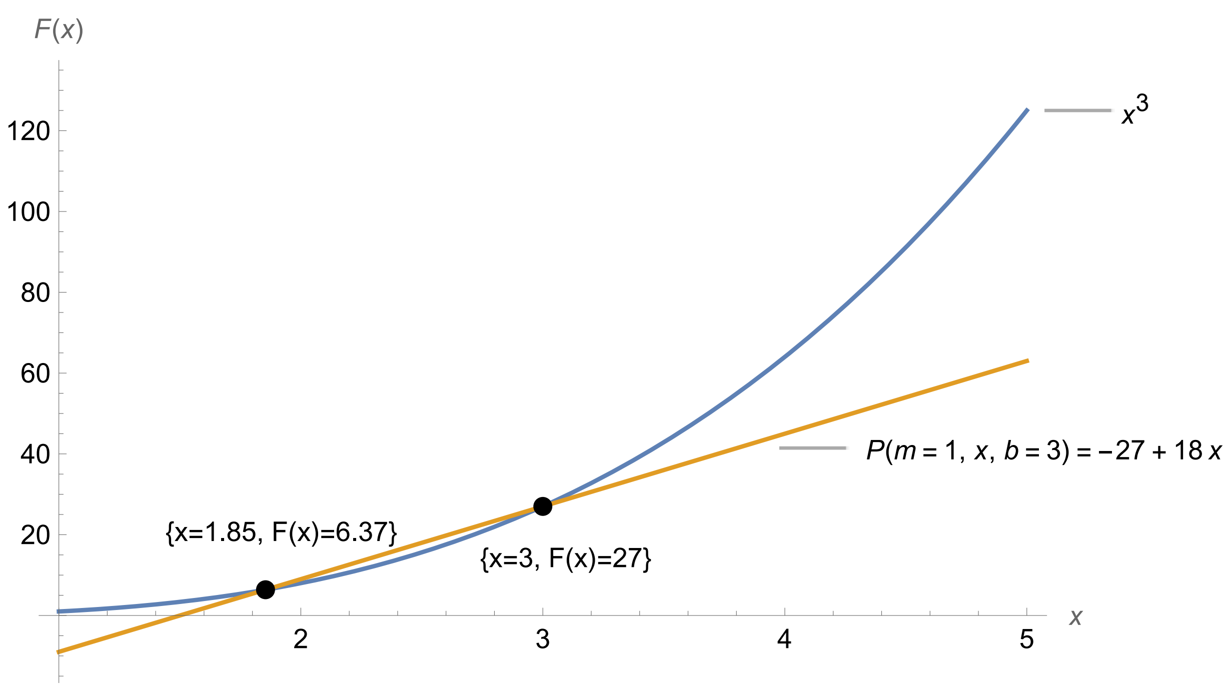

Figure 2. Approximation of x^3.

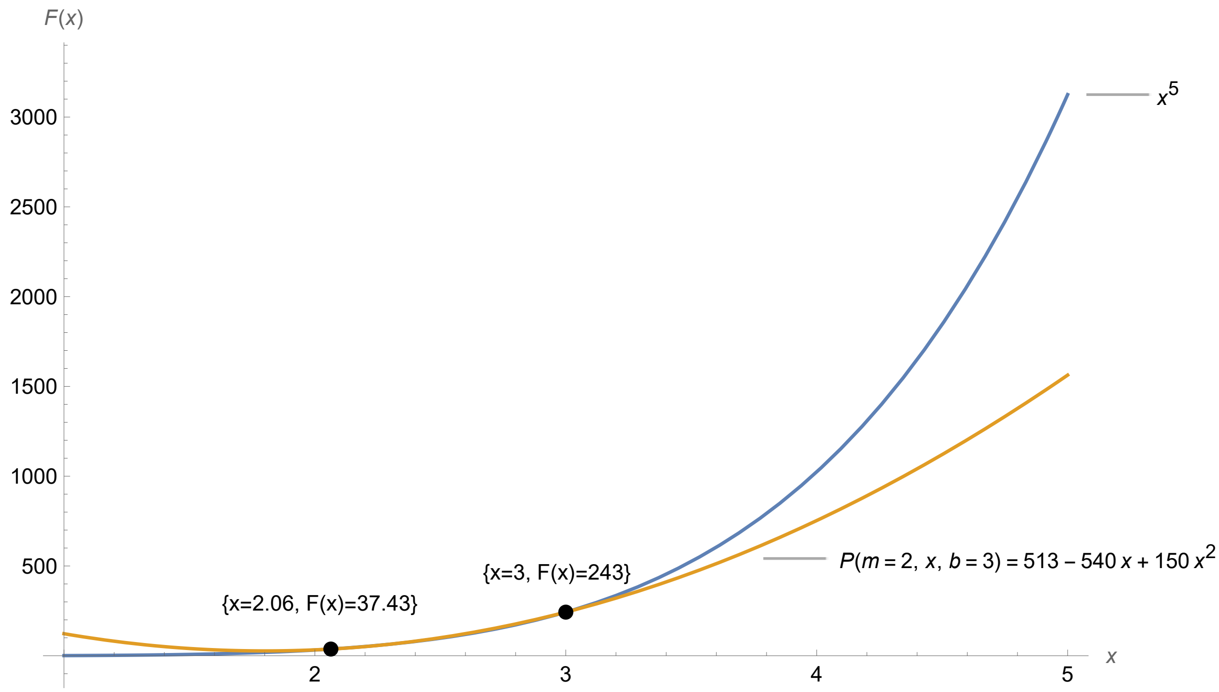

Figure 3. Approximation of x^5.

- English grammar reviews and improvements are welcome.
- Improvements and suggestions to current manuscript under open-source initiatives at https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

4. Conclusions

In this manuscript we have successfully provided a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$ as well as related works such that based onto, for instance various polynomial identities, differential equations etc. In addition, future research directions are proposed and discussed.

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5. Addendum 1: Examples of the polynomial $\mathbf{P}_b^m(x)$

$$\mathbf{P}_b^0(x) = b$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x$$

$$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x + 5bx^2 - 15b^2x^2 + 10b^3x^2$$

$$\begin{aligned}\mathbf{P}_b^3(x) = &-7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x \\ &+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3\end{aligned}$$

$$\begin{aligned}\mathbf{P}_b^4(x) = &-60b^2 + 180b^3 - 294b^5 + 420b^7 - 315b^8 + 70b^9 + 60bx - 270b^2x + 735b^4x - 1470b^6x \\ &+ 1260b^7x - 315b^8x + 90bx^2 - 630b^3x^2 + 1890b^5x^2 - 1890b^6x^2 + 540b^7x^2 + 210b^2x^3 \\ &- 1050b^4x^3 + 1260b^5x^3 - 420b^6x^3 - 21bx^4 + 210b^3x^4 - 315b^4x^4 + 126b^5x^4\end{aligned}$$

$$\begin{aligned}\mathbf{P}_b^5(x) = &-693b^2 + 2068b^3 - 330b^4 - 2640b^5 + 2772b^7 - 2310b^9 + 1386b^{10} - 252b^{11} + 693bx \\ &- 3102b^2x + 660b^3x + 6600b^4x - 9702b^6x + 10395b^8x - 6930b^9x + 1386b^{10}x + 1034bx^2 \\ &- 330b^2x^2 - 5940b^3x^2 + 12936b^5x^2 - 18480b^7x^2 + 13860b^8x^2 - 3080b^9x^2 + 2310b^2x^3 \\ &- 8085b^4x^3 + 16170b^6x^3 - 13860b^7x^3 + 3465b^8x^3 - 330bx^4 + 2310b^3x^4 - 6930b^5x^4 \\ &+ 6930b^6x^4 - 1980b^7x^4 - 231b^2x^5 + 1155b^4x^5 - 1386b^5x^5 + 462b^6x^5\end{aligned}$$

$$\begin{aligned}\mathbf{P}_b^6(x) = &-10920b^2 + 33306b^3 - 9009b^4 - 36036b^5 + 37752b^7 - 22022b^9 + 12012b^{11} - 6006b^{12} + 924b^{13} \\ &+ 10920bx - 49959b^2x + 18018b^3x + 90090b^4x - 132132b^6x + 99099b^8x - 66066b^{10}x + 36036b^{11}x \\ &- 6006b^{12}x + 16653bx^2 - 9009b^2x^2 - 84084b^3x^2 + 180180b^5x^2 - 180180b^7x^2 + 150150b^9x^2 \\ &- 90090b^{10}x^2 + 16380b^{11}x^2 + 36036b^2x^3 - 120120b^4x^3 + 168168b^6x^3 - 180180b^8x^3 \\ &+ 120120b^9x^3 - 24024b^{10}x^3 - 6006bx^4 + 40040b^3x^4 - 84084b^5x^4 + 120120b^7x^4 - 90090b^8x^4 \\ &+ 20020b^9x^4 - 6006b^2x^5 + 21021b^4x^5 - 42042b^6x^5 + 36036b^7x^5 - 9009b^8x^5 + 286bx^6 \\ &- 2002b^3x^6 + 6006b^5x^6 - 6006b^6x^6 + 1716b^7x^6\end{aligned}$$

6. Addendum 2: Derivation of the coefficients $\mathbf{A}_{m,r}$

Consider the definition (1.11) of the coefficients $\mathbf{A}_{m,r}$, it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition of the real coefficient $T(d,r)$

Definition 6.1. *Real coefficient $T(d,r)$*

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Example 6.2. *Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$*

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the coefficient $\mathbf{A}_{2,0}$ is

$$\mathbf{A}_{2,0} = \sum_{d \geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) = \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0)$$

$$= 30 \cdot \frac{1}{30} = 1$$

Example 6.3. *Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$*

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$\mathbf{A}_{3,1} = \sum_{d \geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient $\mathbf{A}_{3,0}$ *is*

$$\mathbf{A}_{3,0} = \sum_{d\geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 6.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 4$. *The value of the coefficient* $\mathbf{A}_{4,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d\geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ *is*

$$\mathbf{A}_{4,0} = \sum_{d\geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 6.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $3 \leq d < 5$. *The value of the coefficient* $\mathbf{A}_{5,2}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d\geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d\geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\begin{aligned}\mathbf{A}_{5,0} &= \sum_{d\geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0) \\ &= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1\end{aligned}$$

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