SUMS OF POWERS VIA CENTRAL FINITE DIFFERENCES AND NEWTON'S FORMULA

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ABSTRACT. In this manuscript, we derive closed-form expressions for multifold sums of powers using Newton's interpolation formula in central differences, evaluated at an arbitrary integer point t. We further show that Knuth's formula for multifold sums of odd powers arises naturally from Newton's interpolation formula in central differences evaluated at zero. Additionally, we provide Wolfram Mathematica programs to validate the main results.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this manuscript, we derive formulas for multifold sums of powers using Newton's formula and central finite differences. The idea of deriving sums of powers using difference operators

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and Newton series is classical and quite general. Formulas for sums of powers using forward and backward differences can be found in the works [1, 2]. We define the recurrence for multifold sums of powers introduced by Donald Knuth [3], which is used throughout the paper.

Proposition 1.1 (Multifold sums of powers recurrence)**.** *For non-negative integers r, n, m*

$$\Sigma^0 n^m = n^m$$

$$\Sigma^1 n^m = \Sigma^0 1^m + \Sigma^0 2^m + \cdots + \Sigma^0 n^m$$

$$\Sigma^{r+1} n^m = \Sigma^r 1^m + \Sigma^r 2^m + \cdots + \Sigma^r n^m$$

Proposition 1.2 (Central factorials)**.** *For integers n, k*

$$n^{[k]} = \begin{cases} 0, & \text{if } k < 0, \\ 1, & \text{if } k = 0, \\ n\left(n + \frac{k}{2} - 1\right)\left(n + \frac{k}{2} - 2\right)\cdots\left(n - \frac{k}{2} + 1\right) = n\prod_{j=1}^{k-1}\left(n + \frac{k}{2} - j\right), & \text{if } k > 0. \end{cases}$$

Consider Newton's interpolation formula [4, 5, 6] in central differences evaluated at zero.

Proposition 1.3 (Newton's formula in central differences at zero)**.**

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k f(0),$$

where $\delta^k f(0) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(\frac{k}{2} - j\right)$ are central finite differences at zero, and $x^{[k]}$ are central factorials, with $x^{[0]} = 1$ for every x.

We observe that central factorials are closely related to falling factorials $(x)_n = x(x-1)(x-2)(x-3)\cdots(x-n+1) = \prod_{k=0}^{n-1}(x-k)$. Therefore,

Proposition 1.4 (Central factorials in terms of falling)**.** *For integers n, k*

$$n^{[k]} = \begin{cases} 0, & \text{if } k < 0 \\ 1, & \text{if } k = 0 \\ n\left(n+\frac{k}{2}-1\right)_{k-1}, & \text{if } k > 0 \end{cases}$$

where $\left(n+\frac{k}{2}-1\right)_{k-1}$ are falling factorials.

To derive a formula for multifold sums of powers, we follow the strategy to express the Newton's formula (1.3) in terms of binomial coefficients, then to reach closed forms of column sum of binomial coefficients by means of hockey stick identity. Therefore,

Proposition 1.5 (Binomial form of central factorials)**.** *For integers n and $k \geq 1$*

$$\frac{n^{[k]}}{k!} = \frac{n}{k}\binom{n+\frac{k}{2}-1}{k-1}$$

Proof. We have

$$\frac{n^{[k]}}{k!} = \frac{n}{k!}\left(n+\frac{k}{2}-1\right)_{k-1} = \frac{n}{k(k-1)!}\left(n+\frac{k}{2}-1\right)_{k-1} = \frac{n}{k}\binom{n+\frac{k}{2}-1}{k-1}$$

because of the identity in falling factorial $\frac{(x)_n}{n!} = \binom{x}{n}$, and Proposition (1.4). □

This yields Newton's formula for powers, in terms of central differences.

Proposition 1.6 (Newton's formula for powers at zero)**.** *For positive integers $n \geq 1$ and $m \geq 1$*

$$n^m = \sum_{k=1}^{m} \frac{n}{k}\binom{n+\frac{k}{2}-1}{k-1}\delta^k 0^m$$

Although it is based on Newton's interpolation formula (1.3), Proposition (1.6) starts the summation at $k = 1$, which avoids division by zero in $\frac{n}{k}$. This is a valid trick, because the central difference $\delta^k 0^n$ is zero for all $n \geq 1$ and $k = 0$. By factoring out and simplifying the term n, we get

$$n^{m-1} = \sum_{k=1}^{m} \frac{1}{k}\binom{n+\frac{k}{2}-1}{k-1}\delta^k 0^m$$

We observe that the central finite difference operator $\delta^k 0^m$ depends on the parity of m and k. In particular,

$$\delta^k 0^m \neq 0 \quad \text{when} \quad m \equiv k \pmod 2,$$

$$\delta^k 0^m = 0 \quad \text{when} \quad m \not\equiv k \pmod 2.$$

Thus, for odd powers, only even-order central differences contribute. By setting $m \to 2m$, we get,

$$n^{2m-1} = \sum_{k=1}^{2m} \frac{1}{k} \binom{n+\frac{k}{2}-1}{k-1} \delta^k 0^{2m}.$$

Since k runs over all integers in the range $0 \leq k \leq 2m$, we can omit odd values of k.

$$n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n+k-1}{2k-1} \delta^{2k} 0^{2m}$$

Hence, we obtain the formula for ordinary sums of odd powers.

Proposition 1.7 (Ordinary sums of odd powers in central differences)**.** *For integers $n \geq 1,\ m \geq 1$*

$$\Sigma^1 n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n+k}{2k} \delta^{2k} 0^{2m}$$

Proof. We have $\Sigma^1 n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \delta^{2k} 0^{2m} \sum_{j=1}^{n} \binom{j+k-1}{k-1}$.

By hockey stick identity $\sum_{j=1}^{n} \binom{j+k-1}{2k-1} = \binom{n+k}{2k}$, thus the statement follows. □

Therefore,

Theorem 1.8 (Multifold sums of odd powers in central differences)**.** *For integers $n, m, r \geq 0$,*

$$\Sigma^r n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \binom{n+k-1+r}{2k-1+r} \delta^{2k} 0^{2m}.$$

Proof. We have $\Sigma^1 n^{2m-1} = \sum_{k=1}^{m} \frac{1}{2k} \delta^{2k} 0^{2m} \sum_{j=1}^{n} \binom{j+k-1}{2k-1}$.

By hockey stick identity $\sum_{j=1}^{n} \binom{j+k-1}{k-1} = \binom{n+k}{2k}$. By induction the claim follows. □

It is quite interesting to notice that the formula for sums of odd-powers n^{2m-1} given by Donald Knuth in *Johann Faulhaber and sums of powers* [3] recovers naturally from the theorem (1.8). The reason is straightforward, instead of using central factorial numbers of the second kind $T(n,k)$, the theorem (1.8) utilizes central differences explicitly, because,

Lemma 1.9 (Central factorial numbers of the second kind)**.** *For integers $n \geq 0,\ k \geq 0$,*

$$k! T(n,k) = \delta^k 0^n,$$

where $T(n,k)$ are central factorial numbers, defined by polynomial identity,

$$x^m = \sum_{k=1}^{m} T(m,k) x^{[k]}.$$

See [7, p. 213]*, and* [8]*.*

It means that Knuth's formula for sums of odd powers,

Proposition 1.10 (Multifold sums of odd powers in central factorial numbers)**.** *For integers $n \geq 1,\ m \geq 1$ and $r \geq 0$*

$$\Sigma^r\, n^{2m-1} = \sum_{k=1}^{m} (2k-1)! \binom{n+k-1+r}{2k-1+r} T(2m,2k).$$

originates from Newton's interpolation formula in central differences (1.3). The non-zero central factorial numbers $T(2m,2k)$ is the sequence A008957 in the OEIS [9]. For example,

$$\Sigma^1\, n^1 = \tbinom{n+1}{2},$$

$$\Sigma^1\, n^3 = 6\tbinom{n+2}{4} + \tbinom{n+1}{2},$$

$$\Sigma^1\, n^5 = 120\tbinom{n+3}{6} + 30\tbinom{n+2}{4} + \tbinom{n+1}{2},$$

$$\Sigma^1\, n^7 = 5040\tbinom{n+4}{8} + 1680\tbinom{n+3}{6} + 126\tbinom{n+2}{4} + \tbinom{n+1}{2}.$$

While multifold sums of odd powers are,

$$\Sigma^r n^1 = \binom{n+1+r}{2+r},$$

$$\Sigma^r n^3 = 6\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r},$$

$$\Sigma^r n^5 = 120\binom{n+3+r}{6+r} + 30\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r},$$

$$\Sigma^r n^7 = 5040\binom{n+4+r}{8+r} + 1680\binom{n+3+r}{6+r} + 126\binom{n+2+r}{4+r} + \binom{n+1+r}{2+r}.$$

The coefficients $1, 6, 1, 120, 30, 1, \ldots$ is the sequence A303675 in the OEIS [9]. This approach can be generalized even further. Consider Newton's interpolation formula around an arbitrary integer t.

Proposition 1.11 (Newton's interpolation formula in central differences)**.**

$$f(x+t) = \sum_{k=0}^{\infty} \frac{x^{[k]}}{k!} \delta^k f(t)$$

Proof. See [5, p. 462]. □

Thus, for powers we have identity,

Proposition 1.12 (Newton's formula for powers)**.** *For integers $n, m \geq 0$, and an arbitrary integer t,*

$$n^m = \sum_{k=0}^{m} \frac{(n-t)^{[k]}}{k!} \delta^k t^m$$

Thus,

Proposition 1.13 (Powers in central binomial form)**.** *For integers n, t and $m \geq 0$*

$$\begin{aligned} n^m &= \frac{(n-t)^{[0]}}{0!} \delta^0 t^m + \sum_{k=1}^{m} \frac{n-t}{k} \binom{n+t+\frac{k}{2}-1}{k-1} \delta^k t^m \\ &= t^m + \sum_{k=1}^{m} (n-t) \binom{n-t+\frac{k}{2}-1}{k-1} \frac{\delta^k t^m}{k} \end{aligned}$$

Now we expand the brackets in central binomial form above,

$$n^m = t^m + \sum_{k=1}^{m} \frac{\delta^k t^m}{k} \left[n\binom{n-t+\frac{k}{2}-1}{k-1} - t\binom{n-t+\frac{k}{2}-1}{k-1} \right].$$

Hence, we get the formula for ordinary sums of powers.

Corollary 1.14 (Centered ordinary sums of powers)**.** *For integers* $t,\ m \geq 0,\ n \geq 0,$

$$\Sigma^1\, n^m = \sum_{j=1}^{n} t^m + \sum_{k=1}^{m} \frac{\delta^k t^m}{k} \left[\sum_{j=1}^{n} j\binom{j-t+\frac{k}{2}-1}{k-1} - t\sum_{j=1}^{n} \binom{j-t+\frac{k}{2}-1}{k-1} \right]$$

Now we notice that,

Proposition 1.15 (Binomial decomposition)**.** *For integers* $n \geq 0,\ r \geq 0,\ m \geq 0,$

$$n\binom{n+r}{m} = (m+1)\binom{n+r}{m+1} - (r-m)\binom{n+r}{m}.$$

Proof. By expanding the brackets yields,

$$\begin{aligned} n\tbinom{n+r}{m} &= (m+1)\tbinom{n+r}{m+1} - (r-m)\tbinom{n+r}{m} \\ &= m\tbinom{n+r}{m+1} + \tbinom{n+r}{m+1} - r\tbinom{n+r}{m} + m\tbinom{n+r}{m}. \end{aligned}$$

Recall the extraction property of binomial coefficients, that is,

$$\tbinom{n}{k+1} = \tfrac{n-k}{k+1}\tbinom{n}{k}.$$

Now we can notice that,

$$\tbinom{n+r}{m+1} = \tfrac{n+r-m}{m+1}\tbinom{n+r}{m},$$

by extraction. Thus,

$$n\tbinom{n+r}{m} = m\tfrac{n+r-m}{m+1}\tbinom{n+r}{m} + \tfrac{n+r-m}{m+1}\tbinom{n+r}{m} - r\tbinom{n+r}{m} + m\tbinom{n+r}{m}.$$

By moving binomial coefficient $\binom{n+r}{m}$ out of the brackets, we get,

$$\begin{aligned} n\tbinom{n+r}{m} &= \tbinom{n+r}{m}\left[m\tfrac{n+r-m}{m+1} + \tfrac{n+r-m}{m+1} - r + m\right] \\ &= \tbinom{n+r}{m}\left[(m+1)\tfrac{n+r-m}{m+1} - r + m\right] \\ &= n\tbinom{n+r}{m}. \end{aligned}$$

This completes the proof. $\square$

Thus, by setting $n = j$, and $r = -t + \frac{k}{2} - 1$, and $m = k-1$ into Proposition (1.15) yields central decomposition identity.

Corollary 1.16 (Central binomial decomposition)**.** *For integers $j, t, k \geq 0$,*

$$j\binom{j-t+\frac{k}{2}-1}{k-1} = k\binom{j-t+\frac{k}{2}-1}{k} + \left[t + \frac{k}{2}\right]\binom{j-t+\frac{k}{2}-1}{k-1}.$$

Proof. By binomial decomposition (1.15), we have,

$$\begin{aligned} j\tbinom{j-t+\frac{k}{2}-1}{k-1} &= (k-1+1)\tbinom{j-t+\frac{k}{2}-1}{k-1+1} - \left[-t + \tfrac{k}{2} - 1 - (k-1)\right]\tbinom{j-t+\frac{k}{2}-1}{k-1} \\ &= k\tbinom{j-t+\frac{k}{2}-1}{k} - \left[-t - \tfrac{k}{2}\right]\tbinom{j-t+\frac{k}{2}-1}{k-1} \\ &= k\tbinom{j-t+\frac{k}{2}-1}{k} + \left[t + \tfrac{k}{2}\right]\tbinom{j-t+\frac{k}{2}-1}{k-1}. \end{aligned}$$

This completes the proof. $\square$

Thus, we have the relation for centered sums of powers,

$$\begin{aligned} \Sigma^1 n^m = \sum_{j=1}^{n} t^m + \sum_{k=1}^{m} \frac{\delta^k t^m}{k}\Bigg[\sum_{j=1}^{n}\left\{k\binom{j-t+\frac{k}{2}-1}{k} + \left[t+\frac{k}{2}\right]\binom{j-t+\frac{k}{2}-1}{k-1}\right\} \\ - t\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k-1}\Bigg]. \end{aligned}$$

By rearranging it, we get,

$$\begin{aligned}\Sigma^1 n^m &= \sum_{j=1}^{n} t^m + \sum_{k=1}^{m} \frac{\delta^k t^m}{k}\left[\left\{k\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k} + \left[t+\frac{k}{2}\right]\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k-1}\right\}\right.\\ &\quad\left. - t\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k-1}\right]\\ &= \sum_{j=1}^{n} t^m + \sum_{k=1}^{m} \frac{\delta^k t^m}{k}\left[k\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k} + \frac{k}{2}\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k-1}\right].\end{aligned}$$

Therefore, formula for centered sums of powers follows.

Proposition 1.17 (Centered decomposition of power sums)**.** *For integers $m, n \geq 0$, and an arbitrary integer t,*

$$\Sigma^1 n^m = \sum_{j=1}^{n} t^m + \sum_{k=1}^{m} \delta^k t^m \left[\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k} + \frac{1}{2}\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k-1}\right].$$

Let us recall generalized hockey stick identity. That is,

Proposition 1.18 (Generalized hockey-stick identity)**.** *For integers a, b and j,*

$$\sum_{k=a}^{b}\binom{k}{j} = \binom{b+1}{j+1} - \binom{a}{j+1}.$$

Proof. We have, $\sum_{k=a}^{b}\binom{k}{j} = \binom{a}{j}+\binom{a+1}{j}+\cdots+\binom{b}{j}$, which means that, $\sum_{k=a}^{b}\binom{k}{j} = \left(\sum_{k=0}^{b}\binom{k}{j}\right) - \left(\sum_{k=0}^{a-1}\binom{k}{j}\right)$. By hockey stick identity $\sum_{k=0}^{n}\binom{k}{j} = \binom{n+1}{j+1}$ yields,

$$\sum_{k=a}^{b}\binom{k}{j} = \left(\sum_{k=0}^{b}\binom{k}{j}\right) - \left(\sum_{k=0}^{a-1}\binom{k}{j}\right) = \binom{b+1}{j+1} - \binom{a}{j+1}.$$

This completes the proof. □

Therefore, by setting $a = -t + \frac{k}{2}$ and $b = n - t - \frac{k}{2} - 1$ yields

Proposition 1.19 (Centered hockey stick identity)**.** *For integers n, j, t, k,*

$$\sum_{j=1}^{n}\binom{j-t+\frac{k}{2}-1}{k} = \sum_{a=-t+\frac{k}{2}}^{n-t-\frac{k}{2}-1}\binom{a}{k} = \binom{n-t+\frac{k}{2}}{k+1} - \binom{-t+\frac{k}{2}}{k+1}$$

Thus, closed form of centered sums of powers yields

Theorem 1.20 (Closed form of centered sums of powers)**.** *For integers $n \geq 0,\ m \geq 0,$ and arbitrary integer t,*

$$\begin{aligned}\Sigma^1 n^m &= \sum_{j=1}^{n} t^m \\ &+ \sum_{k=1}^{m} \delta^k t^m \left[\left(\binom{n-t+\frac{k}{2}}{k+1} - \binom{-t+\frac{k}{2}}{k+1} \right) + \frac{1}{2} \left(\binom{n-t+\frac{k}{2}}{k} - \binom{-t+\frac{k}{2}}{k} \right) \right].\end{aligned}$$

Let $a = n - t + \frac{k}{2}$, then,

$$\begin{aligned}&\left(\binom{a}{k+1} - \binom{a-n}{k+1} \right) + \frac{1}{2} \left(\binom{a}{k} - \binom{a-n}{k} \right) \\ &= \left(\binom{a}{k+1} - \binom{a-n}{k+1} + \frac{1}{2}\binom{a}{k} - \frac{1}{2}\binom{a-n}{k} \right) \\ &= \frac{1}{2} \left(2\binom{a}{k+1} - 2\binom{a-n}{k+1} + \binom{a}{k} - \binom{a-n}{k} \right) \\ &= \frac{1}{2} \left(\binom{a}{k+1} + \binom{a}{k+1} - \binom{a-n}{k+1} - \binom{a-n}{k+1} + \binom{a}{k} - \binom{a-n}{k} \right).\end{aligned}$$

By binomial recurrence $\binom{a+1}{k+1} = \binom{a}{k} + \binom{a}{k+1}$

$$\begin{aligned}&\frac{1}{2} \left(\binom{a}{k+1} + \binom{a}{k+1} - \binom{a-n}{k+1} - \binom{a-n}{k+1} + \binom{a}{k} - \binom{a-n}{k} \right) \\ &= \frac{1}{2} \left(\left[\binom{a}{k+1} + \binom{a}{k} \right] + \binom{a}{k+1} - \binom{a-n}{k+1} - \left[\binom{a-n}{k+1} - \binom{a-n}{k} \right] \right) \\ &= \frac{1}{2} \left(\binom{a+1}{k+1} + \binom{a}{k+1} - \binom{a-n}{k+1} - \binom{a-n+1}{k+1} \right) \\ &= \frac{1}{2} \left(\left[\binom{a+1}{k+1} + \binom{a}{k+1} \right] - \left[\binom{a-n}{k+1} + \binom{a-n+1}{k+1} \right] \right)\end{aligned}$$

Therefore, by setting $a = n - t + \frac{k}{2}$, we get simplified formula for centered sum of powers.

Proposition 1.21 (Simplified centered sums of powers)**.** *For integers $n \geq 0,\ m \geq 0,$ and an arbitrary integer t*

$$\begin{aligned}\Sigma^1 n^m &= \sum_{j=1}^{n} t^m \\ &+ \sum_{k=1}^{m} \frac{\delta^k t^m}{2} \left[\left(\left(\binom{n-t+\frac{k}{2}+1}{k+1} + \binom{n-t+\frac{k}{2}}{k+1} \right) \right) - \left(\left(\binom{-t+\frac{k}{2}}{k+1} + \binom{-t+\frac{k}{2}+1}{k+1} \right) \right) \right]\end{aligned}$$

Continuing similarly, we can derive formulas for multifold sums of powers by using centered hockey stick identity (1.19) repeatedly. For instance, for double sums of powers, we have

$$\begin{aligned}\Sigma^2 n^m &= t^m \Sigma^2 n^0 \\ &\quad + \sum_{k=1}^{m} \frac{\delta^k t^m}{2} \left[\sum_{j=1}^{n} \left(\binom{j-t+\frac{k}{2}+1}{k+1} + \binom{j-t+\frac{k}{2}}{k+1} \right) \right. \\ &\quad \left. - \left(\binom{-t+\frac{k}{2}}{k+1} \Sigma^1 n^0 + \binom{-t+\frac{k}{2}+1}{k+1} \Sigma^1 n^0 \right) \right]\end{aligned}$$

Thus, by generalized hockey stick identity (1.18)

$$\begin{aligned}\sum_{j=1}^{n} \binom{j-t+\frac{k}{2}+1}{k+1} &= \binom{n-t+\frac{k}{2}+2}{k+2} - \binom{-t+\frac{k}{2}+2}{k+2} \\ \sum_{j=1}^{n} \binom{j-t+\frac{k}{2}}{k+1} &= \binom{n-t+\frac{k}{2}+1}{k+2} - \binom{-t+\frac{k}{2}+1}{k+2}\end{aligned}$$

By substituting closed forms above, we get

$$\begin{aligned}\Sigma^2 n^m &= t^m \Sigma^2 n^0 \\ &\quad + \sum_{k=1}^{m} \frac{\delta^k t^m}{2} \left\{ \left[\binom{n-t+\frac{k}{2}+2}{k+2} - \binom{-t+\frac{k}{2}+2}{k+2} \right] \right. \\ &\quad + \left[\binom{n-t+\frac{k}{2}+1}{k+2} - \binom{-t+\frac{k}{2}+1}{k+2} \right] \\ &\quad \left. - \left[\binom{-t+\frac{k}{2}}{k+1} \Sigma^1 n^0 + \binom{-t+\frac{k}{2}+1}{k+1} \Sigma^1 n^0 \right] \right\}\end{aligned}$$

By combining the common terms yields

$$\begin{aligned}\Sigma^2 n^m = t^m \Sigma^2 n^0 + \sum_{k=1}^{m} \frac{\delta^k t^m}{2} &\left\{ \left[\binom{n-t+\frac{k}{2}+2}{k+2} + \binom{n-t+\frac{k}{2}+1}{k+2} \right] \right. \\ &- \left[\binom{-t+\frac{k}{2}+2}{k+2} \Sigma^0 n^0 + \binom{-t+\frac{k}{2}+1}{k+2} \Sigma^0 n^0 \right] \\ &\left. - \left[\binom{-t+\frac{k}{2}+1}{k+1} \Sigma^1 n^0 + \binom{-t+\frac{k}{2}+0}{k+1} \Sigma^1 n^0 \right] \right\}\end{aligned}$$

Thus, formula for double centered sums of powers follows

Proposition 1.22 (Double centered sums of powers)**.** *For integers $n \geq 0$, $m \geq 0$, and an arbitrary integer t*

$$\Sigma^2 n^m = t^m \Sigma^2 n^0 + \sum_{k=1}^{m} \frac{\delta^k t^m}{2} \Bigg\{ \left[\binom{n-t+\frac{k}{2}+2}{k+2} + \binom{n-t+\frac{k}{2}+1}{k+2} \right] - \left[\binom{-t+\frac{k}{2}+2}{k+2} + \binom{-t+\frac{k}{2}+1}{k+2} \right] \Sigma^0 n^0 - \left[\binom{-t+\frac{k}{2}+1}{k+1} + \binom{-t+\frac{k}{2}+0}{k+1} \right] \Sigma^1 n^0 \Bigg\}.$$

Therefore, by continuing similarly, we can derive formula for $r-$fold sums of powers by using centered hockey stick identity (1.19) repeatedly. We have

Theorem 1.23 (Multifold centered sums of powers)**.** *For integers $n \geq 0$, $m \geq 0$, and an arbitrary integer t*

$$\Sigma^r n^m = t^m \Sigma^r n^0 + \sum_{k=1}^{m} \frac{\delta^k t^m}{2} \Bigg\{ \left[\binom{n-t+\frac{k}{2}+r}{k+r} + \binom{n-t+\frac{k}{2}+r-1}{k+r} \right] - \sum_{s=0}^{r-1} \left[\binom{-t+\frac{k}{2}+r-s}{k+r-s} + \binom{-t+\frac{k}{2}+r-s-1}{k+r-s} \right] \Sigma^s n^0 \Bigg\}.$$

Now we notice that

Proposition 1.24 (Multifold sum of zero powers)**.** *For integers $r \geq 0$ and $n \geq 1$*

$$\Sigma^r n^0 = \binom{r+n-1}{r}$$

Proof. (1) Let $r = 0$, then $\Sigma^0 n^0 = n^0 = \binom{n-1}{0} = 1$, by definition (1.1).

(2) Let $r = 1$, then $\Sigma^1 n^0 = \sum_{k=1}^{n} \binom{k-1}{0} = \sum_{k=1}^{n} 1 = \binom{n}{1}$.

(3) Let $r = 2$, then $\Sigma^2 n^0 = \sum_{k=1}^{n} \binom{k}{1} = \sum_{k=1}^{n} k = \binom{n+1}{2}$.

(4) Let $r = 3$, then $\Sigma^3 n^0 = \sum_{k=1}^{n} \binom{k+1}{2} = \binom{n+2}{3}$.

(5) By induction over r and hockey stick identity $\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, the claim follows $\Sigma^r n^0 = \binom{r+n-1}{r}$.

□

Hence, by (1.23) and (1.24), binomial form of multifold sums of powers follows

Proposition 1.25 (Binomial form of multifold centered sums of powers)**.** *For integers $n \geq 0$, $m \geq 0$, and an arbitrary integer t*

$$\Sigma^r n^m = \binom{r+n-1}{r} t^m + \sum_{k=1}^{m} \frac{\delta^k t^m}{2} \Bigg\{ \left[\binom{n-t+\frac{k}{2}+r}{k+r} + \binom{n-t+\frac{k}{2}+r-1}{k+r} \right] - \sum_{s=0}^{r-1} \left[\binom{-t+\frac{k}{2}+r-s}{k+r-s} + \binom{-t+\frac{k}{2}+r-s-1}{k+r-s} \right] \binom{s+n-1}{s} \Bigg\}.$$

We may observe another remarkable result, by setting $t \to -t$ into formula above

Proposition 1.26 (Negated binomial centered sums of powers)**.** *For integers $n \geq 0$, $m \geq 0$, and an arbitrary integer t*

$$\Sigma^r n^m = (-1)^m \binom{r+n-1}{r} t^m + (-1)^m \sum_{k=1}^{m} \frac{(-1)^k \delta^k t^m}{2} \Bigg\{ \left[\binom{n+t+\frac{k}{2}+r}{k+r} + \binom{n+t+\frac{k}{2}+r-1}{k+r} \right] - \sum_{s=0}^{r-1} \left[\binom{t+\frac{k}{2}+r-s}{k+r-s} + \binom{t+\frac{k}{2}+r-s-1}{k+r-s} \right] \binom{s+n-1}{s} \Bigg\}.$$

Proof. We have $\delta^k(-t)^m = (-1)^{m+k}\delta^k t^m$, and $(-t)^m = (-1)^m t^m$. Hence claim follows from (1.25). ☐

Conclusions

In this manuscript, we derive formulas for multifold sums of powers (1.23), (1.25), (1.26), and others, using Newton's formula in central differences, evaluated at an arbitrary integer t. We utilize hockey-stick identities for binomial coefficients, namely (1.18) and (1.19), to compute closed forms of column sums of binomial coefficients. These closed forms are then used in the derivation of formulas for multifold sums of powers. Additionally, we show that Knuth's formula for multifold sums of odd powers n^{2m-1} [3] arises naturally from Newton's formula in central differences, evaluated at $t = 0$. All main results of this manuscript are validated using programs written in Wolfram Mathematica; see Section (2).

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2. Mathematica programs

Use the *Mathematica* package [10] to validate the results

Mathematica Function	Validates / Prints
`MultifoldSumOfPowersRecurrence[r, n, m]`	Computes $\Sigma^r n^m$
`ValidateCentralFactorialsInTermsOfFalling[10]`	Validates Prop. (1.4)
`ValidateBinomialFormOfCentralFactorials[10]`	Validates Prop. (1.5)
`ValidateNewtonsFormulaForPowersInZero[20]`	Validates Prop. (1.6)
`ValidateOrdinarySumsOfOddPowersInCentralDifferences[20]`	Validates Prop. (1.7)
`ValidateMultifoldSumsOfOddPowersInCentralDifferences[5]`	Validates Thm. (1.8)
`ValidateNewtonsFormulaForPowers[10]`	Validates Prop. (1.12)
`ValidatePowersInCentralBinomialForm[10]`	Validates Prop. (1.13)
`ValidateCenteredOrdinarySumsOfPowers[10]`	Validates Cor. (1.14)
`ValidateBinomialDecomposition[5]`	Validates Prop. (1.15)
`ValidateCentralBinomialDecomposition[5]`	Validates Cor. (1.16)
`ValidateCenteredDecompositionOfPowerSums[10]`	Validates Prop. (1.17)
`ValidateCenteredHockeyStickIdentity[10]`	Validates Prop. (1.19)
`ValidateCenteredHockeyStickIdentity[10]`	Validates Prop. (1.19)
`ValidateClosedFormOfCenteredSumsOfPowers[10]`	Validates Thm. (1.20)
`ValidateSimplifiedCenteredSumsOfPowers[10]`	Validates Prop. (1.21)
`ValidateDoubleCenteredSumsOfPowers[10]`	Validates Prop. (1.22)

Mathematica Function	**Validates / Prints**
`ValidateMultifoldCenteredSumsOfPowers[5]`	Validates Theorem (1.23)
`ValidateMultifoldSumOfZeroPowers[10]`	Validates Proposition (1.24)
`ValidateBinomialMultifoldCenteredSumsOfPowers[5]`	Validates Proposition (1.25)
`ValidateNegatedBinomialCenteredSumOfPowers[5]`	Validates Proposition (1.26)

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