MS-EV0004 Vertex operator algebras: Exercise 3

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3.1 Expansion homomorphism

We introduced (Section 5.1.1) the ring

$$\mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}]$$

as the localization of $\mathbb{C}[[x_1, x_2]]$ by the multiplicatively closed set $\{x_1^k x_2^l (x_1 - x_2)^m\}_{k,l,m\geq 0}$. The expansion homomorphism

$$\iota_{12} \colon \mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}] \to \mathbb{C}((x_1))((x_2))$$

is defined by

$$\iota_{12} \colon \frac{g(x_1, x_2)}{x_1^k x_2^l (x_1 - x_2)^m} \mapsto g(x_1, x_2) x_1^{-k} x_2^{-l} \sum_{n=0}^{\infty} \binom{-m}{n} x_1^{-m-n} x_2^n$$

for $g(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ and $k, l, m \geq 0$.

Exercise 3.1.1. Show that ι_{12} is a well-defined injective homomorphism of rings.

Exercise 3.1.2. For

$$q(x_0, x_2) \in \mathbb{C}[[x_0, x_2]][x_0^{-1}, x_2^{-1}, (x_0 + x_2)^{-1}],$$

show the identity

$$(\iota_{02} q(x_0, x_2))|_{x_0 = x_1 - x_2} = \iota_{12} q(x_1 - x_2, x_2)$$

in $\mathbb{C}((x_1))((x_2))$.

3.2 From Jacobi identity to (VA3) and (VA4)

For a triple $(V, \mathbf{1}, Y)$ of

(1) V: vector space

(2) $\mathbf{1} \in V$: distinguished vector

(3) $Y(-, x) \colon V \to \mathrm{End}(V)[[x^{\pm 1}]]$: linear map,

assume that **(VA1)**, **(VA2)** and the Jacobi identities

$$\begin{aligned} & x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(a, x_1) Y(b, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(b, x_2) Y(a, x_1) \\ =& x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(a, x_0) b, x_2) \end{aligned}$$

for all $a, b \in V$ are satisfied.

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Exercise 3.2.1. Define $T \in \mathrm{End}(V)$ by $T(a) := a_{(-2)}\mathbf{1} = [x](Y(a,x)\mathbf{1})$, $a \in V$ and show the identity

$$Y(T(a),x) = \frac{d}{dx}Y(a,x), \quad a \in V.$$

Exercise 3.2.2. With the same $T \in \mathrm{End}(V)$, show the identity (skew-symmetry)

$$Y(a,x)b = e^{xT}Y(b,-x)a, \quad a,b \in V.$$

Remark 3.1. In the lecture, we derived the skew-symmetry assuming **(VA3)** and **(VA4)**, but we cannot use them here.

Hint1: The left-hand side of the Jacobi identity is invariant under exchanging $(a,b;x_0,x_1,x_2)$ and $(b,a;-x_0,x_2,x_1)$, which could relate $Y(a,x_0)b$ and $Y(b,-x_0)a$ in the right-hand sides.

Hint2: $x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)$ replaces x_1 with x_2+x_0 if the result makes sense.

Hint3: From the previous problem, $Y(a,x+y) = Y(e^{yT}a,x)$.

Exercise 3.2.3. Show that **(VA3)** is satisfied, i.e., the identity

$$[T,Y(a,x)] = \frac{d}{dx}Y(a,x), \quad a \in V$$

holds

Hint: Differentiate both sides of the skew-symmetry.

Exercise 3.2.4. Show that **(VA4)** is satisfied, i.e., for any $a,b \in V$, there is $N \gg 0$ such that

$$(x_1-x_2)^N[Y(a,x_1),Y(b,x_2)] = 0.$$

3.3 Goddard's uniqueness theorem

Theorem. Let $(V,\mathbf{1},Y)$ be a vertex algebra. If a field $A(x) \in \mathcal{E}(V)$ on V satisfies

$$A(x)\mathbf{1} = Y(a,x)\mathbf{1}$$

for some $a \in V$ and mutually local with all $Y(b,x)$, $b \in V$, we have $A(x) = Y(a,x)$.

Exercise 3.3.1. Prove the theorem.

Hint: The first step would be applying

$$(x-y)^N A(x)Y(b,y) = (x-y)^N Y(b,y)A(x)$$

with a large enough N (depending on $b \in V$) to $\mathbf{1}$.

3.4 Vacuum property of the n-th product

Let V be a vector space and $\mathbf{1} \in V$ be a distinguished vector. Assume that the fields

$$a^i(x) = \sum_{n\in\mathbb{Z}} a^i_{(n)} x^{-n-1} \in \mathcal{E}(V), \quad i = 1, \ldots, l$$

on V satisfy $a^i_{(n)}\mathbf{1} = 0$ if $n \geq 0$.

Exercise 3.4.1. Show that, for $n_1, \ldots, n_l \in \mathbb{Z}$,

$$a^1(x)_{(n_1)} \left(\cdots (a^l(x)_{(n_l)} \mathrm{Id}_V) \cdots \right) \mathbf{1} \in a^1_{(n_1)} \cdots a^l_{(n_l)} \mathbf{1} + V[[x]]x.$$

3.5 Alternative answer to Exercise 2.3.1

Let $\mathfrak{g}$ be a Lie algebra. A linear map $d \in \mathrm{End}(\mathfrak{g})$ is said to be a Lie derivation if it satisfies

$$d([X,Y]) = [d(X), Y] + [X, d(Y)], \quad X, Y \in \mathfrak{g}.$$

Exercise 3.5.1. Discuss that d naturally induces a derivation on $\mathcal{U}(\mathfrak{g})$.

For a Lie algebra $\overline{\mathfrak{g}}$ and a symmetric invariant[1] bilinear form

$$(\cdot|\cdot) \colon \overline{\mathfrak{g}} \times \overline{\mathfrak{g}} \to \overline{\mathfrak{g}},$$

the associated affine Lie algebra is

$$\mathfrak{g} = \overline{\mathfrak{g}}[t^{\pm 1}] \oplus \mathbb{C}K$$

along with

$$\begin{aligned} [Xt^m, Yt^n] &= [X,Y]t^{m+n} + m(X|Y)\delta_{m+n,0}K, \quad X, Y \in \overline{\mathfrak{g}},\ m, n \in \mathbb{Z}, \\ [K, \mathfrak{g}] &= \{0\}. \end{aligned}$$

Exercise 3.5.2. Find a Lie algebra $\overline{\mathfrak{g}}$ and $(\cdot|\cdot)$ such that the associated affine Lie algebra is isomorphic to the Heisenberg algebra $\widehat{\mathfrak{h}}$.

Exercise 3.5.3. Show that $d \in \mathrm{End}(\mathfrak{g})$ given by

$$d(Xt^m) = -mXt^{m-1} \quad (X \in \overline{\mathfrak{g}},\ m \in \mathbb{Z}), \quad d(K) = 0$$

is a Lie derivation.

The vacuum representation of level $k \in \mathbb{C}$ of the affine Lie algebra $\mathfrak{g}$ is as follows:

$$V_k(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/\mathcal{I}_{k,0},$$

where

$$\mathcal{I}_{k,0} = \mathcal{U}(\mathfrak{g}) \cdot \overline{\mathfrak{g}}[t] + \mathcal{U}(\mathfrak{g}) \cdot (K - k).$$

Exercise 3.5.4. Discuss that the Lie derivation d of the affine Lie algebra $\mathfrak{g}$ naturally induces an operator $T \in \mathrm{End}(V_k(\mathfrak{g}))$ such that

$$[T, Xt^m] = -mXt^{m-1}, \quad X \in \overline{\mathfrak{g}},\ m \in \mathbb{Z}$$

[1] $([X,Y]|Z) + (Y|[X,Z]) = 0$ for all $X, Y, Z \in \overline{\mathfrak{g}}$.