SUMS OF POWERS VIA CENTRAL FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. In this manuscript we derive formula for multifold sums of powers using Newton's formula and central differences.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this manuscript we derive formula for multifold sums of powers using Newton's formula and central differences.

The idea to derive sums of powers using difference operator and Newton's series is quite generic, thus, formulas for sums of powers using forward and backward differences can be found in the works [1, 2].

We define the recurrence for multifold sums of powers introduced by Donald Knuth in [3], which we utilize throughout the paper.

$$\Sigma^0 \, n^m = n^m$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m$$

Consider the Newton's formula [4] in central differences

Proposition 1.1 (Newton's formula in central differences).

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k f(0)$$

where $\delta^k f(0) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} f\left(\frac{k}{2} - j\right)$ are central finite differences in zero, and $x^{[k]} = n\left(n + \frac{k}{2} - 1\right)\left(n + \frac{k}{2} - 2\right) \cdots \left(n - \frac{k}{2} + 1\right)$ are central factorials.

Lemma 1.2 (Central factorial).

$$n^{[k]} = n\left(n + \frac{k}{2} - 1\right)\left(n + \frac{k}{2} - 2\right) \cdots \left(n - \frac{k}{2} + 1\right) = n \prod_{j=1}^{k-1}\left(n + \frac{k}{2} - j\right)$$

We observe that central factorials are closely related to falling factorials $(x)_n = x(x - 1)(x - 2)(x - 3) \cdots (x - n + 1) = \prod_{k=0}^{n-1}(x - k)$. Therefore,

$$n^{[k]} = n\left(n + \frac{k}{2} - 1\right)_{k-1}$$

To derive formula for multifold sums of powers, we follow the strategy to express the Newton's formula (1.1) in terms of binomial coefficients, then to reach closed forms of column sum of binomial coefficients by means of hockey stick identity. Therefore,

Proposition 1.3. *For $k \geq 1$*

$$\frac{n^{[k]}}{k!} = \frac{n}{k!}\left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k(k-1)!}\left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k}\binom{n + \frac{k}{2} - 1}{k - 1}$$

Proof. The identity above is true because $\frac{(x)_n}{n!} = \binom{x}{n}$.   □

Which yields Newton's formula for powers, in terms of central differences. For positive integers $n \geq 1$ and $m \geq 1$

$$n^m = \sum_{k=1}^{m} \frac{n}{k} \binom{n + \frac{k}{2} - 1}{k - 1} \delta^k 0^m$$

Although based on Newton's interpolation series, the formula above begins the summation from $k = 1$ to avoid division by zero in $\frac{n}{k}$. It is a valid trick, because the central difference $\delta^k 0^n$ is zero for all $n \geq 1$ and $k = 0$.

By factoring out and simplifying the term n, we get

$$n^{m-1} = \sum_{k=1}^{m} \frac{1}{k} \binom{n + \frac{k}{2} - 1}{k - 1} \delta^k 0^m$$

We may observe that the operator of central finite difference $\delta^k 0^m$ requires the parity of its arguments m and k meaning that both m and k required to be: $m \bmod 2 = k \bmod 2$, such that finite differences $\delta^k 0^m$ are non-zero.

By setting $m \to 2m$ we get

$$n^{2m-1} = \sum_{k=1}^{2m} \frac{1}{k} \binom{n + \frac{k}{2} - 1}{k - 1} \delta^k 0^{2m}$$

Thus, the central differences $\delta^k 0^{2m}$ are zero for all odd k.

Since k runs over all integers in the range $0 \leq k \leq 2m$, we can omit odd values of k

$$n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n + k - 1}{2k - 1} \delta^{2k} 0^{2m}$$

Hence, formula for ordinary sums of odd powers is

Proposition 1.4 (Ordinary sums of powers)**.**

$$\Sigma^1 \, n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n+k}{2k} \delta^{2k} 0^{2m}$$

Proof. We have $\Sigma^1 \, n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \delta^{2k} 0^{2m} \sum_{j=1}^{n} \binom{j+k-1}{k-1}$.

By hockey stick identity $\sum_{j=1}^{n} \binom{j+k-1}{2k-1} = \binom{n+k}{2k}$, thus the statement follows. $\qquad\square$

Therefore,

Theorem 1.5 (Multifold sums of odd powers)**.**

$$\Sigma^r \, n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n+k-1+r}{2k-1+r} \delta^{2k} 0^{2m}.$$

Proof. We have $\Sigma^1 \, n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \delta^{2k} 0^{2m} \sum_{j=1}^{n} \binom{j+k-1}{2k-1}$.

By hockey stick identity $\sum_{j=1}^{n} \binom{j+k-1}{k-1} = \binom{n+k}{2k}$. By induction the claim follows. $\qquad\square$

It is quite interesting to notice that theorem (1.5) is a central difference form of the formula of sums of odd-powers given by Donald Knuth in *Johann Faulhaber and sums of powers*, see [3].

The reason is straightforward, instead of using Central factorial numbers of the second kind $T(n,k)$, the theorem (1.5) utilizes central differences explicitly, because

Lemma 1.6 (Central factorial numbers of the second kind)**.**

$$k! T(n,k) = \delta^k 0^n$$

Meaning that the Knuth's formula

Proposition 1.7 (Multifold sums of odd powers in central factorial numbers)**.**

$$\Sigma^r \, n^{2m-1} = \sum_{k=1}^{m} (2k-1)! \binom{n+k-1+r}{2k-1+r} T(2m, 2k).$$

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originates from Newton's interpolation formula in central differences (1.1).

The lemma (1.6) is well discussed in [5, 6, 7].

The non-zero central factorial numbers $T(2m, 2k)$ are registered as A008957 in the OEIS [8]. For example,

$$\Sigma^1 n^1 = \binom{n+1}{2}$$

$$\Sigma^1 n^3 = 6\binom{n+2}{4} + \binom{n+1}{2}$$

$$\Sigma^1 n^5 = 120\binom{n+3}{6} + 30\binom{n+2}{4} + \binom{n+1}{2}$$

$$\Sigma^1 n^7 = 5040\binom{n+4}{8} + 1680\binom{n+3}{6} + 126\binom{n+2}{4} + \binom{n+1}{2}$$

While multifold sums of odd powers are

$$\Sigma^r n^1 = \binom{n+1+r}{2+r}$$

$$\Sigma^r n^3 = 6\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r}$$

$$\Sigma^r n^5 = 120\binom{n+3+r}{6+r} + 30\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r}$$

$$\Sigma^r n^7 = 5040\binom{n+4+r}{8+r} + 1680\binom{n+3+r}{6+r} + 126\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r}$$

The coefficients $1, 6, 1, 120, 30, 1, \ldots$ is the sequence A303675 in the OEIS [8].

Conclusions

In this manuscript we derived formula for multifold sums of powers using Newton's formula and central differences. Additionally, we discussed the connection with the classical formulas for sums of powers [3].

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