# IDENTITIES IN ITERATED RASCAL TRIANGLES

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Abstract. In this manuscript, we introduce new binomial identities in iterated Rascal triangles, uncovering a connection between Vandermonde convolution and iterated Rascal numbers. Additionally, we present novel identities involving the finite differences of iterated Rascal numbers and binomial coefficients. The manuscript also offers a proof of the row sums conjecture for iterated Rascal triangles. Furthermore, we establish and explore the relationship between iterated Rascal triangles and $(1, q)$-binomial coefficients, highlighting connections to relevant OEIS sequences. All results are supported by supplementary Mathematica programs for validation.

## Contents



*Date*: September 27, 2024.

2010 *Mathematics Subject Classification.* 11B25, 11B99.

*Key words and phrases.* Pascal's triangle, Rascal triangle, Binomial coefficients, Binomial identities, Binomial theorem, Generalized Rascal triangles, Iterated rascal triangles, Iterated rascal numbers, Vandermonde convolution .

## 1. Introduction

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were challenged to provide the next row for the following number triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & & \cdots & & & \end{array}$$

The teacher anticipated that the next row would match Pascal's triangle, such as "1 4 6 4 1", by applying the binomial coefficient recurrence rule $South = East + West$. However, Anggoro, Liu, and Tulloch proposed that the next row should be "1 4 5 4 1". Instead of using Pascal's triangle rule $South = East + West$, they derived this new row using a relation they termed the diamond formula

$$South = \frac{East \cdot West + 1}{North} \tag{1.1}$$

By applying the recurrence relation from equation (1.1), the students successfully generated an entirely new triangular sequence, now referred to as the Rascal triangle.

Sources: https://github.com/kolosovpetro/IdentitiesInRascalTriangle

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | |
| 4 | 1 | 4 | 5 | 4 | 1 | | | | | |
| 5 | 1 | 5 | 7 | 7 | 5 | 1 | | | | |
| 6 | 1 | 6 | 9 | 10 | 9 | 6 | 1 | | | |
| 7 | 1 | 7 | 11 | 13 | 13 | 11 | 7 | 1 | | |
| 8 | 1 | 8 | 13 | 16 | 17 | 16 | 13 | 8 | 1 | |
| 9 | 1 | 9 | 15 | 19 | 21 | 21 | 19 | 15 | 9 | 1 |

**Table 1.** Rascal triangle. Sequence A077028 in OEIS [2].

For example, the fourth row is "1 4 5 4 1" because $4 = \frac{1\cdot 3+1}{1}$ and $5 = \frac{3\cdot 3+1}{2}$. Moreover, the Rascal triangle, as presented in table (1), represents the first and foundational instance of a new family of Pascal-like triangles. This family, known as *iterated Rascal triangles*, was first introduced by J. Gregory in her master's thesis [3].

We define the $k$-th element in the $n$-th row of an iterated Rascal triangle as $\binom{n}{k}_i$, where $i$ represents the number of iterations. The integer sequence produced by $\binom{n}{k}_i$ is referred to as an *iterated Rascal triangle Ri*, and each $\binom{n}{k}_i$ is termed an *iterated Rascal number*. Therefore, the Rascal triangle shown in table (1) corresponds to the iterated Rascal triangle $R1$, generated by the formula $\binom{n}{k}_1 = k(n-k)+1$. While the iterated Rascal number $\binom{n}{k}_i$ is defined by the diamond rule (1.1), which differs from the standard binomial coefficient recurrence, it still maintains a significant connection with the binomial coefficients $\binom{n}{k}$, as demonstrated by

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} \tag{1.2}$$

For example, $\binom{7}{4}_3 = 35$, $\binom{12}{7}_5 = 792$, $\binom{11}{5}_5 = 462$.

**Example 1.1.** Rascal triangle R2 generated by $\binom{n}{k}_2$

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 6 | 1 | 6 | 15 | 19 | 15 | 6 | 1 | | | |
| 7 | 1 | 7 | 21 | 31 | 31 | 21 | 7 | 1 | | |
| 8 | 1 | 8 | 28 | 46 | 53 | 46 | 28 | 8 | 1 | |
| 9 | 1 | 9 | 36 | 64 | 81 | 81 | 64 | 36 | 9 | 1 |

**Table 2.** Rascal triangle R2. Sequence A374378 in OEIS [4].

**Example 1.2.** Rascal triangle R3 generated by $\binom{n}{k}_3$

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | |
| 8 | 1 | 8 | 28 | 56 | 69 | 56 | 28 | 8 | 1 | |
| 9 | 1 | 9 | 36 | 84 | 121 | 121 | 84 | 36 | 9 | 1 |

**Table 3.** Rascal triangle R3. Sequence A374452 in OEIS [5].

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [6], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [7, 8]. In particular, the concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

## 2. Binomial identities in Iterated Rascal Triangles

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number [3, eq. 3.2]

**Definition 2.1.** *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m} \tag{2.1}$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [9]

$$\binom{a+b}{r} = \sum_{m=0}^{r} \binom{a}{m}\binom{b}{r-m}$$

Implies

$$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m} \tag{2.2}$$

Meaning that iterated rascal number is partial case of Vandermonde convolution of $\binom{n}{k}$ with the upper summation bound equals to $i$. Without further hesitation consider our findings.

**Proposition 2.2.** *(Column identity.) Iterated rascal triangle equals to Pascal's triangle up to i-th column. For every $k \le i$*

$$\binom{n}{k}_i = \binom{n}{k} \tag{2.3}$$

For example, we notice that

- Triangle R1 generated by $\binom{n}{k}_1$ is equivalent to Pascal's triangle for columns $k = 0, 1$. See (1).

- Triangle R2 generated by $\binom{n}{k}_2$ is equivalent to Pascal's triangle for columns $k = 0, 1, 2$. See (2).
- Triangle R3 generated by $\binom{n}{k}_3$ is equivalent to Pascal's triangle for columns $k = 0, 1, 2, 3$. See (3).

Then for every $k \leq i$ binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying the symmetry of binomial coefficients, we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

*Proof.* Proof of proposition (2.2) is given by [8, proposition 6.0.1]. □

**Proposition 2.3.** *(Row identity.) Iterated rascal triangle equals to Pascal's triangle up to $2i+1$-th row. For every $n \leq 2i+1$*

$$\binom{n}{k}_i = \binom{n}{k}$$

For example, we notice that

- Triangle R1 generated by $\binom{n}{k}_1$ is equivalent to Pascal's triangle up to 3-rd row, see (1).
- Triangle R2 generated by $\binom{n}{k}_2$ is equivalent to Pascal's triangle up to 5-th row, see (2).
- Triangle R3 generated by $\binom{n}{k}_3$ is equivalent to Pascal's triangle up to 7-rd row, see (3).

Therefore, for every $i \geq 0$ and $n \geq 0$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k} \tag{2.4}$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over $k$ for each $i$, so that it is true for all cases in $i, k$: $i < k$, $i = k$ and $k > i$. In particular, equation (2.4) implies the row sums identity in iterated rascal triangles

$$\sum_{k=0}^{\infty} \binom{2i+1-n}{k}_i = 2^{2i+1-n}$$

Given $n = 0$ we obtain

$$\sum_{k=0}^{\infty} \binom{2i+1}{k}_i = 2^{2i+1}$$

and so on. Taking $t = 2i + 1$ in (2.4) yields

$$\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$$

Moreover, equation (2.4) gives Vandermonde-like identity, by definition

$$\binom{2i+1-n}{k} = \sum_{m=0}^{i} \binom{2i+1-n-k}{m}\binom{k}{m} \tag{2.5}$$

In particular, given $n = 0, 1$ equation (2.5) yields the following Vandermonde-like identities

$$\binom{2i+1}{k} = \sum_{m=0}^{i} \binom{2i+1-k}{m}\binom{k}{m}$$

$$\binom{2i}{k} = \sum_{m=0}^{i} \binom{2i-k}{m}\binom{k}{m}$$

*Proof.* Proof of proposition (2.3). We have to prove that for every $i, k$

$$\sum_{m=0}^{k} \binom{2i+1-n-k}{m}\binom{k}{m} - \sum_{m=0}^{i} \binom{2i+1-n-k}{m}\binom{k}{m} = 0$$

For the case $k < i$ proof is given by [8, proposition 6.0.1]. For the case $k = i$ proof is trivial. Thus, the remaining case is $k > i$ yields

$$\sum_{m=i+1}^{k} \binom{2i+1-n-k}{m}\binom{k}{m} = 0 \tag{2.6}$$

If (2.6) is true for each $k > i$, then its sum over $k$ should be zero as well. Introducing sum in $k$ to (2.6) we get

$$\sum_{k} \sum_{m=i+1}^{k} \binom{2i+1-n-k}{m}\binom{k}{m} = 0$$

The sum $\sum_k \binom{2i+1-n-k}{m}\binom{k}{m}$ appears to match the equation (5.26) in Concrete mathematics [10, eq. 5.26]

$$\sum_{k=0}^{\ell} \binom{\ell-k}{m}\binom{q+k}{n} = \binom{\ell+q+1}{m+n+1} \tag{2.7}$$

Therefore,

$$\sum_{k} \binom{2i+1-n-k}{m}\binom{k}{m} = \binom{2i+2-n}{2m+1}$$

Thus, our main assumption is equivalent to

$$\sum_{m=i+1}^{k} \sum_{k} \binom{2i+1-n-k}{m}\binom{k}{m} \equiv \sum_{m=i+1}^{k} \binom{2i+2-n}{2m+1}$$

Hence, we have to prove that

$$\sum_{m=i+1}^{k} \binom{2i+2-n}{2m+1} = 0 \tag{2.8}$$

Substituting $m = i+1+m$ into (2.8), we get

$$\sum_{m=0}^{k} \binom{2i+2-n}{2(i+1+m)+1} = \sum_{m=0}^{k} \binom{2i+2-n}{2i+3+2m} = 0$$

Which is indeed true because $\binom{2i+2-n}{2i+3+2m} = 0$ for every $m, n \geq 0$. Thus, the proposition (2.4) is true. □

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences $\binom{n}{k} - \binom{n}{k}_3$

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame → All]
```

k=4

A000332

n=8

Out[67]=

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | | | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 5 | 5 | 0 | 0 | 0 | 0 | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 15 | 26 | 15 | 0 | 0 | 0 | 0 | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 35 | 81 | 81 | 35 | 0 | 0 | 0 | 0 | | | | | | | | | |
| 0 | 0 | 0 | 0 | 70 | 196 | 262 | 196 | 70 | 0 | 0 | 0 | 0 | | | | | | | | |
| 0 | 0 | 0 | 0 | 126 | 406 | 658 | 658 | 406 | 126 | 0 | 0 | 0 | 0 | | | | | | | |
| 0 | 0 | 0 | 0 | 210 | 756 | 1414 | 1716 | 1414 | 756 | 210 | 0 | 0 | 0 | 0 | | | | | | |
| 0 | 0 | 0 | 0 | 330 | 1302 | 2730 | 3830 | 3830 | 2730 | 1302 | 330 | 0 | 0 | 0 | 0 | | | | | |
| 0 | 0 | 0 | 0 | 495 | 2112 | 4872 | 7680 | 8885 | 7680 | 4872 | 2112 | 495 | 0 | 0 | 0 | 0 | | | | |
| 0 | 0 | 0 | 0 | 715 | 3267 | 8184 | 14 232 | 18 525 | 18 525 | 14 232 | 8184 | 3267 | 715 | 0 | 0 | 0 | 0 | | | |
| 0 | 0 | 0 | 0 | 1001 | 4862 | 13 101 | 24 816 | 35 697 | 40 186 | 35 697 | 24 816 | 13 101 | 4862 | 1001 | 0 | 0 | 0 | 0 | | |
| 0 | 0 | 0 | 0 | 1365 | 7007 | 20 163 | 41 217 | 64 713 | 80 587 | 80 587 | 64 713 | 41 217 | 20 163 | 7007 | 1365 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 1820 | 9828 | 30 030 | 65 780 | 111 705 | 152 020 | 168 230 | 152 020 | 111 705 | 65 780 | 30 030 | 9828 | 1820 | 0 | 0 | 0 | 0 |

**Figure 1.** Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $\binom{n}{4}$. Sequence A000332 in the OEIS [11].

We can spot that having $i = 3$ the $k = 4$-th column gives binomial coefficient $\binom{n}{4}$. Indeed, this rule is true for every $i$.

**Proposition 2.4.** *(Row-column difference.) For every* $i \geq 0$

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

*Proof.* We have previously stated that iterated rascal numbers are closely related to Vandermonde convolution (2.2). Thus, proposition (2.4) can be rewritten as

$$\sum_{m=0}^{i} \binom{n+i}{m}\binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m}\binom{i}{i-m} = \binom{n+i}{i}\binom{i}{0}$$

Therefore, $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$ is indeed true. $\square$

Proposition (2.4) yields to few more identities. Applying binomial coefficients symmetry

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking $j = n + i$ gives

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}$$

By symmetry

$$\binom{j+i}{i} - \binom{j+i}{i}_{i-1} = \binom{j}{i}$$

Proposition (2.4) can be generalized even further, for every $i < k$ and $i > k$.

**Proposition 2.5.** *(Finite difference of binomial coefficients and iterated rascal numbers for $i < k$.) For every $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^{k} \binom{n-k}{m}\binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution. □

**Proposition 2.6.** *(Finite difference of binomial coefficients and iterated rascal numbers for $i > k$.) For every $i > k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=k+1}^{i} \binom{n-k}{m}\binom{k}{k-m}$$

*Proof.* It is true by means of Vandermonde convolution. □

## 3. Q-Binomial identities in Iterated Rascal Triangles

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame → All]
```

Out[67]=

k=5

A096943

n=9

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | | | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 5 | 5 | 0 | 0 | 0 | 0 | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 15 | 26 | 15 | 0 | 0 | 0 | 0 | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 35 | 81 | 81 | 35 | 0 | 0 | 0 | 0 | | | | | | | | | |
| 0 | 0 | 0 | 0 | 70 | 196 | 262 | 196 | 70 | 0 | 0 | 0 | 0 | | | | | | | | |
| 0 | 0 | 0 | 0 | 126 | 406 | 658 | 658 | 406 | 126 | 0 | 0 | 0 | 0 | | | | | | | |
| 0 | 0 | 0 | 0 | 210 | 756 | 1414 | 1716 | 1414 | 756 | 210 | 0 | 0 | 0 | 0 | | | | | | |
| 0 | 0 | 0 | 0 | 330 | 1302 | 2730 | 3830 | 3830 | 2730 | 1302 | 330 | 0 | 0 | 0 | 0 | | | | | |
| 0 | 0 | 0 | 0 | 495 | 2112 | 4872 | 7680 | 8885 | 7680 | 4872 | 2112 | 495 | 0 | 0 | 0 | 0 | | | | |
| 0 | 0 | 0 | 0 | 715 | 3267 | 8184 | 14 232 | 18 525 | 18 525 | 14 232 | 8184 | 3267 | 715 | 0 | 0 | 0 | 0 | | | |
| 0 | 0 | 0 | 0 | 1001 | 4862 | 13 101 | 24 816 | 35 697 | 40 186 | 35 697 | 24 816 | 13 101 | 4862 | 1001 | 0 | 0 | 0 | 0 | | |
| 0 | 0 | 0 | 0 | 1365 | 7007 | 20 163 | 41 217 | 64 713 | 80 587 | 80 587 | 64 713 | 41 217 | 20 163 | 7007 | 1365 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 1820 | 9828 | 30 030 | 65 780 | 111 705 | 152 020 | 168 230 | 152 020 | 111 705 | 65 780 | 30 030 | 9828 | 1820 | 0 | 0 | 0 | 0 |

**Figure 2.** Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $(1,5)$-binomial coefficient $\left[{n \atop k}\right]^5$. Sequence A096943 in the OEIS [12].

The $(1,q)$-binomial coefficients $\left[{n \atop k}\right]^q$ are special kind of binomial coefficients defined by

**Definition 3.1.** $(1,q)$-*Binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \left[{n-1 \atop k}\right]^q + \left[{n-1 \atop k-1}\right]^q & \end{cases} \tag{3.1}$$

Indeed, the relation shown in Figure (2) is true for every $i$, so that it establishes a relation between $(1,q)$-binomial coefficients and iterated rascal numbers.

**Proposition 3.2.** *(Relation between iterated rascal numbers and $(1,q)$-binomial coefficients.) For every $i \geq 0$*

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \begin{bmatrix} i+2+j \\ i+2 \end{bmatrix}^{i+2}$$

Taking $t = i + 2$ in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[\begin{matrix} t+j \\ t \end{matrix}\right]^t$$

In particular,

- Having $i = 1$ proposition (3.2) gives the OEIS sequence A006503 [13] such that third column of $(1,3)$-Pascal triangle A095660 [14].
- Having $i = 3$ proposition (3.2) gives the OEIS sequence A096943 [12] such that third column of $(1,5)$-Pascal triangle A096940 [15].
- Having $i = 5$, the proposition (3.2) yields the OEIS sequence A097297 [16] such that seventh column of $(1,6)$-Pascal triangle A096956 [17].
- Having $i = 2$ and $k = 4$: $\binom{n}{k} - \binom{n}{k}_i$ gives Fifth column $(m = 4)$ of $(1,4)$-Pascal triangle https://oeis.org/A095667
- Having $i = 2$ and $k = 3$: $\binom{n}{k} - \binom{n}{k}_i$ gives Tetrahedral (or triangular pyramidal) numbers: $a(n) = C(n+2,3) = n*(n+1)*(n+2)/6$. https://oeis.org/A000292
- Having $i = 1$ and $k = 2$: $\binom{n}{k} - \binom{n}{k}_i$ gives Triangular numbers: $a(n) = binomial(n+1,2) = n*(n+1)/2 = 0+1+2+\cdots+n$ https://oeis.org/A000217
- Having $i = 0$ and $k = 3$: $\binom{n}{k} - \binom{n}{k}_i$ gives Fourth column (r=3) of FS(3) staircase array. https://oeis.org/A062748
- Having $i = 0$ and $k = 6$: $\binom{n}{k} - \binom{n}{k}_i$ gives $a(n) = binomial(n,6) - 1$. https://oeis.org/A124089
- Having $i = 0$ and $k = 7$: $\binom{n}{k} - \binom{n}{k}_i$ gives $a(n) = binomial(n,7) - 1$. https://oeis.org/A124090

## 4. Row sums conjecture

In [8] the authors propose the following conjecture for row sums of iterated rascal triangles.

**Conjecture 4.1.** *(Conjecture 7.5 in [8].) For every $i$*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

*Proof.* Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3}\sum_{m=0}^{i}\binom{4i+3-k}{m}\binom{k}{m}=2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^{i}\sum_{k}\binom{4i+3-k}{m}\binom{k}{m}=2^{4i+2} \tag{4.1}$$

We can observe the pattern (2.7) in equation (4.1). Thus, the sum $\sum_k\binom{4i+3-k}{m}\binom{k}{m}$ equals to

$$\sum_{k}\binom{4i+3-k}{m}\binom{k}{m}=\binom{4i+4}{2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^{i}\binom{4i+4}{2m+1}=2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1}\binom{4i+4}{2m+1}=\sum_{m=0}^{2i+1}\left[\binom{4i+3}{2m+1}+\binom{4i+3}{2m}\right]=2^{4i+3}$$

So that

$$\frac{1}{2}\sum_{m=0}^{2i+1}\binom{4i+4}{2m+1}=\sum_{m=0}^{i}\binom{4i+4}{2m+1}=2^{4i+2}$$

This completes the proof. □

**Proposition 4.2.** *For every $i$*

$$\sum_{k=0}^{4i+3}\binom{4i+3}{k}_i=2^{4i+2}$$

In particular, equation (2.7) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^{n}\binom{n}{k}_i=\sum_{m=0}^{i}\binom{n+1}{2m+1}$$

Decomposing $\binom{n+1}{2m+1}$ in above equation yields

**Proposition 4.3.** *(Iterated rascal triangles row sums.) For every $i$*

$$\sum_{k=0}^{n} \binom{n}{k}_i = \sum_{m=0}^{2i+1} \binom{n}{m}$$

*Proof.*

$$\sum_{k=0}^{n} \binom{n}{k}_i = \sum_{m=0}^{i} \binom{n+1}{2m+1} = \sum_{m=0}^{i} \binom{n}{2m} + \binom{n}{2m+1} = \sum_{m=0}^{2i+1} \binom{n}{m}$$

□

## 5. Conclusions

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.4), (2.5), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.5). Furthermore, we establish a relation between iterated rascal triangles and $(1, q)$-binomial coefficients (3.2). Some of the results accepted for publication in *Mathematical gazette*. Supplementary Mathematica scripts to be found at [18].

## 6. Acknowledgements

Author is grateful to Oleksandr Kulkov, Markus Scheuer, Amelia Gibbs for their valuable feedback and suggestions regarding the conjecture (4.1) at MathStackExchange discussion [19].

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**Version:** `1.2.1-tags-v1-2-0.22+tags/v1.2.0.c7c8936`

## 7. Addendum 1: Mathematica documentation

Mathematica programs documentation. See [18].

- `ColumnIdentity1[20, 20]` validates $\binom{n}{i-k}_i = \binom{n}{i-k}$
- `ColumnIdentity2[20, 20]` validates $\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$
- `RowIdentity1[5]` validates $\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k}$, see (2.4)
- `RowIdentity2[12, 5]]` validates $\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$
- `RowColumnDifferenceIdentity1[10, 20]` validates $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$, see (2.4).
- `RowColumnDifferenceIdentity2[10, 20]` validates $\binom{j+i}{i} - \binom{j+i}{i}_{i-1} = \binom{j}{i}$
- `OneQPascalIdentity1[10, 20]` validates $\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[{i+2+j \atop i+2}\right]^{i+2}$, see (3.2).
- `OneQPascalIdentity2[10, 20]` validates $\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[{t+j \atop t}\right]^{t}$

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