MS-EV0004 Vertex operator algebras: Exercise 5

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5.1 Characterization of modules

Sometimes, the following characterization of modules is convenient:

Proposition 5.1. *Let $(V, \mathbf{1}, Y)$ be a vertex algebra. Suppose that a vector space W and a linear map*

$$Y_W(-,x)\colon V \to \mathrm{End}(W)[[x^{\pm 1}]]; \quad Y_W(a,x) = \sum_{n\in\mathbb{Z}} a^W_{(n)} x^{-n-1}$$

satisfy **(VAmod1)**, **(VAmod2)** *and*

- $Y_W(Ta, x) = \frac{d}{dx} Y_W(a,x)$, $a \in V$,
- $[a^W_{(m)}, b^W_{(n)}] = \sum_{k=0}^{\infty} \binom{m}{k} (a_{(k)}b)^W_{(m+n-k)}$, $a, b \in V$, $m, n \in \mathbb{Z}$,
- $Y_W(a_{(-1)}b, x) = {}^{\circ}_{\circ} Y_W(a,x) Y_W(b,x) {}^{\circ}_{\circ}$, $a, b \in V$.

Then, (W, Y_W) is a V-module.

Exercise 5.1.1. Prove Proposition 5.1.

Hint: It suffices to show

$$Y_W(a_{(n)}b, x) = Y_W(a,x)_{(n)} Y_W(b,x), \quad a, b \in V,\ n \in \mathbb{Z}.$$

Notice that the cases of $n \geq -1$ are assumed.

5.2 Action of the Heisenberg algebra on fields

Exercise 5.2.1. Compete the proof of Proposition 8.6.

5.3 Universality of Fock representations

Exercise 5.3.1. Let M be a representation of $\widehat{\mathfrak{h}}$ and assume that there exists $v \in M$ such that

- $\alpha_n v = 0, \quad n > 0$,
- $\alpha_0 v = \lambda v$ with some $\lambda \in \mathbb{C}$,
- $Kv = v$,
- $M = \mathcal{U}(\widehat{\mathfrak{h}})v$.

Show that there is a unique surjective homomorphism

$$\mathcal{F}_\lambda \to M$$

of representations of $\widehat{\mathfrak{h}}$ such that

$$|\lambda\rangle \mapsto v.$$

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Hint: Given a representation M of $\widehat{\mathfrak{h}}$ and a vector $v \in M$, we always get a $\mathcal{U}(\widehat{\mathfrak{h}})$-module homomorphism

$$\mathcal{U}(\widehat{\mathfrak{h}}) \to M; \quad X \mapsto Xv,$$

but when does this factor through $\mathcal{F}_\lambda$?

5.4 Vertex algebra structure on a space of fields

Exercise 5.4.1. Prove Proposition 8.7.

Hint: Apply the reconstruction theorem.

5.5 Modules over the Virasoro vertex algebra

Exercise 5.5.1. Let $M(c,h)$ be the Verma module of central charge c and conformal weight h and

$$L(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2} \in \mathcal{E}(M(c,h))$$

be the generating series of the action of $\mathfrak{vir}$. Show that

$$\begin{aligned} \mathfrak{vir} &\to \mathrm{End}(\mathcal{E}(M(c,h))) \\ L_n &\mapsto L(x)_{(n+1)}, \quad n \in \mathbb{Z}, \\ C &\mapsto c \cdot \mathrm{Id}_{\mathcal{E}(M(c,h))} \end{aligned}$$

gives a representation of $\mathfrak{vir}$ on $\mathcal{E}(M(c,h))$ and

$$L(x)_{(n+1)}\mathrm{Id}_{M(c,h)} = 0, \quad n \geq -1.$$

Exercise 5.5.2. Find a similar property of the Verma module $M(c,h)$ as Exercise 5.3.1. What about V_c?

Let $(V_c, \mathbf{1}_c, Y)$ be the Virasoro vertex algebra of central charge c. By Exercise 5.5.1, we get a unique homomorphism

$$\Phi_{c,h} \colon V_c \to \mathcal{E}(M(c,h))$$

of representations of $\mathfrak{vir}$ such that

$$\Phi_{c,h}(\mathbf{1}_c) = \mathrm{Id}_{M(c,h)}.$$

Let us write

$$\begin{aligned} &\mathcal{E}_0(M(c,h)) \\ =& \mathrm{Im}\Phi_{c,h} \\ =& \mathrm{Span}\{L(x)_{(-n_1)} \cdots L(x)_{(-n_l)}\mathrm{Id}_{M(c,h)} | n_1 \geq \cdots \geq n_l > 0,\, l \in \mathbb{N}\}. \end{aligned}$$

Exercise 5.5.3. Show that there exists a unique vertex algebra

$$(\mathcal{E}_0(M(c,h)), \mathrm{Id}_{M(c,h)}, Y_{\mathcal{E}})$$

such that

$$Y_{\mathcal{E}}\left(L(x)_{(-n_1)} \cdots L(x)_{(-n_l)}\mathrm{Id}_{M(c,h)}, \zeta\right) = {:}\partial^{(n_1-1)}\mathcal{L}(\zeta) \cdots \partial^{(n_l-1)}\mathcal{L}(\zeta){:}$$

for $n_1 \geq \cdots \geq n_l > 0$, where

$$\mathcal{L}(\zeta) = \sum_{n\in\mathbb{Z}} L(x)_{(n)}\zeta^{-n-1} \in \mathcal{E}(\mathcal{E}_0(M(c,h))).$$

Exercise 5.5.4. Define a linear map

$$Y_{M(c,h)}(-,x)\colon V_c \to \mathrm{End}(M(c,h))[[x^{\pm 1}]]$$

by

$$Y_{M(c,h)}(L_{-n_1}\cdots L_{-n_l}\mathbf{1}_c, x) = {}^{\circ}_{\circ}\, \partial^{(n_1-2)}L(x)\cdots\partial^{(n_l-2)}L(x)\, {}^{\circ}_{\circ}$$

for $n_1 \geq \cdots \geq n_l \geq 2$. Show that $(M(c,h), Y_{M(c,h)})$ is a V_c-module.