DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

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Abstract. In this manuscript we revisit an unexpected identity for odd powers, discussed in [1]. For non-negative integers n and m

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ are rational coefficients. These coefficients are evaluated by solving a system of linear equations. However, for arbitrary integer $m \geq 0$ it might be slightly complicated to build and solve such system of linear equations, especially for large value of m. Thus, this manuscript addresses this problem by providing a recurrence formula for coefficients $\mathbf{A}_{m,r}$, which allows computation with ease.

1. Introduction

In this manuscript we revisit an unexpected identity for odd powers, discussed in [1]. For non-negative integers n and m

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

where $\mathbf{A}_{m,r}$ are rational coefficients. These coefficients are evaluated by solving a system of linear equations. For example,

$$n^5 = \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2$$

By expanding the sums $\sum_{k=1}^{n} k^r (n-k)^r$ using Faulhaber's formula [2], we get

$$\mathbf{A}_{2,0} n + \mathbf{A}_{2,1} \left[\frac{1}{6}(n^3 - n) \right] + \mathbf{A}_{2,2} \left[\frac{1}{30}(n^5 - n) \right] - n^5 = 0$$

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By expanding the brackets and rearranging the terms

$$30\mathbf{A}_{2,0}n + 5\mathbf{A}_{2,1}(n^3 - n) + \mathbf{A}_{2,2}(n^5 - n) - 30n^5 = 0$$

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1}n + 5\mathbf{A}_{2,1}n^3 - \mathbf{A}_{2,2}n + \mathbf{A}_{2,2}n^5 - 30n^5 = 0$$

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1}n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore,

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} & = 0 \\ \mathbf{A}_{2,1} & = 0 \\ \mathbf{A}_{2,2} - 30 & = 0 \end{cases}$$

By solving the system above, we evaluate $\mathbf{A}_{2,2} = 30,\ \mathbf{A}_{2,1} = 0,\ \mathbf{A}_{2,0} = 1$. Thus, the identity for n^5 is

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

However, for arbitrary integer $m \geq 0$ it might be slightly complicated to build and solve such system of linear equations, especially for large value of m. Thus, this manuscript addresses this problem by providing a recurrence formula for coefficients $\mathbf{A}_{m,r}$, which allows computation with ease.

2. Recurrence relation

In 2018, a recurrence formula [3] for coefficients $\mathbf{A}_{m,r}$ was proposed by Dr. Max Alekseyev, George Washington University. The main idea of Alekseyev's approach was to utilize certain generating function to evaluate the set of coefficients $\mathbf{A}_{m,r}$, starting from the base case $\mathbf{A}_{m,m}$, then to compute the previous coefficient $\mathbf{A}_{m,m-1}$ recursively, similarly up to $\mathbf{A}_{m,0}$. We use Binomial theorem and specific version of Faulhaber's formula [2] with upper summation bound set to $p+1$

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} = \frac{1}{p+1}\left[\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - \frac{B_{p+1}}{p+1}$$

The reason we use the Faulhaber's formula above is because we tend to omit summation bounds, for simplicity. This helps us to collapse the common terms across complex sums, because now we can let the sum run over all integers j, while only finitely many terms $\binom{p+1}{j}$ are non-zero, see also [4]. Hence,

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\left[\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - \frac{B_{p+1}}{p+1} \tag{2}$$

Now we expand the sum $\sum_{k=1}^{n} k^r(n-k)^r$ using binomial theorem

$$\sum_{k=1}^{n} k^r(n-k)^r = \sum_{k=1}^{n} k^r \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}k^t = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n} k^{t+r}$$

By applying Faulhaber's formula (2) to $\sum_{k=1}^{n} k^{t+r}$, we get

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\left(\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j}\right) - \frac{B_{t+r+1}}{t+r+1}\right] \\ &= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\left[\left(\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j}\right) - B_{t+r+1}n^{r-t}\right]\end{aligned}$$

By expanding brackets

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

By moving the sum in j and omitting summation bounds in t

$$\sum_{k=1}^{n} k^r(n-k)^r = \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]$$

By rearranging the sums we obtain

$$\begin{aligned}\sum_{k=1}^{n} k^r(n-k)^r = &\left[\sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] \\ &- \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]\end{aligned} \tag{3}$$

We can notice that

Lemma 2.1 (Piecewise Binomial identity)**.** *For non-negative integers r, j*

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \dfrac{1}{(2r+1)\binom{2r}{r}} & \text{if } j = 0 \\ \dfrac{(-1)^r}{j} \dbinom{r}{2r-j+1} & \text{if } j > 0 \end{cases}$$

Proof. For $j = 0$ we have

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} = \sum_t \binom{r}{t} (-1)^t \int_0^1 z^{r+t} \, dz$$

Because $\frac{1}{r+t+1} = \int_0^1 z^{r+t} \, dz$.

$$\sum_t \binom{r}{t} (-1)^t \int_0^1 z^{r+t} \, dz = \int_0^1 z^r \left(\sum_t \binom{r}{t} (-1)^t z^t \right) dz = \int_0^1 z^r (1-z)^r \, dz$$

The work [5] provides the identity $\frac{1}{\binom{n}{k}} = (n+1) \int_0^1 z^k (1-z)^{n-k} \, dz$. By setting $n = 2r$ and $k = r$, for $j = 0$ yields

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \int_0^1 z^r (1-z)^r \, dz = \frac{1}{(2r+1)\binom{2r}{r}}$$

For $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \sum_t \frac{(-1)^t}{j} \binom{r}{t} \binom{r+t}{j-1}$$

Because $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$. Now apply the coefficient extraction $[z^k]$ to represent the coefficient of z^k. For example: $[z^k](1+z)^r = \binom{r}{k}$. Therefore,

$$\sum_t \frac{(-1)^t}{j} \binom{r}{t} \binom{r+t}{j-1} = [z^{j-1}] \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^{r+t}$$

By factoring out $(1+z)^r$ from the sum

$$[z^{j-1}] \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^{r+t} = [z^{j-1}](1+z)^r \sum_t \frac{(-1)^t}{j} \binom{r}{t} (1+z)^t$$

Now apply the binomial theorem to the inner sum

$$\sum_t \binom{r}{t} (-1)^t (1+z)^t = (1-(1+z))^r = (-z)^r = (-1)^r z^r$$

Hence, for $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \frac{(-1)^r}{j} [z^{j-1}](1+z)^r z^r$$

By applying the identity $[z^{p-q}]A(z) = [z^p]z^q A(z)$

$$\frac{(-1)^r}{j}[z^{j-1}](1+z)^r z^r = \frac{(-1)^r}{j}[z^{j-1-r}](1+z)^r = \frac{(-1)^r}{j}\binom{r}{j-1-r}$$

Finally, we use symmetry $\binom{n}{k} = \binom{n}{n-k}$ to show that for $j > 0$

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \frac{(-1)^r}{j}\binom{r}{j-1-r} = \frac{(-1)^r}{j}\binom{r}{2r-j+1}$$

This completes the proof. ☐

To simplify equation (3) using binomial identity (2.1), we have to move $j = 0$ out of summation, to avoid division by zero in $\frac{(-1)^r}{j}$. Therefore,

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_{j=1}^{\infty} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j}\right]$$
$$- \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

Hence, we simplify equation (3) by using binomial identity (2.1)

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_{j=1}^{\infty} \frac{(-1)^r}{j}\binom{r}{2r-j+1} B_j n^{2r-j+1}\right]$$
$$- \left[\sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]$$

By setting $\ell = 2r - j + 1$ to $\sum_{j=1}^{\infty}$, and $\ell = r - t$ to $\sum_t$, we collapse common terms across two sums

$$\sum_{k=1}^{n} k^r(n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_\ell \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^\ell\right]$$
$$- \left[\sum_\ell \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^\ell\right]$$
$$= \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + 2\sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell} B_{2r+1-\ell} n^\ell$$

By replacing odd $\ell = 2k+1$, we get

Proposition 2.2 (Bivariate Faulhaber's Formula)**.** *For integers $r \geq 0$ and $n \geq 1$*

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + (-1)^r \sum_{k=0}^{r-1} \binom{r}{2k+1} \frac{B_{2r-2k}}{r-k} n^{2k+1}$$

Assuming that coefficients $\mathbf{A}_{m,r}$ are defined by odd-power identity (1), we obtain the following relation for polynomials in n

$$R_m = \sum_{r=0}^{\infty} \mathbf{A}_{m,r} \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}} + \left[\sum_{r=0}^{\infty} (-1)^r \sum_{k=0}^{\infty} \mathbf{A}_{m,r} \binom{r}{2k+1} \frac{B_{2r-2k}}{r-k} n^{2k+1} \right] - n^{2m+1} \equiv 0 \quad (4)$$

We now fix the unused values of $\mathbf{A}_{m,r}$ so that $\mathbf{A}_{m,r} = 0$ for every $r < 0$ or $r > m$, thus we can let r running over all integers. By extracting the coefficient of n^{2m+1} in (4) implies

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

because $[n^{2m+1}]R_m = \mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0$. To extract the coefficient of n^{2m+1} from the series (4), we isolate the relevant terms by setting $r = m$ in $\sum_{r=0}^{m} \mathbf{A}_{m,r} \frac{n^{2r+1}}{(2r+1)\binom{2r}{r}}$ and $k = m$ in $\sum_{r=0}^{m} (-1)^r \sum_{k=0}^{\infty} \mathbf{A}_{m,r} \binom{r}{2k+1} \frac{B_{2r-2k}}{r-k} n^{2k+1}$, which gives

$$[n^{2m+1}]R_m = \mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} + \left[\sum_{r=2m+1}^{\infty} (-1)^r \mathbf{A}_{m,r} \binom{r}{2m+1} \frac{B_{2r-2m}}{r-m} \right] - 1 = 0$$

We observe that the sum

$$\sum_{r=2m+1}^{\infty} (-1)^r \mathbf{A}_{m,r} \binom{r}{2m+1} \frac{B_{2r-2m}}{r-m}$$

does not contribute to the determination of the coefficients $\mathbf{A}_{m,r}$, because $\mathbf{A}_{m,r}$ is zero for all $r > m$, by definition. Consequently, all terms in the sum are zero. Thus,

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} - 1 = 0 \implies \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Taking the coefficient of n^{2d+1} for an integer d in range $\frac{m}{2} \leq d \leq m-1$ in (4) gives

$$[n^{2d+1}]R_m = \mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + \sum_{r=2d+1}^{\infty} (-1)^r \mathbf{A}_{m,r} \binom{r}{2d+1} \frac{B_{2r-2d}}{r-d} = 0$$

which implies

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} = 0 \implies \mathbf{A}_{m,d} = 0.$$

because the sum $\sum_{r=2d+1}^{\infty} (-1)^r \mathbf{A}_{m,r} \binom{r}{2d+1} \frac{B_{2r-2d}}{r-d}$ is zero for all d in range $\frac{m}{2} \leq d \leq m-1$. For example: let $d = m-1$, then $r = 2d+1 = 2m-1$, thus $\mathbf{A}_{m,2m-1} = 0$, by definition. Let be $d = \frac{m}{2}$, then $r = 2d+1 = m+1$, thus $\mathbf{A}_{m,m+1} = 0$, by definition. Taking the coefficient of n^{2d+1} for d in range $\frac{m}{4} \leq d \leq \frac{m}{2} - 1$ in (4), we obtain

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0.$$

Solving for $\mathbf{A}_{m,d}$ yields

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!\, d!\, m!\, (m-2d-1)!} \cdot \frac{1}{m-d} B_{2m-2d}.$$

Proceeding recursively, we can compute each coefficient $\mathbf{A}_{m,r}$ for integers r in range $\frac{m}{2^{s+1}} \leq r \leq \frac{m}{2^s} - 1$ for $s = 1, 2, \ldots$, by using previously computed values $\mathbf{A}_{m,d}$

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}.$$

Finally, we define the following recurrence relation for coefficients $\mathbf{A}_{m,r}$

Proposition 2.3. *For integers $m \geq 0$ and r*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d=2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

where B_t are Bernoulli numbers [6]. We assume that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [7, 8].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$.
- $\mathbf{A}_{m,r} = 0$ for $r < 0$ and $r > m$.
- $\mathbf{A}_{m,r} = 0$ for $m < 0$.
- $\mathbf{A}_{m,r} = 0$ for $\lfloor \frac{m}{2} \rfloor \leq r < m$.
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$.
- $\mathbf{A}_{m,r}$ are all integers up to row $m = 11$.
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$.

Proposition 2.4 (Odd power identity)**.** *For non-negative integers m and $n \geq 1$, there is a set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \ldots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r$$

For example,

- $1^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[0^r\right]$
- $2^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[1^r + 0^r\right]$
- $3^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[2^r + 2^r + 0^r\right]$
- $4^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[3^r + 4^r + 3^r + 0^r\right]$
- $5^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[4^r + 6^r + 6^r + 4^r + 0^r\right]$

- $6^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[5^r + 8^r + 9^r + 8^r + 5^r + 0^r\right]$

We define $x^0 = 1$ for all x, see [9, p. 162]. This is because when $k = n$ and $r = 0$ the term $k^r(n-k)^r = n^0 \cdot 0^0$, thus we define $x^0 = 1$ for all x. Thus, the formula for sums of odd powers is straightforward

Proposition 2.5 (Sum of odd powers)**.** *For integers $m \geq 0$ and $n \geq 1$*

$$\begin{aligned}\sum_{k=1}^{n} k^{2m+1} &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\left(\sum_{k=0}^{n-1} k^r\right) + \left(\sum_{k=0}^{n-2} (2k)^r\right) + \left(\sum_{k=0}^{n-3} (3k)^r\right) + \cdots + \left(\sum_{k=0}^{n-n} (nk)^r\right) \right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[\left(\sum_{k=0}^{n-1} k^r\right) + \left(2^r \sum_{k=0}^{n-2} k^r\right) + \left(3^r \sum_{k=0}^{n-3} k^r\right) + \cdots + \left(n^r \sum_{k=0}^{n-n} k^r\right) \right]\end{aligned}$$

Proof. Direct consequence of (2.4). □

Corollary 2.6 (Compact forms)**.** *For integers $m \geq 0$ and $n \geq 1$*

$$\sum_{k=1}^{n} k^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{t=1}^{n} \sum_{k=0}^{n-t} (tk)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{t=1}^{n} \sum_{k=1}^{t} ((n-t)k)^r$$

3. Interesting observations

Interestingly enough that the odd power identity above is a Pascal-type identity in terms of bivariate function $k(n-k)$ and numbers $\mathbf{A}_{m,r}$. We may see it by comparing the Pascal's identity itself [10]

$$(n+1)^{m+1} - 1 = \sum_{r=0}^{m} \binom{m+1}{r} (1^r + 2^r + \cdots + n^r)$$

with identity in terms of bivariate function $k(n-k)$ and numbers $\mathbf{A}_{m,r}$

Corollary 3.1 (Bivariate Pascal's identity)**.** *For integers $n \geq 1$ and $m \geq 0$*

$$\begin{aligned}(n+1)^{2m+1} - 1 &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[1^r n^r + 2^r (n-1)^r + 3^r (n-2)^r + 4^r (n-3)^r + \cdots + n^r (n+1-n)^r\right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \left[n^r + (2n-2)^r + (3n-6)^r + (4n-12)^r + \cdots + n^r\right] \\ &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} (k(n+1-k))^r\end{aligned}$$

Definition 3.2 (Bivariate sum T_m)**.** *For integers n, k and $m \geq 0$*

$$T_m(n,k) = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

Proposition 3.3 (Symmetry of T_m)**.** *For integers n, k and $m \geq 0$*

$$T_m(n,k) = T_m(n, n-k)$$

Proposition 3.4 (Forward Recurrence for T_m)**.** *For integers n, k and $m \geq 0$*

$$T_m(n,k) = \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n+t, k)$$

Proof. The polynomial $T_m(n,k)$ is a polynomial of degree m in n. Thus, the forward difference with respect to n is $\Delta^{m+1} T_m(n,k) = \sum_{t=0}^{m+1} (-1)^t \binom{m+1}{t} T_m(n+t,k) = 0$. By isolating $(-1)^0 \binom{m+1}{0} T_m(n-0,k)$ yields $T_m(n,k) = (-1) \sum_{t=1}^{m+1} (-1)^t \binom{m+1}{t} T_m(n+t,k)$. □

Proposition 3.5 (Odd power forward decomposition)**.** *For non-negative integers m and n*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+1} (-1)^{t+1} \binom{m+1}{t} T_m(n+t, k)$$

Proof. Direct consequence of (2.4) and forward recurrence (3.4). □

For example:

- $3^5 = \binom{3}{1}1023 - \binom{3}{2}2643 + \binom{3}{3}5103$
- $3^5 = \binom{4}{1}1023 - \binom{4}{2}2643 + \binom{4}{3}5103 - \binom{4}{4}8403$

Proposition 3.6 (Forward Recurrence for T_m multifold)**.** *For integers $m \geq 0$, n, k and $s \geq 1$*

$$T_m(n,k) = \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n+t, k)$$

Proposition 3.7 (Odd power forward decomposition multifold)**.** *For non-negative integers m, n and $s \geq 1$*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{t=1}^{m+s} (-1)^{t+1} \binom{m+s}{t} T_m(n+t, k)$$

Proof. Direct consequence of (2.4) and forward recurrence multifold (3.6). □

For example:

- $2^3 = \binom{5}{1}26 - \binom{5}{2}44 + \binom{5}{3}62 - \binom{5}{4}80 + \binom{5}{5}98$
- $5^3 = \binom{3}{1}215 - \binom{3}{2}305 + \binom{3}{3}395$

Proposition 3.8 (Negated binomial form)**.** *For integers n, a and $m \geq 0$ such that $n-2a \geq 0$*

$$(n-2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=a+1}^{n-a} \mathbf{A}_{m,r}(k-a)^r(n-a-k)^r$$

Proof. By observing the summation limits we can see that k runs as $k = a+1, a+2, a+3, \dots, a+n-a$, which implies that $(k-a) = 1, 2, 3, \dots, n$. By observing the term $(n-k-a)$ we see that $(n-k-a) = n-1, n-2, n-3, \dots, 0$. Thus, by reindexing the sum $(n-2a)^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n-2a} \mathbf{A}_{m,r}(a+k-a)^r(n-(a+k)-a)^r$ the statement (3.8) is equivalent to (2.4) with setting $n \to n-2a$. □

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