

IDENTITIES IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In this manuscript, we show new binomial identities in iterated rascal triangles, revealing a connection between the Vandermonde convolution and iterated rascal numbers. We also present Vandermonde-like binomial identities. Furthermore, we establish a relation between iterated rascal triangle and $(1, q)$ -binomial coefficients.

CONTENTS

1. Introduction	2
2. Binomial identities in Iterated Rascal Triangles	3
3. Q-Binomial identities in Iterated Rascal Triangles	7
4. Row sums conjecture	8
5. Conclusions	9
6. Acknowledgements	10
References	10

Date: July 5, 2024.

2010 *Mathematics Subject Classification.* 11B25, 11B99.

Key words and phrases. Pascal's triangle, Rascal triangle, Binomial coefficients, Binomial identities, Binomial theorem, Generalized Rascal triangles, Iterated rascal triangles, Iterated rascal numbers, Vandermonde identity, Vandermonde convolution .

Sources: <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>

1. INTRODUCTION

In 2010, three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1], were challenged to provide the next row for the number triangle shown below

$$\begin{array}{ccccccc}
 & & & & 1 & & & & \\
 & & & & & & 1 & & \\
 & & & 1 & & & & 1 & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 &
 \end{array}$$

The expected answer that matches Pascal's triangle [2] was “1 4 6 4 1”. However, Anggoro, Liu, and Tulloch suggested “1 4 5 4 1” instead. They devised this new row via so-called diamond formula

$$\text{South} = \frac{\text{East} \cdot \text{West} + 1}{\text{North}}$$

So that upcoming rows of the triangle are

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
7	1	7	11	13	13	11	7	1

Table 1. Rascal triangle. See the OEIS sequence [A077028](#) [3].

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [4], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [5, 6]. In particular, the

concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

2. BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number

Definition 2.1. *Iterated rascal number* [6]

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} \quad (2.1)$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [7]. Consider Vandermonde convolution

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (2.2)$$

Therefore, iterated rascal number is partial case of Vandermonde convolution with the upper summation bound equals to i . Without further hesitation consider our findings.

Proposition 2.2. *Iterated rascal triangle equals to Pascal's triangle up to i -th column.*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq k \leq i \quad (2.3)$$

Proof. Proof is given by [6]. □

Then binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying binomial coefficients symmetry principle we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

Proposition 2.3. *Iterated rascal triangle equals to Pascal's triangle up to $2i + 1$ -th row*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i + 1$$

Therefore, for every fixed $i \geq 0$

$$\binom{2i + 1 - n}{k}_i = \binom{2i + 1 - n}{k} \quad (2.4)$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over k for each i , so that it is true for all cases in i, k : $i < k$, $i = k$ and $k > i$.

Taking $t \geq 2i + 1$ for every fixed $i \geq 0$

$$\binom{t - n}{k}_{t-i-1} = \binom{t - n}{k}$$

Proof. Proof of proposition (2.3). We have three possible relations between i, k : $k < i$, $k = i$, $k > i$. So we have to prove that for every i, k

$$\sum_{m=0}^k \binom{2i + 1 - n - k}{m} \binom{k}{m} - \sum_{m=0}^i \binom{2i + 1 - n - k}{m} \binom{k}{m} = 0$$

For the case $k < i$ proof is given in Jenna Gregory et al. [6]. For the case $k = i$ proof is trivial. Thus, the remaining case is $k > i$ yields

$$\sum_{m=i+1}^k \binom{2i + 1 - n - k}{m} \binom{k}{m} = 0$$

Considering the constraints,

$$\begin{cases} n \geq 0 \\ k \geq i + 1 \\ 2i + 1 - n - k \leq i - n \\ m \geq i + 1 \end{cases}$$

Thus,

$$\sum_{m=i+1}^k \binom{2i + 1 - n - k}{m} \binom{k}{m}$$

is indeed equals zero because binomial coefficients $\binom{i-n-s}{i+1+s}$ are zero for each $i, n, s \geq 0$. Therefore, the proposition (2.3) is true. \square

Moreover, equation (2.4) gives Vandermonde-like identity

Proposition 2.4. (*Vandermonde-like identity.*)

$$\binom{2i+1-n}{k} = \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m}$$

In particular, given $n = 0, 1$ proposition (2.4) yields

$$\binom{2i+1}{k} = \sum_{m=0}^i \binom{2i+1-k}{m} \binom{k}{m}$$

$$\binom{2i}{k} = \sum_{m=0}^i \binom{2i-k}{m} \binom{k}{m}$$

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences $\binom{n}{k} - \binom{n}{k}_3$

In[67]:= `Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]`

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Out[67]=

Figure 1. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $\binom{n}{4}$. Sequence A000332 in the OEIS [8].

We can spot that having $i = 3$ the $k = 4$ -th column gives binomial coefficient $\binom{n}{4}$. Indeed, this rule is true for every i .

Proposition 2.5. (*Row-column difference.*) *For every fixed $i \geq 0$*

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

Proof. We have previously stated that iterated rascal numbers are closely related to Vandermonde convolution (2.2). Thus, proposition (2.5) can be rewritten as

$$\sum_{m=0}^i \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{m}$$

Therefore, $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$ is indeed true. \square

Proposition (2.5) yields to few more identities. Applying binomial coefficients symmetry

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking $j = n + i$ gives

$$\begin{aligned} \binom{j+i}{j} - \binom{j+i}{j}_{i-1} &= \binom{j}{j-i} \\ \binom{j+i}{i} - \binom{j+i}{i}_{i-1} &= \binom{j}{i} \end{aligned}$$

Proposition (2.5) can be generalized even further, for every fixed $i < k$.

Proposition 2.6. (*Binomial coefficient difference iterated rascal number.*) *For every fixed $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^k \binom{n-k}{m} \binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. \square

3. Q-BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]
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0	0	0	0	0	1	0	0	0	0											
0	0	0	0	0	5	5	0	0	0	0										
0	0	0	0	0	15	26	15	0	0	0	0									
0	0	0	0	0	35	81	81	35	0	0	0	0								
0	0	0	0	0	70	196	262	196	70	0	0	0	0							
0	0	0	0	0	126	406	658	658	406	126	0	0	0	0						
0	0	0	0	0	210	756	1414	1716	1414	756	210	0	0	0	0					
0	0	0	0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0				
0	0	0	0	0	495	2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0			
0	0	0	0	0	715	3267	8184	14232	18525	18525	14232	8184	3267	715	0	0	0	0		
0	0	0	0	0	1001	4862	13101	24816	35697	40186	35697	24816	13101	4862	1001	0	0	0	0	
0	0	0	0	0	1365	7007	20163	41217	64713	80587	64713	41217	20163	7007	1365	0	0	0	0	
0	0	0	0	0	1820	9828	30030	65780	111705	152020	168230	152020	111705	65780	30030	9828	1820	0	0	0

Figure 2. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $(1, 5)$ -binomial coefficient $\binom{n}{k}^5$. Sequence **A096943** in the OEIS [9].

The $(1, q)$ -binomial coefficients $\binom{n}{k}^q$ are special kind of binomial coefficients defined by

Definition 3.1. $(1, q)$ -Binomial coefficient

$$\binom{n}{k}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \binom{n-1}{k}^q + \binom{n-1}{k-1}^q & \end{cases} \quad (3.1)$$

Indeed, the relation shown in Figure (2) is true for every i , so that it establishes a relation between $(1, q)$ -binomial coefficients and iterated rascal numbers.

Proposition 3.2. *(Relation between iterated rascal numbers and $(1, q)$ -binomial coefficients.)*

For every fixed $i \geq 0$

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[\begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2}$$

Taking $t = i + 2$ in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[\begin{matrix} t+j \\ t \end{matrix} \right]^t$$

In particular, having $i = 1$ proposition (3.2) gives the OEIS sequence [A006503](#) [10] such that third column of $(1, 3)$ -Pascal triangle [A095660](#) [11].

Having $i = 3$ proposition (3.2) gives the OEIS sequence [A096943](#) [9] such that third column of $(1, 5)$ -Pascal triangle [A096940](#) [12].

For $i = 5$, the proposition (3.2) yields the OEIS sequence [A097297](#) [13] such that seventh column of $(1, 6)$ -Pascal triangle [A096940](#) [14].

4. ROW SUMS CONJECTURE

In [6] the authors propose the following conjecture for row sums of iterated rascal triangles.

Conjecture 4.1. *(Conjecture 7.5 in [6].) For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

Proof. Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^i \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^i \sum_k \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2} \quad (4.1)$$

In Concrete mathematics [[15], p. 169, eq (5.26)], Knuth et al. provide the identity for the column sum of binomial coefficients multiplication

$$\sum_{k=0}^l \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1} \quad (4.2)$$

We can observe this pattern in the equation (4.1), thus the sum $\sum_k \binom{4i+3-k}{m} \binom{k}{m}$ equals to

$$\sum_k \binom{4i+3-k}{m} \binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

So that

$$\frac{1}{2} \sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof. \square

Proposition 4.2. *For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

In particular, equation (4.2) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^i \binom{n+1}{2m+1}$$

5. CONCLUSIONS

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.5), (2.6), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.4). Furthermore, we establish a relation between iterated rascal triangles and $(1, q)$ -binomial coefficients (3.2). All the results can be validated using supplementary Mathematica scripts at [16].

6. ACKNOWLEDGEMENTS

Author is grateful to Oleksandr Kulkov, Markus Scheuer, Amelia Gibbs for their valuable feedback and suggestions regarding the conjecture (4.1) at MathStackExchange discussion [17].

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Version: 1.1.2-tags-v1-1-1.2+tags/v1.1.1.d7ef91a

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