COUNTING PRIMES AND TWIN-PRIMES USING MINIMAL GOLDBACH PAIRS

PETRO KOLOSOV

ABSTRACT. Assuming Goldbach's Conjecture holds, every even integer $2N \geq 4$ can be written as $2N = p_i + p_j$ where (p_i, p_j) is called Goldbach pair. The minimal Goldbach pair is a pair (p_i, p_j) having the minimal p_i such that $p_j = 2N - p_i$ is also a prime. We define a function $F_{2N}(P)$ that counts occurrences of $p_j = P$ within the range $6 \leq 2k \leq 2N$. In particular, the function $F_{2N}(P)$ provides the following identities in terms of prime counting $\pi(2N)$ and twin-prime counting $\pi_2(2N)$

$$\pi(2N) = F_{2N}(3) + 1, \quad \pi_2(2N) = F_{2N}(3) - F_{2N}(5)$$

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1. INTRODUCTION

This manuscript provides a comprehensive review of the work [1] done by Michel Yamagishi. The Goldbach conjecture asserts that every even integer $2N \geq 4$ is a sum of two

primes

$$2N = p_i + p_j$$

where (p_i, p_j) is called a Goldbach pair.

A Goldbach pair is not unique for even integers greater than 6, which means that there can be multiple Goldbach pairs for even integer $2N \geq 8$. For example: $10 = 3 + 7$ and $10 = 5 + 5$ and $10 = 7 + 3$, where the Goldbach pairs are $(3, 7)$, $(5, 5)$, $(7, 3)$.

The minimal Goldbach pair is the pair with the smallest p_i among all goldbach pairs for even integer $2N$. For the even integer 10, we have three pairs: $(3, 7)$, $(5, 5)$, $(7, 3)$ and the minimal one is $(3, 7)$ because 3 is the smallest among all p_i values: $3, 5, 7$.

Consider the following minimal Goldbach pairs for even integer $2k$ within the range $6 \leq 2k \leq 50$

$$6 = 3 + 3, \qquad\qquad 8 = 3 + 5,$$

$$10 = 3 + 7, \qquad\qquad 12 = 5 + 7,$$

$$14 = 3 + 11, \qquad\qquad 16 = 3 + 13,$$

$$18 = 5 + 13, \qquad\qquad 20 = 3 + 17,$$

$$22 = 3 + 19, \qquad\qquad 24 = 5 + 19,$$

$$26 = 3 + 23, \qquad\qquad 28 = 5 + 23,$$

$$30 = 7 + 23, \qquad\qquad 32 = 3 + 29,$$

$$34 = 3 + 31, \qquad\qquad 36 = 5 + 31,$$

$$38 = 7 + 31, \qquad\qquad 40 = 3 + 37,$$

$$42 = 5 + 37, \qquad\qquad 44 = 3 + 41,$$

$$46 = 3 + 43, \qquad\qquad 48 = 5 + 43,$$

$$50 = 3 + 47$$

We can notice that minimal Goldbach pairs having $p_i = 3$ produce a sequence of odd prime numbers $p_j = 3, 5, 7, 11, 13, 17 \ldots$ which is quite remarkable:

$$6 = 3 + 3, \qquad\qquad 8 = 3 + 5,$$

$$10 = 3 + 7, \qquad\qquad 14 = 3 + 11,$$

$$16 = 3 + 13, \qquad\qquad 20 = 3 + 17,$$

$$22 = 3 + 19, \qquad\qquad 26 = 3 + 23,$$

$$32 = 3 + 29, \qquad\qquad 34 = 3 + 31,$$

$$40 = 3 + 37, \qquad\qquad 44 = 3 + 41,$$

$$46 = 3 + 43, \qquad\qquad 50 = 3 + 47$$

Another interesting observation is that by selecting the pairs with minimal $p_i = 5$ yields the sequence of primes p_j such that $p_j + 2$ is not prime

$$12 = 5 + 7, \qquad\qquad 18 = 5 + 13,$$

$$24 = 5 + 19, \qquad\qquad 28 = 5 + 23,$$

$$36 = 5 + 31, \qquad\qquad 42 = 5 + 37,$$

$$48 = 5 + 43.$$

To formalize and clarify our discussion, we define a few functions. Let $G_{\min}(2N)$ be a function that returns a set of minimal Goldbach pairs (p_i, p_j) having $\min p_i$ over the range $6 \leq 2k \leq 2N$

$$G_{\min}(2N) = \{(p_i, p_j) \mid p_i + p_j = 2k \mid 6 \leq 2k \leq 2N \mid \min p_i\}.$$

For example,

$$G_{\min}(20) = \{(3,3),\ (3,5),\ (3,7),\ (5,7),\ (3,11),\ (3,13),\ (5,13),\ (3,17)\}$$

Let $W_{2N}(P)$ be a function that returns the set of elements p_j from $G_{\min}(2N)$ having $p_i = P$

$$W_{2N}(P) = \{p_j \mid (p_i, p_j) \in G_{\min}(2N) \text{ and } p_i = P\}$$

Then, the sequence of odd prime numbers [2] is given by $W_{2N}(3)$

$$\{3, 5, 7, 11, \ldots, p \leq 2N - 3\} = W_{2N}(3)$$

Now we can easily count the number of primes within the interval $6 \leq 2k \leq 2N$ because $\pi(2N)$ is equal to the total number of elements inside the set $W_{2N}(3)$, which corresponds to the sequence [3]

$$\pi(2N) = F_{2N}(3) + 1$$

where $F_{2N}(3)$ is the function that counts the number of elements inside the set $W_{2N}(3)$. In general $F_{2N}(P) = |W_{2N}(P)|$.

Taking $P = 5$ in $W_{2N}(P)$ yields a sequence of primes p_j such that $p_j + 2$ is not a prime [4]

$$W_{2N}(5) = \{7, 13, 19, 23, 31, 37, 43, 47, 53, \ldots, p \leq 2N - 5\}$$

This implies that the number of twin primes in range $6 \leq 2k \leq 2N$ can be expressed in terms of $F_{2N}(P)$

$$\pi_2(2N) = F_{2N}(3) - F_{2N}(5)$$

where $2N = 10^k + 2$, $k = 1, 2, 3, 4, \ldots$. For example,

$$\pi_2(12) = F_{12}(3) - F_{12}(5) = 2$$

$$\pi_2(102) = F_{102}(3) - F_{102}(5) = 8$$

$$\pi_2(1002) = F_{1002}(3) - F_{1002}(5) = 35$$

$$\pi_2(10002) = F_{10002}(3) - F_{10002}(5) = 205$$

$$\pi_2(100002) = F_{100002}(3) - F_{100002}(5) = 1224$$

$$\pi_2(1000002) = F_{1000002}(3) - F_{1000002}(5) = 8169$$

These results match the sequence [5].

Having $P = 7$ function $W_{2N}(P)$ yields the sequence of primes such that $p_j - p_i \geq 6$, where p_j is the next prime after p_i, see [6]

$$W_{2N}(7) = \{23, \ 31, \ 47, \ 53, \ 61, \ 73, \ 83, \ 89, \ 113, \ \ldots, \ p \leq 2N - 7\}$$

This allows us to count the primes for which the next-prime gap at least 6: $\delta_6(2N) = F_{2N}(7)$.

2. Conclusions

Assuming Goldbach's Conjecture holds, we introduced a framework based on minimal Goldbach pairs to derive expressions for key prime-related functions. Specifically, we defined the function $F_{2N}(P)$ that counts occurrences of primes p_j in minimal Goldbach pairs (p_i, p_j) where $p_i = P$. Using this framework, we obtained

- The prime-counting function: $\pi(2N) = F_{2N}(3) + 1$
- The twin-prime counting function: $\pi_2(2N) = F_{2N}(3) - F_{2N}(5)$
- The count of primes with next-prime gap at least 6: $\delta_6(2N) = F_{2N}(7)$

These identities establish a novel connection between Goldbach partitions and classical prime number theory. Computational examples confirm alignment with known integer sequences, reinforcing the potential of this approach for analytical and numerical exploration of prime distributions. All the results validated up to $N = 10^8$ with source code available on GitHub [7].

References

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SOFTWARE DEVELOPER, DEVOPS ENGINEER

Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io