Lecture 3 - Discrete Mafematiks

Kori Vernon

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1 Lists

Definition: A list is an ordered sequence of objects. The order does matter.

1.1 Examples:

(1) A student club consists of 10 members. They need to elect a vice-president, secretary, and treasurer.

Q: In how many ways can it be done?

A: If repetitions are not allowed, $10 \cdot 9 \cdot 8 \cdot 7$, however, if a person is allowed to be qualified for each position, it would be 10^4.

Q: If a person is allowed to hold at most two positions?

A: (10,10,9,9), (9,8,9,10), (10,9,8,7) ... the summation of all permutations of the above arrangements.

2 Permutation

- A **Permutation** is an arrangement of n objects $n!$ in a line, and $(n-1)!$ in a circle.
- If we need to fill in n choose k positions when repetitions are allowed, and we choose out of n objects, there are n^k possibilities.
- When repetitions are not allowed, then, we would use what is known as a "falling factorial." $(n)_k = n \cdot (n-1) \cdot ... \cdot (n-k+1)$.

Note: To find the number of lists of length k out of n possible elements, we would use n^k with repetitions, and $(n)_k$ without repetitions.

2.1 Examples:

(1) Airports have codes which consists of 3 letters. How many lists are available (if repetition is allowed)

A: 26^3, because repetition is allowed, there are 26 letters in the English alphabet, and there are three spaces.

(2) A bit string is a list of 0's and 1's. How many length-k bit strings can be made?

A: 2^k, because there are 2 possibilities for every k.

(3) Anna owes different rings. She wears all three rings, but no two rings at the same time.

A: $8 \cdot 7 \cdot 6$. There are 8 possibilities for the first ring, 7 possibilities for the second ring, and 6 possibilities for the third ring.

(3) How many five-digit numbers are there that do not have consecutive digits that are the same?

A: (9,9,9,9,9)

(4) A class contains 10 boys and 10 girls. In how many different way can they stand in a line if they must alternate in gender.

A: $2 \cdot (10!)^2$

(5) In how many ways can 20 books be arranged on a bookshelf?

A: 20! different ways

3 Factorial

- $n! = n \cdot (n-1) \cdot ...3 \cdot 2 \cdot 1$
- $(n)_k = n \cdot (n-1) \cdot (n-1) \cdot ... \cdot (n-k+1)$

Recall: $\sum$ = sigma is used to calculate the summation.

Π = the product. $\Pi_{n=1}^{5} a_n = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$

The product notation of a factorial: $\Pi_{k=1}^{n} k = (1)(2)...(n-1)(n)$

Note: The Scottish Mathematician James Stirling found an approximation for the formula $n!$

$$n! \approx \sqrt{2\pi n} \cdot n^n \cdot e^{-n}$$

n Choose k:

$$\frac{n!}{(n-k)!} = (n)_k$$

Factorion: A number that is equal to the sum of the factorial of its digits. For example, $1! = 1$, and $2! = 2$, as well as $145! = 1! + 4! + 5!$. Initially coined by the author *Clifford A. Pickover.*

Double Fact: *Because two is better than one...* $n!!$ is the factorial of odd positive integers.

$$5!! = 1! \cdot 3! \cdot 5!$$

$$2(n+1)! = \Pi_{k=1}^{n} (2k+1)!$$

Remark: $n!! \neq (n!)!$

3.1 Examples:

(1) There are 6 different French books, 8 different Russian books, and 5 Spanish books. In how many ways can the different books be arranged in a bookshelf if all books in the same language are together.

A: $6! \cdot 8! \cdot 5! \cdot 3!$ An additional $3!$ is needed to account for the different ways that all of the books can be arranged.

(2) For $n = 30$, what is the relative error in the approximation?

$$30! \approx \sqrt{2\pi 30} 30^{30} \cdot e^{-30}$$

$$rel\ err = \frac{|actual\ -\ approx|}{|approx|}$$

$$rel\ err = \frac{|2.653 \cdot 10^{32}\ -\ 2.645 \cdot 10^{32}|}{|2.645 \cdot 10^{32}|} \cdot 100\% = 0.278\%$$

4 Set

Definition: A set is a repetition-free un-ordered collection of objects. A given object is either a member of a set or not. An object cannot be in a set more than once. There is no order to the members of a set.

IN: In order to indicate that an element is a member of a set, we use $\in$.

- $\{x : x \in \mathbb{Z}\}$
- $\{x : x \in \mathbb{Z}, x\ is\ even\}$
- $\{x : x \in \mathbb{Z}, x > 0\}$

- The number of elements in a set A is denoted by $|A|$, and it is called the **cardinality** of set A.
  - $|\mathbb{Z}| = \infty$
  - $A = \{a, b, c\}$, then $|A| = 3$
- An empty set has no elements inside and the notation is $\emptyset$. The cardinality of an empty set is 0 $|\emptyset| = 0$.

- In order to show that two sets are equal, we must consider sets A and B. To show that $A = B$, we must show that:
    1. Suppose $x \in A \rightarrow$ Therefore $x \in B$
    2. Suppose $x \in B \rightarrow$ Therefore $x \in A$

  Note: Must show both ways!!!!

SUBSET: In order for a set to be a **Subset**, we indicate that every element in the **Subset** is also in the **Set**. If we have sets A and B, we say that A is a subset of B if every element of A is also in B. Then we write $A \subseteq B$

  - **Remark:** $\emptyset$ is a subset of anyset.
  - If $A = B$, then $A \subseteq B$ and $B \subseteq A$

POWERSET: A powerset is a set of all possible subsets of a set A. The notation is in the form 2^A.

  - **Finding Cardinality of Powerset:** 2 to the power of the cardinality of the original set.

4.1 Examples:

(1) Let $E = \{x \in \mathbb{Z} : 2|x$ and $F = \{z \in \mathbb{Z} : z = a + b,\ \textit{where a and b are odd}\}$. Show that $E = F$.

Proof: Let $x \in E$, so there exists an integer $y \in \mathbb{Z}$, such that $x = 2y$. Let $a = 2y + 1$, which is an odd integer, and $b = -1$, which is an odd integer as well. Then $x = (2y + 1) + (-1_= a + b \in F$, therefore, $x \in F$.

Let $x \in F$. Then x can be decomposed as $a + b$, where a and b are odd. Let $a = 2u + 1$, and $b = 2u + 1$, where $u, v \in \mathbb{Z}$. Thus, $a + b = (2u + 1) + (2v + 1) = 2u + 2v + 2 = 2(u + v + 1)$, where $2|2(u + v + 1)$, therefore x is even, therefore $x \in E$. $\square$

(2) Given a set $A = \{x, 5, 1\}$, find the power set.

$\{\emptyset, \{x\}, \{1\}, \{5\}, \{x, 1\}, \{1, 5\}, \{x, 5\}, \{x, 1, 5\}\}$

Cardinality: $2^{|A|} = 2^3 = 8$