MS-EV0004 Vertex operator algebras: Exercise 1

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Lecturer: Shinji Koshida*

1.1 Enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra. Its enveloping algebra $\mathcal{U}(\mathfrak{g})$ is given by (Section 2.1)

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}_{\mathfrak{g}},$$

where $T(\mathfrak{g})$ is the tensor algebra over $\mathfrak{g}$ and $\mathcal{I}_{\mathfrak{g}}$ is the two-sided ideal of $T(\mathfrak{g})$ generated by $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$. By definition, the map

$$\sigma \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g});\, x \mapsto x + \mathcal{I}_{\mathfrak{g}}$$

is a homomorphism of Lie algebras.

Exercise 1.1.1. Let A be an associative algebra and $f \colon \mathfrak{g} \to A$ be a homomorphism of Lie algebras. Show that there exists a unique homomorphism of associative algebras $f' \colon \mathcal{U}(\mathfrak{g}) \to A$ such that $f = f' \circ \sigma$.

Exercise 1.1.2. Let U be an associative algebra and $\rho \colon \mathfrak{g} \to U$ be a homomorphism of Lie algebras such that the following property holds:

(Universality) Suppose that A is an associative algebra and $f \colon \mathfrak{g} \to A$ is a homomorphism of Lie algebras. Then there exists a unique homomorphism $f' \colon U \to A$ of associative algebras such that $f = f' \circ \rho$.

Show that $U \simeq \mathcal{U}(\mathfrak{g})$ as associative algebras.

Exercise 1.1.3. Show the isomorphism

$$\mathcal{U}(\mathfrak{g}) \simeq \mathbb{C} \cdot 1 \oplus \mathcal{U}(\mathfrak{g})\mathfrak{g}.$$

1.2 Representation of Heisenberg algebra

Let us recall the infinite dimensional Heisenberg algebra from Section 2.4. It is the vector space

$$\widehat{\mathfrak{h}} = \left(\bigoplus_{n \in \mathbb{Z}} \mathbb{C}\alpha_n \right) \oplus \mathbb{C}K$$

together with the bilinear map $[-,-] \colon \widehat{\mathfrak{h}} \times \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}$ defined by

$$\begin{aligned} [\alpha_m, \alpha_n] &= m\delta_{m+n,0}K, \quad m, n \in \mathbb{Z}, \\ [K, \widehat{\mathfrak{h}}] &= \{0\}. \end{aligned}$$

Exercise 1.2.1. Show that the pair $(\widehat{\mathfrak{h}}, [-,-])$ forms a Lie algebra.

*shinji.koshida@aalto.fi

For $\lambda \in \mathbb{C}$, the Fock representation of $\widehat{\mathfrak{h}}$ is defined by

$$\mathcal{F}_\lambda = \mathcal{U}(\widehat{\mathfrak{h}})/\mathcal{I}_\lambda, \quad \mathcal{I}_\lambda = \sum_{n>0} \mathcal{U}(\widehat{\mathfrak{h}})\alpha_n + \mathcal{U}(\widehat{\mathfrak{h}})(\alpha_0 - \lambda) + \mathcal{U}(\widehat{\mathfrak{h}})(K-1).$$

The representation admits the following basis

$$\alpha_{-n_1}\alpha_{-n_2}\cdots\alpha_{-n_l}\left|\lambda\right\rangle, \quad n_1 \geq n_2 \geq \cdots \geq n_l > 0,\, l \in \mathbb{N},$$

where $\left|\lambda\right\rangle = 1 + \mathcal{I}_\lambda$.

Exercise 1.2.2. Let $n_1 \geq \cdots \geq n_k$ and $n'_1 \geq \cdots \geq n'_l$ be non-increasing sequences of positive integers such that $\sum_{i=1}^k n_i = \sum_{i=1}^l n'_i$. Show that

$$\alpha_{n'_1}\cdots\alpha_{n'_l}\cdot\alpha_{-n_1}\cdots\alpha_{-n_k}\left|\lambda\right\rangle = \begin{cases} z(\boldsymbol{n})\left|\lambda\right\rangle, & (n_1,\ldots,n_k) = (n'_1,\ldots,n'_l), \\ 0, & \text{otherwise,} \end{cases}$$

where

$$z(\boldsymbol{n}) = \prod_{i=1}^{\infty} i^{m_i(\boldsymbol{n})} m_i(\boldsymbol{n})!, \quad m_i(\boldsymbol{n}) = \#\{j = 1,\ldots,k | n_j = i\},\, i \geq 1.$$

Exercise 1.2.3. Show that $\mathcal{F}_\lambda$ is a simple $\mathcal{U}(\widehat{\mathfrak{h}})$-module[1].

1.3 Representation of the Virasoro algebra

The Virasoro algebra $\mathfrak{vir}$ is given by

$$\mathfrak{vir} = \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n\right) \oplus \mathbb{C}C$$

together with

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \quad m,n \in \mathbb{Z}, \\ [C, \mathfrak{vir}] &= \{0\}. \end{aligned}$$

Exercise 1.3.1. Show that $(\mathfrak{vir}, [-,-])$ is a Lie algebra.

For $(c,h) \in \mathbb{C}^2$, the Verma module $M(c,h)$ is defined by

$$M(c,h) = \mathcal{U}(\mathfrak{vir})/\mathcal{I}_{c,h},$$

where

$$\mathcal{I}_{c,h} = \sum_{n>0} \mathcal{U}(\mathfrak{vir})L_n + \mathcal{U}(\mathfrak{vir})(L_0 - h) + \mathcal{U}(\mathfrak{vir})(C - c).$$

Exercise 1.3.2. Show that the following vectors form a basis of $M(c,h)$:

$$L_{-n_1}\cdots L_{-n_k}\left|c,h\right\rangle, \quad n_1 \geq \cdots \geq n_k > 0,\, k \in \mathbb{Z}_{\geq 0},$$

where $\left|c,h\right\rangle = 1 + \mathcal{I}_{c,h}$.

Hint: Similarly to the case of the Heisenberg algebra, the Virasoro algebra is decomposed into Lie subalgebras as a vector space.

[1] i.e., a subspace $W \subset \mathcal{F}_\lambda$ such that $\mathcal{U}(\widehat{\mathfrak{h}})W \subset W$ is either 0 or $\mathcal{F}_\lambda$. In other words, $\mathcal{F}_\lambda$ is an irreducible representation of $\widehat{\mathfrak{h}}$.

Introduce the following formal series:

$$L(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2} \in \mathrm{End}(M(c,h))[[x^{\pm 1}]]$$

where the coefficients L_n are the action of the Virasoro algebra.

Exercise 1.3.3. Show that, for any $v \in M(c,h)$, $L(x)v \in M(c,h)((x))$.

Exercise 1.3.4. Show the formula

$$\begin{aligned}[L(x_1), L(x_2)] = &\frac{c}{2}\partial_{x_2}^{(3)}(x_1^{-1}\delta(x_2/x_1)) + 2L(x_2)\cdot\partial_{x_2}(x_1^{-1}\delta(x_2/x_1)) \\ &+ \partial L(x_2)\cdot x_1^{-1}\delta(x_2/x_1)\end{aligned}$$

in $\mathrm{End}(M(c,h))[[x_1^{\pm 1}, x_2^{\pm 1}]]$.

Exercise 1.3.5. For $t \in \mathbb{C}\backslash\{0\}$, take (c,h) as

$$c = 13 - 6(t + t^{-1}), \quad h = \frac{3-2t}{4t}.$$

Show that, for the vector

$$|\chi\rangle := (L_{-1}^2 - t^{-1}L_{-2})\,|c,h\rangle\,,$$

the following hold:

$$L_0\,|\chi\rangle = (h+2)\,|\chi\rangle\,, \quad L_n\,|\chi\rangle = 0,\ (n>0).$$

Exercise 1.3.6. Under the same choice of (c,h) as Exercise 1.3.5, show that $M(c,h)$ is not a simple $\mathcal{U}(\mathfrak{vir})$-module.

1.4 Paradox in formal calculus

Combining

$$(x_1 - x_2)\cdot\delta(x_2/x_1) = 0$$

and

$$(x_1 - x_2)^{-1}\cdot(x_1 - x_2) = 1$$

in $\mathbb{C}[[x_1^{\pm 1}, x_2^{\pm 1}]]$, we observe the following:

$$\begin{aligned}\delta(x_2/x_1) &= \big((x_1-x_2)^{-1}\cdot(x_1-x_2)\big)\cdot\delta(x_2/x_1) \\ &= (x_1-x_2)^{-1}\cdot((x_1-x_2)\cdot\delta(x_2/x_1)) \\ &= (x_1-x_2)^{-1}\cdot 0 \\ &= 0.\end{aligned}$$

Exercise 1.4.1. Point out what is wrong.

1.5 Formal differential equation

Let V be a vector space and $A(x) \in \mathrm{End}(V)[[x]]$.

Exercise 1.5.1. Show that, for $v \in V$, there is a unique solution $f(x) \in V[[x]]$ of the differential equation

$$\frac{d}{dx}f(x) = A(x)f(x), \quad f(0) = v.$$