Lecture 2 - Discrete Mafematiks

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1 More on Proofs

Proposition: Let x be an integer. Prove that x is odd if and only if there is an integer b such that $x = 2b - 1$.

Proof:

"$\Rightarrow$" need to prove "if x is odd, then $x = 2b - 1$".

Let x be an odd integer. Then, by the definition, then of being odd, there exists an integer a, such that $x = 2a + 1$.

Let $b = a + 1$. Then $2b - 1 = 2(a + 1) - 1 = 2a + 2 - 1 = 2a + 1 = x$.

"$\Leftarrow$" Let $x = 2b - 1$ where b is some integer. *We need to show that:* $2b - 1 = x = 2a + 1$.

Let $a = b - 1$ which is some integer. Then $2a + 1 == 2(b - 1) + 1 = 2b - 2 + 1 = 2b - 1 = x$. Therefore, x is odd $\square$.

Proposition: (5.15) Let x be an integer. Prove that $0|x$ if and only if $x = 0$.

Proof:

"$\Rightarrow$" Let $0|x$, so there exists an integer a so that by the definition $x = 0 \cdot a = 0$

"$\Leftarrow$" Let $x = 0$, so there exists some integer a such that $0 \cdot a = 0$, so $0|x\square$.

Proposition: (5.20) For real numbers a and b, prove that if $0 < a < b$ (a and b are both positive), then $a^2 < b^2$.

Proof: If $a < b$, then we can multiply both sides by a, which is a positive integer.

The inequality will still hold that $a \cdot a < b \cdot a$.

If we multiply both sides by b, we have $a \cdot b < b \cdot b$.

The inequality will still hold that $a \cdot b < b \cdot b$.

Thus, $a^2 < ab \leq ab < b^2 \Rightarrow a^2 < b^2$ $\square$

2 Counterexamples

2.1 Structure for if-then

Refuting a false if-then statement via counterexample where you have $A \Rightarrow B$ requires you to find a statement where A is true, and B is false.

Disprove: If a and b are integers such that $a|b$, then $a \leq b$

Let $a = 2$, and let $b = -4$. We know that 2 divides -4, but $2 \nleq -4$ $\square$

Disprove: If a and b are non-negative integers with $a|b$, then $a \leq b$

If $b = 0$, and $a = 1$. $1|0$, but $1 \nleq 0$ $\square$

Disprove: If a, b, c are positive integers with $a|(bc)$, then $a|b$ and $a|c$.

Let $a = 6$, Let $b = 3$, and Let $c = 4$. $a|(bc)$, but a does not divide b, nor c $\square$.

2.2 Structure for if-and-only-if

A counterexample for AB must be one of the following:

1. If A is true, but b is false
2. If B is true, but B is false

3 Boolean Algebra

Boolean algebra is a framework for dealing with logical statements; every proof or "disproof" is done in the form of a table called "True-False Tables"

"$\wedge$" = "and" (see Table 1)

"$\vee$" = "or" (see Table 2)

"$\neg$" = "not" (see Table 3)

"$\rightarrow$" = "if-then"

"$\leftrightarrow$" = "if-and-only-if"

The only instance that they are true is when they are BOTH true.

Table 1: $X \wedge Y$

X	Y	$X \wedge Y$
True	True	True
True	False	False
False	True	False
False	False	False

Table 2: $X \vee Y$

X	Y	$X \vee Y$
True	True	True
True	False	True
False	True	True
False	False	False

Table 3: X and $\neg X$

X	¬X
True	False
False	True

Show that the boolean expressions $\neg(X \wedge Y$ and $(\neg X) \vee (\neg Y)$ are logically equivalent.

Table 4: $\neg(X \wedge Y$ and $(\neg X) \vee (\neg Y)$

X	Y	(X ∧ Y)	$\neg(X \wedge Y)$	$\neg X$	$\neg Y$	$(\neg X) \vee (\neg Y)$
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
False	False	False	True	True	True	True

Logically Equivalent Statements

1) $X \wedge Y = Y \wedge X$ and $X \vee Y = Y \vee X$ (Cumulative Property)

2) $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$ and $(X \vee Y) \vee Z = X \vee (Y \vee Z)$ (Associative Property)

3) $X \wedge T = X$ and $X \wedge F = X$ (Identity Element)

4) $\neg(\neg X) = X$

5) $X \wedge X = X$ and $X \vee X = X$

6) $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$ and $(X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$

7) $X \wedge (\neg X) = F$ and $X \vee (\neg X) = T$

8) $\neg(X \wedge Y) = (\neg X) \vee (\neg Y)$ and $\neg(X \vee Y) = (\neg X) \wedge (\neg Y)$ (DeMorgan's Law)

3.1 Examples

The result of a boolean algebra expression in which everything is false is called a "Contradiction."

Note: A boolean operation called "nand" is defined as $x \bar{\wedge} y = \neg(x \wedge y)$. The commutative property applies to "Nand", but the associative property does not apply to "Nand".

Prove: Show that $x \to y$ is equivalent to $(\neg x) \vee y$

Table 5: $x \to y$ is equivalent to $(\neg x) \vee y$

x	$\neg x$	y	$(\neg x) \vee y$	x $\to y$
True	False	True	True	False
True	False	False	False	False
False	True	True	True	True
False	True	False	True	True

3.2 Tautology

1. A tautology is a boolean expression that evaluates true for all possible values of its variables.

Now Consider $(T \bar{\wedge} T) \bar{\wedge} F = F \bar{\wedge} F = T$

Show that $\overbrace{(\underbrace{(x \to y) \wedge (y \to z)}_{K}) \to (x \to z)}^{A}$ is a tautology.

X	Y	Z	$(x \to y)$	$(y \to z)$	K	$(x \to z)$	A
T	T	T	T	T	T	T	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T
F	T	F	T	F	F	T	T
T	T	F	T	F	F	F	T

4 Lists

Definition: A list is an ordered sequence of objects. The order does matter.

1. The number of elements in a list is called it's length, and it may contain repeated elements.
2. Two lists are equal if and only if they are the same length and their corresponding elements are equal.
3. Lists can contain different elements, such as matricies, letters, nested lists, numbers, and letters, etc.