SUMS OF POWERS VIA BACKWARD FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. We derive formulas for sums of powers via Newton's interpolation formula based on backward finite differences of powers. In addition, we note that backward differences are closely related to Eulerian numbers, and Stirling numbers of the second kind. Thus, we express formulas for sums of powers in terms of Eulerian numbers, and Stirling numbers of the second kind.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this manuscript, we derive formulas for multifold sums of powers by utilizing Newton's interpolation formula. Furthermore, we provide formulas for multifold sums of powers in

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terms of Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ and Eulerian numbers $\left\langle{n \atop k}\right\rangle$. The main idea is to utilize Newton's interpolation formula combined with hockey stick identities to derive formulas for multifold sums of powers. In this work, we use Newton's interpolation formula in backward finite differences. However, Newton's formula approach for sums of powers is quite generic, as such it does not necessary requires to bind to forward difference operator specifically. Thus, formulas for sums of powers using forward and central differences can be found in [1, 2], respectively. Define multifold sums of powers in Knuth's [3] notation

Proposition 1.1 (Multifold sums of powers recurrence). *For integers $r, n, m \geq 0$,*

$$\Sigma^0 \, n^m = n^m,$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m,$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m.$$

Steffensen gives Newton's formula for backward differences evaluated at zero

$$f(x) = \sum_{k=0}^{n} \binom{x+k-1}{k} \nabla^k f(0)$$

in his book Interpolation [4, chapter 2, eq. (19)]. Thus, Newton's formula for backward differences evaluated at arbitrary integer a follows.

Proposition 1.2 (Newton formula in backward differences). *Let $f \colon \mathbb{Z} \to \mathbb{C}$ be a function, and let $a \in \mathbb{Z}$. Then, for all $x \in \mathbb{Z}$,*

$$f(x) = \sum_{k=0}^{n} \binom{x-a+k-1}{k} \nabla^k f(a),$$

where the k-th backward finite difference of f evaluated at a is defined by

$$\nabla^k f(a) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(a-j).$$

Thus, by setting $f(n) = n^m$, we get interpolation of powers via backward differences evaluated at arbitrary point t.

Corollary 1.3 (Backward Newton's formula for powers)**.** *For integers* $n, m \geq 0$,

$$n^m = \sum_{j=0}^{m} \binom{n - t + j - 1}{j} \nabla^j t^m,$$

where $\nabla^j t^m = \sum_{k=0}^{j} (-1)^k \binom{j}{k} (t - k)^m$.

Therefore, by Corollary (1.3), identity for sums of powers follows.

$$\Sigma^1 \, n^m = \sum_{j=0}^{m} \nabla^j t^m \sum_{k=1}^{n} \binom{k - t + j - 1}{j}.$$

We notice that the sum $\sum_{k=1}^{n} \binom{k-t+j-1}{j}$ is a good candidate for hockey stick identity for binomial coefficients $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$. Thus, by setting $a = j - t$ and $b = j - t - 1 + n$, we get,

$$\sum_{k=1}^{n} \binom{k - t + j - 1}{j} = \sum_{m=j-t}^{j-t-1+n} \binom{m}{j} = \binom{j - t + n}{j + 1} - \binom{j - t}{j + 1}.$$

The closed form above is a partial case of Generalized hockey stick identity. That is,

Lemma 1.4 (Generalized hockey stick identity)**.** *For integers* $a \leq b$ *and* j,

$$\sum_{m=a}^{b} \binom{m}{j} = \binom{b + 1}{j + 1} - \binom{a}{j + 1}.$$

Proof. We have,

$$\sum_{k=a}^{b} \binom{k}{j} = \binom{a}{j} + \binom{a+1}{j} + \cdots + \binom{b}{j},$$

which implies,

$$\sum_{k=a}^{b} \binom{k}{j} = \left(\sum_{k=0}^{b} \binom{k}{j} \right) - \left(\sum_{k=0}^{a-1} \binom{k}{j} \right).$$

Thus, by hockey stick identity $\sum_{k=0}^{n} \binom{k}{j} = \binom{n+1}{j+1}$, we get,

$$\sum_{k=a}^{b} \binom{k}{j} = \left(\sum_{k=0}^{b} \binom{k}{j} \right) - \left(\sum_{k=0}^{a-1} \binom{k}{j} \right) = \binom{b+1}{j+1} - \binom{a}{j+1}.$$

This completes the proof. □

Lemma 1.5 (Shifted hockey stick identity). *For integers $j, r \geq 0$, and an arbitrary integer t,*

$$\sum_{k=1}^{n} \binom{j-t+k+r-1}{j+r} = \sum_{a=j-t+r}^{j-t+n+r-1} \binom{a}{j+r} = \binom{j-t+n+r}{j+r+1} - \binom{j-t+r}{j+r+1}.$$

Proposition 1.6 (Backward sums of powers). *For integers $n, m \geq 0$, and an arbitrary integer t,*

$$\Sigma^1 \, n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n}{j+1} - \binom{j-t}{j+1} \right].$$

Recall the negation property of binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$. Thus, formula for negated backward sums of powers follows.

Proposition 1.7 (Negated backward sums of powers). *For integers $n, m \geq 0$, and an arbitrary integer t,*

$$\Sigma^1 \, n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n}{j+1} + (-1)^j \binom{t}{j+1} \right].$$

For example, by setting $t = 2$ and $m = 1$, we get,

$$\Sigma^1 \, n^1 = 2 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 1 \left[\binom{2}{2} + \binom{n-1}{2} \right].$$

For $m = 2$:

$$\Sigma^1 \, n^2 = 4 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 3 \left[\binom{2}{2} + \binom{n-1}{2} \right] + 2 \left[-\binom{2}{3} + \binom{n}{3} \right].$$

For $m = 3$:

$$\Sigma^1 \, n^3 = 8 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 7 \left[\binom{2}{2} + \binom{n-1}{2} \right] + 6 \left[-\binom{2}{3} + \binom{n}{3} \right] + 6 \left[\binom{2}{4} + \binom{n+1}{4} \right].$$

For $m = 4$:

$$\Sigma^1 \, n^4 = 16 \left[-\binom{2}{1} + \binom{n-2}{1} \right] + 15 \left[\binom{2}{2} + \binom{n-1}{2} \right] + 14 \left[-\binom{2}{3} + \binom{n}{3} \right] + 12 \left[\binom{2}{4} + \binom{n+1}{4} \right]$$

$$+ 24 \left[-\binom{2}{5} + \binom{n+2}{5} \right].$$

The coefficients $1, 2, 1, 4, 3, 2, 8, 7, 6, 6, \ldots$ is the sequence A391068 in the OEIS [5]. The following Lemma connects the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ and backward differences of powers.

Lemma 1.8 (Backward differences in Eulerian numbers). *For integers $j, m \geq 0$, and an arbitrary integer t,*

$$\nabla^j t^m = \sum_{k=0}^{m} \binom{t+k-j}{m-j} \left\langle {m \atop k} \right\rangle.$$

Proof. By Worpitzky identity $t^m = \sum_{k=0}^{m} \binom{t+k}{m} \left\langle {m \atop k} \right\rangle$, and by binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, see [6]. $\qquad\square$

Thus, we get formula for ordinary sums of powers in terms of Eulerian numbers $\left\langle {m \atop k} \right\rangle$.

Proposition 1.9 (Ordinary sums of powers in Eulerian numbers). *For non-negative integers n, m and an arbitrary integer t*

$$\Sigma^1 \, n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t+k-j}{m-j} \left\langle {m \atop k} \right\rangle$$

It is quite remarkable that having $t = 0$ yields formula for sums of powers in double binomial form, involving Eulerian numbers $\left\langle {m \atop k} \right\rangle$.

Proposition 1.10 (Ordinary Eulerian sums of powers in zero). *For non-negative integers n, m,*

$$\Sigma^1 \, n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \binom{j+n}{j+1} \binom{k-j}{m-j} \left\langle {m \atop k} \right\rangle.$$

Lemma 1.11 (Backward differences in Stirling numbers). *For integers $j, m \geq 0$, and an arbitrary integer t,*

$$\nabla^j t^m = \sum_{k=j}^{m} \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

Proof. By the identity $t^m = \sum_{k=0}^{m} \binom{t}{k} \left\{ {m \atop k} \right\} k!$ and binomial recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. $\qquad\square$

Thus, let be a formula for ordinary sums of powers in terms of Stirling numbers $\left\{ {m \atop k} \right\}$

Proposition 1.12 (Ordinary sums of powers in Stirling numbers). *For non-negative integers* n, m *and an arbitrary integer* t,

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \left[(-1)^j \binom{t}{j+1} + \binom{j-t+n}{j+1} \right] \binom{t-j}{k-j} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$$

By setting $t = 0$ yields

Proposition 1.13 (Ordinary Stirling sums of powers in zero). *For non-negative integers* n, m,

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \binom{j+n}{j+1} \binom{-j}{k-j} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$$

By upper negation property of binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, another binomial form for sums of powers follows.

Proposition 1.14 (Ordinary Stirling sums of powers in zero altered). *For non-negative integers* n, m

$$\Sigma^1 n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} (-1)^{k-j} \binom{k-1}{j-1} \binom{j+n}{j+1} \begin{Bmatrix} m \\ k \end{Bmatrix} k!$$

Similarly to Proposition (1.7), we can derive formula for double sums of powers. Thus,

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \binom{j-t+k}{j+1} \right]$$

By applying generalized hockey stick identity (1.4), we obtain,

$$\sum_{k=1}^{n} \binom{j-t+k}{j+1} = \sum_{k=j-t+1}^{j-t+n} \binom{k}{j+1} = \binom{j-t+n+1}{j+2} - \binom{j-t+1}{j+2}.$$

Therefore, we get closed form.

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} n + \left(\binom{j-t+n+1}{j+2} - \binom{j-t+1}{j+2} \right) \right].$$

By applying the identity for negative binomial coefficients $\binom{-k}{j} = (-1)^j \binom{j+k-1}{j}$, we get,

$$\binom{-(t-j-1)}{j+2} = (-1)^{j+2} \binom{t}{j+2}.$$

Hence,

Proposition 1.15 (Double sums of powers via backward differences). *For non-negative integers n, m and an arbitrary integer t,*

$$\Sigma^2 n^m = \sum_{j=0}^{m} \nabla^j t^m \left[(-1)^j \binom{t}{j+1} n + (-1)^{j+1} \binom{t}{j+2} n^0 + \binom{j-t+n+1}{j+2} \right]$$

By applying the hockey stick identities (1.4), and (1.5) repeatedly, yields formula for $r-$fold sums of powers.

Theorem 1.16 (Multifold sums of powers via backward difference). *For non-negative integers r, n, m and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n+r-1}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{t}{j+s+1} \Sigma^{r-1-s} n^0 \right].$$

We may observe that,

Proposition 1.17 (Multifold sum of zero powers). *For integers $r \geq 0$, and $n \geq 1$,*

$$\Sigma^r n^0 = \binom{r+n-1}{r}.$$

Proof.   (1) Let $r = 0$, then $\Sigma^0 n^0 = n^0 = \binom{n-1}{0} = 1$, by definition.

(2) Let $r = 1$, then $\Sigma^1 n^0 = \sum_{k=1}^{n} \binom{k-1}{0} = \sum_{k=1}^{n} 1 = \binom{n}{1}$.

(3) Let $r = 2$, then $\Sigma^2 n^0 = \sum_{k=1}^{n} \binom{k}{1} = \sum_{k=1}^{n} k = \binom{n+1}{2}$.

(4) Let $r = 3$, then $\Sigma^3 n^0 = \sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3}$.

(5) By induction over r and hockey stick identity $\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, the claim follows

$$\Sigma^r n^0 = \binom{r+n-1}{r}.$$

This completes the proof. $\square$

Hence, binomial form of formula for multifold sums of powers (1.16) follows.

Proposition 1.18 (Multifold sums of powers binomial form). *For non-negative integers r, n, m and an arbitrary integer t,*

$$\Sigma^r n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n+r-1}{j+r} + \sum_{s=0}^{r-1} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-2}{r-s-1} \right].$$

By setting $r \to r+1$ yields shifted version of the identity above.

Corollary 1.19 (Multifold sums of powers binomial form shifted)**.** *For non-negative integers* r, n, m *and an arbitrary integer* t,

$$\Sigma^{r+1} \, n^m = \sum_{j=0}^{m} \nabla^j t^m \left[\binom{j-t+n+r}{j+r+1} + \sum_{s=0}^{r} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-1}{r-s} \right].$$

By Lemma (1.11), we get formula for multifold sums of powers in terms of Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$.

Proposition 1.20 (Multifold sums of powers in Stirling numbers)**.** *For non-negative integers* r, n, m *and an arbitrary integer* t,

$$\Sigma^{r+1} \, n^m = \sum_{j=0}^{m} \sum_{k=j}^{m} \left[\binom{j-t+n+r}{j+r+1} + \sum_{s=0}^{r} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-1}{r-s} \right] \binom{t-j}{k-j} \left\{ {m \atop k} \right\} k!$$

By lemma (1.8), we get formula for multifold sums of powers in terms of Eulerian numbers $\left\langle {n \atop k} \right\rangle$.

Proposition 1.21 (Multifold sums of powers in Eulerian numbers)**.** *For non-negative integers* r, n, m *and an arbitrary integer* t

$$\Sigma^{r+1} \, n^m = \sum_{j=0}^{m} \sum_{k=0}^{m} \left[\binom{j-t+n+r}{j+r+1} + \sum_{s=0}^{r} (-1)^{j+s} \binom{t}{j+s+1} \binom{r-s+n-1}{r-s} \right] \binom{t+k-j}{m-j} \left\langle {m \atop k} \right\rangle.$$

CONCLUSIONS

In this manuscript, we derived formulas for multifold sums of powers (1.16), by utilizing Newton's interpolation series in backward differences. In addition, we highlight that backward differences of powers are closely related to Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, and Eulerian numbers $\left\langle {m \atop k} \right\rangle$. Thus, we expressed formulas for multifold sums of powers in terms of Stirling numbers of the second kind (1.20) and Eulerian numbers (1.21). Future research directions are discussed and proposed in [1], which includes development of generalized framework for sums of powers using interpolation formulas combined with hockey stick identities for binomial coefficients. All the results are validated using `Mathematica` programs; see Section (2).

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2. Mathematica programs

Use the *Mathematica* package [7] to validate the results

Mathematica Function	Validates / Prints
MultifoldSumOfPowersRecurrence[r, n, m]	Computes $\Sigma^r\, n^m$
ValidateOrdinarySumsOfPowersViaBackwardDifferences[20]	Validates (1.7)
ValidateBackwardDifferencesInEulerianNumbers[20]	Validates (1.8)
ValidateOrdinarySumsOfPowersInEulerianNumbers[10]	Validates (1.9)
ValidateBackwardDifferencesInStirlingNumbers[20]	Validates (1.11)
ValidateOrdinarySumsOfPowersInStirlingNumbers[20]	Validates (1.12)
ValidateOrdinaryStirlingSumsOfPowersInZero[20]	Validates (1.13)
ValidateOrdinaryStirlingSumsOfPowersInZeroAltered[20]	Validates (1.14)
ValidateDoubleSumsOfPowersViaBackwardDifferences[10]	Validates (1.15)
ValidateMultifoldSumsOfPowersViaBackwardDifference[5]	Validates (1.16)
ValidateMultifoldSumsOfPowersBackwardDiffBinomialForm[5]	Validates (1.18)
ValidateMultifoldSumsOfPowersBinomialFormShifted[5]	Validates (1.19)
ValidateMultifoldSumsOfPowersInStirlingNumbers[5]	Validates (1.20)
ValidateMultifoldSumsOfPowersInEulerianNumbers[5]	Validates (1.21)

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- **Sources:** github/kolosovpetro/SumsOfPowersBackwardDiffNewtonFormula

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