LaTeX Beamer template

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December 19, 2025

Let be the recurrence for R-fold sums of powers, see [1]

$$\begin{aligned}
\Sigma^0 n^m &= n^m \\
\Sigma^1 n^m &= \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m \\
\Sigma^{r+1} n^m &= \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m
\end{aligned}$$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	2						
3	0	1	6	6					
4	0	1	14	36	24				
5	0	1	30	150	240	120			
6	0	1	62	540	1560	1800	720		
7	0	1	126	1806	8400	16800	15120	5040	
8	0	1	254	5796	40824	126000	191520	141120	40320

Table: Triangle $T(0, n, k)$. Sequence A131689.

$$n^0 = 1\binom{n}{0}$$

$$n^1 = 0\binom{n}{0} + 1\binom{n}{1}$$

$$n^2 = 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2}$$

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$

$$n^4 = 0\binom{n}{0} + 1\binom{n}{1} + 14\binom{n}{2} + 36\binom{n}{3} + 24\binom{n}{4}$$

$$n^5 = 0\binom{n}{0} + 1\binom{n}{1} + 30\binom{n}{2} + 150\binom{n}{3} + 240\binom{n}{4} + 120\binom{n}{5}$$

$$\Sigma^1 n^0 = 1\left(\binom{n+1}{1} + \binom{-1}{1}\right)$$

$$\Sigma^1 n^1 = 0\left(\binom{n+1}{1} + \binom{-1}{1}\right) + 1\left(\binom{n+1}{2} - \binom{0}{2}\right)$$

$$\Sigma^1 n^2 = 0\left(\binom{n+1}{1} + \binom{-1}{1}\right) + 1\left(\binom{n+1}{2} - \binom{0}{2}\right) + 2\left(\binom{n+1}{3} + \binom{1}{3}\right)$$

$$\Sigma^1 n^3 = 0\left(\binom{n+1}{1} + \binom{-1}{1}\right) + 1\left(\binom{n+1}{2} - \binom{0}{2}\right) + 6\left(\binom{n+1}{3} + \binom{1}{3}\right) + 6\left(\binom{n+1}{4} - \binom{2}{4}\right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(0,n,k)\left[\binom{n+1}{j+1} + (-1)^j\binom{j-1}{j+1}\right]$$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	3	2						
3	1	7	12	6					
4	1	15	50	60	24				
5	1	31	180	390	360	120			
6	1	63	602	2100	3360	2520	720		
7	1	127	1932	10206	25200	31920	20160	5040	
8	1	255	6050	46620	166824	317520	332640	181440	40320

Table: Triangle $T(1, n, k)$. Sequence A028246.

$$n^0 = 1\binom{n-1}{0}$$

$$n^1 = 1\binom{n-1}{0} + 1\binom{n-1}{1}$$

$$n^2 = 1\binom{n-1}{0} + 3\binom{n-1}{1} + 2\binom{n-1}{2}$$

$$n^3 = 1\binom{n-1}{0} + 7\binom{n-1}{1} + 12\binom{n-1}{2} + 6\binom{n-1}{3}$$

$$n^4 = 1\binom{n-1}{0} + 15\binom{n-1}{1} + 50\binom{n-1}{2} + 60\binom{n-1}{3} + 24\binom{n-1}{4}$$

$$\Sigma^1 n^0 = 1 \left(\binom{n}{1} + \binom{0}{1} \right)$$

$$\Sigma^1 n^1 = 1 \left(\binom{n}{1} + \binom{0}{1} \right) + 1 \left(\binom{n}{2} - \binom{1}{2} \right)$$

$$\Sigma^1 n^2 = 1 \left(\binom{n}{1} + \binom{0}{1} \right) + 3 \left(\binom{n}{2} - \binom{1}{2} \right) + 2 \left(\binom{n}{3} + \binom{2}{3} \right)$$

$$\Sigma^1 n^3 = 1 \left(\binom{n}{1} + \binom{0}{1} \right) + 7 \left(\binom{n}{2} - \binom{1}{2} \right) + 12 \left(\binom{n}{3} + \binom{2}{3} \right) + 6 \left(\binom{n}{4} - \binom{3}{4} \right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(1, m, j) \left[\binom{n}{j+1} + (-1)^j \binom{j}{j+1} \right]$$

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	2	1						
2	4	5	2					
3	8	19	18	6				
4	16	65	110	84	24			
5	32	211	570	750	480	120		
6	64	665	2702	5460	5880	3240	720	
7	128	2059	12138	35406	57120	52080	25200	5040

Table: Triangle $T(2, n, k)$. Sequence A038719.

$$n^0 = 1\binom{n-2}{0}$$

$$n^1 = 2\binom{n-2}{0} + 1\binom{n-2}{1}$$

$$n^2 = 4\binom{n-2}{0} + 5\binom{n-2}{1} + 2\binom{n-2}{2}$$

$$n^3 = 8\binom{n-2}{0} + 19\binom{n-2}{1} + 18\binom{n-2}{2} + 6\binom{n-2}{3}$$

$$n^4 = 16\binom{n-2}{0} + 65\binom{n-2}{1} + 110\binom{n-2}{2} + 84\binom{n-2}{3} + 24\binom{n-2}{4}$$

$$\Sigma^1 n^0 = 1\left(\binom{n-1}{1}+1\right)$$

$$\Sigma^1 n^1 = 2\left(\binom{n-1}{1}+1\right)+1\left(\binom{n-1}{2}-1\right)$$

$$\Sigma^1 n^2 = 4\left(\binom{n-1}{1}+1\right)+5\left(\binom{n-1}{2}-1\right)+2\left(\binom{n-1}{3}+1\right)$$

$$\Sigma^1 n^3 = 8\left(\binom{n-1}{1}+1\right)+19\left(\binom{n-1}{2}-1\right)+18\left(\binom{n-1}{3}+1\right) + 6\left(\binom{n-1}{4}-1\right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(2,m,j)\left[\binom{n-1}{j+1}+(-1)^j\binom{j+1}{j+1}\right]$$

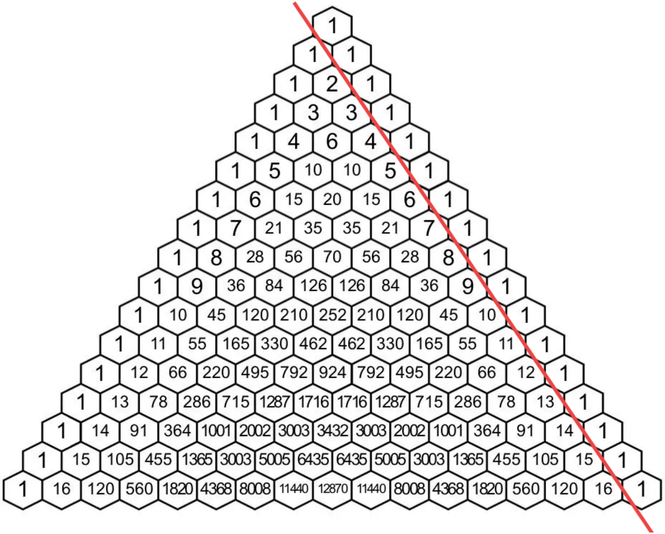

Figure: Pascal's Triangle diagonal 2.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	3	1						
2	9	7	2					
3	27	37	24	6				
4	81	175	194	108	24			
5	243	781	1320	1230	600	120		
6	729	3367	8162	11340	9120	3960	720	
7	2187	14197	47544	92526	109200	77280	30240	5040

Table: Triangle $T(3, n, k)$.

$$
\begin{aligned}
n^0 &= 1\binom{n-3}{0} \\
n^1 &= 3\binom{n-3}{0} + 1\binom{n-3}{1} \\
n^2 &= 9\binom{n-3}{0} + 7\binom{n-3}{1} + 2\binom{n-3}{2} \\
n^3 &= 27\binom{n-3}{0} + 37\binom{n-3}{1} + 24\binom{n-3}{2} + 6\binom{n-3}{3} \\
n^4 &= 81\binom{n-3}{0} + 175\binom{n-3}{1} + 194\binom{n-3}{2} + 108\binom{n-3}{3} + 24\binom{n-3}{4}
\end{aligned}
$$

$$\Sigma^1 n^0 = 1\left(\binom{n-2}{1} + \binom{2}{1}\right)$$

$$\Sigma^1 n^1 = 3\left(\binom{n-2}{1} + \binom{2}{1}\right) + 1\left(\binom{n-2}{2} - \binom{3}{2}\right)$$

$$\Sigma^1 n^2 = 9\left(\binom{n-2}{1} + \binom{2}{1}\right) + 7\left(\binom{n-2}{2} - \binom{3}{2}\right) + 2\left(\binom{n-2}{3} + \binom{4}{3}\right)$$

$$\begin{aligned}\Sigma^1 n^3 = 27\left(\binom{n-2}{1} + \binom{2}{1}\right) + 37\left(\binom{n-2}{2} - \binom{3}{2}\right) \\ + 24\left(\binom{n-2}{3} + \binom{4}{3}\right) + 6\left(\binom{n-2}{4} - \binom{5}{4}\right)\end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(3,m,j)\left[\binom{n-2}{j+1} + (-1)^j\binom{j+2}{j+1}\right]$$

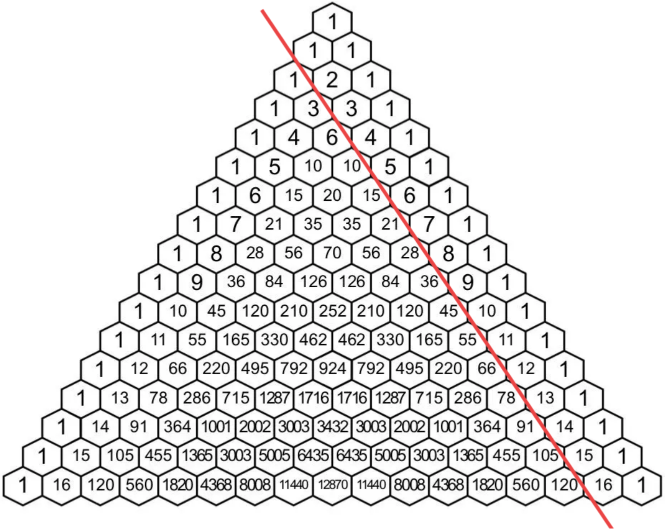

Figure: Pascal's Triangle diagonal 3.

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	4	1						
2	16	9	2					
3	64	61	30	6				
4	256	369	302	132	24			
5	1024	2101	2550	1830	720	120		
6	4096	11529	19502	20460	13080	4680	720	
7	16384	61741	140070	201726	186480	107520	35280	5040

Table: Triangle $T(4, n, k)$.

$$n^0 = 1\binom{n-4}{0}$$
$$n^1 = 4\binom{n-4}{0} + 1\binom{n-4}{1}$$
$$n^2 = 16\binom{n-4}{0} + 9\binom{n-4}{1} + 2\binom{n-4}{2}$$
$$n^3 = 64\binom{n-4}{0} + 61\binom{n-4}{1} + 30\binom{n-4}{2} + 6\binom{n-4}{3}$$
$$n^4 = 256\binom{n-4}{0} + 369\binom{n-4}{1} + 302\binom{n-4}{2} + 132\binom{n-4}{3} + 24\binom{n-4}{4}$$

$$\Sigma^1 n^0 = 1\left(\binom{n-3}{1} + \binom{3}{1}\right)$$

$$\Sigma^1 n^1 = 4\left(\binom{n-3}{1} + \binom{3}{1}\right) + 1\left(\binom{n-3}{2} - \binom{4}{2}\right)$$

$$\Sigma^1 n^2 = 16\left(\binom{n-3}{1} + \binom{3}{1}\right) + 9\left(\binom{n-3}{2} - \binom{4}{2}\right) + 2\left(\binom{n-2}{3} + \binom{5}{3}\right)$$

$$\Sigma^1 n^3 = 64\left(\binom{n-3}{1} + \binom{3}{1}\right) + 61\left(\binom{n-3}{2} - \binom{4}{2}\right)$$
$$+ 30\left(\binom{n-3}{3} + \binom{5}{3}\right) + 6\left(\binom{n-3}{4} - \binom{6}{4}\right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(4,m,j)\left[\binom{n-3}{j+1} + (-1)^j\binom{j+3}{j+1}\right]$$

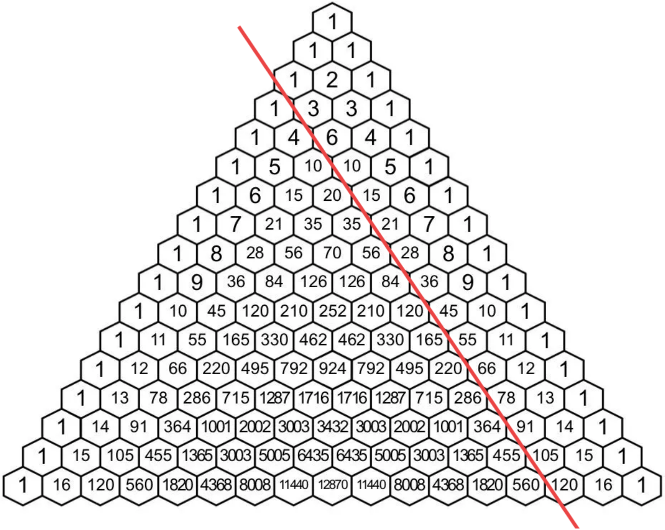

Figure: Pascal's Triangle diagonal 4.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	5	1						
2	25	11	2					
3	125	91	36	6				
4	625	671	434	156	24			
5	3125	4651	4380	2550	840	120		
6	15625	31031	39962	33540	17760	5400	720	
7	78125	201811	341796	388206	294000	142800	40320	5040

Table: Triangle $T(5, n, k)$.

$$n^0 = 1\binom{n-5}{0}$$

$$n^1 = 5\binom{n-5}{0} + 1\binom{n-5}{1}$$

$$n^2 = 25\binom{n-5}{0} + 11\binom{n-5}{1} + 2\binom{n-5}{2}$$

$$n^3 = 125\binom{n-5}{0} + 91\binom{n-5}{1} + 36\binom{n-5}{2} + 6\binom{n-5}{3}$$

$$n^4 = 625\binom{n-5}{0} + 671\binom{n-5}{1} + 434\binom{n-5}{2} + 156\binom{n-5}{3} + 24\binom{n-5}{4}$$

$$\Sigma^1 n^0 = 1\left(\binom{n-4}{1} + \binom{4}{1}\right)$$

$$\Sigma^1 n^1 = 5\left(\binom{n-4}{1} + \binom{4}{1}\right) + 1\left(\binom{n-4}{2} - \binom{5}{2}\right)$$

$$\Sigma^1 n^2 = 25\left(\binom{n-4}{1} + \binom{4}{1}\right) + 11\left(\binom{n-4}{2} - \binom{5}{2}\right) + 2\left(\binom{n-4}{3} + \binom{6}{3}\right)$$

$$\Sigma^1 n^3 = 125\left(\binom{n-4}{1} + \binom{4}{1}\right) + 91\left(\binom{n-4}{2} - \binom{5}{2}\right)$$
$$+ 36\left(\binom{n-4}{3} + \binom{6}{3}\right) + 6\left(\binom{n-4}{4} - \binom{7}{4}\right)$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(5,m,j)\left[\binom{n-4}{j+1} + (-1)^j\binom{j+4}{j+1}\right]$$

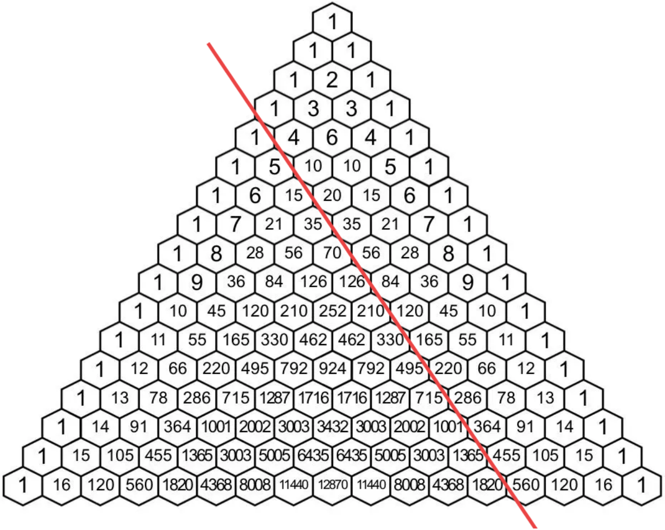

Figure: Pascal's Triangle diagonal 5.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	6	1						
2	36	13	2					
3	216	127	42	6				
4	1296	1105	590	180	24			
5	7776	9031	6930	3390	960	120		
6	46656	70993	73502	51300	23160	6120	720	
7	279936	543607	730002	682206	436800	183120	45360	5040

Table: Triangle $T(6, n, k)$.

$$n^0 = 1\binom{n-6}{0}$$

$$n^1 = 6\binom{n-6}{0} + 1\binom{n-6}{1}$$

$$n^2 = 36\binom{n-6}{0} + 13\binom{n-6}{1} + 2\binom{n-6}{2}$$

$$n^3 = 216\binom{n-6}{0} + 127\binom{n-6}{1} + 42\binom{n-6}{2} + 6\binom{n-6}{3}$$

$$n^4 = 1296\binom{n-6}{0} + 1105\binom{n-6}{1} + 590\binom{n-6}{2} + 180\binom{n-6}{3} + 24\binom{n-6}{4}$$

$$\Sigma^1 n^0 = 1\left(\binom{n-5}{1} + \binom{5}{1}\right)$$

$$\Sigma^1 n^1 = 6\left(\binom{n-5}{1} + \binom{5}{1}\right) + 1\left(\binom{n-5}{2} - \binom{6}{2}\right)$$

$$\Sigma^1 n^2 = 36\left(\binom{n-5}{1} + \binom{5}{1}\right) + 13\left(\binom{n-5}{2} - \binom{6}{2}\right) + 2\left(\binom{n-5}{3} + \binom{7}{3}\right)$$

$$\begin{aligned}\Sigma^1 n^3 = 216\left(\binom{n-5}{1} + \binom{5}{1}\right) + 127\left(\binom{n-5}{2} - \binom{6}{2}\right) \\ + 42\left(\binom{n-5}{3} + \binom{7}{3}\right) + 6\left(\binom{n-5}{4} - \binom{8}{4}\right)\end{aligned}$$

$$\Sigma^1 n^m = \sum_{j=0}^{m} T(6,m,j)\left[\binom{n-5}{j+1} + (-1)^j\binom{j+5}{j+1}\right]$$

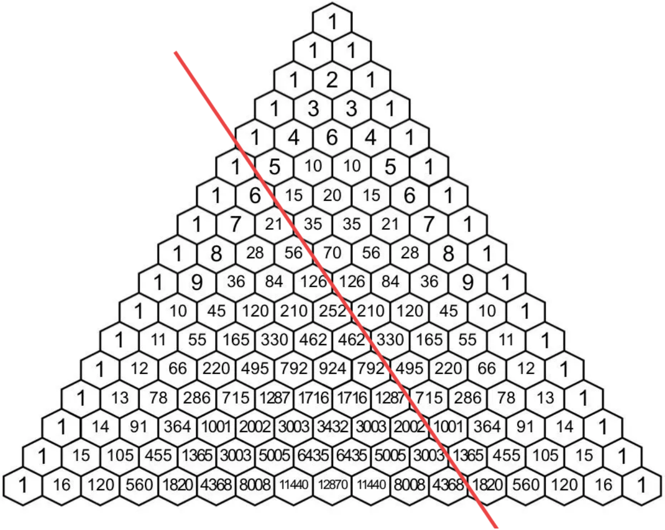

Figure: Pascal's Triangle diagonal 6.

In general,

$$\sum_{k=1}^{n} k^m = \sum_{j=0}^{m} T(t,m,j) \left[\binom{n-t+1}{j+1} + (-1)^j \binom{j+t-1}{j+1} \right]$$

where

$$T(t,m,j) = \sum_{k=0}^{m} \binom{t}{k} \left\{ {m \atop j+k} \right\} (j+k)!$$

Validate as

```
m = 2; Table[SumOfPowers[n, m] - NewSumOfPowers[n, m, t], {n, 1, 20},
    {t, 0, 20}]
```

See github.com/kolosovpetro/FormulasForSumsOfPowers

Knuth, Donald E.
Johann Faulhaber and sums of powers.
Mathematics of Computation, 61(203):277–294, 1993.
https://arxiv.org/abs/math/9207222.

Version: `1.0.3+main.4c60e50`

Sources: github.com/kolosovpetro/FormulasForSumsOfPowers

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