SUMS OF POWERS VIA CENTRAL FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. In this manuscript we derive formula for multifold sums of powers using Newton's formula in central differences. Furthermore, we show that famous Knuth's formula for multifold sums of odd powers originates from Newton's interpolation formula in central differences.

CONTENTS

1. INTRODUCTION

In this manuscript we derive formula for multifold sums of powers using Newton's formula and central differences.

The idea to derive sums of powers using difference operator and Newton's series is quite generic. Thus, formulas for sums of powers using forward and backward differences can be found in the works [1, 2].

We define the recurrence for multifold sums of powers introduced by Donald Knuth in [3], which we utilize throughout the paper.

Proposition 1.1 (Multifold sums of powers recurrence). *For non-negative integers r, n, m*

$$\Sigma^0 \, n^m = n^m$$

$$\Sigma^1 \, n^m = \Sigma^0 \, 1^m + \Sigma^0 \, 2^m + \cdots + \Sigma^0 \, n^m$$

$$\Sigma^{r+1} \, n^m = \Sigma^r \, 1^m + \Sigma^r \, 2^m + \cdots + \Sigma^r \, n^m$$

Proposition 1.2 (Central factorials). *For integers n, k*

$$n^{[k]} = \begin{cases} 0, & \text{if } k < 0 \\[2mm] 1, & \text{if } k = 0 \\[2mm] n\left(n + \frac{k}{2} - 1\right)\left(n + \frac{k}{2} - 2\right) \cdots \left(n - \frac{k}{2} + 1\right) = n \prod_{j=1}^{k-1}\left(n + \frac{k}{2} - j\right), & \text{if } k > 0 \end{cases}$$

Consider Newton's interpolation formula [4, 5] in central differences evaluated in zero

Proposition 1.3 (Newton's formula in central differences in zero).

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k f(0)$$

where $\delta^k f(0) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} f\left(\frac{k}{2} - j\right)$ are central finite differences in zero, and $x^{[k]}$ are central factorials, with $x^{[0]} = 1$ for every x.

We observe that central factorials are closely related to falling factorials $(x)_n = x(x - 1)(x - 2)(x - 3) \cdots (x - n + 1) = \prod_{k=0}^{n-1}(x - k)$. Therefore,

Proposition 1.4 (Central factorials in terms of falling)**.** *For integers n, k*

$$n^{[k]} = \begin{cases} 0, & \text{if } k < 0 \\ 1, & \text{if } k = 0 \\ n\left(n + \frac{k}{2} - 1\right)_{k-1}, & \text{if } k > 0 \end{cases}$$

where $\left(n + \frac{k}{2} - 1\right)_{k-1}$ are falling factorials.

To derive formula for multifold sums of powers, we follow the strategy to express the Newton's formula (1.3) in terms of binomial coefficients, then to reach closed forms of column sum of binomial coefficients by means of hockey stick identity. Therefore,

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Proposition 1.5 (Binomial form of central factorials)**.** *For integers n and $k \geq 1$*

$$\frac{n^{[k]}}{k!} = \frac{n}{k}\binom{n + \frac{k}{2} - 1}{k - 1}$$

Proof. We have

$$\frac{n^{[k]}}{k!} = \frac{n}{k!}\left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k(k-1)!}\left(n + \frac{k}{2} - 1\right)_{k-1} = \frac{n}{k}\binom{n + \frac{k}{2} - 1}{k - 1}$$

because of the identity in falling factorial $\frac{(x)_n}{n!} = \binom{x}{n}$ and (1.4). □

Which yields Newton's formula for powers, in terms of central differences.

Proposition 1.6 (Newton's formula for powers in zero)**.** *For positive integers $n \geq 1$ and $m \geq 1$*

$$n^m = \sum_{k=1}^{m} \frac{n}{k}\binom{n + \frac{k}{2} - 1}{k - 1}\delta^k 0^m$$

Although based on Newton's interpolation formula (1.3), the proposition (1.6) iterates starting fom $k = 1$ to avoid division by zero in $\frac{n}{k}$. This is a valid trick, because the central difference $\delta^k 0^n$ is zero for all $n \geq 1$ and $k = 0$.

By factoring out and simplifying the term n, we get

$$n^{m-1} = \sum_{k=1}^{m} \frac{1}{k} \binom{n + \frac{k}{2} - 1}{k - 1} \delta^k 0^m$$

We may observe that the operator of central finite difference $\delta^k 0^m$ requires the parity of its arguments m and k meaning that both m and k required to be: $m \pmod 2 = k \pmod 2$, such that finite differences $\delta^k 0^m$ are non-zero

$$\delta^k 0^m \neq 0, \quad \text{whether} \quad m \pmod 2 = k \pmod 2,$$

$$\delta^k 0^m = 0, \quad \text{whether} \quad m \pmod 2 \neq k \pmod 2.$$

By setting $m \to 2m$ we get

$$n^{2m-1} = \sum_{k=1}^{2m} \frac{1}{k} \binom{n + \frac{k}{2} - 1}{k - 1} \delta^k 0^{2m}$$

Thus, the central differences $\delta^k 0^{2m}$ are zero for all odd k.

Since that k runs over all integers in the range $0 \leq k \leq 2m$, we can omit odd values of k

$$n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n + k - 1}{2k - 1} \delta^{2k} 0^{2m}$$

Hence, formula for ordinary sums of odd powers yields

Proposition 1.7 (Ordinary sums of odd powers in central differences). *For   integers* $n \geq 1,\ m \geq 1$

$$\Sigma^1 n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n + k}{2k} \delta^{2k} 0^{2m}$$

Proof. We have $\Sigma^1 n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \delta^{2k} 0^{2m} \sum_{j=1}^{n} \binom{j+k-1}{k-1}$.

By hockey stick identity $\sum_{j=1}^{n} \binom{j+k-1}{2k-1} = \binom{n+k}{2k}$, thus the statement follows.   □

Therefore,

Theorem 1.8 (Multifold sums of odd powers in central differences)**.** *For integers* $n \geq 1$, $m \geq 1$ *and* $r \geq 0$

$$\Sigma^r \, n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n+k-1+r}{2k-1+r} \delta^{2k} 0^{2m}.$$

Proof. We have $\Sigma^1 \, n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \delta^{2k} 0^{2m} \sum_{j=1}^{n} \binom{j+k-1}{2k-1}$.

By hockey stick identity $\sum_{j=1}^{n} \binom{j+k-1}{k-1} = \binom{n+k}{2k}$. By induction the claim follows.   □

It is quite interesting to notice that theorem (1.8) is a central difference form of the formula of sums of odd-powers given by Donald Knuth in *Johann Faulhaber and sums of powers*, see [3].

The reason is straightforward, instead of using Central factorial numbers of the second kind $T(n,k)$, the theorem (1.8) utilizes central differences explicitly, because

Lemma 1.9 (Central factorial numbers of the second kind)**.**

$$k! T(n,k) = \delta^k 0^n$$

Meaning that the Knuth's formula

Proposition 1.10 (Multifold sums of odd powers in central factorial numbers)**.** *For integers* $n \geq 1$, $m \geq 1$ *and* $r \geq 0$

$$\Sigma^r \, n^{2m-1} = \sum_{k=1}^{m} (2k-1)! \binom{n+k-1+r}{2k-1+r} T(2m, 2k).$$

originates from Newton's interpolation formula in central differences (1.3).

The lemma (1.9) is well discussed in [6, 7, 8].

The non-zero central factorial numbers $T(2m, 2k)$ is the sequence A008957 in the OEIS [9].

For example,

$$\Sigma^1\, n^1 = \binom{n+1}{2}$$

$$\Sigma^1\, n^3 = 6\binom{n+2}{4} + \binom{n+1}{2}$$

$$\Sigma^1\, n^5 = 120\binom{n+3}{6} + 30\binom{n+2}{4} + \binom{n+1}{2}$$

$$\Sigma^1\, n^7 = 5040\binom{n+4}{8} + 1680\binom{n+3}{6} + 126\binom{n+2}{4} + \binom{n+1}{2}$$

While multifold sums of odd powers are

$$\Sigma^r\, n^1 = \binom{n+1+r}{2+r}$$

$$\Sigma^r\, n^3 = 6\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r}$$

$$\Sigma^r\, n^5 = 120\binom{n+3+r}{6+r} + 30\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r}$$

$$\Sigma^r\, n^7 = 5040\binom{n+4+r}{8+r} + 1680\binom{n+3+r}{6+r} + 126\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r}$$

The coefficients $1, 6, 1, 120, 30, 1, \ldots$ is the sequence A303675 in the OEIS [9].

Conclusions

In this manuscript, we derived formula for multifold sums of powers using Newton's formula in central differences, combined with hockey-stick identity for binomial coefficients. Additionally, we shown that the famous Knuth's formula for multifold sums of powers [3] originates from Newton's formula in central differences. All results of this manuscript are validated using programs in Wolfram Mathematica, see section (2).

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2. Mathematica programs

Use the *Mathematica* package [10] to validate the results

Mathematica Function	Validates / Prints
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r\, n^m$
`ValidateCentralFactorialsInTermsOfFalling[10]`	Validates Proposition (1.4)
`ValidateBinomialFormOfCentralFactorials[10]`	Validates Proposition (1.5)
`ValidateNewtonsFormulaForPowersInZero[20]`	Validates Proposition (1.6)
`ValidateOrdinarySumsOfOddPowersInCentralDifferences[20]`	Validates Prop. (1.7)
`ValidateMultifoldSumsOfOddPowersInCentralDifferences[5]`	Validates Thm. (1.8)

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