# ANOTHER APPROACH TO GET DERIVATIVE OF ODD-POWER

PETRO KOLOSOV

ABSTRACT. This manuscript provides another approach to get derivative of odd-power, that is approach based on partial derivatives of the polynomial function $f_y$ defined as

$$f_y(x, z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x - k)^r$$

where $x, z \in \mathbb{R}$, $y$ is fixed constant $y \in \mathbb{N}$ and $\mathbf{A}_{y,r}$ are real coefficients.

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## 1. INTRODUCTION AND MAIN RESULTS

This manuscript provides another approach to get derivative of odd-power, that is approach based on identity in terms of partial derivatives, extending the main idea explained in [1] that gives an odd-power in a form as follows

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n - k)^r \tag{1}$$

where $m$ is fixed constant $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $\mathbf{A}_{m,r}$ are real coefficients defined recursively, see [2]. Define the function $f_y$ such that based on the identity (1) with the only difference

that values of $n, m$ in the right part of (1) appear to be parameters of the function $f_y$. In contrast to the equation (1), upper bound $n$ of the sum $\sum_{k=1}^{n}$ turned into fixed function's parameter $y$ as well, so that $f_y$ defined as follows

**Definition 1.1.** *(Polynomial function $f_y$.)*

$$f_y(x, z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x - k)^r \tag{2}$$

where $x, z \in \mathbb{R}$ and $y$ is constant $y \in \mathbb{N}$. Note that for every $x \in \mathbb{R}$ and constant $y \in \mathbb{N}$ the polynomial identity satisfies

$$f_y(x, x) = x^{2y+1}$$

At first glance, equation (2) might look complex, so in order to clarify the function $f_y$ and polynomials it produces, let there be a few examples. Substituting the values of $y = 1, 2, 3$ to the function $f$ we get the following polynomials

$f_1(x, z) = 3xz - 3z^2 + 3xz^2 - 2z^3$

$f_2(x, z) = 5x^2 z - 15xz^2 + 15x^2 z^2 + 10z^3 - 30xz^3 + 10x^2 z^3 + 15z^4 - 15xz^4 + 6z^5$

$f_3(x, z) = -7xz + 14x^2 z + 7z^2 - 42xz^2 + 35x^3 z^2 + 28z^3 - 140x^2 z^3 + 70x^3 z^3 + 175xz^4$

$\qquad - 210x^2 z^4 + 35x^3 z^4 - 70z^5 + 210xz^5 - 84x^2 z^5 - 70z^6 + 70xz^6 - 20z^7$

These polynomials are obtained by rearranging the sums in the equation (2) as

$$\sum_{r=0}^{y} \mathbf{A}_{y,r} \left[ \sum_{k=1}^{z} k^r (x - k)^r \right]$$

So that part $\sum_{k=1}^{z} k^r (x - k)^r$ is polynomial in $x, z$ calculated using Faulhaber's formula [3]. According to the main topic of the current manuscript, it provides another approach to get derivative of odd-power. Therefore, we define odd-power function we work in the context of. Odd-power function $g_y$ is a function defined as follows

**Definition 1.2.** *(Odd-power function $g_y$.)*

$$g_y(x) = x^{2y+1}$$

where $x \in \mathbb{R}$ and $y$ is constant $y \in \mathbb{N}$. The Interesting part is that odd-power function $g_y(x)$ may be obtained as a partial case of the function $f$, as well as derivative of odd-power $\frac{d}{dx} g_y$ evaluated in point some $u \in \mathbb{R}$ may be obtained as a sum of partial derivatives of $f$ evaluated in some fixed point, that is explained further in this manuscript. One more important thing is to conclude on partial derivative notation, more precisely the following notation for the partial derivative is used across the manuscript and remains unchanged

**Notation 1.3.** *(Partial derivative.) Let be a function $f(x_1, x_2, \ldots, x_n)$ defined over the real space $\mathbb{R}^n$. We denote partial derivative of the function $f$ with respect to $x_i$ as follows*

$$f'_{x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}$$

*Derivative of the function $f$ with respect to $x_i$ evaluate in point $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is denoted as follows*

$$f'_{x_i}(y_1, y_2, \ldots, y_n)$$

*Moreover, partial derivative $f'_{x_i}$ evaluated at point $(y_1, y_2, \ldots, y_n)$ plus partial derivative $f'_{x_j}$ evaluated at point $(y_1, y_2, \ldots, y_n)$ is equivalent to the sum of partial derivatives $f'_{x_i}$, $f'_{x_j}$ evaluated at point $(y_1, y_2, \ldots, y_n)$ and to be denoted as*

$$f'_{x_i}(y_1, y_2, \ldots, y_n) + f'_{x_j}(y_1, y_2, \ldots, y_n) = [f'_{x_i} + f'_{x_j}](y_1, y_2, \ldots, y_n)$$

Therefore, the following identity in terms of partial derivatives shows the relation between odd-power function $g_y$ and polynomial function $f_y$

**Theorem 1.4.** *Let be a fixed value $v \in \mathbb{N}$, then derivative $g'_v$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluated at point $u$ equals to derivative $(f_v)'_x$ evaluated at point $(u, u)$ plus*

*derivative $(f_v)'_z$ evaluated at point $(u, u)$*

$$g'_v(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u) \tag{3}$$

Particularly, it follows that for every pair $u, v$ an identity holds

$$(2v + 1)u^{2v} = (f_v)'_x(u, u) + (f_v)'_z(u, u)$$

$$= [(f_v)'_x + (f_v)'_z](u, u)$$

that is also an ordinary derivative of odd-power function $t^{2v+1}$, therefore

$$\frac{d}{dt}t^{2v+1}(u) = (f_v)'_x(u, u) + (f_v)'_z(u, u)$$

$$= [(f_v)'_x + (f_v)'_z](u, u)$$

To summarize and clarify all above, we provide a few examples that show an identity (3) in action.

**Example 1.5.** Identity (3) example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 1$. Consider the explicit form of the function $f_1(x, z)$ i.e.,

$$f_1(x, z) = 3xz - 3z^2 + 3xz^2 - 2z^3$$

Therefore, derivative of $f_1$ with respect to $x$ equals to

$$(f_1)'_x = \lim_{d \to 0} \frac{3dz + 3dz^2}{d} = 3z + 3z^2$$

Consider derivative of the function $f_1$ with respect to $z$, that is

$$(f_1)'_z = \lim_{d \to 0} \left[ \frac{-3d^2 - 2d^3 + 3dx + 3d^2x - 6dz - 6d^2z + 6dxz - 6dz^2}{d} \right]$$

$$= \lim_{d \to 0} \left[ -3d - 2d^2 + 3x + 3dx - 6z - 6dz + 6xz - 6z^2 \right]$$

$$= 3x - 6z + 6xz - 6z^2$$

Combining both $(f_1)'_x$ and $(f_1)'_z$ evaluated at the point $(u, u)$ we get

$$(f_1)'_x + (f_1)'_z = 3x - 3z + 6xz - 3z^2$$

$$\frac{d}{dt}t^3(u) = [(f_1)'_x + (f_1)'_z](u, u) = 3u^2$$

that confirms the results of the theorem 1.4.

**Example 1.6.** Identity (3) example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 2$. Consider the explicit form of the function $f_2(x, z)$ i.e.,

$$f_2(x, z) = 5x^2z - 15xz^2 + 15x^2z^2 + 10z^3 - 30xz^3 + 10x^2z^3 + 15z^4 - 15xz^4 + 6z^5$$

Therefore, derivative of $f_2$ with respect to $x$ equals to

$$(f_2)'_x = \lim_{d \to 0} \left[ 5dz + 10xz - 15z^2 + 15dz^2 + 30xz^2 - 30z^3 + 10dz^3 + 20xz^3 - 15z^4 \right]$$

$$= 10xz - 15z^2 + 30xz^2 - 30z^3 + 20xz^3 - 15z^4$$

Consider derivative of the function $f_2$ with respect to $z$, that is

$$(f_2)'_z = 5x^2 - 30xz + 30x^2z + 30z^2 - 90xz^2 + 30x^2z^2 + 60z^3 - 60xz^3 + 30z^4$$

Combining both $(f_2)'_x(x, z)$ and $(f_2)'_z(x, z)$ evaluated at the point $(u, u)$ we get

$$(f_2)'_x + (f_2)'_z = 5x^2 - 20xz + 30x^2z + 15z^2 - 60xz^2 + 30x^2z^2 + 30z^3 - 40xz^3 + 15z^4$$

$$\frac{d}{dt}t^5(u) = [(f_2)'_x + (f_2)'_z](u, u) = 5u^4$$

that confirms the results of the theorem 1.4.

**Example 1.7.** Identity (3) example for $x \in \mathbb{R}$, $z \in \mathbb{R}$ and $y = 3$. Consider the explicit form of the function $f_3(x, z)$ i.e

$$f_3(x, z) = -7xz + 14x^2z + 7z^2 - 42xz^2 + 35x^3z^2 + 28z^3 - 140x^2z^3 + 70x^3z^3 + 175xz^4$$

$$- 210x^2z^4 + 35x^3z^4 - 70z^5 + 210xz^5 - 84x^2z^5 - 70z^6 + 70xz^6 - 20z^7$$

Therefore, derivative of $f_3$ with respect to $x$ equals to

$$(f_3)_x' = -7z + 28xz - 42z^2 + 105x^2z^2 - 280xz^3 + 210x^2z^3 + 175z^4 - 420xz^4$$

$$+ 105x^2z^4 + 210z^5 - 168xz^5 + 70z^6$$

Consider derivative of the function $f_2$ with respect to $z$, that is

$$(f_3)_z' = -7x + 14x^2 + 14z - 84xz + 70x^3z + 84z^2 - 420x^2z^2 + 210x^3z^2 + 700xz^3$$

$$- 840x^2z^3 + 140x^3z^3 - 350z^4 + 1050xz^4 - 420x^2z^4 - 420z^5 + 420xz^5 - 140z^6$$

Combining both $(f_3)_x'(x, z)$ and $(f_3)_z'(x, z)$ evaluated at the point $(u, u)$ we get

$$(f_3)_x' + (f_3)_z' = -7x + 14x^2 + 7z - 56xz + 70x^3z + 42z^2 - 315x^2z^2 + 210x^3z^2$$

$$+ 420xz^3 - 630x^2z^3 + 140x^3z^3 - 175z^4 + 630xz^4 - 315x^2z^4 - 210z^5$$

$$+ 252xz^5 - 70z^6$$

$$\frac{d}{dt}t^3(u) = [(f_3)_x' + (f_3)_z'](u, u) = 7u^6$$

that confirms the results of the theorem 1.4.

## 2. Conclusions

In this manuscript, we have reviewed an approach to get derivative of odd-power using identity in partial derivatives of the function $f$ evaluated at fixed point $(u, u) \in \mathbb{R}^2$. Main results of the manuscript can be validated using Mathematica programs available online at [4].

## 3. Verification of the results

As it is stated in conclusions, there is a possibility to validate the main results of this manuscript using Wolfram Mathematica. Therefore, a complete guide to validate the main results and formulae is attached as well. Mathematica package source file is available online under the folder `mathematica`, see [4]. The following expressions could be verified:

- Function $f_y(x, z)$ for any constant argument $y \in \mathbb{N}$ using mathematica method `f[x, y, z]` e.g.

$$\texttt{f[x, 1, z]} = 3xz - 3z^2 + 3xz^2 - 2z^3$$

- Partial derivative $(f_y)'_x$ for any constant argument $y \in \mathbb{N}$ using mathematica method `DerivativeFByX[x, y, z]`

$$\texttt{DerivativeFByX[x, 1, z]} = 3z + 3z^2$$

- Partial derivative $(f_y)'_z$ for any constant argument $y \in \mathbb{N}$ using mathematica method `DerivativeFByZ[x, y, z]`

$$\texttt{DerivativeFByZ[x, 1, z]} = 3x - 6z + 6xz - 6z^2$$

- Theorem 1.4 for any constant argument $y \in \mathbb{N}$

$$\texttt{DerivativeFByX[x, 1, z]} + \texttt{DerivativeFByZ[x, 1, z]} = 3x - 3z + 6xz - 3z^2$$

$$\texttt{DerivativeFByX[u, 1, u]} + \texttt{DerivativeFByZ[u, 1, u]} = 3u^2$$

## References

[1] Petro Kolosov. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

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[3] Alan F. Beardon. Sums of powers of integers. *The American mathematical monthly*, 103(3):201–213, 1996.

[4] Kolosov, Petro. Another approach to get derivative of odd-power (Source files). Published electronically at https://github.com/kolosovpetro/AnotherApproachToGetDerivativeOfOddPower, 2022.

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*Email address*: kolosovp94@gmail.com

*URL*: https://kolosovpetro.github.io/