MS-EV0004 Vertex operator algebras: Exercise 2

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2.1 Commutative vertex algebra

Recall the setting of Example 4.2: $V = \mathbb{C}[t^{\pm 1}]$ is the space of Laurent polynomials in t, $\mathbf{1} := 1$ and the linear map $Y(-,x)\colon V \to \mathrm{End}(V)[[x^{\pm 1}]]$ is defined by

$$Y(p(t),x)q(t) = p(t+x)q(t) = \sum_{j=0}^{\infty} \partial^{(j)}p(t) \cdot q(t)\, x^j.$$

Exercise 2.1.1. Show that the triple $(V, \mathbf{1}, Y)$ forms a vertex algebra.

2.2 Normally ordered product for the Heisenberg algebra

In Section 4.2, we defined the normally ordered product

$$\mathop{\circ}\limits^{\circ} \partial^{(n_1-1)}\alpha(x) \cdots \partial^{(n_l-1)}\alpha(x) \mathop{\circ}\limits^{\circ}$$

of derivatives of $\alpha(x)$ recursively. Otherwise, we can define normally ordered product of operators by

$$\mathop{\circ}\limits^{\circ} \alpha_{m_1} \cdots \alpha_{m_l} \mathop{\circ}\limits^{\circ} = \alpha_{m_{\sigma(1)}} \cdots \alpha_{m_{\sigma(l)}},$$

where σ is a permutation of $\{1,\dots,l\}$, such that

$$m_{\sigma(1)}, \dots, m_{\sigma(i)} < 0 \le m_{\sigma(i+1)}, \dots, m_{\sigma(l)}$$

with some $i \in \{1,\dots,l\}$.

Exercise 2.2.1. Show the identity

$$\begin{aligned}&\mathop{\circ}\limits^{\circ} \partial^{(n_1-1)}\alpha(x) \cdots \partial^{(n_l-1)}\alpha(x) \mathop{\circ}\limits^{\circ} \\ &= \sum_{m_1,\dots,m_l \in \mathbb{Z}} \binom{-m_1-1}{n_1-1} \cdots \binom{-m_l-1}{n_l-1} \mathop{\circ}\limits^{\circ} \alpha_{m_1} \cdots \alpha_{m_l} \mathop{\circ}\limits^{\circ} x^{-m_1-n_1} \cdots x^{-m_l-n_l}.\end{aligned}$$

2.3 Translation operator for the Heisenberg algebra

The translation operator T on $\mathcal{F}_0$ was defined by

$$T \cdot \alpha_{-n_1} \cdots \alpha_{-n_l} |0\rangle = \sum_{i=1}^{l} n_i\, \alpha_{-n_1} \cdots \alpha_{-n_{i-1}} \alpha_{-n_i-1} \alpha_{-n_{i+1}} \cdots \alpha_{-n_l} |0\rangle$$

for $n_1 \ge \cdots \ge n_l > 0$, $l \in \mathbb{N}$.

Exercise 2.3.1. Show the commutation relations:

$$[T, \alpha_n] = -n\alpha_{n-1}, \quad n \in \mathbb{Z}.$$

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2.4 Wick's theorem for the Heisenberg algebra

Exercise 2.4.1. Prove Wick's theorem for the Heisenberg algebra with as clear combinatorics as possible.

2.5 Formal delta series

Exercise 2.5.1. Show the identity

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) = x_1^{-1}\delta\left(\frac{x_0 + x_2}{x_1}\right)$$

in $\mathbb{C}[[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]]$.

Exercise 2.5.2. Show the identity

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)$$

in $\mathbb{C}[[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]]$.

2.6 Few translation properties

Let $(V, \mathbf{1}, Y)$ be a vertex algebra.

Exercise 2.6.1. Show the identity

$$Y(a, x)\mathbf{1} = e^{xT}a, \quad a \in V.$$

Exercise 2.6.2. Show the identity

$$Y(a, x + y) = e^{yT}Y(a, x)e^{-yT}.$$