

# Lecture 5 - Discrete Mafematiks

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## 1 Combinatorial Proof

In order to prove by **combinatorial proof**, we need to find a question which can be answered by the left-hand side of a given equation as well as the right-hand side. If that question is answered via both approaches, then that would be the proof.

**Note:** *The challenging part is to come up with the question or situation, for which you need to find the answer or explanation.*

### 1.1 Examples:

(1) **Proposition:** Let  $A$  and  $B$  be finite sets. Then  $|A| + |B| = |A \cup B| + |A \cap B|$

- $|A| + |B|$  (*lhs*) =  $|A \cup B| + |A \cap B|$  (*rhs*) where  $A$  and  $B$  are finite sets.

→ **Combinatorial Proof:** Imagine we assign labels to every object in these sets. (*now we ask the question*)

**Question:** How many labels have we assigned?

- Since the cardinalities of sets  $A$  and  $B$  are  $|A|$  and  $|B|$ , respectively, then the total number of labels is  $|A| + |B|$
- On the other hand, we have assigned at least one label to the elements in  $A \cup B$ , so the number of labels in  $|A \cup B|$ . However, elements in the intersection  $A \cap B$  receive 2 labels because they are included in  $|A \cup B|$ .
- Therefore, the total number of labels distributed is  $|A \cup B| + |A \cap B|$  (*rhs*).
- Since both sides of the statement answer the same question,  $|A| + |B| = |A \cup B| + |A \cap B|$   $\square$

(2) **Proposition:** Let  $n$  be a positive integer. Then  $2^0 + 2^1 + 2^2 + \dots + 2^{n-1}$  (*lhs*) =  $2^n - 1$  (*rhs*)  $\square$

→ **Combinatorial Proof:** Consider a set  $N = \{1, 2, 3, \dots, n\}$  (*now we ask the question*)

**Question:** How non-empty subsets does  $N$  have?

- The power set of  $N$  has  $2^n$  subsets, but  $\emptyset$  is a subset of a power set. Thus,  $2^n - 1$  is the total number of non-empty subsets.
- Now, consider subsets with the largest element in that given subset.
- Let  $k$  be the largest element in that given subset. Therefore, the largest  $k$  can be is  $n$  and the smallest is 1, so subsets must look like  $\{\dots, k\}$ , so the largest element will be  $k - 1$  other options for other elements.
- Therefore,  $2^{k-1}$  subsets will look like  $\{\dots, k - 1\}$ , where the largest element is  $k \leq n$ .
- Thus, the number of options is  $2^0 + 2^1 + 2^2 + \dots + 2^{n-1}$  for non-empty subsets.  $\square$

(3) **Proposition:**  $1 + 2 + 3 + \dots + n = \binom{n+1}{2}$

→ **Combinatorial Proof:** Consider a set  $A = \{1, 2, 3, \dots, n + 1\}$  elements. (*now we ask the question*)

**Question:** How many subsets of  $A$  contain exactly 2 elements?

- We need to choose 2 elements out of  $n + 1$  choices, so there are  $\binom{n+1}{2}$  possible subsets with 2 elements.
- Now consider subsets where one element is greater than another.
- Assume the larger element is 2. Then there is 1 choice. Now assume the larger element is 3, then we have 2 choices. ... The larger element is  $n + 1$ , then there are  $n$  choices.
- Since each two element subset must be exactly one of these cases, the total number of possibilities is the sum of all of these choices, which is the left-hand-side.
- Therefore, we obtain both explanations to the same question, so the statement is true.  $\square$

## 2 Relations

**Definition:** A relation is a set of ordered pairs.

- **ex:**  $R = \{(1, 2), (2, 3), (4, 5)\}$
- **Notation:**  $(x, y) \in R \Rightarrow xRy$
- **Remark:**  $R$  is the criteria with respect to which  $x$  and  $y$  are related. It can be "equal to", "less than/greater than" or "less than or equal to/greater than or equal to."

Relations can be established between sets.

- Consider the sets  $A$  and  $B$  and the relations " $\subseteq$ ". We say  $R$  is a relation from  $A$  to  $B$  provided  $R \subseteq A \times B$
- **ex:** Let  $A = \{1, 2, 3, 4\}$  and  $B = \{4, 5, 6, 7\}$
- $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$  relation on set  $A$  with equivalency.
- $S = \{(1, 2), (3, 2)\}$  another relation on  $A$

**Definition:** Let  $R$  be a relation. The inverse of  $R$  is  $R^{-1}$  is the relation. Reversing  $x$  and  $y$ . Thus, if  $(x, y) \in R$  and  $xRy$ , then  $yR^{-1}x$ .

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

- $R = \{(1, 5), (2, 3), (7, 12)\}$ , so  $R^{-1}$  would be the inverse.
- If  $R$  is a relation on set  $A$ , then  $R^{-1}$  is an inverse relation on set  $A$

### 2.1 Properties of Relations:

1. If for all  $x \in A$  we have  $xRx$ , then  $R$  is called **reflexive**.
2. If for all  $x \in A$  we have  $x \not R x$ , then  $R$  is called **irreflexive**.
3. If for all  $x, y \in A$  we have  $xRy \Rightarrow yRx$ , then  $R$  is **symmetric**.
4. If for all  $x, y \in A$  we have  $(xRy \wedge yRx \Rightarrow x = y)$ , then  $R$  is **antisymmetric**.
5. If for all  $x, y, z \in A$  we have  $(xRy \wedge yRz) \Rightarrow xRz$ , then  $R$  is **transitive**.

**Note:** Relations which are reflexive, symmetric, and transitive are called **equivalent relations**.

### 2.2 Examples:

1. Consider the set  $\mathbb{Z}$  with the relation " $=$ ".
  - This is a reflexive relation because we know that  $x = x$ .
  - This is symmetric because  $\forall x, y \in \mathbb{Z}$ , if  $x = y \Rightarrow y = x$
  - This is antisymmetric because  $\forall x, y \in \mathbb{Z}$ ,  $(x = y \wedge y = x) \Rightarrow x = y$
  - This is not irreflexive because  $\forall x \in \mathbb{Z}$ ,  $x = x$ , which not  $x \neq x$ . This is **false**  $\forall x \in \mathbb{Z}$ . The  $=$  relation is not irreflexive.
2. Consider a relation " $\leq$ " on  $\mathbb{Z}$ .
  - It is reflexive because we know that  $x \leq x$ .
  - This is not symmetric because  $x \leq y \Rightarrow y \leq x$  is not necessarily true.
  - This is transitive because  $(x \leq y \wedge y \leq z) \Rightarrow x \leq z$ . This is antisymmetric because  $\forall x, y \in \mathbb{Z}$ ,  $(x \leq y \wedge y \leq x) \Rightarrow x = y$
3. Consider a relation " $<$ " on  $\mathbb{Z}$ 
  - $\forall x \in \mathbb{Z}$ ,  $x < x$  False, so not reflexive.
  - $\forall x, y \in \mathbb{Z}$ , so not symmetric.
  - $\forall x, y, z \in \mathbb{Z}$ ,  $(x < y \wedge y < z) \Rightarrow (x < z)$  so it is transitive.

- $\forall x \in \mathbb{Z}, x < x$ , so it is not irreflexive.
  - $\forall x, y \in \mathbb{Z}, (x < y) \wedge (y < x) \Rightarrow x = y$  is antisymmetric because it is **vacuously true**.
4. Consider the relation  $A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$
- This is reflexive.  $\forall x \in A, xRx$
  - This is not irreflexive.  $\forall x \in A, xRx$
  - This is antisymmetric.  $\forall x, y \in A, (x = y) \wedge (y = x) \Rightarrow x = y$
  - This is symmetric.  $\forall x, y \in A, xRy \Rightarrow yRx$ , so  $x = y \Rightarrow y = x$
  - This is transitive.  $\forall x, y, z \in A$ , then  $(x = y \wedge y = z) \Rightarrow x = z$