

MINIMAL GOLDBACH PAIRS IN PRIME AND TWIN-PRIME COUNTING

PETRO KOLOSOV

ABSTRACT. Goldbach's Conjecture assumes that every even integer $2N \geq 4$ can be written as the sum of two primes $2N = p_i + p_j$, where (p_i, p_j) is called a Goldbach pair. The minimal Goldbach pair is a pair (p_i, p_j) such that p_i is minimal and $p_j = 2N - p_i$ is also a prime. We define a function $F_{2N}(P)$ that counts the occurrences of $p_i = P$ in a set of minimal Goldbach pairs up to $2N$, where P is a fixed prime number. In particular, the function $F_{2N}(P)$ provides the following identities in terms of prime counting $\pi(2N)$, twin-prime counting $\pi_2(2N)$ and cousin prime counting $\pi_4(2N)$

$$\pi(2N) = F_{2N+3}(3) + 1$$

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$$

$$\pi_4(2N) = F_{2N}(5) - F_{2N}(7)$$

CONTENTS

1. Introduction	2
2. Discussion on Twin Primes	6
3. Conclusions	7
4. Acknowledgements	8
References	8

Date: April 22, 2025.

2010 *Mathematics Subject Classification.* 11N05, 11P32.

Key words and phrases. Goldbach conjecture, Goldbach pairs, Minimal Goldbach pairs, Primes, Twin primes .

Sources: <https://github.com/kolosovpetro/MinimalGoldbachPairsInPrimesCounting>

1. INTRODUCTION

This manuscript provides a comprehensive review of the work [1] done by Michel Yamagishi, extending it with additional results. The Goldbach conjecture asserts that every even integer $2N \geq 4$ is a sum of two primes

$$2N = p_i + p_j$$

where (p_i, p_j) is called a Goldbach pair.

A Goldbach pair is not unique for even integers greater than 6, which means that there can be multiple Goldbach pairs for even integer $2N \geq 8$. For example: $10 = 3 + 7$ and $10 = 5 + 5$ and $10 = 7 + 3$, where the Goldbach pairs are $(3, 7)$, $(5, 5)$, $(7, 3)$.

The minimal Goldbach pair is the pair with the smallest p_i among all goldbach pairs for even integer $2N$. For the even integer 10, we have three pairs: $(3, 7)$, $(5, 5)$, $(7, 3)$ and the minimal one is $(3, 7)$ because 3 is the smallest among all p_i values: 3, 5, 7. Consider the following minimal Goldbach pairs for even integer $2k$ within the range $6 \leq 2k \leq 50$

$$6 = 3 + 3,$$

$$24 = 5 + 19,$$

$$40 = 3 + 37,$$

$$8 = 3 + 5,$$

$$26 = 3 + 23,$$

$$42 = 5 + 37,$$

$$10 = 3 + 7,$$

$$28 = 5 + 23,$$

$$44 = 3 + 41,$$

$$12 = 5 + 7,$$

$$30 = 7 + 23,$$

$$46 = 3 + 43,$$

$$14 = 3 + 11,$$

$$32 = 3 + 29,$$

$$48 = 5 + 43,$$

$$16 = 3 + 13,$$

$$34 = 3 + 31,$$

$$50 = 3 + 47,$$

$$18 = 5 + 13,$$

$$36 = 5 + 31,$$

$$20 = 3 + 17,$$

$$38 = 7 + 31,$$

$$22 = 3 + 19,$$

We can notice that minimal Goldbach pairs with minimal $p_i = 3$ produce a sequence of odd prime numbers $p_j = 3, 5, 7, 11, 13, 17 \dots$ which is quite remarkable:

$$6 = 3 + 3, \quad 32 = 3 + 29,$$

$$8 = 3 + 5, \quad 34 = 3 + 31,$$

$$10 = 3 + 7, \quad 40 = 3 + 37,$$

$$14 = 3 + 11, \quad 44 = 3 + 41,$$

$$16 = 3 + 13, \quad 46 = 3 + 43,$$

$$20 = 3 + 17, \quad 50 = 3 + 47,$$

$$22 = 3 + 19,$$

$$26 = 3 + 23,$$

Another interesting observation is that by selecting the pairs with minimal $p_i = 5$ yields the sequence of primes p_j such that $p_j + 2$ is not a prime

$$12 = 5 + 7, \quad 36 = 5 + 31,$$

$$18 = 5 + 13, \quad 42 = 5 + 37,$$

$$24 = 5 + 19, \quad 48 = 5 + 43,$$

$$28 = 5 + 23,$$

To formalize and clarify our discussion, we define a few functions. Let $G_{\min}(2N)$ be a function that returns a set of minimal Goldbach pairs (p_i, p_j) having $\min p_i$ over the range $6 \leq 2k \leq 2N$

$$G_{\min}(2N) = \{(p_i, p_j) \mid p_i + p_j = 2k \mid 6 \leq 2k \leq 2N \mid \min p_i\}$$

For example,

$$G_{\min}(20) = \{(3, 3), (3, 5), (3, 7), (5, 7), (3, 11), (3, 13), (5, 13), (3, 17)\}$$

Let $W_{2N}(P)$ be a function that returns the set of elements p_j from $G_{\min}(2N)$ having $p_i = P$

$$W_{2N}(P) = \{p_j \mid (p_i, p_j) \in G_{\min}(2N) \text{ and } p_i = P\}$$

Then, the sequence of odd prime numbers [2] is given by $W_{2N}(3)$

$$\{3, 5, 7, 11, \dots, p \leq 2N - 3\} = W_{2N}(3)$$

Now we can easily count the number of primes within the interval $6 \leq 2k \leq 2N$ because $\pi(2N)$ is equal to the total number of elements inside the set $W_{2N}(3)$, which corresponds to the sequence [3]

$$\pi(2N) = F_{2N+3}(3) + 1$$

where $F_{2N}(3)$ is the function that counts the number of elements inside the set $W_{2N}(3)$. In general $F_{2N}(P) = |W_{2N}(P)|$.

Taking $P = 5$ in $W_{2N}(P)$ yields a sequence of primes p_j such that $p_j + 2$ is not a prime [4]

$$W_{2N}(5) = \{7, 13, 19, 23, 31, 37, 43, 47, 53, \dots, p \leq 2N - 5\}$$

Hence, by excluding the elements of the set $W_{2N}(5)$ from the set $W_{2N}(3)$ yields the sequence of lesser twin primes [5]

$$W_{2N}(3) \setminus W_{2N}(5) = \{3, 5, 11, 17, 29, 41, 59, 71, 101, 107, 137, \dots, p \leq 2N - 3\}$$

This implies that the number of twin primes in range $6 \leq 2k \leq 2N$ can be expressed in terms of $F_{2N}(P)$

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$$

For example,

$$\pi_2(10) = F_{10+3}(3) - F_{10+5}(5) = 2$$

$$\pi_2(100) = F_{100+3}(3) - F_{100+5}(5) = 8$$

$$\pi_2(1000) = F_{1000+3}(3) - F_{1000+5}(5) = 35$$

$$\pi_2(10000) = F_{10000+3}(3) - F_{10000+5}(5) = 205$$

$$\pi_2(100000) = F_{100000+3}(3) - F_{100000+5}(5) = 1224$$

$$\pi_2(1000000) = F_{1000000+3}(3) - F_{1000000+5}(5) = 8169$$

These results match the sequence [6].

In addition, the functions $W_{2N}(P)$ and $F_{2N}(P)$ provide a way to count cousin primes p such that $p + 4$ is a prime. We can observe this by excluding the elements of the set $W_{2N}(7)$ from the set $W_{2N}(5)$ which gives the sequence of cousin primes [7]

$$W_{2N}(5) \setminus W_{2N}(7) = \{7, 13, 19, 37, 43, 67, 79, 97, 103, 109, 127, 163, \dots, p \leq 2N - 5\}$$

Therefore, the number of cousin primes up to $2N$ can be calculated as

$$\pi_4(2N) = F_{2N}(5) - F_{2N}(7)$$

For instance,

$$\pi_4(10) = F_{10}(5) - F_{10}(7) = 0$$

$$\pi_4(100) = F_{100}(5) - F_{100}(7) = 7$$

$$\pi_4(1000) = F_{1000}(5) - F_{1000}(7) = 40$$

$$\pi_4(10000) = F_{10000}(5) - F_{10000}(7) = 202$$

$$\pi_4(100000) = F_{100000}(5) - F_{100000}(7) = 1215$$

$$\pi_4(1000000) = F_{1000000}(5) - F_{1000000}(7) = 8143$$

These results match the sequence [8].

Having $P = 7$ function $W_{2N}(P)$ yields the sequence of primes such that $p_j - p_i \geq 6$, where p_j is the next prime after p_i , see [9]

$$W_{2N}(7) = \{23, 31, 47, 53, 61, 73, 83, 89, 113, \dots, p \leq 2N - 7\}$$

This allows us to count the primes for which the next-prime gap at least 6: $\delta_6(2N) = F_{2N+7}(7)$.

2. DISCUSSION ON TWIN PRIMES

Twin prime conjecture asserts that there are infinitely many primes p such that $p + 2$ is also a prime. Previously, we have established the following relation in terms of the minimal Goldbach pairs counting function $W_{2N}(P)$. The set of twin primes is the set $W_{2N}(3)$ excluding the set $W_{2N}(5)$

$$\begin{aligned} \text{Lesser twin primes} &= \{3, 5, 11, 17, 29, 41, 59, 71, 101, 107, 137, \dots, p \leq 2N - 3\} \\ &= W_{2N}(3) \setminus W_{2N}(5) \end{aligned}$$

where $W_{2N}(3)$ gives the set of odd primes p , and $W_{2N}(5)$ gives the set of odd primes p such that $p + 2$ is not a prime. Thus, the number of twin primes up to integer $2N$ is

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$$

where $F_{2N}(P)$ counts the elements inside the set $W_{2N}(P)$. Therefore, the twin prime conjecture is identical to the assertion that

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5) = \infty, \quad 2N \rightarrow \infty$$

Thanks to Euclid, we know that there are infinitely many primes, thus there are infinitely many minimal Goldbach pairs $(p_i, p_j) = (3, p_j)$ such that $2N = p_i + p_j$, hence

$$\lim_{2N \rightarrow \infty} F_{2N+3}(3) = \infty$$

Consider the set $W_{2N}(5)$ which gives the sequence of odd primes p such that $p + 2$ is not a prime. Dirichlet's theorem shows that there are infinitely many primes of the form

$$p = 6k + 1$$

Therefore, there are infinitely many primes p such that $p + 2$ is not a prime

$$p + 2 = 6k + 3 = 3(2k + 1)$$

where $p + 2$ is a composite number divisible by 3. Therefore,

$$\lim_{2N \rightarrow \infty} F_{2N+5}(5) = \infty$$

Primes of the form $6k + 1$ are such that either $p + 2$ is composite and $p - 2$ is composite, or they are lesser twin primes. Therefore, the function $W_{2N}(5)$ generates the set of primes $p = 6k + 1$. By excluding this set from the set of odd prime numbers p , we obtain the sequence of lesser twin primes p_l

$$\begin{aligned} \text{Lesser twin primes} &= \{3, 5, 11, 17, 29, 41, 59, 71, 101, 107, 137, \dots, p \leq 2N - 3\} \\ &= W_{2N}(3) \setminus W_{2N}(5) \end{aligned}$$

Above relation holds because we exclude a set of primes p_c

- p_c such that $p_c + 2$ is composite
- p_c such that $p_c - 2$ is composite or lesser twin prime

from the set of odd primes p , so that the result is the set of lesser twin primes. Hence,

$$\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5) = \infty, \quad 2N \rightarrow \infty$$

3. CONCLUSIONS

Assuming Goldbach's Conjecture holds, we introduced a framework based on minimal Goldbach pairs to derive expressions for key prime-related functions. Specifically, we defined the function $F_{2N}(P)$ that counts occurrences of primes p_j in minimal Goldbach pairs (p_i, p_j) where $p_i = P$. Using this framework, we obtained

- The prime-counting function: $\pi(2N) = F_{2N+3}(3) + 1$
- The twin-prime p_2 counting function: $\pi_2(2N) = F_{2N+3}(3) - F_{2N+5}(5)$
- The twin-prime p_4 counting function: $\pi_4(2N) = F_{2N}(5) - F_{2N}(7)$
- The count of primes with next-prime gap at least 6: $\delta_6(2N) = F_{2N+7}(7)$

These identities establish a novel connection between Goldbach partitions and classical prime number theory. Computational examples confirm alignment with known integer sequences, reinforcing the potential of this approach for analytical and numerical exploration of prime distributions. All the results validated up to $N = 10^8$ with source code available on GitHub [10].

4. ACKNOWLEDGEMENTS

I am grateful to Michel Eduardo Beleza Yamagishi for incredible insights and ideas related to minimal Goldbach pairs and their distribution.

REFERENCES

- [1] Michel Yamagishi. Goldbach's conjecture and how to calculate $\pi(n)$ and $\pi_2(n)$, 2025. <http://dx.doi.org/10.13140/RG.2.2.22119.76963>.
- [2] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A065091: Odd primes. <https://oeis.org/A065091>, 2025. Accessed: 2025-04-18.
- [3] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A000720: Number of primes less than or equal to N. <https://oeis.org/A000720>, 2025. Accessed: 2025-04-18.
- [4] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A049591: Odd primes p such that $p+2$ is composite. <https://oeis.org/A049591>, 2025. Accessed: 2025-04-18.
- [5] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A001359: Lesser of twin primes. <https://oeis.org/A001359>, 2025. Accessed: 2025-04-18.
- [6] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A007508: Number of twin prime pairs below 10 to power N . <https://oeis.org/A007508>, 2025. Accessed: 2025-04-18.
- [7] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A023200: Primes p such that $p + 4$ is also prime. <https://oeis.org/A023200>, 2025. Accessed: 2025-04-18.

- [8] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A093737: Number of cousin primes (prime pairs differing by 4) less than or equal to n . <https://oeis.org/A093737>, 2025. Accessed: 2025-04-18.
- [9] The OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, A124582: Primes p such that $q-p$ greater of equal 6, where q is the next prime after p . <https://oeis.org/A124582>, 2025. Accessed: 2025-04-18.
- [10] Petro Kolosov. Minimal Goldbach Pairs (GitHub). <https://github.com/kolosovpetro/MinimalGoldbachPairs>, 2025. Accessed: 2025-04-18.

Version: 1.1.6-MATH-100.5+MATH-100.ca4c34d

SOFTWARE DEVELOPER, DEVOPS ENGINEER

Email address: kolosovp94@gmail.com

URL: <https://kolosovpetro.github.io>