NEWTON'S INTERPOLATION FORMULA AND SUMS OF POWERS

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Abstract. In this manuscript we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind and Eulerian numbers.

Contents

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1. Introduction and main results

In this manuscript, we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ and Eulerian numbers $\left\langle {n \atop k} \right\rangle$. The main idea is to utilize Newton's interpolation formula combined with hockey stick identities to derive formulas for multifold sums of powers. In this manuscript, we use Newton's interpolation formula in forward finite differences. However, the Newton's formula approach for sums of powers is quite generic, as such it does not necessary requires to bind to forward difference operator specifically. Thus, formulas for sums of powers using backward and central differences can be found in [1, 2], respectively. Allow us to start from the definition of multifold sums of powers. We utilize the recurrence proposed by Donald Knuth in his article *Johann Faulhaber and sums of powers*, see [3].

Proposition 1.1 (Multifold sums of powers recurrence)**.** *For integers $r, n, m \geq 0$,*

$$\begin{aligned}\Sigma^0 n^m &= n^m, \\ \Sigma^1 n^m &= \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m, \\ \Sigma^{r+1} n^m &= \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m.\end{aligned}$$

Throughout the paper, we utilize the Newton's interpolation formula [4, Lemma V.] as stated below.

Proposition 1.2 (Newton's interpolation formula)**.** *Let $f\colon \mathbb{Z} \to \mathbb{C}$ be a function, and let $a \in \mathbb{Z}$. Then, for all $x \in \mathbb{Z}$,*

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a),$$

where the kth forward finite difference of f evaluated at a is defined by

$$\Delta^k f(a) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(a+j).$$

Therefore, by setting $f(n) = n^m$ yields

Proposition 1.3 (Newton's formula for powers)**.** *Let $n, m \geq 0$ be integers, and let t be an arbitrary integer. Then,*

$$n^m = \sum_{k=0}^{m} \binom{n-t}{k} \Delta^k t^m.$$

We can see that Proposition (1.3) is indeed true, because,

$$n^3 = 0\tbinom{n-0}{0} + 1\tbinom{n-0}{1} + 6\tbinom{n-0}{2} + 6\tbinom{n-0}{3},$$

$$n^3 = 1\tbinom{n-1}{0} + 7\tbinom{n-1}{1} + 12\tbinom{n-1}{2} + 6\tbinom{n-1}{3},$$

$$n^3 = 8\tbinom{n-2}{0} + 19\tbinom{n-2}{1} + 18\tbinom{n-2}{2} + 6\tbinom{n-2}{3},$$

$$n^3 = 27\tbinom{n-3}{0} + 37\tbinom{n-3}{1} + 24\tbinom{n-3}{2} + 6\tbinom{n-3}{3}.$$

Lemma 1.4 (Generalized hockey stick identity)**.** *For integers $a \leq b$ and j,*

$$\sum_{k=a}^{b} \binom{k}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}.$$

Proof. We have,

$$\sum_{k=a}^{b} \tbinom{k}{j} = \tbinom{a}{j} + \tbinom{a+1}{j} + \cdots + \tbinom{b}{j},$$

which implies,

$$\sum_{k=a}^{b} \tbinom{k}{j} = \left(\sum_{k=0}^{b} \tbinom{k}{j}\right) - \left(\sum_{k=0}^{a-1} \tbinom{k}{j}\right).$$

Thus, by hockey stick identity $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$, we get,

$$\sum_{k=a}^{b} \tbinom{k}{j} = \left(\sum_{k=0}^{b} \tbinom{k}{j}\right) - \left(\sum_{k=0}^{a-1} \tbinom{k}{j}\right) = \tbinom{b+1}{j+1} - \tbinom{a}{j+1}.$$

This completes the proof. $\square$

Therefore, shifted version of Lemma (1.4) follows.

Lemma 1.5 (Shifted hockey stick identity)**.** *For integers n, t, j, r,*

$$\sum_{k=1}^{n} \binom{k-t+r}{j+r} = \sum_{k=1-t+r}^{n-t+r} \binom{k}{j+r} = \binom{n-t+r+1}{j+r+1} - \binom{1-t+r}{j+r+1}.$$

Recall the upper negation property of binomial coefficients.

Lemma 1.6 (Upper negation)**.** *For integers r, k,*

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}.$$

Thus,

$$\begin{aligned}\binom{1-t+r}{j+r+1} &= (-1)^{j+r+1} \binom{(j+r+1)-(1-t+r)-1}{j+r+1} \\ &= (-1)^{j+r+1} \binom{j+t-1}{j+r+1}.\end{aligned}$$

Formula above implies negated version of shifted hockey-stick identity (1.5).

Lemma 1.7 (Negated hockey stick identity)**.** *For integers n, t, j, r,*

$$\sum_{k=1}^{n} \binom{k-t+r}{j+r} = \binom{n-t+r+1}{j+r+1} + (-1)^{j+r} \binom{j+t-1}{j+r+1}.$$

Hence, by Newton's formula for powers (1.3), we get formula for ordinary sums of powers.

$$\Sigma^1 n^m = \sum_{k=1}^{n} \sum_{j=0}^{m} \binom{k-t}{j} \Delta^j t^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \binom{k-t}{j},$$

where t is an arbitrary integer. By shifted hockey stick identity (1.5), the closed form of ordinary sums of powers yields.

Proposition 1.8 (Ordinary power sums)**.** *For integers $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+1}{j+1} - \binom{1-t}{j+1} \right].$$

For example,

$$\Sigma^1 n^3 = 0\left[\tbinom{n+1}{1} - \tbinom{1}{1}\right] + 1\left[\tbinom{n+1}{2} - \tbinom{1}{2}\right] + 6\left[\tbinom{n+1}{3} - \tbinom{1}{3}\right] + 6\left[\tbinom{n+1}{4} - \tbinom{1}{4}\right],$$

$$\Sigma^1 n^3 = 1\left[\tbinom{n}{2} - \tbinom{0}{2}\right] + 7\left[\tbinom{n}{3} - \tbinom{0}{3}\right] + 12\left[\tbinom{n}{4} - \tbinom{0}{4}\right] + 6\left[\tbinom{n}{5} - \tbinom{0}{5}\right],$$

$$\Sigma^1 n^3 = 8\left[\tbinom{n-1}{2} - \tbinom{-1}{2}\right] + 19\left[\tbinom{n-1}{3} - \tbinom{-1}{3}\right] + 18\left[\tbinom{n-1}{4} - \tbinom{-1}{4}\right] + 6\left[\tbinom{n-1}{5} - \tbinom{-1}{5}\right],$$

$$\Sigma^1 n^3 = 27\left[\tbinom{n-2}{2} - \tbinom{-2}{2}\right] + 37\left[\tbinom{n-2}{3} - \tbinom{-2}{3}\right] + 24\left[\tbinom{n-2}{4} - \tbinom{-2}{4}\right] + 6\left[\tbinom{n-2}{5} - \tbinom{-2}{5}\right].$$

By Lemma (1.7), we obtain negated version of Proposition (1.8).

Proposition 1.9 (Negated ordinary power sums)**.** *For integers $n \geq 0$, and $m \geq 0$, and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+1}{j+1} + (-1)^j \binom{j+t-1}{j+1}\right].$$

For example,

$$\Sigma^1 n^3 = 0\left[\tbinom{n+1}{1} + \tbinom{-1}{1}\right] + 1\left[\tbinom{n+1}{2} + \tbinom{-1}{2}\right] + 6\left[\tbinom{n+1}{3} - \tbinom{0}{3}\right] + 6\left[\tbinom{n+1}{4} + \tbinom{1}{4}\right],$$

$$\Sigma^1 n^3 = 1\left[\tbinom{n}{2} + \tbinom{0}{2}\right] + 7\left[\tbinom{n}{3} - \tbinom{1}{3}\right] + 12\left[\tbinom{n}{4} + \tbinom{2}{4}\right] + 6\left[\tbinom{n}{5} - \tbinom{3}{5}\right],$$

$$\Sigma^1 n^3 = 8\left[\tbinom{n-1}{2} + \tbinom{1}{2}\right] + 19\left[\tbinom{n-1}{3} - \tbinom{2}{3}\right] + 18\left[\tbinom{n-1}{4} + \tbinom{3}{4}\right] + 6\left[\tbinom{n-1}{5} - \tbinom{4}{5}\right],$$

$$\Sigma^1 n^3 = 27\left[\tbinom{n-2}{2} + \tbinom{2}{2}\right] + 37\left[\tbinom{n-2}{3} - \tbinom{3}{3}\right] + 24\left[\tbinom{n-2}{4} + \tbinom{4}{4}\right] + 6\left[\tbinom{n-2}{5} - \tbinom{5}{5}\right].$$

By setting $t = 0$ into the Proposition (1.8) yields another well-known identity for sums of powers, see [5, p. 190] and [6]. That is,

Corollary 1.10. *For integers $n \geq 0$, and $m \geq 0$,*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \binom{n+1}{j+1} \Delta^j 0^m.$$

In particular,

$$\Sigma^1 n^2 = 0\tbinom{n+1}{1} + 1\tbinom{n+1}{2} + 2\tbinom{n+1}{3},$$

$$\Sigma^1 n^3 = 0\tbinom{n+1}{1} + 1\tbinom{n+1}{2} + 6\tbinom{n+1}{3} + 6\tbinom{n+1}{4},$$

$$\Sigma^1 n^4 = 0\tbinom{n+1}{1} + 1\tbinom{n+1}{2} + 14\tbinom{n+1}{3} + 36\tbinom{n+1}{4} + 24\tbinom{n+1}{5},$$

$$\Sigma^1 n^5 = 0\tbinom{n+1}{1} + 1\tbinom{n+1}{2} + 30\tbinom{n+1}{3} + 150\tbinom{n+1}{4} + 240\tbinom{n+1}{5} + 120\tbinom{n+1}{6}.$$

The coefficients $0, 1, 2, 0, 1, 6, 6, 0, 1, 14, 36, 24, \ldots$ is the sequence A131689 in the OEIS [7]. By setting $t = 1$ into the Proposition (1.8), we get another well-known special case.

Corollary 1.11. *For non-negative integers n, m,*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \binom{n}{j+1} \Delta^j 1^m.$$

For instance,

$$\Sigma^1 n^2 = 1\tbinom{n}{1} + 3\tbinom{n}{2} + 2\tbinom{n}{3},$$

$$\Sigma^1 n^3 = 1\tbinom{n}{1} + 7\tbinom{n}{2} + 12\tbinom{n}{3} + 6\tbinom{n}{4},$$

$$\Sigma^1 n^4 = 1\tbinom{n}{1} + 15\tbinom{n}{2} + 50\tbinom{n}{3} + 60\tbinom{n}{4} + 24\tbinom{n}{5},$$

$$\Sigma^1 n^5 = 1\tbinom{n}{1} + 31\tbinom{n}{2} + 180\tbinom{n}{3} + 390\tbinom{n}{4} + 360\tbinom{n}{5} + 120\tbinom{n}{6}.$$

The coefficients $1, 3, 2, 1, 7, 12, 6, 1, 15, \ldots$ is the sequence A028246 in the OEIS [7]. It is particularly interesting that the paper [8] give formulas for sums of powers in terms of generalized Stirling numbers of the second kind $\left\{ {k \atop j} \right\}_r$.

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \left\{ {k \atop j} \right\}_r, \tag{1}$$

$$\Sigma^1 n^k = \sum_{j=0}^{k} j! \left[\binom{n+1-r}{j+1} + \binom{r+1}{j+1} \right] \left\{ {k \atop j} \right\}_{-r}. \tag{2}$$

Formula (1) is a special case of Proposition (1.9), in terms of generalized Stirling numbers $\left\{ {k \atop j} \right\}_r$. This implies that finite differences of powers can be expressed in terms of generalized

Stirling numbers of the second kind, $\Delta^j t^m = j!\left\{ {m \atop j} \right\}_t$. Formula (2) is a special case of Proposition (1.8) with $t = -r$. By setting $t = 4$ into Proposition (1.8), we observe rather unusual formulas for sums of powers, namely,

$$\Sigma^1 n^0 = 1\left(\binom{n-3}{1} + \binom{3}{1}\right),$$

$$\Sigma^1 n^1 = 4\left(\binom{n-3}{1} + \binom{3}{1}\right) + 1\left(\binom{n-3}{2} - \binom{4}{2}\right),$$

$$\Sigma^1 n^2 = 16\left(\binom{n-3}{1} + \binom{3}{1}\right) + 9\left(\binom{n-3}{2} - \binom{4}{2}\right) + 2\left(\binom{n-2}{3} + \binom{5}{3}\right),$$

$$\Sigma^1 n^3 = 64\left(\binom{n-3}{1} + \binom{3}{1}\right) + 61\left(\binom{n-3}{2} - \binom{4}{2}\right) + 30\left(\binom{n-3}{3} + \binom{5}{3}\right) + 6\left(\binom{n-3}{4} - \binom{6}{4}\right).$$

The coefficients $1, 4, 1, 16, 9, \ldots$ is the sequence A391633 in the OEIS [7]. In general,

$$\Sigma^1 n^m = \sum_{j=0}^{m} \Delta^j 4^m \left[\binom{n-3}{j+1} + (-1)^j \binom{j+3}{j+1}\right].$$

To derive formula for double sum of powers, we apply summation operator over the ordinary sum of powers (1.8) again. Thus,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{k=1}^{n} \binom{k-t+1}{j+1}\right) - \binom{1-t}{j+1} \sum_{k=1}^{n} 1\right],$$

which implies,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\left(\sum_{k=1}^{n} \binom{k-t+1}{j+1}\right) - \binom{1-t}{j+1} n\right].$$

By applying shifted hockey stick identity (1.5), we get formula for double sums of powers.

Proposition 1.12 (Double sums of powers)**.** *For integers $n \geq 0$, and $m \geq 0$, and an arbitrary integer t*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+2}{j+1} - \binom{1-t}{j+1} n - \binom{2-t}{j+2}\right]$$

By applying Lemma (1.7) on Proposition (1.12), we obtain negated variation of formula for sums of powers.

Proposition 1.13 (Negated double sums of powers)**.** *For integers $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+2}{j+1} + (-1)^j \binom{j+t-1}{j+1} n + (-1)^{j+1} \binom{j+t-1}{j+2} \right].$$

For example, by setting $t = 5$ into Proposition (1.13), we get,

$$\begin{aligned}
\Sigma^2 n^0 &= 1 \left(\tbinom{n-3}{2} + \tbinom{4}{1} n - \tbinom{4}{2} \right), \\
\Sigma^2 n^1 &= 5 \left(\tbinom{n-3}{2} + \tbinom{4}{1} n - \tbinom{4}{2} \right) + 1 \left(\tbinom{n-3}{3} - \tbinom{5}{2} n + \tbinom{5}{3} \right), \\
\Sigma^2 n^2 &= 25 \left(\tbinom{n-3}{2} + \tbinom{4}{1} n - \tbinom{4}{2} \right) + 11 \left(\tbinom{n-3}{3} - \tbinom{5}{2} n + \tbinom{5}{3} \right) + 2 \left(\tbinom{n-3}{4} + \tbinom{6}{3} n - \tbinom{6}{4} \right).
\end{aligned}$$

The coefficients $1, 5, 1, 25, 11, 2, \ldots$ is the sequence A391635 in the OEIS [7]. In general,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \Delta^j 5^m \left[\binom{n-3}{j+2} + (-1)^j \binom{j+4}{j+1} n^1 + (-1)^{j+1} \binom{j+4}{j+2} n^0 \right].$$

By summing up double sums of powers (1.12), we obtain the non-closed expression for triple sums of powers.

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \sum_{k=1}^{n} \left[\binom{k-t+2}{j+1} - \binom{1-t}{j+1} k^1 - \binom{2-t}{j+2} k^0 \right].$$

Note that $\sum_{k=1}^{n} k^1 = \Sigma^2 n^0$ and $\sum_{k=1}^{n} k^0 = \Sigma^1 n^0$ and $1 = \Sigma^0 n^0$. Therefore, by shifted hockey stick identity (1.5), we get the closed form identity for triple sums of powers.

Proposition 1.14 (Triple sums of powers)**.** *For integers $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+2}{j+1} - \binom{1-t}{j+1} \Sigma^2 n^0 - \binom{2-t}{j+2} \Sigma^1 n^0 - \binom{3-t}{j+3} \Sigma^0 n^0 \right].$$

By utilizing Lemma (1.7) on the Proposition (1.14) yields its negated variation.

Proposition 1.15 (Negated triple sums of powers)**.** *For integers $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j t^m \Bigg[\binom{n-t+3}{j+3} + (-1)^j \binom{j+t-1}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+t-1}{j+2} \Sigma^1 n^0 + (-1)^{j+2} \binom{j+t-1}{j+3} \Sigma^0 n^0 \Bigg].$$

For instance, by setting $t = 4$ the Proposition (1.14), for $m = 1$ we have,

$$\Sigma^3 n^1 = 4 \left(\tbinom{n-1}{3} + \tbinom{3}{1} \Sigma^2 n^0 - \tbinom{3}{2} \Sigma^1 n^0 + \tbinom{3}{3} \Sigma^0 n^0 \right),$$
$$+ 1 \left(\tbinom{n-1}{4} - \tbinom{4}{2} \Sigma^2 n^0 + \tbinom{4}{3} \Sigma^1 n^0 - \tbinom{4}{4} \Sigma^0 n^0 \right).$$

For $m = 2$ we have,

$$\Sigma^3 n^2 = 16 \left(\tbinom{n-1}{3} + \tbinom{3}{1} \Sigma^2 n^0 - \tbinom{3}{2} \Sigma^1 n^0 + \tbinom{3}{3} \Sigma^0 n^0 \right),$$
$$+ 9 \left(\tbinom{n-1}{4} - \tbinom{4}{2} \Sigma^2 n^0 + \tbinom{4}{3} \Sigma^1 n^0 - \tbinom{4}{4} \Sigma^0 n^0 \right),$$
$$+ 2 \left(\tbinom{n-1}{5} + \tbinom{5}{3} \Sigma^2 n^0 - \tbinom{5}{4} \Sigma^1 n^0 + \tbinom{5}{5} \Sigma^0 n^0 \right).$$

The coefficients $1, 4, 1, 16, 9, \ldots$ is the sequence A391633 in the OEIS [7]. In general,

$$\Sigma^3 n^m = \sum_{j=0}^{m} \Delta^j 4^m \Bigg[\binom{n-1}{j+3} + (-1)^j \binom{j+3}{j+1} \Sigma^2 n^0 + (-1)^{j+1} \binom{j+3}{j+2} \Sigma^1 n^0 + (-1)^{j+2} \binom{j+3}{j+3} \Sigma^0 n^0 \Bigg].$$

By repeated application of shifted hockey stick identity (1.5), and Newton's formula for powers (1.3), we derive the closed formula for multifold sums of powers.

Theorem 1.16 (Multifold sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \Sigma^{r-s} n^0 \right].$$

By negated hockey stick identity (1.7), another variation of multifold sums of powers follows.

Theorem 1.17 (Negated multifold sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right].$$

In its explicit form,

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \Big[\tbinom{n-t+r}{j+r} + (-1)^j \tbinom{j+t-1}{j+1} \Sigma^{r-1} n^0 + (-1)^{j+1} \tbinom{j+t-1}{j+2} \Sigma^{r-2} n^0 + \cdots$$
$$+ (-1)^{j+r-1} \tbinom{j+t-1}{j+r} \Sigma^0 n^0 \Big].$$

We may notice the identity for closed-form of $\Sigma^{r-1} n^0$.

Proposition 1.18 (Multifold sum of zero powers)**.** *For integers $r \geq 0$, and $n \geq 1$,*

$$\Sigma^r n^0 = \binom{r+n-1}{r}.$$

Proof. (1) Let $r = 0$, then $\Sigma^0 n^0 = n^0 = \binom{n-1}{0} = 1$, by definition.

(2) Let $r = 1$, then $\Sigma^1 n^0 = \sum_{k=1}^{n} \binom{k-1}{0} = \sum_{k=1}^{n} 1 = \binom{n}{1}$.

(3) Let $r = 2$, then $\Sigma^2 n^0 = \sum_{k=1}^{n} \binom{k}{1} = \sum_{k=1}^{n} k = \binom{n+1}{2}$.

(4) Let $r = 3$, then $\Sigma^3 n^0 = \sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3}$.

(5) By induction over r and hockey stick identity $\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, the claim follows $\Sigma^r n^0 = \binom{r+n-1}{r}$.

This completes the proof. □

Hence,

$$\Sigma^{r-s} n^0 = \binom{r-s+n-1}{r-s}.$$

Therefore, we obtain a pure binomial form of the Theorem (1.16).

Proposition 1.19 (Multifold Binomial sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \binom{r-s+n-1}{r-s} \right].$$

By Lemma (1.7) on the Proposition (1.19), we obtain its negated version.

Proposition 1.20 (Negated multifold Binomial sums of powers). *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right].$$

By re-indexing the sum in s in Proposition (1.20), we get its shifted form.

Proposition 1.21 (Shifted multifold Binomial sum of powers). *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \Delta^j t^m \left[\binom{n-t+r}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right].$$

2. Sums of powers in Stirling numbers

Forward finite differences of powers are closely related to Stirling numbers of the second kind $\left\{ n \atop k \right\}$, as the following well-known lemma shows.

Lemma 2.1 (Stirling finite differences). *For non-negative integers j, m, and an arbitrary integer t,*

$$\Delta^j t^m = \sum_{k=0}^{m} \left\{ m \atop j+k \right\} \binom{t}{k} (j+k)!$$

This lemma implies variations for ordinary sums of powers. By utilizing Proposition (1.8), and Lemma (2.1), we obtain formula for sums of powers in terms of Stirling numbers of the second kind $\left\{ n \atop k \right\}$.

Proposition 2.2 (Stirling sums of powers). *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+1}{j+1} - \binom{1-t}{j+1} \right] \left\{ m \atop j+k \right\} \binom{t}{k} (j+k)!$$

In addition, by using the negated formula for sums of powers in Proposition (1.9), with Lemma (2.1) for finite differences in Stirling numbers $\left\{ n \atop k \right\}$, we obtain new result.

Proposition 2.3 (Negated Stirling sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+1}{j+1} + (-1)^j \binom{j+t-1}{j+1} \right] \left\{ {m \atop j+k} \right\} \binom{t}{k} (j+k)!$$

By setting $t = 0$ into the Proposition (2.3) yields well-known identity for sums of powers in terms of Stirling numbers. That is,

Corollary 2.4. *For non-negative integers n, m,*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \binom{n+1}{j+1} \left\{ {m \atop j} \right\} j!$$

Similarly, formula for multifold sums of powers in terms of Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ follows.

Proposition 2.5 (Multifold Stirling sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \Sigma^{r-s} n^0 \right] \left\{ {m \atop j+k} \right\} \binom{t}{k} (j+k)!$$

By utilizing Lemma (1.7) for negated hockey stick identity, we obtain the negated formula for multifold sums of powers in terms of Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$.

Proposition 2.6 (Negated multifold Stirling sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\tbinom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \tbinom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right] \left\{ {m \atop j+k} \right\} \tbinom{t}{k} (j+k)!$$

Propositions (2.5), and (2.6) can be expressed in a pure binomial form, by means of the identity (1.18), which implies that $\Sigma^{r-s} n^0 = \binom{r-s+n-1}{r-s}$. Therefore, we get multifold Stirling-Binomial formula for sums of powers.

Proposition 2.7 (Multifold Stirling-Binomial sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \binom{r-s+n-1}{r-s} \right] \left\{ {m \atop j+k} \right\} \binom{t}{k} (j+k)!$$

Lemma (1.7) yields its negated version.

Proposition 2.8 (Negated Stirling-Binomial sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\tbinom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \tbinom{j+t-1}{j+s} \tbinom{r-s+n-1}{r-s} \right] \left\{ {m \atop j+k} \right\} \tbinom{t}{k} (j+k)!$$

3. Sums of powers in Eulerian numbers

Forward finite differences can be expressed in terms of Eulerian numbers $\left\langle {n \atop k} \right\rangle$, by means of Worpitzky identity [9].

Lemma 3.1 (Worpitzky identity)**.** *For non-negative integers t, m*

$$t^m = \sum_{k=0}^{m} \binom{t+k}{m} \left\langle {m \atop k} \right\rangle,$$

where $\left\langle {n \atop k} \right\rangle$ are Eulerian numbers.

Thus, we obtain formula for finite differences of powers, in terms of Eulerian numbers $\left\langle {n \atop k} \right\rangle$.

Lemma 3.2 (Eulerian finite differences)**.** *For integers $j, m \geq 0$ and an arbitrary integer t,*

$$\Delta^j t^m = \sum_{k=0}^{m} \binom{t+k}{m-j} \left\langle {m \atop k} \right\rangle.$$

Hence, by multifold sums of powers (1.16), and by Lemma (3.2), we get formula for multifold sums of powers in terms of Eulerian numbers $\left\langle {n \atop k} \right\rangle$.

Proposition 3.3 (Multifold Eulerian sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \Sigma^{r-s} n^0 \right] \binom{t+k}{m-j} \left\langle {m \atop k} \right\rangle.$$

In addition, by negated multifold sums of powers (1.17), and by Lemma (3.2), yields negated version of the Proposition (3.3).

Proposition 3.4 (Negated multifold Eulerian sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r\, n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r}(-1)^{j+s-1}\binom{j+t-1}{j+s}\Sigma^{r-s}\, n^0\right]\binom{t+k}{m-j}\left\langle {m \atop k} \right\rangle.$$

In particular,

Proposition 3.5 (Negated Eulerian sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^1\, n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+1}{j+1} + (-1)^j\binom{j+t-1}{j+1}\right]\binom{t+k}{m-j}\left\langle {m \atop k} \right\rangle.$$

By setting $t = 0$ into the Proposition (3.5) yields a well-known identity for sums of powers in terms of Eulerian numbers $\left\langle {n \atop k} \right\rangle$.

Corollary 3.6 (Eulerian sums of powers in zero)**.** *For integers $n \geq 0$, and $m \geq 0$,*

$$\Sigma^1\, n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\binom{n+1}{j+1}\binom{k}{m-j}\left\langle {m \atop k} \right\rangle.$$

By Propositions (1.19), and (1.20), we obtain a formula for multifold sums of powers in Eulerian-Binomial form. That is,

Proposition 3.7 (Multifold Eulerian-Binomial sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r\, n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r}\binom{s-t}{j+s}\binom{r-s+n-1}{r-s}\right]\binom{t+k}{m-j}\left\langle {m \atop k} \right\rangle.$$

While its negated form is,

Proposition 3.8 (Negated multifold Eulerian-Binomial sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r\, n^m = \sum_{j=0}^{m}\sum_{k=0}^{m}\left[\tbinom{n-t+r}{j+r} + \sum_{s=1}^{r}(-1)^{j+s-1}\tbinom{j+t-1}{j+s}\tbinom{r-s+n-1}{r-s}\right]\tbinom{t+k}{m-j}\left\langle {m \atop k} \right\rangle.$$

4. Backward difference form

Formula for multifold sums of powers (1.16) can be expressed in terms of backward differences easily, because,

Proposition 4.1. *For integers j, m, t,*

$$\Delta^j t^m = \nabla^j (t+j)^m.$$

Therefore, by Proposition (4.1), we get formula for sums of powers in terms of backward differences.

Proposition 4.2 (Multifold backward sums of powers)**.** *For integers $r, n, m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \Sigma^{r-s}\, n^0 \right] \nabla^j (t+j)^m.$$

By Theorem (1.17) and by Proposition (4.1) yields negated formula for sums of powers.

Proposition 4.3 (Negated backward sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t*

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s}\, n^0 \right] \nabla^j (t+j)^m.$$

By Proposition (1.19) and by negated Binomial sums of powers (1.20), we obtain multifold sums of powers in Binomial form.

Proposition 4.4 (Multifold backward binomial sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r\, n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} - \sum_{s=1}^{r} \binom{s-t}{j+s} \binom{r-s+n-1}{r-s} \right] \nabla^j (t+j)^m.$$

By negated hockey-stick identity (1.7), the negated formula for sums of powers follows.

Proposition 4.5 (Negated backward binomial sums of powers)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \binom{r-s+n-1}{r-s} \right] \nabla^j (t+j)^m.$$

By re-indexing the sum in s gives its shifted variation.

Proposition 4.6 (Shifted multifold backward Binomial sums)**.** *For integers $r \geq 0$, and $n \geq 0$, and $m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{j+t-1}{j+s+1} \binom{r-s+n-2}{r-s-1} \right] \nabla^j (t+j)^m.$$

5. Central difference form

Theorem (1.16) can be expressed in terms of central differences easily, because,

Proposition 5.1. *For integers j, m, t,*

$$\Delta^j t^m = \delta^j \left(t + \frac{j}{2} \right)^m.$$

Thus, by negated multifold sums of powers (1.17) and by Proposition (5.1), we obtain formula for multifold sums of powers in terms of central differences δ.

Proposition 5.2 (Multifold central sums of powers)**.** *For integers $r, n, m \geq 0$, and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \left[\binom{n-t+r}{j+r} + \sum_{s=1}^{r} (-1)^{j+s-1} \binom{j+t-1}{j+s} \Sigma^{r-s} n^0 \right] \delta^j \left(t + \frac{j}{2} \right)^m.$$

6. Future research

In this manuscript, we focus on derivation of formulas for sums of powers, by combining Newton's formula for powers (1.3) and hockey stick identities (1.4), and (1.5). This idea, however, can be generalized even further. Hence, we are free to utilize interpolation formulas for powers in terms of generalized difference operator D, and binomial coefficients $\binom{f(n)}{k}$. We are not limited to linear difference operators Δ, ∇ and δ. There is a complete theory on dynamic differential equations on time-scales, founded by Bohner and Peterson in [10]. In general, dynamic differential operators assume that the rate of function's change h is represented by another function $g(h)$, which is not necessary a linear one. For example, the Jackson's derivative [11] is as follows.

$$D_q(f(x)) = \frac{f(qx) - f(x)}{qx - x}.$$

Thus, finding interpolation formula for $f(x) = x^n$ in terms of $D_q(f(x))$ implies new formulas for sums of powers. In general, let be an abstract interpolation formula for powers,

$$n^m = \sum_k \binom{f(n)}{k} D(n^m, k).$$

Therefore, we obtain one of formulas for sums of powers, that involves the abstract difference operator D, evaluated at the point k. By hockey stick identity for binomial coefficients $\binom{n}{k}$, we get its closed form.

$$\Sigma^1 n^m = \sum_k D(n^m, k) \sum_{j \le n} \binom{f(j)}{k}.$$

Similarly, for multifold sums of powers,

$$\Sigma^r n^m = \sum_k D(n^m, k) \binom{f(n+r)}{k}.$$

Many interpolation approaches involve rising factorials $x^{(n)}$, or falling factorials $(x)_n$, or regular factorials $n!$, hence can be expressed in terms of binomial coefficients, because,

$$\frac{(x)_n}{n!} = \binom{x}{n}; \quad \frac{x^{(n)}}{n!} = \binom{x+n-1}{n}; \quad x^{[n]} = n\left(x + \frac{n}{2} - 1\right)_{n-1}.$$

In particular, Donald Knuth provides the formula for multifold sums of odd powers in [3].

Proposition 6.1 (Multifold sums of odd powers)**.** *For non-negative integers r, n, m*

$$\begin{aligned}\Sigma^r n^{2m-1} &= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} \frac{1}{2k} \delta^{2k} 0^{2m} \\ &= \sum_{k=1}^{m} \binom{n+k-1+r}{2k-1+r} (2k-1)! T(2m, 2k),\end{aligned}$$

where $T(n,k)$ are the central factorial numbers of the second kind, see [12, section 58] *and* [13, formula (10a)].

This formula originates from Newton's formula in central differences, evaluated at zero, see [2]. The reason is that central factorial numbers can be expressed in term of central differences of nothing.

$$T(n,k) = \frac{1}{k!}\delta^k 0^n = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j}\left(\frac{k}{2}-j\right)^n.$$

In general, the central factorial numbers of the second kind $T(n,k)$ are defined by polynomial identity [14, ch. 6.5, formula (24)]. Thus,

Lemma 6.2 (Riordan power identity)**.** *For integers $n, m \geq 0$,*

$$n^m = \sum_{k=1}^{m} T(m,k) n^{[k]}.$$

where $n^{[k]}$ are central factorials $n^{[k]} = n\prod_{j=0}^{k-1}\left(n+\frac{k}{2}-j\right)$, with $n^{[0]} = 1$ for all n.

The sequence A008957 in the OEIS [7] is the sequence of central factorial numbers of the second kind $T(2n, 2k)$. As future research direction, the Knuth's formula (6.1) utilizes the operator of central finite differences evaluated in zero. Thus, it is worth to investigate the existence of sums of odd powers involving the central differences evaluated at an arbitrary integer point t, similar to Theorems (1.16), and (1.17).

Conclusions

In this manuscript we have derived formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in terms of Stirling numbers of the second kind, and Eulerian numbers. In addition, in (6) we discuss the future research directions that may lead to a complete framework for sums of powers, by means of combining interpolation approaches, binomial coefficients and variations of hockey stick identity for binomial coefficients. The most important results of this manuscript are validated using Mathematica programs; see Section (7).

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7. Mathematica programs

Use the *Mathematica* package [16] to validate the results

Mathematica Function	**Validates / Prints**
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateOrdinarySumsOfPowersViaNewtonsSeries[5]`	Validates Prop. (1.9)
`ValidateDoubleSumsOfPowersViaNewtonsSeries[5]`	Validates Prop. (1.13)
`ValidateMultifoldSumsOfPowersViaNewtonsSeries2[5]`	Validates Thm. (1.16)
`ValidateMultifoldSumsOfPowersViaNewtonsSeries[5]`	Validates Thm. (1.17)
`ValidateFiniteDifferenceViaStirlingNumbers[10]`	Validates Lem. (2.1)
`ValidateFiniteDifferenceViaEulerianNumbers[10]`	Validates Lem. (3.2)
`ValidateMultifoldSumsOfPowersViaCentralDifferences[5]`	Validates Prop. (5.2)
`ValidateMultifoldSumsOfPowersViaBackwardDifferences[5]`	Validates Prop. (4.3)
`ValidateMultifoldSumsOfPowersViaStirlingNumbers[5]`	Validates Prop. (2.6)
`ValidateMultifoldSumsOfPowersViaEulerianNumbers[5]`	Validates Prop. (3.4)
`ValidateMultifoldSumsOfPowersBinomialForm[5]`	Validates Prop. (1.20)
`ValidateMultifoldSumsOfPowersBinomialFormReindexed[5]`	Validates Prop. (1.21)
`ValidateMultifoldSumOfZeroPowers[10]`	Validates Prop. (1.18)

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