SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. We obtain formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we note that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind.

CONTENTS

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1. Introduction and main results

Define multifold sums of powers in Knuth's [1] notation

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

The book Interpolation by Steffensen [2, chapter 2, eq. (19)] gives Newton's formula for backward differences evaluated in zero $f(x) = \sum_{k=0}^{n} \binom{x+k-1}{k} \nabla^k f(0)$.

In general,

Proposition 1.1 (Newton formula via backward differences)**.**

$$f(x) = \sum_{k=0}^{n} \binom{x-a+k-1}{k} \nabla^k f(a)$$

where $\nabla^k f(a) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(a-j)$.

Thus, by setting $f(n) = n^m$

$$n^m = \sum_{j=0}^{m} \binom{n-t+j-1}{j} \nabla^j t^m,$$

where $\nabla^j t^m = \sum_{k=0}^{j} (-1)^k \binom{j}{k} (t-k)^m$. Therefore, ordinary sums of powers is equivalent to

$$\Sigma^1 n^m = \sum_{j=0}^{m} \nabla^j t^m \sum_{k=1}^{n} \binom{k-t+j-1}{j}$$

We notice that the sum $\sum_{k=1}^{n} \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j-t$ and $b = j-t-1+n$, we get

$$\sum_{k=1}^{n} \binom{-t+j-1+k}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j}$$

Thus,

$$\sum_{k=1}^{n} \binom{-t+j-1+k}{j} = \binom{j-t+n}{j+1} - \binom{j-t}{j+1}$$

Because,

$$\sum_{m=a}^{b}\binom{m}{j}=\binom{b+1}{j+1}-\binom{a}{j+1}$$

Applying the identity for binomial coefficients $\binom{-k}{j}=(-1)^j\binom{j+k-1}{j}$, we obtain

Proposition 1.2 (Ordinary sums of powers via backward differences)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m=\sum_{j=0}^{m}\nabla^j t^m\left[(-1)^j\binom{t}{j+1}+\binom{j-t+n}{j+1}\right]$$

For example, by setting $t=2$ and $m=1,2,3,4$, we get formulas for sums of cubes

$$\Sigma^1 n^1=2\left[-\binom{2}{1}+\binom{n-2}{1}\right]+1\left[\binom{2}{2}+\binom{n-1}{2}\right],$$

$$\Sigma^1 n^2=4\left[-\binom{2}{1}+\binom{n-2}{1}\right]+3\left[\binom{2}{2}+\binom{n-1}{2}\right]+2\left[-\binom{2}{3}+\binom{n}{3}\right].$$

$$\Sigma^1 n^3=8\left[-\binom{2}{1}+\binom{n-2}{1}\right]+7\left[\binom{2}{2}+\binom{n-1}{2}\right]+6\left[-\binom{2}{3}+\binom{n}{3}\right]+6\left[\binom{2}{4}+\binom{n+1}{4}\right].$$

$$\Sigma^1 n^4=16\left[-\binom{2}{1}+\binom{n-2}{1}\right]+15\left[\binom{2}{2}+\binom{n-1}{2}\right]+14\left[-\binom{2}{3}+\binom{n}{3}\right]+12\left[\binom{2}{4}+\binom{n+1}{4}\right]+24\left[-\binom{2}{5}+\binom{n+2}{5}\right].$$

The coefficients $1,2,1,4,3,2,8,7,6,6,\cdots$ for $t=2$ is the sequence A391068 in the OEIS [3]. For $t=0$ the coefficients are $1,0,1,0,-1,2,0,1,-6,6,\cdots$ and registered in the OEIS as A278075. For $t=1$ the coefficients are $1,1,1,1,1,2,1,1,0,6,\cdots$ and registered in the OEIS as A389570. For $t = 3$ the coefficients are $1, 3, 1, 9, 5, 2, 27, 19, 12, 6, \cdots$ and registered in the OEIS as A391210.

Lemma 1.3 (Backward differences in Eulerian numbers)**.**

$$\Delta^j t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k-j}{m-j}$$

Proof. By Worpitzky identity $t^m = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{t+k}{m}$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, see [4]. $\square$

Thus, let be a formula for ordinary sums of powers in terms of Eulerian numbers $\left\langle {m \atop k} \right\rangle$

Proposition 1.4 (Ordinary sums of powers in Eulerian numbers)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \left\langle {m \atop k} \right\rangle \binom{t+k-j}{m-j}$$

Lemma 1.5 (Backward differences in Stirling numbers)**.**

$$\nabla^j t^m = \sum_{k=j}^{m} \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

Proof. By the identity $t^m = \sum_{k=0}^{m} \binom{t}{k} \left\{ {m \atop k} \right\} k!$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. $\square$

Thus, let be a formula for ordinary sums of powers in terms of Stirling numbers $\left\{ {m \atop k} \right\}$

Proposition 1.6 (Ordinary sums of powers in Stirling numbers)**.** *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

2. Conclusions

In this manuscript, we derived formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we noticed that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind.

Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind. All the results are validated using `Mathematica` programs, see dedicated section below.

References

[1] Knuth, Donald E. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993. https://arxiv.org/abs/math/9207222.

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[4] J. Worpitzky. Studien über die bernoullischen und eulerschen zahlen. *Journal für die reine und angewandte Mathematik*, 94:203–232, 1883. http://eudml.org/doc/148532.

[5] Petro Kolosov. Mathematica programs for backward finite differences and newton's formula. https://github.com/kolosovpetro/SumsOfPowersViaBackwardFiniteDifferencesAndNewtonFormula/tree/main/mathematica, 2026. GitHub repository, Mathematica source files.

Mathematica programs

Use the *Mathematica* package [5] to validate the results

Mathematica Function	**Validates / Prints**
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]`	Validates Proposition (1.2)
`ValidateBackwardDifferencesInEulerianNumbers[20]`	Validates Lemma (1.3)
`ValidateOrdinarySumsOfPowersInEulerianNumbers[10]`	Validates Proposition (1.4)
`ValidateBackwardDifferencesInStirlingNumbers[20]`	Validates Lemma (1.5)
`ValidateOrdinarySumsOfPowersInStirlingNumbers[20]`	Validates Proposition (1.6)

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Sources: github.com/kolosovpetro/SumsOfPowersViaBackwardDifferences

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