UNEXPECTED POLYNOMIAL IDENTITY

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Abstract. This paper revisits Faulhaber-type identities for odd powers, inspired by Knuth's exposition on power sums. Focusing on central factorial numbers and binomial representations, we derive and analyze an unexpected identity for n^3 involving finite differences and triangular numbers. By reformulating cube identities through nested summations and symmetric expressions like $\sum_{k=1}^{n} 6k(n-k)+1$, we generalize to arbitrary odd powers n^{2m+1} using real coefficients $A_{m,r}$ in the form $\sum_{k=1}^{n} A_{m,r}k^r(n-k)^r$. A table of these coefficients is provided, raising questions about their origin, recurrence, and whether such expressions admit interpretation as a form of polynomial interpolation.

1. Introduction

Recently, I've come back again to one of the prominent articles in area of polynomials, power sums etc. That is *Johann Faulhaber and sums of powers* [1]. Indeed, it is a great source to reach piece of mind in mathematics. The thing that occupied my attention was the odd power identity in terms of Binomial coefficients and Central factorial numbers, which can be found on page 9

$$n^1 = \binom{n}{1}$$

$$n^3 = 6\binom{n+1}{3} + \binom{n}{1}$$

$$n^5 = 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}$$

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It is particularly interesting that well-known identity in terms of triangular numbers and finite differences of cubes becomes clear and obvious

$$\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$$

where $\binom{n+1}{2}$ are triangular numbers. It is true that

$$\Delta n^3 = \left[6\binom{n+2}{3} + \binom{n+1}{1}\right] - \left[6\binom{n+1}{3} + \binom{n}{1}\right] = 6\binom{n+1}{2} + \binom{n}{0}$$

by means of binomial coefficients' recurrence. Moreover, the concept above allows to reach N-order power sum $\sum^N k^{2m+1}$ or finite difference $\Delta^N k^{2m+1}$ of odd powers simply by changing binomial coefficients indexes. Quite strong and impressive.

However, unexpected odd power identity appears when we notice that triangular numbers in $\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$ are equivalent to the sum of first n non-negative integers

$$\binom{n+1}{2} = \sum_{k=0}^{n} k$$

Which leads to identity in terms of finite differences of cubes

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6\sum_{k=0}^{n} k$$

It is obvious that n^3 evaluates to the sum of its $n-1$ finite differences, so that

$$n^3 = \sum_{k=0}^{n-1} \Delta k^3 = \sum_{k=0}^{n-1} \left(1 + 6\sum_{t=0}^{k} t\right) \tag{1}$$

In its explicit form

$$\begin{aligned} n^3 &= [1 + 6\cdot 0] + [1 + 6\cdot 0 + 6\cdot 1] + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2] \\ &+ \cdots + [1 + 6\cdot 0 + 6\cdot 1 + 6\cdot 2 + \cdots + 6\cdot (n-1)] \end{aligned} \tag{2}$$

We could use Faulhaber's formula on $\sum_{t=0}^{k} t$ in equation (1), which leads to well known and expected identity in cubes $n^3 = \sum_{k=0}^{n-1} \sum_{t=0}^{2} \binom{3}{t} k^t$.

Instead, let's rearrange the terms in (2) to get

$$n^3 = n + [(n-0)\cdot 6\cdot 0] + [(n-1)\cdot 6\cdot 1] + [(n-2)\cdot 6\cdot 2] \tag{3}$$
$$+ \cdots + [(n-k)\cdot 6\cdot k] + \cdots + [1\cdot 6\cdot (n-1)]$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1 \tag{4}$$

By inspecting the expression $6k(n-k)+1$ we iterate under summation, we can notice that it is symmetric over k. Let be $T(n,k) = 6k(n-k)+1$ then

$$T(n,k) = T(n,n-k)$$

This symmetry allows us to alter summation bounds easily, so that

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{5}$$

Assume that polynomial identities $n^3 = \sum_{k=0}^{n-1} 6k(n-k)+1$ and $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ have explicit form as follows

$$n^3 = \sum_k A_{1,1}k^1(n-k)^1 + A_{1,0}k^0(n-k)^0$$

where $A_{1,1} = 6$ and $A_{1,0} = 1$, respectively.

It could be generalized even further, for every odd power $2m+1$, by utilizing the set of real coefficients $A_{m,0}, A_{m,1}, A_{m,2}, A_{m,3}, \ldots, A_{m,m}$. Therefore,

$$n^{2m+1} = \sum_{k=1}^{n} A_{m,0}k^0(n-k)^0 + A_{m,1}k^1(n-k)^1 + \cdots + A_{m,m}k^m(n-k)^m \tag{6}$$

It leads us to numerous polynomial identities, including compact form sums

$$n^{2m+1} = \sum_{r=0}^{m}\sum_{k=1}^{n} A_{m,r} k^r (n-k)^r; \quad n^{2m+1} = \sum_{r=0}^{m}\sum_{k=0}^{n-1} A_{m,r} k^r (n-k)^r$$

For example,

$$n^3 = \sum_{k=1}^{n} 6k(n-k)+1$$

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2+1$$

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k)+1$$

$$n^9 = \sum_{k=1}^{n} 630k^4(n-k)^4 - 120k(n-k)+1$$

$$n^{11} = \sum_{k=1}^{n} 2772k^5(n-k)^5 + 660k^2(n-k)^2 - 1386k(n-k)+1$$

Table of these coefficients

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $A_{m,r}$. See OEIS sequences [2, 3].

Recurrence relation for $A_{m,r}$ is given by [4]. One more interesting observation about row sums

$$\sum_{r=0}^{m} A_{m,r} = 2^{2m+1} - 1$$

2. Questions

Question 2.1. *The algorithm we used to obtain identities for cubes* (4) *and* (5) *is quite simple, if not naive. I believe it should be discussed in mathematical literature, as well as identity that gives a set of real coefficients $A_{m,r}$ such that*

$$n^{2m+1} = \sum_{k=1}^{n} A_{m,0}k^0(n-k)^0 + A_{m,1}k^1(n-k)^1 + \cdots + A_{m,m}k^m(n-k)^m$$

However, I was not able to find any widely-known references that mention coefficients $A_{m,r}$, which is one of open questions. If you find any appearance of these identities in widely-known math sources – please let me know.

Question 2.2. *Is the process we used in obtaining the identities* (4) *and* (5) *is an interpolation technique?*

References

[1] Knuth, Donald E. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993. https://arxiv.org/abs/math/9207222.

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