NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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Abstract. In this manuscript we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind and Eulerian numbers.

Contents

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1. Introduction and main results

In this manuscript, we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind and Eulerian numbers.

The main idea is to utilize Newton's interpolation formula combined with hockey-stick family identities to derive formulas for multifold sums of powers.

In this manuscript, we use Newton's interpolation formula in forward finite differences.

However, the Newton's formula approach for sums of powers is quite generic, as such it does not necessary requires to bind to forward difference operator specifically. Thus, formulas for sums of powers using backward and central differences can be found in [1, 2], respectively.

Allow us to start from the definition of multifold sums of powers. We utilize the recurrence proposed by Donald Knuth in his article *Johann Faulhaber and sums of powers*, see [3]

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

Throughout the paper, we utilize the Newton's interpolation formula as stated below

Proposition 1.1. *(Newton's interpolation formula* [4, Lemma V]*.)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

where $\Delta^k f(a) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(a+j)$ *are* k*-degree forward finite differences of function* f *evaluated in point* a*.*

Therefore, by setting $f(n) = n^m$ yields

Proposition 1.2 (Newton's interpolation formula for powers)**.** *For non-negative integers m, n and an arbitrary integer t*

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

Which is indeed true, because

$$n^3 = 0\tbinom{n-0}{0} + 1\tbinom{n-0}{1} + 6\tbinom{n-0}{2} + 6\tbinom{n-0}{3}$$

$$n^3 = 1\tbinom{n-1}{0} + 7\tbinom{n-1}{1} + 12\tbinom{n-1}{2} + 6\tbinom{n-1}{3}$$

$$n^3 = 8\tbinom{n-2}{0} + 19\tbinom{n-2}{1} + 18\tbinom{n-2}{2} + 6\tbinom{n-2}{3}$$

$$n^3 = 27\tbinom{n-3}{0} + 37\tbinom{n-3}{1} + 24\tbinom{n-3}{2} + 6\tbinom{n-3}{3}$$

Proposition 1.3 (Generalized hockey-stick identity)**.** *For integers a, b and j*

$$\sum_{k=a}^{b} \binom{k}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}$$

Proof. We have $\sum_{k=a}^{b} \binom{k}{j} = \binom{a}{j} + \binom{a+1}{j} + \cdots + \binom{b}{j}$, which means that $\sum_{k=a}^{b} \binom{k}{j} = \left(\sum_{k=0}^{b} \binom{k}{j}\right) - \left(\sum_{k=0}^{a-1} \binom{k}{j}\right)$. By hockey stick identity $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$ yields $\sum_{k=a}^{b} \binom{k}{j} = \left(\sum_{k=0}^{b} \binom{k}{j}\right) - \left(\sum_{k=0}^{a-1} \binom{k}{j}\right) = \binom{b+1}{j+1} - \binom{a}{j+1}$. $\square$

Therefore,

Lemma 1.4 (Shifted hockey stick identity)**.** *For integers n, t, j, r*

$$\sum_{k=1}^{n} \binom{k-t+r}{j+r} = \sum_{k=1-t+r}^{n-t+r} \binom{k}{j+r} = \binom{n-t+r+1}{j+r+1} - \binom{1-t+r}{j+r+1}$$

Recall the upper negation property of binomial coefficients

Lemma 1.5 (Upper negation)**.** *For integers r, k*

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$$

Thus,

$$\binom{1-t+r}{j+r+1} = (-1)^{j+r+1}\binom{(j+r+1)-(1-t+r)-1}{j+r+1} = (-1)^{j+r+1}\binom{j+t-1}{j+r+1}$$

Such that negated version of shifted hockey stick identity (1.4) follows

Lemma 1.6 (Negation of shifted hockey stick identity)**.** *For integers n, t, j, r*

$$\sum_{k=1}^{n}\binom{k-t+r}{j+r} = \binom{n-t+r+1}{j+r+1} + (-1)^{j+r}\binom{j+t-1}{j+r+1}$$

Hence, by Newton's interpolation formula for powers (1.2), the ordinary sum of powers is

$$\Sigma^1 n^m = \sum_{k=1}^{n}\sum_{j=0}^{m}\binom{k-t}{j}\Delta^j t^m = \sum_{j=0}^{m}\Delta^j t^m \sum_{k=1}^{n}\binom{k-t}{j}$$

where t is an arbitrary integer. By shifted hockey stick identity (1.4) closed form of ordinary sums of powers yields

Proposition 1.7 (Ordinary sums of powers via Newton's formula)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m}\Delta^j t^m \left[\binom{n-t+1}{j+1} - \binom{1-t}{j+1}\right]$$

By negation of shifted hockey stick identity (1.6) yields

Proposition 1.8 (Negated ordinary sums of powers via Newton's formula)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m}\Delta^j t^m \left[\binom{n-t+1}{j+1} + (-1)^j\binom{j+t-1}{j+1}\right]$$

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently

Corollary 1.9. *By setting $t = 0$ into* (1.7)

$$\Sigma^1 n^m = \sum_{j=0}^{m}\binom{n+1}{j+1}\Delta^j 0^m$$

Corollary 1.10. *By setting $t = 1$ into (1.7)*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \binom{n}{j+1} \Delta^j 1^m$$

In particular, by setting $t = 0$ and $m = 3$ into (1.7), we have the famous identity

$$\Sigma^1 n^3 = 0\tbinom{n+1}{1} + 1\tbinom{n+1}{2} + 6\tbinom{n+1}{3} + 6\tbinom{n+1}{4}$$

which was discussed in [5, p. 190] and in [6].

The coefficients $0, 1, 6, 6, 0, 1, 14, 36, 24, \dots$ is the sequence A131689 in the OEIS [7].

The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were shown in [8]. For instance,

$$\Sigma^1 n^3 = 1\tbinom{n}{1} + 7\tbinom{n}{2} + 12\tbinom{n}{3} + 6\tbinom{n}{4}$$

$$\Sigma^1 n^4 = 1\tbinom{n}{1} + 15\tbinom{n}{2} + 50\tbinom{n}{3} + 60\tbinom{n}{4} + 24\tbinom{n}{5}$$

$$\Sigma^1 n^5 = 1\tbinom{n}{1} + 31\tbinom{n}{2} + 180\tbinom{n}{3} + 390\tbinom{n}{4} + 360\tbinom{n}{5} + 120\tbinom{n}{6}$$

The coefficients $1, 7, 12, 6, 1, 15, \dots$ is the sequence A028246 in the OEIS [7].

Interestingly enough that the paper [8] give formulas for sums of powers in terms of generalized Stirling numbers of the second kind $\left\{ {k \atop j} \right\}_r$

Theorem 1.11 (Cereceda (2022))**.**

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_r \tag{1}$$

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + \binom{r+1}{j+1} \right] \left\{ {k \atop j} \right\}_{-r} \tag{2}$$

The formula (1) is generalized Stirling numbers version of negated ordinary sums of powers via Newton's formula (1.8), which implies that finite forward differences of powers can be expressed in terms of generalized Stirling numbers of the second kind: $\Delta^j t^m = j! \left\{ {m \atop j} \right\}_t$.

The formula (2) is the special case of ordinary sums of powers via Newton's formula (1.7) with $t = -r$.

By considering the case for $t = 4$ of ordinary sums of powers via Newton's formula (1.7), we observe rather unexpected formulas for sums of powers, namely

$$\Sigma^1 n^0 = 1\left(\tbinom{n-3}{1} + \tbinom{3}{1}\right)$$

$$\Sigma^1 n^1 = 4\left(\tbinom{n-3}{1} + \tbinom{3}{1}\right) + 1\left(\tbinom{n-3}{2} - \tbinom{4}{2}\right)$$

$$\Sigma^1 n^2 = 16\left(\tbinom{n-3}{1} + \tbinom{3}{1}\right) + 9\left(\tbinom{n-3}{2} - \tbinom{4}{2}\right) + 2\left(\tbinom{n-2}{3} + \tbinom{5}{3}\right)$$

$$\Sigma^1 n^3 = 64\left(\tbinom{n-3}{1} + \tbinom{3}{1}\right) + 61\left(\tbinom{n-3}{2} - \tbinom{4}{2}\right) + 30\left(\tbinom{n-3}{3} + \tbinom{5}{3}\right) + 6\left(\tbinom{n-3}{4} - \tbinom{6}{4}\right)$$

The coefficients $1, 4, 1, 16, 9, \ldots$ is the sequence A391633 in the OEIS [7]. In general,

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j 4^m \left[\binom{n-3}{j+1} + (-1)^j \binom{j+3}{j+1}\right].$$

To derive formula for double sum of powers, we apply summation operator over the ordinary sum of powers (1.7) again

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{k=1}^{n} \binom{k-t+1}{j+1}\right) - \binom{1-t}{j+1} \sum_{k=1}^{n} 1\right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{k=1}^{n} \binom{k-t+1}{j+1}\right) - \binom{1-t}{j+1} n\right]$$

By applying shifted hockey stick identity (1.4), we get

Proposition 1.12 (Double sums of powers via Newton's formula)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+2}{j+1} - \binom{1-t}{j+1} n - \binom{2-t}{j+2}\right]$$

By negated hockey stick identity (1.6) yields

Proposition 1.13 (Negated double sums of powers via Newton's formula)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+2}{j+1} + (-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2}\right]$$

For example, by setting $t = 5$ into double sums of powers via Newton's interpolation formula (1.12), we get

$$\Sigma^2 n^0 = 1\left(\tbinom{n-3}{2} + \tbinom{4}{1}n - \tbinom{4}{2}\right)$$

$$\Sigma^2 n^1 = 5\left(\tbinom{n-3}{2} + \tbinom{4}{1}n - \tbinom{4}{2}\right) + 1\left(\tbinom{n-3}{3} - \tbinom{5}{2}n + \tbinom{5}{3}\right)$$

$$\Sigma^2 n^2 = 25\left(\tbinom{n-3}{2} + \tbinom{4}{1}n - \tbinom{4}{2}\right) + 11\left(\tbinom{n-3}{3} - \tbinom{5}{2}n + \tbinom{5}{3}\right) + 2\left(\tbinom{n-3}{4} + \tbinom{6}{3}n - \tbinom{6}{4}\right)$$

$$\Sigma^2 n^3 = 125\left(\tbinom{n-3}{2} + \tbinom{4}{1}n - \tbinom{4}{2}\right) + 91\left(\tbinom{n-3}{3} - \tbinom{5}{2}n + \tbinom{5}{3}\right) + 36\left(\tbinom{n-3}{4} + \tbinom{6}{3}n - \tbinom{6}{4}\right) + 6\left(\tbinom{n-3}{5} - \tbinom{7}{4}n + \tbinom{7}{5}\right)$$

The coefficients $1, 5, 1, 25, 11, 2, \ldots$ is the sequence A391635 in the OEIS [7].

In general,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j 5^m \left[\binom{n-3}{j+2} + (-1)^j \binom{j+4}{j+1} n^1 + (-1)^{j+1} \binom{j+4}{j+2} n^0\right]$$

In a similar manner, we are able to derive formula for triple sums of powers, by summing up double sums of powers (1.12), thus

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[\binom{k-t+2}{j+1} - \binom{1-t}{j+1} k^1 - \binom{2-t}{j+2} k^0\right]$$

Note that $\sum_{k=1}^{n} k^1 = \Sigma^2 n^0$ and $\sum_{k=1}^{n} k^0 = \Sigma^1 n^0$ and $1 = \Sigma^0 n^0$. Therefore, by shifted hockey stick identity (1.4), we get

Proposition 1.14 (Triple sums of powers via Newton's formula)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+2}{j+1} - \binom{1-t}{j+1} \Sigma^2 n^0 - \binom{2-t}{j+2} \Sigma^1 n^0 - \binom{3-t}{j+3} \Sigma^0 n^0 \right]$$

By negated hockey stick identity (1.6) yields

Proposition 1.15 (Negated triple sums of powers via Newton's formula)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+3}{j+3} + (-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 \right.$$
$$\left. + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 \right]$$

For example, by setting $t = 4$ into triple sums of powers via Newton's formula (1.14) gives

$$\Sigma^3 n^0 = 1 \left(\tbinom{n-1}{3} + \tbinom{3}{1} \Sigma^2 n^0 - \tbinom{3}{2} \Sigma^1 n^0 + \tbinom{3}{3} \Sigma^0 n^0 \right)$$

$$\Sigma^3 n^1 = 4 \left(\tbinom{n-1}{3} + \tbinom{3}{1} \Sigma^2 n^0 - \tbinom{3}{2} \Sigma^1 n^0 + \tbinom{3}{3} \Sigma^0 n^0 \right)$$
$$+ 1 \left(\tbinom{n-1}{4} - \tbinom{4}{2} \Sigma^2 n^0 + \tbinom{4}{3} \Sigma^1 n^0 - \tbinom{4}{4} \Sigma^0 n^0 \right)$$

$$\Sigma^3 n^2 = 16 \left(\tbinom{n-1}{3} + \tbinom{3}{1} \Sigma^2 n^0 - \tbinom{3}{2} \Sigma^1 n^0 + \tbinom{3}{3} \Sigma^0 n^0 \right)$$
$$+ 9 \left(\tbinom{n-1}{4} - \tbinom{4}{2} \Sigma^2 n^0 + \tbinom{4}{3} \Sigma^1 n^0 - \tbinom{4}{4} \Sigma^0 n^0 \right)$$
$$+ 2 \left(\tbinom{n-1}{5} + \tbinom{5}{3} \Sigma^2 n^0 - \tbinom{5}{4} \Sigma^1 n^0 + \tbinom{5}{5} \Sigma^0 n^0 \right)$$

The coefficients $1, 4, 1, 16, 9, \ldots$ is the sequence A391633 in the OEIS [7]. In general,

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j 4^m \left[\binom{n-1}{j+3} + (-1)^j \binom{j+3}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+3}{j+2} \Sigma^1 n^0 \right.$$
$$\left. + (-1)^{j+2} \binom{j+3}{j+3} \Sigma^0 n^0 \right]$$

By repeated application of shifted hockey stick identity (1.4), and Newton's formula for powers (1.2), we get formula for multifold sums of powers

Theorem 1.16 (Multifold sums of powers via Newton's formula)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \Sigma^{r-s} n^0 \right]$$

By negated hockey stick identity (1.6), another variation of multifold sums of powers follows

Theorem 1.17 (Negated multifold sums of powers via Newton's formula)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right]$$

In its explicit form

$$\begin{aligned}\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \Bigg[\binom{n-t+r}{j+r} &+ (-1)^j \binom{j+t-1}{j+1} \Sigma^{r-1} n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^{r-2} n^0 \\ &+ (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^{r-3} n^0 + \cdots \\ &+ (-1)^{j+r-1} \binom{j+t-1}{j+r} \Sigma^0 n^0 \Bigg]\end{aligned}$$

We may observe that

Proposition 1.18 (Multifold sum of zero powers)**.** *For integers r and n*

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. By hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. □

Thus,

$$\Sigma^{r-s} n^0 = \binom{r-s+n-1}{r-s}$$

Which yields binomial variations of multifold sums of powers via Newton's formula (1.16)

Proposition 1.19 (Multifold sums of powers binomial form). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \binom{r-s+n-1}{r-s} \right]$$

By negated hockey stick identity (1.6) yields

Proposition 1.20 (Negated multifold sums of powers binomial form). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right]$$

By re-indexing the sum in s gives

Proposition 1.21 (Negated multifold sums of powers binomial form re-indexed). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right]$$

2. Sums of powers in Stirling numbers

Finite differences of powers are closely related to Stirling numbers of the second kind

Lemma 2.1 (Finite differences via Stirling numbers). *For non-negative integers j, m and an arbitrary integer t*

$$\Delta^j t^m = \sum_{k=0}^{m} \left\{ {m \atop j+k} \right\} \binom{t}{k} (j+k)!$$

Which implies variations for ordinary sums of powers, by utilizing ordinary sums of powers via Newton's formula (1.7) and finite differences identity (2.1)

Proposition 2.2 (Ordinary sums of powers via Stirling numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+1}{j+1} - \binom{1-t}{j+1}\right]\left\{{m \atop j+k}\right\}\binom{t}{k}(j+k)!$$

In addition, by negated ordinary sums of powers (1.8) and by finite differences (2.1)

Proposition 2.3 (Negated ordinary sums of powers via Stirling numbers). *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+1}{j+1} + (-1)^j\binom{j+t-1}{j+1}\right]\left\{{m \atop j+k}\right\}\binom{t}{k}(j+k)!$$

By setting $t = 0$ into ordinary sums of powers via Stirling numbers (2.2) yields well-known identity for sums of powers in terms of Stirling numbers

Corollary 2.4. *For non-negative integers n, m*

$$\Sigma^1 n^m = \sum_{j=0}^{m}\binom{n+1}{j+1}\left\{{m \atop j}\right\}j!$$

For multifold sums of powers (1.16) and (1.17) we have

Proposition 2.5 (Multifold sums of powers via Stirling numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r}\binom{s-t}{j+s}\Sigma^{r-s} n^0\right]\left\{{m \atop j+k}\right\}\binom{t}{k}(j+k)!$$

Proposition 2.6 (Negated multifold sums of powers via Stirling numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r}(-1)^{j+s-1}\binom{j+t-1}{j+s}\Sigma^{r-s} n^0\right]\left\{{m \atop j+k}\right\}\binom{t}{k}(j+k)!$$

The propositions above can be presented in a pure binomial form as well, by means of the identity (1.18): $\Sigma^{r-s} n^0 = \binom{r-s+n-1}{r-s}$. Therefore,

Proposition 2.7 (Multifold sums of powers Stirling-Binomial form). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \binom{r-s+n-1}{r-s} \right] \left\{ {m \atop j+k} \right\} \binom{t}{k} (j+k)!$$

Proposition 2.8 (Negated multifold sums of powers Stirling-Binomial form). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right] \left\{ {m \atop j+k} \right\} \binom{t}{k} (j+k)!$$

3. Sums of powers in Eulerian numbers

In addition, we are able to express multifold sums of powers via Eulerian numbers, by expressing the forward finite difference via the Worpitzky identity [9]

Lemma 3.1 (Worpitzky identity). *For non-negative integers t, m*

$$t^m = \sum_{k=0}^{m} \binom{t+k}{m} \left\langle {m \atop k} \right\rangle$$

where $\left\langle {n \atop k} \right\rangle$ are Eulerian numbers.

Thus,

Lemma 3.2 (Finite difference via Eulerian numbers). *For non-negative integers j, m and an arbitrary integer t*

$$\Delta^j t^m = \sum_{k=0}^{m} \binom{t+k}{m-j} \left\langle {m \atop k} \right\rangle$$

Hence, by multifold sums of powers via Newton's formula (1.16) and forward finite difference via Eulerian numbers (3.2)

Proposition 3.3 (Multifold sums of powers via Eulerian numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \Sigma^{r-s} n^0 \right] \binom{t+k}{m-j} \left\langle {m \atop k} \right\rangle$$

Therefore, by negated multifold sums of powers via Newton's formula (1.17) and forward finite difference via Eulerian numbers (3.2)

Proposition 3.4 (Negated multifold sums of powers via Eulerian numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right] \binom{t+k}{m-j} \left\langle {m \atop k} \right\rangle$$

Which implies variations for ordinary sums of powers via Newton's formula (1.7)

Proposition 3.5 (Negated ordinary sums of powers via Eulerian numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+1}{j+1} + (-1)^j \binom{j+t-1}{j+1} \right] \binom{t+k}{m-j} \left\langle {m \atop k} \right\rangle$$

By setting $t = 0$ into ordinary sums of powers via Eulerian numbers (3.5) yields a well-known identity for sums of powers in terms of Eulerian numbers

Corollary 3.6 (Sums of powers via Eulerian numbers in zero). *For non-negative integers n, m*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \binom{n+1}{j+1} \binom{k}{m-j} \left\langle {m \atop k} \right\rangle$$

By (1.19) and by (1.20) yield multifold sums of powers in Eulerian-Binomial form

Proposition 3.7 (Multifold sums of powers in Eulerian-Binomial form)**.** *For non negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r}\binom{s-t}{j+s}\binom{r-s+n-1}{r-s}\right]\binom{t+k}{m-j}\left\langle {m \atop k} \right\rangle$$

Proposition 3.8 (Negated multifold sums of powers in Eulerian-Binomial form)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r}(-1)^{j+s-1}\binom{j+t-1}{j+s}\binom{r-s+n-1}{r-s}\right]\binom{t+k}{m-j}\left\langle {m \atop k} \right\rangle$$

4. Backward difference form

Identity for multifold sums of powers via Newton's interpolation formula (1.16) can be expressed in terms of backward differences easily, because

Proposition 4.1. *For integers j, m, t*

$$\Delta^j t^m = \nabla^j (t+j)^m$$

Therefore, by multifold sums of powers via Newton's formula (1.16) and by by proposition (4.1)

Proposition 4.2 (Multifold sums of powers via backward differences)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m}\left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r}\binom{s-t}{j+s}\Sigma^{r-s} n^0\right]\nabla^j (t+j)^m$$

Thus, by negated multifold sums of powers via Newton's formula (1.17) and by proposition (4.1)

Proposition 4.3 (Negated multifold sums of powers via backward differences)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right] \nabla^j (t+j)^m$$

By (1.19) and by (1.20) yield multifold sums of powers in Eulerian-Binomial form

Proposition 4.4 (Multifold sums of powers backward binomial form)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \binom{r-s+n-1}{r-s} \right] \nabla^j (t+j)^m$$

Proposition 4.5 (Negated multifold sums of powers backward binomial form)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right] \nabla^j (t+j)^m$$

By re-indexing the sum in s gives

Proposition 4.6 (Negated multifold sums of powers backward binomial form re-indexed)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right] \nabla^j (t+j)^m$$

5. Central difference form

Identity for multifold sums of powers via Newton's interpolation formula (1.16) can be expressed in terms of central differences easily, because

Proposition 5.1. *For integers j, m, t*

$$\Delta^j t^m = \delta^j \left(t + \frac{j}{2} \right)^m$$

Thus, by negated multifold sums of powers via Newton's formula (1.17) and by proposition (5.1).

Proposition 5.2 (Multifold sums of powers via central differences). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right] \delta^j \left(t + \frac{j}{2}\right)^m$$

6. Future research

In this manuscript we focus on the idea to combine Newton's interpolation formula for forward differences (1.2) and hockey-stick family identities (1.3) and (1.4) to derive formulas for ordinary and multifold sums of powers seamlessly.

This particular idea is great, however it can be generalized even further. Thus, the main idea is to utilize an interpolation formula for power n^m in terms of *abstract difference operator* $D(n^m)$ and binomial coefficients $\binom{f(n)}{k}$.

We are not limited to linear difference operators Δ, ∇, δ, there is a complete theory on dynamic differential equations on time-scales, founded by Bohner and Peterson [10]. For example, Jackson [11] derivative is given by

$$D_q(f(x)) = \frac{f(qx) - f(x)}{qx - x}$$

Thus, finding interpolation formula for $f(x) = x^n$ in terms of $D_q(f(x))$ implies new formulas for sums of powers.

In general, let be an abstract interpolation formula for powers

$$n^m = \sum_{k} \binom{f(n)}{k} D(n^m, k)$$

Thus, the formula of sums of powers involves the abstract difference operator D evaluated at some point k and the hockey-stick family identity over the binomial coefficients $\binom{n}{k}$

$$\Sigma^1 n^m = \sum_k D(n^m, k) \sum_{j \leq n} \binom{f(j)}{k}$$

Similarly, for multifold sums of powers

$$\Sigma^r n^m = \sum_k D(n^m, k) \binom{f(n+r)}{k}$$

Many interpolation approaches involve rising factorials $x^{(n)}$, falling factorials $(x)_n$, or regular factorials $n!$, and thus can be expressed in terms of binomial coefficients, because

$$\frac{(x)_n}{n!} = \binom{x}{n}; \quad \frac{x^{(n)}}{n!} = \binom{x+n-1}{n}; \quad x^{[n]} = n\left(x + \frac{n}{2} - 1\right)_{n-1}$$

In particular, Donald Knuth provides the formula for multifold sums of odd powers [3] based on operator of central finite differences evaluated in zero, and Newton's interpolation formula for central differences [2]

Proposition 6.1 (Multifold sums of odd powers)**.** *For non-negative integers r, n, m*

$$\begin{aligned}\Sigma^r n^{2m-1} &= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} \frac{1}{2k} \delta^{2k} 0^{2m} \\ &= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} (2k-1)! T(2m, 2k)\end{aligned}$$

where $T(n,k)$ are the central factorial numbers of the second kind, see [12, section 58] and [13, formula (10a)]

$$T(n,k) = \frac{1}{k!}\delta^k 0^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n$$

Central factorial numbers of the second kind $T(n,k)$ were defined by Riordan in [14, ch. 6.5, formula (24)] by polynomial identity

Lemma 6.2 (Riordan power identity)**.**

$$n^m = \sum_{k=1}^{m} T(m,k) n^{[k]}$$

where $n^{[k]}$ *are central factorials* $n^{[k]} = n\prod_{j=0}^{k-1}\left(n+\frac{k}{2}-j\right)$.

The sequence A008957 in the OEIS [7] is the sequence of central factorial numbers of the second kind $T(2n, 2k)$.

As future research direction, the Knuth's formula (6.1) utilizes the operator of central finite differences evaluated in zero. Thus, it is worth to investigate the existence of sums of odd powers involving the central differences evaluated at an arbitrary integer point t, similar to multifold sums of powers via Newton's series for central differences.

Conclusions

In this manuscript we have derived formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers. In addition, in (6) we discuss the future research directions that may lead to a complete framework for sums of powers, by means of combining interpolation approaches, binomial coefficients and variations of the hockey-stick identity for binomial coefficients. The most important results of this manuscript are validated using Mathematica programs, see (7).

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References

[1] Petro Kolosov. Sums of powers via backward finite differences and newton's formula. https://doi.org/10.5281/zenodo.18118011, January 2026.

[2] Petro Kolosov. Sums of powers via central finite differences and newton's formula. https://doi.org/10.5281/zenodo.18096789, December 2025.

[3] Knuth, Donald E. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993. https://arxiv.org/abs/math/9207222.

[4] Newton, Isaac and Chittenden, N.W. *Newton's Principia: the mathematical principles of natural philosophy*. New-York, D. Adee, 1850. https://archive.org/details/bub_gb_KaAIAAAAIAAJ/page/466/mode/2up.

[5] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. https://archive.org/details/concrete-mathematics.

[6] Thomas J. Pfaff. Deriving a formula for sums of powers of integers. *Pi Mu Epsilon Journal*, 12(7):425–430, 2007. https://www.jstor.org/stable/24340705.

[7] Sloane, Neil J.A. and others. The on-line encyclopedia of integer sequences. https://oeis.org/, 2003. https://oeis.org/.

[8] Cereceda, José L. Sums of powers of integers and generalized Stirling numbers of the second kind. *arXiv preprint arXiv:2211.11648*, 2022. https://arxiv.org/abs/2211.11648.

[9] J. Worpitzky. Studien über die bernoullischen und eulerschen zahlen. *Journal für die reine und angewandte Mathematik*, 94:203–232, 1883.

[10] Bohner, Martin and Peterson, Allan. *Dynamic equations on time scales: An introduction with applications*. Springer Science & Business Media, 2001. https://web.mst.edu/~bohner/sample.pdf.

[11] Jackson, Frederick H. XI.—on q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, 46(2):253–281, 1909.

[12] Steffensen, Johan Frederik. *Interpolation*. Williams & Wilkins, 1927. https://www.amazon.com/-/de/Interpolation-Second-Dover-Books-Mathematics-ebook/dp/B00GHQVON8.

[13] L. Carlitz and John Riordan. The divided central differences of zero. *Canadian Journal of Mathematics*, 15:94–100, 1963.

[14] John Riordan. *Combinatorial identities*, volume 217. Wiley New York, 1968. https://www.amazon.com/-/de/Combinatorial-Identities-Probability-Mathematical-Statistics/dp/0471722758.

[15] Markus Scheuer. Reference request: identity in central factorial numbers. https://math.stackexchange.com/a/3665722/463487, 2020.

[16] Petro Kolosov. Mathematica programs for finite differences, stirling numbers, and sums of powers. https://github.com/kolosovpetro/NewtonsInterpolationFormulaAndSumsOfPowers/tree/main/mathematica, 2025. GitHub repository, Mathematica source files.

7. Mathematica programs

Use the *Mathematica* package [16] to validate the results

Mathematica Function	**Validates / Prints**
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateOrdinarySumsOfPowersViaNewtonsSeries[5]`	Validates Proposition (1.8)
`ValidateDoubleSumsOfPowersViaNewtonsSeries[5]`	Validates Proposition (1.13)
`ValidateMultifoldSumsOfPowersViaNewtonsSeries2[5]`	Validates Theorem (1.16)
`ValidateMultifoldSumsOfPowersViaNewtonsSeries[5]`	Validates Theorem (1.17)
`ValidateFiniteDifferenceViaStirlingNumbers[10]`	Validates Lemma (2.1)
`ValidateFiniteDifferenceViaEulerianNumbers[10]`	Validates Lemma (3.2)
`ValidateMultifoldSumsOfPowersViaCentralDifferences[5]`	Validates Proposition (5.2)
`ValidateMultifoldSumsOfPowersViaBackwardDifferences[5]`	Validates Proposition (4.3)
`ValidateMultifoldSumsOfPowersViaStirlingNumbers[5]`	Validates Proposition (2.6)
`ValidateMultifoldSumsOfPowersViaEulerianNumbers[5]`	Validates Proposition (3.4)
`ValidateMultifoldSumsOfPowersBinomialForm[5]`	Validates Proposition (1.20)
`ValidateMultifoldSumsOfPowersBinomialFormReindexed[5]`	Validates Proposition (1.21)
`ValidateMultifoldSumOfZeroPowers[10]`	Validates Proposition (1.18)

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