

# Formulas for Sums of Powers

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- 1 Formulas for Sums of Powers
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Let be the recurrence for R-fold sums of powers, see [1]

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	2						
3	0	1	6	6					
4	0	1	14	36	24				
5	0	1	30	150	240	120			
6	0	1	62	540	1560	1800	720		
7	0	1	126	1806	8400	16800	15120	5040	
8	0	1	254	5796	40824	126000	191520	141120	40320

Table: Triangle  $T(0, n, k)$ . Sequence [A131689](#).

$$n^0 = 1 \binom{n}{0}$$

$$n^1 = 0 \binom{n}{0} + 1 \binom{n}{1}$$

$$n^2 = 0 \binom{n}{0} + 1 \binom{n}{1} + 2 \binom{n}{2}$$

$$n^3 = 0 \binom{n}{0} + 1 \binom{n}{1} + 6 \binom{n}{2} + 6 \binom{n}{3}$$

$$n^4 = 0 \binom{n}{0} + 1 \binom{n}{1} + 14 \binom{n}{2} + 36 \binom{n}{3} + 24 \binom{n}{4}$$

$$n^5 = 0 \binom{n}{0} + 1 \binom{n}{1} + 30 \binom{n}{2} + 150 \binom{n}{3} + 240 \binom{n}{4} + 120 \binom{n}{5}$$

$$\Sigma^1 n^0 = 1 \left( \binom{n+1}{1} + \binom{-1}{1} \right)$$

$$\Sigma^1 n^1 = 0 \left( \binom{n+1}{1} + \binom{-1}{1} \right) + 1 \left( \binom{n+1}{2} - \binom{0}{2} \right)$$

$$\Sigma^1 n^2 = 0 \left( \binom{n+1}{1} + \binom{-1}{1} \right) + 1 \left( \binom{n+1}{2} - \binom{0}{2} \right) + 2 \left( \binom{n+1}{3} + \binom{1}{3} \right)$$

$$\begin{aligned} \Sigma^1 n^3 = & 0 \left( \binom{n+1}{1} + \binom{-1}{1} \right) + 1 \left( \binom{n+1}{2} - \binom{0}{2} \right) + 6 \left( \binom{n+1}{3} + \binom{1}{3} \right) \\ & + 6 \left( \binom{n+1}{4} - \binom{2}{4} \right) \end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^m T(0, n, k) \left[ \binom{n+1}{j+1} + (-1)^j \binom{j-1}{j+1} \right]$$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	3	2						
3	1	7	12	6					
4	1	15	50	60	24				
5	1	31	180	390	360	120			
6	1	63	602	2100	3360	2520	720		
7	1	127	1932	10206	25200	31920	20160	5040	
8	1	255	6050	46620	166824	317520	332640	181440	40320

Table: Triangle  $T(1, n, k)$ . Sequence [A028246](#).

$$n^0 = 1 \binom{n-1}{0}$$

$$n^1 = 1 \binom{n-1}{0} + 1 \binom{n-1}{1}$$

$$n^2 = 1 \binom{n-1}{0} + 3 \binom{n-1}{1} + 2 \binom{n-1}{2}$$

$$n^3 = 1 \binom{n-1}{0} + 7 \binom{n-1}{1} + 12 \binom{n-1}{2} + 6 \binom{n-1}{3}$$

$$n^4 = 1 \binom{n-1}{0} + 15 \binom{n-1}{1} + 50 \binom{n-1}{2} + 60 \binom{n-1}{3} + 24 \binom{n-1}{4}$$



$$\Sigma^1 n^0 = 1 \left( \binom{n}{1} + \binom{0}{1} \right)$$

$$\Sigma^1 n^1 = 1 \left( \binom{n}{1} + \binom{0}{1} \right) + 1 \left( \binom{n}{2} - \binom{1}{2} \right)$$

$$\Sigma^1 n^2 = 1 \left( \binom{n}{1} + \binom{0}{1} \right) + 3 \left( \binom{n}{2} - \binom{1}{2} \right) + 2 \left( \binom{n}{3} + \binom{2}{3} \right)$$

$$\Sigma^1 n^3 = 1 \left( \binom{n}{1} + \binom{0}{1} \right) + 7 \left( \binom{n}{2} - \binom{1}{2} \right) + 12 \left( \binom{n}{3} + \binom{2}{3} \right) + 6 \left( \binom{n}{4} - \binom{3}{4} \right)$$

$$\Sigma^1 n^m = \sum_{j=0}^m T(1, m, j) \left[ \binom{n}{j+1} + (-1)^j \binom{j}{j+1} \right]$$

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	2	1						
2	4	5	2					
3	8	19	18	6				
4	16	65	110	84	24			
5	32	211	570	750	480	120		
6	64	665	2702	5460	5880	3240	720	
7	128	2059	12138	35406	57120	52080	25200	5040

Table: Triangle  $T(2, n, k)$ . Sequence [A038719](#).

$$n^0 = 1 \binom{n-2}{0}$$

$$n^1 = 2 \binom{n-2}{0} + 1 \binom{n-2}{1}$$

$$n^2 = 4 \binom{n-2}{0} + 5 \binom{n-2}{1} + 2 \binom{n-2}{2}$$

$$n^3 = 8 \binom{n-2}{0} + 19 \binom{n-2}{1} + 18 \binom{n-2}{2} + 6 \binom{n-2}{3}$$

$$n^4 = 16 \binom{n-2}{0} + 65 \binom{n-2}{1} + 110 \binom{n-2}{2} + 84 \binom{n-2}{3} + 24 \binom{n-2}{4}$$

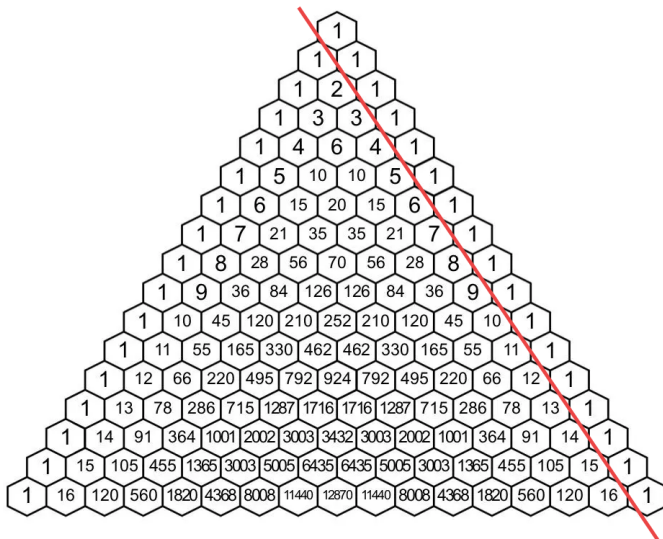
$$\Sigma^1 n^0 = 1 \left( \binom{n-1}{1} + 1 \right)$$

$$\Sigma^1 n^1 = 2 \left( \binom{n-1}{1} + 1 \right) + 1 \left( \binom{n-1}{2} - 1 \right)$$

$$\Sigma^1 n^2 = 4 \left( \binom{n-1}{1} + 1 \right) + 5 \left( \binom{n-1}{2} - 1 \right) + 2 \left( \binom{n-1}{3} + 1 \right)$$

$$\begin{aligned} \Sigma^1 n^3 = & 8 \left( \binom{n-1}{1} + 1 \right) + 19 \left( \binom{n-1}{2} - 1 \right) + 18 \left( \binom{n-1}{3} + 1 \right) \\ & + 6 \left( \binom{n-1}{4} - 1 \right) \end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^m T(2, m, j) \left[ \binom{n-1}{j+1} + (-1)^j \binom{j+1}{j+1} \right]$$



$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	3	1						
2	9	7	2					
3	27	37	24	6				
4	81	175	194	108	24			
5	243	781	1320	1230	600	120		
6	729	3367	8162	11340	9120	3960	720	
7	2187	14197	47544	92526	109200	77280	30240	5040

Table: Triangle  $T(3, n, k)$ .

$$n^0 = 1 \binom{n-3}{0}$$

$$n^1 = 3 \binom{n-3}{0} + 1 \binom{n-3}{1}$$

$$n^2 = 9 \binom{n-3}{0} + 7 \binom{n-3}{1} + 2 \binom{n-3}{2}$$

$$n^3 = 27 \binom{n-3}{0} + 37 \binom{n-3}{1} + 24 \binom{n-3}{2} + 6 \binom{n-3}{3}$$

$$n^4 = 81 \binom{n-3}{0} + 175 \binom{n-3}{1} + 194 \binom{n-3}{2} + 108 \binom{n-3}{3} + 24 \binom{n-3}{4}$$

$$\Sigma^1 n^0 = 1 \left( \binom{n-2}{1} + \binom{2}{1} \right)$$

$$\Sigma^1 n^1 = 3 \left( \binom{n-2}{1} + \binom{2}{1} \right) + 1 \left( \binom{n-2}{2} - \binom{3}{2} \right)$$

$$\Sigma^1 n^2 = 9 \left( \binom{n-2}{1} + \binom{2}{1} \right) + 7 \left( \binom{n-2}{2} - \binom{3}{2} \right) + 2 \left( \binom{n-2}{3} + \binom{4}{3} \right)$$

$$\begin{aligned} \Sigma^1 n^3 = & 27 \left( \binom{n-2}{1} + \binom{2}{1} \right) + 37 \left( \binom{n-2}{2} - \binom{3}{2} \right) \\ & + 24 \left( \binom{n-2}{3} + \binom{4}{3} \right) + 6 \left( \binom{n-2}{4} - \binom{5}{4} \right) \end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^m T(3, m, j) \left[ \binom{n-2}{j+1} + (-1)^j \binom{j+2}{j+1} \right]$$



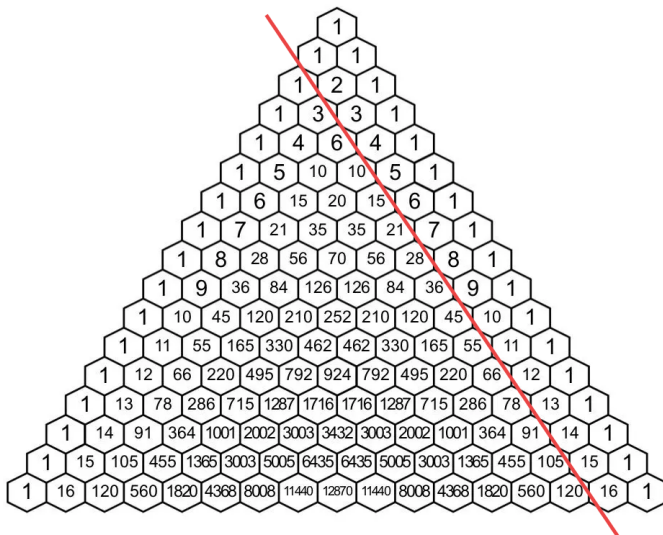


Figure: Pascal's Triangle diagonal 3.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	4	1						
2	16	9	2					
3	64	61	30	6				
4	256	369	302	132	24			
5	1024	2101	2550	1830	720	120		
6	4096	11529	19502	20460	13080	4680	720	
7	16384	61741	140070	201726	186480	107520	35280	5040

Table: Triangle  $T(4, n, k)$ .

$$n^0 = 1 \binom{n-4}{0}$$

$$n^1 = 4 \binom{n-4}{0} + 1 \binom{n-4}{1}$$

$$n^2 = 16 \binom{n-4}{0} + 9 \binom{n-4}{1} + 2 \binom{n-4}{2}$$

$$n^3 = 64 \binom{n-4}{0} + 61 \binom{n-4}{1} + 30 \binom{n-4}{2} + 6 \binom{n-4}{3}$$

$$n^4 = 256 \binom{n-4}{0} + 369 \binom{n-4}{1} + 302 \binom{n-4}{2} + 132 \binom{n-4}{3} + 24 \binom{n-4}{4}$$

$$\Sigma^1 n^0 = 1 \left( \binom{n-3}{1} + \binom{3}{1} \right)$$

$$\Sigma^1 n^1 = 4 \left( \binom{n-3}{1} + \binom{3}{1} \right) + 1 \left( \binom{n-3}{2} - \binom{4}{2} \right)$$

$$\Sigma^1 n^2 = 16 \left( \binom{n-3}{1} + \binom{3}{1} \right) + 9 \left( \binom{n-3}{2} - \binom{4}{2} \right) + 2 \left( \binom{n-2}{3} + \binom{5}{3} \right)$$

$$\begin{aligned} \Sigma^1 n^3 = & 64 \left( \binom{n-3}{1} + \binom{3}{1} \right) + 61 \left( \binom{n-3}{2} - \binom{4}{2} \right) \\ & + 30 \left( \binom{n-3}{3} + \binom{5}{3} \right) + 6 \left( \binom{n-3}{4} - \binom{6}{4} \right) \end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^m T(4, m, j) \left[ \binom{n-3}{j+1} + (-1)^j \binom{j+3}{j+1} \right]$$

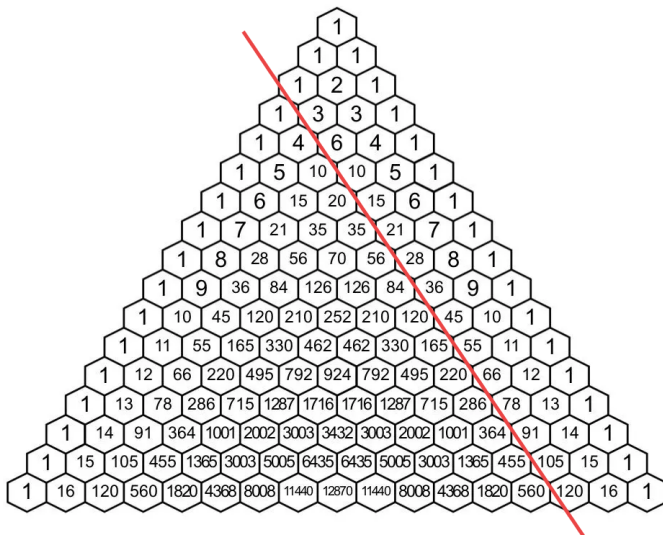


Figure: Pascal's Triangle diagonal 4.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	5	1						
2	25	11	2					
3	125	91	36	6				
4	625	671	434	156	24			
5	3125	4651	4380	2550	840	120		
6	15625	31031	39962	33540	17760	5400	720	
7	78125	201811	341796	388206	294000	142800	40320	5040

Table: Triangle  $T(5, n, k)$ .

$$n^0 = 1 \binom{n-5}{0}$$

$$n^1 = 5 \binom{n-5}{0} + 1 \binom{n-5}{1}$$

$$n^2 = 25 \binom{n-5}{0} + 11 \binom{n-5}{1} + 2 \binom{n-5}{2}$$

$$n^3 = 125 \binom{n-5}{0} + 91 \binom{n-5}{1} + 36 \binom{n-5}{2} + 6 \binom{n-5}{3}$$

$$n^4 = 625 \binom{n-5}{0} + 671 \binom{n-5}{1} + 434 \binom{n-5}{2} + 156 \binom{n-5}{3} + 24 \binom{n-5}{4}$$

$$\Sigma^1 n^0 = 1 \left( \binom{n-4}{1} + \binom{4}{1} \right)$$

$$\Sigma^1 n^1 = 5 \left( \binom{n-4}{1} + \binom{4}{1} \right) + 1 \left( \binom{n-4}{2} - \binom{5}{2} \right)$$

$$\Sigma^1 n^2 = 25 \left( \binom{n-4}{1} + \binom{4}{1} \right) + 11 \left( \binom{n-4}{2} - \binom{5}{2} \right) + 2 \left( \binom{n-4}{3} + \binom{6}{3} \right)$$

$$\begin{aligned} \Sigma^1 n^3 = & 125 \left( \binom{n-4}{1} + \binom{4}{1} \right) + 91 \left( \binom{n-4}{2} - \binom{5}{2} \right) \\ & + 36 \left( \binom{n-4}{3} + \binom{6}{3} \right) + 6 \left( \binom{n-4}{4} - \binom{7}{4} \right) \end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^m T(5, m, j) \left[ \binom{n-4}{j+1} + (-1)^j \binom{j+4}{j+1} \right]$$



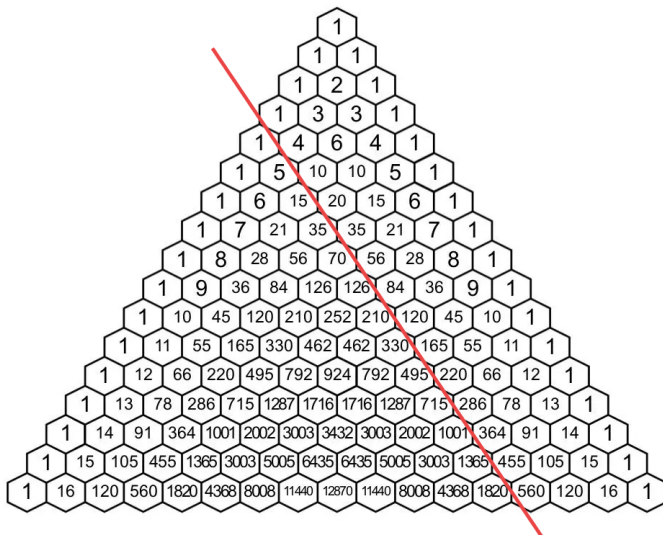


Figure: Pascal's Triangle diagonal 5.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	6	1						
2	36	13	2					
3	216	127	42	6				
4	1296	1105	590	180	24			
5	7776	9031	6930	3390	960	120		
6	46656	70993	73502	51300	23160	6120	720	
7	279936	543607	730002	682206	436800	183120	45360	5040

Table: Triangle  $T(6, n, k)$ .

$$n^0 = 1 \binom{n-6}{0}$$

$$n^1 = 6 \binom{n-6}{0} + 1 \binom{n-6}{1}$$

$$n^2 = 36 \binom{n-6}{0} + 13 \binom{n-6}{1} + 2 \binom{n-6}{2}$$

$$n^3 = 216 \binom{n-6}{0} + 127 \binom{n-6}{1} + 42 \binom{n-6}{2} + 6 \binom{n-6}{3}$$

$$n^4 = 1296 \binom{n-6}{0} + 1105 \binom{n-6}{1} + 590 \binom{n-6}{2} + 180 \binom{n-6}{3} + 24 \binom{n-6}{4}$$

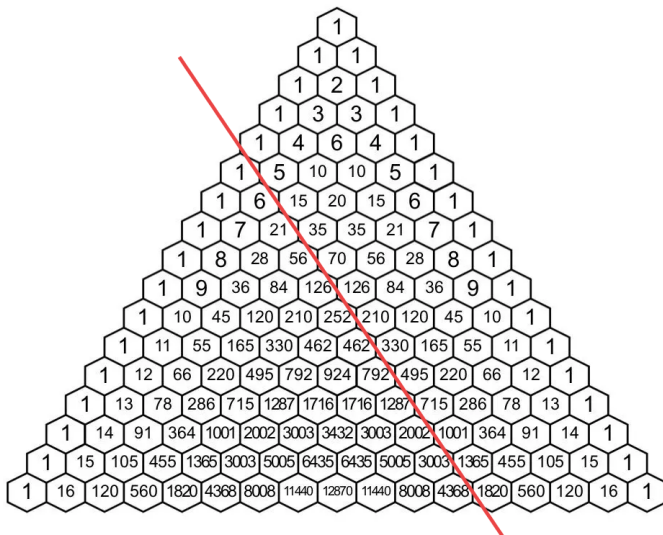
$$\Sigma^1 n^0 = 1 \left( \binom{n-5}{1} + \binom{5}{1} \right)$$

$$\Sigma^1 n^1 = 6 \left( \binom{n-5}{1} + \binom{5}{1} \right) + 1 \left( \binom{n-5}{2} - \binom{6}{2} \right)$$

$$\Sigma^1 n^2 = 36 \left( \binom{n-5}{1} + \binom{5}{1} \right) + 13 \left( \binom{n-5}{2} - \binom{6}{2} \right) + 2 \left( \binom{n-5}{3} + \binom{7}{3} \right)$$

$$\begin{aligned} \Sigma^1 n^3 = & 216 \left( \binom{n-5}{1} + \binom{5}{1} \right) + 127 \left( \binom{n-5}{2} - \binom{6}{2} \right) \\ & + 42 \left( \binom{n-5}{3} + \binom{7}{3} \right) + 6 \left( \binom{n-5}{4} - \binom{8}{4} \right) \end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^m T(6, m, j) \left[ \binom{n-5}{j+1} + (-1)^j \binom{j+5}{j+1} \right]$$



In general,

$$\sum_{k=1}^n k^m = \sum_{j=0}^m T(t, m, j) \left[ \binom{n-t+1}{j+1} + (-1)^j \binom{j+t-1}{j+1} \right]$$

where

$$T(t, m, j) = \sum_{k=0}^m \binom{t}{k} \left\{ j+k \atop m \right\} (j+k)!$$

Validate as

```
m = 2; Table[SumOfPowers[n, m] - NewSumOfPowers[n, m, t], {n, 1, 20},  
             {t, 0, 20}]
```

See [github.com/kolosovpetro/FormulasForSumsOfPowers](https://github.com/kolosovpetro/FormulasForSumsOfPowers)



Knuth, Donald E.

Johann Faulhaber and sums of powers.

*Mathematics of Computation*, 61(203):277–294, 1993.

<https://arxiv.org/abs/math/9207222>.



**Version:** 1.0.0+main.4f4511d

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**Sources:** [github.com/kolosovpetro/FormulasForSumsOfPowers](https://github.com/kolosovpetro/FormulasForSumsOfPowers)

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