MS-EV0004 Vertex operator algebras: Exercise 4

. **. **

Lecturer: Shinji Koshida*

4.1 Heisenberg vertex algebra from the reconstruction theorem

Exercise 4.1.1. Apply the reconstruction theorem to construct the Heisenberg vertex algebra. (Identify generators and check the assumptions.)

4.2 Few properties of a vertex (operator) algebra

Exercise 4.2.1. Let $(V, \mathbf{1}, Y, \omega)$ be a vertex operator algebra. Show that $\mathbf{1} \in V_0$ and $\omega \in V_2$.

Exercise 4.2.2. Let $(V, \mathbf{1}, Y)$ be a vertex algebra. For $a, b \in V$ and $m, n \in \mathbb{Z}$, show the following commutator formula:

$$[a_{(m)}, b_{(n)}] = \sum_{k=0}^{\infty} \binom{m}{k} \left(a_{(k)} b\right)_{(m+n-k)}.$$

Exercise 4.2.3. Let $(V, \mathbf{1}, Y, \omega)$ be a vertex operator algebra. Show that, for $a \in V_m$,

$$a_{(n)} V_d \subset V_{m+d-n-1}, \quad n, d \in \mathbb{Z}.$$

4.3 Taylor's theorem in another form

Let V be a vector space, and $a(x) \in V[[x^{\pm 1}]]$.

Exercise 4.3.1. For $m \in \mathbb{N}$, show the identity

$$a(x) \cdot \partial_y^{(m)} (x^{-1} \delta(y/x)) = \sum_{j=0}^{m} \partial^{(j)} a(y) \cdot \partial_y^{(m-j)} (x^{-1} \delta(y/x))$$

in $V[[x^{\pm 1}, y^{\pm 1}]]$.

4.4 Conformal vectors of the Heisenberg vertex algebra

Let $(\mathcal{F}_0, \mathbf{1}, Y)$ be the Heisenberg vertex algebra and set

$$\omega^\eta := \left(\frac{1}{2}\alpha_{-1}^2 + \eta \alpha_{-2}\right)\mathbf{1}, \quad \eta \in \mathbb{C}.$$

Exercise 4.4.1. Show that, for $n \geq 0$,

$$\omega^\eta_{(n)} \omega^\eta = \begin{cases} c^\eta \mathbf{1}, & n = 3, \\ 2\omega^\eta, & n = 1, \\ T\omega^\eta, & n = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $c^\eta = 1 - 12\eta^2$.

*shinji.koshida@aalto.fi

Exercise 4.4.2. Show that, for $n \geq 0$,

$$\omega^{\eta}_{(n)}\alpha_{-1}\mathbf{1} = \begin{cases} -2\eta\mathbf{1}, & n = 2, \\ \alpha_{-1}\mathbf{1}, & n = 1, \\ T\alpha_{-1}\mathbf{1}, & n = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Exercise 4.4.3. Show that $(\mathcal{F}_0, \mathbf{1}, Y, \omega^{\eta})$ is a vertex operator algebra of central charge c^{η}.