School of Mathematical Sciences
2016 [S2]

MTH2015 Advanced Multivariable Calculus

Assignment 3

To be submitted in support class the week of 10 October

Please attach an Assessment Coversheet to your assignment showing your full name and student identity number, the day and time of your support class and the name of your demonstrator. Staple pages together. The unit guide (on the unit homepage) contains details on coversheets, the policy on late assignments and university policy regarding special consideration, plagiarism, etc...

Three marks for this assignment will be allocated to the presentation of mathematics.

1. Convert the integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{(x^2+y^2)/2} (x^2+y^2)^{-1/2} z \, dz dy dx$$

in both cylindrical and spherical coordinates. Evaluate the integral.

[6 marks]

2. Evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \mathrm{e}^{\sqrt{xy}} \sqrt{y/x} \, dx dy$$

by making the coordinate transformation $x = u/v$ and $y = uv$. Sketch the domain of the integral in both the xy-plane and the uv-plane (restrict to positive values for all variables involved).

[7 marks]

3. Consider the vector field in $\mathbb{R}^3$:

$$\vec{F} \;:=\; (2xy)\vec{i} + (x^2 + 2yz - 1)\vec{j} + y^2\vec{k} \,.$$

(a) Compute $\operatorname{curl} \vec{F}$.

(b) Compute $\operatorname{div} \vec{F}$.

(c) Find a potential function $f(x, y, z)$ such that $\vec{F} = \nabla f$.

(d) Evaluate, **first directly and then using your answer to (c)**, the integral

$$\int_C \vec{F} \cdot d\vec{r}\,, \qquad \text{where} \quad C \;:=\; \left\{ \vec{r}(t) := (t, t^2, t^3)\;;\; t \in [0, 1] \right\}.$$

[8 marks]

4. An *isometry (of the plane)* is a function $f : \mathbb{C} \to \mathbb{C}$ such that

$$|f(z_1) - f(z_2)| \;=\; |z_1 - z_2| \qquad \forall\; z_1, z_2 \in \mathbb{C}\,.$$

Define

$$g(z) \;:=\; \frac{f(z) - f(0)}{f(1) - f(0)}\,.$$

(a) Show that g is an isometry if f is.

(b) By observing that $g(0) = 0$ and $g(1) = 1$, show that the real parts of $g(z)$ and of z are equal for all $z \in \mathbb{C}$; and moreover that $g(i) = \pm i$.

(c) Show that

(i) if $g(i) = i$, then $g(z) = z$;

(ii) if $g(i) = -i$, then $g(z) = \bar{z}$.

(d) Using your answers to (a)-(c), prove that an isometry f must be of the form

$$f(z) \;=\; \mathrm{e}^{i\alpha} z + b \qquad \text{or} \qquad f(z) \;=\; \mathrm{e}^{i\alpha} \bar{z} + b\,,$$

where $\alpha \in [0, 2\pi)$ and $b \in \mathbb{C}$ are constants.

In other words, an isometry of the plane is composed of a translation, a rotation, and a reflection. These are the only transformations of the plane that preserve distances (and thus that preserve areas).

[7 marks]