Contents

1. Finite difference formulas

Proposition 1.1 (Forward finite difference)**.**

$$\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j)$$

Corollary 1.2 (Forward finite difference of power)**.**

$$\Delta^k x^m = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^m$$

Corollary 1.3 (Forward difference of power in zero)**.**

$$\Delta^n 0^m = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^m$$

2. Newton series

Proposition 2.1. *(Newton's series around zero* [1, p. 190]*.)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x}{j} \Delta^j f(0)$$

Proposition 2.2. *(Newton's series around arbitrary point* [2, Lemma V]*.)*

$$f(x+a) = \sum_{j=0}^{\infty} \binom{x}{j} \Delta^j f(a)$$

3. Partial sums of powers

Example 3.1 (Newton series for cubes monomial).

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$

$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$

$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

$$n^3 = \Delta^0 t^3\binom{n-t}{0} + \Delta^1 t^3\binom{n-t}{1} + \Delta^2 t^3\binom{n-t}{2} + \Delta^3 t^3\binom{n-t}{3}$$

Example 3.2 (Newton series for cubes binomial).

$$(n+0)^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$

$$(n+1)^3 = 1\binom{n}{0} + 7\binom{n}{1} + 12\binom{n}{2} + 6\binom{n}{3}$$

$$(n+2)^3 = 8\binom{n}{0} + 19\binom{n}{1} + 18\binom{n}{2} + 6\binom{n}{3}$$

$$(n+t)^3 = \Delta^0 t^3\binom{n}{0} + \Delta^1 t^3\binom{n}{1} + \Delta^2 t^3\binom{n}{2} + \Delta^3 t^3\binom{n}{3}$$

Example 3.3 (Partial sums of cubes 1).

$$\sum_{k=0}^{n+0} k^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$

$$\sum_{k=1}^{n+1} k^3 = 1\binom{n+1}{1} + 7\binom{n+1}{2} + 12\binom{n+1}{3} + 6\binom{n+1}{4}$$

$$\sum_{k=2}^{n+2} k^3 = 8\binom{n+1}{1} + 19\binom{n+1}{2} + 18\binom{n+1}{3} + 6\binom{n+1}{4}$$

Corollary 3.4 (Partial sums of cubes 1).

$$\begin{aligned}\sum_{k=t}^{n+t} k^3 &= \Delta^0 t^3\binom{n+1}{1} + \Delta^1 t^3\binom{n+1}{2} + \Delta^2 t^3\binom{n+1}{3} + \Delta^3 t^3\binom{n+1}{4}\\ &= \sum_{j=0}^{3}\binom{n+1}{j+1}\Delta^j t^3\end{aligned}$$

Corollary 3.5 (Partial sums of powers 1)**.**

$$\sum_{k=t}^{n+t} k^m = \Delta^0 t^m \binom{n+1}{1} + \Delta^1 t^m \binom{n+1}{2} + \Delta^2 t^m \binom{n+1}{3} + \Delta^3 t^m \binom{n+1}{4}$$
$$= \sum_{j=0}^{m} \binom{n+1}{j+1} \Delta^j t^m$$

Example 3.6 (Partial power sums of cubes 2)**.**

$$\sum_{k=0}^{n} k^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$
$$\sum_{k=1}^{n} k^3 = 1\binom{n-1}{1} + 7\binom{n-1}{2} + 12\binom{n-1}{3} + 6\binom{n-1}{4}$$
$$\sum_{k=2}^{n} k^3 = 8\binom{n-2}{1} + 19\binom{n-2}{2} + 18\binom{n-2}{3} + 6\binom{n-2}{4}$$

Corollary 3.7 (Partial power sums of cubes 2)**.**

$$\sum_{k=t}^{n} k^3 = \Delta^0 t^3 \binom{n-t+1}{1} + \Delta^1 t^3 \binom{n-t+1}{2} + \Delta^2 t^3 \binom{n-t+1}{3} + \Delta^3 t^3 \binom{n-t+1}{4}$$
$$= \sum_{j=0}^{3} \binom{n-t+1}{j+1} \Delta^j t^3$$

Corollary 3.8 (Partial power sums 2)**.**

$$\sum_{k=t}^{n} k^m = \Delta^0 t^m \binom{n-t+1}{1} + \Delta^1 t^m \binom{n-t+1}{2} + \Delta^2 t^m \binom{n-t+1}{3} + \Delta^3 t^m \binom{n-t+1}{4}$$
$$= \sum_{j=0}^{m} \binom{n-t+1}{j+1} \Delta^j t^m$$

4. Polynomial identities

Example 4.1 (Newton series for cubes monomial)**.**

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$
$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$
$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

In general,

$$n^3 = \Delta^0 t^3\binom{n-t}{0} + \Delta^1 t^3\binom{n-t}{1} + \Delta^2 t^3\binom{n-t}{2} + \Delta^3 t^3\binom{n-t}{3}$$

Example 4.2 (Newton series for cubes binomial)**.**

$$(n+0)^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$
$$(n+1)^3 = 1\binom{n}{0} + 7\binom{n}{1} + 12\binom{n}{2} + 6\binom{n}{3}$$
$$(n+2)^3 = 8\binom{n}{0} + 19\binom{n}{1} + 18\binom{n}{2} + 6\binom{n}{3}$$
$$(n+t)^3 = \Delta^0 t^3\binom{n}{0} + \Delta^1 t^3\binom{n}{1} + \Delta^2 t^3\binom{n}{2} + \Delta^3 t^3\binom{n}{3}$$

Corollary 4.3 (Newton series for power in zero)**.**

$$n^m = \sum_{k=0}^{m} \binom{n}{m-k} \Delta^{m-k} 0^m$$

Corollary 4.4 (Newton series for power in zero reversed)**.**

$$n^m = \sum_{k=0}^{m} \binom{n}{k} \Delta^{k} 0^m$$

Corollary 4.5 (Newton series for binomial)**.**

$$(n+t)^m = \sum_{k=0}^{m} \binom{n}{m-k} \Delta^{m-k} t^m$$

Corollary 4.6 (Commutativity of Newton series for binomial)**.**

$$(n+t)^m = \sum_{k=0}^{m} \binom{t}{m-k} \Delta^{m-k} n^m$$

Corollary 4.7 (Newton series for binomial reversed)**.**

$$(n+t)^m = \sum_{k=0}^{m} \binom{n}{k} \Delta^{k} t^m$$

Corollary 4.8 (Commutativity of Newton series for binomial reversed)**.**

$$(n+t)^m = \sum_{k=0}^{m} \binom{t}{k} \Delta^{k} n^m$$

Corollary 4.9 (Newton series for monomial)**.**

$$n^m = \sum_{k=0}^{m} \binom{n-t}{m-k} \Delta^{m-k} t^m$$

Proof. By setting $n \to n-t$ into (4.5). $\square$

Proposition 4.10 (Newton series for monomial reversed)**.**

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^{k} t^m$$

Proof. By setting $n \to n-t$ into (4.7). $\square$

Proposition 4.11 (Newton series for monomial reindexed)**.**

$$n^m = \sum_{k=0}^{m} \binom{t}{k} \Delta^{k} (n-t)^m$$

Proposition 4.12 (Newton series for monomial reindexed with $b = m$)**.**

$$n^m = \sum_{k=0}^{m} \binom{m}{k} \Delta^{k} (n-m)^m$$

Proposition 4.13 (Newton series for monomial reversed with $b = m$)**.**

$$n^m = \sum_{k=0}^{m} \binom{n-m}{k} \Delta^{k} m^m$$

Proof. By setting $t = m$ into (4.10). $\square$

5. Stirling numbers finite differences

Example 5.1 (Differences of Cubes via Stirling Numbers)**.**

$$T(3,0) = 0!\left\{ 3 \atop 0 \right\} = \Delta^0(0^3) = 0$$

$$T(3,1) = 1!\left\{ 3 \atop 1 \right\} = \Delta^1(0^3) = 1$$

$$T(3,2) = 2!\left\{ 3 \atop 2 \right\} = \Delta^2(0^3) = 6$$

$$T(3,3) = 3!\left\{ 3 \atop 3 \right\} = \Delta^3(0^3) = 6$$

Corollary 5.2 (Newton series via Stirling numbers)**.**

$$n^m = \sum_{k=0}^{m} \binom{n}{k} \left\{ m \atop k \right\} k!$$

Proposition 5.3 (Finite differences via Stirling numbers)**.**

$$T_1(n,k,x) = \sum_{t=0}^{n} \binom{x}{t-k} \left\{ n \atop t \right\} t! = \Delta^k x^n$$

Proposition 5.4 (Finite differences via Stirling numbers reindexed)**.**

$$T_2(x,n,k) = \sum_{t=0}^{n} \binom{x}{t} \left\{ n \atop k+t \right\} (k+t)! = \Delta^k x^n$$

Corollary 5.5 (For OEIS)**.**

$$T_2(t,n,k) = \sum_{j=0}^{n} \binom{t}{j} \left\{ n \atop k+j \right\} (k+j)! = \Delta^k t^n$$

For example

- $T_2(0,n,k) = \binom{0}{0}\left\{ n \atop k \right\} k!$
- $T_2(1,n,k) = \binom{1}{0}\left\{ n \atop k \right\} k! + \binom{1}{1}\left\{ n \atop k+1 \right\} (k+1)!$
- $T_2(2,n,k) = \binom{2}{0}\left\{ n \atop k \right\} k! + \binom{2}{1}\left\{ n \atop k+1 \right\} (k+1)! + \binom{2}{2}\left\{ n \atop k+2 \right\} (k+2)!$
- $T_2(3,n,k) = \binom{3}{0}\left\{ n \atop k \right\} k! + \binom{3}{1}\left\{ n \atop k+1 \right\} (k+1)! + \binom{3}{2}\left\{ n \atop k+2 \right\} (k+2)! + \binom{3}{3}\left\{ n \atop k+3 \right\} (k+3)!$
- $T_2(4,n,k) = \binom{4}{0}\left\{ n \atop k \right\} k! + \binom{4}{1}\left\{ n \atop k+1 \right\} (k+1)! + \binom{4}{2}\left\{ n \atop k+2 \right\} (k+2)! + \binom{4}{3}\left\{ n \atop k+3 \right\} (k+3)! + \binom{4}{4}\left\{ n \atop k+4 \right\} (k+4)!$

6. Segmented Hockey stick identity

Lemma 6.1 (Negative binomial coefficient)**.**

$$\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$$

Proposition 6.2 (Segmented Hockey stick identity)**.** *For integers n, t and j*

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Proof. First we split the sum $\sum_{k=0}^{n} \binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n} \binom{-t+k}{j} = \sum_{k=0}^{t-1} \binom{-t+k}{j} + \sum_{k=t}^{n} \binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0, 1, 2, \cdots, t, \cdots, n\}$ such that $t < n$. Considering the sum $\sum_{k=0}^{t-1} \binom{-t+k}{j}$ we notice that

$$\begin{aligned}\sum_{k=0}^{t-1} \binom{-t+k}{j} = \binom{-t}{j} + \binom{-t+1}{j} + \binom{-t+2}{j} + \cdots +\\ + \binom{-t+t-2}{j} + \binom{-t+t-1}{j}\end{aligned}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = \sum_{k=1}^{t} \binom{-k}{j} = \sum_{k=0}^{t-1} \binom{-k-1}{j}$$

By lemma (6.1)

$$\binom{-k-1}{j} = \binom{-(k+1)}{j} = (-1)^j \binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1} \binom{-t+k}{j} = (-1)^j \sum_{k=0}^{t-1} \binom{j+k}{j} = (-1)^j \binom{j+t}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n} \binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j}$$

Thus

$$\sum_{k=t}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j} = \binom{n-t+1}{j+1}$$

By means of Hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. Thus

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof. $\square$

Lemma 6.3. *For integers*

$$\sum_{k=1}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$$

Proof. By rearranging

$$\sum_{k=1}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-1} \binom{-t+k+1}{j}$$

By segmented hockey stick identity

$$\sum_{k=0}^{n-1} \binom{-t+k+1}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$$

$\square$

7. Multifold sums of powers

Definition 7.1 (Multifold sum of powers recurrence).

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

Theorem 7.2 (Sums of powers via finite difference). *For non-negative integers n, m and arbitrary integer t*

$$\sum_{k=0}^{n} k^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1} \right]$$

Theorem 7.3 (Sums of powers via finite difference 2). *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

Theorem 7.4 (Double sums of powers via finite difference 2).

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2} \right]$$

Validation (Mathematica). See the function t = 121; DoubleSumsOfPowersViaFiniteDifference5[10, t, 3] □

Proof.

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[(-1)^j \binom{j+t-1}{j+1} + \binom{k-t+1}{j+1} \right]$$

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \sum_{k=1}^{n} \binom{j+t-1}{j+1} + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

□

Theorem 7.5 (Triple sums of powers via finite difference 24)**.**

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + \right.$$
$$\left. + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3} \right]$$

Validation (Mathematica). See t = 126; TripleSumsOfPowersViaFiniteDifference24[10, t, 3]

□

Proof.

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[(-1)^j \binom{j+t-1}{j+1} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} k^0 + \binom{k-t+2}{j+2} \right]$$
$$= \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \sum_{k=1}^{n} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} \sum_{k=1}^{n} k^0 + \sum_{k=1}^{n} \binom{k-t+2}{j+2} \right]$$
$$= \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + \right.$$
$$\left. + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3} \right]$$

□

Theorem 7.6 (R-Fold Sum via Alternating Binomial Correction Term)**.**

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By Newton series for monomial reversed (4.10) and repeated segmented hockey stick identity (6.2). □

Validation (Mathematica). r = 3; Table[MultifoldSumOfPowersRecurrence[r, n, m] - RFoldSumViaAlternatingBinomialCorrectionTerm[r, n, m, t], n, 0, 10, m, 1, 10, t, 0, n] // Flatten □

Proposition 7.7. *For integers r and n*

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. By hockey stick identity. ◻

Corollary 7.8.

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right) + \binom{n-t+r}{j+r} \right]$$

Validation (Mathematica). r = 3; Table[MultifoldSumOfPowersRecurrence[r, n, m] - RFoldSumViaAlternatingBinomialCorrectionTerm3[r, n, m, t], n, 0, 10, m, 0, 10, t, 0, n] // Flatten ◻

Corollary 7.9.

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right) + \binom{n-t+r}{j+r} \right]$$

Validation (Mathematica). r = 3; Table[MultifoldSumOfPowersRecurrence[r, n, m] - RFoldSumViaAlternatingBinomialCorrectionTerm4[r, n, m, t], n, 0, 10, m, 0, 10, t, 0, n] // Flatten ◻

8. Examples for single sums

Example 8.1 (Sums of powers via finite difference 2)**.**

$$\sum_{k=1}^{n} k^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+1}{j+1} + (-1)^j \binom{j+t-1}{j+1} \right]$$

8.1. **For t=0.**

Example 8.2 (For $t = 0$)**.**

$$\sum_{k=1}^{n} k^m = \sum_{j=0}^{m} \Delta^j 0^m \left[\binom{n+1}{j+1} + (-1)^j \binom{j-1}{j+1} \right]$$

Example 8.3 (For t=0)**.**

$$\Sigma^1 \, n^0 = 1 \left(\binom{n+1}{1} + \binom{-1}{1} \right)$$

$$\Sigma^1 \, n^1 = 0 \left(\binom{n+1}{1} + \binom{-1}{1} \right) + 1 \left(\binom{n+1}{2} - \binom{0}{2} \right)$$

$$\Sigma^1 \, n^2 = 0 \left(\binom{n+1}{1} + \binom{-1}{1} \right) + 1 \left(\binom{n+1}{2} - \binom{0}{2} \right) + 2 \left(\binom{n+1}{3} + \binom{1}{3} \right)$$

$$\Sigma^1 \, n^3 = 0 \left(\binom{n+1}{1} + \binom{-1}{1} \right) + 1 \left(\binom{n+1}{2} - \binom{0}{2} \right) + 6 \left(\binom{n+1}{3} + \binom{1}{3} \right) + 6 \left(\binom{n+1}{4} - \binom{2}{4} \right)$$

$$\Sigma^1 \, n^m = \sum_{j=0}^{m} T(0,n,k) \left[\binom{n+1}{j+1} + (-1)^j \binom{j-1}{j+1} \right]$$

8.2. **For t=1.**

Example 8.4 (For $t = 1$)**.**

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j 1^m \left[(-1)^j \binom{j}{j+1} + \binom{n}{j+1}\right]$$

Example 8.5 (For $t = 1$)**.**

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(1,m,j) \left[(-1)^j \binom{j}{j+1} + \binom{n}{j+1}\right]$$

Example 8.6 (For t=1)**.**

$$\Sigma^1 n^0 = 1\left(\binom{n}{1} + \binom{0}{1}\right)$$

$$\Sigma^1 n^1 = 1\left(\binom{n}{1} + \binom{0}{1}\right) + 1\left(\binom{n}{2} - \binom{1}{2}\right)$$

$$\Sigma^1 n^2 = 1\left(\binom{n}{1} + \binom{0}{1}\right) + 3\left(\binom{n}{2} - \binom{1}{2}\right) + 2\left(\binom{n}{3} + \binom{2}{3}\right)$$

$$\Sigma^1 n^3 = 1\left(\binom{n}{1} + \binom{0}{1}\right) + 7\left(\binom{n}{2} - \binom{1}{2}\right) + 12\left(\binom{n}{3} + \binom{2}{3}\right) + 6\left(\binom{n}{4} - \binom{3}{4}\right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(1,m,j) \left[\binom{n}{j+1} + (-1)^j \binom{j}{j+1}\right]$$

8.3. For t=2.

Example 8.7 (For $t=2$)**.**

$$\sum_{k=1}^{n} k^m = \sum_{j=0}^{m} \Delta^j 2^m \left[\binom{n-1}{j+1} + (-1)^j\right]$$

Example 8.8 (For t=2)**.**

$$\Sigma^1 n^0 = 1\left(\binom{n-1}{1} + \binom{1}{1}\right)$$

$$\Sigma^1 n^1 = 2\left(\binom{n-1}{1} + \binom{1}{1}\right) + 1\left(\binom{n-1}{2} - \binom{2}{2}\right)$$

$$\Sigma^1 n^2 = 4\left(\binom{n-1}{1} + \binom{1}{1}\right) + 5\left(\binom{n-1}{2} - \binom{2}{2}\right) + 2\left(\binom{n-1}{3} + \binom{3}{3}\right)$$

$$\Sigma^1 n^3 = 8\left(\binom{n-1}{1} + \binom{1}{1}\right) + 19\left(\binom{n-1}{2} - \binom{2}{2}\right) + 18\left(\binom{n-1}{3} + \binom{3}{3}\right) + 6\left(\binom{n-1}{4} - \binom{4}{4}\right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(2,m,j)\left[\binom{n-1}{j+1} + (-1)^j \binom{j+1}{j+1}\right]$$

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- $\Sigma^1 n^0 = 1\left[\binom{n-1}{1} + 1\right]$
- $\Sigma^1 n^1 = 2\left[\binom{n-1}{1} + 1\right] + 1\left[\binom{n-1}{2} - 1\right]$
- $\Sigma^1 n^2 = 4\left[\binom{n-1}{1} + 1\right] + 5\left[\binom{n-1}{2} - 1\right] + 2\left[\binom{n-1}{3} + 1\right]$
- $\Sigma^1 n^3 = 8\left[\binom{n-1}{1} + 1\right] + 19\left[\binom{n-1}{2} - 1\right] + 18\left[\binom{n-1}{3} + 1\right] + 6\left[\binom{n-1}{4} - 1\right]$
- $\Sigma^1 n^m = \sum_{j=0}^{m} T(m,j)\left[\binom{n-1}{j+1} + (-1)^j\right]$

8.4. **For t=3.**

Example 8.9 (For $t = 3$)**.**

$$\sum_{k=1}^{n} k^m = \sum_{j=0}^{m} \Delta^j 3^m \left[\binom{n-2}{j+1} + (-1)^j \binom{j+2}{j+1} \right]$$

Example 8.10 (For t=3)**.**

$$\Sigma^1 n^0 = 1 \left(\binom{n-2}{1} + \binom{2}{1} \right)$$

$$\Sigma^1 n^1 = 3 \left(\binom{n-2}{1} + \binom{2}{1} \right) + 1 \left(\binom{n-2}{2} - \binom{3}{2} \right)$$

$$\Sigma^1 n^2 = 9 \left(\binom{n-2}{1} + \binom{2}{1} \right) + 7 \left(\binom{n-2}{2} - \binom{3}{2} \right) + 2 \left(\binom{n-2}{3} + \binom{4}{3} \right)$$

$$\Sigma^1 n^3 = 27 \left(\binom{n-2}{1} + \binom{2}{1} \right) + 37 \left(\binom{n-2}{2} - \binom{3}{2} \right)$$

$$+ 24 \left(\binom{n-2}{3} + \binom{4}{3} \right) + 6 \left(\binom{n-2}{4} - \binom{5}{4} \right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(3,m,j) \left[\binom{n-2}{j+1} + (-1)^j \binom{j+2}{j+1} \right]$$

8.5. **For t=4.**

Example 8.11 (For $t = 4$)**.**

$$\sum_{k=1}^{n} k^m = \sum_{j=0}^{m} \Delta^j 4^m \left[\binom{n-3}{j+1} + (-1)^j \binom{j+3}{j+1} \right]$$

Example 8.12 (For t=4)**.**

$$\begin{aligned}
\Sigma^1 n^0 &= 1\left(\binom{n-3}{1} + \binom{3}{1}\right) \\
\Sigma^1 n^1 &= 4\left(\binom{n-3}{1} + \binom{3}{1}\right) + 1\left(\binom{n-3}{2} - \binom{4}{2}\right) \\
\Sigma^1 n^2 &= 16\left(\binom{n-3}{1} + \binom{3}{1}\right) + 9\left(\binom{n-3}{2} - \binom{4}{2}\right) + 2\left(\binom{n-2}{3} + \binom{5}{3}\right) \\
\Sigma^1 n^3 &= 64\left(\binom{n-3}{1} + \binom{3}{1}\right) + 61\left(\binom{n-3}{2} - \binom{4}{2}\right) \\
&\quad + 30\left(\binom{n-3}{3} + \binom{5}{3}\right) + 6\left(\binom{n-3}{4} - \binom{6}{4}\right) \\
\Sigma^1 n^m &= \sum_{j=0}^{m} T(4,m,j) \left[\binom{n-3}{j+1} + (-1)^j \binom{j+3}{j+1} \right]
\end{aligned}$$

8.6. **For t=5.**

Example 8.13 (For $t = 5$)**.**

$$\sum_{k=1}^{n} k^m = \sum_{j=0}^{m} \Delta^j 5^m \left[\binom{n-4}{j+1} + (-1)^j \binom{j+4}{j+1} \right]$$

Example 8.14 (For t=5)**.**

$$\Sigma^1 n^0 = 1 \left(\binom{n-4}{1} + \binom{4}{1} \right)$$

$$\Sigma^1 n^1 = 5 \left(\binom{n-4}{1} + \binom{4}{1} \right) + 1 \left(\binom{n-4}{2} - \binom{5}{2} \right)$$

$$\Sigma^1 n^2 = 25 \left(\binom{n-4}{1} + \binom{4}{1} \right) + 11 \left(\binom{n-4}{2} - \binom{5}{2} \right) + 2 \left(\binom{n-4}{3} + \binom{6}{3} \right)$$

$$\Sigma^1 n^3 = 125 \left(\binom{n-4}{1} + \binom{4}{1} \right) + 91 \left(\binom{n-4}{2} - \binom{5}{2} \right)$$

$$+ 36 \left(\binom{n-4}{3} + \binom{6}{3} \right) + 6 \left(\binom{n-4}{4} - \binom{7}{4} \right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(5, m, j) \left[\binom{n-4}{j+1} + (-1)^j \binom{j+4}{j+1} \right]$$

9. Examples for double sums

Theorem 9.1 (Double sums of powers via finite difference 2)**.**

$$\Sigma^2\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+2}{j+2} + (-1)^j \binom{j+t-1}{j+1} n^1 + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 \right]$$

9.1. **For t=0.**

Example 9.2 (For t=0)**.**

$$\Sigma^2\, n^m = \sum_{j=0}^{m} \Delta^j 0^m \left[\binom{n+2}{j+2} + (-1)^j \binom{j-1}{j+1} n + (-1)^{j+1} \binom{j-1}{j+2} n^0 \right]$$

Example 9.3 (For t=0)**.**

$$\Sigma^2\, n^m = \sum_{j=0}^{m} T(0,m,j) \left[\binom{n+2}{j+2} + (-1)^j \binom{j-1}{j+1} n + (-1)^{j+1} \binom{j-1}{j+2} n^0 \right]$$

Example 9.4 (For t=0)**.**

$$\begin{aligned}
\Sigma^2\, n^0 &= 1 \left(\binom{n+2}{2} + \binom{-1}{1} n - \binom{-1}{2} \right) \\
\Sigma^2\, n^1 &= 0 \left(\binom{n+2}{2} + \binom{-1}{1} n - \binom{-1}{2} \right) + 1 \left(\binom{n+2}{3} - \binom{0}{2} n + \binom{0}{3} \right) \\
\Sigma^2\, n^2 &= 0 \left(\binom{n+2}{2} + \binom{-1}{1} n - \binom{-1}{2} \right) + 1 \left(\binom{n+2}{3} - \binom{0}{2} n + \binom{0}{3} \right) \\
&\quad + 2 \left(\binom{n+2}{4} + \binom{1}{3} n - \binom{1}{4} \right) \\
\Sigma^2\, n^3 &= 0 \left(\binom{n+2}{2} + \binom{-1}{1} n - \binom{-1}{2} \right) + 1 \left(\binom{n+2}{3} - \binom{0}{2} n + \binom{0}{3} \right) \\
&\quad + 6 \left(\binom{n+2}{4} + \binom{1}{3} n - \binom{1}{4} \right) + 6 \left(\binom{n+2}{5} - \binom{2}{4} n + \binom{2}{5} \right)
\end{aligned}$$

9.2. **For t=1.**

Example 9.5 (For t=1)**.**

$$\Sigma^2\, n^m = \sum_{j=0}^{m} T(1,m,j) \left[\binom{n+1}{j+2} + (-1)^j \binom{j}{j+1} n^1 + (-1)^{j+1} \binom{j}{j+2} n^0 \right]$$

Example 9.6 (For t=1)**.**

$$\begin{aligned}
\Sigma^2\, n^0 &= 1 \left(\binom{n+1}{2} + \binom{0}{1} n - \binom{0}{2} \right) \\
\Sigma^2\, n^1 &= 1 \left(\binom{n+1}{2} + \binom{0}{1} n - \binom{0}{2} \right) + 1 \left(\binom{n+1}{3} - \binom{1}{2} n + \binom{1}{3} \right) \\
\Sigma^2\, n^2 &= 1 \left(\binom{n+1}{2} + \binom{0}{1} n - \binom{0}{2} \right) + 3 \left(\binom{n+1}{3} - \binom{1}{2} n + \binom{1}{3} \right) \\
&\quad + 2 \left(\binom{n+1}{4} + \binom{2}{3} n - \binom{2}{4} \right) \\
\Sigma^2\, n^3 &= 1 \left(\binom{n+1}{2} + \binom{0}{1} n - \binom{0}{2} \right) + 7 \left(\binom{n+1}{3} - \binom{1}{2} n + \binom{1}{3} \right) \\
&\quad + 12 \left(\binom{n+1}{4} + \binom{2}{3} n - \binom{2}{4} \right) + 6 \left(\binom{n+1}{5} - \binom{3}{4} n + \binom{3}{5} \right)
\end{aligned}$$

9.3. **For t=2.**

Example 9.7 (For t=2)**.**

$$\Sigma^2\, n^m = \sum_{j=0}^{m} T(2,m,j)\left[\binom{n}{j+2} + (-1)^j\binom{j+1}{j+1}n^1 + (-1)^{j+1}\binom{j+1}{j+2}n^0\right]$$

Example 9.8 (For t=2)**.**

$$\Sigma^2\, n^0 = 1\left(\binom{n}{2} + \binom{1}{1}n - \binom{0}{2}\right)$$

$$\Sigma^2\, n^1 = 2\left(\binom{n}{2} + \binom{1}{1}n - \binom{0}{2}\right) + 1\left(\binom{n}{3} - \binom{2}{2}n + \binom{1}{3}\right)$$

$$\Sigma^2\, n^2 = 4\left(\binom{n}{2} + \binom{1}{1}n - \binom{0}{2}\right) + 5\left(\binom{n}{3} - \binom{2}{2}n + \binom{1}{3}\right) + 2\left(\binom{n}{4} + \binom{3}{3}n - \binom{2}{4}\right)$$

$$\begin{aligned}\Sigma^2\, n^3 = {} & 8\left(\binom{n}{2} + \binom{1}{1}n - \binom{0}{2}\right) + 19\left(\binom{n}{3} - \binom{2}{2}n + \binom{1}{3}\right) + 18\left(\binom{n}{4} + \binom{3}{3}n - \binom{2}{4}\right) \\ & + 6\left(\binom{n}{5} - \binom{4}{4}n + \binom{3}{5}\right)\end{aligned}$$

9.4. **For t=3.**

Example 9.9 (For t=3)**.**

$$\Sigma^2\, n^m = \sum_{j=0}^{m} T(3,m,j)\left[\binom{n+1}{j+2} + (-1)^j\binom{j+2}{j+1}n^1 + (-1)^{j+1}\binom{j+2}{j+2}n^0\right]$$

Example 9.10 (For t=3)**.**

$$\Sigma^2\, n^0 = 1\left(\binom{n+1}{2} + \binom{2}{1}n - \binom{2}{2}\right)$$

$$\Sigma^2\, n^1 = 3\left(\binom{n+1}{2} + \binom{2}{1}n - \binom{2}{2}\right) + 1\left(\binom{n+1}{3} - \binom{3}{2}n + \binom{3}{3}\right)$$

$$\begin{aligned}\Sigma^2\, n^2 &= 9\left(\binom{n+1}{2} + \binom{2}{1}n - \binom{2}{2}\right) + 7\left(\binom{n+1}{3} - \binom{3}{2}n + \binom{3}{3}\right)\\ &\quad + 2\left(\binom{n+1}{4} + \binom{4}{3}n - \binom{4}{4}\right)\end{aligned}$$

$$\begin{aligned}\Sigma^2\, n^3 &= 27\left(\binom{n+1}{2} + \binom{2}{1}n - \binom{2}{2}\right) + 37\left(\binom{n+1}{3} - \binom{3}{2}n + \binom{3}{3}\right)\\ &\quad + 24\left(\binom{n+1}{4} + \binom{4}{3}n - \binom{4}{4}\right) + 6\left(\binom{n+1}{5} - \binom{5}{4}n + \binom{5}{5}\right)\end{aligned}$$

9.5. **For t=4.**

Example 9.11 (For t=4)**.**

$$\Sigma^2\, n^m = \sum_{j=0}^{m} T(4,m,j)\left[\binom{n-2}{j+2} + (-1)^j\binom{j+3}{j+1}n^1 + (-1)^{j+1}\binom{j+3}{j+2}n^0\right]$$

Example 9.12 (For t=4)**.**

$$\Sigma^2\, n^0 = 1\left(\binom{n-2}{2} + \binom{3}{1}n - \binom{3}{2}\right)$$

$$\Sigma^2\, n^1 = 4\left(\binom{n-2}{2} + \binom{3}{1}n - \binom{3}{2}\right) + 1\left(\binom{n-2}{3} - \binom{4}{2}n + \binom{4}{3}\right)$$

$$\begin{aligned}\Sigma^2\, n^2 &= 16\left(\binom{n-2}{2} + \binom{3}{1}n - \binom{3}{2}\right) + 9\left(\binom{n-2}{3} - \binom{4}{2}n + \binom{4}{3}\right)\\ &\quad + 2\left(\binom{n-2}{4} + \binom{5}{3}n - \binom{5}{4}\right)\end{aligned}$$

$$\begin{aligned}\Sigma^2\, n^3 &= 64\left(\binom{n-2}{2} + \binom{3}{1}n - \binom{3}{2}\right) + 61\left(\binom{n-2}{3} - \binom{4}{2}n + \binom{4}{3}\right)\\ &\quad + 30\left(\binom{n-2}{4} + \binom{5}{3}n - \binom{5}{4}\right) + 6\left(\binom{n-2}{5} - \binom{6}{4}n + \binom{6}{5}\right)\end{aligned}$$

9.6. **For t=5.**

Example 9.13 (For t=5)**.**

$$\Sigma^2 n^m = \sum_{j=0}^{m} T(5,m,j)\left[\binom{n-3}{j+2} + (-1)^j\binom{j+4}{j+1}n^1 + (-1)^{j+1}\binom{j+4}{j+2}n^0\right]$$

Example 9.14 (For t=5)**.**

$$\begin{aligned}
\Sigma^2 n^0 &= 1\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right)\\
\Sigma^2 n^1 &= 5\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 1\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)\\
\Sigma^2 n^2 &= 25\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 11\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)\\
&\quad + 2\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right)\\
\Sigma^2 n^3 &= 125\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 91\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right)\\
&\quad + 36\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right) + 6\left(\binom{n-3}{5} - \binom{7}{4}n + \binom{7}{5}\right)
\end{aligned}$$

10. Coefficients of finite diffrence in sums of powers

Theorem 10.1.

$$\boxed{\Sigma^r\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s}\, n^0\right) + \binom{n-t+r}{j+r}\right]}$$

Validation (Mathematica). See t = 121213; RFoldSumViaAlternatingBinomialCorrection-Term[5, 10, 4, t] □

Corollary 10.2.

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s}\, n^0\right) + \binom{n-t+r}{j+r}\right]$$

$$\Sigma^r\, n^m = \sum_{j=0}^{m} R(t,n,j,r) \cdot \Delta^j t^m$$

Validation (Mathematica). See t = 121213; RFoldSumViaAlternatingBinomialCorrection-Term2[5, 10, 4, t] □

Proposition 10.3.

$$R(t,n,j,r) = \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s}\, n^0\right) + \binom{n-t+r}{j+r}\right]$$

11. OEIS TABLE

N	Sequence data	Formula	Offset	OEIS ID
0	1, 0, 1, 0, 1, 2, 0, 1, 6, 6, 0, 1, 14, 36, 24, 0, 1, 30, 150, 240, 120	$T(n,k) = \Delta^k 0^n$	$0 \le k \le n$	A131689 in [3]
1	1, 1, 1, 1, 3, 2, 1, 7, 12, 6, 1, 15, 50, 60, 24, 1, 31, 180, 390, 360, 120	$T(n,k) = \Delta^k 1^n$	$1 \le k \le n$	A028246 in [3]
1	1, 1, 1, 1, 3, 2, 1, 7, 12, 6, 1, 15, 50, 60, 24, 1, 31, 180, 390, 360, 120	$T(n,k) = \Delta^k 1^n$	$0 \le k \le n$	NA
2	1, 2, 1, 4, 5, 2, 8, 19, 18, 6, 16, 65, 110, 84, 24, 32, 211, 570, 750, 480, 120	$T(n,k) = \Delta^k 2^n$	$0 \le k \le n$	A038719 in [3]
3	1, 3, 1, 9, 7, 2, 27, 37, 24, 6, 81, 175, 194, 108, 24, 243, 781, 1320, 1230, 600, 120	$T(n,k) = \Delta^k 3^n$	$0 \le k \le n$	A391552 in [3]
4	1, 4, 1, 16, 9, 2, 64, 61, 30, 6, 256, 369, 302, 132, 24	$T(n,k) = \Delta^k 4^n$	$0 \le k \le n$	A391633 in [3]
5	1, 5, 1, 25, 11, 2, 125, 91, 36, 6, 625, 671, 434, 156, 24	$T(n,k) = \Delta^k 5^n$	$0 \le k \le n$	A391635 in [3]

12. Mathematica programs

Mathematica Function	Validates / Prints
`FiniteDifferenceOfPowerOrderN[var, exp, n]`	Corollary 1.2
`NewtonSeriesForMonomialReindexed[n, t, m]`	Proposition 4.11
`NewtonSeriesForPowerInZero[n, m]`	Corollary 4.3
`NewtonSeriesForPowerInZeroReversed[n, m]`	Corollary 4.4
`NewtonSeriesForBinomial[n, t, m]`	Corollary 4.5
`CommutativityOfNewtonSeriesForBinomial[n, t, m]`	Corollary 4.6
`NewtonSeriesForBinomialReversed[n, t, m]`	Corollary 4.7
`CommutativityOfNewtonSeriesForBinomialReversed[n, t, m]`	Corollary 4.8
`NewtonSeriesForMonomial[n, t, m]`	Corollary 4.9
`NewtonSeriesForMonomialReversed[n, t, m]`	Corollary **??**
`PartialPowerSumLHS[t, n, exp]`	Corollary 3.8 (LHS)
`PartialPowerSumRHS[t, n, exp]`	Corollary 3.8 (RHS)

References

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[2] Newton, Isaac and Chittenden, N.W. *Newton's Principia: the mathematical principles of natural philosophy*. New-York, D. Adee, 1850. https://archive.org/details/bub_gb_KaAIAAAAIAAJ/page/466/mode/2up.

[3] Sloane, Neil JA and others. The on-line encyclopedia of integer sequences, 2003. https://oeis.org/.