

# DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

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ABSTRACT. <https://mathoverflow.net/a/297916/113033>

## 1. INTRODUCTION

Assuming that following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^m \sum_{k=1}^n \mathbf{A}_{m,r} k^r (n-k)^r \quad (1)$$

Our main goal is to identify the set of coefficients  $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$  such that identity above is true.

Although, the recurrence relation is already given at [1], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the  $\mathbf{A}_{m,r}$  recursively, taking the base case  $\mathbf{A}_{m,m}$  and then evaluating the next coefficient  $\mathbf{A}_{m,m-1}$  by using backtracking, continuing similarly up to  $\mathbf{A}_{m,0}$ .

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By applying Binomial theorem  $(n - k)^r = \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} k^t$  and Faulhaber's formula  $\sum_{k=1}^n k^p = \left[ \frac{1}{p+1} \sum_j \binom{p+1}{j} B_j n^{p+1-j} \right] - B_{p+1}$ , we get

$$\begin{aligned}
\sum_{k=1}^n k^r (n - k)^r &= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^n k^{t+r} \\
&= \sum_{t=0}^r (-1)^t \binom{r}{t} n^{r-t} \left[ \frac{1}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{t+r+1-j} - B_{t+r+1} \right] \\
&= \sum_{t=0}^r \binom{r}{t} \left[ \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} - B_{t+r+1} n^{r-t} \right] \\
&= \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} \sum_j \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \\
&= \left[ \sum_{j,t} \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} B_j n^{2r+1-j} \right] - \left[ \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]
\end{aligned}$$

Rearranging terms yields

$$\left[ \sum_j B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] - \left[ \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right] \quad (2)$$

We can notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1) \binom{2r}{r}} & \text{if } j = 0 \\ \frac{(-1)^r}{j} \binom{r}{2r-j+1} & \text{if } j > 0 \end{cases} \quad (3)$$

An elegant proof of the binomial identity (3) is presented in [2].

In particular, equation (3) is zero for  $0 < t \leq j$ . In order to apply (3), we have to move  $j = 0$  out of summation in (2) to avoid division by zero in  $\frac{(-1)^r}{j}$ , which yields

$$\begin{aligned}
\sum_{k=1}^n k^r (n - k)^r &= \frac{1}{(2r+1) \binom{2r}{r}} n^{2r+1} + \left[ \sum_{j \geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] \\
&\quad - \left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]
\end{aligned}$$

Now we do not care about division by zero in  $\frac{(-1)^r}{j}$  so that simplifying above equation by using (3) yields

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[ \sum_{j \geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]}_{(\star)} \\ &\quad - \underbrace{\left[ \sum_{t=0}^r \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)} \end{aligned}$$

Hence, introducing  $\ell = 2r - j + 1$  to  $(\star)$  and  $\ell = r - t$  to  $(\diamond)$  we collapse the common terms across two sums

$$\begin{aligned} \sum_{k=1}^n k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[ \sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[ \sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Assuming that  $\mathbf{A}_{m,r}$  is defined by (1), we obtain the following relation for polynomials in  $n$

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r, \text{ odd } \ell} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd  $\ell$  by  $k$  we get

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r, k} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1} \quad (4)$$

Taking the coefficient of  $n^{2m+1}$  we get

$$\mathbf{A}_{m,m} = (2m+1) \binom{2m}{m} \quad (5)$$

because  $\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1$ .

Taking the coefficient of  $n^{2d+1}$  for an integer  $d$  in the range  $\frac{m}{2} \leq d < m$ , we get

$$\mathbf{A}_{m,d} = 0 \quad (6)$$

because we focus on sum  $2 \sum_{r,k} \mathbf{A}_{m,r} \frac{(-1)^r}{2^{r-2k}} \binom{r}{2k+1} B_{2r-2k} n^{2k+1}$ , in particular on  $n^{2k+1}$  and binomial coefficient  $\binom{r}{2k+1}$ . For instance, if we have to get coefficient of  $n^{2d+1}$  in range  $\frac{m}{2} \leq d < m$ , we set  $d = m-1$ , thus we have to get coefficient of  $m-1$  in  $2 \sum_{r,k} \mathbf{A}_{m,r} \frac{(-1)^r}{2^{r-2k}} \binom{r}{2k+1} B_{2r-2k} n^{2k+1}$ . Therefore, we set  $k = m-1$  and  $r = m-1$  which leads that  $\binom{r}{2k+1} = \binom{m-1}{2m-1} = 0$ , so that  $\mathbf{A}_{m,m-1} \frac{1}{(2m-1) \binom{2m-2}{m-1}} n^{2m-1} = 0$ . Same applies for every  $d$  in the range  $\frac{m}{2} \leq d < m$ , because  $r = \frac{m}{2}$  and  $k = \frac{m}{2}$  means that  $\binom{r}{2k+1} = \binom{\frac{m}{2}}{m+1} = 0$ .

To summarize, the value of  $k$  should be in range  $k \leq \frac{d-1}{2}$  so that binomial coefficient  $\binom{d}{2k+1}$  is non-zero.

Taking the coefficient of  $n^{2d+1}$  for  $d$  in the range  $\frac{m}{4} \leq d < \frac{m}{2}$  we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1) \binom{2d}{d}} + 2(2m+1) \binom{2m}{m} \binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0 \quad (7)$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can compute  $\mathbf{A}_{m,r}$  for each integer  $r$  in range  $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$ , iterating consecutively over  $s = 1, 2, \dots$  by using previously determined values of  $\mathbf{A}_{m,d}$  as follows

$$\mathbf{A}_{m,r} = (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient  $\mathbf{A}_{m,r}$

**Definition 1.1.** (*Definition of coefficient  $\mathbf{A}_{m,r}$ .*)

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1) \binom{2r}{r} & \text{if } r = m \\ (2r+1) \binom{2r}{r} \sum_{d \geq 2r+1}^m \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \quad (8)$$

where  $B_t$  are Bernoulli numbers [3]. It is assumed that  $B_1 = \frac{1}{2}$ .

For example,

$m/r$	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

**Table 1.** Coefficients  $\mathbf{A}_{m,r}$ . See OEIS sequences [4, 5].

Properties of the coefficients  $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$  for  $m < 0$  and  $r > m$
- $\mathbf{A}_{m,r} = 0$  for  $r < 0$
- $\mathbf{A}_{m,r} = 0$  for  $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$  for  $m \geq 0$
- $\mathbf{A}_{m,r}$  are integers for  $m \leq 11$
- Row sums:  $\sum_{r=0}^m \mathbf{A}_{m,r} = 2^{2m+1} - 1$

## 2. QUESTIONS

**Question 2.1.** Although, a proof of combinatorial identity (3) is already present, it is good to point out literature or more context on it. Reference to a book or article with deeper discussion.

**Question 2.2.** I have struggle to understand the equation (5), it takes the coefficient of  $n^{2m+1}$  meaning that we substitute  $r = m$  into (4) evaluating it, if I understand it properly. So that coefficient of  $n^{2m+1}$  is

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} n^{2m+1} + 2 \sum_k \mathbf{A}_{m,m} \frac{(-1)^m}{2m-2k} \binom{m}{2k+1} B_{2m-2k} n^{2k+1} = 1$$

It implies that coefficient of  $n^{2m+1}$  in following sum is zero

$$2 \sum_k \mathbf{A}_{m,m} \frac{(-1)^m}{2m-2k} \binom{m}{2k+1} B_{2m-2k} n^{2k+1} = 0$$

So that

$$\mathbf{A}_{m,m} \frac{1}{(2m+1) \binom{2m}{m}} = 1; \quad \mathbf{A}_{m,m} = (2m+1) \binom{2m}{m}$$

Which is indeed true because  $\binom{m}{2k+1} = 0$  as  $k = m$ .

**Question 2.3.** Almost the same problem with equation (6), taking the coefficient of  $n^{2d+1}$  for an integer  $d$  in the range  $\frac{m}{2} \leq d < m$ , we get

$$\mathbf{A}_{m,d} = 0$$

Let be  $r = d$  in (4)

$$\mathbf{A}_{m,d} \frac{1}{(2d+1) \binom{2d}{d}} n^{2d+1} + 2 \sum_k \mathbf{A}_{m,d} \frac{(-1)^d}{2d-2k} \binom{d}{2k+1} B_{2d-2k} n^{2k+1} = 0$$

Let be  $d = m - 1$  then again same principle

$$2 \sum_k \mathbf{A}_{m,d} \frac{(-1)^d}{2d-2k} \binom{d}{2k+1} B_{2d-2k} n^{2k+1} = 0$$

because  $\binom{m-1}{2k+1} = 0$  as  $k = m - 1$ .

To summarize, the value of  $k$  should be in range  $k \leq \frac{d-1}{2}$  so that binomial coefficient  $\binom{d}{2k+1}$  is non-zero.

## REFERENCES

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