COUNTING PRIMES AND TWIN-PRIMES USING MINIMAL GOLDBACH PAIRS

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Abstract. Assuming Goldbach's Conjecture holds, every even integer $2N \geq 4$ can be written as $2N = p_i + p_j$ where (p_i, p_j) is called Goldbach pair. The minimal Goldbach pair is a pair (p_i, p_j) having the minimal p_i such that $p_j = 2N - p_i$ is also a prime. We define a function $F_{2N}(P)$ that counts occurrences of $p_j = P$ within the range $6 \leq 2k \leq 2N$. In particular, the function $F_{2N}(P)$ provides the following identities in terms of prime counting $\pi(2N)$ and twin-prime counting $\pi_2(2N)$

$$\pi(2N) = F_{2N}(3) + 1, \quad \pi_2(2N) = F_{2N}(3) - F_{2N}(5)$$

Contents

1. Introduction

This manuscript provides a comprehensive review of the work [1] done by Michel Yamagishi. Goldbach conjecture asserts that every even integer $2N \geq 4$ is a sum of two primes

$$2N = p_i + p_j$$

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Sources: https://github.com/kolosovpetro/github-latex-template

where (p_i, p_j) is called Goldbach pair.

Goldbach pair is not unique for even integers greater than 6, meaning that there can be multiple goldbach pairs for even integer $2N \geq 8$.

For example: $10 = 3 + 7$ and $10 = 5 + 5$ and $10 = 7 + 3$ where goldbach pairs are $(3, 7),\ (5, 5),\ (7, 3)$.

Minimal goldbach pair is the pair having minimal p_i across all goldbach pairs for even integer $2N$.

For even integer 10 we have three pairs $(3, 7),\ (5, 5),\ (7, 3)$ while the minimal is $(3, 7)$ because 3 is the minimal value across all the p_i values: $3, 5, 7$.

Consider the following minimal goldbach pairs for even integer $2k$ within the range $6 \leq 2k \leq 50$

$$
\begin{aligned}
6 &= 3 + 3, & 8 &= 3 + 5, \\
10 &= 3 + 7, & 12 &= 5 + 7, \\
14 &= 3 + 11, & 16 &= 3 + 13, \\
18 &= 5 + 13, & 20 &= 3 + 17, \\
22 &= 3 + 19, & 24 &= 5 + 19, \\
26 &= 3 + 23, & 28 &= 5 + 23, \\
30 &= 7 + 23, & 32 &= 3 + 29, \\
34 &= 3 + 31, & 36 &= 5 + 31, \\
38 &= 7 + 31, & 40 &= 3 + 37, \\
42 &= 5 + 37, & 44 &= 3 + 41, \\
46 &= 3 + 43, & 48 &= 5 + 43, \\
50 &= 3 + 47
\end{aligned}
$$

We can notice that minimal Goldbach pairs having $p_i = 3$ produce a sequence of odd prime numbers $p_j = 3, 5, 7, 11, 13, 17\ldots$ which is quite remarkable

$$\begin{aligned}
6 &= 3+3, & 8 &= 3+5,\\
10 &= 3+7, & 14 &= 3+11,\\
16 &= 3+13, & 20 &= 3+17,\\
22 &= 3+19, & 26 &= 3+23,\\
32 &= 3+29, & 34 &= 3+31,\\
40 &= 3+37, & 44 &= 3+41,\\
46 &= 3+43, & 50 &= 3+47
\end{aligned}$$

One more interesting observation is that by selecting the pairs with minimal $p_i = 5$ yields the sequence of primes p_j such that $p_j + 2$ is not a prime.

$$\begin{aligned}
12 &= 5+7, & 18 &= 5+13,\\
24 &= 5+19, & 28 &= 5+23,\\
36 &= 5+31, & 42 &= 5+37,\\
48 &= 5+43
\end{aligned}$$

For our conversation to be more formal and objective, we define a few functions. Let $G_{\min}(2N)$ be a function that returns a set of minimal Goldbach pairs (p_i, p_j) having $\min p_i$ over the range $6 \leq 2k \leq 2N$

$$G_{\min}(2N) = \{(p_i, p_j) \mid p_i + p_j = 2k \mid 6 \leq 2k \leq 2N \mid \min p_i\}.$$

For example,

$$G_{\min}(20) = \{(3,3),\ (3,5),\ (3,7),\ (5,7),\ (3,11),\ (3,13),\ (5,13),\ (3,17)\}$$

Let $W_{2N}(P)$ be a function that returns the set of elements p_j from $G_{\min}(2N)$ having $p_i = P$

$$W_{2N}(P) = \{p_j \mid (p_i, p_j) \in G_{\min}(2N) \text{ and } p_i = P\}$$

Then sequence of odd prime numbers [2] is given by $W_{2N}(3)$

$$\{3,\ 5,\ 7,\ 11,\ \ldots,\ p \leq 2N - 3\} = W_{2N}(3)$$

Now we can easily count the number of primes within the interval $6 \leq 2k \leq 2N$ because $\pi(2N)$ equals to the total number of elements inside the set $W_{2N}(3)$

$$\pi(2N) = F_{2N}(3) + 1$$

where $F_{2N}(3)$ is the function that counts the number of elements inside the set $W_{2N}(3)$, in general $F_{2N}(P) = |W_{2N}(3)|$.

Taking $P = 5$ in $W_{2N}(P)$ yields a sequence of primes p_j such that $p_j + 2$ is not a prime [3]

$$W_{2N}(5) = \{7,\ 13,\ 19,\ 23,\ 31,\ 37,\ 43,\ 47,\ 53,\ \ldots,\ p \leq 2N - 5\}$$

Which implies that the number of twin primes in range $6 \leq 2k \leq 2N$ can be expressed in terms of $F_{2N}(P)$

$$\pi_2(2N) = F_{2N}(3) - F_{2N}(5)$$

where $2N = 10^k + 2,\ k = 1, 2, 3, 4, \ldots$. For example,

$$\pi_2(12) = F_{12}(3) - F_{12}(5) = 2$$

$$\pi_2(102) = F_{102}(3) - F_{102}(5) = 8$$

$$\pi_2(1002) = F_{1002}(3) - F_{1002}(5) = 35$$

$$\pi_2(10002) = F_{10002}(3) - F_{10002}(5) = 205$$

$$\pi_2(100002) = F_{100002}(3) - F_{100002}(5) = 1224$$

$$\pi_2(1000002) = F_{1000002}(3) - F_{1000002}(5) = 8169$$

Which matches the sequence [4].

Having $P = 7$ function $W_{2N}(P)$ yields the sequence of primes such that $p_j - p_i \geq 6$ where p_j is the next prime after p_i, see [5]

$$W_{2N}(7) = \{23,\ 31,\ 47,\ 53,\ 61,\ 73,\ 83,\ 89,\ 113,\ \ldots,\ p \leq 2N - 7\}$$

Allowing us to count the primes with next-prime gap at least 6: $\delta_6(2N) = F_{2N}(7)$.

2. Conclusions

Assuming Goldbach's Conjecture holds, we introduced a framework based on minimal Goldbach pairs to derive expressions for key prime-related functions. Specifically, we defined the function $F_{2N}(P)$ that counts occurrences of primes p_j in minimal Goldbach pairs (p_i, p_j) where $p_i = P$. Using this framework, we obtained

- The prime-counting function: $\pi(2N) = F_{2N}(3) + 1$
- The twin-prime counting function: $\pi_2(2N) = F_{2N}(3) - F_{2N}(5)$
- The count of primes with next-prime gap at least 6: $\delta_6(2N) = F_{2N}(7)$

These identities establish a novel connection between Goldbach partitions and classical prime number theory. Computational examples confirm alignment with known integer sequences, reinforcing the potential of this approach for analytical and numerical exploration of prime distributions. All the results validated up to $N = 10^8$ with source code available on GitHub

References

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