

IDENTITIES IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In this manuscript, we show new binomial identities in iterated rascal triangles, revealing a connection between the Vandermonde convolution and iterated rascal numbers. We also present Vandermonde-like binomial identities. Furthermore, we establish a relation between iterated rascal triangle and $(1, q)$ -binomial coefficients.

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Sources: <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>

1. INTRODUCTION

In 2010, three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1], were challenged to provide the next row for the number triangle shown below

$$\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & & & & 1 & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \end{array}$$

The expected answer that matches Pascal’s triangle [2] was “1 4 6 4 1”. However, Anggoro, Liu, and Tulloch suggested “1 4 5 4 1” instead. They devised this new row via so-called diamond formula

$$\text{South} = \frac{\text{East} \cdot \text{West} + 1}{\text{North}}$$

So that upcoming rows of the triangle are

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
7	1	7	11	13	13	11	7	1

Table 1. Rascal triangle. See the OEIS sequence [A077028](#) [3].

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [4], in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [5, 6]. In particular, the

concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

2. BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number

Definition 2.1. *Iterated rascal number* [6]

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} \quad (2.1)$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [7]. Consider Vandermonde convolution

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (2.2)$$

Therefore, iterated rascal number is partial case of Vandermonde convolution with the upper summation bound equals to i . Without further hesitation consider our findings.

Proposition 2.2. *Iterated rascal triangle equals to Pascal's triangle up to i -th column.*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq k \leq i \quad (2.3)$$

Proof. Proof is given by [6]. □

Then binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying binomial coefficients symmetry principle we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

Proposition 2.3. *Iterated rascal triangle equals to Pascal's triangle up to $2i + 1$ -th row*

$$\binom{n}{k}_i = \binom{n}{k}, \quad 0 \leq n \leq 2i + 1$$

Therefore, for every fixed $i \geq 0$

$$\binom{2i + 1 - n}{k}_i = \binom{2i + 1 - n}{k} \quad (2.4)$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over k for each i , so that it is true for all cases in i, k : $i < k$, $i = k$ and $k > i$.

Taking $t \geq 2i + 1$ for every fixed $i \geq 0$

$$\binom{t - n}{k}_{t-i-1} = \binom{t - n}{k}$$

Proof. Proof of proposition (2.3). We have three possible relations between i, k : $k < i$, $k = i$, $k > i$. So we have to prove that for every i, k

$$\sum_{m=0}^k \binom{2i + 1 - n - k}{m} \binom{k}{m} - \sum_{m=0}^i \binom{2i + 1 - n - k}{m} \binom{k}{m} = 0$$

For the case $k < i$ proof is given in Jenna Gregory et al. [6]. For the case $k = i$ proof is trivial. Thus, the remaining case is $k > i$ yields

$$\sum_{m=i+1}^k \binom{2i + 1 - n - k}{m} \binom{k}{m} = 0$$

Considering the constraints,

$$\begin{cases} n \geq 0 \\ k \geq i + 1 \\ 2i + 1 - n - k \leq i - n \\ m \geq i + 1 \end{cases}$$

Thus,

$$\sum_{m=i+1}^k \binom{2i + 1 - n - k}{m} \binom{k}{m}$$

is indeed equals zero because binomial coefficients $\binom{i-n-s}{i+1+s}$ are zero for each $i, n, s \geq 0$. Therefore, the proposition (2.3) is true. \square

Moreover, equation (2.4) gives Vandermonde-like identity

Proposition 2.4. (*Vandermonde-like identity.*)

$$\binom{2i+1-n}{k} = \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m}$$

In particular, given $n = 0, 1$ proposition (2.4) yields

$$\binom{2i+1}{k} = \sum_{m=0}^i \binom{2i+1-k}{m} \binom{k}{m}$$

$$\binom{2i}{k} = \sum_{m=0}^i \binom{2i-k}{m} \binom{k}{m}$$

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences $\binom{n}{k} - \binom{n}{k}_3$

In[67]:= `Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]`

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Out[67]=

Figure 1. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $\binom{n}{4}$. Sequence A000332 in the OEIS [8].

We can spot that having $i = 3$ the $k = 4$ -th column gives binomial coefficient $\binom{n}{4}$. Indeed, this rule is true for every i .

Proposition 2.5. (*Row-column difference.*) *For every fixed $i \geq 0$*

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$

Proof. We have previously stated that iterated rascal numbers are closely related to Vandermonde convolution (2.2). Thus, proposition (2.5) can be rewritten as

$$\sum_{m=0}^i \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{m}$$

Therefore, $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$ is indeed true. \square

Proposition (2.5) yields to few more identities. Applying binomial coefficients symmetry

$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking $j = n + i$ gives

$$\begin{aligned} \binom{j+i}{j} - \binom{j+i}{j}_{i-1} &= \binom{j}{j-i} \\ \binom{j+i}{i} - \binom{j+i}{i}_{i-1} &= \binom{j}{i} \end{aligned}$$

Proposition (2.5) can be generalized even further, for every fixed $i < k$.

Proposition 2.6. (*Binomial coefficient difference iterated rascal number.*) *For every fixed $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^k \binom{n-k}{m} \binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. \square

3. Q-BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

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In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 2. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $(1, 5)$ -binomial coefficient $\binom{n}{k}^5$. Sequence [A096943](#) in the OEIS [9].

The $(1, q)$ -binomial coefficients $\binom{n}{k}^q$ are special kind of binomial coefficients defined by

Definition 3.1. $(1, q)$ -Binomial coefficient

$$\binom{n}{k}^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \binom{n-1}{k}^q + \binom{n-1}{k-1}^q & \end{cases} \quad (3.1)$$

Indeed, the relation shown in Figure (2) is true for every i , so that it establishes a relation between $(1, q)$ -binomial coefficients and iterated rascal numbers.

Proposition 3.2. (*Relation between iterated rascal numbers and $(1, q)$ -binomial coefficients.*)

For every fixed $i \geq 0$

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[\begin{matrix} i+2+j \\ i+2 \end{matrix} \right]^{i+2}$$

Taking $t = i + 2$ in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[\begin{matrix} t+j \\ t \end{matrix} \right]^t$$

In particular, having $i = 1$ proposition (3.2) gives the OEIS sequence A006503 [10] such that third column of $(1, 3)$ -Pascal triangle A095660 [11].

Having $i = 3$ proposition (3.2) gives the OEIS sequence A096943 [9] such that third column of $(1, 5)$ -Pascal triangle A096940 [12].

For $i = 5$, the proposition (3.2) yields the OEIS sequence A097297 [13] such that seventh column of $(1, 6)$ -Pascal triangle A096940 [14].

4. CONCLUSIONS

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.5), (2.6), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.4). Furthermore, we establish a relation between iterated rascal triangles and $(1, q)$ -binomial coefficients (3.2). All the results can be validated using supplementary Mathematica scripts at [15].

REFERENCES

- [1] Anggoro, Alif and Liu, Eddy and Tulloch, Angus. The Rascal Triangle. *The College Mathematics Journal*, 41(5):393–395, 2010. <https://doi.org/10.4169/074683410X521991>.
- [2] Sloane, N. J. A. Pascal’s triangle read by rows. Entry A007318 in The On-Line Encyclopedia of Integer Sequences, 1994. <https://oeis.org/A007318>.
- [3] Sloane, N. J. A. The Rascal triangle read by rows. Entry A077028 in The On-Line Encyclopedia of Integer Sequences, 2002. <https://oeis.org/A077028>.

- [4] Amelia Gibbs and Brian K. Miceli. Two Combinatorial Interpretations of Rascal Numbers. *arXiv preprint arXiv:2405.11045*, 2024. <https://arxiv.org/abs/2405.11045>.
- [5] Hotchkiss, Philip K. Generalized Rascal Triangles. *arXiv preprint arXiv:1907.11159*, 2019. <https://arxiv.org/abs/1907.11159>.
- [6] Gregory, Jena and Kronholm, Brandt and White, Jacob. Iterated rascal triangles. *Aequationes mathematicae*, pages 1–18, 2023. <https://doi.org/10.1007/s00010-023-00987-6>.
- [7] Andrews, George E. and Askey, Richard and Roy, Ranjan and Roy, Ranjan and Askey, Richard. *Special functions*, volume 71. Cambridge university press Cambridge, 2000.
- [8] Sloane, N. J. A. Binomial coefficient binomial(n,4). Entry A000332 in The On-Line Encyclopedia of Integer Sequences, 2009. <https://oeis.org/A000332>.
- [9] Sloane, N. J. A. Sixth column of (1,5)-Pascal triangle. Entry A096943 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096943>.
- [10] Sloane, N. J. A. Entry A006503 in The On-Line Encyclopedia of Integer Sequences, 1995. <https://oeis.org/A006503>.
- [11] Sloane, N. J. A. Pascal (1,3) triangle. Entry A095660 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A095660>.
- [12] Sloane, N. J. A. Pascal (1,5) triangle. Entry A096940 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096940>.
- [13] Sloane, N. J. A. Seventh column (m=6) of (1,6)-Pascal triangle. Entry A097297 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A097297>.
- [14] Sloane, N. J. A. Pascal (1,6) triangle. Entry A096940 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096940>.
- [15] Kolosov, Petro. Identities in Iterated Rascal Triangles Mathematica programs, 2024. <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>.

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