A NOVEL PROOF OF POWER RULE IN CALCULUS

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Abstract. In Calculus, the power rule is the fundamental result stating that the derivative of power function x^n is nx^{n-1}. The most common way to prove it is by using limit form of derivative combined with binomial theorem, such that binomial theorem expresses the function growth. In this manuscript, we discuss a new approach to express function growth in derivatives to prove the power rule.

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1. Introduction and main results

In Calculus, the power rule is the fundamental result stating that the derivative of power function x^n is nx^{n-1}

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}$$

where n is constant. The most common way to prove it is by using limit form of derivative combined with binomial theorem, such that binomial theorem expresses the function growth.

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Sources: https://github.com/kolosovpetro/ANovelProofOfPowerRuleInCalculus

The limit form of derivative is

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

where $f(x)$ is a function defined over the real set $\mathbb{R}$.

Let be $f(x) = x^n$ then its derivative in limit form is

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{(x+h)^n - x^n}{h}\right]$$

The function growth $(x+h)^n$ can be expressed in terms of binomial theorem

$$(x+h)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k$$

So that

$$\begin{aligned}\frac{\mathrm{d}x^n}{\mathrm{d}x} &= \lim_{h\to 0}\left[\frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k\right] = \lim_{h\to 0}\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n}x^0h^n\right]\\ &= \binom{n}{1}x^{n-1}\end{aligned}$$

Therefore, it is clear that binomial theorem plays central role to express function growth.

However, there is a new approach to show the growth rate of power function. More precisely, by using the identity for odd powers [1, 2, 3, 4]

$$(x-2a)^{2m+1} = \sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=a+1}^{x-a}(k-a)^r(x-k-a)^r \tag{1.1}$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} = \begin{cases}(2r+1)\binom{2r}{r} & \text{if } r = m\\ (2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0 \leq r < m\\ 0 & \text{if } r < 0 \text{ or } r > m\end{cases} \tag{1.2}$$

where B_t are Bernoulli numbers [5] such that $B_1 = \frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [6, 7].

Properties of coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

Therefore, by setting $a = -\frac{h}{2}$ in (1.1) we can express the growth rate of odd powers as

$$(x+h)^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=-\frac{h}{2}+1}^{x+\frac{h}{2}} \left(k + \frac{h}{2}\right)^r \left(x - k + \frac{h}{2}\right)^r$$

By rearranging summation bounds

$$(x+h)^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{x+h} k^r (x-k+h)^r$$

Thus, the derivative of odd power yields

$$\frac{\mathrm{d}x^{2m+1}}{\mathrm{d}x} = \lim_{h \to 0} \frac{1}{h} \left[\left(\sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=-\frac{h}{2}+1}^{x+\frac{h}{2}} \left(k + \frac{h}{2}\right)^r \left(x - k + \frac{h}{2}\right)^r \right) - x^{2m+1} \right]$$

For example,

$$
\begin{aligned}
\frac{\mathrm{d}x^3}{\mathrm{d}x} &= \lim_{h\to 0} \frac{h+x+(-1+h+x)(h+x)(1+h+x)-x^3}{h} \\
&= \lim_{h\to 0} \frac{h^3+3h^2x+3hx^2}{h} \\
\frac{\mathrm{d}x^5}{\mathrm{d}x} &= \lim_{h\to 0} \frac{(h+x)(h+x+1)\left(h^3+3h^2x-h^2+3hx^2-2hx+h+x^3-x^2+x-1\right)+h+x-x^5}{h} \\
&= \lim_{h\to 0} \frac{h^5+5h^4x+10h^3x^2+10h^2x^3+5hx^4}{h}
\end{aligned}
$$

Which further simplifies to binomial theorem.

2. Conclusions

In this manuscript, we discuss a new approach to express function growth in derivatives to prove the power rule. As further research, the approach discussed in this manuscript can be generalized for backward and central derivatives additionally. Results can be validated using Mathematica programs at GitHub repository.

References

[1] Kolosov, Petro. History and overview of the polynomial P(m,b,x), 2024. https://kolosovpetro.github.io/pdf/HistoryAndOverviewOfPolynomialP.pdf.

[2] Kolosov, Petro. On the link between binomial theorem and discrete convolution. *arXiv preprint arXiv:1603.02468*, 2016. https://arxiv.org/abs/1603.02468.

[3] Kolosov, Petro. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

[4] Petro Kolosov. Polynomial identity involving Binomial Theorem and Faulhaber's formula, 2023. https://kolosovpetro.github.io/pdf/PolynomialIdentityInvolvingBTandFaulhaber.pdf.

[5] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

[6] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A302971, 2018.

[7] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A304042, 2018.

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