UNEXPECTED POLYNOMIAL IDENTITIES ARISING FROM A CLASSICAL INTERPOLATION PROBLEM

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ABSTRACT. This manuscript originates from a classical interpolation problem: how to reconstruct the cubes n^3 from their finite differences. The investigation leads to an unexpected identity expressing n^3 as sum of bivariate terms $k(n-k)$, such that $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$. This identity serves as the base case for a more general identity for odd powers, involving rational numbers $\mathbf{A}_{m,r}$ that is $n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$. We evaluate the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \cdots, \mathbf{A}_{m,m}$ by solving a system of linear equations. Furthermore, this work provides a recurrence relation for coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \cdots, \mathbf{A}_{m,m}$, by utilizing generating functions. The main results include odd power identities, identities for binomial forms, and identities for sums of powers. Apart that, we discuss the similarities between our findings and well-known results like Pascal's identity etc. Afterward, the manuscript continues with related works that are based on our findings, including approximation for powers, derivatives, Faulhaber-like formulas. This manuscript concludes with discussion of future research directions that include the topics of integration into mathematical literature, approximation methods, combinatorial interpretations, and q-derivatives.

CONTENTS

1. Discussion on interpolation of cubes

This is the story of unexpected results obtained by an amateur student with a deep curiosity for mathematics. Although not a specialist in mathematics, the young explorer always possessed a strong sense of mathematical beauty and aesthetics. The student's mathematical knowledge was limited to undergraduate-level courses, which included the basics of matrix operations, basic calculus, and elementary linear algebra. One day, the student found himself observing tables of finite differences, in particular, finite differences of cubes. That was the year 2014.

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of n^3.

Thus, the following problem was stated

Problem 1.1. *How to reconstruct the value of n^3 from the values of its finite differences?*

Precisely, the inquiry was to find a way to reconstruct the values of the sequence $\{0, 1, 8, 27, 64, \ldots\}$ given the values of its finite differences. Clearly, the inquiry refers to the process of interpolation.

The problem of interpolation is quite old and can be traced back to ancient Babylonian and Greek times, several centuries BC and first centuries AD [1]. Interpolation – is the process of finding new data points based on the range of a discrete set of known data points. Interpolation, as we know it today, has been developed between 1674–1684 by Sir

Isaac Newton. His works in interpolation are frequently referenced as the foundation of classical interpolation theory [2]. For instance, Newton's interpolation formula addresses the problem (1.1) immediately, because

$$27 = 6\binom{3}{3} + 6\binom{3}{2} + 1\binom{3}{1} + 0\binom{3}{0} = 6 + 18 + 3$$

where $6, 6, 1, 0$ correspond to the first row in table (1). In general,

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + 1\binom{n}{1} + 0\binom{n}{0}$$

because $f(x) = \sum_{k=0}^{d} \Delta^{d-k} f(0)\binom{x}{d-k}$, see [3, p. 190].

Great! But there is one thing: the student who stated the problem (1.1) had no clue about interpolation theory at all. What was the action taken next? Exactly, our inquirer decided to re-invent the solution himself, being driven by the pure feeling of mystery. The student's mind was occupied by only one thought: *All mathematical truths exist timelessly, we only reveal and describe them.* That mindset inspired our young researcher to start his own mathematical journey.

By observing the finite differences from table (1), we can notice that the first order finite difference of cubes may be expressed in terms of its third order finite difference $\Delta^3(n^3) = 6$, as follows

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

$$\vdots$$

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6n$$

By using sigma notation, we get

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k$$

However, there is a more beautiful way to prove that $\Delta(n^3) = 1 + 6\sum_{k=0}^{n} k$. We refer to one of the finest articles in the area of polynomials and power sums, that is *Johann Faulhaber and sums of powers* written by Donald Knuth [4]. We now focus on the odd power identities shown at [4, p. 9]

$$n^1 = \binom{n}{1}$$

$$n^3 = 6\binom{n+1}{3} + \binom{n}{1}$$

$$n^5 = 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}$$

It is quite interesting that the identity in terms of triangular numbers $\binom{n+1}{2}$ and finite differences of cubes becomes more obvious

$$\Delta n^3 = (n+1)^3 - n^3 = 6\binom{n+1}{2} + \binom{n}{0}$$

It easy to see that

$$\Delta n^3 = \left[6\binom{n+2}{3} + \binom{n+1}{1}\right] - \left[6\binom{n+1}{3} + \binom{n}{1}\right] = 6\binom{n+1}{2} + \binom{n}{0}$$

because $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Moreover, the concept above allows us to reach r-fold power sums $\sum^r k^{2m+1}$ or finite differences $\Delta^r k^{2m+1}$ by altering binomial coefficients indices. Quite strong and impressive.

$$\Sigma^r n^3 = 6\binom{n+1+r}{3+r} + \binom{n+r}{1+r}$$

$$\Delta^r n^5 = 120\binom{n+2-r}{5-r} + 30\binom{n+1-r}{3-r} + \binom{n-r}{1-r}$$

We can observe that triangular numbers $\binom{n+1}{2}$ are equivalent to

$$\binom{n+1}{2} = \sum_{k=0}^{n}\binom{k}{1} = \sum_{k=0}^{n} k$$

because $\binom{n+1}{m+1} = \sum_{k=0}^{n}\binom{k}{m}$. This leads us to the fact that

$$\Delta n^3 = (n+1)^3 - n^3 = 1 + 6\sum_{k=0}^{n} k$$

An experienced mathematician would immediately notice a spot to apply Faulhaber's formula [5] to get the closed form of the sum $\sum_{k=0}^{n} k$

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Thus, the finite difference $\Delta(n^3)$ takes the well-known form, which matches Binomial theorem [6]

$$\Delta(n^3) = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2}\binom{3}{k}n^k$$

And... that could be the end of the story, isn't it? Because all what remains is to say that

$$n^3 = \sum_{k=0}^{n-1}(k+1)^3 - k^3 = \sum_{k=0}^{n-1}\left(1 + 6\sum_{t=0}^{k} t\right) = \sum_{k=0}^{n-1} 1 + 3k + 3k^2$$

Thus, the formula for cubes n^3 was derived, and thus, our protégée's problem (1.1) was successfully solved, because we have found a formula that recovers the cubes n^3 from their finite differences. It is indeed straightforward that $5^3 = \sum_{k=0}^{4} 1 + 3k + 3k^2 = 1 + 7 + 19 + 37 + 61 = 125$ which is exactly the sum of terms of the first column of table (1).

However, not this time. Luckily enough (say), the student who stated the problem (1.1) was not really aware of the approaches above either. What a lazy student! Probably, that is exactly the case when unawareness leads to a fresh sight into classical problems, leading to unexpected results and new insights. Instead, our investigator observed a different pattern in $\Delta n^3 = 6\binom{n+1}{2} + \binom{n}{0} = 1 + 6\sum_{k=0}^{n} k$.

Consider the polynomial n^3 as sum of its finite differences $\Delta n^3 = 1 + 6\sum_{k=0}^{n} k$

$$n^3 = [1 + 6 \cdot 0]$$

$$+ [1 + 6 \cdot 0 + 6 \cdot 1]$$

$$+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots$$

$$+ [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]$$

We can observe that the term 1 appears n times, while the term $6 \cdot 0$ appears $n - 0$ times. The term $6 \cdot 1$ appears $n - 1$ times and so on. By combining the recurring common terms,

we get

$$n^3 = n + [(n - 0) \cdot 6 \cdot 0]$$

$$+ [(n - 1) \cdot 6 \cdot 1]$$

$$+ [(n - 2) \cdot 6 \cdot 2] + \cdots$$

$$+ [(n - k) \cdot 6 \cdot k] + \cdots$$

$$+ [1 \cdot 6 \cdot (n - 1)]$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n - k)$$

We can freely move the term n under the summation, because there are exactly n iterations. Therefore,

$$n^3 = \sum_{k=0}^{n-1} 6k(n - k) + 1$$

By inspecting the expression $6k(n - k) + 1$, we can notice that it is symmetric over k. Let be $T_1(n, k) = 6k(n - k) + 1$ then

$$T_1(n, k) = T_1(n, n - k)$$

This symmetry allows us to alter summation bounds easily. Hence,

$$n^3 = \sum_{k=1}^{n} 6k(n - k) + 1$$

By arranging the values of $T_1(n, k)$ in a table, we see that the identities above are indeed true.

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $T_1(n,k) = 6k(n-k)+1$. See the sequence A287326 in OEIS [7].

The following recurrence holds for $T_1(n,k)$

$$T_1(n,k) = 2T_1(n-1,k) - T_1(n-2,k)$$

Which is indeed true, because

$$T_1(5,2) = 2 \cdot 25 - 13 = 37$$

Finally, our curious learner has reached the first milestone, by finding an unusual solution to the problem (1.1). What an excitement it was! However, the excitement shall not last too long. Indeed, curiosity is not the thing that can be fulfilled completely, and thus new questions arise. Somehow, the inquirer got a strong feeling that more general law hides behind the identity $n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$. Quite intuitive it was. Fair enough that the next problem was

Problem 1.2. *Is there a generalization of identity $n^3 = \sum_{k=1}^{n} 6k(n-k)+1$ for higher-degree powers?*

However, it was not so easy for our young explorer to generalize cube identity for higher powers n^4 or n^5 by observing the tables of finite differences, as before. The previous approach to express the difference of cubes Δn^3 in terms of $\Delta^3 n^3 = 6$, reaching the identity $n^3 = \sum_k 6k(n-k)+1$ — was not successful. Moreover, it was not even clear what is the general

form of polynomial identity to search for. Thus, the problem (1.2) was shared with the mathematical community, and there was an answer.

2. System of linear equations

In 2018, Albert Tkaczyk shared two preprints [8, 9] addressing the problem (1.2). These preprints discussed the cases of n^5, n^7 and n^9 in context of problem (1.2). Furthermore, these results were refined and published in *Mathematical gazette* [10] in 2021. Tkaczyk assumed that the identity for n^5 takes the following explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C \right]$$

where A, B, C are unknown coefficients. We denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to get the compact form of double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

By observing the equation above, the general form of the odd-power identity becomes clearer. We define $x^0 = 1$ for all x, see [3, p. 162], because when $k = n$ and $r = 0$ the term $k^r(n-k)^r = n^0 \cdot 0^0$, thus we define $x^0 = 1$ for all x.

To evaluate the set of coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ we have to build and solve a system of linear equations

$$n^5 = \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0(n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1(n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2(n-k)^2$$

By expanding the sums $\sum_{k=1}^{n} k^r(n-k)^r$ using Faulhaber's formula [5], we get

$$\mathbf{A}_{2,0} n + \mathbf{A}_{2,1} \left[\frac{1}{6}(n^3 - n) \right] + \mathbf{A}_{2,2} \left[\frac{1}{30}(n^5 - n) \right] - n^5 = 0$$

By multiplying both sides by 30, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(n^3 - n) + \mathbf{A}_{2,2}(n^5 - n) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms yields

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} n + 5\mathbf{A}_{2,1} n^3 - \mathbf{A}_{2,2} n + \mathbf{A}_{2,2} n^5 - 30n^5 = 0$$

By combining the common terms, we obtain

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore,

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} & = 0 \\\\ \mathbf{A}_{2,1} & = 0 \\\\ \mathbf{A}_{2,2} - 30 & = 0 \end{cases}$$

By solving the system above, we evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$

$$\begin{cases} \mathbf{A}_{2,2} & = 30 \\\\ \mathbf{A}_{2,1} & = 0 \\\\ \mathbf{A}_{2,0} & = 1 \end{cases}$$

Thus, the identity for n^5

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

Again, the terms $30k^2(n-k)^2 + 1$ are symmetric over k. Let be $T_2(n,k) = 30k^2(n-k)^2 + 1$ then

$$T_2(n,k) = T_2(n, n-k)$$

By arranging the values of $T_2(n,k)$ in a table, we may observe that the identity for n^5 above is indeed true

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Values of $T_2(n,k) = 30k^2(n-k)^2 + 1$. See the sequence A300656 in OEIS [11].

The following recurrence holds for $T_2(n,k)$

$$T_2(n,k) = 3T_2(n-1,k) - 3T_2(n-2,k) + T_2(n-3,k)$$

Which is indeed true because

$$T_2(6,2) = 3 \cdot 1081 - 3 \cdot 481 + 271 = 1921$$

Thus, our curious learner obtained the solution for the problem (1.2). This time, the solution contained even more than a methodology to evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \ldots, \mathbf{A}_{2,2}$ — it revealed a general form of identity for odd powers.

Hence, let be a conjecture

Conjecture 2.1. *For non-negative integers* m *and* $n \geq 1$*, there is a set of coefficients* $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ *such that*

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r$$

As we already know, we have to build and solve a system of linear equations to evaluate the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$. However, the conjecture (2.1) cannot be proven by building and solving countless systems of linear equations for any integer $m \geq 0$. There

must be a formula that evaluates the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ for every non-negative integer m — our young investigator thought. Yet again, the intuition was correct.

3. RECURRENCE RELATION

In 2018, a recurrence formula [12] for coefficients $\mathbf{A}_{m,r}$ was proposed by Dr. Max Alekseyev, George Washington University. The main idea of Alekseyev's approach was to utilize a certain generating function to evaluate the set of coefficients $\mathbf{A}_{m,r}$, starting from the base case $\mathbf{A}_{m,m}$, then to compute the previous coefficient $\mathbf{A}_{m,m-1}$ recursively, similarly up to $\mathbf{A}_{m,0}$. We use a specific version of Faulhaber's formula [5] with upper summation bound set to $p+1$

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} \binom{p+1}{j} B_j n^{p+1-j} = \frac{1}{p+1} \left[\sum_{j=0}^{p+1} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1}$$

The reason we use the Faulhaber's formula above is that we tend to omit summation bounds, for simplicity. This helps us to collapse the common terms across complex sums, because now we can let the sum run over all integers j, while only finitely many terms $\binom{p+1}{j}$ are non-zero, see also [13]. Hence,

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \left[\sum_{j} \binom{p+1}{j} B_j n^{p+1-j} \right] - \frac{B_{p+1}}{p+1} \tag{1}$$

Now we expand the sum $\sum_{k=1}^{n} k^r (n-k)^r$ using binomial theorem

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{k=1}^{n} k^r \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} k^t = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \sum_{k=1}^{n} k^{t+r}$$

By applying Faulhaber's formula (1) to $\sum_{k=1}^{n} k^{t+r}$, we get

$$\sum_{k=1}^{n} k^r (n-k)^r = \sum_{t=0}^{r} (-1)^t \binom{r}{t} n^{r-t} \left[\left(\frac{1}{t+r+1} \sum_{j} \binom{t+r+1}{j} B_j n^{t+r+1-j} \right) - \frac{B_{t+r+1}}{t+r+1} \right]$$

$$= \sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} \left[\left(\sum_{j} \binom{t+r+1}{j} B_j n^{2r+1-j} \right) - B_{t+r+1} n^{r-t} \right]$$

By expanding brackets

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j}\right]$$
$$-\left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

By moving the sum in j and omitting summation bounds in t

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right]$$
$$-\left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

By rearranging the sums we obtain

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] \tag{2}$$
$$-\left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

We can notice that

Lemma 3.1 (Piecewise Binomial identity). *For non-negative integers r, j*

$$\sum_{t}\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases} \dfrac{1}{(2r+1)\binom{2r}{r}} & \text{if } j = 0 \\[2ex] \dfrac{(-1)^r}{j}\dbinom{r}{2r-j+1} & \text{if } j > 0 \end{cases}$$

Proof. For $j = 0$ we have

$$\sum_{t}\binom{r}{t}\frac{(-1)^t}{r+t+1} = \sum_{t}\binom{r}{t}(-1)^t\int_{0}^{1}z^{r+t}\,dz$$

Because $\frac{1}{r+t+1} = \int_{0}^{1}z^{r+t}\,dz$.

$$\sum_{t}\binom{r}{t}(-1)^t\int_{0}^{1}z^{r+t}\,dz = \int_{0}^{1}z^r\left(\sum_{t}\binom{r}{t}(-1)^tz^t\right)dz = \int_{0}^{1}z^r(1-z)^r\,dz$$

The work [14] provides the identity $\frac{1}{\binom{n}{k}} = (n+1) \int_0^1 z^k (1-z)^{n-k} \, dz$. By setting $n = 2r$ and $k = r$, for $j = 0$ yields

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \int_0^1 z^r (1-z)^r \, dz = \frac{1}{(2r+1)\binom{2r}{r}}$$

For $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \sum_t \frac{(-1)^t}{j} \binom{r}{t} \binom{r+t}{j-1}$$

Because $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$. Now apply the coefficient extraction $[z^k]$ to represent the coefficient of z^k. For example: $[z^k](1+z)^r = \binom{r}{k}$. Therefore,

$$\sum_t \frac{(-1)^t}{j} \binom{r}{t} \binom{r+t}{j-1} = [z^{j-1}] \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^{r+t}$$

By factoring out $(1+z)^r$ from the sum

$$[z^{j-1}] \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^{r+t} = [z^{j-1}](1+z)^r \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^t$$

Now apply the binomial theorem to the inner sum

$$\sum_t \binom{r}{t} (-1)^t (1+z)^t = (1 - (1+z))^r = (-z)^r = (-1)^r z^r$$

Hence, for $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \frac{(-1)^r}{j} [z^{j-1}](1+z)^r z^r$$

By applying the identity $[z^{p-q}]A(z) = [z^p]z^q A(z)$

$$\frac{(-1)^r}{j} [z^{j-1}](1+z)^r z^r = \frac{(-1)^r}{j} [z^{j-1-r}](1+z)^r = \frac{(-1)^r}{j} \binom{r}{j-1-r}$$

Finally, we use symmetry $\binom{n}{k} = \binom{n}{n-k}$ to show that for $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \frac{(-1)^r}{j} \binom{r}{j-1-r} = \frac{(-1)^r}{j} \binom{r}{2r-j+1}$$

This completes the proof. $\qquad \square$

To simplify equation (2) using binomial identity (3.1), we have to move $j = 0$ out of the summation, to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_{j=1}^{\infty} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right]$$
$$- \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Hence, we simplify equation (2) by using binomial identity (3.1)

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_{j=1}^{\infty} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]$$
$$- \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

By setting $\ell = 2r - j + 1$ to $\sum_{j=1}^{\infty}$, and $\ell = r - t$ to $\sum_t$, we collapse common terms across two sums

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right]$$
$$- \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right]$$
$$= \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}$$

By replacing odd $\ell = 2k + 1$, we get

Proposition 3.2 (Bivariate Faulhaber's Formula). *For integers $r \geq 0$ and $n \geq 1$*

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + (-1)^r \sum_{k=0}^{r-1} \binom{r}{2k+1} \frac{B_{2r-2k}}{r-k} n^{2k+1}$$

Assuming that coefficients $\mathbf{A}_{m,r}$ are defined by odd-power identity (2.1), we obtain the following relation for polynomials in n

$$R_m = \sum_{r=0}^{\infty} \mathbf{A}_{m,r} \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_{r=0}^{\infty} (-1)^r \sum_{k=0}^{\infty} \mathbf{A}_{m,r} \binom{r}{2k+1} \frac{B_{2r-2k}}{r-k} n^{2k+1} \right] - n^{2m+1} \equiv 0 \quad (3)$$

We now fix the unused values of $\mathbf{A}_{m,r}$ so that $\mathbf{A}_{m,r} = 0$ for every $r < 0$ or $r > m$, thus we can let r running over all integers. By extracting the coefficient of n^{2m+1} in (3) implies

$$\mathbf{A}_{m,m} = (2m + 1)\binom{2m}{m}$$

because $[n^{2m+1}]R_m = \mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0$. To extract the coefficient of n^{2m+1} from the series (3), we isolate the relevant terms by setting $r = m$ in $\sum_{r=0}^{m} \mathbf{A}_{m,r}\frac{n^{2r+1}}{(2r+1)\binom{2r}{r}}$ and $k = m$ in $\sum_{r=0}^{m}(-1)^r \sum_{k=0}^{\infty} \mathbf{A}_{m,r}\binom{r}{2k+1}\frac{B_{2r-2k}}{r-k}n^{2k+1}$, which gives

$$[n^{2m+1}]R_m = \mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} + \left[\sum_{r=2m+1}^{\infty}(-1)^r\mathbf{A}_{m,r}\binom{r}{2m+1}\frac{B_{2r-2m}}{r-m}\right] - 1 = 0$$

We observe that the sum

$$\sum_{r=2m+1}^{\infty}(-1)^r\mathbf{A}_{m,r}\binom{r}{2m+1}\frac{B_{2r-2m}}{r-m}$$

does not contribute to the determination of the coefficients $\mathbf{A}_{m,r}$, because $\mathbf{A}_{m,r}$ is zero for all $r > m$, by definition. Consequently, all terms in the sum are zero. Thus,

$$\mathbf{A}_{m,m}\frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0 \implies \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Taking the coefficient of n^{2d+1} for an integer d in range $\frac{m}{2} \leq d \leq m - 1$ in (3) gives

$$[n^{2d+1}]R_m = \mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + \sum_{r=2d+1}^{\infty}(-1)^r\mathbf{A}_{m,r}\binom{r}{2d+1}\frac{B_{2r-2d}}{r-d} = 0$$

which implies

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} = 0 \implies \mathbf{A}_{m,d} = 0.$$

because the sum $\sum_{r=2d+1}^{\infty}(-1)^r\mathbf{A}_{m,r}\binom{r}{2d+1}\frac{B_{2r-2d}}{r-d}$ is zero for all d in range $\frac{m}{2} \leq d \leq m - 1$. For example: let $d = m - 1$, then $r = 2d + 1 = 2m - 1$, thus $\mathbf{A}_{m,2m-1} = 0$, by definition. Let be $d = \frac{m}{2}$, then $r = 2d + 1 = m + 1$, thus $\mathbf{A}_{m,m+1} = 0$, by definition. Taking the coefficient of n^{2d+1} for d in range $\frac{m}{4} \leq d \leq \frac{m}{2} - 1$ in (3), we obtain

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0.$$

Solving for $\mathbf{A}_{m,d}$ yields

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!\,d!\,m!\,(m-2d-1)!}\cdot\frac{1}{m-d}B_{2m-2d}.$$

Proceeding recursively, we can compute each coefficient $\mathbf{A}_{m,r}$ for integers r in the range $\frac{m}{2^{s+1}} \le r \le \frac{m}{2^s} - 1$ for $s = 1, 2, \ldots$, by using previously computed values $\mathbf{A}_{m,d}$

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}.$$

Finally, we define the following recurrence relation for the coefficients $\mathbf{A}_{m,r}$

Proposition 3.3. *For integers $m \ge 0$ and r*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\[2mm] (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \le r < m \\[2mm] 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

where B_t are Bernoulli numbers [15]. We assume that $B_1 = \frac{1}{2}$ because [16, 17].

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 4. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [18, 19].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$.

- $\mathbf{A}_{m,r} = 0$ for $r < 0$ and $r > m$.

- $\mathbf{A}_{m,r} = 0$ for $m < 0$.

- $\mathbf{A}_{m,r} = 0$ for $\lfloor \frac{m}{2} \rfloor \leq r < m$.

- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$.

- $\mathbf{A}_{m,r}$ are all integers up to row $m = 11$.

- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$.

4. MAIN RESULTS

Thus, the conjecture (2.1) is true

Theorem 4.1 (Odd power identity). *For non-negative integers m and $n \geq 1$, there is a set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r$$

In explicit form

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} (kn - k^2)^r$$

$$= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[1^r \cdot (n-1)^r + 2^r \cdot (n-2)^r + 3^r \cdot (n-3)^r + \cdots + n^r \cdot (n-n)^r \right]$$

$$= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[(n-1)^r + (2n-4)^r + (3n-9)^r + (4n-16)^r + \cdots + (n^2 - n^2)^r \right]$$

For example,

- $1^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[0^r \right]$

- $2^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[1^r + 0^r \right]$

- $3^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[2^r + 2^r + 0^r \right]$

- $4^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[3^r + 4^r + 3^r + 0^r \right]$

- $5^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[4^r + 6^r + 6^r + 4^r + 0^r \right]$

- $6^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[5^r + 8^r + 9^r + 8^r + 5^r + 0^r \right]$

We define $x^0 = 1$ for all x, see [3, p. 162]. This is because when $k = n$ and $r = 0$ the term $k^r(n-k)^r = n^0 \cdot 0^0$, thus we define $x^0 = 1$ for all x.

Thus, the formula for sums of odd powers is straightforward

Proposition 4.2 (Sum of odd powers). *For integers $m \geq 0$ and $n \geq 1$*

$$\sum_{k=1}^{n} k^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\left(\sum_{k=0}^{n-1} k^r \right) + \left(\sum_{k=0}^{n-2}(2k)^r \right) + \left(\sum_{k=0}^{n-3}(3k)^r \right) + \cdots + \left(\sum_{k=0}^{n-n}(nk)^r \right) \right]$$

$$= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\left(\sum_{k=0}^{n-1} k^r \right) + \left(2^r \sum_{k=0}^{n-2} k^r \right) + \left(3^r \sum_{k=0}^{n-3} k^r \right) + \cdots + \left(n^r \sum_{k=0}^{n-n} k^r \right) \right]$$

Proof. Direct consequence of (4.1). □

Corollary 4.3 (Compact forms). *For integers $m \geq 0$ and $n \geq 1$*

$$\sum_{k=1}^{n} k^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{t=1}^{n} \sum_{k=0}^{n-t}(tk)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{t=1}^{n} \sum_{k=1}^{t}((n-t)k)^r$$

Interestingly enough that the odd power identity above is a Pascal-type identity in terms of bivariate function $k(n-k)$ and numbers $\mathbf{A}_{m,r}$. We may see it by comparing the Pascal's identity itself [20]

$$(n+1)^{m+1} - 1 = \sum_{r=0}^{m} \binom{m+1}{r} (1^r + 2^r + \cdots + n^r)$$

with identity in terms of bivariate function $k(n-k)$ and numbers $\mathbf{A}_{m,r}$

Corollary 4.4 (Bivariate Pascal's identity). *For integers $n \geq 1$ and $m \geq 0$*

$$(n+1)^{2m+1} - 1 = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[1^r n^r + 2^r(n-1)^r + 3^r(n-2)^r + 4^r(n-3)^r + \cdots + n^r(n+1-n)^r \right]$$

$$= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[n^r + (2n-2)^r + (3n-6)^r + (4n-12)^r + \cdots + n^r \right]$$

$$= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n}(k(n+1-k))^r$$

Definition 4.5 (Bivariate sum T_m). *For integers n, k and $m \geq 0$*

$$T_m(n,k) = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

Proposition 4.6 (Symmetry of T_m). *For integers n, k and m*

$$T_m(n, k) = T_m(n, n - k)$$

4.1. **Forward decompositions.**

Proposition 4.7 (Forward Recurrence for T_m). *For integers $m \geq 0, n, k$*

$$T_m(n, k) = \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n + t, k)$$

Proof. The polynomial $T_m(n, k)$ is a polynomial of degree m in n. Thus, the forward difference with respect to n is $\Delta^{m+1} T_m(n, k) = \sum_{t=0}^{m+1} (-1)^t \binom{m+1}{t} T_m(n + t, k) = 0$. By isolating $(-1)^0 \binom{m+1}{0} T_m(n - 0, k)$ yields $T_m(n, k) = (-1) \sum_{t=1}^{m+1} (-1)^t \binom{m+1}{t} T_m(n + t, k)$. $\qquad \square$

Proposition 4.8 (Odd power forward decomposition). *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n + t, k)$$

Proof. Direct consequence of (4.1) and forward recurrence (4.7). $\qquad \square$

For example: $3^5 = \binom{3}{1} 1023 - \binom{3}{2} 2643 + \binom{3}{3} 5103$. Interesting to note that by swapping the signs yields $(-3)^5 = -\binom{3}{1} 1023 + \binom{3}{2} 2643 - \binom{3}{3} 5103$.

Proposition 4.9 (Odd power forward decomposition $m - 1$). *For non-negative integers m and n*

$$n^{2m-1} = \sum_{k=1}^{n} \sum_{t=1}^{m} (-1)^{t+1} \binom{m}{t} T_{m-1}(n + t, k)$$

Proof. By setting $m \to m - 1$ to (4.8). $\qquad \square$

Proposition 4.10 (Odd power forward decomposition shifted). *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n + t, k)$$

Proof. Direct consequence of (4.1), forward recurrence (4.7), and symmetry (4.6). $\qquad \square$

Proposition 4.11 (Odd power forward decomposition $m - 1$ shifted)**.** *For integers $m \geq 1$ and $n \geq 0$*

$$n^{2m-1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m} (-1)^{t+1} \binom{m}{t} T_{m-1}(n+t, k)$$

Proof. By setting $m \to m - 1$ to (4.8) and by symmetry (4.6). $\qquad\square$

4.2. **Forward decompositions multifold.**

Proposition 4.12 (Forward Recurrence for T_m multifold)**.** *For non-negative integers m, n, k and $s \geq 1$*

$$T_m(n, k) = \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n+t, k)$$

Proposition 4.13 (Odd power forward decomposition multifold)**.** *For non-negative integers m, n and $s \geq 1$*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n+t, k)$$

Proof. Direct consequence of (4.1) and forward recurrence multifold (4.12). $\qquad\square$

Proposition 4.14 (Odd power forward decomposition $m - 1$ multifold)**.** *For non-negative integers m, n and $s \geq 0$*

$$n^{2m-1} = \sum_{k=1}^{n} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_{m-1}(n+t, k)$$

Proof. By setting $m \to m - 1$ to (4.13). $\qquad\square$

Proposition 4.15 (Odd power forward decomposition shifted multifold)**.** *For non-negative integers m, n and $s \geq 1$*

$$n^{2m+1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n+t, k)$$

Proof. Direct consequence of (4.1), forward recurrence (4.12), and symmetry (4.6). $\qquad\square$

Proposition 4.16 (Odd power forward decomposition $m - 1$ shifted multifold). *For non-negative integers m, n and $s \geq 0$*

$$n^{2m-1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_{m-1}(n+t, k)$$

Proof. By setting $m \to m - 1$ to (4.13) and by symmetry (4.6). $\qquad\square$

4.3. **Backward decompositions.**

Proposition 4.17 (Backward Recurrence for T_m). *For non-negative integers m, n and k*

$$T_m(n, k) = \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n-t, k)$$

Proof. The polynomial $T_m(n, k)$ is a polynomial of degree m in n. Thus, the backward difference with respect to n is $\nabla^{m+1} T_m(n, k) = \sum_{t=0}^{m+1} (-1)^t \binom{m+1}{t} T_m(n - t, k) = 0$. By isolating $(-1)^0 \binom{m+1}{0} T_m(n - 0, k)$ yields $T_m(n, k) = (-1) \sum_{t=1}^{m+1} (-1)^t \binom{m+1}{t} T_m(n-t, k)$. $\qquad\square$

Proposition 4.18 (Odd power backward decomposition). *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n-t, k)$$

Proof. Direct consequence of (4.1) and backward recurrence (4.17). $\qquad\square$

Proposition 4.19 (Odd power backward decomposition shifted). *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n-t, k)$$

Proof. Direct consequence of (4.1), backward recurrence (4.17), and symmetry (4.6). $\qquad\square$

Corollary 4.20 (Odd power backward decomposition $m - 1$). *For non-negative integers $m \geq 1$ and n*

$$n^{2m-1} = \sum_{k=1}^{n} \sum_{t=1}^{m} (-1)^{t+1} \binom{m}{t} T_{m-1}(n-t, k)$$

Proof. By setting $m \to m - 1$ to (4.18). $\qquad\square$

Corollary 4.21 (Odd power backward decomposition $m - 1$ shifted). *For non-negative integers $m \geq 1$ and n*

$$n^{2m-1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m} (-1)^{t+1} \binom{m}{t} T_{m-1}(n-t, k)$$

Proof. By setting $m \to m - 1$ to (4.19) and by symmetry (4.6). $\qquad\square$

4.4. **Backward decompositions multifold.**

Proposition 4.22 (Backward Recurrence for T_m multifold). *For non-negative integers m, n, k and $s \geq 1$*

$$T_m(n, k) = \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n-t, k)$$

Proposition 4.23 (Odd power backward decomposition multifold). *For non-negative integers m, n and $s \geq 1$*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n-t, k)$$

Proof. Direct consequence of (4.1) and backward recurrence (4.22). $\qquad\square$

Proposition 4.24 (Odd power backward decomposition shifted multifold). *For non-negative integers m, n and $s \geq 1$*

$$n^{2m+1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n-t, k)$$

Proof. Direct consequence of (4.1), backward recurrence (4.22), and symmetry (4.6). $\qquad\square$

Corollary 4.25 (Odd power backward decomposition $m - 1$ multifold). *For non-negative integers m, n and $s \geq 0$*

$$n^{2m-1} = \sum_{k=1}^{n} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_{m-1}(n-t, k)$$

Proof. By setting $m \to m - 1$ to (4.23). $\qquad\square$

Corollary 4.26 (Odd power backward decomposition $m-1$ shifted multifold). *For non-negative integers m, n and $s \geq 0$*

$$n^{2m-1} = \sum_{k=0}^{n-1} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_{m-1}(n-t,k)$$

Proof. By setting $m \to m-1$ to (4.24) and by symmetry (4.6). $\qquad\square$

4.5. Binomial forms.

Corollary 4.27 (Binomial form). *For integers n and a such that $n + 2a \geq 0$*

$$(n+2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=-a+1}^{n+a} \mathbf{A}_{m,r}(k+a)^r(n+a-k)^r$$

Corollary 4.28 (Shifted binomial form). *For integers n and a such that $n + 2a \geq 0$*

$$(n+2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=-a}^{n+a-1} \mathbf{A}_{m,r}(k+a)^r(n+a-k)^r$$

Corollary 4.29 (Centered binomial form). *For integers n and a such that $n + a \geq 0$*

$$(n+a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=-\frac{a}{2}+1}^{n+\frac{a}{2}} \mathbf{A}_{m,r}\left(k+\frac{a}{2}\right)^r\left(n+\frac{a}{2}-k\right)^r$$

Corollary 4.30 (Shifted centered binomial form). *For integers n and a such that $n - 2a \geq 0$*

$$(n+a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=-\frac{a}{2}}^{n+\frac{a}{2}-1} \mathbf{A}_{m,r}\left(k+\frac{a}{2}\right)^r\left(n+\frac{a}{2}-k\right)^r$$

Proposition 4.31 (Negated binomial form). *For integers n and a such that $n - 2a \geq 0$*

$$(n-2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=a+1}^{n-a} \mathbf{A}_{m,r}(k-a)^r(n-a-k)^r$$

Proof. By observing the summation limits we can see that k runs as $k = a+1, a+2, a+3, \ldots, a+n-a$, which implies that $(k-a) = 1, 2, 3, \ldots, n$. By observing the term $(n-k-a)$ we see that $(n-k-a) = n-1, n-2, n-3, \ldots, 0$. Thus, by reindexing the sum $(n-2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n-2a} \mathbf{A}_{m,r}(a+k-a)^r(n-(a+k)-a)^r$ the statement (4.31) is equivalent to (4.1) with setting $n \to n-2a$. $\qquad\square$

Corollary 4.32 (Shifted negated binomial form)**.** *For integers n and a such that $n - 2a \geq 0$*

$$(n - 2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=a}^{n-a-1} \mathbf{A}_{m,r}(k-a)^r(n-a-k)^r$$

Corollary 4.33 (Centered negated binomial form)**.** *For integers n and a such that $n - a \geq 0$*

$$(n - a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=\frac{a}{2}}^{n-\frac{a}{2}-1} \mathbf{A}_{m,r}\left(k - \frac{a}{2}\right)^r\left(n - \frac{a}{2} - k\right)^r$$

Corollary 4.34 (Shifted centered negated binomial form)**.** *For integers n and a such that $n - a \geq 0$*

$$(n - a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=\frac{a}{2}+1}^{n-\frac{a}{2}} \mathbf{A}_{m,r}\left(k - \frac{a}{2}\right)^r\left(n - \frac{a}{2} - k\right)^r$$

4.6. **Sums of powers.**

Proposition 4.35 (Sums of odd powers)**.** *For non-negative integers n and m*

$$\sum_{n=1}^{p} n^{2m+1} = \sum_{n=1}^{p} \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r}k^r(n-k)^r$$

In explicit view

$$\sum_{n=1}^{p} n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r}\bigg\{ [0^r + 1^r + 2^r + 3^r + \cdots (n-1)^r]$$

$$+[0^r + 2^r + 4^r + 6^r + \cdots (2n-4)^r]$$

$$+[0^r + 3^r + 6^r + 9^r + \cdots (3n-9)^r]$$

$$+[0^r + 4^r + 8^r + 12^r + \cdots (4n-16)^r]$$

$$+ \cdots + (p^2 - p^2)^r \bigg\}$$

Proposition 4.36 (Sums of odd powers forward decomposition)**.** *For non-negative integers m and n*

$$\sum_{n=1}^{p} n^{2m+1} = \sum_{n=1}^{p} \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1}\binom{m+1}{t}T_m(n+t,k)$$

Proposition 4.37 (Sum of odd powers backward decomposition). *For non-negative integers*

m and n

$$\sum_{n=1}^{p} n^{2m+1} = \sum_{n=1}^{p}\sum_{k=1}^{n}\sum_{t=1}^{m+1}(-1)^{t+1}\binom{m+1}{t}T_m(n-t,k)$$

4.7. Double bivariate identities.

Definition 4.38 (Double bivaraite sum R_m). *For non-negative integers m, n and t*

$$R_m(n,t) = \sum_{k=1}^{n}\sum_{r=0}^{m}\mathbf{A}_{m,r}k^r(n+t-k)^r$$

Proposition 4.39 (Odd power double bivariate). *For non-negative integers m, n*

$$n^{2m+1} = \sum_{t=1}^{m+1}(-1)^{t+1}\binom{m+1}{t}R_m(n,t)$$

In explicit view

$$n^{2m+1} = \binom{m+1}{1}R_m(n,1) - \binom{m+1}{2}R_m(n,2) + \binom{m+1}{3}R_m(n,3)$$
$$- \binom{m+1}{4}R_m(n,4) + \cdots$$

For example,

- $2^3 = \binom{2}{1}26 - \binom{2}{2}44$

- $2^5 = \binom{3}{1}242 - \binom{3}{2}752 + \binom{3}{3}1562$

- $2^7 = \binom{4}{1}2186 - \binom{4}{2}12644 + \binom{4}{3}39062 - \binom{4}{4}89000$

- $2^9 = \binom{5}{1}19682 - \binom{5}{2}211472 + \binom{5}{3}976562 - \binom{5}{4}2972672 + \binom{5}{4}7114562$

Proposition 4.40 (Odd power double bivariate negated). *For non-negative integers m, n*

$$n^{2m+1} = \sum_{t=1}^{m+1}(-1)^{t+1}\binom{m+1}{t}R_m(n,-t)$$

For example,

- $2^3 = -\binom{2}{1}10 + \binom{2}{2}28$

- $2^5 = \binom{3}{1}122 - \binom{3}{2}512 + \binom{3}{3}1202$

- $2^7 = -\binom{4}{1}1090 + \binom{4}{2}9028 - \binom{4}{3}31246 + \binom{4}{4}75304$

- $2^9 = \binom{5}{1}10322 - \binom{5}{2}162512 + \binom{5}{3}827522 - \binom{5}{4}2632832 + \binom{5}{4}6462962$

Proposition 4.41 (Odd power double bivariate multifold)**.** *For non-negative integers m, n and $s \geq 1$*

$$n^{2m+1} = \sum_{t=1}^{m+s}(-1)^{t+1}\binom{m+s}{t}R_m(n, t)$$

Proposition 4.42 (Odd power double bivariate negated multifold)**.** *For non-negative integers m, n and $s \geq 1$*

$$n^{2m+1} = \sum_{t=1}^{m+s}(-1)^{t+1}\binom{m+s}{t}R_m(n, -t)$$

5. Related research

5.1. **Spline approximation for power function.** The paper [21] describes a remarkable result that follows from the odd power identity (4.1). As revealed, by introducing an additional parameter for upper summation bound in k to (4.1), the resulting family of polynomials approximate the odd power in some neighborhood of a fixed point.

$$P(m, X, N) = \sum_{r=0}^{m}\sum_{k=1}^{N}\mathbf{A}_{m,r}k^r(X - k)^r$$

For example,

$$P(2, X, 0) = 0$$

$$P(2, X, 1) = 30X^2 - 60X + 31$$

$$P(2, X, 2) = 150X^2 - 540X + 512$$

$$P(2, X, 3) = 420X^2 - 2160X + 2943$$

$$P(2, X, 4) = 900X^2 - 6000X + 10624$$

The reason behind this behavior lies in the implicit form of the polynomial $P(m, X, N)$, meaning that

$$P(m, X, N) = \sum_{r=0}^{m}(-1)^{m-r}U(m, N, r) \cdot X^r$$

where $U(m, N, r)$ is a polynomial defined as follows

$$U(m, N, r) = (-1)^m \sum_{k=1}^{N} \sum_{j=r}^{m} \binom{j}{r} \mathbf{A}_{m,j} k^{2j-r} (-1)^j$$

which grows as N increases. Few cases of coefficients $U(m, N, r)$ are registered as OEIS sequences [22, 23, 24].

The following plot demonstrates the approximation of fifth power X^5 by $P(2, X, 4) = 900X^2 - 6000X + 10624$
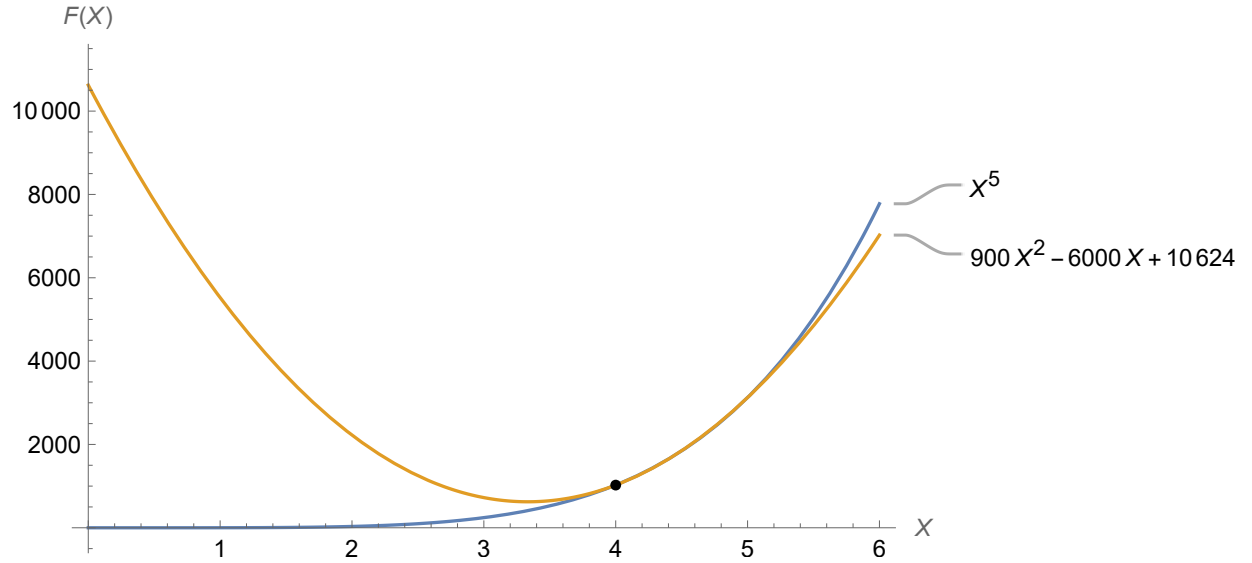

Figure 1. Approximation of fifth power X^5 by $P(2, X, 4)$. Convergence interval is $4.0 \leq X \leq 5.1$ with a percentage error $E < 1\%$.

Another remarkable observation is One more interesting observation arises by increasing the value of N in $P(m, X, N)$ while keeping m fixed. As N increases, the length of the convergence interval with the odd-power X^{2m+1} also increases. For instance,

- For $P(2, X, 4)$ and X^5, the convergence interval with a percentage error less than 1% is $4.0 \leq X \leq 5.1$, with a length $L = 1.1$

- For $P(2, X, 20)$ and X^5, the convergence interval with a percentage error less than 1% is $18.7 \leq X \leq 22.9$, with a length $L = 4.2$

- For $P(2, X, 120)$ and X^5, the convergence interval with a percentage error less than 1% is $110.0 \leq X \leq 134.7$, with a length $L = 24.7$

5.2. **Two-sided Faulhaber's formulas.** The paper [25] generalizes the proposition (3.2) to a new family of polynomials, namely two-sided Faulhaber-like formulas involving Bernoulli polynomials.

$$\sum_{k=1}^{N-1} k^m (N-k)^m = \frac{N^{2m+1}}{(2m+1)\binom{2m}{m}} + 2(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+1}}{m+k+1} N^{m-k}$$

5.3. **Derivatives.** The paper [26] reveals a connection between ordinary derivatives of odd power and partial derivatives of the function

$$f_y(x, z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r$$

Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx} g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)'_x(u, u)$ evaluate in point (u, u) plus partial derivative $(f_v)'_z(u, u)$ evaluate in point (u, u)

$$\frac{d}{dx} g_v(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u)$$

5.4. **Discrete convolution of powers.** The paper [27] reveals the connection between binomial theorem and discrete convolution of powers, via (4.1). The works [28, 29, 30] are useful because of discussion of convoluted power sums, for instance the sums

$$C(i, j; N) = \sum_{k=0}^{N} k^i (N-k)^j \quad \text{and} \quad C(i, j; N) = \sum_{k=0}^{N} (k+a)^i (N-k-a)^j,$$

which are used in odd power identities

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r$$

and binomial form

$$(n+2a)^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=-a+1}^{n+a} (k+a)^r (n+a-k)^r.$$

6. Future research

Several promising directions emerge from the findings of this manuscript:

6.1. **Integration into mathematical literature.** The identities presented in this work do not appear in standard mathematical references, despite their elementary nature and apparent classical flavor. Notably, related sequences are absent from major repositories such as the OEIS. Future work should investigate the originality of these results and aim to contextualize them within the broader mathematical framework.

6.2. **Extension of approximation methods.** The approximation technique developed in [21] is generalizable to a broader class of polynomials. In particular, by leveraging the symmetry property (4.6), one could explore alternative summation domains for the polynomials $P(m, X, N)$.

6.3. **Combinatorial interpretations.** The polynomial family $T_m(n, k)$, introduced in (4.5), currently lacks a clear combinatorial interpretation. Understanding its structural or enumerative significance would deepen insight into the algebraic identities presented.

6.4. **Connection with finite differences and derivatives.** The binomial form of the odd power identity (4.27) offers a mechanism to express both finite differences and classical derivatives of odd powers in terms of the coefficients $\mathbf{A}_{m,r}$.

6.5. **Q-derivatives.** The general identity (4.1) also suggests a natural expression for q-derivatives via the coefficients $\mathbf{A}_{m,r}$, potentially leading to a generalized notion of differentiation through limiting procedures.

7. Conclusions

This work began with a seemingly elementary interpolation problem and evolved into a broader investigation of polynomial identities involving odd powers. Starting from the finite differences of the cubic function, we uncovered a nontrivial identity that served as a base case for a family of structured decompositions of n^{2m+1}. These identities were expressed

in terms of symmetric bivariate sums with recursively defined coefficients. By employing systems of linear equations and a generating function approach, we derived both closed-form expressions and recurrence relations for these identities. The results were further extended to include binomial forms of odd power identities and formulas for the sums of odd powers. Computational experiments in *Mathematica* confirmed all theoretical claims and provided a toolkit for further exploration. These findings not only contribute novel results to the theory of polynomial identities but also open pathways to related domains, such as approximation theory and calculus.

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Sources: github.com/kolosovpetro/surprising-polynomial-identities

Application 1: Mathematica programs

We support our theoretical findings with Wolfram Mathematica programs that verify the main results of this manuscript. All source code and computational notebooks are available in the GitHub repository. The repository includes the following files:

- `unexpected-polynomial-identities-classical-interpolation.m` — the package file where all Mathematica functions are defined. Load it into your session using `filename.m` or by evaluating the file with `Shift+Enter`.
- `unexpected-polynomial-identities-classical-interpolation.nb` — a working notebook that demonstrates the usage of these functions to validate the manuscript's results.

Definitions

Mathematica Function	Validates / Prints
`A[m, r]`	Coefficient $\mathbf{A}_{m,r}$ (Definition 3.3)
`PrintTriangleA[m]`	Triangle of $\mathbf{A}_{m,r}$ values
`OddPowerIdentity[n, m]`	Theorem 4.1
`OddPowerIdentitySimplified[n, m]`	Simplified form of Theorem 4.1
`BivariateSumT[m, n, k]`	Bivariate sum $T_m(n,k)$ (Definition 4.5)
`TableFormBivariateSumT[m, rows]`	Triangle view of $T_m(n,k)$

Forward decompositions

Mathematica Function	Validates / Prints
`ForwardRecurrenceForT[m, n, k]`	Proposition 4.7
`OddPowerForwardDecomposition[n, m]`	Proposition 4.8
`OddPowerForwardDecompositionMMinus1[n, m]`	Proposition 4.9
`OddPowerForwardDecompositionShifted[n, m]`	Proposition 4.10
`OddPowerForwardDecompositionMMinus1Shifted[n, m]`	Proposition 4.11
`TableFormForwardRecurrenceForT[m, rows]`	Triangle view of 4.7

Forward decompositions multifold

Mathematica Function	Validates / Prints
`ForwardRecurrenceForTMultifold[m, n, k, s]`	Proposition 4.12
`OddPowerForwardDecompositionMultifold[n, m, s]`	Proposition 4.13
`OddPowerForwardDecompositionMMinus1Multifold[n, m, s]`	Proposition 4.14
`OddPowerForwardDecompositionShiftedMultifold[n, m, s]`	Proposition 4.15
`OddPowerForwardDecompositionMMinus1ShiftedMultifold[n, m, s]`	Proposition 4.16

Backward decompositions

Mathematica Function	Validates / Prints
`BackwardRecurrenceForT[m, n, k]`	Proposition 4.17
`OddPowerBackwardDecomposition[n, m]`	Proposition 4.18
`OddPowerBackwardDecompositionShifted[n, m]`	Proposition 4.19
`OddPowerBackwardDecompositionMMinus1[n, m]`	Corollary 4.20
`OddPowerBackwardDecompositionMMinus1Shifted[n, m]`	Corollary 4.21
`TableFormBackwardRecurrenceForT[m, rows]`	Triangle view 4.17

Backward decompositions multifold

Mathematica Function	Validates / Prints
`BackwardRecurrenceForTMultifold[m, n, k, s]`	Proposition 4.22
`OddPowerBackwardDecompositionMultifold[n, m, s]`	Proposition 4.23
`OddPowerBackwardDecompositionShiftedMultifold[n, m, s]`	Proposition 4.24
`OddPowerBackwardDecompositionMMinus1Multifold[n, m, s]`	Corollary 4.25
`OddPowerBackwardDecompositionMMinus1ShiftedMultifold[n, m, s]`	Corollary 4.26

Binomial forms

Mathematica Function	Validates / Prints
`BinomialForm[m, n, a]`	Corollary 4.27
`ShiftedBinomialForm[m, n, a]`	Corollary 4.28
`CenteredBinomialForm[m, n, a]`	Corollary 4.29
`ShiftedCenteredBinomialForm[m, n, a]`	Corollary 4.30
`NegatedBinomialForm[m, n, a]`	Proposition 4.31
`ShiftedNegatedBinomialForm[m, n, a]`	Corollary 4.32
`CenteredNegatedBinomialForm[m, n, a]`	Corollary 4.33
`ShiftedCenteredNegatedBinomialForm[m, n, a]`	Corollary 4.34

Double bivariate identities

Mathematica Function	Validates / Defines
DoubleBivariateSumR[m, n, t]	Definition 4.38
OddPowerDoubleBivariate[n, m]	Proposition 4.39
OddPowerDoubleBivariateMultifold[n, m, s]	Proposition 4.41
OddPowerDoubleBivariateNegated[n, m]	Proposition 4.40
OddPowerDoubleBivariateNegatedMultifold[n, m, s]	Proposition 4.42

APPLICATION 2: EXAMPLES OF COEFFICIENTS A

Consider the proposition (3.3) of the coefficients $\mathbf{A}_{m,r}$, it can be written as

$$
\mathbf{A}_{m,r} = \begin{cases}
(2r+1)\binom{2r}{r} & \text{if } r = m \\
\sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)} & \text{if } 0 \leq r < m \\
0 & \text{if } r < 0 \text{ or } r > m
\end{cases}
$$

Let be the definition of the polynomial $T(d,r)$

Definition 7.1. *For integers d, r such that $r \neq d$*

$$
T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}
$$

Example 7.2. *Let be $m = 2$ so first we get $\mathbf{A}_{2,2}$*

$$
\mathbf{A}_{2,2} = 5\binom{4}{2} = 30
$$

Then $\mathbf{A}_{2,1} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $1 \leq d < 2$. Finally, the coefficient $\mathbf{A}_{2,0}$ is

$$
\mathbf{A}_{2,0} = \sum_{d\geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) = \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0)
$$

$$
= 30 \cdot \frac{1}{30} = 1
$$

Example 7.3. *Let be $m = 3$ so that first we get $\mathbf{A}_{3,3}$*

$$
\mathbf{A}_{3,3} = 7\binom{6}{3} = 140
$$

Then $\mathbf{A}_{3,2} = 0$ because $\mathbf{A}_{m,d}$ is zero in the range $m/2 \leq d < m$ means that zero for d in $2 \leq d < 3$. The $\mathbf{A}_{3,1}$ coefficient is non-zero and calculated as

$$
\mathbf{A}_{3,1} = \sum_{d\geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14
$$

Finally, the coefficient $\mathbf{A}_{3,0}$ *is*

$$\mathbf{A}_{3,0} = \sum_{d \geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 7.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9 \binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for d in* $2 \leq d < 4$. *The value of the coefficient* $\mathbf{A}_{4,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d \geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ *is*

$$\mathbf{A}_{4,0} = \sum_{d \geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 7.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11 \binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for d in* $3 \leq d < 5$. *The value of the coefficient* $\mathbf{A}_{5,2}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d \geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d \geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\mathbf{A}_{5,0} = \sum_{d \geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0)$$

$$= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1$$

APPLICATION 3: SYSTEMS OF LINEAR EQUATIONS EXAMPLES

Example 7.6. *Let be $m = 3$ so that we have the following relation defined by* (4.1)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n + n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n + n^5)\right] + \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n + 7n^3 + 3n^7)\right] - n^7 = 0$$

Multiplying by 420 right-hand side and left-hand side, we get

$$420\mathbf{A}_{3,0}n + 70\mathbf{A}_{2,1}(-n + n^3) + 14\mathbf{A}_{2,2}(-n + n^5) + \mathbf{A}_{3,3}(-10n + 7n^3 + 3n^7) - 420n^7 = 0$$

Opening brackets and rearranging the terms gives

$$420\mathbf{A}_{3,0}n - 70\mathbf{A}_{3,1} + 70\mathbf{A}_{3,1}n^3 - 14\mathbf{A}_{3,2}n + 14\mathbf{A}_{3,2}n^5$$

$$- 10\mathbf{A}_{3,3}n + 7\mathbf{A}_{3,3}n^3 + 3\mathbf{A}_{3,3}n^7 - 420n^7 = 0$$

Combining the common terms yields

$$n(420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3})$$

$$+ n^3(70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3}) + n^5 14\mathbf{A}_{3,2} + n^7(3\mathbf{A}_{3,3} - 420) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 420\mathbf{A}_{3,0} - 70\mathbf{A}_{3,1} - 14\mathbf{A}_{3,2} - 10\mathbf{A}_{3,3} & = 0 \\[2mm] 70\mathbf{A}_{3,1} + 7\mathbf{A}_{3,3} & = 0 \\[2mm] \mathbf{A}_{3,2} - 30 & = 0 \\[2mm] 3\mathbf{A}_{3,3} - 420 & = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{3,3} & = 140 \\[2mm] \mathbf{A}_{3,2} & = 0 \\[2mm] \mathbf{A}_{3,1} = -\frac{7}{70}\mathbf{A}_{3,3} & = -14 \\[2mm] \mathbf{A}_{3,0} = \frac{(70\mathbf{A}_{3,1} + 10\mathbf{A}_{3,3})}{420} & = 1 \end{cases}$$

So that odd-power identity (4.1) *holds*

$$n^7 = \sum_{k=1}^{n} 140k^3(n-k)^3 - 14k(n-k) + 1$$

It is also clearly seen why the above identity is true evaluating the terms $140k^3(n-k)^3 - 14k(n-k) + 1$ over $0 \leq k \leq n$ as the OEIS sequence A300785 [32] *shows*

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	127	1					
3	1	1093	1093	1				
4	1	3739	8905	3739	1			
5	1	8905	30157	30157	8905	1		
6	1	17431	71569	101935	71569	17431	1	
7	1	30157	139861	241753	241753	139861	30157	1

Table 5. Values of $T_3(n,k) = 140k^3(n-k)^3 - 14k(n-k) + 1$. See the sequence A300785 in OEIS [32].

Example 7.7. *Let be $m = 4$ so that we have the following relation defined by* (4.1)

$$\mathbf{A}_{m,0}n + \mathbf{A}_{m,1}\left[\frac{1}{6}(-n + n^3)\right] + \mathbf{A}_{m,2}\left[\frac{1}{30}(-n + n^5)\right]$$

$$+ \mathbf{A}_{m,3}\left[\frac{1}{420}(-10n + 7n^3 + 3n^7)\right]$$

$$+ \mathbf{A}_{m,4}\left[\frac{1}{630}(-21n + 20n^3 + n^9)\right] - n^9 = 0$$

Multiplying by 630 *right-hand side and left-hand side, we get*

$$630\mathbf{A}_{4,0}n + 105\mathbf{A}_{4,1}(-n + n^3) + 21\mathbf{A}_{4,2}(-n + n^5)$$

$$+ \frac{3}{2}\mathbf{A}_{4,3}(-10n + 7n^3 + 3n^7)$$

$$+ \mathbf{A}_{4,4}(-21n + 20n^3 + n^9) - 630n^9 = 0$$

Opening brackets and rearranging the terms gives

$$630\mathbf{A}_{4,0}n - 105\mathbf{A}_{4,1}n + 105\mathbf{A}_{4,1}n^3 - 21\mathbf{A}_{4,2}n + 21\mathbf{A}_{4,2}n^5$$

$$- \frac{3}{2}\mathbf{A}_{4,3} \cdot 10n + \frac{3}{2}\mathbf{A}_{4,3} \cdot 7n^3 + \frac{3}{2}\mathbf{A}_{4,3} \cdot 3n^7$$

$$- 21\mathbf{A}_{4,4}n + 20\mathbf{A}_{4,4}n^3 + \mathbf{A}_{4,4}n^9 - 630n^9 = 0$$

Combining the common terms yields

$$n(630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4})$$

$$+ n^3\left(105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4}\right) + n^5(21\mathbf{A}_{4,2})$$

$$+ n^7\left(\frac{9}{2}\mathbf{A}_{4,3}\right) + n^9(\mathbf{A}_{4,4} - 630) = 0$$

Therefore, the system of linear equations follows

$$
\begin{cases}
630\mathbf{A}_{4,0} - 105\mathbf{A}_{4,1} - 21\mathbf{A}_{4,2} - 15\mathbf{A}_{4,3} - 21\mathbf{A}_{4,4} & = 0 \\[2mm]
105\mathbf{A}_{4,1} + \frac{21}{2}\mathbf{A}_{4,3} + 20\mathbf{A}_{4,4} & = 0 \\[2mm]
\mathbf{A}_{4,2} & = 0 \\[2mm]
\mathbf{A}_{4,3} & = 0 \\[2mm]
\mathbf{A}_{4,4} - 630 & = 0
\end{cases}
$$

Solving it, we get

$$
\begin{cases}
\mathbf{A}_{4,4} & = 630 \\[2mm]
\mathbf{A}_{4,3} & = 0 \\[2mm]
\mathbf{A}_{4,2} & = 0 \\[2mm]
\mathbf{A}_{4,1} = -\frac{20}{105}\mathbf{A}_{4,4} & = -120 \\[2mm]
\mathbf{A}_{4,0} = \frac{105\mathbf{A}_{4,1} + 21\mathbf{A}_{4,4}}{630} & = 1
\end{cases}
$$

So that odd-power identity (4.1) *holds*

$$n^9 = \sum_{k=1}^{n} 630k^4(n-k)^4 - 120k(n-k) + 1$$

Application 4: Related OEIS sequences

OEIS ID	Description	Citation
A287326	Numerical triangle, row sums give third power	[7]
A300656	Numerical triangle, row sums give fifth power	[11]
A300785	Numerical triangle, row sums give seventh power	[32]
A302971	Numerators of the coefficient $\mathbf{A}_{m,r}$	[18]
A304042	Denominators of the coefficient $\mathbf{A}_{m,r}$	[19]
A320047	Coefficients $U(m,l,k)$ for $m=1$ defined by identity (4.1)	[22]
A316349	Coefficients $U(m,l,k)$ for $m=2$ defined by identity (4.1)	[23]
A316387	Coefficients $U(m,l,k)$ for $m=3$ defined by identity (4.1)	[24]
A094053	Triangle read by rows: $T(n,k) = k(n-k)$	[33]
A000217	Triangular numbers: $\binom{n+1}{2}$	[34]
A306548	Table of self-convolution of powers	[35]

APPLICATION 5: DOUBLE BIVARIATE IDENTITIES EXAMPLES

For constant $N = 2$ and $m = 0, 1, 2, 3, 4 \ldots$

- $2^1 = \binom{1}{1}2$

- $2^3 = \binom{2}{1}26 - \binom{2}{2}44$

- $2^5 = \binom{3}{1}242 - \binom{3}{2}752 + \binom{3}{3}1562$

- $2^7 = \binom{4}{1}2186 - \binom{4}{2}12644 + \binom{4}{3}39062 - \binom{4}{4}89000$

- $2^9 = \binom{5}{1}19682 - \binom{5}{2}211472 + \binom{5}{3}976562 - \binom{5}{4}2972672 + \binom{5}{4}7114562$

For $N = 0, 1, 2, 3, 4 \ldots$ and $m = 3$

- $0^7 = \binom{4}{1}0 - \binom{4}{2}0 + \binom{4}{3}0 - \binom{4}{4}0$

- $1^7 = \binom{4}{1}127 - \binom{4}{2}1093 + \binom{4}{3}3739 - \binom{4}{4}8905$

- $2^7 = \binom{4}{1}2186 - \binom{4}{2}12644 + \binom{4}{3}39062 - \binom{4}{4}89000$

- $3^7 = \binom{4}{1}16383 - \binom{4}{2}69219 + \binom{4}{3}190935 - \binom{4}{4}411771$

- $4^7 = \binom{4}{1}78124 - \binom{4}{2}262504 + \binom{4}{3}653524 - \binom{4}{4}1335184$

- $5^7 = \binom{4}{1}279935 - \binom{4}{2}793385 + \binom{4}{3}1807475 - \binom{4}{4}3511205$

For $N = 2$ and $m = 1$ such that $s = 1, 2, 3, 4, 5 \ldots$

- $2^3 = \binom{2}{1}26 - \binom{2}{2}44$

- $2^3 = \binom{3}{1}26 - \binom{3}{2}44 + \binom{3}{3}62$

- $2^3 = \binom{4}{1}26 - \binom{4}{2}44 + \binom{4}{3}62 - \binom{4}{4}80$

- $2^3 = \binom{5}{1}26 - \binom{5}{2}44 + \binom{5}{3}62 - \binom{5}{4}80 + \binom{5}{5}98$

For $m = 1$ such that $s = 2, 3, 4, 5 \ldots$

- For $N = 2$ and $m = 1$ the sequence is $26, 44, 62, 80, 98, 116, 134, 152, 170, 188 \ldots$

- For $N = 3$ and $m = 1$ the sequence is $63, 99, 135, 171, 207, 243, 279, 315, 351, 387 \ldots$

- For $N = 4$ and $m = 1$ the sequence is $124, 184, 244, 304, 364, 424, 484, 544, 604, 664 \ldots$

For constant $N = 2$ and $m = 1, 2, 3, 4 \ldots$

- $2^1 = \binom{1}{1}2$

- $2^3 = -\binom{2}{1}10 + \binom{2}{2}28$

- $2^5 = \binom{3}{1}122 - \binom{3}{2}512 + \binom{3}{3}1202$
- $2^7 = -\binom{4}{1}1090 + \binom{4}{2}9028 - \binom{4}{3}31246 + \binom{4}{4}75304$

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