MS-EV0004 Vertex operator algebras: Exercise 6

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Lecturer: Shinji Koshida*

6.1 Jacobi identity in another way

Let V be a vector space and

$$Y(-,x)\colon V \to \mathrm{End}(V)[[x^{\pm 1}]]$$

be a linear map such that $Y(a,x) \in \mathcal{E}(V)$ for any $a \in V$.

Exercise 6.1.1. Assume that $Y(-,x)$ satisfies the Jacobi identity:

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(a,x_1)Y(b,x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(b,x_2)Y(a,x_1) \\ =\,& x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(a,x_0)b,x_2) \end{aligned}$$

for any $a,b \in V$. Then, show the identity

$$\begin{aligned} &[x_1^{-1}]\left(Y(a,x_1)Y(b,x_2)\iota_{12}F(x_1,x_2) - Y(b,x_2)Y(a,x_1)\iota_{21}F(x_1,x_2)\right) \\ =\,&[x_0^{-1}]Y(Y(a,x_0)b,x_2)\iota_{20}F(x_0+x_2,x_2) \end{aligned}$$

holds for all $a,b \in V$ and rational functions $F(x_1,x_2) = x_1^m x_2^n (x_1-x_2)^l$, $m,n,l \in \mathbb{Z}$.

Hint: Multiply $x_0^l x_1^m x_2^n$ on both sides of the Jacobi identity. Then what do the delta distributions do?

Exercise 6.1.2. Show the converse of Exercise 6.1.1.

6.2 Simple $\mathbb{N}$-gradable modules

The following is Theorem 9.2 from the lecture note:

Theorem 6.1. *Let $M = \bigoplus_{n=0}^{\infty} M_n$ be a simple $\mathbb{N}$-gradable V-module. Then, there exists $h \in \mathbb{C}$ such that*

$$L_0^M|_{M_n} = (h+n)\mathrm{Id}_{M_n}, \quad n \in \mathbb{Z}.$$

Exercise 6.2.1. Show that any vector in M is a linear sum of vectors of the form

$$(a_1)^M_{(n_1)} \dots (a_k)^M_{(n_k)} w, \quad a_1,\dots,a_k \in V,\ w \in M_0$$

with

$$\deg a_i - n_i - 1 \geq 0, \quad i = 1,\dots,k.$$

Exercise 6.2.2. Show that L_0^M acts on M_0 by $h \cdot \mathrm{Id}_{M_0}$ with some $h \in \mathbb{C}$.

Hint: As M_0 is finite-dimensional, there is at least one eigenvector of L_0^M in M_0, i.e.,

$$N_0 := \ker(L_0^M|_{M_0} - h \cdot \mathrm{Id}_{M_0})$$

is a non-zero subspace of M_0 for some $h \in \mathbb{C}$. Consider the submodule of M generated by N_0.

Exercise 6.2.3. Prove the theorem.

*shinji.koshida@aalto.fi

6.3 Action of the Zhu algebra on the top space

Let $M = \bigoplus_{n=0}^{\infty} M_n$ be an $\mathbb{N}$-gradable V-module. For a homogeneous $a \in V$, we write $o(a) = a^M_{(\deg a - 1)}$ and extend the symbol linearly.

Exercise 6.3.1. Show that $o(a * b)|_{M_0} = o(a)o(b)|_{M_0}$ for $a, b \in V$.

Hint: If we can compare $o(a * b)$ with $o(a)o(b)$, we can do it because of the Jacobi identity. The alternative way in Exercise 6.1.1 might be useful.

Exercise 6.3.2. Show that

$$o(L_{-1}a) + (\deg a)o(a) = 0$$

for any homogeneous $a \in V$.

Exercise 6.3.3. Show that $o(a)|_{M_0} = 0$ for $a \in O(V)$.

Hint: Start with, for instance,

$$o(L_{-1}a)o(b)|_{M_0} = o((L_{-1}a) * b)|_{M_0} = \cdots .$$

Exercise 6.3.4. The above exercises combined altogether,

$$V \to \mathrm{End}(M_0); \quad a \mapsto o(a)$$

induces an action of $A(V)$ on M_0. Show that, if M is a simple $\mathbb{N}$-gradable V-module, then M_0 is a simple $A(V)$-module.

6.4 Zhu algebras of minimal Virasoro VOAs

The minimal Virasoro VOAs are labeled by

$$p, q \in \{2, 3, \dots\} : \text{coprime}, \quad p < q, \quad (p, q) \neq (2, 3).$$

Exercise 6.4.1. Show that

$$G_{p,q}(\mathsf{h}) = \Big(\prod_{r=1}^{p-1}\prod_{s=1}^{q-1}(\mathsf{h} - h_{r,s})\Big)^{1/2}$$

is a polynomial of h, where

$$h_{r,s} = \frac{(sp - rq)^2 - (p - q)^2}{4pq}, \quad r, s \in \mathbb{Z}.$$

6.5 Recursive formula of matrix elements

The following is Lemma 9.15 from the lecture note:

Lemma 6.2. *For any $a^1, \dots, a^n \in V$, $w \in M_0$, and $\varphi \in M_0^*$, we have*

$$\begin{aligned}
&\big\langle \varphi, Y(a^1, x_1)Y(a^2, x_2)\dots Y(a^n, x_n)w \big\rangle \\
=& \big\langle o(a^1)^*\varphi, Y(a^2, x_2)\dots Y(a^n, x_n)w \big\rangle \\
+& \sum_{k=2}^{n}\sum_{i=0}^{\infty} \iota_{1k} F_{\deg a^1, i}(x_1, x_k) \cdot \big\langle \varphi, Y(a^2, x_2)\cdots Y(a^1_{(i)}a^k, x_k)\cdots Y(a^n, x_n)w \big\rangle ,
\end{aligned}$$

where

$$F_{n,i}(x,y) = x^{-n}\partial_y^{(i)}\frac{y^n}{x-y} \in \mathbb{C}[x,y][x^{-1},y^{-1},(x-y)^{-1}]$$

for $n, i \in \mathbb{N}$.

Exercise 6.5.1. Prove the following formula:

$$\sum_{k=1}^{\infty}[a_{(n-1+k)}, Y(b,y)]x^{-n-k} = \sum_{i=0}^{\infty} \iota_{x,y}F_{n,i}(x,y)\cdot Y(a_{(i)}b,y)$$

Exercise 6.5.2. Prove the lemma.

6.6 Eisenstein series

Let $k > 2$ be an even integer and $\tau \in \mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$. The k-th Eisenstein series is given by

$$G_k(\tau) = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} (m\tau+n)^{-k}.$$

Then it absolutely converges to a holomorphic function on $\mathbb{H}$.

Exercise 6.6.1. Show the identity

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau)$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$