NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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Abstract. In this manuscript we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind and Eulerian numbers.

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1. Introduction and main results

In this manuscript we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind and Eulerian numbers.

The idea to derive sums of powers using difference operator and Newton's series is quite generic, thus, formulas for sums of powers using backward and central differences can be found in the works [1, 2].

Allow us to start from the definition of multifold sums of powers. We utilize the recurrence proposed by Donald Knuth in his article *Johann Faulhaber and sums of powers*, see [3]

$$\Sigma^0 \, n^m = n^m$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m$$

Throughout the paper, we utilize the Newton's interpolation formula as stated below

Proposition 1.1. *(Newton's series around arbitrary point [4, Lemma V].)*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a)$$

where $\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j)$ *is* k*-degree forward finite difference of* f*.*

Which indeed holds, because

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$
$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$
$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

Proposition 1.2 (Newton's series for power)**.** *For non-negative integers m, n and an arbitrary integer t*

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m$$

Thus, for an arbitrary integer t, the ordinary sum of powers is

$$\Sigma^1 n^m = \sum_{k=1}^{n} \sum_{j=0}^{m} \binom{-t+k}{j} \Delta^j t^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$$

Proposition 1.3 (Segmented Hockey stick identity)**.** *For integers n, t and j*

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

Therefore,

Proposition 1.4 (Ordinary sums of powers via Newton's series)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right]$$

Proof. Ordinary sum of powers is given by $\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{-t+k}{j}$, where $\sum_{k=1}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1}$ by means of segmented hockey stick identity (1.3). $\square$

The special cases for $t = 0$ and $t = 1$ are widely known and appear in literature quite frequently. For $t = 0$ and $m = 3$ we have the famous identity

$$\Sigma^1 n^3 = 0\binom{n+1}{1} + 1\binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}$$

which was discussed in [5, p. 190] and in [6]. The coefficients $0, 1, 6, 6, 0, 1, 14, 36, 24, \ldots$ are given by the sequence A131689 in the OEIS [7]. The special cases for $t = 1$ and $m = 2, 3, 4, 5$ were discussed in [8]. For instance,

$$\Sigma^1 n^3 = 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4}$$

$$\Sigma^1 n^4 = 1\binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5}$$

The coefficients $1, 7, 12, 6, 1, 15, \ldots$ are given by the sequence A028246 in the OEIS [7]. Interestingly enough that the paper [8] gives the formula for sums of powers

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_r$$

where $\left\{ {k \atop j} \right\}_r$ are generalized Stirling numbers of the second kind. The formula above is identical to the proposition (1.4), which implies that finite differences can be expressed in terms of generalized Stirling numbers of the second kind, that is $\Delta^j t^m = j! \left\{ {m \atop j} \right\}_t$.

By considering the special cases of the proposition (1.4) for $t = 4$, we observe rather unexpected formulas for sums of powers, namely

$$\begin{aligned}
\Sigma^1 n^0 &= 1 \left(\binom{n-3}{1} + \binom{3}{1} \right) \\
\Sigma^1 n^1 &= 4 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 1 \left(\binom{n-3}{2} - \binom{4}{2} \right) \\
\Sigma^1 n^2 &= 16 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 9 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 2 \left(\binom{n-2}{3} + \binom{5}{3} \right) \\
\Sigma^1 n^3 &= 64 \left(\binom{n-3}{1} + \binom{3}{1} \right) + 61 \left(\binom{n-3}{2} - \binom{4}{2} \right) + 30 \left(\binom{n-3}{3} + \binom{5}{3} \right) \\
&\quad + 6 \left(\binom{n-3}{4} - \binom{6}{4} \right)
\end{aligned}$$

The coefficients $1, 4, 1, 16, 9, \ldots$ are given by the sequence A391633 in the OEIS [7]. In general,

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j 4^m \left[\binom{n-3}{j+1} + (-1)^j \binom{j+3}{j+1} \right].$$

To obtain the formula for double sum of powers, we apply the summation operator over the ordinary sum of powers again, thus

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \sum_{k=1}^{n} \binom{j+t-1}{j+1} + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

which yields

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1} \right]$$

Thus,

Proposition 1.5 (Double sums of powers via Newton's series)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}\right]$$

Proof. We have $\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} n + \sum_{k=1}^{n} \binom{k-t+1}{j+1}\right]$, where $\sum_{k=1}^{n} \binom{k-t+1}{j+1} = (-1)^{j+1}\binom{j+t-1}{j+2} n^0 + \binom{n-t+2}{j+2}$ by means of segmented hockey stick identity (1.3). ☐

For example, given $t = 5$, the double sums of powers are

$$\begin{aligned}
\Sigma^2 n^0 &= 1\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) \\
\Sigma^2 n^1 &= 5\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 1\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right) \\
\Sigma^2 n^2 &= 25\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 11\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right) \\
&\quad + 2\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right) \\
\Sigma^2 n^3 &= 125\left(\binom{n-3}{2} + \binom{4}{1}n - \binom{4}{2}\right) + 91\left(\binom{n-3}{3} - \binom{5}{2}n + \binom{5}{3}\right) \\
&\quad + 36\left(\binom{n-3}{4} + \binom{6}{3}n - \binom{6}{4}\right) + 6\left(\binom{n-3}{5} - \binom{7}{4}n + \binom{7}{5}\right)
\end{aligned}$$

The coefficients $1, 5, 1, 25, 11, 2, \ldots$ are given by the sequence A391635 in the OEIS [7]. In general,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j 5^m \left[\binom{n-3}{j+2} + (-1)^j \binom{j+4}{j+1} n^1 + (-1)^{j+1} \binom{j+4}{j+2} n^0\right]$$

Similarly, we obtain the formula for the triple sums of powers

Proposition 1.6 (Triple sums of powers via Newton's series)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \Bigg[(-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + \\ + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3} \Bigg]$$

Proof. By summing up the double powers sums, we get

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[(-1)^j \binom{j+t-1}{j+1} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} k^0 + \binom{k-t+2}{j+2} \right] \\ = \sum_{j=0}^{m} \Delta^j t^m \left[(-1)^j \binom{j+t-1}{j+1} \sum_{k=1}^{n} k^1 + (-1)^{j+1} \binom{j+t-1}{j+2} \sum_{k=1}^{n} k^0 + \sum_{k=1}^{n} \binom{k-t+2}{j+2} \right]$$

Note that $\sum_{k=1}^{n} k^1 = \Sigma^2 n^0$ and $\sum_{k=1}^{n} k^0 = \Sigma^1 n^0$. Thus,

$$\sum_{k=1}^{n} \binom{k-t+2}{j+2} = (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 + \binom{n-t+3}{j+3}$$

by segmented hockey stick identity (1.3). This completes the proof. $\square$

For example, given $t = 4$, the triple sums of powers are

$$\Sigma^3 n^0 = 1 \left(\binom{n-1}{3} + \binom{3}{1} \Sigma^2 n^0 - \binom{3}{2} \Sigma^1 n^0 + \binom{3}{3} \Sigma^0 n^0 \right)$$

$$\Sigma^3 n^1 = 4 \left(\binom{n-1}{3} + \binom{3}{1} \Sigma^2 n^0 - \binom{3}{2} \Sigma^1 n^0 + \binom{3}{3} \Sigma^0 n^0 \right) \\ + 1 \left(\binom{n-1}{4} - \binom{4}{2} \Sigma^2 n^0 + \binom{4}{3} \Sigma^1 n^0 - \binom{4}{4} \Sigma^0 n^0 \right)$$

$$\Sigma^3 n^2 = 16 \left(\binom{n-1}{3} + \binom{3}{1} \Sigma^2 n^0 - \binom{3}{2} \Sigma^1 n^0 + \binom{3}{3} \Sigma^0 n^0 \right) \\ + 9 \left(\binom{n-1}{4} - \binom{4}{2} \Sigma^2 n^0 + \binom{4}{3} \Sigma^1 n^0 - \binom{4}{4} \Sigma^0 n^0 \right) \\ + 2 \left(\binom{n-1}{5} + \binom{5}{3} \Sigma^2 n^0 - \binom{5}{4} \Sigma^1 n^0 + \binom{5}{5} \Sigma^0 n^0 \right)$$

In general,

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j 4^m \left[(-1)^j \binom{j+3}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+3}{j+2} \Sigma^1 n^0 + \right.$$
$$\left. + (-1)^{j+2} \binom{j+3}{j+3} \Sigma^0 n^0 + \binom{n-1}{j+3} \right]$$

Continuing similarly, we are able to derive the formula for multifold sums of powers, which is

Theorem 1.7 (Multifold sums of powers via Newton's series)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By Newton's series for power (1.2) and repeated applications of the segmented hockey stick identity (1.3). □

In its explicit form

Example 1.8 (Explicit expansion of the s–sum in the r-fold case)**.** *For non-negative integers r, n, m and an arbitrary integer t, the r-fold sum $\Sigma^r n^m$ can be written as*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \Big[(-1)^j \binom{j+t-1}{j+1} \Sigma^{r-1} n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^{r-2} n^0$$
$$+ (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^{r-3} n^0 + \cdots + (-1)^{j+r-1} \binom{j+t-1}{j+r} \Sigma^0 n^0$$
$$+ \binom{n-t+r}{j+r} \Big]$$

We may observe that

Proposition 1.9 (Multifold sum of zero powers)**.** *For integers r and n*

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. By hockey stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. □

Which yields the following binomial variations of the multifold sums of powers (1.7)

Proposition 1.10 (Multifold sums of powers binomial form)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right) + \binom{n-t+r}{j+r} \right]$$

Proposition 1.11 (Multifold sums of powers binomial form re-indexed)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right) + \binom{n-t+r}{j+r} \right]$$

Finite differences of powers are closely related to Stirling numbers of the second kind

Lemma 1.12 (Finite differences via Stirling numbers)**.** *For non-negative integers j, m and an arbitrary integer t*

$$\Delta^j t^m = \sum_{k=0}^{m} \binom{t}{k} \left\{ {m \atop j+k} \right\} (j+k)!$$

Which implies variations of the formulas for sums of powers

Proposition 1.13 (Ordinary sums of powers via Stirling numbers)**.** *For non-negative integers n, m and arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right] \binom{t}{k} \left\{ {m \atop j+k} \right\} (j+k)!$$

Proof. By ordinary sums of powers via Newton's series (1.4) and forward finite difference via Stirling numbers of the second kind (1.12). ☐

By setting $t = 0$ into (1.13) yields well-known identity for sums of powers in terms of Stirling numbers

Corollary 1.14. *For non-negative integers n, m*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \binom{n+1}{j+1} \left\{ {m \atop j} \right\} j!$$

In general,

Proposition 1.15 (Multifold sums of powers via Stirling numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right] \binom{t}{k} \left\{ {m \atop j+k} \right\} (j+k)!$$

Proof. By multifold sums of powers via Newton's series (1.7) and forward finite difference via Stirling numbers of the second kind (1.12). □

The proposition above can be presented in a pure binomial form as well, by means of the identity (1.9): $\Sigma^r n^0 = \binom{r+n-1}{r}$.

In addition, we are able to express multifold sums of powers via Eulerian numbers, by expressing the forward finite difference via the Worpitzky identity [9]

Lemma 1.16 (Worpitzky identity). *For non-negative integers t, m*

$$t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k}{m}$$

where $\left\langle {n \atop k} \right\rangle$ are Eulerian numbers. Thus,

Lemma 1.17 (Finite difference via Eulerian numbers). *For non-negative integers j, m and an arbitrary integer t*

$$\Delta^j t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k}{m-j}$$

Therefore,

Proposition 1.18 (Multifold sums of powers via Eulerian numbers). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right] \left\langle {m \atop k} \right\rangle \binom{t+k}{m-j}$$

Proof. By multifold sums of powers via Newton's series (1.7) and forward finite difference via Eulerian numbers (1.17). □

Which implies variations of the formulas for sums of powers

Proposition 1.19. *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[(-1)^j \binom{j+t-1}{j+1} + \binom{n-t+1}{j+1} \right] \left\langle {m \atop k} \right\rangle \binom{t+k}{m-j}$$

By setting $t = 0$ into (1.19) yields a well-known identity for sums of powers in terms of Eulerian numbers

Corollary 1.20 (Sums of powers via Eulerian numbers). *For non-negative integers n, m*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \binom{n+1}{j+1} \binom{k}{m-j} \left\langle {m \atop k} \right\rangle$$

2. Backward difference form

The formula for multifold sums of powers via Newton's series (1.7) can be altered to be in terms of backward differences easily, because

Proposition 2.1. *For integers j, m, t*

$$\Delta^j t^m = \nabla^j (t+j)^m$$

Thus,

Proposition 2.2 (Multifold sums of powers via backward differences). *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r n^m = \sum_{j=0}^{m} \nabla^j (t+j)^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right) + \binom{n-t+r}{j+r} \right]$$

Proof. By multifold sums of powers via Newton's series (1.7) and by proposition (2.1). □

3. Central difference form

The formula for multifold sums of powers via Newton's series (1.7) can be altered to be in terms of central differences easily, because

Proposition 3.1. *For integers j, m, t*

$$\Delta^j t^m = \delta^j \left(t + \frac{j}{2}\right)^m$$

Thus,

Proposition 3.2 (Multifold sums of powers via central differences)**.** *For non-negative integers r, n, m and an arbitrary integer t*

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \delta^j \left(t + \frac{j}{2}\right)^m \left[\left(\sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s}\, n^0\right) + \binom{n-t+r}{j+r}\right]$$

Proof. By multifold sums of powers via Newton's series (1.7) and by proposition (3.1). $\square$

4. Future research

In this manuscript we focus on the idea to combine Newton's interpolation formula for forward differences (1.2) and the hockey-stick family identities (e.g (1.3)) to express the sums of powers seamlessly.

This particular idea is great, however it can be generalized even further. Thus, the main idea is to utilize an interpolation formula for power n^m in terms of *abstract difference operator* $D(n^m)$ and binomial coefficients $\binom{f(n)}{k}$.

We are not limited to linear difference operators Δ, ∇, δ, there is a complete theory on dynamic differential equations on time-scales, founded by Bohner and Peterson [10]. For example, Jackson [11] derivative is given by

$$D_q(f(x)) = \frac{f(qx) - f(x)}{qx - x}$$

Thus, finding interpolation formula for $f(x) = x^n$ in terms of $D_q(f(x))$ implies new formulas for sums of powers.

In general, let be an abstract interpolation formula for powers

$$n^m = \sum_{k} \binom{f(n)}{k} D(n^m, k)$$

Thus, the formula of sums of powers involves the abstract difference operator D evaluated at some point k and the hockey-stick family identity over the binomial coefficients $\binom{n}{k}$

$$\Sigma^1 n^m = \sum_k D(n^m, k) \sum_{j \leq n} \binom{f(j)}{k}$$

Similarly, for multifold sums of powers

$$\Sigma^r n^m = \sum_k D(n^m, k) \binom{f(n+r)}{k}$$

Many interpolation approaches involve rising factorials $x_{(n)}$, falling factorials $(x)_n$, or regular factorials $n!$, and thus can be expressed in terms of binomial coefficients, because

$$\frac{(x)_n}{n!} = \binom{x}{n}; \quad \frac{x_{(n)}}{n!} = \binom{x+n-1}{n}.$$

In particular, Donald Knuth provides the formula for multifold sums of odd powers [3] based on operator of central finite differences evaluated in zero, and Newton's interpolation formula for central differences [2]

Proposition 4.1 (Multifold sums of odd powers)**.**

$$\begin{aligned}
\Sigma^r n^{2m-1} &= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} \frac{1}{2k} \delta^{2k} 0^{2m} \\
&= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} (2k-1)! T(2m, 2k)
\end{aligned}$$

where $T(n,k)$ are the central factorial numbers of the second kind, see [12, section 58] and [13, formula (10a)]

$$T(n,k) = \frac{1}{k!} \delta^k 0^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n$$

Central factorial numbers of the second kind $T(n,k)$ were defined by Riordan in [14, ch. 6.5, formula (24)] by polynomial identity

Lemma 4.2 (Riordan power identity)**.**

$$n^m = \sum_{k=1}^{m} T(m,k) n^{[k]}$$

where $n^{[k]}$ are central factorials $n^{[k]} = n\prod_{j=0}^{k-1}\left(n+\frac{k}{2}-j\right)$.

The sequence A008957 in the OEIS [7] provides non-zero central factorial numbers of the second kind $T(2n,2k)$.

As future research direction, the Knuth's formula (4.1) utilizes the operator of central finite differences evaluated in zero. Thus, it is worth to investigate the existence of sums of odd powers involving the central differences evaluated at an arbitrary integer point t, similar to multifold sums of powers via Newton's series for central differences.

5. Conclusions

In this manuscript we have derived formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers. In addition, in (4) we discuss the future research directions that may lead to a complete framework for sums of powers, by means of combining interpolation approaches, binomial coefficients and variations of the hockey-stick identity for binomial coefficients. The most important results of this manuscript are validated using Mathematica programs, see (8).

6. Acknowledgements

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References

[1] Petro Kolosov. Sums of powers via backward finite differences and newton's formula, January 2026. https://doi.org/10.5281/zenodo.18118011.

[2] Petro Kolosov. Sums of powers via central finite differences and newton's formula, December 2025. https://doi.org/10.5281/zenodo.18096789.

[3] Knuth, Donald E. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993. https://arxiv.org/abs/math/9207222.

[4] Newton, Isaac and Chittenden, N.W. *Newton's Principia: the mathematical principles of natural philosophy*. New-York, D. Adee, 1850. https://archive.org/details/bub_gb_KaAIAAAAIAAJ/page/466/mode/2up.

[5] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. https://archive.org/details/concrete-mathematics.

[6] Thomas J. Pfaff. Deriving a formula for sums of powers of integers. *Pi Mu Epsilon Journal*, 12(7):425–430, 2007. https://www.jstor.org/stable/24340705.

[7] Sloane, Neil J.A. and others. The on-line encyclopedia of integer sequences, 2003. https://oeis.org/.

[8] Cereceda, José L. Sums of powers of integers and generalized Stirling numbers of the second kind. *arXiv preprint arXiv:2211.11648*, 2022. https://arxiv.org/abs/2211.11648.

[9] J. Worpitzky. Studien über die bernoullischen und eulerschen zahlen. *Journal für die reine und angewandte Mathematik*, 94:203–232, 1883. http://eudml.org/doc/148532.

[10] Bohner, Martin and Peterson, Allan. *Dynamic equations on time scales: An introduction with applications*. Springer Science & Business Media, 2001. https://web.mst.edu/~bohner/sample.pdf.

[11] Jackson, Frederick H. XI.—on q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, 46(2):253–281, 1909.

[12] Steffensen, Johan Frederik. *Interpolation*. Williams & Wilkins, 1927. https://www.amazon.com/-/de/Interpolation-Second-Dover-Books-Mathematics-ebook/dp/B00GHQVON8.

[13] L. Carlitz and John Riordan. The divided central differences of zero. *Canadian Journal of Mathematics*, 15:94–100, 1963. https://doi.org/10.4153/CJM-1963-010-8.

[14] John Riordan. *Combinatorial identities*, volume 217. Wiley New York, 1968. https://www.amazon.com/-/de/Combinatorial-Identities-Probability-Mathematical-Statistics/dp/0471722758.

[15] Markus Scheuer. Reference request: identity in central factorial numbers, 2020. https://math.stackexchange.com/a/3665722/463487.

[16] Petro Kolosov. Mathematica programs for finite differences, stirling numbers, and sums of powers. https://github.com/kolosovpetro/NewtonsInterpolationFormulaAndSumsOfPowers/tree/main/mathematica, 2025. GitHub repository, Mathematica source files.

Version: `1.4.0+main.4541448`

Sources: github.com/kolosovpetro/NewtonsInterpolationFormulaAndSumsOfPowers

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7. Proof of Segmented hockey stick identity

First we split the sum $\sum_{k=0}^{n}\binom{-t+k}{j}$ into two sub-sums so that we discuss them separately

$$\sum_{k=0}^{n}\binom{-t+k}{j} = \sum_{k=0}^{t-1}\binom{-t+k}{j} + \sum_{k=t}^{n}\binom{-t+k}{j}$$

We assume that the two sums above run over the partition $\{0, 1, 2, \cdots, t, \cdots, n\}$, with $t < n$. Considering the sum $\sum_{k=0}^{t-1}\binom{-t+k}{j}$ we notice that

$$\sum_{k=0}^{t-1}\binom{-t+k}{j} = \binom{-t}{j} + \binom{-t+1}{j} + \binom{-t+2}{j} + \cdots + \\ + \binom{-t+t-2}{j} + \binom{-t+t-1}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j} = \sum_{k=1}^{t}\binom{-k}{j} = \sum_{k=0}^{t-1}\binom{-k-1}{j}$$

By using the identity $\binom{-k}{j} = (-1)^j\binom{j+k-1}{j}$, we obtain

$$\binom{-k-1}{j} = \binom{-(k+1)}{j} = (-1)^j\binom{j+k}{j}$$

Thus

$$\sum_{k=0}^{t-1}\binom{-t+k}{j} = (-1)^j\sum_{k=0}^{t-1}\binom{j+k}{j} = (-1)^j\binom{j+t}{j+1}$$

by the hockey-stick identity $\sum_{k=0}^{t}\binom{j+k}{j} = \binom{j+t+1}{j+1}$.

Considering the sum $\sum_{k=t}^{n}\binom{-t+k}{j}$ we notice that

$$\sum_{k=t}^{n}\binom{-t+k}{j} = \sum_{k=0}^{n-t}\binom{k}{j}$$

Thus

$$\sum_{k=t}^{n} \binom{-t+k}{j} = \sum_{k=0}^{n-t} \binom{k}{j} = \binom{n-t+1}{j+1}$$

again by the hockey-stick identity $\sum_{k=0}^{t} \binom{j+k}{j} = \binom{j+t+1}{j+1}$. Combining the two parts, we obtain

$$\sum_{k=0}^{n} \binom{-t+k}{j} = (-1)^j \binom{j+t}{j+1} + \binom{n-t+1}{j+1}$$

This completes the proof.

8. Mathematica programs

Use the *Mathematica* package [16] to validate the results

Mathematica Function	**Validates / Prints**
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateMultifoldSumsOfPowersViaNewtonsSeries[r]`	Validates Theorem (1.7)
`ValidateFiniteDifferenceViaStirlingNumbers[t]`	Validates Lemma (1.12)
`ValidateFiniteDifferenceViaEulerianNumbers[t]`	Validates Lemma (1.17)
`ValidateMultifoldSumsOfPowersViaCentralDifferences[r]`	Validates Proposition (3.2)
`ValidateMultifoldSumsOfPowersViaBackwardDifferences[r]`	Validates Proposition (2.2)
`ValidateMultifoldSumsOfPowersViaStirlingNumbers[r]`	Validates Proposition (1.15)
`ValidateMultifoldSumsOfPowersViaEulerianNumbers[r]`	Validates Proposition (1.18)
`ValidateMultifoldSumsOfPowersBinomialForm[r]`	Validates Proposition (1.10)
`ValidateMultifoldSumsOfPowersBinomialFormReindexed[r]`	Validates Proposition (1.11)

DevOps Engineer

URL: https://kolosovpetro.github.io