ANOTHER APPROACH TO GET DERIVATIVE OF ODD-POWER

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ABSTRACT. This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives of the polynomial function f_y defined as

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r$$

where $x, z \in \mathbb{R}$, y is fixed constant $y \in \mathbb{N}$ and $\mathbf{A}_{y,r}$ are real coefficients.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives identity in terms of partial derivatives, extending the main idea explained in [1] that gives an odd-power identity in a form as follows

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

where m is fixed constant $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [2]. Define the function f_y such that based on the identity (1) with the only difference that values of n, m in the right part of (1) appear to be parameters of the function f_y. In contrast to the equation (1), upper bound n of the sum $\sum_{k=1}^{n}$ turned into fixed function's parameter y as well, so that f_y defined as follows

Definition 1.1. *(Polynomial function f_y.)*

$$f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r \tag{2}$$

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where $x, z \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. Note that for every $x \in \mathbb{R}$ and constant $y \in \mathbb{N}$ the polynomial identity satisfies

$$f_y(x, x) = x^{2y+1}$$

At first glance, equation (2) might look complex, so in order to clarify the function f_y and polynomials it produces let's show few examples. Substituting the values of $y = 1, 2, 3$ to the function f we get the following polynomials

$$\begin{aligned}
f_1(x,z) &= 3xz - 3z^2 + 3xz^2 - 2z^3 \\
f_2(x,z) &= 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5 \\
f_3(x,z) &= -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4 \\
&\quad - 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7
\end{aligned}$$

These polynomials are obtained rearranging the sums in the equation (2) as

$$\sum_{r=0}^{y} \mathbf{A}_{y,r} \left[\sum_{k=1}^{z} k^r (x-k)^r \right]$$

so that part $\sum_{k=1}^{z} k^r (x-k)^r$ is polynomial in x, z calculated using Faulhaber's formula [3]. According to the main topic of the current manuscript, it provides another approach to get derivative of odd-power. Therefore, we define odd-power function we work in context of. Odd-power function g_y is a function defined as follows

Definition 1.2. *(Odd-power function g_y.)*

$$g_y(x) = x^{2y+1}$$

where $x \in \mathbb{R}$ and y is constant $y \in \mathbb{N}$. Interesting part is that odd-power function $g_y(x)$ may be obtained as partial case of the function f, as well as derivative of odd-power $\frac{d}{dx} g_y$ evaluated in point some $u \in \mathbb{R}$ may be obtained as sum of partial derivatives of f evaluated in some fixed point, that is explained further in this manuscript. One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

Notation 1.3. *(Partial derivative.) Let be a function $f(x_1, x_2, \ldots, x_n)$ defined over the real space $\mathbb{R}^n$. We denote partial derivative of the function f with respect to x_i as follows*

$$f'_{x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}$$

Derivative of the function f with respect to x_i evaluated in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is denoted as follows

$$f'_{x_i}(y_1, y_2, \ldots, y_n)$$

Moreover, partial derivative f'_{x_i} evaluated at point $(y_1, y_2, \ldots, y_n)$ plus partial derivative f'_{x_j} evaluated at point $(y_1, y_2, \ldots, y_n)$ is equivalent to the sum of partial derivatives f'_{x_i}, f'_{x_j} evaluated at point $(y_1, y_2, \ldots, y_n)$ and to be denoted as

$$f'_{x_i}(y_1, y_2, \ldots, y_n) + f'_{x_j}(y_1, y_2, \ldots, y_n) = [f'_{x_i} + f'_{x_j}](y_1, y_2, \ldots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between odd-power function g_y and polynomial function f_y

Theorem 1.4. *Let be a fixed value $v \in \mathbb{N}$, then derivative g'_v of the odd-power function $g_v(x) = x^{2v+1}$ evaluated at point u equals to derivative $(f_v)'_x$ evaluated at point (u, u) plus derivative $(f_v)'_z$ evaluated at point (u, u)*

$$g'_v(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u) \tag{3}$$

Particularly, it follows that for every pair u, v an identity holds

$$\begin{aligned}(2v+1)u^{2v} &= (f_v)'_x(u, u) + (f_v)'_z(u, u)\\ &= [(f_v)'_x + (f_v)'_z](u, u)\end{aligned}$$

that is also an ordinary derivative of odd-power function t^{2v+1}, therefore

$$\begin{aligned}\frac{d}{dt}t^{2v+1}(u) &= (f_v)'_x(u, u) + (f_v)'_z(u, u)\\ &= [(f_v)'_x + (f_v)'_z](u, u)\end{aligned}$$

To summarize and clarify all above, we provide a few examples that show an identity (3) in action.

Example 1.5. Identity (3) example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 1$. Consider the explicit form of the function $f_1(x, z)$ i.e

$$f_1(x, z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

Therefore, derivative of f_1 with respect to x equals to

$$(f_1)'_x = \lim_{d \to 0} \frac{3dz + 3dz^2}{d} = 3z + 3z^2$$

Consider derivative of the function f_1 with respect to z, that is

$$\begin{aligned}(f_1)'_z &= \lim_{d \to 0} \left[\frac{-3d^2 - 2d^3 + 3dx + 3d^2x - 6dz - 6d^2z + 6dxz - 6dz^2}{d}\right]\\ &= \lim_{d \to 0} \left[-3d - 2d^2 + 3x + 3dx - 6z - 6dz + 6xz - 6z^2\right]\\ &= 3x - 6z + 6xz - 6z^2\end{aligned}$$

Combining both $(f_1)'_x$ and $(f_1)'_z$ evaluated at point (u,u) we get

$$\begin{aligned}(f_1)'_x + (f_1)'_z &= 3x - 3z + 6xz - 3z^2\\ \frac{d}{dt}t^3(u) &= [(f_1)'_x + (f_1)'_z](u,u) = 3u^2\end{aligned}$$

that confirms results of the theorem 1.4.

Example 1.6. Identity (3) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 2$. Consider the explicit form of the function $f_2(x,z)$ i.e

$$f_2(x,z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

Therefore, derivative of f_2 with respect to x equals to

$$\begin{aligned}(f_2)'_x &= \lim_{d\to 0}\left[5dz + 10xz - 15z^2 + 15dz^2 + 30xz^2 - 30z^3 + 10dz^3 + 20xz^3 - 15z^4\right]\\ &= 10xz - 15z^2 + 30xz^2 - 30z^3 + 20xz^3 - 15z^4\end{aligned}$$

Consider derivative of the function f_2 with respect to z, that is

$$(f_2)'_z = 5x^2 - 30xz + 30x^2z + 30z^2 - 90xz^2 + 30x^2z^2 + 60z^3 - 60xz^3 + 30z^4$$

Combining both $(f_2)'_x(x,z)$ and $(f_2)'_z(x,z)$ evaluated at point (u,u) we get

$$\begin{aligned}(f_2)'_x + (f_2)'_z &= 5x^2 - 20xz + 30x^2z + 15z^2 - 60xz^2 + 30x^2z^2 + 30z^3 - 40xz^3 + 15z^4\\ \frac{d}{dt}t^5(u) &= [(f_2)'_x + (f_2)'_z](u,u) = 5u^4\end{aligned}$$

that confirms results of the theorem 1.4.

Example 1.7. Identity (3) example for $x \in \mathbb{R},\ z \in \mathbb{R}$ and $y = 3$. Consider the explicit form of the function $f_3(x,z)$ i.e

$$\begin{aligned}f_3(x,z) = &-7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4\\ &- 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7\end{aligned}$$

Therefore, derivative of f_3 with respect to x equals to

$$\begin{aligned}(f_3)'_x = &-7z + 28xz - 42z^2 + 105x^2z^2 - 280xz^3 + 210x^2z^3 + 175z^4 - 420xz^4\\ &+ 105x^2z^4 + 210z^5 - 168xz^5 + 70z^6\end{aligned}$$

Consider derivative of the function f_2 with respect to z, that is

$$\begin{aligned}(f_3)'_z = &-7x + 14x^2 + 14z - 84xz + 70x^3z + 84z^2 - 420x^2z^2 + 210x^3z^2 + 700xz^3\\ &- 840x^2z^3 + 140x^3z^3 - 350z^4 + 1050xz^4 - 420x^2z^4 - 420z^5 + 420xz^5 - 140z^6\end{aligned}$$

Combining both $(f_3)'_x(x,z)$ and $(f_3)'_z(x,z)$ evaluated at point (u,u) we get

$$\begin{aligned}
(f_3)'_x + (f_3)'_z &= -7x + 14x^2 + 7z - 56xz + 70x^3z + 42z^2 - 315x^2z^2 + 210x^3z^2 \\
&\quad + 420xz^3 - 630x^2z^3 + 140x^3z^3 - 175z^4 + 630xz^4 - 315x^2z^4 - 210z^5 \\
&\quad + 252xz^5 - 70z^6 \\
\frac{d}{dt}t^3(u) &= [(f_3)'_x + (f_3)'_z](u,u) = 7u^6
\end{aligned}$$

that confirms results of the theorem 1.4.

2. Conclusions

In this manuscript we have reviewed an approach to get derivative of odd-power using identity in partial derivatives of the function f evaluated at fixed point $(u,u) \in \mathbb{R}^2$. Main results of the manuscript can be validated using Mathematica programs available online at [4].

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